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FULL LENGTH PAPER

Calmness modulus of fully perturbed linear programs

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Abstract This paper provides operative point-based formulas (only involving the nominal data, and not data in a neighborhood) for computing or estimating the calmness modulus of the optimal set (argmin) mapping in linear optimization under uniqueness of nominal optimal solutions. Our analysis is developed in two different parametric settings. First, in the framework of canonical perturbations (i.e., perturbations of the objective function and the right-hand-side of the constraints), the paper provides a computationally tractable formula for the calmness modulus, which goes beyond some preliminary results of the literature. Second, in the framework of full perturbations (perturbations of all coefficients), after characterizing the calmness property for the optimal set mapping, the paper provides an operative upper bound for the corresponding calmness modulus, as well as some illustrative examples. We provide

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two applications related to algorithms traced out from the literature: the first one to a descent method in LP, and the second to a regularization method for linear programs with complementarity constraints.

Keywords Variational analysis · Calmness · Linear programming · Calmness modulus · Descent methods · Complementarity

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1 Introduction

The present paper is focussed on quantifying the stability of (finite) linear optimization problems, through the analysis of the *calmness* property of the *optimal set mapping* (also called *argmin mapping*) and the computation or estimation of the corresponding *calmness modulus*. Our linear optimization problem is expressed in the form

$$\begin{array}{lll} P(c,a,b): & \text{minimize} & c'x \\ & \text{subject to} & a_t'x \le b_t, \quad t \in T := \{1,2,\ldots,m\}, \end{array} \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables, and $c \in \mathbb{R}^n,\ a \equiv (a_t)_{t\in T} \in (\mathbb{R}^n)^T$, and $b \equiv (b_t)_{t\in T} \in \mathbb{R}^T$ are the problem's data. All elements in $\mathbb{R}^n$ are regarded as column-vectors and y' denotes the transpose of $y \in \mathbb{R}^n$.

We consider two parameterized families of linear optimization problems: the first one, $\left\{P(c,\overline{a},b) : (c,b) \in \mathbb{R}^n \times \mathbb{R}^T\right\}$, corresponds to the framework of *canonical perturbations*; i.e., perturbations fall on the objective function coefficient vector, c, together with the right-hand-side of the constraints, b, while the left hand side, $\overline{a} \equiv (\overline{a}_t)_{t\in T}$, is considered to be fixed at its nominal value. The second family, which corresponds to the context of perturbations of *all data*—also called *full* perturbations—is of the form $\{P(c,a,b) : (c,a,b) \in \mathbb{R}^n \times (\mathbb{R}^n)^T \times \mathbb{R}^T\}$. Associated with this second family, we consider the corresponding *optimal set mapping*, $\mathcal{S} : \mathbb{R}^n \times (\mathbb{R}^n)^T \times \mathbb{R}^T \rightrightarrows \mathbb{R}^n$, defined by

$$\mathcal{S}(c,a,b) := \left\{x \in \mathbb{R}^n \mid x \text{ is an optimal solution of } P(c,a,b)\right\}.$$

Here, the parameter space $\mathbb{R}^n \times (\mathbb{R}^n)^T \times \mathbb{R}^T$ is endowed with the norm

$$\|(c,a,b)\| := \max\{\|c\|_*, \|(a,b)\|_\infty\}, \tag{2}$$

where $\mathbb{R}^n$ is equipped with an arbitrary norm, $\|\cdot\|$, with *dual norm* given by $\|u\|_* = \max_{\|x\|\le 1}\left|u'x\right|$, and $\|(a,b)\|_\infty := \max_{t\in T}\left\|\binom{a_t}{b_t}\right\|$, where

$$\left\|\begin{pmatrix} a_t \\ b_t \end{pmatrix}\right\| = \max\{\|a_t\|_*, |b_t|\}.$$

When confined to the particular case of canonical perturbations, the associated optimal set mapping $\mathcal{S}_{\overline{a}} : \mathbb{R}^n \times \mathbb{R}^T \rightrightarrows \mathbb{R}^n$ is given by

$$\mathcal{S}_{\overline{a}}(c, b) = \mathcal{S}(c, \overline{a}, b), \quad \text{for all } (c, b) \in \mathbb{R}^n \times \mathbb{R}^T,$$

with $\|(c, b)\| := \max\{\|c\|_*, \max_{t\in T}|b_t|\}$.

This paper provides, in Theorem 3.1, an operative formula for the calmness modulus of $\mathcal{S}_{\overline{a}}$ under uniqueness of the nominal optimal solution by combining some results traced out from [5] and [7]. However, the main difficulties tackled in the paper are related to the context of perturbations of all data. At this moment we point out that mapping $\mathcal{S}_{\overline{a}}$ is always calm at any point of its graph, as a consequence of a classical result by Robinson [25], since the Karush–Kunh–Tucker conditions allow us to express the graph of $\mathcal{S}_{\overline{a}}$ as a finite union of polyhedral sets. This is no longer the case for $\mathcal{S}$ in the framework of perturbations of all data. In this last framework, and under the assumption of the uniqueness of nominal optimal solution, namely $\mathcal{S}\left(\overline{c}, \overline{a}, \overline{b}\right) = \{\overline{x}\}$, which will stand throughout the paper, a characterization of the calmness of $\mathcal{S}$ at $\left(\left(\overline{c}, \overline{a}, \overline{b}\right), \overline{x}\right)$ can be traced out from two results by Robinson [24]. Theorem 4.1 points out this characterization and adds a new equivalent condition of geometrical type, which is stated in terms of the nominal data. Moreover, the paper establishes in Theorem 4.2, an operative upper bound for the corresponding calmness modulus, again exclusively in terms of the nominal data.

In the next paragraphs we recall some definitions related to a generic mapping $\mathcal{M} : Y \rightrightarrows X$ between metric spaces (with distances denoted indistinctly by d). $\mathcal{M}$ is said to be *calm* at $(\overline{y}, \overline{x}) \in \mathrm{gph}\mathcal{M}$ (the graph of $\mathcal{M}$) if there exist a constant $\kappa \geq 0$ and neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that

$$d(x, \mathcal{M}(\overline{y})) \leq \kappa d(y, \overline{y}) \tag{3}$$

whenever $x \in \mathcal{M}(y) \cap U$ and $y \in V$; where, as usual, $d(x, \Omega)$ is defined as $\inf\{d(x, z) \mid z \in \Omega\}$ for $\Omega \subset \mathbb{R}^n$, and $d(x, \emptyset) := +\infty$.

It is well-known that the calmness of $\mathcal{M}$ at $(\overline{y}, \overline{x})$ is equivalent to the *metric subregularity* of $\mathcal{M}^{-1}$ at $(\overline{x}, \overline{y})$ (see, for instance, [9, Theorem 3H.3 and Exercise 3H.4]). Recall that $\mathcal{M}^{-1}$ (given by $y \in \mathcal{M}^{-1}(x) \Leftrightarrow x \in \mathcal{M}(y)$) is *metrically subregular* at $(\overline{x}, \overline{y})$ if there exist a constant $\kappa \geq 0$ and a (possibly smaller) neighborhood U of $\overline{x}$ such that

$$d(x, \mathcal{M}(\overline{y})) \leq \kappa d\left(\overline{y}, \mathcal{M}^{-1}(x)\right), \quad \text{for all } x \in U. \tag{4}$$

The infimum of those $\kappa \geq 0$ for which (3)—or (4)—holds (for some associated neighborhoods) is called the *calmness modulus* of $\mathcal{M}$ at $(\overline{y}, \overline{x})$ and it is denoted by $\mathrm{clm}\mathcal{M}(\overline{y}, \overline{x})$. The case when $\mathcal{M}$ is not calm at $(\overline{y}, \overline{x})$ corresponds to $\mathrm{clm}\mathcal{M}(\overline{y}, \overline{x}) = +\infty$.

For comparative purposes, recall that $\mathcal{M}$ satisfies the *Aubin* property (also called pseudo-Lipschitz or Lipschitz-like) at $(\overline{y}, \overline{x})$ when (3)—or (4)—is valid when replacing $\overline{y}$ with an arbitrary $\widetilde{y}$ in some neighborhood V of $\overline{y}$. The corresponding infimum of all κ's is then called Lipschitz modulus and denoted by $\mathrm{lip}\mathcal{M}(\overline{y}, \overline{x})$. Obviously,

$$\operatorname{clm}\mathcal{M}\left(\overline{y},\overline{x}\right)\leq\operatorname{lip}\mathcal{M}\left(\overline{y},\overline{x}\right). \tag{5}$$

Calmness and Aubin properties play an important role in relation to issues from optimization (theory and algorithms). Comprehensive studies of these properties can be traced out from the monographs [9,17,22,27]. One can find in the literature deep contributions to the analysis of calmness for constraint systems in the context of canonical perturbations (see, e.g., [10,13,18,19]). The reader is addressed to [1,20] for the analysis of this property in relation to *local error bounds*. Subdifferential approaches to calmness/local error bounds can be found in [1,12,14,20].

The structure of the paper is as follows: Sect. 2 provides the necessary notation and preliminary results. In Sect. 3, by assembling [5, Proposition 4.1] and [7, Corollary 8], we obtain for the first time a formula for the calmness modulus of $\mathcal{S}_{\overline{a}}$ exclusively in terms of the nominal data $\overline{c}$, $\overline{a}$, $\overline{b}$, and $\overline{x}$ (see Theorem 3.1). This formula has a clear geometrical flavor, as Examples 3.1 and 3.2 show. Moreover, a comparative analysis between calmness and Lipschitz moduli is carried out. Section 4 is concerned with the framework of full perturbations, and tackles, in a first stage, the characterization of the calmness property of $\mathcal{S}$ at a given point of its graph, again under the assumption of uniqueness of nominal optimal solution by using some ideas of Robinson [24]. In a second stage, Sect. 4 provides an upper bound on the calmness modulus of $\mathcal{S}$, as well as some examples showing that this upper bound may be attained or not. Example 4.1 turns out to be particularly technical. In order to preserve the rhythm of the paper, a sketch of these technicalities is given as an "Appendix" in Sect. 6. Nevertheless, these details have their own interest as far as they show some perturbation strategies underlying the referred upper bound, and may be used in future research to investigate in which cases the upper bound becomes the exact modulus. Finally, Sect. 5 provides two applications related to certain algorithms traced out from [18] (Sect. 5.1) and [15] (Sect. 5.2). The first one concerns a descent method in linear programming, and the second refers to a regularization method for linear programs with complementarity constraints.

2 Preliminaries

In this section we introduce some additional notation and preliminary results which are needed later on. Given $X \subset \mathbb{R}^k$, $k \in \mathbb{N}$, we denote by $\operatorname{conv}X$ and $\operatorname{cone}X$ the *convex hull* and the *conical convex hull* of X, respectively. It is assumed that $\operatorname{cone}X$ always contains the zero-vector 0_k, in particular $\operatorname{cone}(\emptyset) = \{0_k\}$. If X is a subset of any topological space, $\operatorname{int}X$, $\operatorname{cl}X$ and $\operatorname{bd}X$ stand, respectively, for the interior, the closure, and the boundary of X.

We begin this section with a proposition which comes straightforwardly from [8, Theorem 5]. This result allows us to develop Sect. 4 under perturbations of *all* parameters, using as a starting point some results given in Sect. 3 in the framework of canonical perturbations. From now on, we denote by $\mathcal{F}_{\overline{a}}$ and $\mathcal{F}$ the *feasible set mappings* corresponding, respectively, to the settings of canonical perturbations and perturbations of all data. Formally,

$$\mathcal{F}_{\overline{a}}\left(b\right):=\mathcal{F}\left(\overline{a},b\right),\quad b\in\mathbb{R}^{T},$$

where $\mathcal{F} : (\mathbb{R}^n)^T \times \mathbb{R}^T \rightrightarrows \mathbb{R}^n$ is given by

$$\mathcal{F}(a, b) := \left\{x \in \mathbb{R}^n \mid a_t' x \leq b_t, \quad t \in T\right\}, \quad (a, b) \in \left(\mathbb{R}^n\right)^T \times \mathbb{R}^T.$$

Proposition 2.1 (see [8, Theorem 5]) *Let* $(\left(\overline{a}, \overline{b}\right), \overline{x}) \in \text{gph}\mathcal{F}$. *Then*

$$\text{clm}\mathcal{F}(\left(\overline{a}, \overline{b}\right), \overline{x}) = (\|\overline{x}\| + 1)\, \text{clm}\mathcal{F}_{\overline{a}}(\overline{b}, \overline{x}).$$

Remark 2.1 As a consequence of the previous proposition, the involved calmness moduli are both finite (i.e., both mappings are calm at the corresponding points of their graphs) due to the finiteness of T, since the calmness modulus in the right-hand-side is finite according to the above mentioned result by Robinson [25] (because $\mathcal{F}_{\overline{a}}$ has a polyhedral graph). Observe that gph$\mathcal{F}$ may not be written as a finite union of polyhedral sets (just consider the case of a single inequality in $\mathbb{R}$), so that the calmness of $\mathcal{F}$ at $(\left(\overline{a}, \overline{b}\right), \overline{x})$ does not follow from the aforementioned result. In summary, at this moment we know that mappings $\mathcal{F}_{\overline{a}}$, $\mathcal{F}$, and $\mathcal{S}_{\overline{a}}$ are calm at any point of their grahps. We will see in Sect. 4 that this is not the case for $\mathcal{S}$.

Throughout the paper, we appeal to the *set of active indices* at $x \in \mathcal{F}(a, b)$, denoted by $T_{a,b}(x)$ and defined as

$$T_{a,b}(x) := \left\{t \in T \mid a_t' x = b_t\right\}.$$

The next result follows directly from [5, Proposition 4.1] and constitutes a key tool in the present paper since it provides a point-based expression (i.e., just involving the nominal elements, and not elements in a neighborhood) for $\text{clm}\mathcal{F}_{\overline{a}}(\overline{b}, \overline{x})$ assuming $\mathcal{F}\left(\overline{a}, \overline{b}\right) = \{\overline{x}\}$. Such an assumption may seem too restrictive, but it is not so when applied to mappings $\mathcal{L}_D$ defined later. The reader can easily check that the assumption $\mathcal{F}\left(\overline{a}, \overline{b}\right) = \{\overline{x}\}$ entails $0_n \in \text{int conv}\left\{\overline{a}_t, t \in T_{\overline{a},\overline{b}}(\overline{x})\right\}$, since otherwise the separation theorem would provide a nonzero feasible direction of $\mathcal{F}\left(\overline{a}, \overline{b}\right)$ at $\overline{x}$.

Proposition 2.2 [5, Proposition 4.1] *Let* $\left(\overline{a}, \overline{b}\right) \in (\mathbb{R}^n)^T \times \mathbb{R}^T$ *and assume* $\mathcal{F}\left(\overline{a}, \overline{b}\right) = \{\overline{x}\}$. *Then*

$$\text{clm}\mathcal{F}_{\overline{a}}(\overline{b}, \overline{x}) = \frac{1}{d_*\left(0_n, \text{bd conv}\left\{\overline{a}_t, t \in T_{\overline{a},\overline{b}}(\overline{x})\right\}\right)},$$

where d_* stands for the distance in $\mathbb{R}^n$ associated with $\|\cdot\|_*$.

Remark 2.2 More in detail, the previous expression for $\text{clm}\mathcal{F}_{\overline{a}}(\overline{b}, \overline{x})$ comes from [20, Theorem 1] and [5, Theorem 3.1] (which are the basis for the referred [5, Proposition 4.1]). The first of these results provides a subdifferential approach to the computation of local error bounds (closely related to calmness moduli) and the second establishes a key result for deriving a point-based formula; specifically, [5, Theorem 3.1] states, for any convex finite function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ at any given $\overline{x} \in \mathbb{R}^n$,

$$\text{bd}\partial f(\overline{x}) = \limsup_{x \to \overline{x},\, x \neq \overline{x}} \partial f(x).$$

For the sake of simplicity, from now on we abbreviate our nominal parameter as $\overline{p}$; i.e.,

$$\overline{p} := \left(\overline{c}, \overline{a}, \overline{b}\right) \in \mathbb{R}^n \times \left(\mathbb{R}^n\right)^T \times \mathbb{R}^T.$$

The next proposition comes directly from [7, Corollary 8] and constitutes our starting point in Sect. 3. In it, associated with a given $(\overline{p}, \overline{x}) \in \mathrm{gph}\mathcal{S}$, we appeal to the following family of index subsets associated with the Karush–Kuhn–Tucker (KKT) conditions (hereafter referred to as *KKT index sets*)

$$\mathcal{K}_{\overline{p}}\left(\overline{x}\right) = \left\{ D \subset T_{\overline{a},\overline{b}}\left(\overline{x}\right) \middle| \, |D| \leq n \text{ and } -\overline{c} \in \operatorname{cone}\left\{\overline{a}_t, \quad t \in D\right\} \right\},$$

where $|D|$ stands for the cardinality of D and condition $|D| \leq n$ comes from Carathéodory's Theorem. For any $D \in \mathcal{K}_{\overline{p}}\left(\overline{x}\right)$ we consider the mapping $\mathcal{L}_D : (\mathbb{R}^n)^T \times \mathbb{R}^T \times (\mathbb{R}^n)^D \times \mathbb{R}^D \rightrightarrows \mathbb{R}^n$ given by

$$\mathcal{L}_D\left(a, b, u, d\right) := \left\{x \in \mathbb{R}^n \mid a_t' x \leq b_t,\ t \in T;\ u_t' x \leq d_t,\ t \in D\right\}, \tag{6}$$

and, using the notation $\overline{a}_D = (\overline{a}_t)_{t \in D}$, $\overline{b}_D = \left(\overline{b}_t\right)_{t \in D}$, we also define

$$\mathcal{L}_{D,\overline{a},-\overline{a}_D}\left(b, d\right) := \mathcal{L}_D\left(\overline{a}, b, -\overline{a}_D, d\right) \quad \text{for } (b, d) \in \mathbb{R}^T \times \mathbb{R}^D. \tag{7}$$

Observe that all preliminary results for feasible set mappings $\mathcal{F}$ and $\mathcal{F}_{\overline{a}}$ may be specified for $\mathcal{L}_D$ and $\mathcal{L}_{D,\overline{a},-\overline{a}_D}$, respectively, which are nothing else but feasible set mappings associated with enlarged systems.

Proposition 2.3 [7, Corollary 8] *Let* $\overline{p} = \left(\overline{c}, \overline{a}, \overline{b}\right) \in \mathbb{R}^n \times (\mathbb{R}^n)^T \times \mathbb{R}^T$ *and assume* $\mathcal{S}\left(\overline{p}\right) = \{\overline{x}\}$. *Then*

$$\mathrm{clm}\mathcal{S}_{\overline{a}}(\left(\overline{c}, \overline{b}\right), \overline{x}) = \max_{D \in \mathcal{K}_{\overline{p}}(\overline{x})} \mathrm{clm}\mathcal{L}_{D,\overline{a},-\overline{a}_D}\left(\left(\overline{b}, -\overline{b}_D\right), \overline{x}\right).$$

Remark 2.3 Observe that $\mathcal{L}_D\left(\overline{a}, \overline{b}, -\overline{a}_D, -\overline{b}_D\right)$ is the set of KKT points of problem $P\left(\overline{c}, \overline{a}, \overline{b}\right)$ associated with D as the KKT index set. Under our current assumption $\mathcal{S}\left(\overline{p}\right) = \{\overline{x}\}$, we have

$$\mathcal{L}_D\left(\overline{a}, \overline{b}, -\overline{a}_D, -\overline{b}_D\right) = \{\overline{x}\} \quad \text{for all } D \in \mathcal{K}_{\overline{p}}\left(\overline{x}\right).$$

3 Calmness modulus versus Lipschitz modulus under canonical perturbations

The following theorem provides the announced expression for $\mathrm{clm}\mathcal{S}_{\overline{a}}$, only involving the nominal point $\overline{x}$ and the nominal problem's data $\left(\overline{c}, \overline{a}, \overline{b}\right)$.

Theorem 3.1 *Let* $\overline{p} = \left(\overline{c}, \overline{a}, \overline{b}\right) \in \mathbb{R}^n \times (\mathbb{R}^n)^T \times \mathbb{R}^T$ *and assume* $\mathcal{S}(\overline{p}) = \{\overline{x}\}$. *Then*

$$\operatorname{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) = \max_{D \in \mathcal{K}_{\overline{p}}(\overline{x})} \frac{1}{d_*\left(0_n, \operatorname{bd}\operatorname{conv}\left\{\overline{a}_t,\ t \in T_{\overline{a},\overline{b}}(\overline{x})\,;\, -\overline{a}_t,\ t \in D\right\}\right)}.$$

Proof The result follows by combining Propositions 2.2 and 2.3, and Remark 2.3. □

Remark 3.1 For problem (1), given any $(\overline{p}, \overline{x}) \in \operatorname{gph}\mathcal{S}$ without requiring $\mathcal{S}(\overline{p}) = \{\overline{x}\}$, and denoting $\mathcal{S}_{\overline{c},\overline{a}}(b) := \mathcal{S}_{\overline{a}}(\overline{c}, b)$ (which equals $\mathcal{S}(\overline{c}, \overline{a}, b)$) for $b \in \mathbb{R}^T$, [7, Theorem 7] establishes

$$\operatorname{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) = \operatorname{clm}\mathcal{S}_{\overline{c},\overline{a}}\left(\overline{b}, \overline{x}\right);$$

i.e., perturbations of $\overline{c}$ are negligible when computing the calmness modulus of $\mathcal{S}_{\overline{a}}$ at $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$, and, therefore, only perturbations of $\overline{b}$ are needed.

For comparative purposes, in the following proposition we recall the expression of $\operatorname{lip}\mathcal{S}_{\overline{a}}(\left(\overline{c}, \overline{b}\right), \overline{x})$, provided that it is finite, where we use the notation

$$\mathcal{T}_{\overline{p}}(\overline{x}) = \left\{D \in \mathcal{K}_{\overline{p}}(\overline{x}) \,\|D| = n \text{ and } A_D \text{ is nonsingular}\right\},$$

with A_D denoting the matrix whose rows are $\overline{a}_t'$, $t \in D$ (given in some prefixed order).

Remark 3.2 The assumption $\mathcal{S}(\overline{p}) = \{\overline{x}\}$ entails $-\overline{c} \in \operatorname{int}\operatorname{cone}\{\overline{a}_t,\ t \in T_{\overline{a},\overline{b}}(\overline{x})\}$, which implies $\mathcal{T}_{\overline{p}}(\overline{x}) \neq \emptyset$. The same assumption also implies that 0_n belongs to $\operatorname{int}\operatorname{cone}\{\overline{a}_t,\ t \in T_{\overline{a},\overline{b}}(\overline{x});\ \overline{c}\}$, which is contained in $\operatorname{int}\operatorname{cone}$ $\{\overline{a}_t,\ t \in T_{\overline{a},\overline{b}}(\overline{x});\ -\overline{a}_t,\ t \in D\}$ for any $D \in \mathcal{K}_{\overline{p}}(\overline{x})$, and, consequently,

$$0_n \in \operatorname{int}\operatorname{conv}\left\{\overline{a}_t,\ t \in T_{\overline{a},\overline{b}}(\overline{x})\,;\, -\overline{a}_t,\ t \in D\right\} \textit{ for all } D \in \mathcal{K}_{\overline{p}}(\overline{x}).$$

Accordingly, the denominator appearing in Theorem 3.1 is always positive.

We also appeal to the following concepts:

- The *Slater constraint qualification* (SCQ) holds at parameter $\left(\overline{a}, \overline{b}\right) \in (\mathbb{R}^n)^T \times \mathbb{R}^T$ if there exists $\widehat{x} \in \mathbb{R}^n$ (called Slater point) such that $\overline{a}_t'\widehat{x} < \overline{b}_t$ for all $t \in T$.
- The *Nürnberger condition* holds at $(\left(\overline{c}, \overline{b}\right), \overline{x}) \in \operatorname{gph}\mathcal{S}_{\overline{a}}$ if the SCQ verifies at $\left(\overline{a}, \overline{b}\right)$ and
$$\mathcal{T}_{\overline{p}}(\overline{x}) = \mathcal{K}_{\overline{p}}(\overline{x}).$$

Proposition 3.1 *Let* $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) \in \operatorname{gph}\mathcal{S}_{\overline{a}}$. *Then* $\mathcal{S}_{\overline{a}}$ *satisfies the Aubin property at* $(\left(\overline{c}, \overline{b}\right), \overline{x})$, *i.e.,* $\operatorname{lip}\mathcal{S}_{\overline{a}}(\left(\overline{c}, \overline{b}\right), \overline{x}) < +\infty$, *if and only if the Nürnberger condition holds at* $(\left(\overline{c}, \overline{b}\right), \overline{x})$. *In this case,*

$$\operatorname{lip}\mathcal{S}_{\overline{a}}(\left(\overline{c}, \overline{b}\right), \overline{x}) = \max_{D \in \mathcal{T}_{\overline{p}}(\overline{x})} \left\|A_D^{-1}\right\|. \tag{8}$$

Remark 3.3 The previous characterization of the Aubin property for $\mathcal{S}_{\overline{a}}$ can be found in [6, Theorem 16], although the name 'Nürnberger condition' appeared for the first time in [4] (extended to the convex case). It can be easily seen that the Nürnberger condition at $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$ entails $\mathcal{S}(\overline{p}) = \{\overline{x}\}$. Expression (8) comes from [3, Corollary 2]. For $D \in \mathcal{T}_{\overline{p}}(\overline{x})$, we can identify matrix A_D with the 'endomorphism' $\mathbb{R}^n \ni x \mapsto A_D x \in \mathbb{R}^D$, where $\mathbb{R}^n$ is equipped with an arbitrary norm $\|\cdot\|$ and $\mathbb{R}^D$ is endowed with the supremum norm $\|\cdot\|_\infty$. For our choice of norms we have

$$\left\| A_D^{-1} \right\| := \max_{\|y\|_\infty \leq 1} \left\| A_D^{-1} y \right\| = \frac{1}{d_* \left(0_n, \operatorname{bd} \operatorname{conv} \left\{ \pm \overline{a}_t, t \in D \right\}\right)}, \tag{9}$$

where the last equality is a straightforward consequence of [2, Corollary 3.2] together with the fact that $\left\| A_D^{-1} \right\|$ coincides with the Lipschitz modulus of A_D^{-1} at any point of its graph. Moreover, with the only assumption that $\mathcal{S}(\overline{p}) = \{\overline{x}\}$, [7, Theorem 13] shows that

$$\operatorname{clm} \mathcal{S}_{\overline{a}} \left(\left(\overline{c}, \overline{b}\right), \overline{x} \right) \leq \max_{D \in \mathcal{T}_{\overline{p}}(\overline{x})} \left\| A_D^{-1} \right\| \tag{10}$$

without requiring the Nürnberger condition; i.e., the right-hand-side of (10) is finite and still constitutes an upper bound on the calmness modulus when the Lipschitz modulus is infinite.

The next example comes from [7, Example 2] and shows that inequality (10) may be strict even under the Nürnberger condition. In [7, Example 2], ad hoc arguments were used to obtain $\operatorname{clm} \mathcal{S}_{\overline{a}} \left(\left(\overline{c}, \overline{b}\right), \overline{x} \right)$. Now Theorem 3.1 provides a direct way to compute that modulus, as the subsequent figure (Fig. 1) illustrates.

Example 3.1 Consider the linear optimization problem $P\left(\overline{c}, \overline{a}, \overline{b}\right)$, in $\mathbb{R}^2$ endowed with the Euclidean norm,

$$\begin{array}{lll} \text{minimize} & x_1 + \frac{1}{3} x_2 & \\ \text{subject to} & -x_1 \leq 0, & (t = 1) \\ & -x_1 - \frac{1}{2} x_2 \leq 0, & (t = 2) \\ & -x_1 - x_2 \leq 0, & (t = 3) \\ & -x_1 + x_2 \leq 0, & (t = 4) \end{array}$$

whose unique optimal solution is $\overline{x} = 0_2$, and where

$$\mathcal{K}_{\overline{p}}(\overline{x}) = \mathcal{T}_{\overline{p}}(\overline{x}) = \{\{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}\}.$$

According to Theorem 3.1, the reader can easily check that the corresponding maximum over $D \in \mathcal{K}_{\overline{p}}(\overline{x})$ is attained at both $D = \{1, 2\}$ and $D = \{1, 3\}$, and therefore $\operatorname{clm} \mathcal{S}_{\overline{a}} \left(\left(\overline{c}, \overline{b}\right), \overline{x} \right)$ coincides, if for instance we choose $D = \{1, 2\}$, with

$$\frac{1}{d_* \left(0_2, \operatorname{bd} \operatorname{conv} \left\{ \pm \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} -1 \\ -1/2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} \right)} = \sqrt{5}.$$

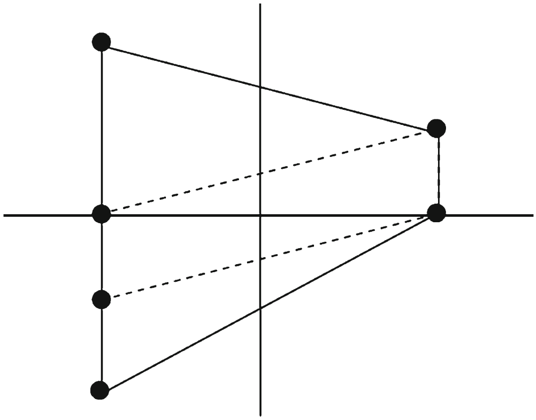

Fig. 1 Illustration of Example 3.1

The previous figure illustrates this example. We can see that the distance in the previous denominator is attained at $(1/5, -2/5)'$ and equals $1/\sqrt{5}$. In the same figure we can also check that the distance from the origin to the segments with discontinuous trace is $1/\sqrt{17}$, which coincides with $\left\| A_{\{1,2\}}^{-1} \right\|^{-1}$ according to (9). Hence,

$$\mathrm{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) = \sqrt{5} < \mathrm{lip}\mathcal{S}_{\overline{a}}\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) = \sqrt{17}.$$

The following example can be traced out from [6, Example 6]. In this example we can see that $\mathrm{lip}\mathcal{S}_{\overline{a}}\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) = +\infty$. Now we are able to compute the calmness modulus.

Example 3.2 Consider $P\left(\overline{c}, \overline{a}, \overline{b}\right)$, in $\mathbb{R}^2$ endowed with the Euclidean norm,

$$\begin{array}{lll} \text{minimize} & x_1 & \\ \text{subject to} & -x_1 + x_2 \leq 0, & (t = 1) \\ & -x_1 - x_2 \leq 0, & (t = 2) \\ & -x_1 \leq 0, & (t = 3) \end{array}$$

whose unique optimal solution is $\overline{x} = 0_2$, and where

$$\mathcal{K}_{\overline{p}}\left(\overline{x}\right) = \left\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3\}\right\},$$

whereas $\mathcal{T}_{\overline{p}}\left(\overline{x}\right) = \mathcal{K}_{\overline{p}}\left(\overline{x}\right) \setminus \{\{3\}\}$. In this case, the maximum considered in Theorem 3.1 is attained at any element of $\left\{\{1, 3\}, \{2, 3\}, \{3\}\right\}$, yielding $\mathrm{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) = \sqrt{5}$. Also observe that in this case $\max_{D \in \mathcal{T}_{\overline{p}}(\overline{x})} \left\| A_D^{-1} \right\| = \sqrt{5}$, although $\mathrm{lip}\mathcal{S}_{\overline{a}}\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) = +\infty$ since the Nürnberger condition fails (see Proposition 3.1).

4 Calmness under perturbations of all coefficients

As we show in this section, devoted to characterize the calmness of $\mathcal{S}$ under the uniqueness of optimal solution and to provide an upper estimate of $\text{clm}\mathcal{S}\,(\overline{p},\overline{x})$, the case of perturbations of all data is notably different from the one of canonical perturbations.

To start with, observe that the finiteness of T is no longer a sufficient condition for the calmness of $\mathcal{S}$. In fact, by combining two results traced out from the seminal paper by Robinson [24], we derive condition (ii) in the next theorem as a characterization of the calmness of $\mathcal{S}$ at $(\overline{p},\overline{x})$, provided that $\mathcal{S}\,(\overline{p})=\{\overline{x}\}$.

Theorem 4.1 *Let* $\overline{p}=\left(\overline{c},\overline{a},\overline{b}\right)\in\mathbb{R}^n\times(\mathbb{R}^n)^T\times\mathbb{R}^T$ *and assume that* $\mathcal{S}\,(\overline{p})=\{\overline{x}\}$. *Then the following statements are equivalent:*

(i) $\mathcal{S}$ *is calm at* $(\overline{p},\overline{x})$;
(ii) Either the SCQ holds at $\left(\overline{a},\overline{b}\right)$ *or* $\mathcal{F}\left(\overline{a},\overline{b}\right)=\{\overline{x}\}$;
(iii) $0_n\notin\text{bd}\,\text{conv}\left\{\overline{a}_t,\ t\in T_{\overline{a},\overline{b}}\,(\overline{x})\right\}$.

Proof $(i)\Rightarrow(ii)$. Reasoning by contradiction, assume that the SCQ fails at $\left(\overline{a},\overline{b}\right)$ and that $\mathcal{S}\,(\overline{p})\subsetneq\mathcal{F}\left(\overline{a},\overline{b}\right)$. Let us consider some sequence $\{x^r\}_{r\in\mathbb{N}}\subset\mathcal{F}\left(\overline{a},\overline{b}\right)\setminus\mathcal{S}\,(\overline{p})$ converging to $\overline{x}$. Applying [24, Theorem 2], for each r, we can construct a perturbed problem associated to a parameter $(\overline{c},a^r,b^r)$ such that

$$\left\|\left(\overline{c},a^r,b^r\right)-\left(\overline{c},\overline{a},\overline{b}\right)\right\|\leq\left\|x^r-\overline{x}\right\|^2,\quad\text{and}\quad x^r\in\mathcal{S}\left(\overline{c},a^r,b^r\right)$$

(observe that the objective function remains unchanged). Then,

$$\begin{aligned}\text{clm}\mathcal{S}\,(\overline{p},\overline{x})&\geq\lim_{r\to\infty}\frac{\|x^r-\overline{x}\|}{\left\|(\overline{c},a^r,b^r)-\left(\overline{c},\overline{a},\overline{b}\right)\right\|}\\&\geq\lim_{r\to\infty}\frac{\|x^r-\overline{x}\|}{\|x^r-\overline{x}\|^2}=+\infty.\end{aligned}$$

$(ii)\Rightarrow(i)$. The case when the SCQ holds at $\left(\overline{a},\overline{b}\right)$ follows from [24, Theorem 1], taking into account the fact that, under the current uniqueness assumption, calmness and Robinson's upper Lipschitz property coincide.

The calmness of $\mathcal{S}$ at $(\overline{p},\overline{x})$ when $\mathcal{F}\left(\overline{a},\overline{b}\right)=\{\overline{x}\}$ follows from the calmness of $\mathcal{F}$ at $\left(\left(\overline{a},\overline{b}\right),\overline{x}\right)$ together with the obvious fact that $\mathcal{S}\,(c,a,b)\subset\mathcal{F}\,(a,b)$ for every $(c,a,b)\in\mathbb{R}^n\times(\mathbb{R}^n)^T\times\mathbb{R}^T$.

$(ii)\Leftrightarrow(iii)$ comes from the facts that: the SCQ holds at $\left(\overline{a},\overline{b}\right)$ if and only if $0_n\notin\text{conv}\left\{\overline{a}_t,\ t\in T_{\overline{a},\overline{b}}\,(\overline{x})\right\}$ (see, e.g., [11, Theorem 6.9(v)]), and $\mathcal{F}\left(\overline{a},\overline{b}\right)=\{\overline{x}\}$ if and only if $0_n\in\text{int}\,\text{conv}\left\{\overline{a}_t,\ t\in T_{\overline{a},\overline{b}}\,(\overline{x})\right\}$ (see the paragraph preceding Proposition 2.2 for the direct implication; the converse is evident). □

Remark 4.1 The proof of $(i)\Rightarrow(ii)$ appeals to [24, Theorem 2]. The reader is addressed to that paper for a precise description of (a^r,b^r). At this moment, we comment that such a construction only requires a single constraint to be perturbed, while the remaining ones keep unaltered.

The following Lemma isolates a key step for establishing an upper bound on the calmness modulus of $\mathcal{S}$ under the SCQ.

Lemma 4.1 *Let $(\overline{p}, \overline{x}) \in \operatorname{gph}\mathcal{S}$, with $\overline{p} = \left(\overline{c}, \overline{a}, \overline{b}\right)$, and assume that the SCQ holds at $\left(\overline{a}, \overline{b}\right)$. Consider any sequence $\{(p^r, x^r)\} \subset \operatorname{gph}\mathcal{S}$, with $p^r = (c^r, a^r, b^r)$, converging to $(\overline{p}, \overline{x})$. For each r, let $D^r \subset T_{a^r,b^r}(x^r)$ be such that $|D^r| \leq n$ and*

$$-c^r \in \operatorname{cone}\left\{a_t^r,\ t \in D^r\right\}. \tag{11}$$

Then, there exists a subsequence $\{(p^{r_k}, x^{r_k})\}$ of $\{(p^r, x^r)\}$ such that the corresponding $\{D^{r_k}\}$ is constant and, denoting $D^{r_k} = \widehat{D}$ for all k, we have

$$\widehat{D} \in \mathcal{K}_{\overline{p}}(\overline{x}).$$

Proof Consider sequences $\{(p^r, x^r)\} \subset \operatorname{gph}\mathcal{S}$ and $\{D^r\}$ as in the statement of the lemma. The finiteness of T allows us to consider a constant subsequence $\{D^{r_k}\}$. Write $D^{r_k} = \widehat{D}$ for all k. Our assumption $\widehat{D} \subset T_{a^{r_k},b^{r_k}}(x^{r_k})$ clearly implies $\widehat{D} \subset T_{\overline{a},\overline{b}}(\overline{x})$ by letting $k \to \infty$. From (11), we can write, for each k,

$$-c^{r_k} = \sum_{t \in \widehat{D}} \lambda_t^{r_k} a_t^{r_k}, \tag{12}$$

for some $\lambda_t^{r_k} \geq 0, t \in \widehat{D}$. Note that the sequence $\{\gamma_k\}$, where $\gamma_k := \sum_{t \in \widehat{D}} \lambda_t^{r_k}$ for all k, is bounded; since otherwise, dividing both members of (12) by γ_k (assuming $\gamma_k \to \infty$ without loss of generality) and letting $k \to \infty$, we would obtain

$$0_n \in \operatorname{conv}\left\{\overline{a}_t,\ t \in T_{\overline{a},\overline{b}}(\overline{x})\right\},$$

which represents a contradiction with the SCQ (see, e.g., [11, Theorem 6.9(v)]).

The boundedness of $\{\gamma_k\}$ yields the existence of some subsequence of k's (denoted in the same way for simplicity) such that, for each $t \in \widehat{D}$, the sequence $\left\{\lambda_t^{r_k}\right\}_{k \in \mathbb{N}}$ converges to some $\lambda_t \geq 0$. Then, from (12) we conclude

$$-\overline{c} = \sum_{t \in \widehat{D}} \lambda_t \overline{a}_t \in \operatorname{cone}\left\{\overline{a}_t,\ t \in \widehat{D}\right\}.$$

This implies $\widehat{D} \in \mathcal{K}_{\overline{p}}(\overline{x})$. □

Now we are able to provide an upper bound on the calmness modulus of $\mathcal{S}$ under the uniqueness of nominal optimal solution. We point out the fact that the right-hand-side of both inequalities in (i) and (ii) below is always finite when $\mathcal{S}(\overline{p}) = \{\overline{x}\}$ (see Remark 3.2 for (i)), although obviously these inequalities are not true when $\mathcal{S}$ is not calm at $(\overline{p}, \overline{x})$.

Theorem 4.2 *Let $\overline{p} = \left(\overline{c}, \overline{a}, \overline{b}\right) \in \mathbb{R}^n \times (\mathbb{R}^n)^T \times \mathbb{R}^T$. Assume that $\mathcal{S}(\overline{p}) = \{\overline{x}\}$ and that $\mathcal{S}$ is calm at $(\overline{p}, \overline{x})$. The following assertions are true:*

(i) If the SCQ holds at $\left(\overline{a}, \overline{b}\right)$, then

$$\mathrm{clm}\mathcal{S}(\overline{p}, \overline{x}) \leq \max_{D \in \mathcal{K}_{\overline{p}}(\overline{x})} \frac{\|\overline{x}\| + 1}{d_* \left(0_n, \mathrm{bd}\,\mathrm{conv}\{\overline{a}_t,\ t \in T_{\overline{a},\overline{b}}(\overline{x});\ -\overline{a}_t,\ t \in D\}\right)}. \tag{13}$$

(ii) If $\mathcal{F}\left(\overline{a}, \overline{b}\right) = \{\overline{x}\}$, then

$$\mathrm{clm}\mathcal{S}(\overline{p}, \overline{x}) \leq \mathrm{clm}\mathcal{F}\left(\left(\overline{a}, \overline{b}\right), \overline{x}\right) = \frac{\|\overline{x}\| + 1}{d_* \left(0_n, \mathrm{bd}\,\mathrm{conv}\{\overline{a}_t,\ t \in T_{\overline{a},\overline{b}}(\overline{x})\}\right)}.$$

Proof (i) First note that the right-hand-side of (13) may be written as

$$\max_{D \in \mathcal{K}_{\overline{p}}(\overline{x})} \mathrm{clm}\mathcal{L}_D\left(\left(\overline{a}, \overline{b}, -\overline{a}_D, -\overline{b}_D\right), \overline{x}\right)$$

as an application of Propositions 2.1 and 2.2. Recall from its definition (6) that $\mathcal{L}_D$ is nothing else but the feasible set mapping associated with a certain enlarged system, whose parameter size is measured by

$$\|(a, b, u, d)\| := \max\left\{\max_{t \in T} \left\|\begin{pmatrix} a_t \\ b_t \end{pmatrix}\right\|, \max_{t \in D} \left\|\begin{pmatrix} u_t \\ d_t \end{pmatrix}\right\|\right\}.$$

Set

$$\mathrm{clm}\mathcal{S}(\overline{p}, \overline{x}) = \lim_{r \to \infty} \frac{\|x^r - \overline{x}\|}{\|p^r - \overline{p}\|}, \tag{14}$$

for certain sequences of parameters $p^r = (c^r, a^r, b^r)$ and points $x^r \in \mathcal{S}(p^r)$ such that $(p^r, x^r) \to (\overline{p}, \overline{x})$ with $p^r \neq \overline{p}$. By the KKT conditions (together with Carathéodory's Theorem) , for each r there exists $D^r \subset T_{a^r,b^r}(x^r)$ such that $|D^r| \leq n$ and

$$-c^r \in \mathrm{cone}\left\{a_t^r,\ t \in D^r\right\}.$$

Applying the previous lemma we may assume w.l.o.g. that $D^r = \widehat{D} \in \mathcal{K}_{\overline{p}}(\overline{x})$ for all r.

Since $\widehat{D} \subset T_{a^r,b^r}(x^r)$ and $x^r \in \mathcal{F}(a^r, b^r)$, we have

$$x^r \in \mathcal{L}_{\widehat{D}}\left(a^r, b^r, -a^r_{\widehat{D}}, -b^r_{\widehat{D}}\right), \quad r = 1, 2, \ldots$$

Then, since $\left\| \left(a^r, b^r, -a^r_{\widehat{D}}, -b^r_{\widehat{D}}\right) - \left(\overline{a}, \overline{b}, -\overline{a}_{\widehat{D}}, -\overline{b}_{\widehat{D}}\right) \right\| = \left\| (a^r, b^r) - \left(\overline{a}, \overline{b}\right) \right\|$, we have (applying the convention $0/0 := 0$ if necessary)

$$\begin{aligned}\operatorname{clm}\mathcal{L}_{\widehat{D}}\left(\left(\overline{a}, \overline{b}, -\overline{a}_{\widehat{D}}, -\overline{b}_{\widehat{D}}\right), \overline{x}\right) &\geq \limsup_{r\to\infty} \frac{\|x^r - \overline{x}\|}{\left\|(a^r, b^r) - \left(\overline{a}, \overline{b}\right)\right\|} \\ &\geq \lim_{r\to\infty} \frac{\|x^r - \overline{x}\|}{\|p^r - \overline{p}\|} = \operatorname{clm}\mathcal{S}\left(\overline{p}, \overline{x}\right).\end{aligned}$$

Note that $\mathcal{L}_{\widehat{D}}\left(\overline{a}, \overline{b}, -\overline{a}_{\widehat{D}}, -\overline{b}_{\widehat{D}}\right) = \{\overline{x}\}$, which comes from the uniqueness of nominal optimal solution, has been appealed to in the first inequality of the previous chain.

(ii) It follows from the facts that $\mathcal{S}\left(\overline{p}\right) = \mathcal{F}\left(\overline{a}, \overline{b}\right) = \{\overline{x}\}$ and $\mathcal{S}\left(p\right) \subset \mathcal{F}\left(a, b\right)$ for every $p = (c, a, b) \in \mathbb{R}^n \times (\mathbb{R}^n)^T \times \mathbb{R}^T$. More specifically, if $\operatorname{clm}\mathcal{S}\left(\overline{p}, \overline{x}\right)$ is written as (14), then

$$\operatorname{clm}\mathcal{S}\left(\overline{p}, \overline{x}\right) \leq \limsup_{r\to\infty} \frac{\|x^r - \overline{x}\|}{\left\|(a^r, b^r) - \left(\overline{a}, \overline{b}\right)\right\|} \leq \operatorname{clm}\mathcal{F}\left(\left(\overline{a}, \overline{b}\right), \overline{x}\right).$$

Finally, the expression of $\operatorname{clm}\mathcal{F}\left(\left(\overline{a}, \overline{b}\right), \overline{x}\right)$ comes directly from Propositions 2.1 and 2.2. □

Remark 4.2 In case (i) of the previous theorem, and recalling Theorem 3.1, inequality (13) may be read as

$$\operatorname{clm}\mathcal{S}\left(\overline{p}, \overline{x}\right) \leq \left(\|\overline{x}\| + 1\right) \operatorname{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right), \tag{15}$$

which in the case $\overline{x} = 0_n$ holds as an equality; i.e.,

$$\begin{aligned}\operatorname{clm}\mathcal{S}\left(\overline{p}, 0_n\right) &= \operatorname{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c}, \overline{b}\right), 0_n\right) \\ &= \max_{D\in\mathcal{K}_{\overline{p}}(0_n)} \frac{1}{d_*\left(0_n, \operatorname{bd}\operatorname{conv}\{\overline{a}_t,\ t \in T_{\overline{a},\overline{b}}\left(0_n\right); -\overline{a}_t,\ t \in D\}\right)}\end{aligned}$$

as a direct consequence of the fact that $\operatorname{clm}\mathcal{S}_{\overline{a}}(\left(\overline{c}, \overline{b}\right), \overline{x}) \leq \operatorname{clm}\mathcal{S}(\overline{p}, \overline{x})$, which follows immediately from the definitions.

The next example shows that the upper bound on $\operatorname{clm}\mathcal{S}\left(\overline{p}, \overline{x}\right)$ provided in Theorem 4.2(i) may be strict when $\overline{x} \neq 0_n$ (see Remark 4.2). Accordingly, inequality (15) may be strict for $\overline{x} \neq 0_n$. The technical details are postponed to Sect. 6 in order to avoid breaking the rhythm of the paper.

Example 4.1 Let us consider, in the context of parameterized linear optimization problems of the form (1), the nominal problem, in $\mathbb{R}^2$ endowed with the Euclidean norm,

$$\begin{array}{llrl} P\left(\overline{c}, \overline{a}, \overline{b}\right): & \text{minimize} & 10x_1 & \\ & \text{subject to} & -x_1 + x_2 \leq -1 & (t = 1), \\ & & -2x_1 - 2x_2 \leq -6 & (t = 2), \\ & & -x_1 \leq -2 & (t = 3), \end{array} \tag{16}$$

whose unique optimal solution is $\overline{x} = (2, 1)'$. Set once more $\overline{p} = \left(\overline{c}, \overline{a}, \overline{b}\right)$. The reader can check the following:

$D \in \mathcal{K}_{\overline{p}}(\overline{x})$	$\mathrm{clm}\mathcal{L}_D\left(\left(\overline{a}, \overline{b}, -\overline{a}_D, -\overline{b}_D\right), \overline{x}\right)$
$\{3\},\ \{1, 3\}$	$5 + \sqrt{5} \approx 7.2361$
$\{1, 2\}$	$\sqrt{10}\left(1 + \sqrt{5}\right)/4 \approx 2.5583$
$\{2, 3\}$	$\sqrt{13}\left(1 + \sqrt{5}\right)/2 \approx 5.8339$

Hence, the maximum in the right-hand-side of (13) equals $5 + \sqrt{5} \approx 7.2361$ and is attained at both $D = \{3\}$ and $D = \{1, 3\}$. However, ad hoc arguments (see Sect. 6 for details) show that

$$\mathrm{clm}\mathcal{S}\left(\overline{p}, \overline{x}\right) \leq \frac{1}{10}\sqrt{820\sqrt{5} + 3142} \approx 7.0538, \tag{17}$$

and therefore (13) holds as a strict inequality in this example.

The next example shows that the upper bound on $\mathrm{clm}\mathcal{S}\left(\overline{p}, \overline{x}\right)$ provided in Theorem 4.2(i) may be attained for $\overline{x} \neq 0_n$. It is basically the same nominal problem as (16), with the only difference that the objective function coefficient vector $\overline{c}$ is shorter now. Observe that in the case of canonical perturbations the size of $\overline{c}$ has no effect on $\mathrm{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$ (see Theorem 3.1).

Example 4.2 Consider the nominal problem obtained from (16) by just replacing $\overline{c}$ therein with $(1, 0)'$. Then (13) holds as an equality. Just consider the perturbed parameter $p^r = (c^r, a^r, b^r)$ given by

$$\begin{gathered}\begin{pmatrix} a_1^r \\ b_1^r \end{pmatrix} = \begin{pmatrix} \overline{a}_1 - \frac{1}{r}\frac{\overline{x}}{\|\overline{x}\|} \\ \overline{b}_1 + \frac{1}{r} \end{pmatrix}, \begin{pmatrix} a_2^r \\ b_2^r \end{pmatrix} = \begin{pmatrix} \overline{a}_2 \\ \overline{b}_2 \end{pmatrix}, \\ \begin{pmatrix} a_3^r \\ b_3^r \end{pmatrix} = \begin{pmatrix} \overline{a}_3 + \frac{1}{r}\frac{\overline{x}}{\|\overline{x}\|} \\ \overline{b}_3 - \frac{1}{r} \end{pmatrix}, c^r = -a_3^r. \end{gathered} \tag{18}$$

The reader can check that

$$x^r := \frac{1}{1 - 4/\left(r\sqrt{5}\right)} \begin{pmatrix} 2 - 3/\left(r\sqrt{5}\right) + 1/r \\ 1 + 6/\left(r\sqrt{5}\right) + 2/r \end{pmatrix} \in \mathcal{S}\left(p^r\right), \tag{19}$$

indeed $x^r \in \mathcal{L}_{\{3\}}\left(a^r, b^r, -a^r_{\{3\}}, -b^r_{\{3\}}\right)$, and

$$\lim_{r \to \infty} \frac{\|x^r - \overline{x}\|}{\|p^r - \overline{p}\|} = 5 + \sqrt{5}.$$

The upper bound given in Theorem 4.2(ii) may also not be attained, even with $\overline{x} = 0_n$ as the following example shows.

Example 4.3 Consider the nominal problem $P\left(\overline{c}, \overline{a}, \overline{b}\right)$, in $\mathbb{R}^2$ endowed with the Euclidean norm, given by

$$\begin{array}{lll} \text{minimize} & x_1 + x_2 & \\ \text{subject to} & -x_1 \leq 0, & (t = 1) \\ & -x_2 \leq 0, & (t = 2) \\ & x_1 + x_2 \leq 0, & (t = 3) \end{array}$$

so that $\mathcal{F}\left(\overline{a}, \overline{b}\right) = \mathcal{S}\left(\overline{c}, \overline{a}, \overline{b}\right) = \{\overline{x}\}$ with $\overline{x} = 0_2$. Then, appealing to [8, Theorems 4 and 5], the reader can check that $\mathrm{clm}\mathcal{F}\left(\left(\overline{a}, \overline{b}\right), \overline{x}\right) = \sqrt{5}$. On the other hand, if $\widetilde{\mathcal{S}}$ denotes the optimal set mapping obtained by removing the last constraint $(t = 3)$ from the parameterized problem, and we denote as $\left(a_{\widetilde{T}}, b_{\widetilde{T}}\right)$ the restriction of parameter (a, b) to $\widetilde{T} := \{1, 2\}$, then the reader can easily check that

$$\mathrm{clm}\mathcal{S}\left(\left(\overline{c}, \overline{a}, \overline{b}\right), \overline{x}\right) \leq \mathrm{clm}\widetilde{\mathcal{S}}\left(\left(\overline{c}, \overline{a}_{\widetilde{T}}, \overline{b}_{\widetilde{T}}\right), \overline{x}\right) = \sqrt{2},$$

where the first inequality comes from the fact that $\mathcal{S}\left(\overline{c}, \overline{a}, \overline{b}\right) = \widetilde{\mathcal{S}}\left(\overline{c}, \overline{a}_{\widetilde{T}}, \overline{b}_{\widetilde{T}}\right) = \{\overline{x}\}$ and $\mathcal{S}(c, a, b) \subset \widetilde{\mathcal{S}}\left(c, a_{\widetilde{T}}, b_{\widetilde{T}}\right)$ for (c, a, b) close enough to $\left(\overline{c}, \overline{a}, \overline{b}\right)$, and the last equality comes straightforwardly from Remark 4.2.

We finish this section with a last example, which shows that the upper bound given in Theorem 4.2(ii) may be attained and be strictly larger than the right-hand-side of (13).

Example 4.4 Consider $P\left(\overline{c}, \overline{a}, \overline{b}\right)$, in $\mathbb{R}^2$ endowed with the Euclidean norm, given by

$$\begin{array}{lll} \text{minimize} & x_1 & \\ \text{subject to} & -x_1 \leq 0, & (t = 1) \\ & -x_2 \leq 0, & (t = 2) \\ & x_2 \leq 0, & (t = 3) \\ & \frac{1}{2}x_1 \leq 0. & (t = 4) \end{array}$$

whose unique feasible solution is $\overline{x} = 0_2$. Then, the reader can check that $\{1\} \in \mathcal{K}_{\overline{p}}(\overline{x})$ and $t = 1$ must belong to any other $D \in \mathcal{K}_{\overline{p}}(\overline{x})$, entailing that the right-hand-side of (13) equals $\sqrt{2}$. On the other hand, let us consider, for each $r = 1, 2, \ldots,$ the perturbed problem $P\left(\overline{c}, a^r, b^r\right)$ given by

$$\begin{array}{ll} \text{minimize} & x_1 \\ \text{subject to} & -x_1 \leq 0, \\ & -x_2 \leq -\frac{1}{r}, \\ & -\frac{1}{r^2}x_1 + x_2 \leq -\frac{2}{r^3} + \frac{1}{r}, \\ & \frac{1}{2}x_1 \leq \frac{1}{r}. \end{array}$$

Then we have $\mathcal{S}(\overline{c}, a^r, b^r) = \{x^r\}$, with $x^r := \left(\frac{2}{r}, \frac{1}{r}\right)'$, and

$$\begin{aligned}\mathrm{clm}\mathcal{S}(\overline{p}, \overline{x}) &\geq \lim_{r\to\infty} \frac{\|x^r - \overline{x}\|}{\left\|(\overline{c}, a^r, b^r) - \left(\overline{c}, \overline{a}, \overline{b}\right)\right\|} \\ &= \lim_{r\to\infty} \frac{\sqrt{5}/r}{1/r} = \sqrt{5} = \mathrm{clm}\mathcal{F}\left(\left(\overline{a}, \overline{b}\right), \overline{x}\right),\end{aligned}$$

where the last equality comes from [8, Theorems 4 and 5].

5 Applications: calmness modulus and algorithms

It is well known that Lipschitz type properties (as Aubin or calmness properties) for generic multifunctions $\mathcal{M} : Y \rightrightarrows X$ between Banach spaces have a close relationship with the behavior of methods for solving the generalized equation

$$y \in \mathcal{M}^{-1}(x). \tag{20}$$

The use of generalized equations as a unified framework for several aspects of optimization and variational analysis (such as stationarity or complementarity) goes back to Robinson (see, e.g., [26]). The reader is addressed to [18] (and references therein) for the analysis of this interrelation between the calmness and the Aubin propety of $\mathcal{M}$, assumed to be closed (i.e., its graph is closed), and the linear convergence of descent methods and approximate projection methods.

This section is devoted to present two specific applications of the previous sections to the computation of some constants related to the convergence of certain optimization methods. So, the section is divided into two subsections. The first one is focussed on a specific procedure described in [18, Sect. 3.1], applied here in the context of linear programming (LP), and the second deals with a concrete regularization scheme for mathematical programs with complementarity constraints (MPCCs) introduced in [15], applied here in the context of linear MPCCs.

5.1 Calmness modulus and a descent method in LP

To begin with, for completeness purposes, we recall a specific version of the algorithm ALG1 introduced in [18] for solving system (20), which is closely related to the calmness of $\mathcal{M}$ (see the same paper for its counterpart in relation to the Aubin property of $\mathcal{M}$). For the sake of simplicity, in the following paragraphs we refer to this method as ALG1 (calmness). As said in [18], under the view point of methods, we have some $y^0 \in Y$ and $y^0 \in \mathcal{M}^{-1}(\cdot)$ is the 'equation' we want to solve with start at some $\left(y^1, x^1\right) \in \mathrm{gph}\mathcal{M}$.

ALG1 (calmness):

- Suppose that $0 < \lambda < 1$ and initial point $\left(y^1, x^1\right) \in \mathrm{gph}\mathcal{M}$ are given.

- Choose $\left(y^{k+1}, x^{k+1}\right) \in \mathrm{gph}\mathcal{M}$ such that

$$\left\|y^{k+1}-y^{0}\right\|-\left\|y^{k}-y^{0}\right\| \leq -\lambda \max\left\{\left\|x^{k+1}-x^{k}\right\|, \left\|y^{k}-y^{0}\right\|\right\}. \tag{21}$$

Recall from [18] that ALG1 (calmness) is said to be *applicable* if related $\left(y^{k+1}, x^{k+1}\right)$ exist in each step (for some fixed $\lambda > 0$).

Remark 5.1 The interpretation of ALG1 (calmness) as a descent method comes from identifying y^k with some element $f\left(x^k\right) \in \mathcal{M}^{-1}\left(x^k\right)$ and then writing (21) as

$$\frac{\|f\left(x^{k+1}\right)-y^{0}\|-\|f\left(x^{k}\right)-y^{0}\|}{\left\|x^{k+1}-x^{k}\right\|} \leq -\lambda, \; \textit{if } x^{k+1} \neq x^{k},$$

together with

$$\left\|f\left(x^{k+1}\right)-y^{0}\right\| \leq (1-\lambda)\left\|f\left(x^{k}\right)-y^{0}\right\|,$$

which entails $\lim_{k\to\infty}\|f\left(x^k\right)-y^0\|=0$. In this way, ALG1 (calmness) is a descent method for the function $x \mapsto \|f(x)-y^0\|$.

The following two results come from applying [18, Theorem 1] to our optimal set mappings $\mathcal{S}_{\overline{a}}$ and $\mathcal{S}$; so, they are concerned with ALG1 (calmness) when applied for solving the respective equations

$$\left(\overline{c}, \overline{b}\right) \in \mathcal{S}_{\overline{a}}^{-1}(\cdot) \quad \text{and} \quad \left(\overline{c}, \overline{a}, \overline{b}\right) \in \mathcal{S}^{-1}(\cdot).$$

In both cases, the algorithm is devoted to find optimal solutions of our LP problem (1) for given nominal data. Recall that $\mathcal{S}_{\overline{a}}$ is always calm, while the characterization of the calmness of $\mathcal{S}$ is given in Theorem 4.1.

The original contribution of these results consists of providing an explicit (constructive) expression for $\overline{\lambda}$ as a consequence of the knowledge about the calmness modulus of these mappings.

Theorem 5.1 *Consider ALG1 (calmness) dealing with $\mathcal{S}_{\overline{a}}$. Assume that $\mathcal{S}_{\overline{a}}\left(\overline{c}, \overline{b}\right) = \{\overline{x}\}$ and consider any $0 < \overline{\lambda} < 1$ such that*

$$\overline{\lambda} < \min_{D \in \mathcal{K}_{\overline{p}}(\overline{x})} d_*\left(0_n, \mathrm{bd\,conv}\left\{\overline{a}_t,\; t \in T_{\overline{a},\overline{b}}(\overline{x});\, -\overline{a}_t,\; t \in D\right\}\right).$$

Then:

(i) ALG1 (calmness) is applicable for this $\overline{\lambda}$ and all initial points $\left(\left(c^1, b^1\right), x^1\right)$ near $((\overline{c}, \overline{b}), \overline{x})$ (in some neighborhood);
(ii) For given initial points $\left(\left(c^1, b^1\right), x^1\right)$ near $((\overline{c}, \overline{b}), \overline{x})$, the sequence $\left\{\left(\left(c^k, b^k\right), x^k\right)\right\}$ converges to $((\overline{c}, \overline{b}), \overline{x})$.

Proof (i) comes from the proof of [18, Theorem 1], taking into account the fact that $1/\overline{\lambda}$ is a calmness constant for $\mathcal{S}_{\overline{a}}$, associated with some neighborhood of $((\overline{c},\overline{b}),\overline{x})$, as far as

$$\left(\overline{\lambda}\right)^{-1} > \mathrm{clm}\mathcal{S}_{\overline{a}}((\overline{c},\overline{b}),\overline{x}),$$

where we have applied Theorem 3.1.

(ii) comes straightforwardly from [18, Theorem 1].

□

Theorem 5.2 *Consider ALG1 (calmness) dealing with $\mathcal{S}$. Assume that the SCQ holds at $(\overline{a},\overline{b})$ and $\mathcal{S}(\overline{c},\overline{a},\overline{b})=\{\overline{x}\}$. Consider any $0<\overline{\lambda}<1$ such that*

$$\overline{\lambda} < (\|\overline{x}\|+1)^{-1} \min_{D\in\mathcal{K}_{\overline{p}}(\overline{x})} d_*\left(0_n, \mathrm{bd\,conv}\left\{\overline{a}_t,\ t\in T_{\overline{a},\overline{b}}(\overline{x})\,;-\overline{a}_t,\ t\in D\right\}\right).$$

Then:

(i) ALG1 (calmness) is applicable for this $\overline{\lambda}$ and all initial points $\left(\left(c^1,a^1,b^1\right),x^1\right)$ near $((\overline{c},\overline{a},\overline{b}),\overline{x})$ (in some neighborhood);

(ii) For given initial points $\left(\left(c^1,a^1,b^1\right),x^1\right)$ near $((\overline{c},\overline{a},\overline{b}),\overline{x})$, the sequence $\left\{\left(\left(c^k,a^k,b^k\right),x^k\right)\right\}$ converges to $((\overline{c},\overline{a},\overline{b}),\overline{x})$.

Proof The proof is analogous to the previous one; it comes from [18, Theorem 1], appealing now to Theorem 4.2. □

5.2 On the convergence of a regularization method for linear MPCCs

Complementarity constraints naturally appear in numerous applications in economics and engineering. Generically, solving a MPCC via classical technics of nonlinear programming theory presents serious difficulties, as far as its feasible set has a very special structure and violates most of standard constraint qualifications. See the monograph [21] (and references therein) for details on theory and applications of MPCCs.

One prominent class of specialized algorithms for solving MPCCs are the regularization (or relaxation) methods, which are devoted to relax the difficult constraints in different ways. From the first regularization scheme, introduced by Scholtes [28] in 2001, one can find in the literature new procedures as well as different contributions to the analysis of the convergence of these methods (see, e.g., [16,23]).

This subsection provides an application of previous results on the calmness modulus of the optimal set mapping to the analysis of convergence of a concrete regularization scheme introduced by Kadrani et al. [15]. This application is developed in the framework of linear MPCCs given in the form

$$\begin{array}{lll}
\pi: \text{minimize} & c'x & \\
\text{subject to} & \left(a_t^1\right)'x-b_t^1\le 0, & t=1,\ldots,m; \\
& \left(a_t^2\right)'x-b_t^2\le 0,\quad \left(a_t^3\right)'x-b_t^3\le 0, & t=1,\ldots,s; \\
& \left(\left(a_t^2\right)'x-b_t^2\right)\left(\left(a_t^3\right)'x-b_t^3\right)=0, & t=1,\ldots,s;
\end{array}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables, all the a_t^i belong to $\mathbb{R}^n$ and all the b_t^i belong to $\mathbb{R}$. We shall assume that problem π has a unique solution, denoted by $\overline{x}$.

Applied to our problem π, the regularization of Kadrani et al. consists of replacing the complementarity conditions (last two rows in the description of π) with

$$\begin{cases} \left(a_t^2\right)' x - b_t^2 \leq \varepsilon, \quad \left(a_t^3\right)' x - b_t^3 \leq \varepsilon, \quad t = 1, \ldots, s; \\ \left(\left(a_t^2\right)' x - b_t^2 + \varepsilon\right)\left(\left(a_t^3\right)' x - b_t^3 + \varepsilon\right) \leq 0, \quad t = 1, \ldots, s; \end{cases}$$

for some parameter $\varepsilon > 0$. In this way, for $\varepsilon_r \downarrow 0$, we consider a sequence of (nonlinear) optimization problems $\{\pi_r\}$, where for each $r \in \mathbb{N}$,

$$\begin{array}{lll} \pi_r : \text{minimize} & c'x & \\ \text{subject to} & \left(a_t^1\right)' x - b_t^1 \leq 0, \quad t = 1, \ldots, m; \\ & \left(a_t^2\right)' x - b_t^2 \leq \varepsilon_r, \quad \left(a_t^3\right)' x - b_t^3 \leq \varepsilon_r, \quad t = 1, \ldots, s; \\ & \left(\left(a_t^2\right)' x - b_t^2 + \varepsilon_r\right)\left(\left(a_t^3\right)' x - b_t^3 + \varepsilon_r\right) \leq 0, \quad t = 1, \ldots, s. \end{array}$$

In order to apply our results on calmness modulus in linear programming to analyze the converge of optimal solutions of $\{\pi_r\}$ to the solution of π, for each $J \subset \{1, \ldots, s\}$, whose cardinality is denoted by $|J|$, we set

$$\mathcal{P} := \mathbb{R}^m \times \mathbb{R}^s \times \mathbb{R}^s \times \mathbb{R}^{|J|} \times \mathbb{R}^{|J|} \times \mathbb{R}^{s-|J|} \times \mathbb{R}^{s-|J|}$$

and consider the following associated multifunctions:

- The feasible set mapping $\mathcal{F}^J : \mathcal{P} \rightrightarrows \mathbb{R}^n$, given by

$$\mathcal{F}^J(p) := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \left(a_t^1\right)' x \leq p_t^1, \quad t = 1, \ldots, m; \\ \left(a_t^2\right)' x \leq p_t^2, \quad \left(a_t^3\right)' x \leq p_t^3, \quad t = 1, \ldots, s; \\ \left(a_t^2\right)' x \leq p_t^4, \quad -\left(a_t^3\right)' x \leq p_t^5, \quad t \in J; \\ -\left(a_t^2\right)' x \leq p_t^6, \quad \left(a_t^3\right)' x \leq p_t^7, \quad t \in \{1, \ldots, s\} \backslash J. \end{array} \right\},$$

  for $p := \left(p^1, \ldots, p^7\right) \in \mathcal{P}$. Note that in the cases $J = \emptyset$ and $J = \{1, \ldots, s\}$ parameter p is, respectively, of the form $\left(p^1, p^2, p^3, p^6, p^7\right)$ and $\left(p^1, p^2, p^3, p^4, p^5\right)$.
- The optimal set mapping $\mathcal{S}^J : \mathcal{P} \rightrightarrows \mathbb{R}^n$, given by

$$\mathcal{S}^J(p) := \arg\min\{c'x \mid x \in \mathcal{F}^J(p)\}, \quad p \in \mathcal{P}.$$

  (Observe that c remains fixed in our discussion.)

From now on, let us denote by F and F_r the feasible sets of π and π_r, respectively, and by S and S_r the corresponding sets of optimal solutions. Moreover, for the sake of simplicity, we use the following notation:

$$\beta^J := \left(b^1, b^2, b^3, b_J^2, -b_J^3, -b_{\{1,\ldots,s\}\setminus J}^2, b_{\{1,\ldots,s\}\setminus J}^3\right)',$$
$$\beta_r^J := (b^1, b^2 + \varepsilon_r 1_{\{1,\ldots,s\}}, b^3 + \varepsilon_r 1_{\{1,\ldots,s\}}, b_J^2 - \varepsilon_r 1_J, -b_J^3 + \varepsilon_r 1_J,$$
$$-b_{\{1,\ldots,s\}\setminus J}^2 + \varepsilon_r 1_{\{1,\ldots,s\}\setminus J}, b_{\{1,\ldots,s\}\setminus J}^3 - \varepsilon_r 1_{\{1,\ldots,s\}\setminus J})', \quad r \in \mathbb{N},$$

where 1_X stands for the constant function defined as 1 at every point of set X. Then, a standard argument yields the following lemma.

Lemma 5.1 *With the preceding notation one has, for each $r \in \mathbb{N}$:*

(i) $F = \bigcup_{J \subset \{1,\ldots,s\}} \mathcal{F}^J\left(\beta^J\right)$,

(ii) $F_r = \bigcup_{J \subset \{1,\ldots,s\}} \mathcal{F}^J\left(\beta_r^J\right), r \in \mathbb{N}$,

(iii) $\bigcap_{J \subset \{1,\ldots,s\}} \mathcal{S}^J\left(\beta^J\right) \subset S \subset \bigcup_{J \subset \{1,\ldots,s\}} \mathcal{S}^J\left(\beta^J\right)$,

(iv) $\bigcap_{J \subset \{1,\ldots,s\}} \mathcal{S}^J\left(\beta_r^J\right) \subset S_r \subset \bigcup_{J \subset \{1,\ldots,s\}} \mathcal{S}^J\left(\beta_r^J\right)$.

The following result relates the calmness modulus of mappings $\mathcal{S}^J$ with the convergence speed of the regularization method presented in this subsection.

Theorem 5.3 *Assume that $S = \{\overline{x}\}$, and let $\{x^r\} \subset \mathbb{R}^n$ be a sequence converging to $\overline{x}$ and such that $x^r \in S_r$ for all r. Then,*

$$\limsup_{r\to\infty} \frac{\|x^r - \overline{x}\|}{\varepsilon_r} \leq \max_{J \subset \{1,\ldots,s\}} \operatorname{clm}\mathcal{S}^J\left(\beta^J, \overline{x}\right).$$

Proof According to statement (iv) in the previous lemma, for each r there exists $J_r \subset \{1,\ldots,s\}$ such that $x^r \in \mathcal{S}^{J_r}\left(\beta_r^{J_r}\right)$. Due to the finiteness of $\{1,\ldots,s\}$, we may assume (by taking a suitable subsequence if necessary) that $\{J_r\}$ is constant, say $J_r = J$ for all r.

Let us see that $\{\overline{x}\} = \mathcal{S}^J\left(\beta^J\right)$. First, let us prove that $\overline{x} \in \mathcal{S}^J\left(\beta^J\right)$. By the closedness of $\mathcal{F}^J$, it is clear that

$$\overline{x} \in \mathcal{F}^J\left(\beta^J\right),$$

since $x^r \in \mathcal{F}^J\left(\beta_r^J\right)$ and $\left\{\beta_r^J\right\}$ converges to β^J. Reasoning by contradiction, if $\overline{x} \notin \mathcal{S}^J\left(\beta^J\right)$, there would exist $y \in \mathcal{F}^J\left(\beta^J\right)$ such that

$$c'y < c'\overline{x}.$$

In this way, by applying statement (i) in the previous lemma, $y \in F$ and, then, we attain the contradiction $\overline{x} \notin S$.

Now, we establish the opposite inclusion $\mathcal{S}^J\left(\beta^J\right) \subset \{\overline{x}\}$. Reasoning again by contradiction, let us assume the existence of $z \in \mathcal{S}^J\left(\beta^J\right)$, with $z \neq \overline{x}$; in particular

$$c'z = c'\overline{x}. \tag{22}$$

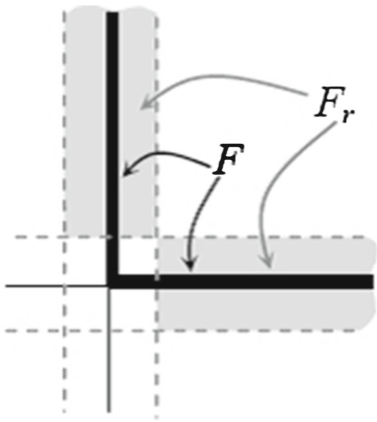

Fig. 2 Feasible sets of π and π_r

Then, $z \in \mathcal{F}^J\left(\beta^J\right) \subset F$ (again by Lemma 5.1(i)) and applying (22) we obtain $z \in S$, which represents a contradiction with the current uniqueness assumption.

Therefore,

$$\limsup_{r\to\infty} \frac{\|x^r - \overline{x}\|}{\varepsilon_r} = \limsup_{r\to\infty} \frac{d\left(x^r, \mathcal{S}^J\left(\beta^J\right)\right)}{\left\|\beta_r^J - \beta^J\right\|_\infty} \leq \mathrm{clm}\mathcal{S}^J\left(\beta^J, \overline{x}\right).$$

□

The next example shows that inequality in Theorem 5.3 may be strict.

Example 5.1 Let us consider the linear MPCC problem in $\mathbb{R}^2$,

$$\begin{array}{llll} \pi: \text{minimize} & x_1 + x_2 & \\ \quad \text{subject to} & -x_1 - x_2 \leq 0, & (m=1) \\ & -x_1 \leq 0, -x_2 \leq 0, & (s=1) \\ & x_1\, x_2 = 0. & \end{array}$$

Consider any $\{\varepsilon_r\} \downarrow 0$. Then, for each $r \in \mathbb{N}$, we have

$$\begin{array}{llll} \pi_r: \text{minimize} & x_1 + x_2 & \\ \quad \text{subject to} & -x_1 - x_2 \leq 0, & (m=1) \\ & -x_1 \leq \varepsilon_r, -x_2 \leq \varepsilon_r, & (s=1) \\ & (-x_1 + \varepsilon_r) \quad (-x_2 + \varepsilon_r) \leq 0. & \end{array}$$

Figure 2 graphically illustrates the feasible sets of these problems.

In this case, there are two possible choices for J: either $\emptyset$ or $\{1\}$. So, by considering multifunctions $\mathcal{S}^\emptyset$ and $\mathcal{S}^{\{1\}}$, in this example the upper bound on the rate of convergence provided by the previous theorem is

$$\max\left\{\mathrm{clm}\mathcal{S}^\emptyset\left(\beta^\emptyset, 0_2\right),\ \mathrm{clm}\mathcal{S}^{\{1\}}\left(\beta^{\{1\}}, 0_2\right)\right\} = \sqrt{5}, \tag{23}$$

with $\beta^{\emptyset} \equiv \beta^{\{1\}} \equiv 0_5$, whereas for any choice of $x^r \in S_r$, $r \in \mathbb{N}$, one has

$$\limsup_{r\to\infty} \frac{\|x^r - \overline{x}\|}{\varepsilon_r} = \sqrt{2},$$

as we can easily see from Fig. 2.

Let us justify (23) a bit more in detail: We can view $\mathcal{S}^{\{1\}}$ as the argmin mapping associated with problem

$$\begin{array}{lll} \pi^{\{1\}}: \text{minimize} & x_1 + x_2 \\ \quad\quad\ \text{subject to} & -x_1 - x_2 \leq p^1, \\ & -x_1 \leq p^2, -x_2 \leq p^3, \\ & -x_1 \leq p^4,\ x_2 \leq p^5, \\ & p = \left(p^1, p^2, p^3, p^4, p^5\right) \in \mathbb{R}^5, \end{array} \tag{24}$$

whereas $\mathcal{S}^{\emptyset}$ refers to the problem obtained by replacing the last two rows of (24) with

$$\begin{array}{l} x_1 \leq p^6, \quad -x_2 \leq p^7, \\ p = \left(p^1, p^2, p^3, p^6, p^7\right). \end{array}$$

In both cases perturbations fall just on the right-hand-side of the constraint system, and accordingly $\mathcal{S}^{\emptyset}$ and $\mathcal{S}^{\{1\}}$ fit in the format $\mathcal{S}_{\overline{c},\overline{a}}$ dealt with in Remark 3.1. Now Theorem 3.1 (together with Remark 3.1) entails

$$\mathrm{clm}\mathcal{S}^{\emptyset}\left(\beta^{\emptyset}, 0_2\right) = \mathrm{clm}\mathcal{S}^{\{1\}}\left(\beta^{\{1\}}, 0_2\right) = \sqrt{5}.$$

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6 Appendix: Geometrical perturbation ideas for improved bounds

The primary goal of this section is to provide a sketch of the technical details ensuring that inequality (13) is strict in Example 4.1. The underlying idea is that the norm of $\overline{c}$ is too large in this example to let the strategy followed in Example 4.2 work in Example 4.1 (see Remark 6.1 below). The question of investigating in general to what extent the norm of $\overline{c}$ affects $\mathrm{clm}\mathcal{S}\left(\left(\overline{c}, \overline{a}, \overline{b}\right), \overline{x}\right)$ is left to future research.

Now let us go back to problem (16) and the subsequent table in Example 4.1. Let us write $\mathrm{clm}\mathcal{S}\left(\overline{p}, \overline{x}\right)$ in the form (14) and assume w.l.o.g. that the associated sequence $\{D^r\}_{r\in\mathbb{N}}$ is constant, say $D^r = \widehat{D} \in \mathcal{K}_{\overline{p}}\left(\overline{x}\right)$ for all r, according to the lines after (14) in the proof of Theorem 4.2. Next we are going to prove (17).

Looking at the end of the proof of Theorem 4.2(i), we realize that $\mathrm{clm}\mathcal{S}(\overline{p}, \overline{x}) \leq \mathrm{clm}\mathcal{L}_{\widehat{D}}\left(\left(\overline{a}, \overline{b}, -\overline{a}_{\widehat{D}}, -\overline{b}_{\widehat{D}}\right), \overline{x}\right)$, so that our claim (17) holds automatically if $\widehat{D}$ equals either $\{1, 2\}$ or $\{2, 3\}$ (see the table of Example 4.1).

Let us consider now the case $\widehat{D} = \{3\}$ and set $\varepsilon_r := \|p^r - \overline{p}\|$ for all r, with $p^r = (c^r, a^r, b^r)$. Next, we relax the constraints determining $\mathcal{L}_{\{3\}}\left(a^r, b^r, -a^r_{\{3\}}, -b^r_{\{3\}}\right)$ (which contains x^r) around x^r in an appropriate way. Specifically, after some calculations one can check that x^r is a solution of the following system:

$$\left.\begin{aligned} \left(\overline{a}_t - \alpha_r \overline{x}/\|\overline{x}\|\right)' x &\leq \overline{b}_t + \alpha_r, \text{ for } t = 1, 2, 3, \\ (1 - \alpha_r)\, x_1 - \frac{\alpha_r (1 + \alpha_r)}{10} x_2 &\leq 2 + \alpha_r, \end{aligned}\right\} \tag{25}$$

with $\alpha_r := \dfrac{\|\overline{x}\| + \beta\varepsilon_r}{\|\overline{x}\| - \beta\varepsilon_r}\varepsilon_r$, for a scalar $\beta > 5 + \sqrt{5}$ arbitrarily chosen, and r being assumed to be large enough to ensure $\|\overline{x}\| - \beta\varepsilon_r > 0$ and $\|x^r - \overline{x}\| < \beta\varepsilon_r$. The last inequality of (25) is inspired by the fact that $\varepsilon_r \geq \|c^r - \overline{c}\| \geq d\left(\overline{c}, \mathbb{R}a^r_3\right)$.

The reader can check via a routinary computation that, if $\widetilde{x}^r$ stands for the furthest solution of (25) with respect to $\overline{x}$, then one has

$$\left\|\widetilde{x}^r - \overline{x}\right\| \approx \frac{\sqrt{820\sqrt{5} + 3142}}{10}\alpha_r \tag{26}$$

(i.e., $\lim_{r\to\infty} \|\widetilde{x}^r - \overline{x}\| / \alpha_r = (1/10)\sqrt{820\sqrt{5} + 3142} \approx 7.0538$), which together with the obvious fact that $\|x^r - \overline{x}\| \leq \|\widetilde{x}^r - \overline{x}\|$—since x^r is a solution of (25)—yields (17) by taking into account that $\alpha_r \approx \varepsilon_r$ as $r \to \infty$.

The remaining case $\widehat{D} = \{1, 3\}$ is very similar to $\widehat{D} = \{3\}$. Indeed (25) still holds at $x = x^r$ in the subcase $a^r_{32} > 0$, with a^r_{32} standing for the second coordinate of a^r_3. The subcase $a^r_{32} \leq 0$ is also very similar, but replacing the fourth inequality of (25) with $(1 - \alpha_r)\, x_1 \leq 2 + \alpha_r$. In this subcase the corresponding counterpart of (26) reads as

$$\left\|\widetilde{x}^r - \overline{x}\right\| \approx \sqrt{8\sqrt{5} + 30}\,\alpha_r$$

with $\sqrt{8\sqrt{5} + 30} \approx 6.9202$, leading again to (17).

Remark 6.1 Coming back to example 4.1, if we perturbed there the constraint system as in (18), then the minimum perturbation of $\overline{c} = (10, 0)'$ (yielding a perturbed c^r) making point x^r in (19) belong to $\mathcal{S}(p^r)$ would satisfy

$$\left\|c^r - \overline{c}\right\| = d\left(-\overline{c}, \mathrm{cone}\{a^r_1, a^r_3\}\right) \approx 2\sqrt{5}/r,$$

while $\left\|(a^r, b^r) - \left(\overline{a}, \overline{b}\right)\right\| = 1/r$ and, accordingly we would obtain the smaller ratio $\|x^r - \overline{x}\| / \|p^r - \overline{p}\| \approx \left(5 + \sqrt{5}\right) / \left(2\sqrt{5}\right) \approx 1.618$.

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