Computational hardness of enumerating groundstates of the antiferromagnetic Ising model in triangulations

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ARTICLE INFO

ABSTRACT

Satisfying spin-assignments of triangulations of a surface are states of minimum energy of the antiferromagnetic Ising model on triangulations which correspond (via geometric duality) to perfect matchings in cubic bridgeless graphs. In this work we show that it is NP-complete to decide whether or not a triangulation of a surface admits a satisfying spin-assignment, and that it is #P-complete to determine the number of such assignments. Our results imply that the determination of even the entropy of the Ising model on triangulations at the thermodynamical limit is already #P-hard.

The aforementioned claims are derived via elaborate (and atypical) reductions that map a Boolean formula in conjunctive normal form into triangulations of orientable closed surfaces. Moreover, the novel reduction technique enables us to prove that even very constrained versions of #MaxCut are already #P-hard.

1. Introduction

The Ising model is one of the most studied models in statistical physics. Characterizing its behavior on a system (graph) helps to understand physical phenomena associated to its thermodynamic properties [16]. The Ising model has been widely studied in lattices and regular structures (see for example [2,12] and references therein). In contrast, irregular systems have received much less attention, probably due to the difficulty of deriving meaningful analytic results. Our work in part justifies why this should be so, by showing how some quite restricted problems of interest concerning irregular systems are special cases of a class of problems which are, in a well defined sense, regarded as the most intractable enumeration problems that can be sensibly posed.

The number of distinct groundstates of the antiferromagnetic (negative coupling constant) Ising model of a system is called groundstate degeneracy and is typically exponentially large as a function of a parameter that measures the system's size (the number of nodes of the underlying graph). The latter translates to nonzero entropy[1] at zero temperature when the system size goes to infinity, which in physical terms means that in the thermodynamical limit the spin arrangements of particles in the system is disordered. This partly explains the considerable attention physicists have given to developing techniques for approximating the groundstate degeneracy of a system.

One research line has focused on developing tools, for example the Transfer Matrix Method [11, Section 6.6], for bounding the groundstate degeneracy of a system.

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[1] Here, the entropy at the thermodynamical limit is proportional to the logarithm of the number of groundstates divided by the system's size.
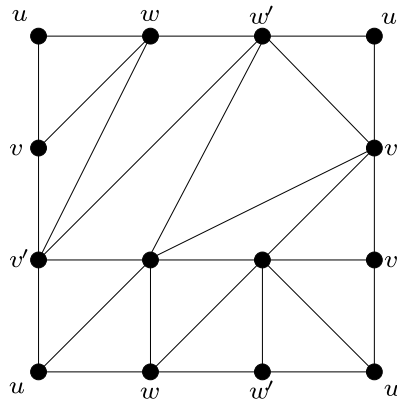
Fig. 1. A triangulation of the torus with no satisfying spin-assignment.

Another significant line of research tackles the computational complexity of problems arising from statistical physics models (see [8,18]). A great variety of these problems are computationally hard (see for example [1,3,8]). This work is part of the latter research direction. Informally speaking, we first focus on the following two associated computational complexity problems; (1) hardness of deciding whether or not a given system admits a satisfying state, and (2) hardness of enumerating a system's satisfying states (equivalently, computing the groundstate degeneracy of systems that admit satisfying states). We show that the former problem is NP-complete and the latter is #P-complete. But in fact, we do more. We develop a novel and atypical reduction technique that maps Boolean formulas in conjunctive normal form to triangulations of orientable closed surfaces. We illustrate the usefulness of our reduction technique and address the following additional computational complexity problems; (3) hardness of determining the entropy of the antiferromagnetic Ising model in the thermodynamical limit (when temperature goes to zero),[2] and (4) hardness of enumerating the number of maximum size edge cuts of triangulations of surfaces.[3]

1.1. Background

We now make precise the notions discussed above and then, in the following section, formally state our main results. First, we describe the antiferromagnetic Ising model. We say that an embedding of a graph in an orientable closed surface is a *surface triangulation* if each face is bounded by a cycle of length 3 (in particular there is no loop)—multiple edges allowed. Given a triangulation T, let $V(T)$ and $E(T)$ denote the node and edge set of T. A mapping $s : V(T) \rightarrow \{-1, +1\}$ will be called a *spin-assignment* (state) to T. We refer to -1 and $+1$ as *spins*. The *energy* of a spin-assignment s of the antiferromagnetic Ising model is defined as

$$\mathcal{E}_T(\sigma) = \sum_{uv \in E(T)} \sigma(u) \cdot \sigma(v). \tag{1}$$

A *groundstate* is a spin-assignment of minimum energy. The number of distinct groundstates that a triangulation T admits is often referred to as the *groundstate degeneracy* of T. Clearly, under any spin-assignment to T the ends of at least one edge of each face of a surface triangulation T are both assigned either -1 or $+1$. Moreover, a spin-assignment is a groundstate if it has the smallest possible number of edges with both its ends being assigned the same spin. A face $\triangle$ of a surface triangulation T is said to be *frustrated* under assignment s, if s restricted to $V(\triangle)$ is not identically -1 or $+1$. A spin-assignment s to T is said to be *satisfying* (or *feasible*) if every face of T is frustrated under s. Obviously a satisfying spin-assignment is a groundstate. The converse is true for triangulations that can be embedded in the plane [10]. Nevertheless, the equivalence does not hold in general (the reader can verify that the toroidal triangulation depicted in Fig. 1 does not have satisfying spin-assignments). However, note that when satisfying spin-assignments exist, then a groundstate is necessarily a satisfying spin-assignment.

In [10] a relation was established between the groundstate degeneracy of the antiferromagnetic Ising model and the number of perfect matchings in cubic bridgeless graphs. Specifically, let T be triangulation of a orientable closed surface and T^* its geometric dual. In [10], it is shown that the set of edges whose ends are assigned the same spin under a given satisfying spin-assignment to T correspond to a perfect matching of the cubic bridgeless graph T^*. Moreover, it is shown that if T admits a satisfying spin-assignment, then the groundstate degeneracy of T is at most twice the number of perfect matchings of T^*. Thus, lower bounds on the groundstate degeneracy of T provide lower bounds on the number of distinct perfect matchings of the cubic bridgeless graph T^*. An old and famous conjecture of Lovász and Plummer, recently positively

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[2] We remark that this is a restricted instance of a known #P-hard problem. Indeed, computing the so called Ising Partition Function of a graph is #P-hard since it corresponds to evaluating the so called Tutte polynomial of G at a particular (non-trivial) point of the complex plane [18, Theorem 2].

[3] It is known that over general graphs #MaxCut is #P-hard [9, Lemma 13].

settled [6], claimed that the number of distinct perfect matchings of T^* is exponential in the size (number of nodes) of T^*. The relation between the number of satisfying spin-assignments of a surface triangulation and the number of perfect matchings of bridgeless cubic graphs is another one of our motivations for considering the problem of computing the groundstate degeneracy of surface triangulations.

To formally state this work's main results we need to introduce one additional concept. For a surface triangulation T and v in $V(T)$, let $\delta(v)$ denote the set of edges incident to v. The map $\pi_v : \delta(v) \rightarrow \delta(v)$ is called *cyclic permutation of the edges incident to v* if for every e incident to v the edge $\pi_v(e)$ is the successor of e in the clockwise ordering around v defined by the surface embedding of T. The tuple $\pi = (\pi_v : v \in V(T))$ is called the *rotation system* of T. A direct consequence of the Heffter–Edmonds–Ringel rotation principle, is that every surface triangulation is uniquely determined, up to homeomorphism, by its rotation system [13, Section 3.2].

1.2. Contributions

We start by considering the computational complexity problem of deciding membership in SatAssign—the collection of (encodings of) rotation systems of surface triangulations that admit a satisfying spin-assignment. We then study the computational hardness of calculating #SatAssign—the function mapping (encodings of) rotation systems of surface triangulations to its number of satisfying spin-assignments.

To see that SatAssign is in NP, first recall that in order to check that $\pi = (\pi_v : v \in V(T))$ is an instance of SatAssign we need not start with a surface. Indeed, it suffices to check for every $v \in V(T)$ that π_v is a cyclic permutation of $\delta(v) = \{uv : uv$ is in π_u's domain$\}$, a task that can be performed in time quadratic in $|V(T)|$ time in the Random Access Model. Then, observe that a certificate of membership in SatAssign of a rotation system of a surface triangulation T is simply a spin-assignment $s : V(T) \rightarrow \{-1, +1\}$ and that verifying that such an assignment is satisfying amounts to checking that each face Δ of the surface triangulation T is frustrated under s (which can be checked in $O(1)$ time per face in the Random Access Model).

In this work we show that the problem of deciding if a triangulation admits a satisfying spin-assignment is NP-complete. We also establish that computing the groundstate degeneracy of surface triangulations that admit satisfying spin-assignments is #P-complete. Formally, we establish the following results.

Theorem 1. SatAssign *is* NP-*complete.*

Theorem 2. #SatAssign *is* #P-*complete.*

Both of the stated results follow from an elaborate weakly parsimonious reduction [5, Definition 2.27] that maps a Boolean function in 3-conjunctive normal form to a rotation system of a triangulation (equivalently, to a triangulation embedded on a surface). As far as we are aware, this seems to be an atypical reduction, whose underlying ideas (specifically, gadgets) might be of independent interest due to their potential usefulness in the study of the computational hardness of other related spin glass problems. To illustrate this last claim, we adapt our reduction technique and consider the computational complexity problem involved in calculating:

- #GroundSt—the function mapping (encodings of) rotation systems of triangulations to its number of groundstates (in other words, to its groundstate degeneracy), and
- #TriangMaxCut—the function mapping (encodings of) rotation systems of triangulations to its number of maximum size edge cuts.

In particular, we derive the following two results.

Theorem 3. #GroundSt *is* #P-*hard.*

Corollary 4. #TriangMaxCut *is* #P-*hard.*

We remark that the arbitrariness (specifically, unbounded genus) of the surface where the triangulations generated by our many-to-one reductions are embedded plays an essential role in establishing the last stated result. Indeed, it is known [7] that computing the number of maximum size edge-cuts of graphs of a given fixed genus g can be done in polynomial time.

1.3. Organization

In Section 2, we give an overview of a mapping (reduction) from Boolean formulas in 3-conjunctive normal form to (rotation systems of) triangulations. Specifically, we describe all gadgets on which the reduction relies as well as establish several properties that the gadgets satisfy. In Section 3, we delineate and analyze the reduction through which we establish Theorems 1 and 2. In Section 4, we provide an alternative to one of the gadgets we introduce in Section 2 and come up with a variant of the reduction of Section 3 which establishes Theorem 3, from which we then derive Corollary 4. Finally, in Section 5, we discuss some open questions and future research directions.

PROBLEM	Pos-NAE-3SAT
INPUT	A Boolean formula φ in 3-conjunctive normal form such that each of its clauses $C_1, \ldots, C_m$ has exactly three (all non-negated) literals.
OUTPUT	TRUE if there is a truth assignment to φ such that for each clause C_i not all of its variables are assigned the same truth value.

Fig. 2. Pos-NAE-3SAT.

PROBLEM	#Pos-NAE-3SAT
INPUT	A Boolean formula φ in 3-conjunctive normal form such that each of its clauses $C_1, \ldots, C_m$ has exactly three (all non-negated) literals.
OUTPUT	A (binary encoding) of the number of distinct truth value assignments to φ such for each clause C_i not all of its variables are assigned the same truth value.

Fig. 3. #Pos-NAE-3SAT.

2. Main reduction idea and gadgets

The two main results of this work follow from reductions from a well known variant of the standard not-all-equal 3-satisfiability (NAE-3SAT for short) problem which is known to be NP-complete even in the absence of negated variables [15], a variant we denote Pos-NAE-3SAT. For completeness sake, we recall in Fig. 2 the precise definition of Pos-NAE-3SAT. Moreover, the counting version of Pos-NAE-3SAT, namely #Pos-NAE-3SAT, is #P-complete [4]. See Fig. 3 for the precise definition of #Pos-NAE-3SAT.

The overall strategy we follow in proving Theorems 1 and 2 is fairly standard, i.e. we design gadgets where truth values of variables are set (choice gadgets) and gadgets where the truth value of clauses are evaluated (clause gadgets). We need to "carry" truth values from choice gadgets to clause gadgets, and make as many copies of the truth values taken by a literal as times they appear in all clauses. To achieve this task we build the so called replicator gadgets. However, the construction of the aforementioned gadgets is quite delicate and non-obvious. In general, the main aspects we take care of in the construction of each gadget are existence and uniqueness of satisfying spin-assignments. However, there are subtle issues that need to be properly handled when building and piecing together the different gadgets. Below, we describe in separate sections each of the gadgets we will require for the reduction and establish that they satisfy certain properties. First, we introduce some additional terminology and conventions we use throughout the remaining part of this work.

2.1. Preliminaries

Note that given a triangulation T, a spin-assignment s to T is satisfying if and only if $-s$ is also a satisfying spin-assignment to T. We shall refer to this fact as *duality*. We repeatedly use it in order to reduce the number of cases that need to be analyzed in order to establish some of the claims we make. If T admits exactly two satisfying spin-assignments s and $-s$, we say that s ($-s$ respectively) is unique up to duality.

A 3-cycle in a triangulation will be called *positive* for a spin-assignment if at least two of its vertices are assigned spin $+1$. Otherwise, it will be called *negative*. This concept will be referred to as the *sign* of a 3-cycle.

Henceforth, if s assigns the same spin to all nodes of a subgraph H of T (respectively all elements of $S \subseteq V(T)$), we say that H (respectively a subset S) is *monochromatic* under s. Similarly, we say that an edge is monochromatic (respectively non-monochromatic) under s if s assigns the same (respectively distinct) spins to both ends of the edge. Monochromatic and non-monochromatic faces are defined analogously depending on whether or not its circumscribing cycle is either monochromatic or non-monochromatic. An edge e in $E(T)$ will be called *serious* if and only if e is monochromatic under every satisfying spin-assignment to T.

The gadgets we build in this work are embedded graphs in orientable closed surfaces with some removed disks (with holes) so that each face is bounded by a 3-cycle and each hole is circumscribed by a 3-cycle. In other words, every gadget may be obtained from a triangulation by cutting along the boundary of some of its faces (triangles). Thence, every term defined for surface triangulations is naturally adapted to gadgets so they will be reformulated only in case it is needed.

Throughout this work, serious edges are depicted as thicker lines and surface holes are depicted as gray areas.

2.2. Choice gadget

In this section we describe a gadget (a triangulation of a surface) that we associate to Boolean variables in such a way that satisfying spin-assignments can be unambiguously interpreted as truth assignments to the Boolean variables. A *choice gadget* is a triangulation as depicted in Fig. 4 embedded in a toroidal surface with one hole. The cycle with node set $\{u, v, w\}$
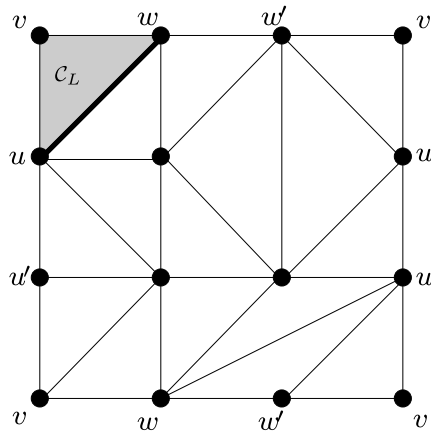
Fig. 4. Choice gadget (region in gray depicts a surface hole).

circumscribing the removed triangle of the choice gadget L, henceforth denoted by $\mathcal{C}_L$, will be referred to as the *variable cycle* of L and the edge uw will be called the *fundamental edge* of L (see Fig. 4).

Our reduction will associate to each variable x_i a choice gadget L_i. A satisfying spin-assignment will be interpreted as setting x_i to TRUE if $\mathcal{C}_{L_i}$ ends up being monochromatic, and FALSE otherwise. The key functionality that we show a choice gadget provides is that it has a unique up to duality satisfying spin-assignment where the variable cycle is monochromatic (respectively, non-monochromatic). Furthermore, choice gadgets will also be used as auxiliary building blocks in the construction of another type of gadget we will soon encounter.

The following result encapsulates the most relevant properties of choice gadgets.

Proposition 5. *Let L be a choice gadget. The fundamental edge of the choice gadget is serious. Moreover, there exists a unique up to duality feasible spin-assignment to L where the variable cycle $\mathcal{C}_L$ of L is monochromatic (respectively, non-monochromatic).*

Proof. To prove the first statement, by duality, it suffices to show that there is no feasible spin-assignment extension to L when node v (node labels as in Fig. 4) is assigned spin $+1$ and the fundamental edge uw is assigned spins $+-$ or $-+$. In Fig. 5(a) we work out the case where uw is assigned spins $+-$; a subindex i accompanying a $+$ or $-$ sign indicates that the spin is forced by the spin-assignments with smaller indices in order for the assignment to be satisfiable—if spins assigned to the vertices of a triangle are forced to be all of the same sign, then no satisfying assignment can exist under the given initial conditions. The case when uw is assigned spins $-+$ is dealt with in the same way and worked out in Fig. 5(b). This establishes that the fundamental edge of L is serious.

We now establish the claimed existence and uniqueness. Since the fundamental edge of L is serious, if s is a satisfying spin-assignment to L, then L's fundamental edge is monochromatic under s. Therefore, again by duality, it is enough to prove that in the following two cases there exist exactly one feasible spin-assignment extension: (a) when the variable cycle $\mathcal{C}_L$ (i.e. uwv) of L is assigned spin $+++$ (the monochromatic case), and (b) when it is assigned $++-$ (the non-monochromatic case). In Fig. 6(a), the unique satisfying spin-assignment to L when nodes of its variable cycle are assigned $+1$ is exhibited.

For the non-monochromatic case, by duality and since L's fundamental edge is serious, it suffices to consider the situation where L's fundamental edge is assigned spins $++$. Two subcases arise, depending on whether or not the spin $+1$ is assigned to node u' (node labels as in Fig. 4)—each subcase is worked out separately in Fig. 6(b) and (c). □

2.3. Replicator gadget

A variable may appear several times in different clauses (or even multiple times in the same clause) of a Boolean formula in three conjunctive normal form. Given that the truth value a variable, say x_i, will be unambiguously set by the values taken by a satisfying spin-assignment at the associated choice gadget, say L_i, we require a way of "replicating" the encoding of the truth value of x_i as many times as x_i appears in the collection of formula clauses. To achieve this goal, to every choice gadget we connect a special gadget, namely a *k-replicator gadget*. When the value k is clear from context or is not relevant, we just write replicator gadget instead of k-replicator gadget. For each $k > 0$, a k-replicator gadget will be a triangulation embedded in a surface of large genus (depending on k) with $2^k + 1$ holes. The purpose of a k-replicator gadget is to generate 2^k copies of the truth value encoded by a satisfying spin-assignment to the choice gadget to which the replicator gadget is connected.

To construct a k-replicator gadget we glue together (in a particular way soon to be discussed) $2^k - 1$ so called block-replicator gadgets. A *block-replicator gadget* is a triangulation R of an orientable closed surface of genus 4 with three holes. A block-replicator gadget is built by gluing together three choice gadgets and the surface triangulation of the torus with six
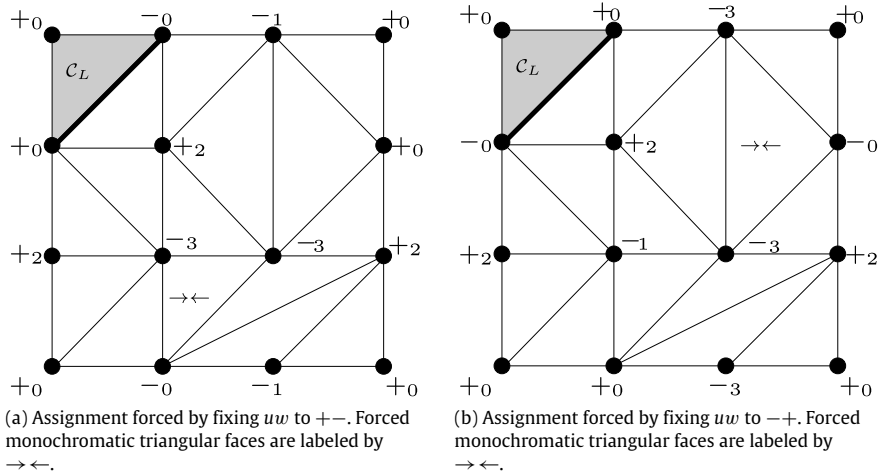
(a) Assignment forced by fixing uw to $+-$. Forced monochromatic triangular faces are labeled by $\rightarrow\leftarrow$.

(b) Assignment forced by fixing uw to $-+$. Forced monochromatic triangular faces are labeled by $\rightarrow\leftarrow$.

Fig. 5. Cases considered to establish that the fundamental edge of the choice gadget is serious.

(a) Unique satisfying spin-assignment forced by the spin-assignment $+++$ to the variable cycle of L.

(b) Unique satisfying spin-assignment to L, up to duality, when L's variable cycle is non-monochromatic.

(c) Spin assignments forced by the assignment of spin $+1$ to u' (forced monochromatic faces are labeled by $\rightarrow\leftarrow$).
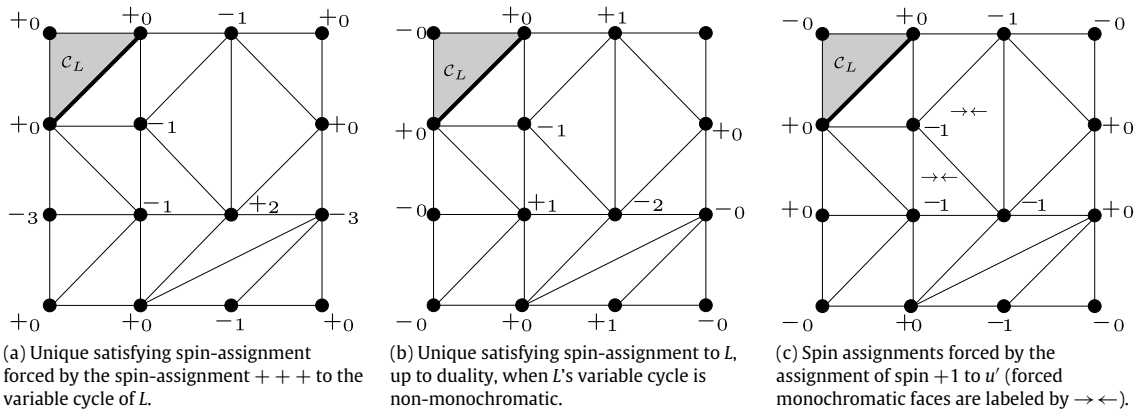
Fig. 6. Cases considered to establish the existence and uniqueness (up to duality) of feasible monochromatic and non-monochromatic spin-assignments to the variable cycle of choice gadgets.

removed triangles depicted in Fig. 7.[4] Specifically, the construction takes three choice gadgets, say $\hat{L}$, $\bar{L}$ and $\tilde{L}$ (see Fig. 4), and identifies the variable cycle $\mathcal{C}_{\hat{L}}$ (respectively, $\mathcal{C}_{\bar{L}}$ and $\mathcal{C}_{\tilde{L}}$) of the choice gadget $\hat{L}$ (respectively, $\bar{L}$ and $\tilde{L}$) with the cycle $\hat{c} = u'xu$ (respectively, $\bar{c} = xyw$ and $\tilde{c} = zw'v$) of the block-replicator gadget R as depicted in Fig. 7. The identification is done in such a way that edge $u'x$ (respectively, xy and zw') of the cycle $\hat{c}$ (respectively, $\bar{c}$ and $\tilde{c}$) coincides with the fundamental edge of the choice gadget $\hat{L}$ (respectively, $\bar{L}$ and $\tilde{L}$). Clearly, under this construction, each block-replicator gadget has exactly three holes, because after gluing the choice gadgets to the surface triangulation depicted in Fig. 7, the holes circumscribed by the cycles $\hat{c}$, $\bar{c}$ and $\tilde{c}$ disappear. The length 3 cycle uvw circumscribing one of R's hole is referred to as the *incoming cycle* (node labels as in Fig. 7). The length 3 cycles circumscribing the other two holes of R will be called the *outgoing cycles*. Moreover, edges vw, $v'u'$ and xy will be referred to as *fundamental edges* of R (see again Fig. 7 for node labeling scheme).

The attentive reader might wonder whether the described block-replicator gadget is indeed a surface triangulation. Specifically, whether indeed every "surface" point has an open neighborhood homeomorphic to some open subset of the Euclidean plane. This is indeed the case. Moreover, a block-replicator gadget has the following key property, henceforth referred to as *intersection property*: the incoming and outgoing cycles of a block-replicator gadget do not share vertices. The intersection property implies that the surface on which the block-replicator gadget is embedded can be smoothly deformed into the one depicted in Fig. 8.

The key purpose of each block-replicator gadget is to enforce that whenever the incoming cycle is monochromatic (respectively, non-monochromatic), both outgoing cycles are monochromatic (respectively, non-monochromatic). In fact, we will show that the block-replicator gadget inverts the sign of the incoming cycle, namely if the incoming cycle is positive

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[4] Here, choice gadgets are used as auxiliary gadgets. This auxiliary gadgets will not be associated to Boolean variables. The reason why we rely on this auxiliary choice gadgets is solely because of one of the properties we have shown they exhibit. Specifically, the fact that fundamental edges of choice gadgets are serious.
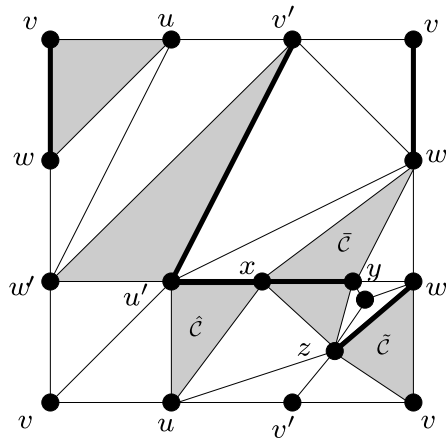
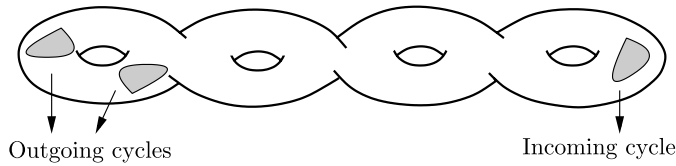**Fig. 7.** Block-replicator gadget.

Fig. 8. Block-replicator gadget sketch.

(respectively, negative) the outgoing cycles are negative (respectively, positive). Actually, much more is true. Formally, we have the following results concerning block-replicator gadgets.

Proposition 6. *Let R be a block replicator gadget. Fundamental edges of R are serious.*

Proof. Given that xy is the fundamental edge of the variable cycle $\mathcal{C}_{\bar{L}} = xyw$ of the choice gadget $\bar{L}$, by Proposition 5, we have that xy is serious. To prove that vw is serious, by duality, it suffices to show that when vw is assigned $+-$ and u is assigned $+$ and $-$, there is no feasible spin-assignment extension to R. These two situations are worked out in Fig. 9.

We proceed as above to prove now that $u'v'$ is serious. In Fig. 10, we show that in the cases where $u'v'$ is assigned $-+$ and w' is assigned $+$ and $-$ there is no feasible spin-assignment extension to R. $\square$

Proposition 7. *Let R be a block-replicator gadget. Then, there exists a unique up to duality satisfying spin-assignment to R for which the incoming cycle is monochromatic (respectively, non-monochromatic). Moreover, if s is a satisfying spin-assignment to R, one of the two following statements hold:*

1. *the incoming and outgoing cycles are all monochromatic with the incoming cycle positive (respectively, negative) and both outgoing cycles negative (respectively, positive), or*
2. *the incoming and outgoing cycles are all non-monochromatic with the incoming cycle positive (respectively, negative) and both outgoing cycles negative (respectively, positive).*

Proof. To prove existence and uniqueness of the satisfying spin-assignment when the incoming cycle is monochromatic, by duality, it is enough to prove that if all nodes in the incoming cycle uvw are assigned $+1$, then there exists a unique feasible spin-assignment extension to R (node labels as in Fig. 7). This situation is worked out in Fig. 11(a). On the other hand, by Proposition 6, if s is a satisfying spin-assignment to R, the edge vw belonging to the incoming cycle is monochromatic (because it is serious). Thus, by duality, we can assume that s assigns to the incoming cycle uvw spins $+++$ or $-++$. Therefore, to establish existence and uniqueness of the satisfying spin-assignment when the incoming cycle is non-monochromatic, it will suffice to show that there exists unique satisfying spin-assignment extension to R when the incoming cycle uvw is assigned spin $-++$. This case is studied in Fig. 11(b). We need to check that each of the spin-assignments depicted in Fig. 11(a) and (b) has a unique extension to the block-replicator gadget, even when the auxiliary choice gadgets are glued to the block-replicator gadget via proper identification of $\hat{c}$, $\bar{c}$, $\tilde{c}$ (see labels in Fig. 8) and the variable cycles of the auxiliary choice gadgets. Proposition 5 and the fact that the spin-assignments depicted in Fig. 11(a) and (b) completely determine the spins of the nodes of $\hat{c}$, $\bar{c}$, $\tilde{c}$ imply that the spin-assignment extensions to the whole block-replicator gadget are indeed feasible and unique.

The remaining part of the claimed result can be ascertained by inspecting in Fig. 11(a) and (b) the satisfying spin-assignments forced by the (non) monochromaticity of the incoming cycles. $\square$
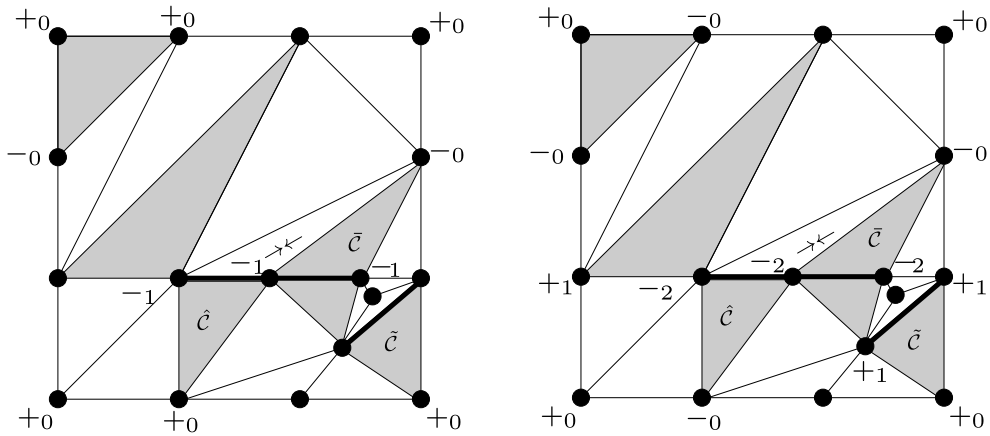
Fig. 9. In (a), satisfying spin-assignment forced by fixing the outgoing cycle uvw to $++-$ and in (b) to $-+-$. Serious edges are shown as thick lines. Note that fixing the spin of an end of a serious edge immediately forces the spin of its other end.
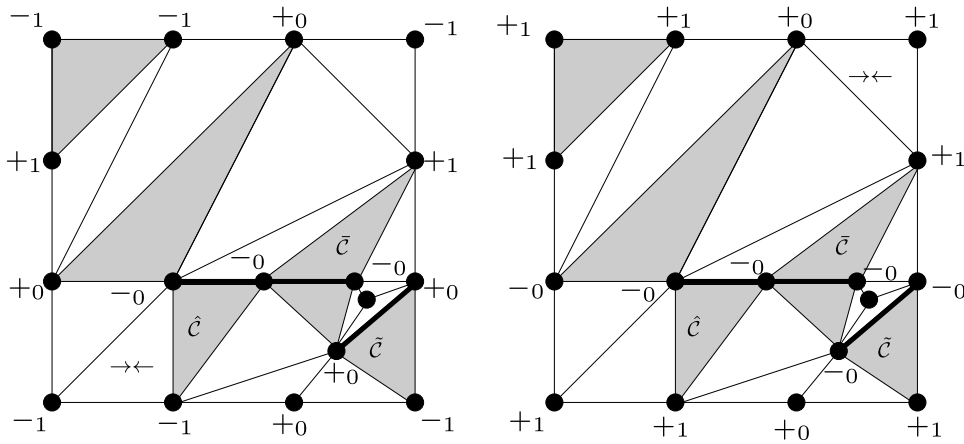
Fig. 10. In (a), satisfying spin-assignment forced by fixing the outgoing cycle $u'v'w'$ to $-++$ and in (b) to $-+-$. Serious edges are shown as thick lines. Note that fixing the spin of an end of a serious edge immediately forces the spin of its other end.
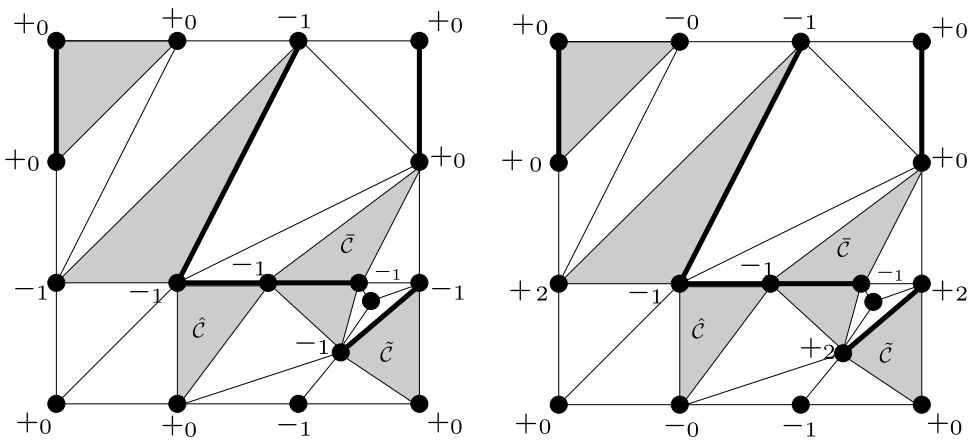
(a) Case where the incoming cycle is positive and monochromatic. Note that the outgoing cycles are forced to be negative and monochromatic.

(b) Case where the incoming cycle is positive and non-monochromatic. Note that outgoing cycles are forced to be negative and non-monochromatic.

Fig. 11. Cases considered to establish the existence and uniqueness (up to duality) of satisfying spin-assignments to the block replicator gadget which extend monochromatic and non-monochromatic spin-assignments of the incoming cycle.
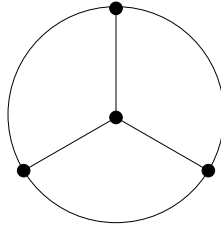
Fig. 12. A cap.

We are now ready to describe the construction of a k-replicator gadget. Take $2^k - 1$ block-replicator gadgets R_1, R_2, $R_3, \ldots, R_{2^k-1}$. Identify the outgoing cycles of R_1 with the incoming cycles of the block replicator gadgets R_2 and R_3 so that the fundamental edges of R_1 that belong to the outgoing cycles and the fundamental edges of R_2 and R_3 that belong to the incoming cycles coincide. Continue in this way piecing together new block replicator gadgets and identifying fundamental edges, and construct a "rooted binary tree of depth k" type structure of block-replicator gadgets. Let R_1 denote the block-replicator gadget at the "root" of the tree, and let $R_{2^{k-1}}, R_{2^{k-1}+1}, \ldots, R_{2^k-1}$ denote the block-replicator gadgets at the "leaves" of the tree. The incoming cycle of R_1 will be referred to as the *starting cycle* of the k-replicator gadget R^k and the outgoing cycles of the block replicator gadgets $R_{2^{k-1}}, R_{2^{k-1}+1}, \ldots, R_{2^k-1}$ will be called *end cycles* of R^k. Moreover, fundamental edges of the block-replicator gadget belonging to the incoming cycle of R_1 and to the outgoing cycles of $R_{2^{k-1}}, R_{2^{k-1}+1}, \ldots, R_{2^k-1}$ will be referred to as fundamental edges of the k-replicator gadget.

Note that each k-replicator gadget is a triangulation of an orientable closed surface of genus $4 \cdot (2^k - 1)$ with $2^k + 1$ holes. Furthermore, the intersection property of the block-replicator gadgets is trivially transferred to k-replicator gadgets; namely, the starting and end cycles of a k-replicator gadget do not share vertices.

In our reduction, the starting cycle of each k-replicator gadget R^k will be identified with the variable cycle of a choice gadget, say L. By Proposition 7, this guarantees that the end cycles of R^k will be monochromatic if and only if the variable cycle of L is monochromatic. It is somewhat unfortunate that the block-replicator gadgets generate, at its outgoing cycles, encodings of opposite signs as the one of its incoming cycle. By taking k even, we can guarantee that each of the end cycles of R^k will have the same chromaticity (monochromatic or non-monochromatic) and sign as the variable cycle of the choice gadget L. The following result captures all relevant properties we will need that are satisfied by replicator gadgets. The reader can easily check that the claimed properties are immediately inherited from those satisfied by block-replicator gadgets.

Corollary 8. *Let k be an even positive integer and let R^k be a k-replicator gadget. The following statements hold:*

1. *Fundamental edges of R^k are serious.*
2. *For any satisfying spin-assignment to R^k, the starting cycle and the end cycles have the same sign.*
3. *For any satisfying spin-assignment to R^k, the starting cycle and the end cycles are all either monochromatic or non-monochromatic.*
4. *There is a unique up duality satisfying spin-assignment to R^k so that the starting cycle and the end cycles are all monochromatic (respectively, non-monochromatic).*

As we have already mentioned, the starting cycle of a replicator gadget, say R, will be identified with a variable cycle of a choice gadget, say L. Assuming that L is in turn associated to a formula variable, say x, it follows that in any satisfying spin-assignment all end cycles of R encode the same truth value of x encoded by the variable cycle of L. Eventually, some end cycles of R will be identified with cycles of the (next to be described) clause gadgets associated to formula clauses where x appears. If the total number of appearances of x in formula clauses is t, then R will be a k-replicator gadget where k is the smallest positive even integer greater or equal than $\log_2 t$. Thus, after identifying end cycles of R with cycles in clause gadgets, we might end up with non-identified end cycles (a situation that occurs whenever $\log_2 t$ is not a positive even integer). The holes circumscribed by such end cycles will need to be "capped" in order so at the end of our reduction we do indeed generate a surface triangulation. Moreover, holes will need to be "capped" in such a way that the properties satisfied by replicator gadgets are preserved. To achieve this goal, when necessary, we identify an end cycle with the edges of the outer face of the complete graph on 4 nodes embedded in the plane, henceforth referred to as *cap gadget* (depicted in Fig. 12).

The following statement is trivial.

Proposition 9. *For any spin-assignment to the outer cycle of a cap gadget, there exists a unique satisfying spin-assignment extension to the whole cap.*

Before we proceed, it is worth mentioning that the described replicator gadget regular binary tree like structure is somewhat wasteful. We could instead have relied on alternative non-regular binary tree like structures as long as they generate the appropriate number of end cycles. However, this yields at most a factor of 2 saving in the time and space needed to generate and store a description of a replicator gadget. For this reason, and simplicity of exposition, we only work with replicator gadgets that have a regular binary tree like structure.
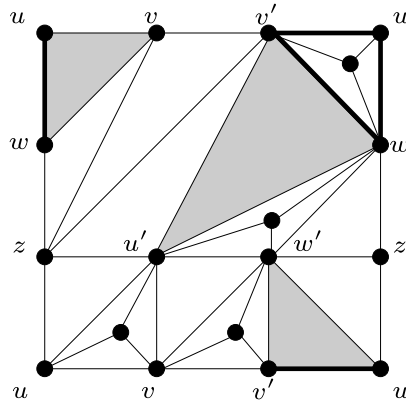
Fig. 13. Clause gadget.

2.4. Clause gadget

A *clause gadget* is a toroidal triangulation with three holes as depicted in Fig. 13. The cycles circumscribing the holes of the clause gadget will be called *literal cycles*. Moreover, edges uw, $v'w$ and $v'u$ will be referred to as *fundamental edges* of the clause gadget (depicted as thicker lines in Fig. 13).

As already mentioned in the preceding section, we will eventually identify end cycles of replicator gadgets with literal cycles in such a way that fundamental edges of the replicator and clause gadgets coincide. In our reduction, replicator gadgets will "carry" from choice gadgets towards clause gadgets the encodings of the truth values of formula variables. The clause gadget is built in such a way as to allow a unique up to duality satisfying spin-assignment extension if and only if not all the truth value encodings "arriving" to the clause gadget represent the same truth value.

Unfortunately, fundamental edges of clause gadgets are not serious. However, once every literal cycle of a clause gadget is identified with an end cycle of a replicator gadget, fundamental edges of the clause gadget will become serious in the triangulation thus formed—since fundamental edges of the replicator gadgets are serious, and because fundamental edges of the clause and replicator gadgets will be identified. This explains why when stating the following claims we assume seriousness of fundamental edges of the clause gadgets. The functionality provided by a clause gadget is summarized by the next results, the first of which is obvious.

Proposition 10. *Let C be a clause gadget. Assume that fundamental edges of C are serious. Then, for any satisfying spin-assignment, all literal cycles of C have the same sign.*

Proposition 11. *Let C be a clause gadget. Assume that fundamental edges of C are serious. In the following cases there is no satisfying spin-assignment extension to C:*

1. *when all literal cycles of C are monochromatic, and*
2. *when all literal cycles of C are non-monochromatic.*

Proof. To prove the first claim, by duality and Proposition 10, it suffices to show that if all nodes in each literal cycle are assigned $+1$, then there is no feasible spin-assignment extension to C. This case is worked out in Fig. 14(a).

To establish the second claim, it suffices to show that the same conclusion holds when the literal cycles uwv, $v'wu'$ and $v'uw'$ are all assigned $++-$ (node labels as in Fig. 13). This case is worked out in Fig. 14(b). □

Proposition 12. *Let C be a clause gadget. Assume that the fundamental edges of C are serious. If exactly one literal cycle of C is monochromatic (respectively, non-monochromatic) there is a unique up to duality satisfying spin-assignment extension to C.*

Proof. By duality and Proposition 10 the monochromatic case holds if we show that in each of the following situations there is exactly one feasible spin-assignment extension to C (node labels as in Fig. 13): (a) when the literal cycle uwv is assigned $+++$ and spins $++-$ are assigned to cycles $v'wu'$ and $v'uw'$, (b) when the literal cycle $v'wu'$ is assigned $+++$ and spins $++-$ are assigned to cycles uwv and $v'uw'$, and (c) when the literal cycle $v'uw'$ is assigned $+++$ and spins $++-$ are assigned to cycles uwv and $v'wu'$. Each case is worked out separately in Fig. 15(a), (b), and (c).

For the non-monochromatic case, we proceed in the same way. Again, by duality and Proposition 10, it suffices to examine the following situations (node labels as in Fig. 13): (a) when the literal cycle uwv is assigned $++-$ and spins $+++$ are assigned to cycles $v'wu'$ and $v'uw'$, (b) when the literal cycle $v'wu'$ is assigned $++-$ and spins $+++$ are assigned to cycles uwv and $v'uw'$, and (c) when the literal cycle $v'uw'$ is assigned $++-$ and spins $+++$ are assigned to cycles uwv and $v'wu'$. Each case is worked out in Fig. 16(a), (b), and (c). □

(a) Case where all literal cycles are monochromatic. (b) Case where all literal cycles are non-monochromatic.
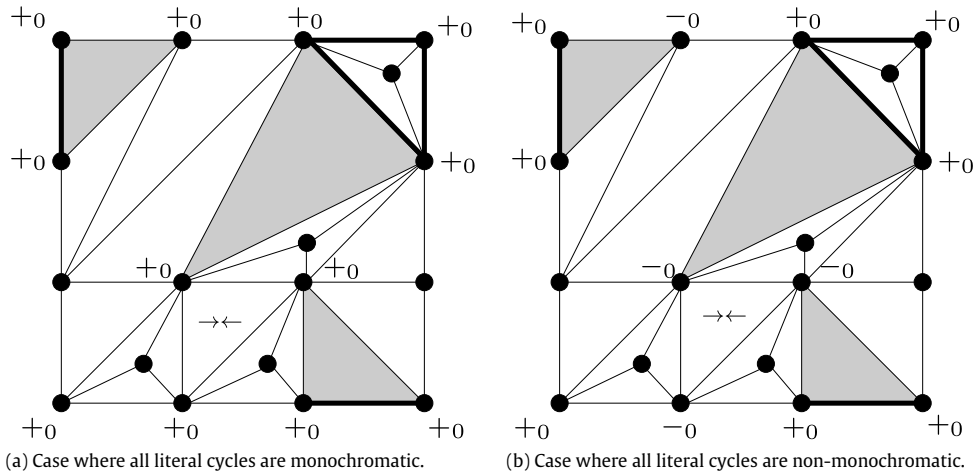
Fig. 14. Cases analyzed to establish that there are no satisfying spin-assignment extensions to the clause gadget when all literal cycles are either monochromatic or non-monochromatic.
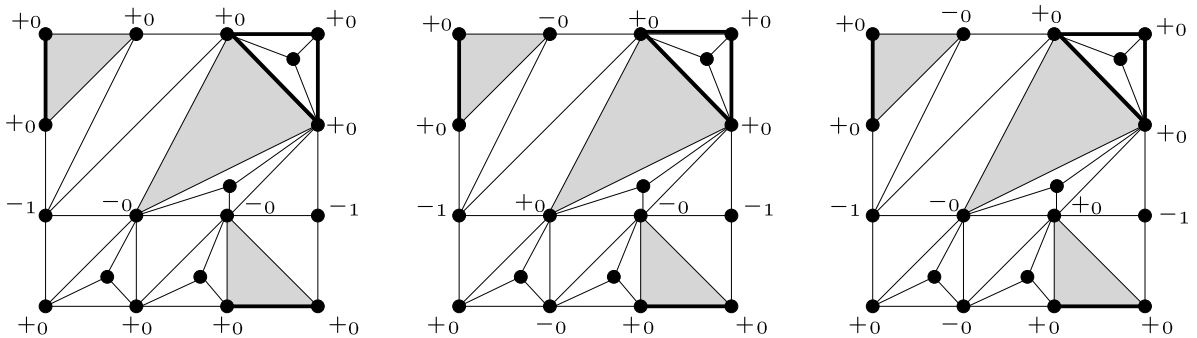

Fig. 15. Unique forced satisfying spin-assignments to a clause gadget when exactly one literal cycle is monochromatic. Each shown spin-assignment encodes a truth value assignment to the variables of a clause where not all truth values are equal.
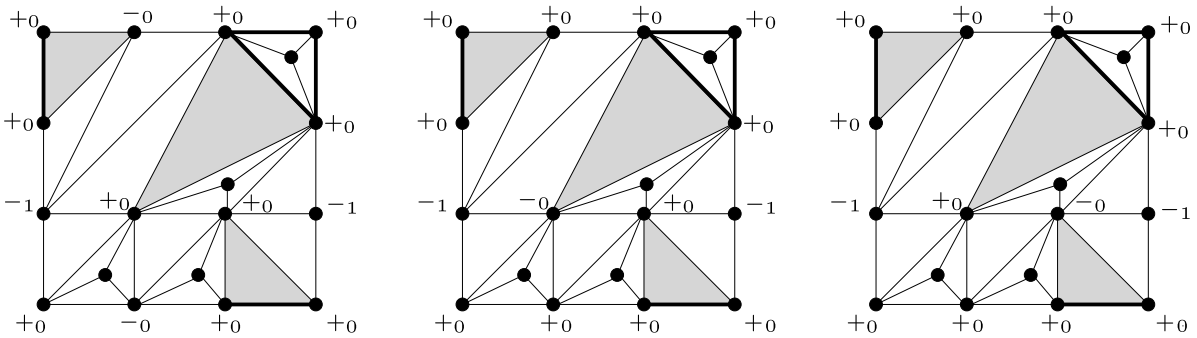

Fig. 16. Unique forced satisfying spin-assignments to a clause gadget when exactly one literal cycle is non-monochromatic. Each shown spin-assignment encodes a truth value assignment to the variables of a clause where not all truth values are equal.

3. The main reduction

We now describe the reduction from Pos-NAE-3SAT to SatAssign. Let φ be a Boolean formula in conjunctive normal form, where each clause has exactly three (non-negated) literals. Let $x_1, \ldots, x_n$ be the variables and $Cl_1, \ldots, Cl_m$ be the clauses of φ. Let t_i denote the number of times variable x_i appears in the collection of clauses (multiple occurrences are counted multiple times). Define

$$k_i = \begin{cases} 2, & \text{if } t_i = 1, \\ 2\lceil (1/2) \log_2(t_i) \rceil, & \text{if } t_i > 1. \end{cases}$$

To each variable x_i we associate a choice gadget L_i. To each clause Cl_j we associate a clause gadget C_j. For $i = 1, \ldots, n$, we identify the starting cycle of a k_i-replicator gadget R^{k_i} with the variable cycle of the choice gadget L_i so that the fundamental edge of L_i that belongs to L_i's variable cycle and the fundamental edge of R^{k_i} that belongs to its starting cycle coincide. Note that the number of end cycles of R^{k_i} is at least t_i. For $i = 1, \ldots, n$, identify t_i end cycles of R^{k_i} with literal cycles of the clause gadgets $C_1, \ldots, C_m$ where variable x_i appears in such a way that the fundamental edges of the end cycles of R^{k_i} and the fundamental edges of the literal cycles of the clause gadgets coincide. Identify the remaining $2^{k_i} - t_i$ end cycles of R^{k_i} (if any) with the outer cycle of a cap gadget. Denote by T_φ the surface triangulation thus obtained.

We first make a simple observation.

Lemma 13. *The fundamental edges of each clause gadget of T_φ are serious.*

Proof. Just observe that by construction of T_φ, fundamental edges of clause gadgets are identified with fundamental edges of replicator gadgets which are known to be serious, as established by Corollary 8(1). □

Note that, by construction, the surface on which T_φ is embedded is an orientable closed surface of genus at least $m + n + 4 \sum_{i=1}^{n} (2^{k_i} - 1)$ (1 due to each choice gadget $L_1, \ldots, L_n$, another 1 due to each clause gadget $C_1, \ldots, C_m$, and $4(2^{k_i} - 1)$ due to each replicator gadget R^{k_i}, $i = 1, \ldots, n$). Clearly, since each 3-cycle circumscribing gadget holes were identified with a 3-cycle circumscribing another gadget hole, the surface on which T_φ is embedded *does not* have holes, i.e. it is a closed surface. Moreover, since the construction process does not create additional faces, each face of T_φ is a face of some gadget. Thence, each face is bounded by a 3-cycle, so T_φ is a triangulation. Finally, note that since each of the gadgets used in the construction of T_φ is embeddable in an orientable surface with holes, the resulting surface on which T_φ is embedded is also an orientable surface. Summarizing, T_φ is a triangulation of an orientable closed surface.

We say that φ is *connected* if for every non-trivial partition $\{S, \bar{S}\}$ of the clauses of φ (i.e. $S, \bar{S} \neq \emptyset$, $S \cap \bar{S} = \emptyset$, and $S \cup \bar{S}$ equals the set of clauses of φ) there is at least one variable that appears in one of the clauses in S and in one of the clauses of $\bar{S}$.

We now make a couple of useful observations.

Lemma 14. *Let φ be an instance of Pos-NAE-3SAT. If φ is connected, then the surface in which T_φ is embedded is also connected.*

Proof. Assume T_φ is embedded in a non-connected surface. Consider the set S of clauses whose associated clause gadgets are embedded in one of the connected surface components, say $\mathcal{S}$. Let $\bar{S}$ be the collection of clauses not in S. Note that $\{S, \bar{S}\}$ is non-trivial. Moreover, the set of variables that appear in clauses in S (respectively, in $\bar{S}$) correspond to those variables associated to choice gadgets embedded in $\mathcal{S}$ (respectively, not in $\mathcal{S}$). Both of these collection of variables must be disjoint, contradicting the fact that φ is connected. □

Lemma 15. *Consider an instance φ of Pos-NAE-3SAT. Let Cl be a clause of φ and x_1, x_2, and x_3 the (not necessarily distinct) variables appearing in Cl. Let L_1, L_2, and L_3 be the (not necessarily distinct) choice gadgets of T_φ associated to x_1, x_2 and x_3, respectively. Let C be the clause gadget of T_φ associated to Cl. Then, for every satisfying spin-assignment s to T_φ, the literal cycles of C and the variable cycles of L_1, L_2, and L_3 have the same sign. Moreover, if φ is connected, then all literal cycles and variable cycles of T_φ have the same sign.*

Proof. By Proposition 10 and Lemma 13, for any satisfying spin-assignment to T_φ all literal cycles of C have the same sign. Without loss of generality, we can assume that all literal cycles of C are positive. Hence, the end cycles of the replicator gadgets, say R^{k_1}, R^{k_2}, and R^{k_3}, which are identified with the literal cycles of C, must all be positive. Since k_1, k_2 and k_3 are even, by Corollary 8(2), the starting cycles of R^{k_1}, R^{k_2} and R^{k_3} are positive. Given that the variable cycles of L_1, L_2, and L_3 are identified in T_φ with the starting cycles of R^{k_1}, R^{k_2} and R^{k_3}, the first stated claim follows.

The last statement follows trivially from Lemma 14. □

Theorem 16. *If φ is a connected instance of Pos-NAE-3SAT, then*

1. *For each truth value assignment that witnesses membership of φ in Pos-NAE-3SAT, there is exactly one (up to duality) satisfying spin-assignment to T_φ.*
2. *For every pair of duality related satisfying spin-assignment to T_φ there is exactly one truth value assignment that witnesses membership of φ in Pos-NAE-3SAT.*

Proof. Let $a_1, \ldots, a_n$ be a truth value assignment to the variables $x_1, \ldots, x_n$ that is a witness of membership of φ in Pos-NAE-3SAT. We claim that there is a unique up to duality satisfying spin-assignment to T_φ. As usual, let $Cl_1, \ldots, Cl_m$ be the clauses of φ, let $C_1, \ldots, C_m$ denote the associated clause gadgets, and let $L_1, \ldots, L_n$ and $R^{k_1}, \ldots, R^{k_n}$ be the choice and replicator gadgets associated to variables $x_1, \ldots, x_n$.

If a_i is TRUE, fix the spins of all nodes of the variable cycle of L_i in such a way that the cycle ends up being positive and monochromatic (observe that this can be done in a unique way). Otherwise, a_i is FALSE, fix the spins of all nodes of the variable cycle of L_i in such a way that the cycle ends up being positive and non-monochromatic (observe that since fundamental edges of choice gadgets are serious, this can again be done in a unique way). Extend the so far partially defined

satisfying spin-assignment to the union of choice gadgets (by Proposition 5, such a satisfying assignment extension exists and is unique).

Similarly, fix the spins of the nodes of the literal cycles of each clause gadget C_j according to the truth value taken by the associated Boolean formula variable, i.e. if the variable's value is TRUE, make the literal cycle positive and monochromatic, and positive and non-monochromatic otherwise (observe again that such spin-assignments can be done in a unique way). Extend once more the so far partially defined spin-assignment to the union of clause gadgets. We claim that such an extension exists and is unique. Indeed, given that $a_1, \ldots, a_n$ is a witness of membership of φ in Pos-NAE-3SAT, the variables in each clause Cl_j do not take the same truth value under the assignment $a_1, \ldots, a_n$. Thus, the aforementioned spin-assignment to the literal cycles of C_j is such that not all literal cycles end up having the same chromaticity. By Proposition 12, each clause gadget has a unique satisfying spin assignment extension, thus establishing our claim.

Recall that variable cycles of choice gadgets (respectively, literal cycles of clause gadgets) are identified with starting cycles (respectively, end cycles) of replicator gadgets. Hence, for all $i = 1, \ldots, n$, the partial spin-assignment thus far defined makes the start and end cycle of the replicator gadget R^{k_i} positive and monochromatic (respectively, non-monochromatic) if and only if a_i is TRUE (respectively, FALSE). Given that k_i is even and positive for all $i = 1, \ldots, n$, by Corollary 8(4) and Proposition 9, there is a unique extension of the previously defined partial spin-assignment to all nodes of replicator gadgets so the resulting spin-assignment is a satisfying spin-assignment for T_φ.

We now prove the second part of the claimed result. Assume there is a satisfying spin-assignment s to T_φ. By Lemma 15 and since φ is connected, all literal and variable cycles of T_φ have the same sign, say positive. For $i = 1, \ldots, n$, let a_i be TRUE if the variable cycle of L_i is positive and monochromatic, and FALSE if the variable cycle of L_i is positive and non-monochromatic. We claim that $a_1, \ldots, a_n$ is a witness of membership of φ in Pos-NAE-3SAT. Indeed, assume x_{j_1}, x_{j_2}, and x_{j_3} are the (not necessarily distinct) variables appearing in clause Cl_j. Let $s \in \{1, 2, 3\}$. Since the start cycle of the replicator gadgets $R^{k_{j_s}}$ is identified with the variable cycle of the choice gadget L_{j_s}, then by Corollary 8 the end cycles of $R^{k_{j_s}}$ must be monochromatic if and only if a_{j_s} is TRUE. Since the non-capped end cycles of replicator gadgets are identified with the literal cycles of clause gadgets, we have that the literal cycle of the clause gadget C_j associated to the variable x_{j_s} is monochromatic if and only if a_{j_s} is TRUE. Moreover, all literal cycles of clause gadgets are positive. Since s is a satisfying spin-assignment, by Propositions 11 and 12, the literal cycles of C_j cannot all be either monochromatic or non-monochromatic. This implies that a_{j_1}, a_{j_2}, and a_{j_3} are not all equal, as we wanted to establish. $\square$

Corollary 17. *Let φ be an instance of Pos-NAE-3SAT. Then, φ is satisfiable if and only if there is a satisfying spin-assignment to T_φ.*

Proof. If φ is connected, the result is immediate from Theorem 16. Assume φ is not connected. Then, there are $\varphi_1, \ldots, \varphi_c$ instances of Pos-NAE-3SAT such that each φ_i is connected and $\varphi = \wedge_{i=1}^{c} \varphi_i$. Moreover, φ belongs to Pos-NAE-3SAT if and only if $\varphi_1, \ldots, \varphi_c$ belong to Pos-NAE-3SAT. By Theorem 16, this is equivalent to saying that φ belongs to Pos-NAE-3SAT if and only the union of the surface triangulations $T_{\varphi_1}, \ldots, T_{\varphi_c}$, i.e. T_φ, admits a satisfying spin assignment. $\square$

The next result allows us to successfully handle, in our reduction, instances of Pos-NAE-3SAT which are not connected.

Lemma 18. *Let φ be an instance of Pos-NAE-3SAT. Then, there exists a log-space (hence, polynomial time) computable instance φ' of Pos-NAE-3SAT such that φ' is connected and the following equality holds:*

$$\left| \{\vec{a} : \text{if } \vec{a} \text{ is a witness of membership of } \varphi \text{ in Pos-NAE-3SAT} \} \right|$$
$$= \frac{1}{2} \left| \{\vec{a'} : \text{if } \vec{a'} \text{ is a witness of membership of } \varphi' \text{ in Pos-NAE-3SAT} \} \right|.$$

Proof. Assume $x_1, \ldots, x_n$ are the variables and $C_1, \ldots, C_m$ the clauses of φ. Consider two additional Boolean variables y and z and define n additional clauses $C_1', \ldots, C_n'$ such that $C_i' = x_i \wedge y \wedge z$. Let φ' be the conjunction of $C_1, \ldots, C_m, C_1', \ldots, C_n'$. Clearly, φ' is a connected instance of Pos-NAE-3SAT which is log-space computable given φ. Note that a membership witness $a_1, \ldots, a_s$ of an instance of Pos-NAE-3SAT cannot be such that all a_i's are equal. This immediately implies that $x_1, \ldots, x_n$ is a witness of membership of φ in Pos-NAE-3SAT if and only if $x_1, \ldots, x_n, y = 0, z = 1$ and $x_1, \ldots, x_n, y = 1, z = 0$ are witnesses of membership of φ' in Pos-NAE-3SAT. $\square$

Corollary 19. *Let φ be an instance of Pos-NAE-3SAT. Then, there exists a log-space (hence, polynomial time) computable instance φ' of Pos-NAE-3SAT such that*

$$\left| \{s : s \text{ is a satisfying spin-assignments to } T_{\varphi'} \} \right| = 4 \cdot \left| \{\vec{a} : \text{if } \vec{a} \text{ is a witness of membership of } \varphi \text{ in Pos-NAE-3SAT} \} \right|.$$

To conclude, note that given an instance φ of Pos-NAE-3SAT with n variables and m clauses each of the necessary gadgets can be constructed in $O(\log |V(T_\varphi')|)$ space, i.e. in log-space in the size of the encoding of φ. The number of choice, block-replicator, and clause gadgets that need to be built are $n + 2$, $\sum_{i=1}^{n} (2^{k_i} - 1) = O(m)$, and $m + n$, respectively. Hence, an encoding of a rotation system for $T_{\varphi'}$ can be computed in log-space, thence in polynomial time. Then, Corollaries 17 and 19 imply Theorems 1 and 2, respectively.

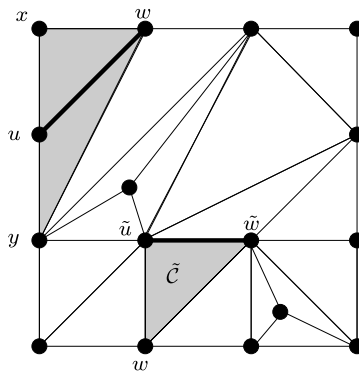| PROBLEM	#Monotone-2SAT
INPUT	A Boolean formula φ in 2-conjunctive normal form such that each of its clauses $C_1, \ldots, C_m$ has exactly two (all non-negated) literals.
OUTPUT	A (binary encoding) of the number of distinct truth value assignments to φ such that for each clause C_i at least one term is TRUE.

Fig. 17. #Monotone-2SAT.

Fig. 18. New clause gadget.

4. Additional hardness results

We now establish the #P-hardness of #GroundSt (Theorem 3). The result will follow by a weakly parsimonious reduction from the standard counting version of monotone 2-satisfiability, formally defined in Fig. 17 and usually abbreviated as #Monotone-2SAT. It is known that #Monotone-2SAT is #P-complete [17] even though every instance of #Monotone-2SAT is trivially satisfiable (by setting all variables to TRUE).

The strategy to prove Theorem 3 is based on a modification of the reduction presented in the preceding section. Indeed all gadgets except one, the clause gadget, on which the new reduction relies are identical to those on which the proof of the #P-hardness of #SatAssign (Theorem 2) relies. The key new ingredient we use to derive hardness result for #GroundSt is that every instances of #Monotone-2SAT is satisfiable. Therefore, since the new reduction will be weakly parsimonious, any surface triangulation T to which an instance of #Monotone-2SAT ends up being mapped will admit a satisfying spin-assignment. Hence, the groundstates of T will correspond exactly to satisfying states, and vice versa.

The *new clause gadget* will be a punctured triangulation of an orientable closed surface of genus two. Each new clause gadget is built by gluing together a choice gadget[5] and the punctured toroidal triangulation with three removed faces depicted in Fig. 18. Specifically, the construction takes a choice gadget, say $\tilde{L}$, (see Fig. 4), and identifies the literal cycle $c_{\tilde{l}}$ of the choice gadget with the cycle $\tilde{c} = \{\tilde{u}\tilde{w}w\}$ of the new clause gadget, as depicted in Fig. 18. The identification is done in such a way that edge $\tilde{u}\tilde{w}$ of the cycle $\tilde{c}$ coincides with the fundamental edge of the choice gadget $\tilde{L}$. As usual, the cycles circumscribing the holes of the clause gadget (uxw and uyw in the new clause gadget) will be called *literal cycles*. Moreover, edge uw will be referred to as the *fundamental edge* of the new clause gadget.

The functionality provided by the new clause gadget is summarized by the next results, the first of which is obvious and the proof of the second one is analogous to Proposition 12.

Proposition 20. *Let C be a new clause gadget. Assume that the fundamental edge of C is serious. Then, for any satisfying spin-assignment to C, both literal cycles of C have the same sign.*

Proposition 21. *Let C be a new clause gadget. Assume that the fundamental edge of C is serious. In each of the following cases there is a unique up to duality pair of satisfying spin-assignment extensions to C:*

1. *when all literal cycles of C are monochromatic,*
2. *when exactly one literal cycle of C is monochromatic.*

Moreover, when all literal cycles of C are non-monochromatic, there is no satisfying spin-assignment extension to C.

---

[5] Again, the choice gadget will not be associated to Boolean variables, we only rely on it because of one of its properties. In particular, because its fundamental edge is serious.

The reduction from #Monotone-2SAT to #GroundSt maps an instance φ of #Monotone-2SAT to a surface triangulation T_φ in the same way as the reduction of the previous section associates to an instance of #Pos-NAE-3SAT and instance of #SatAssign, except that new clause gadgets are used instead of the clause gadgets defined in the previous section. It follows that if φ is connected, then

1. For each truth-value assignment that satisfies φ there is a unique up to duality pair of groundstates of T_φ.
2. For every pair of duality related groundstates of T_φ there is exactly one truth-value assignment that satisfies φ.

Since an instance of #Monotone-2SAT may not be connected we require the following result.

Lemma 22. *For any φ instance of #Monotone-2SAT, there is a log-space (hence, polynomial time) computable instance φ' of #Monotone-2SAT such that φ' is connected and the following equality holds:*

$$\left|\{\vec{a} : \vec{a} \text{ is a truth-value assignment that satisfies } \varphi\}\right| = \left|\{\vec{a'} : \vec{a'} \text{ is a truth-value assignment that satisfies } \varphi'\}\right| - 1.$$

Proof. Assume $x_1, \ldots, x_n$ are the variables and $C_1, \ldots, C_m$ the clauses of φ. Consider one additional Boolean variable y and define n additional clauses $C'_1, \ldots, C'_n$ such that $C'_i = x_i \wedge y$. Let φ' be the conjunction of $C_1, \ldots, C_m, C'_1, \ldots, C'_n$. Clearly, φ' is an instance of #Monotone-2SAT which is connected and log-space computable given φ. Assume φ is satisfied when $x_1, \ldots, x_n$ take the truth-value $a_1, \ldots, a_n$. If one among $a_1, \ldots, a_n$, say a_i, is FALSE, then y must take the truth-value TRUE in order for C'_i to evaluate to TRUE. If all of $a_1, \ldots, a_n$ are equal to TRUE, then φ' will evaluate to TRUE at $a_1, \ldots, a_n, y$ independently of the value taken by y. Conversely, if φ' is satisfied for a given truth-value assignment of $x_1, \ldots, x_n, y$, then φ is satisfied for the same assignment to $x_1, \ldots, x_n$. The desired conclusion immediately follows. □

Note now that given an instance φ of #Monotone-2SAT with n variables and m clauses each of the necessary reduction gadgets can be constructed in $O(\log |V(T_{\varphi'})|)$ space, i.e. in log-space in the size of the encoding of φ. The number of choice, block-replicator, and clause gadgets that need to be built are $n + 1$, $\sum_{i=1}^{n}(2^{k_i} - 1) = O(m)$, and $m + n$, respectively. Hence, an encoding of a rotation system for $T_{\varphi'}$ can be computed in log-space, thence in polynomial time (in fact, in linear time). In other words, the following result holds:

Corollary 23. *Let φ be an instance of #Monotone-2SAT. Then, there exists a log-space (hence, polynomial time) computable instance φ' also of #Monotone-2SAT such that*

$$\left|\{\sigma : \sigma \text{ is a groundstate of } T_{\varphi'}\}\right| = 2 \cdot \left(\left|\{\vec{a} : \vec{a} \text{ is a truth-value assignment that satisfies } \varphi\}\right| + 1\right).$$

Corollary 23 immediately implies Theorem 3. Recall that in the thermodynamical limit at zero temperature, the entropy of the antiferromagnetic Ising model for a triangulation T is proportional to the logarithm of its number of groundstates normalized by the number of nodes of T. Hence, Corollary 23 says that already the determination of the entropy of such systems, at zero temperature, is an extremely hard problem, specifically it is #P-hard. This latter result is complementary to that of [1], which establishes that for the general Ising model (not just the antiferromagnetic version), it is NP-hard to compute its partition function—a quantity from which the system's entropy in the thermodynamical limit can be efficiently computed.

We conclude this section with the proof of the last of the main results of this article.

Proof of Corollary 4. First, recall the definition of energy (refer to (1)) of a spin assignment σ of a triangulation T and observe that for the edge cut C_σ consisting of the edges of T whose extremes are assigned distinct spins, the following holds:

$$\mathcal{E}_T(\sigma) = \sum_{uv \in E(T)} \sigma(u) \cdot \sigma(v) = |E(T)| - 2|C_\sigma|.$$

Hence, since groundstates (i.e. states of minimum energy) of the antiferromagnetic Ising model over triangulations are non-constant, they correspond to edge cuts of maximum size.

A second crucial observation is that implicit in the proof of Theorem 3 is the fact that #GroundSt remains #P-hard even when restricted to connected triangulations.

Clearly, if σ is a groundstate of the antiferromagnetic Ising model over a triangulation T, then it is non-constant and the corresponding edge-cut C_σ is of maximum size. On the other hand, an edge-cut C of a connected triangulation T corresponding to the set of edges with one of its extremes in $\emptyset \neq S \subsetneq V(T)$ gives rise to two duality related spin assignments; (1) the spin assignment σ that is identically -1 and $+1$ over S and $V(T) \setminus S$, respectively, and (2) the spin assignment $-\sigma$. Since T is connected, σ is uniquely determined (up to duality) by the edge-cut C of T. It follows that if C is a maximum size edge-cut and T is a connected triangulation, then

$$|\{\sigma : \sigma \text{ is a groundstate of } T\}| = 2 \cdot |\{C : C \text{ is a maximal edge-cut of } T\}|.$$

The #P-hardness of #TriangMaxCut follows immediately from Theorem 3. □

5. Final comments

We believe that this work raises several interesting problems. First, characterize the class of surface triangulations that do not exhibit groundstate degeneracy. Second, to determine the groundstate degeneracy of random triangulations in the antiferromagnetic Ising model. Solutions to this latter problem could help understand the behavior of geometrically frustrated systems, which is a central problem in condensed matter physics.

On the other hand, spin glass models have attracted considerable attention over recent years, both in solid state and in statistical physics (see for instance [14]). In the Ising spin glass model, coupling constants are randomly distributed, in fact they are typically set to either $+1$ or -1 uniformly at random. (The antiferromagnetic Ising model corresponds to the particular case where the coupling constant is set to -1 with probability equal to one.) In this random setting, it would be interesting to study the behavior of (the random variable corresponding to) the number of groundstates of the Ising model on plane triangulations (where groundstates are satisfying states) and toroidal triangulations (where groundstates are not necessarily satisfying states).

Acknowledgments

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