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Sharp lower bounds for regulators of small-degree number fields ☆

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Minimal discriminants of number fields are presently known for 22 signatures. For 20 of these we give the minimal regulator. Except in the totally complex case, in each signature we find that the field with the minimal discriminant has the minimal regulator.

Contents

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1. Introduction and table of lower bounds

Remak had the idea that big regulators and big discriminants should go together [Re, pp. 245–246]. In support of this he referred to Landau's inequality [Re, footnote 2] $R_k < C_1\sqrt{|D_k|}(\log|D_k|)^{n-1}$, and showed $|D_k| < C_2\exp(C_3R_k)$. Here k is a number field of degree n, discriminant D_k, regulator R_k, and the C_i are explicit constants depending only on n.[1] It followed that there was a minimal regulator for each signature (r_1, r_2), where r_1 and r_2 denote the number of real and complex places of k, respectively.

Few of these minimal regulators seem to have been explicitly calculated. The real quadratic case is an easy exercise. Pohst [Po2, p. 491] found the minimal regulator among all totally real cubic fields. Zimmert [Zi], relying on work of Pohst [Po2] and a new analytic method, showed that $\log\big((1+\sqrt{5})/2\big)$ is the minimal regulator among all totally real fields. Friedman [Fr1, p. 599] found the minimal regulator for totally complex sextic fields and showed that it was the smallest regulator among all number fields.

To find minimal regulators for a given signature, one needs general lower bounds for R_k in terms of $|D_k|$, and lists of all fields up to a certain discriminant. Rigorous lists of initial number fields, ordered by discriminant, are known for all signatures in degree $n \le 7$. We find the smallest regulator for all but one of these signatures, as shown in Table 1 below. In signature $(r_1, r_2) = (5, 1)$ we are only able to prove $R_k > 2.11$.[2] This failure is shown in Table 1 with a "?!".

In degree 8 initial discriminants are presently known only if $r_2 = 0$ or $r_1 = 0$, so we have no hope of finding the smallest regulator in the remaining three octic signatures.[3] These are shown in Table 1 with a question mark. A rigorous list of small discriminants

[1] Remak made the necessary exception for CM fields, and proved that their unit index is 1 or 2 [Re, p. 250], [Wa, Th. 4.12], well before Hasse. A CM field k has a proper subfield with the same unit rank as k, and is therefore a totally complex quadratic extension of a totally real number field.

[2] The sharp lower bound is likely to be $R_k \ge 2.8846...$, corresponding to the regulator of the field of discriminant $-2\,306\,599$, the first for this signature.

[3] For totally complex octic fields, the 15 fields of smallest discriminant are described in [Di3], but no equations or regulators are given. We give equations and regulators in §7, apparently for the first time in print.

Table 1
Sharp regulator lower bound for degree n and unit-rank r.

n	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$
2	0.481211						
3	0.281199	0.525454					
4	0.337377	0.369184	0.825068				
5		0.268355	0.628579	1.635694			
6		0.205216	0.478924	1.262710	3.277562		
7			0.380447	1.004348	?!	14.446932	
8			0.313539	?	?	?	22.446870

is also known for totally real fields of degree 9 [Ta,Vo]. In this case we are only able to prove $R_k > 37.2$.[4]

Our lower bounds are actually more detailed than Table 1. A typical result reads as follows.

Theorem. (Octics)

- *With three exceptions, all totally real octic fields have regulator greater than* 28.43*. The exceptions are the unique totally real octic fields of discriminant* 282 300 416*,* 309 593 125 *and* 324 000 000*, with respective regulators* 22.446870...*,* 23.696789... *and* 24.388406... *.*
- *With four exceptions, all totally complex octic fields have regulator greater than* 0.345*. The exceptions are the unique totally complex octic fields of discriminant* 1 282 789*,* 1 361 513*,* 1 385 533 *and* 1 424 293*, with respective regulators* 0.313539...*,* 0.326412...*,* 0.331112... *and* 0.336709... *.*

We prove our bounds along the lines of [Fr1], using geometric and analytic techniques. For degrees $n > 6$, however, we need geometric refinements by Pohst for totally real fields [Po2,Po4], and we need to improve some analytic and geometric estimates used in [Fr1] (see Lemmas 3 and 5 below). The signatures $(5,1)$ and $(9,0)$, where we fail to obtain a sharp lower bound for the regulator, are a sign that our techniques are still too coarse to handle high unit-ranks.

A recent application of lower bounds for regulators of totally real fields is given by Katok, Katok and Rodriguez Hetz [KKR], who conjecture that the Fried average entropy attached to certain dynamical systems in dimension $n-1$ is bounded below by $0.3301\ldots$, independently of n. They show that the Fried average entropy is bounded below by $2^{n-1}R_k/\binom{2n-2}{n-1}$, where k is some totally real number field of degree n and $\binom{2n-2}{n-1}$ is a binomial coefficient. Using lower bounds for regulators they are able to prove their conjecture, except when $8 \le n \le 16$ [KKR]. Our results in degree 8 (*i.e.* $R_k > 22.44$) and 9 (*i.e.* $R_k > 37.2$) narrow the gap to $10 \le n \le 16$.

[4] The sharp bound is probably $R_k \ge 62.3871\ldots$, corresponding to the field of discriminant 9 685 993 193, again minimal for its signature.

2. Geometric methods

To obtain a lower bound for the regulator R_k in terms of the discriminant D_k, we will follow Remak for the general case [Re], and Pohst–Bertin for the totally real case [Po2,Po4,Be]. For a unit $\varepsilon \in k$, we will use the Euclidean length $m_k(\varepsilon)$ in the logarithmic embedding

$$m_k(\varepsilon)^2 := \sum_{\omega\in\infty_k} \left(\log \|\varepsilon\|_\omega\right)^2. \tag{1}$$

Here ∞_k denotes the set of archimedean places of k and $\| \ \|_\omega$ the absolute value corresponding to $\omega \in \infty_k$, normalized so $\left|\mathrm{Norm}_{k/\mathbb{Q}}(a)\right| = \prod_{\omega\in\infty_k} \|a\|_\omega$ for all $a \in k$.

Lemma 1. (*Remak, Pohst, Bertin*) *Suppose $H = L(\varepsilon)$ is a number field generated over L by a unit ε of H. Then the discriminant D_H of H satisfies*

$$\log|D_H| \le [H:L]\log|D_L| + [H:\mathbb{Q}]\log([H:L]) + m_H(\varepsilon)A(H/L), \tag{2}$$

where

$$A(H/L) := \sqrt{\frac{1}{3}\sum_{v\in\infty_L}\Big([H:L]^3 - [H:L] - 4r_2(v)^3 - 2r_2(v)\Big)}, \tag{3}$$

and $r_2(v) = 0$ unless v is real, in which case $r_2(v)$ is the number of complex places of H lying above v.

Suppose now that $H = \mathbb{Q}(\varepsilon)$ is a totally real number field generated over $\mathbb{Q}$ by a unit $\varepsilon \in H$. Then

$$\log|D_H| \le [n/2]\log(4) + m_H(\varepsilon)\sqrt{(n^3-n)/3}, \tag{4}$$

where n is the degree of $H/\mathbb{Q}$ and $[n/2] := n/2$ if n is even, $[n/2] := (n-1)/2$ if n is odd.

Proof. The first part of the lemma is proved in [Fr1, Lemma 3.4], drawing on the case $L = \mathbb{Q}$ due to Remak [Re]. The second part is proved by combining an inequality due to Pohst and Bertin with one of Remak's. Namely, order the n conjugates $\varepsilon^{(i)}$ $(1 \le i \le n)$ of ε so $\left|\varepsilon^{(1)}\right| \ge \left|\varepsilon^{(2)}\right| \ge \ldots \ge \left|\varepsilon^{(n)}\right|$. Then the discriminant $D(\varepsilon)$ of the order $\mathbb{Z}[\varepsilon] \subset H$ satisfies

$$|D_H| \le |D(\varepsilon)| = \prod_{i=1}^{n-1}\left|\varepsilon^{(i)}\right|^{2(n-i)} \prod_{1\le i<j\le n}\left(1 - \frac{\left|\varepsilon^{(j)}\right|}{\left|\varepsilon^{(i)}\right|}\right)^2.$$

The logarithm of the first product was bounded by Remak [Re, §6] from above by $m_H(\varepsilon)\sqrt{(n^3-n)/3}$.[5] As regards the second product, for $n \le 11$ Pohst [Po2, Satz IV] improved Schur's old upper bound (namely n^n, reproved in [Be]) to $4^{[n/2]}$. A different proof was later found by Bertin [Be], who was able to allow any n. □

Let $0 < \mu_1 \le \mu_2 \le \ldots \le \mu_r$ be the successive minima of m_k on the unit lattice, attained at the independent units ε_i $(1 \le i \le r)$, so $\mu_i := m_k(\varepsilon_i)$ and $r = r_1 + r_2 - 1$. Minkowski's theorem on successive minima [Ca, pp. 120, 205, 332] yields

$$\mu_1^r \le \prod_{i=1}^{r} \mu_i \le \sqrt{r+1} R_k \gamma_r^{r/2}, \qquad (\mu_i := m_k(\varepsilon_i),\ \gamma_r := \text{Hermite's constant}). \tag{5}$$

We shall use the known values [Ca, p. 332]

$$\gamma_1 = 1,\ \gamma_2 = \frac{2}{\sqrt{3}},\ \gamma_3 = \sqrt[3]{2},\ \gamma_4 = \sqrt{2},\ \gamma_5 = \sqrt[5]{8},\ \gamma_6 = \frac{2}{\sqrt[6]{3}},\ \gamma_7 = \sqrt[7]{64},\ \gamma_8 = 2. \tag{6}$$

Lemma 1 will presently lead us to bounds of the form $\log|D_k| \le A_0 + \sum_{i=1}^{T} A_i \mu_i$, where $T \le r$ and the $A_i \ge 0$ will depend on the subfields generated by the ε_i. To bound such sums we will need the following maximization result.

Lemma 2. *If $r \in \mathbb{N}$, $\delta > 0$ and $\Delta \ge \delta^r$, let*

$$B = B(r,\delta,\Delta) := \left\{ \kappa = (\kappa_1, \ldots, \kappa_r) \in \mathbb{R}^r \,\middle|\, \delta \le \kappa_1 \le \kappa_2 \le \ldots \le \kappa_r \text{ and } \prod_{j=1}^{r} \kappa_j \le \Delta \right\},$$

let $A_j \ge 0$ be given for $1 \le j \le r$ and suppose that not all the A_j vanish. Then $F(\kappa) := \sum_{\ell=1}^{r} A_\ell \kappa_\ell$ assumes its maximum value M on B at one (or more) of the r points $\nu^{(0)}, \nu^{(1)}, \ldots, \nu^{(r-1)}$ defined by

$$\nu_\ell^{(t)} := \begin{cases} \delta & \text{if } 1 \le \ell \le t, \\ \left(\Delta/\delta^t\right)^{1/(r-t)} & \text{if } t < \ell \le r. \end{cases}$$

If $\kappa \in B$ and $\kappa \ne \nu^{(0)}$, $\kappa \ne \nu^{(1)}$, …, $\kappa \ne \nu^{(r-1)}$, then $F(\kappa) < M$.

Proof. We first dispose of an extreme case. If $A_j = 0$ for all $j \ge 2$, so $A_1 > 0$, then $\sum_{\ell=1}^{r} A_\ell \kappa_\ell = A_1 \kappa_1$ assumes its maximum value if and only if κ_1 is maximal over all $\kappa \in B(r,\delta,\Delta)$. This implies $\kappa_j = \Delta^{1/r}$ for $1 \le j \le r$, *i.e.* $\kappa = \nu^{(0)}$. Hence we may assume that $A_j > 0$ for some $j \ge 2$.

We proceed by induction on r. For $r = 1$ the lemma reduces to the extreme case treated above. So assume the lemma applies to r and any $B(r,\delta',\Delta')$ with $0 < \delta' \le (\Delta')^{1/r}$.

[5] Remak's proof is also found in [Fr1]. Namely, take $L = \mathbb{Q}$ in equations (3.6) to (3.11) in [Fr1].

Any $\kappa \in B(r+1,\delta,\Delta)$ can be written as $\kappa = (\kappa_1, \kappa')$ where $\kappa' \in B(r, \kappa_1, \Delta/\kappa_1)$ and $\kappa'_\ell := \kappa_{\ell+1}$ $(1 \le \ell \le r)$. Hence

$$\begin{aligned} M := \max_{\kappa \in B(r+1,\delta,\Delta)} \Big\{ \sum_{\ell=1}^{r+1} A_\ell \kappa_\ell \Big\} &= \max_{\kappa_1 \in [\delta, \Delta^{1/(r+1)}]} \Big\{ \max_{\kappa' \in B(r,\kappa_1,\Delta/\kappa_1)} \Big\{ A_1\kappa_1 + \sum_{\ell=1}^{r} A_{\ell+1}\kappa'_\ell \Big\} \Big\} \\ &= \max_{\kappa_1 \in [\delta, \Delta^{1/(r+1)}]} \Big\{ A_1\kappa_1 + \max_{\kappa' \in B(r,\kappa_1,\Delta/\kappa_1)} \Big\{ \sum_{\ell=1}^{r} A_{\ell+1}\kappa'_\ell \Big\} \Big\} \\ &= \max_{\kappa_1 \in [\delta, \Delta^{1/(r+1)}]} \big\{ A_1\kappa_1 + H(\kappa_1) \big\}, \end{aligned}$$

where

$$H(\kappa_1) := \max_{\kappa' \in B(r,\kappa_1,\Delta/\kappa_1)} \Big\{ \sum_{\ell=1}^{r} A_{\ell+1}\kappa'_\ell \Big\}.$$

Moreover, F assumes its maximum at $\nu = (\nu_1, \nu') \in B(r+1,\delta,\Delta)$ if and only if $x \to A_1 x + H(x)$ assumes its maximum for $x \in [\delta, \Delta^{1/(r+1)}]$ at $x = \nu_1$ and $\kappa' \to \sum_{\ell=1}^{r} A_{\ell+1}\kappa'_\ell$ assumes its maximum on $B(r, \nu_1, \Delta/\nu_1)$ at $\kappa' = \nu'$. Note that H is not identically 0 since $A_j > 0$ for some $j \ge 2$.

By induction, for a fixed $\kappa_1 \in [\delta, \Delta^{1/(r+1)}]$ we know that $\sum_{\ell=1}^{r} A_{\ell+1}\kappa'_\ell$ assumes its maximum on $B(r, \kappa_1, \Delta/\kappa_1)$ only at some κ' whose first t coordinates are equal to κ_1 and the remaining $r-t$ coordinates are equal to $\big((\Delta/\kappa_1)/\kappa_1^t\big)^{1/(r-t)}$ $(0 \le t < r)$. Thus

$$A_1\kappa_1 + H(\kappa_1) = A\kappa_1 + B\kappa_1^{-\gamma},$$

where

$$A := \sum_{\ell=1}^{t+1} A_\ell \ge 0, \qquad B := \Delta^{1/(r-t)} \sum_{\ell=t+2}^{r+1} A_\ell \ge 0, \qquad \gamma := \frac{t+1}{r-t} > 0.$$

But the function (of one real variable) $\kappa_1 \to A\kappa_1 + B\kappa_1^{-\gamma}$ has a positive second derivative if $B > 0$, and so assumes no interior maximum. If $B = 0$, then $A > 0$, and again there is no interior maximum. Thus, the maximum for $\kappa_1 \in [\delta, \Delta^{1/(r+1)}]$ occurs only at $\kappa_1 = \delta$ or at $\kappa_1 = \Delta^{1/(r+1)}$.

Suppose first that the maximum occurs at $\kappa_1 = \delta$. By induction we know that $\sum_{\ell=1}^{r} A_{\ell+1}\kappa'_\ell$ assumes its maximum for $\kappa' \in B(r,\delta,\Delta/\delta)$ only at one (or more) of the r points, parameterized by $t = 0, 1, ..., r-1$, where $\kappa'_\ell = \delta$ for $1 \le \ell \le t$, and for $r \ge \ell > t$

$$\kappa'_\ell = \big((\Delta/\delta)/\delta^t\big)^{1/(r-t)} = \big(\Delta/\delta^{t+1}\big)^{1/(r+1-(t+1))}.$$

This means that $F(\kappa) := \sum_{\ell=1}^{r+1} A_\ell \kappa_\ell$ assumes its maximum at $\nu^{(t+1)} \in B(r+1,\delta,\Delta)$.

If the maximum of $A\kappa_1 + B\kappa_1^{-\gamma}$ is assumed at $\kappa_1 = \Delta^{1/(r+1)}$, then $B(r, \kappa_1, \Delta/\kappa_1) = B(r, \Delta^{1/(r+1)}, \Delta^{r/(r+1)})$ reduces to the single point where all coordinates are equal to $\Delta^{1/(r+1)}$, *i.e.* F assumes its maximum only at $\nu^{(0)} \in B(r+1, \delta, \Delta)$. □

The next result improves on [Fr1, Lemma 3.6].

Lemma 3. *Let k be a number field and let $\mu_1 \leq \mu_2 \leq ... \leq \mu_r$ be the successive minima of m_k on the units of k, as in (5). Suppose we are given $R_0 > 0$, $\delta > 0$ and $A_j \geq 0$ $(0 \leq j \leq r)$ such that*

$$R_k \leq R_0, \qquad \mu_1 \geq \delta, \qquad \textit{and} \quad \log|D_k| \leq A_0 + \sum_{j=1}^{r} A_j \mu_j. \tag{7}$$

Then, $\log|D_k| \leq A_0 + \max_{0 \leq t \leq r-1} \{M_t\}$, *where for* $0 \leq t \leq r-1$, $t \in \mathbb{Z}$,

$$M_t := \delta \sum_{\ell=1}^{t} A_\ell \ + \ \left(\frac{\Delta}{\delta^t}\right)^{\frac{1}{r-t}} \sum_{\ell=t+1}^{r} A_\ell, \qquad \qquad \Delta := \sqrt{r+1} R_0 \gamma_r^{r/2}. \tag{8}$$

Proof. We may assume $A_j > 0$ for some $1 \leq j \leq r$, for otherwise the lemma holds trivially. Since $0 < \delta \leq \mu_1 \leq \mu_2 \leq ... \leq \mu_r$, and we have from Minkowski's inequality $\prod_{j=1}^{r} \mu_j \leq \Delta$ (see (5)), it follows that $\delta^r \leq \Delta$. Hence, $(\mu_1, ..., \mu_r) \in B(r, \delta, \Delta)$, in the notation of Lemma 2. Now,

$$\log|D_k| \leq A_0 + \sum_{j=1}^{r} A_j \mu_j \leq A_0 + \max_{\kappa \in B(r,\delta,\Delta)} \left\{ \textstyle\sum_{j=1}^{r} A_j \kappa_j \right\} = A_0 + \max_{0 \leq t \leq r-1} \{M_t\},$$

by Lemma 2. □

When the first minimum $\mu_1 = m_k(\varepsilon)$ occurs at a unit ε contained in a proper subfield $L \subsetneq k$, we will need a lower bound for μ_1 in order to apply Lemma 3. This amounts to finding a lower bound for $m_L(\varepsilon)$ valid for all units $\varepsilon \in L$ (excepting roots of unity, of course). We note the following lower bounds.

- For totally real fields L, Pohst [Po4, p. 98] proved the optimal lower bound

$$m_L(\varepsilon) \geq \sqrt{[L:\mathbb{Q}]} \log\big((1+\sqrt{5})/2\big) \qquad \qquad (L \text{ totally real}, \ \varepsilon \in L). \tag{9}$$

- For fields L of unit-rank one, the regulator gives the sharp lower bound

$$m_L(\varepsilon) \geq \sqrt{2} R_L \qquad \qquad (L = \mathbb{Q}(\varepsilon) \text{ of unit-rank } 1). \tag{10}$$

- Boyd's lower bounds on the height [Bo] give [Fr1, (3.18) with $k = L$]

$$m_L(\varepsilon) \geq \sqrt{3/2} \cdot 0.28 > 0.3429 \qquad (L \text{ of unit-rank } 2,\ \varepsilon \in L,\ [L : \mathbb{Q}] \leq 4). \tag{11}$$

- There is an elementary lower bound [Fr1, p. 613]

$$m_k(\varepsilon) \geq \sqrt{[k : L]}\, m_L(\varepsilon) \qquad (\varepsilon \in L \subset k). \tag{12}$$

3. Analytic methods

3.1. Regulator inequalities

Our main analytic tool will be the absolutely convergent series [Fr1, p. 599]

$$\frac{R_k}{w_k} = \sum_{\mathfrak{a}} g\big(\mathrm{Norm}(\mathfrak{a}^2)/|D_k|\big) + \sum_{\mathfrak{b}} g\big(\mathrm{Norm}(\mathfrak{b}^2)/|D_k|\big). \tag{13}$$

Here R_k is the regulator of the number field k with discriminant D_k, w_k is the number of roots of unity in k, $\mathfrak{a}$ runs over the principal integral ideals in k, $\mathfrak{b}$ runs over the integrals ideals in the ideal class $\overline{\mathfrak{d}}_k$ of the different of k, and $g : (0, \infty) \mapsto \mathbb{R}$ is defined by

$$g(x) = g_{r_1,r_2}(x) := \frac{1}{2^{r_1} 4\pi i} \int_{2-i\infty}^{2+i\infty} (\pi^n 4^{r_2} x)^{-s/2}(2s-1)\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}\, ds, \tag{14}$$

where $r_1 = r_1(k)$ and $r_2 = r_2(k)$ are the number of real and complex places of k, respectively.

Note that the right-hand side of (13) always contains the term $g(1/|D_k|)$, and more generally, the terms $g(j^{2[k:\mathbb{Q}]}/|D_k|)$ for all $j \in \mathbb{N}$. These terms come from the principal ideals generated by $j \in \mathbb{N}$. The other terms are of the form $g(m^2/|D_k|)$ for various integers $m \geq 2$ over which we have no control in general. To turn the formula into an inequality we will therefore need to drop all unknown terms, after insuring that they are positive.

Using the theory of total positivity, in [Fr2] it was shown that g has a unique zero $t_0 \in (0, \infty)$, is negative for $0 < t < t_0$, positive for $t > t_0$, has a unique critical point $t_1 \in (0, \infty)$, is monotone increasing for $0 < t < t_1$ and monotone decreasing for $t > t_1$. In particular, g has no minimum in any open interval (a, b), and so

$$g(t) \geq \min\big(g(a), g(b)\big) \qquad (a \leq t \leq b). \tag{15}$$

The next lemma gives a version of the inequality $R_k/w_k \geq g(1/|D_k|)$ [Fr2] suited to a range of discriminants.

Lemma 4. *If* $0 < d_1 \le |D_k| \le d_2$ *and* $g = g_{r_1,r_2}$ *is as in* (14), *then*

$$\frac{R_k}{w_k} \ge g(1/|D_k|) \ge \min\bigl(g(1/d_1), g(1/d_2)\bigr). \tag{16}$$

If the ideal class $\bar{\mathfrak{d}}_k$ *of the different of* k *is trivial, then*

$$\frac{R_k}{w_k} \ge 2g(1/|D_k|) \ge 2\min\bigl(g(1/d_1), g(1/d_2)\bigr). \tag{17}$$

Proof. The right-most inequalities are just (15). The inequality $R_k/w_k \ge g(1/|D_k|)$ is trivial if $g(1/|D_k|) \le 0$, so we may assume $g(1/|D_k|) > 0$. Then $g(m^2/|D_k|) > 0$ for all $m \ge 1$. Hence, we obtain (16) by dropping all terms in (13), except the one coming from the unit ideal. When $\bar{\mathfrak{d}}_k$ is trivial, the two sums in (13) coincide. Hence we double our estimate. □

Since $w_k \ge 2$, we have in general

$$R_k \ge 2\min\bigl(g(1/d_1), g(1/d_2)\bigr) \qquad \text{if } 0 < d_1 \le |D_k| \le d_2, \tag{18}$$

and

$$R_k \ge 4\min\bigl(g(1/d_1), g(1/d_2)\bigr) \quad \text{if } \bar{\mathfrak{d}}_k \text{ is trivial and } 0 < d_1 \le |D_k| \le d_2. \tag{19}$$

In a few signatures the contribution from the unit ideal will not suffice for our purposes. The next lemma incorporates N ideals.

Lemma 5. *Let* $0 < d_1 \le |D_k| \le d_2 \le d_3$, *let* $N \in \mathbb{N}$, *and assume* $g_{r_1,r_2}(4/d_3) \ge 0$. *Then* $R_k/w_k \ge G(d_1, d_2, N)$, *where* $G = G_{r_1,r_2}$ *is defined by*

$$G(d_1, d_2, N) := \sum_{j=1}^{N} \min\bigl(g(j^{2n}/d_1), g(j^{2n}/d_2)\bigr) \qquad (g := g_{r_1,r_2},\ \ n := r_1 + 2r_2).$$

Proof. Except for the term coming from the unit ideal, all terms in formula (13) are of the form $g(m^2/|D_k|)$, where $m \ge 2$. From [Fr2] we know that g has a unique zero $t_0 \in (0, \infty)$. Furthermore, $g(t) < 0$ for $t < t_0$, while $g(t) > 0$ for $t > t_0$. Since by assumption $|D_k| \le d_3$ and $g(4/d_3) > 0$, we have $t_0 \le 4/d_3 \le m^2/|D_k|$ for $m \ge 2$. Hence $g(m^2/|D_k|) > 0$ for $m \ge 2$. We therefore obtain a lower bound on R_k/w_k by dropping all terms from the series (13), except those coming from the principal ideals (j) generated by rational integers j satisfying $1 \le j \le N$. Thus,

$$\frac{R_k}{w_k} \ge \sum_{j=1}^{N} g(j^{2n}/|D_k|) \ge \sum_{j=1}^{N} \min\bigl(g(j^{2n}/d_1), g(j^{2n}/d_2)\bigr),$$

where we used $j^{2n}/d_2 \le j^{2n}/|D_k| \le j^{2n}/d_1$ and (15). □

Table 2
Lower bounds for $|D_k|^{1/n}$ for $n = [k : \mathbb{Q}]$, assuming $h_k \geq 3$.

n	r_1	$\|D\|^{1/n} >$	γ	n	r_1	$\|D\|^{1/n} >$	γ
2	2	7.941	1.2	6	2	11.968	0.74
3	1	7.558	1.1	6	4	15.536	0.68
3	3	11.823	0.92	6	6	20.314	0.64
4	0	7.412	1.02	7	1	11.204	0.73
4	2	10.468	0.88	7	3	14.036	0.67
4	4	15.121	0.78	7	5	17.686	0.62893
5	1	9.747	0.85	7	7	22.389	0.59
5	3	13.136	0.76	8	0	10.668	0.71
5	5	17.919	0.7	8	8	24.206	0.558
6	0	9.305	0.82	9	9	25.811	0.53

3.2. The class of the different

The analytic regulator inequalities (19) and (18) differ by a factor of 2, which we cannot afford to lose for discriminants near the minimum for a given signature. Fortunately, Odlyzko's discriminant bounds [Poi] (surveyed in [Od]) imply that $|D_k|$ cannot be small if the class $\overline{\mathfrak{d}}_k$ of the different is not trivial. Indeed, Hecke showed that $\overline{\mathfrak{d}}_k$ is the square of an ideal class [He, p. 234], [We, p. 291]. Therefore, if $\overline{\mathfrak{d}}_k$ is not trivial, it follows that the class number $h_k \geq 3$. By classfield theory, k possesses an unramified extension K/k of degree at least 3. As in any unramified extension K/k, we have

$$\frac{\log |D_K|}{[K:\mathbb{Q}]} = \frac{\log |D_k|}{[k:\mathbb{Q}]}, \qquad \frac{r_1(K)}{[K:\mathbb{Q}]} = \frac{r_1(k)}{[k:\mathbb{Q}]}.$$

Odlyzko's lower bounds [Poi] for the root discriminant are monotone increasing in the degree n, for a fixed ratio r_1/n. Hence, if $\overline{\mathfrak{d}}_k$ is not trivial, we can bound $\log |D_k|/[k : \mathbb{Q}]$ by the corresponding bound in degree $3[k : \mathbb{Q}]$. We do this in Table 2 above.[6]

3.3. Unconditional computation of regulators

All our regulator calculations rely on PARI [GP], which would seem to make their correctness conditional on the Generalized Riemann Hypothesis [Co, p. 353]. Fortunately, when PARI returns a (GRH-assuming) regulator $\tilde{R}_k$ for the number field k, it means unconditionally that it has found a system of independent units and that $\tilde{R}_k = mR_k$, where $m \in \mathbb{N}$ and R_k is the true regulator. GRH is only needed to ensure $m = 1$.

[6] In Table 2, D is the discriminant of the number field k of degree n, r_1 is the number of real places of k and we assume that the class number $h_k \geq 3$. The bound was obtained using the parameter γ in Table 2, which defines the auxiliary function $f(x) := T(x\gamma)$, where

$$T(x) := \left(3(\sin(x) - x\cos(x))/x^3\right)^2$$

is the Tartar function. The function $f(x) = T(x\gamma)$ is inserted in (13) of [Poi] to obtain a lower bound for $\frac{1}{n}\log |D|$ corresponding to degree $3[k : \mathbb{Q}]$.

All numerical values of regulators we will use will correspond to fields for which the ideal class of the different $\bar{\mathfrak{d}}_k$ is unconditionally known to be trivial. Then the analytic lower bound (17) gives

$$m = \frac{\tilde{R}_k/w_k}{R_k/w_k} \leq \frac{\tilde{R}_k/w_k}{2g(1/|D_k|)},$$

provided $g(1/|D_k|) > 0$. Since $\tilde{R}_k$, w_k and $2g(1/|D_k|)$ can be computed without GRH, we show $m = 1$ unconditionally by verifying that $0 < \tilde{R}_k/\big(2w_k g(1/|D_k|)\big) < 2$. We check this below every time we use a numerical approximation of the regulator.

4. Prime degrees

All proofs in this section proceed along the following lines. We fix a signature and begin by assuming $R_k \leq R_0$, where R_0 is the regulator lower bound that we seek to establish for that signature. We apply Lemma 1, and the fact that $k = \mathbb{Q}(\varepsilon_1)$ if $[k : \mathbb{Q}]$ is prime, to find an upper bound $|D_k| \leq d_2$. Here and below ε_1 is always a unit of k at which the minimum μ_1 of m_k is attained (see (1) and (5)).

Our first aim is to show that the ideal class $\bar{\mathfrak{d}}_k$ of the different is trivial. If $\bar{\mathfrak{d}}_k$ is not trivial, Table 2 gives a lower bound $|D_k| \geq d_1$. If $d_1 > d_2$ we conclude that $\bar{\mathfrak{d}}_k$ is trivial. If $d_1 \leq d_2$, we verify that $2g(1/d_2) > R_0$ and $2g(1/d_1) > R_0$. The analytic lower bound (18) then contradicts $R_k \leq R_0$.

Having thus shown that $\bar{\mathfrak{d}}_k$ is trivial, we find a (small) $d_1' < d_2$ such that $4g(1/d_1') > R_0$. This implies $|D_k| < d_1'$, because of (19) and the previously verified $2g(1/d_2) > R_0$. Lastly, we resort to tables to inspect all fields in the discriminant range $|D_k| < d_1'$. We find the (usually three or four) exceptions to $R_k > R_0$ amongst these. While we give references to the published tables in each case, the reader can save time by downloading number field tables for all signatures up to degree 7 in http://pari.math.u-bordeaux1.fr/pub/pari/packages/nftables/, which includes regulators.

4.1. Degree 2

Although this case is very easy, we include it for later reference and as an outline of the general proof. We deliberately do not exploit special facts about quadratic fields, such as the triviality of $\bar{\mathfrak{d}}_k$.

Theorem 6. (Quadratics) *With three exceptions, all real quadratic fields k satisfy $R_k >$ 1.31. The exceptions are the real quadratic fields of discriminant* 5, 8 *and* 13, *with respective regulators* 0.481211..., 0.881373... *and* 1.194763... .

Proof. Assume $R_k \leq 1.31$ and let $\mu_1 = m_k(\varepsilon_1)$ be the minimum of the quadratic form m_k on the unit lattice. Then (5), (4) and $k = \mathbb{Q}(\varepsilon_1)$ show that $|D_k| < 54.95$.[7]

Table 2 shows that $\bar{\mathfrak{d}}_k$ is trivial, for otherwise $|D_k| > 63$. Since $4g_{2,0}(1/20.2) = 1.3108...$ and $4g_{2,0}(1/54) = 1.6183...$, the analytic bounds (19) show that $|D_k| \leq 20$. Inspecting the five real quadratic fields with $|D_k| \leq 20$ we find the three fields in the theorem. □

Of course, we could go on with a much longer ordered list of regulators. We will discuss this in §6 below.

4.2. Degree 3

Regulator lower bounds for cubic fields are not new. The totally real case is explicitly proved by Pohst [Po4, p. 491], and the complex case is nearly as easy as the quadratic case given Remak's inequality (2) and the old lists by Delone and Faddeev [DF] of cubic fields of small discriminant.

Theorem 7. (Cubics)

- *With three exceptions, all totally real cubic fields k satisfy $R_k > 1.66$. The exceptions are the unique cubic fields of discriminant* 49, 81 *and* 169*, with respective regulators* 0.525454..., 0.849287... *and* 1.365049... .
- *With three exceptions, all complex cubic fields k satisfy $R_k > 0.79$. The exceptions are the unique cubic fields of discriminant* -23, -31 *and* -44*, with respective regulators* 0.281199..., 0.382245... *and* 0.609377... .

Proof. Suppose k is a totally real cubic field, assume $R_k \leq 1.66$, and let $\mu_1 = m_k(\varepsilon_1)$ as in (5). Since $k = \mathbb{Q}(\varepsilon_1)$, Pohst's bound (4) and (5) yield $|D_k| < 692.25$. Table 2 shows that $\bar{\mathfrak{d}}_k$ is trivial, for otherwise $|D_k| > 1\,652$. Since $4g_{3,0}(1/332) = 1.661...$ and $4g_{3,0}(1/692) = 2.144...$, the analytic bounds (19) show that $|D_k| \leq 331$. Inspecting the eight cubic fields in the range $0 < D_k \leq 331$, we find the three fields in the theorem.[8]

Suppose now that k is a complex cubic field and assume $R_k \leq 0.79$. Remak's bound (2) and (5) give $|D_k| < 416.74$. Table 2 shows that $\bar{\mathfrak{d}}_k$ is trivial, for otherwise $|D_k| > 431$. Since $4g_{1,1}(1/121) = 0.791...$ and $4g_{1,1}(1/416) = 1.077...$, we find $|D_k| \leq 120$. Inspecting the 11 cubic fields with $-120 \leq D_k < 0$, we arrive at the three fields in the theorem. □

[7] We nearly always write $|D_k|$ even if the sign of D_k is obvious.

[8] The cubic fields of discriminant 49 and 81 are the first two by discriminant, but that of discriminant 169 is the fourth. The third one, of discriminant 148, has regulator 1.662... . Except in the totally complex case, it will turn out that the field of smallest discriminant has the smallest regulator in all signatures that we examine. However the second and third regulator occasionally do not correspond to the second and third discriminant. In the totally complex case, the discriminant and regulator order are quite different, as the reader can see for octic fields in Table 4 below. To order the totally complex case nicely, we should replace R_k by R_k/w_k, the regulator divided by the number of roots of unity.

4.3. Degree 5

Minimal quintic discriminants were found by Hunter [Hu]. Lists of initial discriminants can be found in [Po1,Ri,Di5,Vo,SPD].

Theorem 8. (Quintics)

- *With three exceptions, all totally real quintic fields k satisfy $R_k > 3.55$. The exceptions are the unique totally real quintic fields of discriminant* $14\,641$, $24\,217$ *and* $38\,569$*, with respective regulators* $1.635694...$, $2.399421...$ *and* $3.155437...$.
- *With three exceptions, all fields k of signature $(r_1, r_2) = (3, 1)$ satisfy $R_k > 0.75$. The exceptions are the unique fields of signature $(3, 1)$ with discriminant* $-4\,511$, $-4\,903$ *and* $-5\,519$*, with respective regulators* $0.628579...$, $0.668925...$ *and* $0.732128...$.
- *With three exceptions, all fields k of signature $(1, 2)$ satisfy $R_k > 0.34$. The exceptions are the unique fields of signature $(1, 2)$ with discriminant* $1\,609$, $1\,649$ *and* $1\,777$*, with respective regulators* $0.268355...$, $0.273599...$ *and* $0.290415...$.

Proof. Suppose k is a totally real quintic field with $R_k \leq 3.55$. Pohst's bound (4) and (5) give $|D_k| < 4\,862\,856.55$. If $\overline{\mathfrak{d}}_k$ is not trivial, Table 2 gives $|D_k| > 1\,847\,433.6$. However, $2g_{5,0}(1/1\,847\,434) = 6.509...$, $2g_{5,0}(1/4\,862\,856) = 8.315...$, and (18) yield $R_k > 6.509$, contradicting our assumption that $R_k \leq 3.55$. Hence $\overline{\mathfrak{d}}_k$ is trivial. But then $4g_{5,0}(1/60\,470) = 3.5501...$, $4g_{5,0}(1/4\,862\,856) = 16.63...$ and (19) show that $|D_k| < 60\,470$. Checking the four totally real quintic fields in this discriminant range we arrive at the three fields in the theorem.

Suppose now that k has signature $(3, 1)$ and $R_k \leq 0.75$. Remak's bound (2) and (5) yield $|D_k| < 8\,604\,833.12$. If $\overline{\mathfrak{d}}_k$ is not trivial, Table 2 shows $|D_k| > 391\,125.11$. As before, $2g_{3,1}(1/391\,126) = 2.1588...$ and $2g_{3,1}(1/8\,604\,833) = 3.4264...$ lead to $R_k > 2.1588$, so $\overline{\mathfrak{d}}_k$ must be trivial. Now $4g_{3,1}(1/6\,055) = 0.75007...$ shows $|D_k| < 6\,055$. Inspecting the four fields of signature $(3, 1)$ with $|D_k| < 6\,055$, we arrive at the three fields in the theorem.

Lastly, suppose k has signature $(1, 2)$ and $R_k \leq 0.34$. As before, we obtain $|D_k| < 245\,407.26$. Table 2 gives $|D_k| > 87\,974.09$ if $\overline{\mathfrak{d}}_k$ is not trivial. But $2g_{1,2}(1/87\,975) = 0.751...$ and $2g_{1,2}(1/245\,407) = 0.902...$ show that $\overline{\mathfrak{d}}_k$ is trivial. Since $4g_{1,2}(1/2\,352) = 0.3401...$, checking the five fields in signature $(1, 2)$ with $|D_k| < 2\,352$, we arrive at the three fields in the theorem. □

4.4. Degree 7

In degree 7 we encounter our first difficulties. Signatures $(7, 0)$ and $(1, 3)$ are easy sailing. In signature $(3, 2)$ we manage to prove a sharp lower bound using Lemma 5. In signature $(5, 1)$ the regulator of the first field (by discriminant) is $2.8846...$, but we fail to prove $R_k > 2.88$. The best we can do is to prove $R_k > 2.11$. Tables of small discriminants in degree 7 can be found in [Po3,Di1,Di2,Di4,Let,Vo].

Theorem 9. (Heptics)

- *With three exceptions, all totally real fields k of degree 7 satisfy $R_k > 19.19$. The exceptions are the unique totally real fields in degree 7 of discriminant* 20 134 393, 25 367 689 *and* 28 118 369, *with respective regulators* 14.446932..., 16.005863... *and* 18.127843... .
- *All fields k of signature $(r_1, r_2) = (5, 1)$ satisfy $R_k > 2.11$.*
- *With four exceptions, all fields k of signature $(r_1, r_2) = (3, 2)$ satisfy $R_k > 1.055$. The exceptions are the unique fields of signature $(3, 2)$ with discriminant* 612 233, 612 569, 640 681 *and* 649 177, *with regulators* 1.004348..., 1.004731, 1.035721... *and* 1.044578..., *respectively.*
- *With four exceptions, all fields k of signature $(1, 3)$ satisfy $R_k > 0.4$. The exceptions are the unique fields of signature $(1, 3)$ with discriminant* −184 607, −193 327, −193 607 *and* −196 127, *with regulators* 0.380447..., 0.393017..., 0.393408... *and* 0.396915..., *respectively.*

Proof. Suppose k is a totally real field of degree 7 with $R_k \leq 19.19$. Applying (4) to $k = \mathbb{Q}(\varepsilon_1)$, we get $|D_k| < \mathrm{e}^{30.44}$. Table 2 tells us that $|D_k| > \mathrm{e}^{21.75}$ if $\overline{\mathfrak{d}}_k$ is not trivial. But (18), $2g_{7,0}(\mathrm{e}^{-21.75}) = 45.01...$, and $2g_{7,0}(\mathrm{e}^{-30.44}) = 191.81...$ give $R_k > 45.01$. Hence $\overline{\mathfrak{d}}_k$ is trivial. Calculating $4g_{7,0}(1/49\,890\,000) = 19.191...$, by (19) we see that $|D_k| < 49\,890\,000$. On calculating the regulators of the 20 totally real fields of degree 7 in this discriminant range [Vo], we arrive at the three fields in the theorem. This concludes the proof in the totally real case.

Leaving signature $(5, 1)$ for the end of the proof, we now assume that k has signature $(3, 2)$ and $R_k \leq 1.055$. Then $|D_k| < \mathrm{e}^{28.3595}$, by (2). Table 2 shows that $\overline{\mathfrak{d}}_k$ is trivial if $|D_k| < \mathrm{e}^{18.49}$. We compute

$$2g_{3,2}(\mathrm{e}^{-18.49}) = 4.2823..., \qquad 2g_{3,2}(\mathrm{e}^{-27.6}) = 1.3283...,$$

and conclude from (18) that $\mathrm{e}^{18.49} \leq |D_k| \leq \mathrm{e}^{27.6}$ is impossible. We consider first the case $|D_k| < \mathrm{e}^{18.49}$, so that $\overline{\mathfrak{d}}_k$ is trivial. Then (19) and $4g_{3,2}(1/701\,100) = 1.055026...$ show that $|D_k| < 701\,100$. Inspection of the 8 fields in this signature and discriminant range leads to the four fields in the theorem.

The remaining possibility in signature $(3, 2)$ is $\mathrm{e}^{27.6} \leq |D_k| \leq \mathrm{e}^{28.3595}$. Since $g_{3,2}(4\mathrm{e}^{-28.3595}) = 2.255... > 0$, Lemma 5 applies in any range $0 < d_1 \leq |D_k| \leq d_2$ for $d_2 \leq \mathrm{e}^{28.3595}$. Subdividing $[\mathrm{e}^{27.6}, \mathrm{e}^{28.3595}]$ into a succession of short intervals, and letting $G := G_{3,2}$ be as defined in Lemma 5, we compute

$$2G(\mathrm{e}^{28.355}, \mathrm{e}^{28.3595}, 5) = 1.05638..., \qquad 2G(\mathrm{e}^{28.35}, \mathrm{e}^{28.355}, 5) = 1.08413...,$$
$$2G(\mathrm{e}^{28.34}, \mathrm{e}^{28.35}, 5) = 1.10820..., \qquad 2G(\mathrm{e}^{28.32}, \mathrm{e}^{28.34}, 5) = 1.15621...,$$
$$2G(\mathrm{e}^{28.2}, \mathrm{e}^{28.32}, 5) = 1.13342..., \qquad 2G(\mathrm{e}^{27.6}, \mathrm{e}^{28.2}, 5) = 1.21149... .$$

Lemma 5 now shows $R_k > 1.056$ if $\mathrm{e}^{27.6} \leq |D_k| \leq \mathrm{e}^{28.3595}$, concluding the proof for signature $(3, 2)$.

Suppose now that k has signature $(1, 3)$ and $R_k \leq 0.4$. Then $|D_k| < \mathrm{e}^{22.59}$, as follows from (2). From Table 2 we have $|D_k| > \mathrm{e}^{16.91}$ if $\overline{\mathfrak{d}}_k$ is not trivial. But (18), $2g_{1,3}(\mathrm{e}^{-16.91}) = 1.373...$, and $2g_{1,3}(\mathrm{e}^{-22.59}) = 2.205...$ give $R_k > 1.373$. Hence we may assume that $\overline{\mathfrak{d}}_k$ is trivial. Then (18) and $4g_{1,3}(1/211\,000) = 0.4003...$ show that $|D_k| < 211\,000$. On examining the nine fields with $|D_k| < 211\,000$ and signature $(1, 3)$ we find the four fields in the theorem. This concludes the proof in signature $(1, 3)$.

Lastly, assume k has signature $(5, 1)$ and $R_k \leq 2.11$. We proceed as in signature $(3, 2)$. Remak's inequality (2) gives $|D_k| < \mathrm{e}^{31.5554}$. Table 2 gives $|D_k| > \mathrm{e}^{20.1}$ if $\overline{\mathfrak{d}}_k$ is not trivial. From

$$2g_{5,1}(\mathrm{e}^{-20.1}) = 13.705..., \qquad 2g_{5,1}(\mathrm{e}^{-30.7}) = 3.656...,$$

and (18), we conclude that $\mathrm{e}^{20.1} \leq |D_k| \leq \mathrm{e}^{30.7}$ is not possible. If $|D_k| \leq \mathrm{e}^{20.1}$, then $\overline{\mathfrak{d}}_k$ is trivial. Odlyzko's lower bounds yield $|D_k| \geq 1\,702\,492$ in signature $(5, 1)$ (use $\gamma = 1.16$ in Tartar's function, as explained in the proof of Table 2). Since $4g_{5,1}(1/1\,702\,492) = 2.3409...$, we see that the range $|D_k| \leq \exp(20.1)$ is ruled out. To handle the remaining case, *i.e.* $\mathrm{e}^{30.7} \leq |D_k| < \mathrm{e}^{31.5554}$, we check $g_{5,1}(4\mathrm{e}^{-31.5554}) = 7.9293... > 0$ and conclude by applying Lemma 5 to subintervals. The computations

$$2G_{5,1}(\mathrm{e}^{31.5}, \mathrm{e}^{31.5554}, 5) = 2.1737..., \qquad 2G_{5,1}(\mathrm{e}^{31.3}, \mathrm{e}^{31.5}, 5) = 2.8234...,$$
$$2G_{5,1}(\mathrm{e}^{30.7}, \mathrm{e}^{31.3}, 5) = 5.9206...,$$

showing $R_k > 2.17$ for $\mathrm{e}^{30.7} \leq |D_k| < \mathrm{e}^{31.5554}$, rule out this final possibility. □

5. Composite degrees

In composite degree, $k \neq \mathbb{Q}(\varepsilon_1)$ is possible, so we may need several independent units to generate k. This forces us to find a lower bound for $m_k(\varepsilon_1)$ in order to apply Lemma 3.

In the totally complex case, we will have to give CM fields separate consideration. If k is a CM field with maximal totally real subfield k_+, then [Re, p. 250], [Wa, p. 39]

$$R_k = 2^{[k_+:\mathbb{Q}]-1} R_{k_+}/Q \geq 2^{[k_+:\mathbb{Q}]-2} R_{k_+}, \tag{20}$$

where $Q = 1$ or 2 is the unit index of k. Fortunately, the regulator lower bounds already proved for totally real fields will easily suffice to dispose of the CM case.

5.1. Degree 4

The original lists of quartics were found by Delone and Faddeev [DF] and Godwin [Go1,Go2,Go3]. Long lists can be found in [BF,Fo,BFP].

Theorem 10. (Quartics)

- *With three exceptions, all totally real quartic fields k satisfy $R_k > 1.85$. The exceptions are the unique totally real quartic fields of discriminant* 725, 1 125 *and* 1 600, *with respective regulators* 0.825068..., 1.165455... *and* 1.542505... .
- *With three exceptions, all fields k of signature $(r_1, r_2) = (2, 1)$ satisfy $R_k > 0.51$. The exceptions are the unique fields of signature* $(2, 1)$ *with discriminant* -275, -283 *and* -331, *with respective regulators* 0.369184..., 0.378199... *and* 0.432203... .
- *With three exceptions, all totally complex quartic fields k satisfy $R_k > 0.61$. The exceptions are the unique totally complex quartic fields with discriminant* 229, 257 *and* 117, *with respective regulators* 0.337377..., 0.442137... *and* 0.543535... .

Proof. We leave the CM case for the end of the proof. If k is a non-CM quartic field, then either $k = L := \mathbb{Q}(\varepsilon_1)$, or L is a real quadratic field and $k = L(\varepsilon_2)$. In the latter case, (4) and (12) imply

$$\log|D_L| \le \log 4 + \sqrt{2} m_L(\varepsilon_1) \le \log 4 + m_k(\varepsilon_1) = \log 4 + \mu_1. \tag{21}$$

By Lemma 1 and (21),

$$\begin{aligned} \log|D_k| &\le 2\log|D_L| + 4\log 2 + A(k/L) m_k(\varepsilon_2) \le 4\log 4 + 2\mu_1 + A(k/L)\mu_2 \\ &= \begin{cases} 4\log 4 + 2\mu_1 + 2\mu_2 & \text{if } \big(r_1(k), r_2(k)\big) = (4, 0), \\ 4\log 4 + 2\mu_1 + \sqrt{2}\mu_2 & \text{if } \big(r_1(k), r_2(k)\big) = (2, 1). \end{cases} \end{aligned} \tag{22}$$

Note that the case $\big(r_1(k), r_2(k)\big) = (0, 2)$ and $k \ne L$ is the (for now) excluded CM case. If $k \ne L$, then $m_L(\varepsilon_1) \ge \sqrt{2} R_L \ge \sqrt{2}\log\big((1+\sqrt{5})/2\big)$, by (10) and Theorem 6. Since $\mu_1 := m_k(\varepsilon_1) \ge \sqrt{2} m_L(\varepsilon_1)$, we find $\mu_1 > 0.962$. We now examine each signature separately.

Suppose k is a totally real quartic with $R_k \le 1.85$. If $k = L$, then (4) gives $|D_k| \le 37\,670$. If $k \ne L$, then (22) and Lemma 3, with $\delta = 0.962$ and $R_0 = 1.85$, yield $|D_k| < 265\,544.46$. In either case, $|D_k| \le 265\,544$. If the ideal class of the different $\bar{\mathfrak{d}}_k$ is not trivial, Table 2 gives $|D_k| > 52\,278$. Since $2g_{4,0}(1/265\,544) = 3.68...$, and $2g_{4,0}(1/52\,279) = 2.74...$, we conclude that $\bar{\mathfrak{d}}_k$ is trivial. We obtain $|D_k| < 2\,775$ from $4g_{4,0}(1/2\,775) = 1.851...$. There are ten such totally real quartic fields, of which only the three in the theorem satisfy $R_k \le 1.85$.

Suppose now that $\big(r_1(k), r_2(k)\big) = (2, 1)$ and $R_k \le 0.51$. If $k = L$, then (2) gives $|D_k| < 18\,583.6$. If $k \ne L$, (22) and Lemma 3 yield $|D_k| < 8\,049.5$. In both cases we have $|D_k| \le 18\,583$. Table 2 gives $|D_k| > 12\,007.56$ if $\bar{\mathfrak{d}}_k$ is not trivial. Since $2g_{2,1}(1/18\,583) = 1.065...$, and $2g_{2,1}(1/12\,008) = 0.983..$, we again find that $\bar{\mathfrak{d}}_k$ is trivial. From $4g_{2,1}(1/443) = 0.511...$, we obtain $|D_k| < 443$, leading to four number fields, of which only the three in the theorem satisfy $R_k \le 0.51$.

Lastly, assume that k is a totally complex quartic field with $R_k \le 0.61$. If $k = L$, then (2) gives $|D_k| < 2937.1$. We note that this is true even if k is CM. We must have that $\overline{\mathfrak{d}}_k$ is trivial, since otherwise Table 2 gives $|D_k| > 3\,018.15$. From $4g_{0,2}(1/2937) = 0.759...$, and $4g_{2,1}(1/1\,177) = 0.611..$, we find $|D_k| < 1\,177$. There are 46 totally complex quartic fields with discriminants in this range. Only the three in the theorem satisfy $R_k \le 0.61$.

If $k \ne L$, then k is necessarily a CM field and L is its maximal totally real subfield. From (20) we have $R_k \ge 2R_L/Q$, where Q is the unit index of k. If $Q = 1$, Theorem 6 shows $R_k > 0.96$, a contradiction. As k has unit-rank 1, it is clear that if $Q = 2$, then a shortest unit $\varepsilon_1 \notin L$, again a contradiction. □

5.2. Degree 6

Tables for sextic fields can be found in [Po1,Po5,BMO,Ol1,Ol2,Vo].

Theorem 11. (Sextics)

- *With three exceptions, all totally real sextic fields k satisfy $R_k > 4.39$. The exceptions are the unique totally real sextic fields of discriminant* 300 125, 371 293 *and* 434 581, *with respective regulators* 3.277562..., 3.774500... *and* 4.187943... .
- *With three exceptions, all fields k of signature $(r_1, r_2) = (4, 1)$ satisfy $R_k > 1.37$. The exceptions are the unique fields of signature $(4, 1)$ with discriminant* −92 779, −94 363 *and* −103 243, *with respective regulators* 1.262710..., 1.277066... *and* 1.359897... .
- *With three exceptions, all fields k of signature $(r_1, r_2) = (2, 2)$ satisfy $R_k > 0.50$. The exceptions are the unique fields of signature $(2, 2)$ with discriminant* 28 037, 29 077 *and* 29 189, *with respective regulators* 0.478924..., 0.491602... *and* 0.492916... .
- *With three exceptions, all totally complex sextic fields k satisfy $R_k > 0.27$. The exceptions are the unique totally complex sextic fields of discriminant* −10 051, −10 571 *and* −12 167, *with respective regulators* 0.205216..., 0.213209... *and* 0.237219... .

Proof. Let k be a sextic field of signature (r_1, r_2) with

$$R_k \le \begin{cases} 4.39 & \text{if } (r_1, r_2) = (6, 0), \\ 1.37 & \text{if } (r_1, r_2) = (4, 1), \\ 0.5 & \text{if } (r_1, r_2) = (2, 2), \\ 0.27 & \text{if } (r_1, r_2) = (0, 3). \end{cases} \tag{23}$$

The CM case is easily dismissed, since (20) and Theorem 7 yield

$$R_k \ge 2R_{k_+} \ge 2 \cdot 0.525 > 0.27.$$

We will obtain upper bounds for $|D_k|$ according to the different possibilities for $L := \mathbb{Q}(\varepsilon_1) \subset k$. Suppose first that $k = L$. Then from (2), or from (4) in the totally real case, we find

$$|D_k| < \begin{cases} \exp(20.723) & \text{if } (r_1, r_2) = (6, 0), \\ \exp(23.73) & \text{if } (r_1, r_2) = (4, 1), \\ \exp(19.3) & \text{if } (r_1, r_2) = (2, 2), \\ \exp(14.91) & \text{if } (r_1, r_2) = (0, 3). \end{cases} \qquad \big(k = \mathbb{Q}(\varepsilon_1)\big)$$

Next we assume that L is a real quadratic field. Then k cannot be totally complex. As in (21) and (22), we find

$$\log |D_L| \leq \log 4 + \sqrt{2}\, m_L(\varepsilon_1) \leq \log 4 + \sqrt{2/3}\, m_k(\varepsilon_1) = \log 4 + \sqrt{2/3}\, \mu_1.$$

Therefore, from (2) in Lemma 1,

$$\log |D_k| \leq \begin{cases} 3 \log 4 + 6 \log 3 + \sqrt{6}\, \mu_1 + 4\mu_2 & \text{if } (r_1, r_2) = (6, 0), \\ 3 \log 4 + 6 \log 3 + \sqrt{6}\, \mu_1 + \sqrt{14}\, \mu_2 & \text{if } (r_1, r_2) = (4, 1), \\ 3 \log 4 + 6 \log 3 + \sqrt{6}\, \mu_1 + 2\sqrt{3}\, \mu_2 & \text{if } (r_1, r_2) = (2, 2). \end{cases}$$

By (9) and (12), we have $m_k(\varepsilon_1) \geq \sqrt{6} \log\big((1 + \sqrt{5})/2\big) > 1.178$. From this, (5) and Lemma 3, we obtain

$$|D_k| < \begin{cases} \exp(23.52) & \text{if } (r_1, r_2) = (6, 0), \\ \exp(20.5) & \text{if } (r_1, r_2) = (4, 1), \\ \exp(17.72) & \text{if } (r_1, r_2) = (2, 2). \end{cases} \qquad \big(\mathbb{Q}(\varepsilon_1) \text{ real quadratic}\big)$$

Now assume that L is a complex cubic field. Then k must have signature $(2, 2)$ or $(0, 3)$. From (2) and (12) we get

$$\log |D_L| \leq 3 \log 3 + \sqrt{6}\, m_L(\varepsilon_1) \leq 3 \log 3 + \sqrt{3}\, \mu_1.$$

Hence,

$$\log |D_k| \leq \begin{cases} 6 \log 3 + 6 \log 2 + 2\sqrt{3}\, \mu_1 + 2\mu_2 & \text{if } (r_1, r_2) = (2, 2), \\ 6 \log 3 + 6 \log 2 + 2\sqrt{3}\, \mu_1 + \sqrt{2}\, \mu_2 & \text{if } (r_1, r_2) = (0, 3). \end{cases}$$

From (10) and Theorem 7, $m_L(\varepsilon_1) \geq \sqrt{2} \cdot 0.281$, whence $\mu_1 := m_k(\varepsilon_1) \geq 2 \cdot 0.281$. Lemma 3 now gives

$$|D_k| < \begin{cases} \exp(16.89) & \text{if } (r_1, r_2) = (2, 2), \\ \exp(14.34) & \text{if } (r_1, r_2) = (0, 3). \end{cases} \qquad \big(\mathbb{Q}(\varepsilon_1) \text{ complex cubic}\big)$$

Lastly, suppose L is a totally real cubic field. In this case $k = L(\varepsilon_j)$, where $j = 2$ or 3. Since we have already ruled out the CM case, k cannot be totally complex. From (4) we have

$$\log |D_L| \leq \log 4 + \sqrt{8}\, m_L(\varepsilon_1) \leq \log 4 + 2m_k(\varepsilon_1) = \log 4 + 2\mu_1.$$

Hence, for $j = 2$ or 3 and $\mu_j := m_k(\varepsilon_j)$,

$$\log |D_k| \leq \begin{cases} 2\log 4 + 6\log 2 + 4\mu_1 + \sqrt{6}\,\mu_j & \text{if } (r_1, r_2) = (6, 0), \\ 2\log 4 + 6\log 2 + 4\mu_1 + 2\mu_j & \text{if } (r_1, r_2) = (4, 1), \\ 2\log 4 + 6\log 2 + 4\mu_1 + \sqrt{2}\,\mu_j & \text{if } (r_1, r_2) = (2, 2). \end{cases}$$

By (12) and (9),

$$\mu_1 := m_k(\varepsilon_1) \geq \sqrt{2}\, m_L(\varepsilon_1) \geq \sqrt{6}\, \log\big((1 + \sqrt{5})/2\big) > 1.178,$$

so Lemma 3 yields, for $j = 2$ or 3,

$$|D_k| < \begin{cases} \exp(19.71) & \text{if } (r_1, r_2) = (6, 0), \\ \exp(16.38) & \text{if } (r_1, r_2) = (4, 1), \\ \exp(13.31) & \text{if } (r_1, r_2) = (2, 2). \end{cases} \qquad \big(\mathbb{Q}(\varepsilon_1) \text{ totally real cubic}\big)$$

In all cases we have

$$|D_k| < \begin{cases} \exp(23.52) & \text{if } (r_1, r_2) = (6, 0), \\ \exp(23.73) & \text{if } (r_1, r_2) = (4, 1), \\ \exp(19.3) & \text{if } (r_1, r_2) = (2, 2), \\ \exp(14.91) & \text{if } (r_1, r_2) = (0, 3). \end{cases} \tag{24}$$

If $\overline{\mathfrak{d}}_k$ is not trivial, Table 2 yields

$$|D_k| > \begin{cases} \exp(18.06) & \text{if } (r_1, r_2) = (6, 0), \\ \exp(16.45) & \text{if } (r_1, r_2) = (4, 1), \\ \exp(14.89) & \text{if } (r_1, r_2) = (2, 2), \\ \exp(13.38) & \text{if } (r_1, r_2) = (0, 3). \end{cases} \tag{25}$$

From (23), (18) and

$$\begin{aligned} 2g_{6,0}\big(1/\exp(18.06)\big) &= 16.66..., & 2g_{6,0}\big(1/\exp(23.52)\big) &= 47.36..., \\ 2g_{4,1}\big(1/\exp(16.45)\big) &= 5.25..., & 2g_{4,1}\big(1/\exp(23.73)\big) &= 8.81..., \\ 2g_{2,2}\big(1/\exp(14.89)\big) &= 1.71..., & 2g_{2,2}\big(1/\exp(19.3)\big) &= 2.90..., \\ 2g_{0,3}\big(1/\exp(13.38)\big) &= 0.58..., & 2g_{0,3}\big(1/\exp(14.91)\big) &= 0.76..., \end{aligned}$$

we conclude that $\bar{\mathfrak{d}}_k$ is trivial. From (23), (19) and

$$4g_{6,0}(1/517\,500) = 4.3904..., \qquad 4g_{4,1}(1/110\,200) = 1.3704...,$$
$$4g_{2,2}(1/30\,890) = 0.5001..., \qquad 4g_{0,3}(1/16\,420) = 0.2701...,$$

we find

$$|D_k| < \begin{cases} 517\,500 & \text{if } (r_1, r_2) = (6, 0) \quad \text{(5 fields in this range)}, \\ 110\,200 & \text{if } (r_1, r_2) = (4, 1) \quad \text{(5 fields in this range)}, \\ 30\,890 & \text{if } (r_1, r_2) = (2, 2) \quad \text{(4 fields in this range)}, \\ 16\,420 & \text{if } (r_1, r_2) = (0, 3) \quad \text{(8 fields in this range)}. \end{cases}$$

Inspection of tables results in the lists of regulators in the theorem. □

5.3. Degree 8 totally real and totally complex fields

Tables for totally real octic fields can be fount in [PMD,Vo]. For totally complex octic fields see §7.

Theorem 12. (Octics)

- *With three exceptions, all totally real octic fields have regulator greater than* 28.43. *The exceptions are the unique totally real octic fields of discriminant* 282 300 416, 309 593 125 *and* 324 000 000, *with respective regulators* 22.446870..., 23.696789... *and* 24.388406... .
- *With four exceptions, all totally complex octic fields have regulator greater than* 0.345. *The exceptions are the unique totally complex octic fields of discriminant* 1 282 789, 1 361 513, 1 385 533 *and* 1 424 293, *with respective regulators* 0.313539..., 0.326412..., 0.331112... *and* 0.336709... .

Proof. We assume first that k is a totally real octic field with regulator $R_k \leq 28.43$. As in degree 6, we consider the various possibilities for $L := \mathbb{Q}(\varepsilon_1)$. If $[L : \mathbb{Q}] = 8$, so $k = L$, Pohst's bound (4) and (5) give $\log |D_k| < 38.2$. We shall show that $\log |D_k| < 38.2$ for the remaining values of $[L : \mathbb{Q}]$.

If $[L : \mathbb{Q}] = 4$, (4) gives

$$\log |D_L| \leq 2 \log 4 + m_L(\varepsilon_1)\sqrt{20} = 2 \log 4 + \sqrt{10}\, \mu_1. \tag{26}$$

Since the unit rank of L is three, we must have $k = L(\varepsilon_j)$, where $j = 2,\ 3$ or 4. Lemma 1 and (26) yield

$$\log |D_k| \leq 2 \log |D_L| + 8 \log(2) + \sqrt{8}\, \mu_j \leq 8 \log(4) + 2\sqrt{10}\, \mu_1 + \sqrt{8}\, \mu_j.$$

Table 3
All totally real octic fields k with discriminant $D_k < 582\,918\,125$.

D_k	Polynomial	R_k
282 300 416	$x^8 - 4x^7 + 14x^5 - 8x^4 - 12x^3 + 7x^2 + 2x - 1$	22.446870
309 593 125	$x^8 - 4x^7 - x^6 + 17x^5 - 5x^4 - 23x^3 + 6x^2 + 9x - 1$	23.696789
324 000 000	$x^8 - 7x^6 + 14x^4 - 8x^2 + 1$	24.388406
410 338 673	$x^8 - x^7 - 7x^6 + 6x^5 + 15x^4 - 10x^3 - 10x^2 + 4x + 1$	28.437595
432 640 000	$x^8 - 2x^7 - 7x^6 + 16x^5 + 4x^4 - 18x^3 + 2x^2 + 4x - 1$	28.989022
442 050 625	$x^8 - 2x^7 - 12x^6 + 26x^5 + 17x^4 - 36x^3 - 5x^2 + 11x - 1$	29.638515
456 768 125	$x^8 - 2x^7 - 7x^6 + 11x^5 + 14x^4 - 18x^3 - 8x^2 + 9x - 1$	30.339822
483 345 053	$x^8 - x^7 - 7x^6 + 4x^5 + 15x^4 - 3x^3 - 9x^2 + 1$	31.405649
494 613 125	$x^8 - x^7 - 7x^6 + 4x^5 + 13x^4 - 4x^3 - 7x^2 + x + 1$	31.437552

Pohst's inequality (9) and Lemma 3 give $\log|D_k| < 34.15$ for each of the three possible values of j.

If $[L:\mathbb{Q}] = 2$, from (5) we find $m_k(\varepsilon_1) < 2.52$. Hence $m_L(\varepsilon_1) = m_k(\varepsilon_1)/2 < 1.26$. But $m_L(\varepsilon_1) = \sqrt{2}R_L$. Hence $R_L < 1.26/\sqrt{2} < 0.9$. Theorem 6 shows that $L = \mathbb{Q}(\sqrt{5})$ or $L = \mathbb{Q}(\sqrt{2})$, showing that $\mu_1 = m_k(\varepsilon_1) = \sqrt{8}\log\big((1+\sqrt{5})/2\big)$ or $\mu_1 = \sqrt{8}\log(1+\sqrt{2})$.

We now consider $L_2 := L(\varepsilon_2) = \mathbb{Q}(\sqrt{5}, \varepsilon_2)$ (resp., $L_2 = \mathbb{Q}(\sqrt{2}, \varepsilon_2)$). Since L_2 contains at least two independent units, we have $[L_2 : \mathbb{Q}] = 4$ or 8. We consider first the case $[L_2 : \mathbb{Q}] = 8$, *i.e.* $L_2 = k$. Lemma 1 shows

$$\log|D_k| \le 4\log|D_L| + 8\log(4) + 2\sqrt{10}\,\mu_2.$$

We now take $|D_L| = 5$ and $\delta = \sqrt{8}\log\big((1+\sqrt{5})/2\big)$ (resp., $|D_L| = 8$, $\delta = \sqrt{8}\log(1+\sqrt{2})$) and apply Lemma 3 with $A_j = 0$ except for $A_0 = 4\log|D_L| + 8\log(4)$, $A_2 = 2\sqrt{10}$. The result is $\log|D_k| < 35.18$ (resp., $\log|D_k| < 35.37$).

The remaining case is $[L_2 : \mathbb{Q}] = 4$. In this case Lemma 1 yields

$$\log|D_{L_2}| \le 2\log|D_L| + 4\log(2) + 2m_{L_2}(\varepsilon_2) = 2\log|D_L| + 4\log(2) + \sqrt{2}\,\mu_2.$$

Since the unit rank of L_2 is three, we must have $k = L_2(\varepsilon_j)$ for $j = 3$ or 4. Again by Lemma 1 and the above bound we obtain

$$\log|D_k| \le 2\log|D_{L_2}| + 8\log(2) + \sqrt{8}\,\mu_j \le 4\log|D_L| + 8\log(4) + \sqrt{8}\,\mu_2 + \sqrt{8}\,\mu_j.$$

As above, Lemma 3 gives $\log|D_k| < 33.32$ (resp., $\log|D_k| < 33.69$) for $j = 3$ or 4.

We have in all cases $\log|D_k| < 38.2$. From Table 2 we see that if the class $\overline{\mathfrak{d}}_k$ of the different ideal is not trivial, then $\log|D_k| > 25.49$. Since $2g_{8,0}\big(1/\exp(25.49)\big) = 126.5\ldots$ and $2g_{8,0}\big(1/\exp(38.2)\big) = 425.9\ldots$, (18) shows that $\overline{\mathfrak{d}}_k$ is trivial. By (19) we have $|D_k| < 518\,000\,000$, since $4g_{8,0}(1/518\,000\,000) = 28.434$. From [PMD,Vo] we get a list of polynomials corresponding to the nine totally real octic fields with discriminant below 518 000 000. We calculate their regulators R_k (see Table 3 above) and arrive at the three totally real fields in the theorem, concluding the proof in this case.

Suppose now that k is a totally complex octic field with $R_k \le 0.345$. The CM case is impossible since (20) and Theorem 10 yield $R_k \ge 4R_{k_+} \ge 4 \cdot 0.824 > 0.345$. We again consider the possibilities for $L := \mathbb{Q}(\varepsilon_1)$. If $L = k$, we find $\log|D_k| < 25.51$. We shall show that this is the largest possible value of $\log|D_k|$.

If $[L : \mathbb{Q}] = 4$, (2) and (12) give

$$\log|D_L| \le \begin{cases} 4\log 4 + \sqrt{8}\, m_L(\varepsilon_1) = 4\log 4 + 2\mu_1 & \text{if } r_2(L) = 2, \\ 4\log 4 + \sqrt{18}\, m_L(\varepsilon_1) \le 4\log 4 + 3\mu_1 & \text{if } r_2(L) = 1. \end{cases}$$

Since, $k = L(\varepsilon_j)$, where $j = 2$ if $r_2(L) = 2$ and $j = 2$ or 3 if $r_2(L) = 1$, Lemma 1 yields

$$\log|D_k| \le \begin{cases} 12\log 4 + 4\mu_1 + 2\mu_2 & \text{if } r_2(L) = 2, \\ 12\log 4 + 6\mu_1 + \sqrt{2}\,\mu_j & \text{if } r_2(L) = 1. \end{cases}$$

If $r_2(L) = 2$, then $\mu_1 := m_k(\varepsilon_1) = \sqrt{2}\, m_L(\varepsilon_1) = 2R_L > 0.674$, by Theorem 10. Lemma 3 then gives $\log|D_k| < 22.59$. If $r_2(L) = 1$, then (11) and (12) give $\mu_1 := m_k(\varepsilon_1) \ge \sqrt{2}\, m_L(\varepsilon_1) > \sqrt{2}\cdot 0.3429$. Now Lemma 3 yields $\log|D_k| < 25.42$.

We actually cannot have $[L : \mathbb{Q}] = 2$ since (5) gives $m_k(\varepsilon_1) < 0.992$. If ε_1 were real quadratic, we would have $m_k(\varepsilon_1) = \sqrt{8}\, m_L(\varepsilon_1)$, as k is totally complex. But $\sqrt{8} m_L(\varepsilon_1) = 4R_L > 4\cdot 0.48 > 0.992$. Hence in all cases $\log|D_k| < 25.51$.

Table 2 gives $\log|D_k| > 18.93$ if $\bar{\mathfrak{d}}_k$ is not trivial. From (18) and

$$2g_{0,4}(\mathrm{e}^{-18.93}) = 1.1..., \qquad 2g_{0,4}(\mathrm{e}^{-25.51}) = 1.6...,$$

we see that $\bar{\mathfrak{d}}_k$ is trivial. From $4g_{0,4}(1/1\,652\,000) = 0.34506...$ and (19) we conclude $|D_k| < 1\,652\,000$. Inspecting Table 4 in §7, we arrive at the four totally complex fields in the theorem. □

5.4. *Degree 9 totally real fields*

Tables of totally real fields of degree 9 can be found in [Ta,Vo].

Theorem 13. *All totally real fields of degree 9 have regulator greater than* 37.2.

Proof. Assume k is a totally real field of degree 9 with regulator $R_k \le 37.2$. If $L := \mathbb{Q}(\varepsilon_1) = k$, then Pohst's bound (4) gives $\log|D_k| < 45.044$.

If L is a cubic extension of $\mathbb{Q}$, (4) yields

$$\log|D_L| \le \log 4 + \sqrt{8}\, m_L(\varepsilon_1) = \log 4 + \sqrt{8/3}\,\mu_1. \tag{27}$$

As L has unit rank 2, we have $k = L(\varepsilon_j)$ for $j = 2$ or 3. Lemma 1 and (27) give

$$\log|D_k| \le 3\log|D_L| + 9\log(3) + \sqrt{24}\,\mu_j \le 3\log(4) + 9\log(3) + \mu_1\sqrt{24} + \mu_j\sqrt{24}.$$

By (9), $m_k(\varepsilon_1) \ge 3\log\big((1+\sqrt{5})/2\big)$. Lemma 3 gives $\log|D_k| < 39.03$.

In any case we have $\log|D_k| < 45.044$. Also, $|D_k| \geq 9\,685\,993\,193$, the minimal discriminant [Ta,Vo]. Since $2g_{9,0}(\mathrm{e}^{-29.257}) = 365.26...$ and $2g_{9,0}(\mathrm{e}^{-44.6}) = 51.95...$, from (18) we conclude that either $\log|D_k| \leq 29.256$ or $44.6 \leq \log|D_k| \leq 45.044$. In the first case, Table 2 shows that $\overline{\mathfrak{d}}_k$ is trivial. If $9\,685\,993\,193 \leq |D_k| \leq \mathrm{e}^{29.256}$, we note $4g_{9,0}(1/9\,685\,993\,193) = 56.55...$. Hence, by (19), this range of discriminant is impossible.

We now rule out the one case left, *i.e.* $\exp(44.6) \leq \log|D_k| \leq 45.044$. Since $g_{9,0}\big(4/\exp(45.044)\big) = 847.136 > 0$, and

$$2G_{9,0}(\mathrm{e}^{-45.04}, \mathrm{e}^{-45.044}, 6) = 43.5796..., \quad 2G_{9,0}(\mathrm{e}^{-45.02}, \mathrm{e}^{-45.04}, 6) = 48.3153...,$$

$$2G_{9,0}(\mathrm{e}^{-45.02}, \mathrm{e}^{-44.92}, 6) = 72.0797..., \quad 2G_{9,0}(\mathrm{e}^{-44.6}, \mathrm{e}^{-44.92}, 6) = 239.1207...,$$

Lemma 5 allows us to conclude $R_k > 43.57$, contradicting our assumption that $R_k \leq 37.2$. □

6. Numerical limits of current methods

The reader will have observed that our choice of giving only the first three or four regulators for a signature is usually just to keep the list short. For example, for non-CM totally complex quartics, the geometric method could list all k with $R_k \leq 2.6$ using the known list [CDO] of all k with $|D_k| < \mathrm{e}^{15.95} \approx 8.45 \cdot 10^6$. On the other hand, already $R_k \leq 0.62$ is unattainable by the analytic method. Indeed, if $R_k = 0.62$, Remak's bound (3) gives $|D_k| \leq 3056$, but Table 2 shows that the ideal class $\overline{\mathfrak{d}}_k$ of the different is guaranteed to be trivial only up to $|D_k| \leq 3018$. The loss of the factor of 2 that this entails kills the analytic approach.

For totally real quartics k, the analytic method can handle listing $R_k \leq 2.8$, but not $R_k \leq 2.9$. The limitation is again imposed by Table 2 and the possible non-triviality of $\overline{\mathfrak{d}}_k$. The geometric method could list all k with $R_k \leq 9.3$, needing just the list of all k with $|D_k| < 9.6 \cdot 10^6$ [CDO].[9]

For quintics of signature $(1, 2)$, the analytic method can easily handle listing $R_k \leq 0.75$. Equipped with the list of k with $|D_k| \leq 510\,000$, the analytic method could handle up to $R_k \leq 1.02$, failing at $R_k \leq 1.03$. The upper limit of 1.02 comes from Lemma 5, and the need for such a long table of fields comes from the possibly nontrivial $\overline{\mathfrak{d}}_k$. On the other hand, Remak's inequality (2) shows that a list of all fields in this signature with $|D_k| < 3 \cdot 10^7$ would suffice to list all $R_k \leq 1.5$. We do not know of such a list of quintics, but it seems to us that it could be constructed.

For totally real quintics the geometric method still fares well. Using nearly all quintics in Malle's list [Ma] of totally real primitive fields with $|D_k| < 10^9$, the geometric method yields all totally real quintics k with $R_k \leq 14.5$. The analytic method easily lists all

[9] Admittedly, computing regulators in this range without GRH would need a bit of care, but is feasible.

$R_k \leq 6.6$, needing only a list of k with $|D_k| < 240\,000$. Using the list of $|D_k| < 4.5 \cdot 10^8$, less than half of Malle's list, the analytic method could list all totally real quintics k with $R_k \leq 17.8$. The long list of fields is needed because of the possible non-triviality of $\bar{\mathfrak{d}}_k$. The analytic method could not get to $R_k \leq 17.9$ because Lemma 5 fails to help at that point.

In short, for degrees $[k : \mathbb{Q}] \leq 5$ extensive use of existing or feasible tables would allow us to go far beyond our lists of small regulators, in most signatures even without using analytic methods.

Totally complex sextics can still be treated geometrically, although extensive lists are required. To equal our list of $R_k \leq 0.27$ (Theorem 11) by geometric methods, we would need a list of k with $|D_k| < 3 \cdot 10^6$. On the other hand, the analytic method works easily up to listing $R_k \leq 0.58$, for which the existing list of k with $|D_k| < 77\,000$ suffices. It even works up to $R_k \leq 0.76$, but then requires a list of k with $|D_k| < 2.7 \cdot 10^6$. To list $R_k \leq 0.76$, the geometric approach would need a list of k with $|D_k| < 5 \cdot 10^7$.

Totally real sextics seem just at the limit of the geometric method. A list of k with $R_k \leq 4.39$, as given in Theorem 11, by purely geometric techniques requires almost all of Malle's list of primitive k with $|D_k| < 10^9$, and a (probably feasible) list of imprimitive totally real sextics with $|D_k| < 10^9$. By analytic techniques one can list all k with $R_k \leq 13.3$ (but not $R_k \leq 13.4$) using the existing list [Vo] of all k satisfying $|D_k| < 5.3 \cdot 10^6$. The limit of the analytic method (Lemma 5) is $R_k \leq 31.6$, but this would require completing Malle's list to include imprimitive sextics of discriminant up to $7.5 \cdot 10^8$.

Degree 7 is out of reach of current geometric methods. If we wanted all totally real fields k of degree 7 with $R_k \leq 14.45$ (the minimum is 14.44...), Pohst's inequality (4) would require a list of all k with $|D_k| < 4.927 \cdot 10^{12}$, which is probably unfeasible. Using Lemma 5, for totally real fields of degree 7 we could find all fields k with $R_k \leq 44.8$ using the existing list [Ma] of all fields with $|D_k| < 3.8 \cdot 10^8$. However, $R_k \leq 44.9$ is beyond our methods, as Lemma 5 is useless.

In signature $(1, 3)$ the geometric method needs all k with $|D_k| < 6.44 \cdot 10^9$ to equal our list of $R_k \leq 0.4$. Using Lemma 5 we could list all k with $R_k \leq 0.79$ using a list of k with $|D_k| < 762\,000$, but $R_k \leq 0.8$ is beyond the analytic method. The geometric method alone would need all $|D_k| < 6.308 \cdot 10^{10}$ to list all $R_k \leq 0.79$.

In signatures $(5, 1)$ and $(3, 2)$, we cannot go beyond Theorem 9. We showed $R_k > 2.11$ for signature $(5, 1)$, but cannot prove $R_k > 2.12$. In signature $(3, 2)$ we found all $R_k \leq 1.055$, but would fail with $R_k \leq 1.056$. Both of these signatures are well beyond present geometric methods.

Octics are also currently far beyond the reach of geometric methods. For totally real octics, the limit of the analytic method is listing all $R_k \leq 39.8$, for which the current list [Vo] of discriminants is amply sufficient. For totally complex octics, we can hardly go beyond Theorem 12 with the current table of discriminants, but a table listing all k with $|D_k| < 4.1 \cdot 10^6$ would allow us to list all $R_k \leq 0.55$, which is the limit of the analytic method.

Table 4
All totally complex octic fields k with discriminant $D_k < 1\,656\,110$.

D_k	Polynomial	R_k	R_k/w_k
1 257 728	$x^8 - 2x^7 + 4x^5 - 4x^4 + 3x^2 - 2x + 1$	0.618886	0.15472
1 265 625	$x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$	4.661820	0.15539
1 282 789	$x^8 - x^7 + 2x^6 - 3x^5 + 3x^4 - 3x^3 + 3x^2 - 2x + 1$	0.313539	0.15676
1 327 833	$x^8 - x^7 + x^6 - 2x^5 + 3x^4 - 4x^3 + 4x^2 - 2x + 1$	0.963083	0.16051
1 342 413	$x^8 - x^7 + x^6 + x^4 - 2x^3 + 3x^2 - 3x + 1$	0.970727	0.16178
1 361 513	$x^8 - x^7 + 2x^6 - 3x^5 + 3x^4 - 3x^3 + 2x^2 - x + 1$	0.326412	0.16320
1 385 533	$x^8 - 2x^7 + 3x^6 - 3x^5 + x^4 + 1$	0.331112	0.16555
1 424 293	$x^8 - 2x^7 + 2x^6 - x^5 + x^4 - x^3 + 2x^2 - 2x + 1$	0.336709	0.16835
1 474 013	$x^8 - x^7 + x^6 - x^4 + x^3 - x^2 + 1$	0.345105	0.17255
1 492 101	$x^8 - x^7 + x^6 - 3x^5 + x^4 - 2x^3 + 3x^2 + 1$	1.043255	0.17387
1 513 728	$x^8 - 2x^7 + x^6 + 2x^5 - 3x^4 - 2x^3 + x^2 + 2x + 1$	2.106944	0.17557
1 520 789	$x^8 - x^7 - x^6 + x^4 - x^2 + x + 1$	0.353845	0.17692
1 578 125	$x^8 - 2x^7 + 3x^5 - x^4 - 3x^3 + 2x + 1$	1.811959	0.18119
1 590 773	$x^8 - x^7 + 2x^5 - 2x^4 + x^3 + x^2 - 2x + 1$	0.363609	0.18180
1 601 613	$x^8 - 2x^6 - 3x^5 + 3x^4 + 3x^3 - 2x^2 + 1$	1.100958	0.18349

7. Equations and regulators for totally complex octic fields

In [Di3] it was shown that there are 15 totally complex octic fields k with discriminant $D_k < 1\,656\,110$, corresponding to 15 distinct discriminants. Unfortunately, no equations were given, and they were subsequently lost. We reconstruct them here from their description as class fields [Di3,Le]. As we have checked that the 15 polynomials in Table 4 above are irreducible, have no real roots, and have field discriminant as tabulated, it follows that we have a rigorous list of all totally complex octic fields k with discriminant $D_k < 1\,656\,110$.

The first column of Table 4 gives the discriminant of the totally complex octic number field k, the second one gives a polynomial $p(x)$ such that $k = \mathbb{Q}(\alpha)$ and $p(\alpha) = 0$, the third column gives the (first six decimal digits of the) regulator R_k, and the last column gives R_k/w_k, where w_k is the number of roots of unity in k. All of these were calculated with PARI [GP].

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