Math. Ann. (2016) 365:923–967
DOI 10.1007/s00208-015-1283-z

**Mathematische Annalen**

# The NSLUC property and Klee envelope

**A. Jourani[1] · L. Thibault[2,3] · D. Zagrodny[4]**

Received: 7 October 2014 / Revised: 26 March 2015 / Published online: 29 August 2015


**Abstract** A notion called norm subdifferential local uniform convexity (NSLUC) is introduced and studied. It is shown that the property holds for all subsets of a Banach space whenever the norm is either locally uniformly convex or $k$-fully convex. The property is also valid for all subsets of the Banach space if the norm is Kadec-Klee and its dual norm is Gâteaux differentiable off zero. The NSLUC concept allows us to obtain new properties of the Klee envelope, for example a connection between attainment sets of the Klee envelope of a function and its convex hull. It is also proved that the Klee envelope with unit power plus an appropriate distance function is equal to some constant on an open convex subset as large as we need. As a consequence of obtained results, the subdifferential properties of the Klee envelope can be inherited from subdifferential properties of the opposite of the distance function to the complement of the bounded convex open set. Moreover the problem of singleton property of

This research was supported by Math-AmSud 13math-01 and ECOS projects.

✉ L. Thibault
lionel.thibault@univ-montp2.fr

A. Jourani
jourani@u-bourgogne.fr

D. Zagrodny
d.zagrodny@uksw.edu.pl

1 Institut de Mathématiques de Bourgogne, UMR 5584 CNRS, Université de Bourgogne, BP 47 870, 21078 Dijon Cedex, France

2 Institut de Mathématiques, UMR 5149 CNRS, Université Montpellier II, Case Courier 051, Place Eugène Bataillon, 34095 Montpellier Cedex, France

3 Centro de Modelamiento Matematico, Universidad de Chile, Santiago, Chile

4 Faculty of Mathematics and Natural Sciences, College of Science, Institute of Mathematics, Cardinal Stefan Wyszynski University, Dewajtis 5, Warsaw, Poland

sets with unique farthest point is reduced to the problem of convexity of Chebyshev sets.

**Mathematics Subject Classification** Primary 49J52 · 46B20; Secondary 52A41 · 41A50

## 1 Introduction

Let $(X, \|\cdot\|)$ be a real normed vector space. At the beginning of thirties of tweentieth century S. Mazur considered the following property:

*Every closed bounded convex subset of X, say D, is the intersection of closed balls containing it,*

See [28]. There are several papers where this is investigated. We refer to [3,8] for surveys on achievements and historical information of the property, which is called nowadays the Mazur Intersection Property, MIP for short. In other words, denoting by $B[x,r]$ [resp. $B(x,r)$] the closed (resp. open) ball centered at $x$ with radius $r > 0$, we can express the Mazur Intersection Property as follows

$$D = \bigcap_{x\in X, r>0, D\subset B[x,r]} B[x,r].$$

Moreover we expect that $D \cap (B[x,r]\backslash B(x,r)) \neq \emptyset$ for every ball with the smallest radius in the right-hand side of the equality. With this geometry we can relate an analytic reasoning. Namely, for every bounded subset $S \subset X$ we can define the *farthest distance function* from the set $S$ (also called antidistance function)

$$\Delta_S(x) := \sup_{y\in S} \|x - y\|$$

and the set $Q_S(x)$ of farthest points from $S$ to $x$, that is,

$$Q_S(x) := \{y \in S : \|x - y\| = \Delta_S(x)\}.$$

Using the notions defined above the Mazur Intersection Property can be rewritten in the form

$$D = \bigcap_{x\in X} B[x, \Delta_D(x)].$$

The nonemptiness of the intersection $D\cap(B[x,r]\backslash B(x,r)) \neq \emptyset$ with $r = \Delta_D(x)$ is in fact a question on nonemptiness of the set of farthest points. The above consideration can be embedded into a more general set up. Namely we can use the Klee envelope instead of the farthest distance function, that is, for $\lambda > 0$, $p \geq 1$ and an extended real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$ we put

$$\kappa_{\lambda,p} f(x) := \sup_{y \in X} \left( \frac{1}{p\lambda} \|x - y\|^p - f(y) \right). \tag{1}$$

Then the associated attainment set is denoted by $Q_{\lambda,p} f(x)$, that is,

$$Q_{\lambda,p} f(x) := \left\{ y \in X : \frac{1}{p\lambda} \|x - y\|^p - f(y) = \kappa_{\lambda,p} f(x) \right\}. \tag{2}$$

Of course, the Klee function with $p = 1$ and $\lambda = 1$ coincides with the farthest distance function, that is, $\kappa_{1,1}(\psi_S)(x) = \Delta_S(x)$ and $Q_{\lambda,p}(\psi_S)(x)$ coincides with $Q_S(x)$, whenever $f$ is the indicator function $\psi_S$ of a subset $S$ of $X$, that is, $\psi_S(x) = 0$ if $x \in S$ and $\psi_S(x) = +\infty$ otherwise.

We shall see that results on the Klee envelope add new knowledge to the old problems recalled below concerning sets of farthest points. Let us also point out that using this set up we do not take care about the convexity of the set $S$, more generally about the convexity of the function $f$ that we transform through the Klee envelope. However, it is worth considering whether the convexity is important or not in Definitions (1) and (2). On the one hand, we show that it does not matter if we take $f$ or $\overline{\text{co}} f$ in (1), we get the same value, see Proposition 7. On the other hand, in some spaces we have also the following implication

$$d \in Q_{\lambda,p}(\overline{\text{co}}\, f)(x) \implies d \in \text{co}(Q_{\lambda,p} f(x) \cap (x + \text{span}(d - x))),$$

which entails the inclusions

$$Q_{\lambda,p} f(x) \subset Q_{\lambda,p}(\overline{\text{co}}\, f)(x) \subset \text{co}\, Q_{\lambda,p} f(x), \tag{3}$$

See Theorem 2. Thus the convex hull of the associated attainment set for $\overline{\text{co}}\, f$ can be recovered as the convex hull from the associated attainment set for $f$. When $f$ is the indicator function $\psi_S$, we obtain the equality

$$Q_S(x) = Q_{\overline{\text{co}} S}(x),$$

in several important cases increasing the range of its use, see Sect. 5 and Theorem 2. Of course, if $Q_S(x)$ is single-valued, then $Q_{\overline{\text{co}} S}(x)$ is single-valued too in this case. So, natural is the question: *Which sets and points have this property?* There is also the old question posed by V. Klee (see [26] or [23]), closely related to this question, namely:

Klee's question: *Suppose that $Q_S(x)$ is a singleton for every $x \in X$. Must $S$ consist of a single point?*

In fact, if we look carefully at [26, Theorem1.2] we notice that an affirmative answer to the above question by V. Klee implies the convexity of Chebyshev sets and vice versa, whenever $X$ is a real Hilbert space; we recall that *a set $S$ is called Chebyshev provided that every point of the space $X$ admits a unique nearest point in $S$*. So far, both questions are unsolved in Hilbert spaces, see for example [23]. Let us also point out that not always investigating farthest points from a set $S$ can be changed into finding farthest points from its closed convex hull. For example, if $X := c_0$, $x := (1, 2^{-1}, 3^{-1}, \ldots)$, $s_i := (i^{-1}, 0, \ldots, 0, 1, 0, \ldots)$ with 1 being the $i$th component for every $i = 2, 3, \ldots$, $S := \{s_2, s_3, \ldots\}$, then $Q_S(x) = \emptyset$ but $Q_{\overline{\text{co}}S}(x) = \{0\}$ since $0 \in \text{cl}_{w(c_0, \ell^1)} S \subset \overline{\text{co}}\, S$. This example realizes the need of finding a large class of spaces and sets for which (3) holds true. For this reason we propose a new notion, called norm subdifferential local uniform convexity (NSLUC), see Definition 1, which allows us to get in particular (3) in several cases, for example:

1. The norm of $X$ has the LUR property (that is, the local uniform rotundity/convexity property);
2. The norm of $X$ is fully $k-$convex;
3. The norm of $X$ is strictly convex and has the Kadec-Klee property, and $S$ is relatively weakly sequentially compact;
4. $X$ is a Banach space whose norm has the Kadec-Klee property and the dual norm is Gâteaux differentiable off the origin;
5. The norm of $X$ is strictly convex and $S$ is relatively norm-compact,

See Sect. 5 for details on relations among NSLUC properties and properties of the norm, and Theorem 3 where (3) is established in the listed cases. As an example of application of obtained results we relate the Klee envelope with a distance function, namely the sum of these two functions is constant on some open convex set, see Theorem 4. The two obvious consequence of the relationship between the Klee envelope and the distance functions are: (1) Subdifferential properties of one function can be inherited from subdifferential properties of the other one; (2) Another proof of the Klee idea (that is, the Klee question is in fact a question on the convexity of Chebyshev sets) is obtained, see Theorem 5 and Remark 10.

For some results concerning differential properties of Klee envelopes we refer to the paper [38] in the finite dimensional setting, and to [9] under the strong attainment.

## 2 Background

Throughout we shall assume that $(X, \|\cdot\|)$ is a (real) normed vector space, $X^*$ is its topological dual and $\langle \cdot, \cdot \rangle$ is the pairing between $X$ and $X^*$. We denote by $\mathbb{B}_X$, $\mathbb{S}_X$ and $B[x, r]$ (resp. $B(x, r)$) the closed unit ball, the unit sphere and the closed (resp. open) ball of $X$ centered at $x$ with radius $r > 0$. By co and $\overline{\text{co}}$, we denote the convex hull and the closed convex hull.

The *metric projection mapping* on a subset $S$ of $X$ is defined by

$$P_S(x) := \{s \in S : d(x, S) = \|x - s\|\}, \quad \forall x \in X, \tag{4}$$

where $d(\cdot, S)$ is the distance function from the set $S$, that is,

$$d(x, S) := \inf_{y \in S} \|x - y\|.$$

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. We recall that the *Legendre-Fenchel conjugate* of $f$ is the function $f^* : X^* \to \mathbb{R} \cup \{-\infty, +\infty\}$ with

$$f^*(x^*) := \sup_{y \in X} \Big( \langle x^*, y \rangle - f(y) \Big) \quad \text{for all } x^* \in X^*.$$

The *subdifferential in the sense of convex analysis* of $f$ at a point $x \in X$ where $f$ is finite is defined by

$$\partial f(x) := \{x^* \in X^* : \langle x^*, y - x \rangle \leq f(y) - f(x), \quad \forall y \in X\}.$$

If $f$ is finite at $x$, then $x^* \in \partial f(x)$ if and only if

$$f(x) + f^*(x^*) = \langle x^*, x \rangle;$$

so if the function $f : X \to \mathbb{R} \cup \{+\infty\}$ is convex and lower semicontinuous, one has

$$x^* \in \partial f(x) \Longleftrightarrow x \in \partial f^*(x^*), \tag{5}$$

since under that condition $f$ coincides with the restriction of $(f^*)^*$ to $X$. If $f$ is convex, the nonvacuity of $\partial f(x)$ at any point $x$ where $f$ is both finite and continuous is well known and this will be frequently used in the paper.

The *Fréchet subdifferential* of $f$ at the point $x$ where $f$ is finite is given by

$$\partial_F f(x) := \left\{ x^* \in X^* : \liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle x^*, h \rangle}{\| h \|} \geq 0 \right\}.$$

We adopt the convention $\partial_F f(x) = \emptyset$ when $f(x) = +\infty$. Again with $f(x)$ finite, the lower Dini directional derivative of $f$ at $x$ is given by

$$d^- f(x; h) := \liminf_{w \to h; t \downarrow 0} t^{-1} \big( f(x + tw) - f(x) \big)$$

and when $f$ is Lipschitz continuous near $x$ we obviously have for all $h \in X$

$$d^- f(x; h) = \liminf_{t \downarrow 0} t^{-1} \big( f(x + th) - f(x) \big).$$

The *Dini subdifferential* of $f$ at $x$ is then the set

$$\partial^- f(x) := \{x^* \in X^* : \langle x^*, h \rangle \leq d^- f(x; h), \quad \forall h \in X\}$$

for $x \in \operatorname{dom} f$ and $\partial^- f(x) = \emptyset$ if $x \notin \operatorname{dom} f$, where $\operatorname{dom} f := \{u \in X : f(u) < +\infty\}$ denotes the effective domain of $f$. When the set $\operatorname{dom} f$ is nonempty, one says that the function $f$ is proper.

Given any set-valued mapping $M : X \rightrightarrows Y$ (which can be $P_S, \partial f, \partial_F f, \partial^- f$), it will be convenient as usual to denote by $\operatorname{Dom} M$ its (effective) domain and by $\operatorname{Rge} M$ its range, that is,

$$\operatorname{Dom} M := \{x \in X : M(x) \neq \emptyset\} \quad \text{and} \quad \operatorname{Rge} M := \bigcup_{x \in X} M(x).$$

The graph of $M$ is the set

$$\text{gph}\, M := \{(x, y) \in X \times Y : y \in M(x)\}.$$

## 3 Properties of the Klee envelope

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. We recall that for any reals $\lambda > 0$ and $p \geq 1$, the *Klee envelope* of $f$ with index $\lambda$ and power $p$ was defined in (1) and the associated attainment set $Q_{\lambda,p} f(x)$ in (2). If $\kappa_{\lambda,p} f \equiv +\infty$ the study of $\kappa_{\lambda,p} f$ and $Q_{\lambda,p} f$ is trivial. It is also worth pointing out that if $\kappa_{\lambda,p} f(x_0)$ is finite for some $x_0$, then there is some real $\beta$ such that

$$\frac{1}{p2^{p-1}\lambda}\|y\|^p - \beta \leq f(y) \quad \text{for all } y \in X. \tag{6}$$

Indeed, putting $\mu := \kappa_{\lambda,p} f(x_0)$ (so, $\frac{1}{p\lambda}\|y - x_0\|^p - \mu \leq f(y)$) and noting that

$$\|y\|^p \leq \big(\|y - x_0\| + \|x_0\|\big)^p \leq 2^{p-1}\|y - x_0\|^p + 2^{p-1}\|x_0\|^p,$$

we see that for all $y \in X$

$$\frac{1}{p2^{p-1}\lambda}\|y\|^p - \frac{1}{p\lambda}\|x_0\|^p - \mu \leq \frac{1}{p\lambda}\|y - x_0\|^p - \mu \leq f(y).$$

The Klee envelope is a particular important case of supremal convolutions. Given functions $g_i : X \to \mathbb{R} \cup \{-\infty\}$ with $i = 1, \ldots, n$, the *(Moreau) supremal convolution* (or sup-convolution) of $g_1, \ldots, g_n$ is the function (see [32])

$$x \mapsto \varphi(x) := \sup\{g_1(x_1) + \cdots + g_n(x_n) : x_1 + \cdots + x_n = x\},$$

where the supremum is taken over all $n$- tuples ($n$-vectors) $(x_1, \ldots, x_n)$ in $X^n$ such that $x_1 + \cdots + x_n = x$; so the Klee envelope $\kappa_{\lambda,p} f$ is the supremal convolution of $-f$ with the kernel function $\frac{1}{p\lambda}\|\cdot\|^p$. If at least one of the functions $g_1, \ldots, g_n$, say $g_1$, is convex, then $\varphi$ is convex since the equality

$$\varphi(x) = \sup_{(u_2,\cdots,u_n)\in X^{n-1}} \big(g_1(x - u_2 - \cdots - u_n) + g_2(u_2) + \cdots + g_n(u_n)\big)$$

ensures that $\varphi$ is the pointwise supremum of a family of convex functions. In particular, the Klee envelope $\kappa_{\lambda,p} f$ is *always convex.*

For every real $\varepsilon \geq 0$, it will be convenient to put

$$Q^{\varepsilon}_{\lambda,p} f(x) := \left\{ y \in X : \kappa_{\lambda,p} f(x) - \varepsilon \leq \frac{1}{p\lambda}\|x - y\|^p - f(y) \right\},$$

so $Q^{\varepsilon}_{\lambda,p} f(x) \neq \emptyset$ whenever $\kappa_{\lambda,p} f(x)$ is finite and $\varepsilon > 0$.

If $f$ is finite at some point, say $\bar{y}$, then the inequality $\frac{1}{p\lambda}\|x-\bar{y}\|^p - f(\bar{y}) \leq \kappa_{\lambda,p} f(x)$ tells us that $\kappa_{\lambda,p} f$ is coercive in the sense that

$$\kappa_{\lambda,p} f(x) \to +\infty \quad \text{as } \|x\| \to +\infty. \tag{7}$$

In addition to that coercivity property, the next proposition establishes the local boundedness of the set-valued mapping $Q^{\varepsilon}_{\lambda,p} f$ and some Lipschitz property of the function $\kappa_{\lambda,p} f$. In view of the proof of the proposition, we must recall (and this is not difficult to verify) that the subdifferential of the convex function $(1/p)\|\cdot\|^p$ is described for any $x \in X$ by

$$\partial\left(\frac{1}{p}\|\cdot\|^p\right)(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x^*\|\,\|x\| \quad \text{and} \\ \|x^*\| = \|x\|^{p-1}\} \ \text{ if } \ p > 1, \tag{8}$$

and with $p = 1$

$$\partial\|\cdot\|(x) = \{x^* \in X^* : \|x^*\| \leq 1 \quad \text{and} \quad \langle x^*, x\rangle = \|x\|\}. \tag{9}$$

The latter means that $\partial\|\cdot\|(0) = \mathbb{B}_{X^*}$ and $\partial\|\cdot\|(x) = \{x^* \in \mathbb{S}_{X^*} : \langle x^*, x\rangle = \|x\|\}$ for all $x \neq 0$.

**Proposition 1** *Let $(X, \|\cdot\|)$ be a normed vector space and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function for which there exist two real numbers $\alpha$, $\beta$ with $\alpha > 0$ such that*

$$\alpha\|x\|^p - \beta \leq f(x) \ \ \textit{for all } x \in X. \tag{10}$$

*The following hold:*

(a) *With $p = 1$ and $\lambda \geq 1/\alpha$ the function $\kappa_{\lambda,1} f$ is finite-valued and globally Lipschitz on $X$ with $1/\lambda$ as a Lipschitz constant.*
(b) *If $p > 1$ and $\lambda > 1/(p\alpha)$, the function $\kappa_{\lambda,p} f$ is finite-valued on $X$ and Lipschitz continuous on each ball $r\mathbb{B}_X$ of $X$ with some Lipschitz contant $L \geq r^{p-1}/\lambda$ therein.*
(c) *With $p \geq 1$ and $\lambda > 1/(p\alpha)$, for each pair of reals $\varepsilon \geq 0$ and $r > 0$, the set-valued mapping $Q^{\varepsilon}_{\lambda,p}(\cdot)$ is bounded over the ball $r\mathbb{B}_X$.*

*Proof* Let $\alpha$, $\beta$ be as given in the statement.

First, assume that $p = 1$ and $\lambda \geq 1/\alpha$. Fixing $x \in X$, we have for all $y \in X$,

$$\frac{1}{\lambda}\|x-y\| - f(y) \leq \frac{1}{\lambda}\|x\| + \frac{1}{\lambda}\|y\| - \alpha\|y\| + \beta \leq \frac{1}{\lambda}\|x\| + \beta,$$

so $\kappa_{\lambda,1} f(x)$ is finite. Further, with $x, x' \in X$ taking the supremum over $y \in X$ in the inequality

$$\frac{1}{\lambda}\|x-y\|-f(y)\le\frac{1}{\lambda}\|x-x'\|+\left(\frac{1}{\lambda}\|x'-y\|-f(y)\right)$$

gives $\kappa_{\lambda,1}f(x)\le\kappa_{\lambda,1}f(x')+\lambda^{-1}\|x-x'\|$, hence $\kappa_{\lambda,1}f$ is Lipschitz on $X$ with $\lambda^{-1}$ as a Lipschitz constant.

Now assume that $p\ge 1$ and $\lambda>1/(p\alpha)$. Take any $x\in X$ and write, for every $y\in X$,

$$\frac{1}{p\lambda}\|x-y\|^p-f(y)\le\frac{1}{p\lambda}\|x-y\|^p-\alpha\|y\|^p+\beta. \tag{11}$$

On the one hand, observing that $\frac{1}{p\lambda}\|x-y\|^p-\alpha\|y\|^p+\beta\to-\infty$ as $\|y\|\to+\infty$ since $\frac{1}{p\lambda}-\alpha<0$, it results that $\frac{1}{p\lambda}\|x-\cdot\|^p-\alpha\|\cdot\|^p+\beta$ is bounded from above over $X$. From (11) the function $\frac{1}{p\lambda}\|x-\cdot\|^p-f(\cdot)$ is also bounded from above over $X$ and finite at some point according to the properness of $f$, hence $\kappa_{\lambda,p}f$ is finite-valued over $X$.

On the other hand, fixing any reals $\varepsilon\ge 0$ and $r>0$, we note by (11) that, for all $x\in X$ and $y\in Q^{\varepsilon}_{\lambda,p}f(x)$,

$$\kappa_{\lambda,p}f(x)-\varepsilon\le\frac{1}{p\lambda}\|x-y\|^p-\alpha\|y\|^p+\beta.$$

Choose some element $y_0\in X$ with $f(y_0)$ finite and consider any $x\in r\mathbb{B}_X$ and $y\in Q^{\varepsilon}_{\lambda,p}f(x)$, that is, $\kappa_{\lambda,p}f(x)-\varepsilon\le\frac{1}{p\lambda}\|x-y\|^p-f(y)$. We have

$$-f(y_0)\le\frac{1}{p\lambda}\|x-y_0\|^p-f(y_0)\le\kappa_{\lambda,p}f(x)\le\frac{1}{p\lambda}\|x-y\|^p-\alpha\|y\|^p+\beta+\varepsilon,$$

so choosing some $y^*\in\partial\left(\frac{1}{p}\|\cdot\|^p\right)(y-x)$ we obtain by (8)

$$\begin{aligned}-f(y_0)&\le\left(\frac{1}{p\lambda}-\alpha\right)\|y\|^p+\frac{1}{p\lambda}(\|y-x\|^p-\|y\|^p)+\beta+\varepsilon\\&\le\left(\frac{1}{p\lambda}-\alpha\right)\|y\|^p+\frac{1}{\lambda}\langle y^*,-x\rangle+\beta+\varepsilon\\&\le\left(\frac{1}{p\lambda}-\alpha\right)\|y\|^p+\frac{1}{\lambda}\|y-x\|^{p-1}\|x\|+\beta+\varepsilon,\end{aligned}$$

and this entails

$$-f(y_0)\le\left(\frac{1}{p\lambda}-\alpha\right)\|y\|^p+\frac{r}{\lambda}(\|y\|+r)^{p-1}+\beta+\varepsilon,$$

or equivalently

$$\left(\alpha-\frac{1}{p\lambda}\right)\|y\|^p-\frac{r}{\lambda}(\|y\|+r)^{p-1}\le\beta+f(y_0)+\varepsilon,$$

with $\alpha - \frac{1}{p\lambda} > 0$. It ensues that there exists some real $\gamma > 0$ depending only on $\lambda, r, \varepsilon$ such that $\|y\| \leq \gamma$ for all $y \in Q^{\varepsilon}_{\lambda,p}(x)$ (since otherwise for a sequence $(y_i)_i$ of elements of $Q^{\varepsilon}_{\lambda,p}(x)$ with $\|y_i\| \to +\infty$ as $i \to +\infty$, the first member of the latter inequality would tend to $\infty$, contradicting the inequality). This means that $Q^{\varepsilon}_{\lambda,p}(x) \subset \gamma\mathbb{B}_X$, and the set-valued mapping $Q^{\varepsilon}_{\lambda,p} f(\cdot)$ is bounded on $r\mathbb{B}_X$ as desired.

Finally, assume that $p > 1$ and $\lambda > 1/(p\alpha)$. Fix $\varepsilon > 0$ and $r > 0$, and let $\gamma > 0$ given as above. Take any $x, x' \in r\mathbb{B}_X$. Considering any $y \in \gamma\mathbb{B}_X$ and choosing $x^* \in \partial\big(\frac{1}{p}\|\cdot\|^p\big)(x-y)$ we see from (8) again that

$$\begin{aligned}\frac{1}{p\lambda}\|x-y\|^p - \frac{1}{p\lambda}\|x'-y\|^p &\leq \frac{1}{\lambda}\langle x^*, x-x'\rangle\\ &\leq \frac{1}{\lambda}\|x^*\|\,\|x'-x\|\\ &= \frac{1}{\lambda}\|x-y\|^{p-1}\|x'-x\|.\end{aligned}$$

Consequently,

$$\frac{1}{p\lambda}\|x-y\|^p - \frac{1}{p\lambda}\|x'-y\|^p \leq \frac{(r+\gamma)^{p-1}}{\lambda}\|x-x'\|,$$

then for every $y \in \gamma\mathbb{B}_X$,

$$\frac{1}{p\lambda}\|x-y\|^p - f(y) \leq \frac{(r+\gamma)^{p-1}}{\lambda}\|x-x'\| + \left(\frac{1}{p\lambda}\|x'-y\|^p - f(y)\right).$$

Taking the supremum over all $y \in \gamma\mathbb{B}_X$ and noting by what precedes that

$$\kappa_{\lambda,p} f(u) = \sup_{y\in\gamma\mathbb{B}_X}\left(\frac{1}{p\lambda}\|u-y\|^p - f(y)\right) \quad \text{for all } u \in r\mathbb{B}_X,$$

we obtain $\kappa_{\lambda,p} f(x) \leq \kappa_{\lambda,p} f(x') + \frac{(r+\gamma)^{p-1}}{\lambda}\|x-x'\|$. This finishes the proof. □

*Remark 1* (a) If $\kappa_{\lambda_0,p} f(x_0)$ is finite for some $\lambda_0 > 0$ and $x_0 \in X$, then by (6) there is some real $\beta$ such that

$$\frac{1}{p2^{p-1}\lambda_0}\|y\|^p - \beta \leq f(y) \quad \text{for all } y \in X,$$

so the inequality assumption (10) in the proposition is fulfilled.

In particular, the Klee envelope $\kappa_{\lambda,1} f$ is finite-valued and $(1/\lambda)$-Lipschitz on $X$ if and only it is finite at some point in $X$.

(b) If $\operatorname{dom} f$ is bounded and $f$ is bounded from below, then for any real $\alpha > 0$ there exists some $\beta \in \mathbb{R}$ such that (10) is satisfied. Indeed, considering a lower bound $\gamma$ of $f$ and putting $\mu := \sup_{u\in\operatorname{dom} f}\|u\|$, we see that

$$\alpha \|x\|^p + \gamma - \alpha\mu^p \le \gamma + \psi_{\mathrm{dom}\, f}(x) \le f(x) \quad \text{for all } x \in X.$$

(c) For a nonempty subset $S$ of the normed space $(X, \|\cdot\|)$, the existence of $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $\alpha\|x\|^p - \beta \le \psi_S(x)$ for all $x \in X$ amounts to requiring that the set $S$ is bounded, which in turn is equivalent the property that, for each real $\alpha > 0$, there exists some $\beta \in \mathbb{R}$ such that (10) holds true with $f = \psi_S$. Indeed, under the minorization assumption (10) for $f = \psi_S$ with some $\alpha > 0$ and $\beta \in \mathbb{R}$, for all $x \in S$ we have $\|x\|^p \le \beta\alpha^{-1}$, hence $S$ is bounded. On the other hand, by (b) the boundedness of $S$ is equivalent to the fact that, for each real $\alpha > 0$, there exists $\beta \in \mathbb{R}$ such that $\alpha\|x\|^p - \beta \le \psi_S(x)$ for all $x \in X$. □

The set-valued mapping $Q^{(\cdot)}_{\lambda,p} f(\cdot)$ satisfies a closedness property.

**Proposition 2** *Let $(X, \|\cdot\|)$ be a normed vector space and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function for which there exist two real numbers $\alpha, \beta$ with $\alpha > 0$ such that (10) is fulfilled. Assume either $p = 1$ and $\lambda \ge 1/\alpha$ or $p > 1$ and $\lambda > 1/(p\alpha)$. Then the graph*

$$\mathrm{gph}\, Q^{(\cdot)}_{\lambda,p} f(\cdot) := \{(\varepsilon, x, y) \in [0, +\infty[\times X \times X : y \in Q^{\varepsilon}_{\lambda,p} f(x)\}$$

*in $[0, +\infty[\times X \times X$ of the set-valued mapping $(\varepsilon, x) \mapsto Q^{\varepsilon}_{\lambda,p} f(x)$ defined on $[0, +\infty[\times X$ is closed in $[0, +\infty[\times X \times X$.*

*Proof* Let $(\varepsilon_i, x_i, y_i)_i$ be a sequence of elements of $[0, +\infty[\times X \times X$ converging to $(\varepsilon, x, y)$ with $y_i \in Q^{\varepsilon_i}_{\lambda,p} f(x_i)$. Then for every $i \in \mathbb{N}$,

$$\kappa_{\lambda,p} f(x_i) - \varepsilon_i \le \frac{1}{p\lambda}\|x_i - y_i\|^p - f(y_i),$$

so from the lower semicontinuity of $f$ and the continuity of $\kappa_{\lambda,p} f$ (see the previous proposition) we obtain

$$\kappa_{\lambda,p} f(x) - \varepsilon \le \frac{1}{p\lambda}\|x - y\|^p - f(y).$$

This means that $y \in Q^{\varepsilon}_{\lambda,p} f(x)$ as required. □

The following lemma prepares the next result related to the behavior of $Q_{\lambda,p} f$ when both the norm $\|\cdot\|$ and its dual norm are differentiable.

**Lemma 1** *Let $(X, \|\cdot\|)$ be a normed space, let $g_i : X \to \mathbb{R} \cup \{-\infty\}$, $i = 1, \ldots, n$, be extended real-valued functions and let $\varphi$ be their supremal convolution, that is,*

$$\varphi(x) = \sup\{g_1(x_1) + \cdots + g_n(x_n) : x_1 + \cdots + x_n = x\}.$$

*Assume that $\varphi(\bar{x})$ is finite and attained at $(\bar{x}_1, \ldots, \bar{x}_n)$, that is, $\varphi(\bar{x}) = g_1(\bar{x}_1) + \cdots + g_n(\bar{x}_n)$ with $\bar{x}_1 + \cdots + \bar{x}_n = \bar{x}$. Then*

$$\mathrm{co}\big(\partial_D g_1(\bar{x}_1) \cup \cdots \cup \partial_D g_n(\bar{x}_n)\big) \subset \partial_D \varphi(\bar{x}),$$

*and*

$$\overline{\text{co}}\big(\partial_F g_1(\bar{x}_1) \cup \cdots \cup \partial_F g_n(\bar{x}_n)\big) \subset \partial_F \varphi(\bar{x}),$$

*where* $\overline{\text{co}}$ *denotes the norm-closure in* $X^*$ *of the convex hull.*

*Proof* Fix any $x^* \in \partial_F g_1(\bar{x}_1)$ and consider any real $\varepsilon > 0$. There exists some real $\delta > 0$ such that $\langle x^*, u\rangle - \varepsilon\|u\| \leq g_1(\bar{x}_1 + u) - g_1(\bar{x}_1)$, for all $u \in \delta\mathbb{B}_X$. Since $\varphi(\bar{x}) = g_1(\bar{x}_1) + \cdots + g_n(\bar{x}_n)$, it follows that

$$\langle x^*, u\rangle - \varepsilon\|u\| \leq \varphi(\bar{x} + u) - \varphi(\bar{x}) \quad \forall u \in \delta\mathbb{B}_X,$$

thus $x^* \in \partial_F \varphi(\bar{x})$. It results that (repeating the reasoning with $i = 2, \ldots, n$)

$$\partial_F g_1(\bar{x}_1) \cup \cdots \cup \partial_F g_n(\bar{x}_n) \subset \partial_F \varphi(\bar{x}),$$

and the latter entails the required inclusion since the Fréchet subdifferential of a function at any point of $X$ is known to be convex and norm closed in $X^*$ (see, e.g., [30, Definition 1.83 and comments]).

The inclusion concerning the Dini subdifferential is obtained in a similar and easier way. □

*Remark 2* With the same arguments, the first inclusion also holds with several other subdifferentials, for example with proximal subdifferential (see, e.g., [37, Definition 9.1.1 (b)]) in place of the Dini subdifferential. □

Below Lemma 1 is applied to get relations between subdifferentials of the norm in power $p \geq 1$ and the Klee envelope at points where the Klee envelope is attained. The assertions (b)–(c) involve the Gâteaux derivative $D\varphi$ of a function, see, e.g., [11, p. 2] for the definition and properties. The case $p = 2$ was first obtained in [9, Proposition 8].

**Proposition 3** *Let* $(X, \|\cdot\|)$ *be a normed vector space and* $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *be a proper function, and let* $p \in [1, +\infty[$. *For any* $x \in X$ *such that* $\kappa_{\lambda,p} f(x)$ *is finite, the following hold:*

(a) *For every* $y \in Q_{\lambda,p} f(x)$, *one has*

$$\frac{1}{\lambda}\partial\left(\frac{1}{p}\|\cdot\|^p\right)(x - y) \subset \partial\kappa_{\lambda,p} f(x).$$

(b) *If* $\kappa_{\lambda,p} f$ *is Gâteaux differentiable at* $x$ *and* $Q_{\lambda,p} f(x) \neq \emptyset$, *then for all* $y \in Q_{\lambda,p} f(x)$

$$\langle D\big(\kappa_{\lambda,p} f\big)(x), x - y\rangle = \frac{1}{\lambda}\|x - y\|^p.$$

(c) *If $\kappa_{\lambda,1} f$ is Gâteaux differentiable at $x$ and $Q_{\lambda,1} f(x) \neq \emptyset$, then*

$$\|D(\kappa_{\lambda,1} f)(x)\| = 1/\lambda.$$

(d) *If $p > 1$ and both the norm $\|\cdot\|$ and its dual norm are Gâteaux differentiable off zero, then at any $x \in X$ where $\kappa_{\lambda,p} f$ is Gâteaux differentiable, the set $Q_{\lambda,p} f(x)$ is at most a singleton.*

*Proof* The inclusion in (a) follows directly from the above lemma and (b) is a consequence of (a) and the descriptions (8) and (9).

Concerning (c) with $p = 1$, the assumption on $Q_{\lambda,1} f(x)$ allows us to choose some $y \in Q_{\lambda,1} f(x)$. It follows from (9) and the Gâteaux differentiability of $\kappa_{\lambda,1} f$ at $x$ that $y \neq x$. Using this in the equality in (b) and noting that $\|D(\kappa_{\lambda,1} f)(x)\| \leq 1/\lambda$ because of the $(1/\lambda)$-Lipschitz property of $\kappa_{\lambda,1} f$, see Proposition 1 and Remark 1, we see that $\|D(\kappa_{\lambda,1} f)(x)\| = 1/\lambda$ as required.

Now to justify (d), suppose that both the norm $\|\cdot\|$ and its dual norm are Gâteaux differentiable off zero. Then $\partial\left(\frac{1}{p}\|\cdot\|^p\right)(x)$ is a singleton for every $x \in X$, say $\{J_p(x)\} = \partial\left(\frac{1}{p}\|\cdot\|^p\right)(x)$, and the mapping $J_p : X \to X^*$ is one-to-one. Consequently at any $x \in X$ where $\kappa_{\lambda,p} f$ is Gâteaux differentiable, we have $\lambda^{-1} J_p(x - y) = D(\kappa_{\lambda,p} f)(x)$ whenever $y \in Q_{\lambda,p} f(x)$. Since $J_p$ is one-to-one, $Q_{\lambda,p} f(x)$ is at most a singleton. □

The result of [20, Theorem 2.9] says that the differentiability of the farthest distance function at a point $x$ entails the same differentiability of the norm at an appropriate point. The next proposition extends the result to the function $\kappa_{\lambda,p} f$. We refer also to [29] for some results on differentiability of the farthest distance function.

**Proposition 4** *Let $(X, \|\cdot\|)$ be a normed space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function with $\kappa_{\lambda,p} f$ finite at some point. Assume that $x \in X$ is a point where $\kappa_{\lambda,p} f$ is Gâteaux (resp. Fréchet) differentiable. Then for any $y \in Q_{\lambda,p} f(x)$, the function $\|\cdot\|^p$ is Gâteaux (resp. Fréchet) differentiable at $x - y$ with $p\lambda D(\kappa_{\lambda,p} f)(x)$ as derivative at $x - y$.*

*Proof* Suppose that $\kappa_{\lambda,p} f$ is Gâteaux (resp. Fréchet) differentiable at $x$ and denote $\zeta^* := D(\kappa_{\lambda,p} f)(x)$. Suppose also that $Q_{\lambda,p} f(x)$ is nonempty. Put $q := (1/p\lambda)\|\cdot\|^p$. Fix any $y \in Q_{\lambda,p} f(x)$ and take $x^* \in \partial q(x - y)$. For any real $t > 0$ and any $h \in X$, we have

$$\begin{aligned} q(x - y + th) - q(x - y) &= (q(x - y + th) - f(y)) - (q(x - y) - f(y)) \\ &\leq \kappa_{\lambda,p} f(x + th) - \kappa_{\lambda,p} f(x), \end{aligned}$$

hence we see through the inclusion $x^* \in \partial q(x - y)$ that

$$\langle x^*, h\rangle \leq t^{-1}(q(x - y + th) - q(x - y)) \leq t^{-1}(\kappa_{\lambda,p} f(x + th) - \kappa_{\lambda,p} f(x)). \quad (12)$$

Since the last member tends to $\langle \zeta^*, h\rangle$ as $t \downarrow 0$, it ensues that $\langle x^*, h\rangle \leq \langle \zeta^*, h\rangle$, for all $h \in X$, thus $x^* = \zeta^*$. Replacing $x^*$ by $\zeta^*$ in the first member of (12) yields

$$\begin{aligned}0 &\leq t^{-1}(q(x-y+th)-q(x-y)-t\langle\zeta^*,h\rangle)\\&\leq t^{-1}(\kappa_{\lambda,p}f(x+th)-\kappa_{\lambda,p}f(x)-t\langle\zeta^*,h\rangle).\end{aligned}$$

Since the last member tends to zero (resp. tends to zero uniformly with respect to $h \in \mathbb{B}_X$), we conclude that $q$ is Gâteaux (resp. Fréchet) differentiable at $x-y$ with $D\big(\kappa_{\lambda,p}f\big)(x)$ as derivative there. □

Our next aim is to show that the set Dom $Q_{\lambda,1}$ is quite large. We then start with the analysis of the nonemptiness of $Q_{\lambda,1}f(x)$, by noting that such a nonemptiness implies the equalities

$$\kappa_{\lambda,1}f(x) = \sup_{z\in X, y^*\in\mathbb{S}_{X^*}} (\lambda^{-1}\langle y^*, x-z\rangle - f(z)) = \lambda^{-1}\langle x^*, x-y\rangle - f(y)$$

for all $y \in Q_{\lambda,1}f(x)$ and $x^* \in \partial\|\cdot\|(x-y)$. Moreover if (10) is fulfilled with $\alpha > \lambda^{-1}$ and the space $X$ is finite dimensional, then the emptiness of $Q_{\lambda,1}f(x)$ implies the inequality

$$\kappa_{\lambda,1}f(x) > \sup_{z\in X, y^*\in\mathbb{S}_{X^*}} (\lambda^{-1}\langle y^*, x-z\rangle - f(z)),$$

whenever $f$ is proper and lower semicontinuous. Indeed, if the equality holds true, then

$$\kappa_{\lambda,1}f(x) = \lim_{i\to+\infty} (\lambda^{-1}\|x-z_i\| - f(z_i))$$

for some sequence $(z_i)_i$ in $X$, then it follows from Proposition 1(c) that the sequence $(z_i)_i$ is bounded, thus we may assume that it converges to a certain $\bar{y}$. It results that

$$\kappa_{\lambda,1}f(x) = \lim_{i\to+\infty} (\lambda^{-1}\|x-z_i\| - f(z_i)) = \lambda^{-1}\|x-\bar{y}\| - f(\bar{y}) \leq \kappa_{\lambda,1}f(x),$$

so $\kappa_{\lambda,1}f(x) = \lambda^{-1}\|x-\bar{y}\| - f(\bar{y})$, which means that $\bar{y} \in Q_{\lambda,1}f(x)$, a contradiction. This indicates the role of the set

$$\left\{x \in X : \exists x^* \in \partial\kappa_{\lambda,1}f(x),\ \sup_{y\in X}(\lambda^{-1}\langle x^*, x-y\rangle - f(y)) < \kappa_{\lambda,1}f(x)\right\}$$

in investigating the set of points for which the attainment set is nonempty. Below it is shown that this set is not too large, namely it is of first category. Let us also recall that the $G_\delta$ density property of Dom $Q_{\lambda,2}f$ was studied in [9, Theorem 5] with $p=2$, in fact the strong attainment on a dense $G_\delta$ subset was proved. We consider in the foregoing theorem the case $p=1$. The proof of the theorem as well as that of the following lemma use the main ideas of [27, Lemma 2.2, Theorem 2.3].

**Lemma 2** *Let* $(X, \|\cdot\|)$ *be a normed space and* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper function for which there are reals* $\alpha > 0$ *and* $\beta \in \mathbb{R}$ *such that*

$$\alpha\|x\| - \beta \leq f(x) \quad \text{for all } x \in X.$$

*Then for any real* $\lambda > 1/\alpha$ *the set*

$$\left\{x \in X : \exists x^* \in \partial\kappa_{\lambda,1} f(x),\ \sup_{y\in X}\big(\langle x^*, x - y\rangle - f(y)\big) < \kappa_{\lambda,1} f(x)\right\}$$

*is a countable union of closed sets with empty interior, so it is a set of first category in the space* $X$.

*Proof* For each integer $i \in \mathbb{N}$ denote

$$A_i := \left\{x \in X : \exists x^* \in \partial\kappa_{\lambda,1} f(x),\ \sup_{y\in X}(\langle x^*, x - y\rangle - f(y)) \leq \kappa_{\lambda,1} f(x) - \frac{1}{i}\right\},$$

so clearly the set of the lemma coincides with $\bigcup_{i\in\mathbb{N}} A_i$.

Let us first fix $i \in \mathbb{N}$ and show that $A_i$ is closed. Consider any sequence $(x_n)_n$ of elements of $A_i$ converging to some $x \in X$, and for each $n \in \mathbb{N}$ choose by definition of $A_i$ some $x_n^* \in \partial\kappa_{\lambda,1} f(x_n)$ satisfying

$$\sup_{y\in X}(\langle x_n^*, x_n - y\rangle - f(y)) \leq \kappa_{\lambda,1} f(x_n) - \frac{1}{i}.$$

The $(1/\lambda)$-Lipschitz property of $\kappa_{\lambda,1} f$ ensures that $\|x_n^*\| \leq 1/\lambda$, so extracting a subnet we may suppose that $(x_n^*)_n$ converges weakly* to some $x^*$ in $X^*$. The norm×weak* closedness of gph $\partial\kappa_{\lambda,1} f$ guarantees that $x^* \in \partial\kappa_{\lambda,1} f(x)$. Further, for every $y \in X$, since

$$\langle x_n^*, x_n - y\rangle - f(y) \leq \kappa_{\lambda,1} f(x_n) - \frac{1}{i}$$

and $\kappa_{\lambda,1} f$ is continuous, we also have $\langle x^*, x - y\rangle - f(y) \leq \kappa_{\lambda,1} f(x) - 1/i$. It ensues that $x \in A_i$, justifying the closedness of $A_i$.

It remains to prove that all $A_i$ have empty interior. Suppose, for some $i \in \mathbb{N}$, that int $A_i \neq \emptyset$ and take some $\bar{x} \in X$ and $r > 0$ such that $B[\bar{x}, r] \subset A_i$. We know by Proposition 1 that the set $Q^1_{\lambda,1} f(\bar{x})$ is nonempty and bounded, hence we can define the real $\gamma := \sup\{\|\bar{x} - y\| : y \in Q^1_{\lambda,1} f(\bar{x})\}$. For $\varepsilon := \big(2i(\gamma + r)\big)^{-1} r$, there exists by definition of $\kappa_{\lambda,1} f$ some $\bar{y} \in Q^\varepsilon_{\lambda,1} f(\bar{x}) \subset Q^1_{\lambda,1} f(\bar{x}) \subset \operatorname{dom} f$ satisfying

$$\kappa_{\lambda,1} f(\bar{x}) - \varepsilon < \frac{1}{\lambda}\|\bar{x} - \bar{y}\| - f(\bar{y}) \leq \kappa_{\lambda,1} f(\bar{x}). \tag{13}$$

Define $t := r/\gamma$ and $u := \bar{x} + t(\bar{x} - \bar{y}) \in B[\bar{x}, r]$, so $u \in A_i$. From the definition of $A_i$ there is some $u^* \in \partial\kappa_{\lambda,1} f(u)$ such that

$$\sup_{y\in X}(\langle u^*, u-y\rangle - f(y)) \le \kappa_{\lambda,1} f(u) - \frac{1}{i}. \tag{14}$$

On the other hand, by (13) and the equality $u-\bar{y}=(1+t)(\bar{x}-\bar{y})$ we also have

$$\begin{aligned}\kappa_{\lambda,1}f(\bar{x})-\kappa_{\lambda,1}f(u) &< \frac{1}{\lambda}\|\bar{x}-\bar{y}\| - f(\bar{y}) + \varepsilon - \kappa_{\lambda,1}f(u)\\ &= \frac{1}{(1+t)\lambda}\|u-\bar{y}\| - f(\bar{y}) + \varepsilon - \kappa_{\lambda,1}f(u)\\ &\le \frac{1}{1+t}\left(\frac{1}{\lambda}\|u-\bar{y}\| - f(\bar{y})\right) - \frac{t}{1+t}f(\bar{y}) + \varepsilon - \kappa_{\lambda,1}f(u),\end{aligned}$$

which ensures by the definition of $\kappa_{\lambda,1}f$ and by (14)

$$\begin{aligned}\kappa_{\lambda,1}f(\bar{x})-\kappa_{\lambda,1}f(u) &< \frac{1}{1+t}\kappa_{\lambda,1}f(u) - \frac{t}{1+t}f(\bar{y}) + \varepsilon - \kappa_{\lambda,1}f(u)\\ &= \frac{-t}{1+t}\kappa_{\lambda,1}f(u) - \frac{t}{1+t}f(\bar{y}) + \varepsilon\\ &\le \frac{t}{1+t}\Big(\langle u^*, \bar{y}-u\rangle + f(\bar{y})\Big) - \frac{t}{1+t}f(\bar{y}) + \varepsilon - \frac{t}{i(1+t)}\\ &= \frac{t}{1+t}\langle u^*, \bar{y}-u\rangle + \varepsilon - \frac{r}{i(\gamma+r)},\end{aligned}$$

hence taking the equality $\bar{y}-u = \frac{1+t}{t}(\bar{x}-u)$ into account we obtain

$$\kappa_{\lambda,1}f(\bar{x})-\kappa_{\lambda,1}f(u) < \langle u^*, \bar{x}-u\rangle + \varepsilon - \frac{r}{i(\gamma+r)}.$$

Since $\varepsilon < (i(\gamma+r))^{-1}r$, we deduce that

$$\kappa_{\lambda,1}f(\bar{x})-\kappa_{\lambda,1}f(u) < \langle u^*, \bar{x}-u\rangle,$$

which contradicts the inclusion $u^* \in \partial\kappa_{\lambda,1}f(u)$, completing the proof. □

**Theorem 1** *Let $(X, \|\cdot\|)$ be a Banach space and $f : X \to \mathbb{R}\cup\{+\infty\}$ be a proper function for which there are reals $\alpha > 0$ and $\beta \in \mathbb{R}$ such that*

$$\alpha\|x\| - \beta \le f(x) \quad \text{for all } x \in X,$$

*and let $\lambda > 1/\alpha$. Assume that, for each $x^* \in \lambda^{-1}\mathbb{B}_{X^*}$, the infimum of the function $f + \langle x^*, \cdot\rangle$ is attained. Then the set* Dom $Q_{\lambda,1}f$ *contains a dense $G_\delta$ subset of $X$.*

*Proof* Denote by $M$ the set of first category in the statement of Lemma 2 above. Fix any $x \in X\backslash M$ and by Lemma 2 choose some $x^* \in \partial\kappa_{\lambda,1}f(x)$ such that

$$\sup_{y\in X}(\langle x^*, x-y\rangle - f(y)) \ge \kappa_{\lambda,1}f(x).$$

Since $\partial\kappa_{\lambda,1}f(x) \subset \lambda^{-1}\mathbb{B}_{X^*}$ [according to the $(1/\lambda)$-Lipschitz property of $\kappa_{\lambda,1}f$], we can take thanks to the attainement assumption some $\bar{y} \in X$ satisfying

$$\langle x^*, \bar{y}\rangle + f(\bar{y}) = \inf_{y\in X}(\langle x^*, y\rangle + f(y)),$$

or equivalently

$$\langle x^*, x - \bar{y}\rangle - f(\bar{y}) = \sup_{y\in Y}(\langle x^*, x - y\rangle - f(y)).$$

It results that

$$\kappa_{\lambda,1}f(x) \le \langle x^*, x - \bar{y}\rangle - f(\bar{y}) \le \frac{1}{\lambda}\|x - \bar{y}\| - f(\bar{y}) \le \kappa_{\lambda,1}f(x),$$

and this justifies the inclusion $\bar{y} \in Q_{\lambda,1}f(x)$, concluding the proof of the theorem. □

Let us notice that, assuming that the effective domain of $f$ is a compact set and $f$ is lower semicontinuous, we guarantee the assumption that *the infimum of the function* $f + \langle x^*, \cdot\rangle$ *is attained for every* $x^* \in \lambda^{-1}\mathbb{B}_{X^*}$. It is also worth observing that if $f$ is constant on its effective domain dom $f$ which is additionally assumed to be weakly closed and bounded, then (by James theorem, see [24]) the attainment assumption in Theorem 1 means that dom $f$ is weakly compact. In particular, if the effective domain of $f$ is $\mathbb{B}_X$, then $X$ has to be a reflexive Banach space, whenever the attainement assumption in Theorem 1 is satisfied.

The next corollary is a direct consequence of Theorem 1. Before stating the corollary, let us recall that, given a topology $\theta$ on $X$, a function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is $\theta$-inf-compact provided that all lower sections $\{x : \varphi(x) \le r\}$ are $\theta$-compact, for all $r \in \mathbb{R}$.

**Corollary 1** *Let* $(X, \|\cdot\|)$ *be a Banach space and* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper function for which there are reals* $\alpha > 0$ *and* $\beta \in \mathbb{R}$ *such that*

$$\alpha\|x\| - \beta \le f(x) \quad \textit{for all } x \in X,$$

*and let* $\lambda > 1/\alpha$. *Assume, for some topology* $\theta$ *on* $X$, *that the function* $f + \langle x^*, \cdot\rangle$ *is* $\theta$*-inf-compact for all* $x^* \in \lambda^{-1}\mathbb{B}_{X^*}$. *Then the set* Dom $Q_{\lambda,1}f$ *contains a dense* $G_\delta$ *subset of* $X$.

The second corollary assumes the weak inf-compactness of $f$.

**Corollary 2** *Let* $(X, \|\cdot\|)$ *be a Banach space and* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper function for which there are reals* $\alpha > 0$ *and* $\beta \in \mathbb{R}$ *such that*

$$\alpha\|x\| - \beta \le f(x) \quad \textit{for all } x \in X,$$

*and let* $\lambda > 1/\alpha$. *Assume that* $f$ *is weakly inf-compact. Then the set* Dom $Q_{\lambda,1}f$ *contains a dense* $G_\delta$ *subset of* $X$.

*Proof* Fix any $x^* \in \lambda^{-1}\mathbb{B}_{X^*}$ and note that

$$f(x) + \langle x^*, x\rangle \geq (\alpha - \lambda^{-1})\|x\| - \beta.$$

This entails that $\mu := \inf_X(f + \langle x^*, \cdot\rangle)$ is finite and coincides with $\inf_{\rho\mathbb{B}_X}(f + \langle x^*, \cdot\rangle)$ for some real $\rho > 0$. Putting $r := 1+\mu$ and $C := \{x \in X : f(x)+\langle x^*, x\rangle \leq r\}\cap\rho\mathbb{B}_X$, we see that $\mu = \inf_C(f + \langle x^*, \cdot\rangle)$. Further, for any $x \in C$ we have

$$f(x) \leq r - \langle x^*, x\rangle \leq r + \rho\|x^*\|,$$

hence $C \subset \{x \in X : f(x) \leq r + \rho\|x^*\|\}$. The latter set being weakly compact (by the weak inf-compactness of $f$), the weakly closed subset $C$ is weakly compact too. Consequently, the infimum $\mu$ is attained on the set $C$ according to the weak lower semicontinuity of $f + \langle x^*, \cdot\rangle$. The corollary then follows from Theorem 1. □

Taking for $f$ the indicator function of a nonempty weakly compact subset of $X$ in the latter corollary yields the following result:

**Corollary 3** (Theorem 2.3 in [27]) *Let* $(X, \|\cdot\|)$ *be a Banach space and* $S$ *be a nonempty weakly compact subset of* $X$*. Then* Dom $Q_S$ *contains a dense* $G_\delta$ *subset of* $X$*.*

The next corollary makes use of the Mackey topology on the topological dual space $X^*$; we refer to [36, IV Duality, 3. Locally Convex Topologies Consistent with a Given Duality. The Mackey-Arens Theorem] for the definition and properties of that topology.

**Corollary 4** *Let* $(X, \|\cdot\|)$ *be a Banach space and* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous convex function for which there are reals* $\alpha > 0$ *and* $\beta \in \mathbb{R}$ *such that*

$$\alpha\|x\| - \beta \leq f(x) \quad \text{for all } x \in X,$$

*and let* $\lambda > 1/\alpha$*. Assume that at every* $x^* \in \lambda^{-1}\mathbb{B}_{X^*}$ *the Legendre-Fenchel conjugate* $f^*$ *is finite and continuous with respect to the Mackey topology* $\tau(X^*, X)$*. Then the set* Dom $Q_{\lambda,1} f$ *contains a dense* $G_\delta$ *set of* $X$*.*

*Proof* The function $f$ being proper, lower semicontinuous and convex, the continuity of $f^*$ at an element $x^* \in \operatorname{dom} f^*$ with respect to the Mackey topology $\tau(X^*, X)$ is known to entail that $f - \langle x^*, \cdot\rangle$ is weakly inf-compact (see [33, Corollary 8.2]). The assertion of the corollary then follows from Corollary 1. □

*Remark 3* For the proper lower semicontinuous convex function $f : X \to \mathbb{R}\cup\{+\infty\}$, we know that there always exist reals $\alpha, \beta$ satisfying the inequality $\alpha\|x\| - \beta \leq f(x)$ for every $x \in X$. The assumption of the latter corollary requires the positiveness of such a real $\alpha$. □

Since $\kappa_{\lambda,1} f$ is Lipschitz with $1/\lambda$ as Lipschitz constant, we know that $\mathrm{Rge}(\partial\kappa_{\lambda,1} f) \subset \lambda^{-1}\mathbb{B}_{X^*}$. Adapting the proofs of Lemma 4.1 and Theorem 4.2 in [39], we can show more.

**Lemma 3** *Let $(X, \|\cdot\|)$ be a normed space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function with $\kappa_{\lambda,1} f$ finite at some point. Let also $(x_i)_i$ be a sequence of elements of $X$ with $\lim_{i\to+\infty} \|x_i\| = +\infty$. Then for any $x \in \mathrm{dom}\, f$ and any sequence $(x_i^*)_i$ with $x_i^* \in \partial\kappa_{\lambda,1} f(x_i)$, one has*

$$\lim_{i\to+\infty} \left\langle x_i^*, \frac{x_i - x}{\|x_i - x\|} \right\rangle = \lim_{i\to+\infty} \|x_i^*\| = \frac{1}{\lambda}.$$

*Proof* Since the function $\kappa_{\lambda,1} f$ is assumed to be finite at some point, we know that it is finite on $X$ and Lipschitz with $1/\lambda$ as a Lipschitz constant. Fix any $x \in \mathrm{dom}\, f$. Observe that on the one hand by definition of $\kappa_{\lambda,1} f(x_i)$

$$\frac{1}{\lambda}\|x_i - x\| - f(x) - \kappa_{\lambda,1} f(x) \le \kappa_{\lambda,1} f(x_i) - \kappa_{\lambda,1} f(x),$$

and on the other hand by the inclusion $x_i^* \in \partial\kappa_{\lambda,1} f(x_i)$

$$\langle x_i^*, x - x_i\rangle \le \kappa_{\lambda,1} f(x) - \kappa_{\lambda,1} f(x_i).$$

It follows that, for large $i \in \mathbb{N}$,

$$\frac{1}{\lambda} - \frac{f(x) + \kappa_{\lambda,1} f(x)}{\|x_i - x\|} \le \frac{\kappa_{\lambda,1} f(x_i) - \kappa_{\lambda,1} f(x)}{\|x_i - x\|} \le \left\langle x_i^*, \frac{x_i - x}{\|x_i - x\|} \right\rangle \le \|x_i^*\| \le \frac{1}{\lambda},$$

hence passing to the limit as $i \to +\infty$ (keeping in mind that $\|x_i\| \to +\infty$ as $i \to +\infty$) gives

$$\lim_{i\to+\infty} \left\langle x_i^*, \frac{x_i - x}{\|x_i - x\|} \right\rangle = \lim_{i\to+\infty} \|x_i^*\| = \frac{1}{\lambda}.$$

□

**Proposition 5** *Let $(X, \|\cdot\|)$ be a reflexive Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function with $\kappa_{\lambda,p} f$ finite-valued.*

(a) *If $p > 1$, then* $\mathrm{Rge}(\partial\kappa_{\lambda,p} f) = X^*$.
(b) *If the norm $\|\cdot\|$ is smooth and strictly convex (or equivalently both norms $\|\cdot\|$ and its dual norm are smooth, that is, Gâteaux differentiable off zero), then* $\mathrm{Rge}(\partial\kappa_{\lambda,1} f)$ *is convex and*

$$B(0, 1/\lambda) \subset \mathrm{Rge}(\partial\kappa_{\lambda,1}) \subset B[0, 1/\lambda] = \mathrm{cl}(\mathrm{Rge}(\partial\kappa_{\lambda,1})).$$

*Proof* (a) The assertion (a) follows from a classical result. We sketch the arguments. Fixing any $x^* \in X^*$, since $\kappa_{\lambda,p} f(x) - \langle x^*, x\rangle \to +\infty$ as $\|x\| \to +\infty$ [see (6)], the weakly lower semicontinuous function $\kappa_{\lambda,p} f(\cdot) - \langle x^*, \cdot\rangle$ has a global minimizer $\bar{x} \in X$, hence the Moreau-Rockafellar sum rule yields $x^* \in \partial\kappa_{\lambda,p} f(\bar{x})$.

(b) The Gâteaux differentiability assumption on $\|\cdot\|^2$ entails that (the duality mapping) $J := \partial(\frac{1}{2}\|\cdot\|^2)$ is single-valued and norm-weak* continuous. Let $\bar{x} \in X$ be a point where $f$ is finite. Fix any $x^* \in X^*$ with $\|x^*\| = 1/\lambda$. Choose some $x \in \mathbb{S}_X$ with $\langle x^*, x\rangle = 1/\lambda$. By (8) we have $x^* = \lambda^{-1} J(x)$. For each integer $i \in \mathbb{N}$, choosing $x_i^* \in \partial\kappa_{\lambda,1} f(ix + \bar{x})$ the above lemma ensures that

$$\lim_i \langle x_i^*, x\rangle = \lim_i \|x_i^*\| = 1/\lambda.$$

Consider a subsequence (that we do not relabel) of $(x_i^*)_i$ converging weakly* to some $z^*$, it is clear that $\|z^*\| \le 1/\lambda$ and $\langle z^*, x\rangle = 1/\lambda$, so $\|z^*\| = 1/\lambda$. It results that $z^* = \lambda^{-1} J(x)$, hence $z^* = x^*$. On the other hand, denoting by $\varphi$ the Legendre-Fenchel conjugate of $\kappa_{\lambda,1} f$, by (5) one has $\mathrm{cl}(\mathrm{Rge}(\partial\kappa_{\lambda,1} f)) = \mathrm{cl}(\mathrm{dom}\,\varphi)$, thus $\mathrm{cl}(\mathrm{Rge}(\partial\kappa_{\lambda,1} f))$ is a weakly closed convex set in $X^*$. Since $x_i^* \in \mathrm{Rge}(\partial\kappa_{\lambda,1} f)$, it results that $x^* \in \mathrm{cl}(\mathrm{Rge}(\partial\kappa_{\lambda,1} f))$. Consequently, $\lambda^{-1}\mathbb{S}_{X^*}$ is included in the closed convex set $\mathrm{cl}(\mathrm{Rge}(\partial\kappa_{\lambda,1} f))$, which entails $B[0, 1/\lambda] \subset \mathrm{cl}(\mathrm{Rge}(\partial\kappa_{\lambda,1} f))$, and this inclusion is an equality since the converse follows from the Lipschitz property of $\kappa_{\lambda,1} f$ with constant $1/\lambda$.

Now let us show that $B(0, 1/\lambda) \subset \mathrm{Rge}(\partial\kappa_{\lambda,1} f)$. Fix any $x^* \in B(0, 1/\lambda)$. Since $B[0, 1/\lambda] = \mathrm{cl}(\mathrm{Rge}(\partial\kappa_{\lambda,1} f))$ (as seen above), we can take a sequence $(x_i, x_i^*)_i$ of elements of $\mathrm{gph}\,\partial\kappa_{\lambda,1} f$ such that $\lim\limits_{i\to+\infty} \|x_i^* - x^*\| = 0$. The above lemma ensures that the sequence $(x_i)_i$ is bounded since $\|x^*\| < \frac{1}{\lambda}$ in this case, hence a subsequence converges weakly to some $x$. By the weak-norm closedness of the graph of the subdifferential of the continuous convex function $\kappa_{\lambda,1} f$, we obtain $x^* \in \partial\kappa_{\lambda,1} f(x)$, which justifies the desired inclusion $B(0, 1/\lambda) \subset \mathrm{Rge}(\partial\kappa_{\lambda,1} f)$.

Finally, the strict convexity of $\|\cdot\|$ and the inclusions

$$B(0, 1/\lambda) \subset \mathrm{Rge}(\partial\kappa_{\lambda,1} f) \subset B[0, 1/\lambda]$$

guarantee the convexity of $\mathrm{Rge}(\partial\kappa_{\lambda,1} f)$. □

The statement (a) is in fact a simple consequence of [41, Proposition 3.5], where an equivalent condition to the reflexivity of the space was given.

In view of Proposition 3(c) the Gâteaux differentiablity of $\kappa_{\lambda,1} f$ at $x$ implies that $D(\kappa_{\lambda,1} f)(x) \in \lambda^{-1}\mathbb{S}_{X^*}$ whenever $Q_{\lambda,1} f(x) \neq \emptyset$. Thus it is natural to investigate the set

$$\mathcal{C}_\lambda f := \{x \in X : \partial\kappa_{\lambda,1} f(x) \subset \lambda^{-1}\mathbb{S}_{X^*}\}.$$

It follows from Proposition 5(b) that there exists at least one point from the domain of $\kappa_{\lambda,1} f$ which is not in $\mathcal{C}_\lambda f$. As a consequence, the Klee envelope $\kappa_{\lambda,1} f$ is not a smooth function. In other words, there must exist a point in its domain where it is not

Gâteaux differentiable, whenever $X$ is a reflexive Banach space with both norm $\|\cdot\|$ and its dual norm being smooth.

Additional properties of the Klee envelope can be obtained in the case when $(X, \|\cdot\|)$ is a Hilbert space and $p = 2$. Indeed, writing in that case

$$\frac{1}{2\lambda}\|x-y\|^2 - f(y) = \frac{1}{2\lambda}\|x\|^2 + \left(\langle -\lambda^{-1}x, y\rangle - f(y) + \frac{1}{2\lambda}\|y\|^2\right),$$

and taking the supremum over $y \in X$ gives

$$\kappa_{\lambda,2} f(x) = \frac{1}{2\lambda}\|x\|^2 + \left(f - \frac{1}{2\lambda}\|\cdot\|^2\right)^* (-\lambda^{-1}x).$$

Further, given $(-\lambda x, y) \in \operatorname{gph}(Q_{\lambda,2} f)$, we have

$$\begin{aligned}\kappa_{\lambda,2} f(-\lambda x) - \frac{1}{2\lambda}\|-\lambda x\|^2 &= \frac{1}{2\lambda}\|-\lambda x - y\|^2 - f(y) - \frac{1}{2\lambda}\|-\lambda x\|^2\\ &= \frac{1}{2\lambda}\|y\|^2 + \langle y, x\rangle - f(y).\end{aligned}$$

Putting $\varphi(u) := (f - \frac{1}{2\lambda}\|\cdot\|^2)^*(u)$ for all $u \in X$, we deduce on the one hand that $\kappa_{\lambda,2} f(-\lambda x)$ as well as $\varphi(x)$ are finite, and on the other hand that

$$\varphi(x) + \langle y, u - x\rangle = \langle u, y\rangle + \frac{1}{2\lambda}\|y\|^2 - f(y) \le \varphi(u), \quad \text{for all } u \in X.$$

This says that $y \in \partial\varphi(x) = \partial(f - \frac{1}{2\lambda}\|\cdot\|^2)^*(x)$.

We have then proved the following:

**Proposition 6** *Assume that $(X, \|\cdot\|)$ is a Hilbert space and $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper function. Then for all $x \in X$*

$$\kappa_{\lambda,2} f(x) = \frac{1}{2\lambda}\|x\|^2 + \left(f - \frac{1}{2\lambda}\|\cdot\|^2\right)^* (-\lambda^{-1}x)$$

*and*

$$Q_{\lambda,2} f(-\lambda x) \subset \partial\left(f - \frac{1}{2\lambda}\|\cdot\|^2\right)^* (x).$$

**Corollary 5** *Assume that $(X, \|\cdot\|)$ is a Hilbert space and $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper function with $\kappa_{\lambda,2} f$ finite at some point in $X$. Then the inverse subdifferential $(\partial(\kappa_{\lambda,2} f))^{-1}$ is single-valued and $\lambda$-Lipschitz on $X$.*

*Proof* By the first equality in Proposition 6 the lower semicontinuous convex function $g$, defined by

$$g(x) := \left(f - \frac{1}{2\lambda}\|\cdot\|^2\right)^*(-\lambda^{-1}x) \quad \text{for all } x \in X,$$

is proper, hence the resolvent mapping $J := (Id_X + \lambda\partial g)^{-1}$ is single-valued and 1-Lipschitz on $X$ (see, e.g., [31, Propositions 5.b and 6.a] or [5, Corollary 23.10]). This means by the first equality in Proposition 6 again that $(\partial(\lambda\kappa_{\lambda,2} f))^{-1}$ is single-valued and 1-Lipschitz on $X$ or equivalently $(\partial(\kappa_{\lambda,2} f))^{-1}$ is single-valued and $\lambda$-Lipschitz on $X$, as easily seen. □

Finishing this Section we would like to emphasize, that there are other papers concerning subdifferential or differential properties of the Klee envelope. For example Proposition 4.4 and Theorem 4.7 in [38] provide such some results in the finite dimensional setting. In Proposition 3 and Theorem 7 of [9], these properties were investigated under the strong attainment assumption with $p = 2$.

## 4 Klee envelope of the lower semicontinuous convex hull function

For a set $S$ of the normed space $(X, \|\cdot\|)$, according to the Mazur Intersection Property, we define the *Mazur hull* Maz $S$ of $S$ as the intersection of all closed balls containing $S$, so a closed set fulfills the Mazur intersection property if and only if it coincides with its Mazur hull. Clearly, from the very definition

$$\text{Maz}\, S = \bigcap_{x \in X} B[x, \Delta_S(x)],$$

so $\Delta_{\text{Maz}\, S}(x) \le \Delta_S(x)$ for any $x \in X$. This combined with the inclusions $S \subset \text{co}\, S \subset \overline{\text{co}}\, S \subset \text{Maz}\, S$ entails that for any bounded set $S$

$$\Delta_S(x) = \Delta_{\text{co}\, S}(x) = \Delta_{\overline{\text{co}}\, S}(x) = \Delta_{\text{Maz}\, S}(x), \quad \text{for all } x \in X. \tag{15}$$

Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, its *convex hull* co $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is defined by

$$\text{co}\, f(x) = \inf\{r \in \mathbb{R} : (x, r) \in \text{co}\,(\text{epi}\, f)\},$$

where epi $f$ denotes the epigraph of $f$, that is,

$$\text{epi}\, f = \{(y, r) \in X \times \mathbb{R} : f(y) \le r\}.$$

Clearly, it is the greatest convex function majorized by $f$ and

$$\text{co}\, f(x) = \inf\left\{\sum_{i=1}^m t_i f(y_i) : y_i \in X,\ t_i > 0,\ \sum_{i=1}^m t_i y_i = x,\ \sum_{i=1}^m t_i = 1\right\}.$$

Similarly, the *lower semicontinuous convex hull* (or *closed convex hull*) $\overline{\text{co}}\, f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ of $f$ is defined by

$$\overline{\text{co}}\, f(x) = \inf\{r \in \mathbb{R} : (x, r) \in \overline{\text{co}}\,(\text{epi}\, f)\}.$$

It follows from the construction that $\overline{\text{co}}\, f$ is convex and lower semicontinuous and it is the greatest lower semicontinuous convex function less or equal to $f$. It also satisfies the following properties

$$\overline{\text{co}}\,[\text{epi}\, f] = \text{epi}\,(\overline{\text{co}}\, f), \quad \text{co}\,[\text{dom}\, f] \subset \text{dom}\, \overline{\text{co}}\, f.$$

This allows us to express this closed convex hull function in the following manner in the case where $f$ is lower semicontinuous: For all $x \in X$ there exist sequences of elements $(m_n)_n$ in $\mathbb{N}$, $(t_1^n)_n, \ldots, (t_{m_n}^n)_n$ in $]0, 1]$, with $\sum_{i=1}^{m_n} t_i^n = 1$, and $(y_1^n)_n, \ldots, (y_{m_n}^n)_n$ in $\text{dom}\, f$ such that

$$\lim_{n\to+\infty} \sum_{i=1}^{m_n} t_i^n y_i^n = x, \quad \overline{\text{co}}\, f(x) = \lim_{n\to+\infty} \sum_{i=1}^{m_n} t_i^n f(y_i^n).$$

The foregoing proposition extends the two first equalities in (15) to Klee envelopes of functions.

**Proposition 7** *Let $(X, \|\cdot\|)$ be a normed space and let $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be an extended real-valued function. For all $x \in X$,*

$$\kappa_{\lambda,p} f(x) = \kappa_{\lambda,p}(\text{co}\, f)(x) = \kappa_{\lambda,p}(\overline{\text{co}}\, f)(x). \tag{16}$$

*Proof* Since $\overline{\text{co}}\, f \le \text{co}\, f \le f$, we see that $\kappa_{\lambda,p} f \le \kappa_{\lambda,p}(\text{co}\, f) \le \kappa_{\lambda,p}(\overline{\text{co}}\, f)$.

Now fix $x \in X$ and take any $y \in X$. Consider any convex combination $\sum_{i=1}^m t_i y_i = y$, that is, $t_i > 0$ and $\sum_{i=1}^m t_i = 1$. For every $i = 1, \ldots, m$, we have

$$\frac{1}{p\lambda}\|x - y_i\|^p - f(y_i) \le \kappa_{\lambda,p} f(x),$$

hence

$$\sum_{i=1}^m t_i \frac{1}{p\lambda}\|x - y_i\|^p - \sum_{i=1}^m t_i f(y_i) \le \kappa_{\lambda,p} f(x),$$

which combined with the convexity of $\|x - \cdot\|^p$ yields

$$\frac{1}{p\lambda}\|x - y\|^p - \sum_{i=1}^m t_i f(y_i) \le \kappa_{\lambda,p} f(x).$$

Taking the supremum of both members over all convex combinations of $y$, we obtain

$$\frac{1}{p\lambda}\|x-y\|^p - \inf\left\{\sum_{i=1}^m t_i f(y_i) : \sum_{i=1}^m t_i y_i = y,\ t_i > 0, \sum_{i=1}^m t_i = 1\right\} \le \kappa_{\lambda,p} f(x),$$

in other words $\frac{1}{p\lambda}\|x-y\|^p - \operatorname{co} f(y) \le \kappa_{\lambda,p} f(x)$.

For every $u \in X$, taking the limit superior of both members of the latter inequality as $y \to u$ gives

$$\frac{1}{p\lambda}\|x-u\|^p - \overline{\operatorname{co}}\, f(u) \le \kappa_{\lambda,p} f(x),$$

hence taking now the supremum over all $u \in X$ guarantees that $\kappa_{\lambda,p}\big(\overline{\operatorname{co}}\, f\big)(x) \le \kappa_{\lambda,p} f(x)$. This and what precedes justifies the desired equalities in (16). □

## 5 The NSLUC property

We begin this section by defining a class of closed sets which will be involved in the next section for the study of relationship between the attainment sets of $\kappa_{\lambda,p} f$ and $\kappa_{\lambda,p}\big(\overline{\operatorname{co}}\, f\big)$.

**Definition 1** Let $S$ be a subset of the normed space $(X, \|\cdot\|)$. We say that $S$ has norm subdifferential local uniform convexity property, NSLUC in short, if for every bounded subset $S' \subset S$ with $0 \notin \operatorname{cl}_{\|\cdot\|} S'$ and every $u \in \mathbb{S}_X$ for which there is a continuous linear functional $u^* \in \partial\|\cdot\|(u)$ satisfying

$$\inf_{s' \in S'} \|s' - \langle u^*, s'\rangle u\| > 0, \tag{17}$$

one can find a real $\beta > 0$ such that

$$\forall s' \in S', \quad \|s'\| \ge |\langle u^*, s'\rangle| + \beta\|s' - \langle u^*, s'\rangle u\|. \tag{18}$$

Let us point out that if we omit the restriction that $S'$ must be bounded, then even in Hilbert spaces (or simply in the Euclidean space $\mathbb{R}^2$), we cannot ensure that (18) holds true. As an example, in a Hilbert space $H$ endowed with the inner product $(\cdot|\cdot)$ let us take the subset $S' = \{v + nu : n \ge 0\}$, where $\|u\| = \|v\| = 1$ and $(u|v) = 0$. Then, $u^* := (u|\cdot) \in \partial\|\cdot\|(u)$ and $\inf_{s' \in S'} d(s', \operatorname{span} u) = 1$, thus (17) is satisfied. So, if there exists $\beta > 0$ for which relation (18) is fulfilled, otherwise stated, for all $n \in \mathbb{N}$, $\beta + n \le \|v + nu\|$ or equivalently $\beta^2 + 2\beta n + n^2 \le n^2 + 1$, that is, $\beta^2 + 2\beta n \le 1$, then we arrive at a contradiction.

Below we provide another characterization of the NSLUC property by means of distances from the kernel of the functional $u^* \in \partial\|\cdot\|(u)$ and the space generated by $u$, whenever $u \in \mathbb{S}_X$. For this reason, fix any $u \in \mathbb{S}_X$ and $u^* \in \partial\|\cdot\|(u)$. Since $\langle u^*, u\rangle = 1 \ne 0$, we know that the space $X$ is the algebraic (even topological) direct sum of the vector spaces $\ker u^*$ and $\operatorname{span} u$, that is,

$$X = \ker u^* \oplus \operatorname{span} u.$$

Further, for every $x \in X$, noticing by the well-known distance formula from a closed hyperplane (see, e.g., [17, Exercice 2.12, p. 71]) that

$$d(x, \ker u^*) = \frac{|\langle u^*, x\rangle|}{\|u^*\|} = |\langle u^*, x\rangle|,$$

we see that $p_1(x) := x - \langle u^*, x\rangle u$ is a nearest point (with respect to $\|\cdot\|$) of $x$ in $\ker u^*$. Putting $p_2(x) := \langle u^*, x\rangle u \in \operatorname{span} u$, it is known and obvious that $p_2$ is the projector onto $\operatorname{span} u$, that is, $x = p_1(x) + p_2(x)$ for all $x \in X$.

Let $S' \subset S$ be a bounded subset, with $0 \notin \operatorname{cl}_{\|\cdot\|} S'$ such that $S'$ keeps the space $\operatorname{span} u$ (uniformly) far, that is,

$$\inf_{s' \in S'} d(s',\ \operatorname{span} u) > 0, \tag{19}$$

then assuming NSLUC property, there exists $\beta > 0$, such that

$$\forall s' \in S', \quad \|s'\| \geq \|s' - p_1(s')\| + \beta\|s' - p_2(s')\|$$

or equivalently (for the proof see Proposition 8 below)

$$\forall s' \in S', \quad \|s'\| \geq d(s', \ker u^*) + \gamma d(s',\ \operatorname{span} u), \tag{20}$$

for some $\gamma > 0$. When $(X, \|\cdot\|)$ is a Hilbert space and $S'$ keeps the set $\operatorname{span} u$ (uniformly) far, such a real $\gamma$ can be precised. Let such a bounded subset $S'$ of a Hilbert space $(X, \|\cdot\|)$ be given. Put $M := \sup_{s' \in S'} \|s'\|$ and $r := \inf_{s' \in S'} d(s', \operatorname{span} u)$. Note that, for all $s' \in S'$,

$$\begin{aligned}\|s'\|^2 &= \|s' - p_1(s')\|^2 + \|p_1(s')\|^2 \\ &= d(s', \ker u^*)^2 + \|s' - \langle u^*, s'\rangle u\|^2 \geq d(s', \ker u^*)^2 + d(s', \operatorname{span} u)^2.\end{aligned}$$

So, to ensure the desired inequality it suffices to choose $\gamma$ so that for all $s' \in S'$

$$\begin{aligned}&d(s', \ker u^*)^2 + 2\gamma d(s', \ker u^*) d(s', \operatorname{span} u) + \gamma^2 d(s', \operatorname{span} u)^2 \\ &\quad \leq d(s', \ker u^*)^2 + d(s', \operatorname{span} u)^2,\end{aligned}$$

and (by definitions of $M$ and $r$) this holds true if in particular

$$2\gamma M^2 + \gamma^2 M^2 \leq r^2 \ \Leftrightarrow\ M^2(\gamma^2 + 2\gamma + 1) \leq M^2 + r^2 \ \Leftrightarrow\ M^2(\gamma + 1)^2 \leq M^2 + r^2,$$

which in turn is satisfied in particular for $\gamma := \frac{-M + \sqrt{M^2 + r^2}}{M}$.

In fact, as it will be established hereafter, in several normed vector spaces the implication (19) $\Rightarrow$ (20) is satisfied. This can be easily seen whenever the characterization of the NSLUC property given below is used.

**Proposition 8** *Let $S$ be a subset in the normed space $(X, \|\cdot\|)$ and $S' \subset S$ with $0 \notin \mathrm{cl}_{\|\cdot\|} S'$ be a bounded subset. Then for every $u \in \mathbb{S}_X$ and $u^* \in \partial\|\cdot\|(u)$ relation (17) holds if and only if (19) holds. Moreover, the following assertions are equivalent:*

1. *$S$ has NSLUC property,*
2. *For every bounded subset $S' \subset S$ with $0 \notin \mathrm{cl}_{\|\cdot\|} S'$ and every $u \in \mathbb{S}_X$ satisfying relation (19) and every continuous linear functional $u^* \in \partial\|\cdot\|(u)$ one can find a real $\gamma > 0$ such that relation (20) holds true.*

*Proof* We start our proof by establishing the following equivalence: relation (19) is equivalent to relation (17) for every $u^* \in \partial\|\cdot\|(u)$. Since for every $u^* \in \partial\|\cdot\|(u)$, $d(s', \text{ span } u) \leq \|s' - \langle u^*, s'\rangle u\|$, then the implication (19) $\Rightarrow$ (17) is trivial. Now, assume that relation (17) holds for every $u^* \in \partial\|\cdot\|(u)$ and suppose that relation (19) does not hold. Then there exists a bounded sequence $(s_i)_i$ of elements of $S'$ such that $\lim_{i\to+\infty} d(s_i, \text{ span } u) = 0$. Since the space span $u$ is finite dimensional, extracting subsequence, we may assume that $\lim_{i\to+\infty} s_i = \lambda u$ and hence $\lim_{i\to+\infty} \langle u^*, s_i\rangle = \lambda$ and $\lim_{i\to+\infty} \|s_i - \langle u^*, s_i\rangle u\| = 0$, which contradicts relation (17).

The implication (1) $\Rightarrow$ (2) is easy to observe, because of the previous part of the proof and for all $s \in X$, $d(s, \ker u^*) = |\langle u^*, s\rangle|$ and $d(s, \text{ span } u) \leq \|s - \langle u^*, s\rangle u\|$.

Now, we prove the implication (2) $\Rightarrow$ (1). Fix any bounded set $S' \subset S$ and $u \in \mathbb{S}_X$ for which there is $u^* \in \partial\|\cdot\|(u)$ fulfilling (17) and for each $i \in \mathbb{N}$ there exists $s_i \in S'$ such that

$$\|s_i\| < |\langle u^*, s_i\rangle| + \frac{1}{i}\|s_i - \langle u^*, s_i\rangle u\|. \tag{21}$$

By (2), there exists $\gamma > 0$ depending only on $S'$ [and not on $(s_i)_i$] such that

$$\|s_i\| \geq d(s_i, \ker u^*) + \gamma d(s_i, \text{ span } u),$$

and hence, using the equality $d(\cdot, \ker u^*) = |\langle u^*, \cdot\rangle|$, we see that relation (21) ensures that

$$\gamma d(s_i, \text{ span } u) < \frac{1}{i}\|s_i - \langle u^*, s_i\rangle u\|.$$

Since the sequence $(s_i)_i$ is bounded, we obtain that $\lim_{i\to+\infty} d(s_i, \text{ span } u) = 0$, which contradicts relation (19). □

*Remark 4* Let $S$ be a subset of the normed space $(X, \|\cdot\|)$ having the NSLUC property. Let $(u, u^*) \in \operatorname{gph} \partial\|\cdot\|$ with $u \in \mathbb{S}_X$ and let $\alpha > 0$. If $(s_i)_i$ is a sequence of elements of $S$, then the following implication

$$\lim_{i\to+\infty} \|s_i\| = \lim_{i\to+\infty} \langle u^*, s_i\rangle = \alpha \Longrightarrow \lim_{i\to+\infty} \|s_i - \alpha u\| = 0$$

holds true. □

Let us also observe that, if $X$ has the NSLUC property, then any subset $S$ of $X$ has the NSLUC property too. Below it is proved that if the unit sphere has the NSLUC property, then the whole set $X$ has the property too.

*Remark 5* The sphere $\mathbb{S}_X$ of the normed space $(X, \|\cdot\|)$ has the NSLUC property if and only if the whole set $X$ has the NSLUC property.

*Proof* Suppose that $\mathbb{S}_X$ has the NSLUC property. Let us fix a bounded set $S' \subset X$ with $0 \notin \mathrm{cl}_{\|\cdot\|} S'$ and $(u, u^*) \in \mathrm{gph}\, \partial\|\cdot\|$ such that

$$\inf_{s' \in S'} \|s' - \langle u^*, s'\rangle u\| > 0. \tag{22}$$

Put

$$A := \{\|s'\|^{-1} s' : s' \in S'\}$$

and notice that $A \subset \mathbb{S}_X$. Since

$$\Delta_{S'}(0) \inf_{a \in A} \|a - \langle u^*, a\rangle u\| \geq \inf_{s' \in S'} \|s' - \langle u^*, s'\rangle u\|,$$

it follows from (22) and the NSLUC property for $\mathbb{S}_X$ that there is $\beta > 0$ such that

$$\forall a \in A, \quad \|a\| \geq |\langle u^*, a\rangle| + \beta\|a - \langle u^*, a\rangle u\|. \tag{23}$$

It is a simple consequence of (23) that

$$\forall s' \in S', \quad \|s'\| \geq |\langle u^*, s'\rangle| + \beta\|s' - \langle u^*, s'\rangle u\|,$$

which gives the NSLUC property of the whole set $X$.

The converse implication is a direct consequence of the observation preceding the remark. □

In the following proposition, we shall show that if a subset $S$ of $X$ has the NSLUC property, then the norm of the space is strictly convex on $S$ and the set $S$ has the Kadec-Klee property with respect to the norm. Let us recall that the norm $\|\cdot\|$ is *strictly convex on a subset* $S$ of $X$ if the following implication holds true:

$$x, y \in S,\ \|x + y\| = \|x\| + \|y\|,\ x \neq 0, y \neq 0 \Rightarrow x = \frac{\|x\|}{\|y\|} y. \tag{24}$$

If (24) holds true for $S = \mathbb{S}_X$, then one just says that the norm is *strictly convex* or the space $(X, \|\cdot\|)$ is *stricly convex*.

We say that a set $S \subset X$ has Kadec-Klee property with respect to the norm $\|\cdot\|$ whenever any sequence $(x_i)_i$ of elements of $S$ converging weakly to $x \in X$ along with $\lim_{i \to +\infty} \|x_i\| = \|x\|$ converges strongly to $x$ (that is, $\|x_i - x\| \to 0$ as $i \to +\infty$). So, the norm $\|\cdot\|$ has the Kadec-Klee property if and only if the whole set $X$ has the Kadec-Klee property with respect to $\|\cdot\|$.

**Proposition 9** *Let $S$ be a set in the normed space $(X, \|\cdot\|)$ having the NSLUC property. Then the norm $\|\cdot\|$ is strictly convex on $S$ and $S$ has the Kadec-Klee property with respect to $\|\cdot\|$.*

*Proof* *Strict convexity*: Let $x, y \in S$, with $\|x+y\| = \|x\| + \|y\|$, $x \neq 0$ and $y \neq 0$. We shall show that $x = \lambda y$, where $\lambda := \frac{\|x\|}{\|y\|}$. Suppose that

$$\|x - \lambda y\| > 0. \tag{25}$$

Put $S' = \{x, y\}$ and $u = \frac{x+y}{\|x\|+\|y\|}$. Then $\|u\| = 1$. We claim that, for all $u^* \in \partial\|\cdot\|(u)$

$$\|y - \langle u^*, y\rangle u\| > 0, \quad \text{and} \quad \|x - \langle u^*, x\rangle u\| > 0.$$

Indeed, if $y = \langle u^*, y\rangle u$, then $|\langle u^*, y\rangle| = \|y\|$. If $\langle u^*, y\rangle = \|y\|$, then $y = \lambda^{-1}x$, and this contradicts relation (25). If $\langle u^*, y\rangle = -\|y\|$, then $y = -\frac{\|y\|}{\|x\|+2\|y\|}x$, and hence $y = 0$, and this contradicts $y \neq 0$ and hence the claim is justified. Then, by the NSLUC property of $S$, there exists $\beta > 0$ such that

$$\|x\| \geq |\langle u^*, x\rangle| + \beta\|x - \langle u^*, x\rangle u\|, \quad \text{and} \quad \|y\| \geq |\langle u^*, y\rangle| + \beta\|y - \langle u^*, y\rangle u\|,$$

and adding these two inequalities gives

$$\|x\| + \|y\| \geq |\langle u^*, x+y\rangle| + \beta(\|x - \langle u^*, x\rangle u\| + \|y - \langle u^*, y\rangle u\|).$$

Since $\|x\| + \|y\| = \|x+y\|$ and $\langle u^*, x+y\rangle = \|x+y\|$, it ensues that

$$\|x+y\| \geq \|x+y\| + \beta(\|x - \langle u^*, x\rangle u\| + \|y - \langle u^*, y\rangle u\|),$$

which contradicts the inequality $\|x - \langle u^*, x\rangle u\| + \|y - \langle u^*, y\rangle u\| > 0$.

*Kadec-Klee property*: Let $(x_i)_i$ be a sequence of elements of $S$ and $x \in X$ be such that $(x_i)_i$ converges weakly to $x$ and $\lim_{i\to+\infty} \|x_i\| = \|x\|$. We may assume that $x \neq 0$, otherwise we are done. Put $\alpha := \|x\|$, $u := \alpha^{-1}x$ and take $u^* \in \partial\|\cdot\|(u)$. It follows from Remark 4 that $\lim_{i\to+\infty} \|x_i - x\| = 0$. □

Now we give examples of sets satisfying the NSLUC property. Let us recall first that the norm $\|\cdot\|$ of $X$ is locally uniformly rotund (LUR), or simply $(X, \|\cdot\|)$ is LUR (see, e.g., [11]), if the following condition holds:

$$\lim_{i\to+\infty} \|x_i\| = 1, \; \lim_{i\to+\infty} \|x_i + x\| = 2 \implies \lim_{i\to+\infty} \|x_i - x\| = 0.$$

**Proposition 10** *If the normed space $(X, \|\cdot\|)$ is LUR, then $X$ has the NSLUC property.*

*Proof* Fix any bounded set $S' \subset X$ with $0 \notin \mathrm{cl}_{\|\cdot\|} S'$ and any $u \in \mathbb{S}_X$ for which there is $u^* \in \partial\|\cdot\|(u)$ fulfilling (17). Suppose that there is no real $\beta > 0$ satisfying relation (18). Then for each $i \in \mathbb{N}$ there exists $s_i \in S'$ such that

$$\|s_i\| < |\langle u^*, s_i\rangle| + \frac{1}{i}\|s_i - \langle u^*, s_i\rangle u\|. \tag{26}$$

Further by (17), for $\delta := \inf_{s'\in S'} \|s' - \langle u^*, s'\rangle u\| > 0$, we have

$$\|s_i - \langle u^*, s_i\rangle u\| \geq \delta, \quad \text{for all } i \in \mathbb{N}. \tag{27}$$

As $S'$ is bounded, the sequence $(s_i)_i$ is bounded too and hence

$$\lim_{i\to+\infty} \frac{1}{i}\|s_i - \langle u^*, s_i\rangle u\| = 0.$$

Note that, because of (26) and the relation $0 \notin \mathrm{cl}_{\|\cdot\|} S'$, each convergent subsequence (which exists because of the boundedness of $S'$) of the sequence $(\langle u^*, s_i\rangle)_i$ has a nonzero limit. Put $\alpha_i = \langle u^*, s_i\rangle$ for all $i \in \mathbb{N}$. We may suppose that

$$\alpha_i \neq 0, \quad \forall i \quad \text{and} \quad \lim_{i\to+\infty} \alpha_i = \alpha \neq 0.$$

Using (26), we get

$$\lim_{i\to+\infty} \|\alpha_i^{-1} s_i\| = 1 \quad \text{and} \quad \langle u^*, \alpha_i^{-1} s_i\rangle = 1, \quad \text{for all } i \in \mathbb{N}. \tag{28}$$

Now, note that

$$2 = \|u\| + \langle u^*, \alpha_i^{-1} s_i\rangle = \langle u^*, u + \alpha_i^{-1} s_i\rangle \leq \|u + \alpha_i^{-1} s_i\| \leq \|u\| + \|\alpha_i^{-1} s_i\| \tag{29}$$

and hence $\lim_{i\to+\infty} \|u + \alpha_i^{-1} s_i\| = 2$. So the LUR property of $X$ implies that $\lim_{i\to+\infty} \|u - \alpha_i^{-1} s_i\| = 0$. Because of (17), the last equality contradicts the following inequality

$$\|\alpha_i^{-1} s_i - u\| \geq \frac{1}{|\alpha_i|} \inf_{s'\in S} \|s' - \langle u^*, s'\rangle u\|.$$

□

For any fixed integer $k \geq 2$, a normed space $(X, \|\cdot\|)$ is called *fully $k$-convex* (see, e.g., [18] and [19]), if every sequence $(x_n)_n$ of elements of $X$ satisfying $\lim_{n\to+\infty} \|x_n\| = 1$, and $\frac{1}{k}\|\sum_{i=1}^{k} x_{\nu_i}\| \geq 1$ for any $k$ indices $\nu_1 \leq \cdots \leq \nu_k$ is a Cauchy sequence. Fan and Glicksberg proved that $(X, \|\cdot\|)$ is fully $k$-convex if and only if every sequence $(x_n)_n$ of elements of $X$ satisfying

$$\lim_{\nu_1,\cdots,\nu_k\to\infty} \frac{1}{k}\left\|\sum_{i=1}^{k} x_{\nu_i}\right\| = 1,$$

is a Cauchy sequence. It is also shown in [19], that for any integer $k \geq 2$, every fully $k$-convex normed space is strictly convex and has the Kadec-Klee property. Further, Polak and Sims exhibited in [35] an example of a Banach space which is fully 2−convex but not locally uniformly rotund.

In the following proposition, we shall show that all subsets of a fully $k$-convex Banach space have the NSLUC property. This allows us to say that there is no equivalence between the LUR property of a Banach space and the NSLUC property of the space.

**Proposition 11** *If $(X, \|\cdot\|)$ is a fully $k$-convex Banach space, for some integer $k \geq 2$, then $X$ has the NSLUC property.*

*Proof* Fix any bounded subset $S' \subset X$ with $0 \notin \mathrm{cl}_{\|\cdot\|} S'$ and any $u \in \mathbb{S}_X$ for which there is $u^* \in \partial\|\cdot\|(u)$ fulfilling (17). We repeat the proof of Proposition 10, to get the existence of $\delta > 0$ and a sequence $(s_i)_i$ of elements of the bounded subset $S'$ such that

$$\|s_i - \langle u^*, s_i\rangle u\| \geq \delta \quad \forall i \in \mathbb{N}, \tag{30}$$

the sequence $(\langle u^*, s_i\rangle)_i$ converging to $\alpha \neq 0$ and $\lim\limits_{i\to+\infty} \dfrac{\|s_i\|}{|\langle u^*, s_i\rangle|} = 1$ as well as (26). Putting $\varepsilon_\alpha := \mathrm{sign}(\alpha)$, there is some $i_0$ such that $|\langle u^*, s_i\rangle| = \varepsilon_\alpha\langle u^*, s_i\rangle$ for all $i \geq i_0$. Let $\nu_1 \leq \cdots \leq \nu_k$ be any $k$ indices greater than $i_0$. Using relation (26) gives

$$\begin{aligned}\varepsilon_\alpha\langle u^*, s_{\nu_1} + \cdots + s_{\nu_k}\rangle &\leq \|s_{\nu_1} + \cdots + s_{\nu_k}\| \\ &< \varepsilon_\alpha\langle u^*, s_{\nu_1} + \cdots + s_{\nu_k}\rangle + \sum_{j=1}^{k} \frac{1}{\nu_j}\|s_{\nu_j} - \langle u^*, s_{\nu_j}\rangle u\|,\end{aligned}$$

and hence noting that $\sum_{j=1}^{k} \frac{1}{\nu_j}\|s_{\nu_j} - \langle u^*, s_{\nu_j}\rangle u\| \to 0$ as $\nu_j \to \infty$ (for $j = 1, \ldots, k$) we get

$$\lim_{\nu_1,\cdots,\nu_k\to\infty} \frac{1}{k}\|\sum_{i=1}^{k} \frac{s_{\nu_i}}{\alpha}\| = 1.$$

Since $X$ is fully $k$-convex, the sequence $(s_i)_i$ converges to some $s$, and $s \neq 0$ because $0 \notin \mathrm{cl}_{\|\cdot\|} S'$. Consequently, $\langle u^*, \varepsilon_\alpha s\rangle = \|\varepsilon_\alpha s\|$ or equivalently $u^* \in \partial\|\cdot\|(\varepsilon_\alpha s)$. Since $u^* \in \partial\|\cdot\|(u)$, we obtain $\|u + \frac{\varepsilon_\alpha s}{\|\varepsilon_\alpha s\|}\| = 2$. The strict convexity of $X$ (because of the full $k$-convexity of $(X, \|\cdot\|)$) ensures that $u = \frac{\varepsilon_\alpha s}{\|\varepsilon_\alpha s\|}$. Thus $s = \langle u^*, s\rangle u$, and this contradicts relation (30) [by passing to the limit in (30)]. □

Recall that one says that a subset $S$ of the normed space $(X, \|\cdot\|)$ is *relatively ball-compact* (resp. *relatively weakly sequentially ball-compact*) whenever the intersection of $S$ with any closed ball is relatively compact (resp. relatively weakly sequentially compact). Below it is established that weakly sequentially ball-compact sets have the NSLUC property, whenever the norm of the space is strictly convex and has the Kadec-Klee property, compare to Proposition 9.

**Proposition 12** *Let $S$ be a relatively weakly sequentially ball-compact subset of the normed space $(X, \|\cdot\|)$. If the norm $\|\cdot\|$ is strictly convex (or equivalently the norm is stricly convex on $\mathbb{S}_X$) and the set $S$ has the Kadec-Klee property, then $S$ has the NSLUC property.*

*Proof* We repeat the proof of Proposition 10, to get the existence of the sequence $(\alpha_i)_i$ converging to $\alpha \neq 0$ and a sequence $(s_i)_i$ of $S$ satisfying $\lim_{i\to+\infty} \|\alpha_i^{-1} s_i\| = 1$ as well as (27) and (28). This says in particular that the sequence $(s_i)_i$ is bounded, and hence according to the fact that $S$ is relatively weakly sequentially ball-compact, we may suppose that $(s_i - \alpha_i u)_i$ converges weakly to some $z$, hence $(s_i)_i$ converges weakly to $z + \alpha u$.

If $z = 0$, then $(s_i)_i$ converges weakly to $\alpha u$. As $\|u\| = 1$ and $\lim_{i\to+\infty} \|\alpha_i^{-1} s_i\| = 1$, the Kadec-Klee property of $S$ ensures that

$$\lim_{i\to+\infty} \|s_i - \alpha_i u\| = 0,$$

and this contradicts relation (27).

Suppose $z \neq 0$. Since the sequence $(u+\alpha_i^{-1} s_i)_i$ converges weakly to $2u+\alpha^{-1} z$ and $\lim_{i\to+\infty} \langle u^*, u+\alpha_i^{-1} s_i\rangle = 2$ by (28), we have $\langle u^*, u+\alpha^{-1} z\rangle = 1$, that is, $\langle u^*, \alpha u+z\rangle = \alpha$. We then obtain

$$|\alpha| = |\langle u^*, \alpha u + z\rangle| \leq \|\alpha u + z\| \leq \liminf_{i\to+\infty} \|s_i\| \leq |\alpha|,$$

and hence $\|\alpha u + z\| = |\alpha|$ and $|\alpha| > 0$ (since, as said above, $\alpha \neq 0$). Noting that the sequence $(s_i)_i$ converges weakly to $\alpha u + z$, $\lim_{i\to+\infty} \|s_i\| = |\alpha|$ and $\|\alpha u + z\| = |\alpha|$, by the Kadec-Klee property of $S$, we get $\lim_{i\to+\infty} \|s_i - \alpha u - z\| = 0$.

The strict convexity of the norm $\|\cdot\|$ together with the equalities $\|u\| = 1$, $\|u + \alpha^{-1} z\| = 1$ and the following relations

$$\begin{aligned} 2 &= \langle u^*, u + u + \alpha^{-1} z\rangle \leq \|u + u + \alpha^{-1} z\| \\ &\leq \liminf_{i\to+\infty} \|u + \alpha_i^{-1} s_i\| \leq 2 \end{aligned}$$

ensure that $u + \alpha^{-1} z = u$. Thus $z = 0$ and this second contradiction completes the proof. □

When we look closely at the proof of Proposition 12, then we see that the assumption that the set $S$ is relatively weakly sequentially ball-compact is too strong to get the statement of the Proposition. What we need in fact is a possibility to choose a weakly converging subsequence from any sequence $(s_i)_i$ satisfying inequality (26). Of course the problem does not occur, whenever the space $X$ is reflexive. Moreover the strict convexity of the norm, which is one of assumptions in Proposition 12, implies the Gâteaux differentiability of the dual norm in this case, see [13, Corollary 1, p. 24]. So it seems that smoothness of the dual norm is a suitable assumption to preserve the weak sequential compactness whenever $X$ is not a reflexive Banach space. We present details in Proposition 13 below.

**Proposition 13** *Let $(X, \|\cdot\|)$ be a Banach space whose norm has the Kadec-Klee property and the dual norm is Gâteaux differentiable off the origin. Then $X$ has the NSLUC property.*

*Proof* We repeat the proof of Proposition 10, to get the existence of $\delta > 0$ and a bounded sequence $(s_i)_i$ such that relation (27) holds, the sequence $(\langle u^*, s_i\rangle)_i$ converging to $\alpha \neq 0$ and $\lim_{i\to+\infty} \frac{\|s_i\|}{|\langle u^*, s_i\rangle|} = 1$ as well as (26). So, $\langle u^*, u\rangle = 1$ and $\lim_{i\to+\infty} \langle u^*, \alpha^{-1}s_i\rangle = 1$ with $\lim_{i\to+\infty} \|\alpha^{-1}s_i\| = 1 = \|u\|$. Since the dual norm is Gâteaux differentiable off the origin, the Šmulyan theorem [11, Theorem 1.4] asserts that the sequence $(\alpha^{-1}s_i)_i$ converges weakly to $u$ and by the Kadec-Klee property, $(\alpha^{-1}s_i)_i$ norm-converges to $u$ and this contradicts relation (27). □

As a corollary we obtain that the Kadec-Klee property of the norm and the Gâteaux differentiability of the dual norm characterize the NSLUC property in the reflexive Banach space setting.

**Corollary 6** *Let $(X, \|\cdot\|)$ be a reflexive Banach space. Then the following assertions are equivalent:*

(a) *The norm $\|\cdot\|$ has the Kadec-Klee property and its dual norm is Gâteaux differentiable off the origin;*
(b) *$X$ has the NSLUC property;*
(c) *The sphere $\mathbb{S}_X$ of $(X, \|\cdot\|)$ has the NSLUC property;*
(d) *The norm $\|\cdot\|$ is strictly convex and has the Kadec-Klee property.*

*Proof* The equivalence (b) $\Leftrightarrow$ (c) is established in Remark 5. The implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (d) follow from Propositions 13 and 9 respectively. On the other hand, it is known that the strict convexity of a dual norm entails the Gâteaux differentiability of the corresponding initial norm; this and the reflexivity of $(X, \|\cdot\|)$ justifies the last implication (d) $\Rightarrow$ (a). □

*Remark 6* Even in a reflexive Banach space, the NSLUC property is weaker than the LUR one. Indeed, with the help of Corollary 6, [7, Remark 6.7] ensures the existence of a reflexive Banach space with NSLUC property which is not LUR. Namely, there are reflexive Banach spaces with the norm strictly convex and having the Kadec-Klee property but not being LUR, see [7, Remark 6.7]. □

The strict convexity of a normed space can be also characterized through the NSLUC property for some class of sets.

**Proposition 14** *A normed space $(X, \|\cdot\|)$ is strictly convex if and only if every relatively ball-compact subset $S$ of $X$ has the NSLUC property.*

*Proof* The "if" part follows from Proposition 9. Let us establish the "only if" part. As above, we repeat the proof of Proposition 10, to get the existence of the sequence

$(\alpha_i)_i$ converging to $\alpha \neq 0$ and a sequence $(s_i)_i$ of $S$ satisfying $\lim_{i\to+\infty} \|\alpha_i^{-1}s_i\| = 1$ as well as (28) and (29). Because of the boundedness of the sequence $(s_i)_i$ and the relative ball-compactness of $S$, we may suppose that $(s_i - \alpha_i u)_i$ converges to some $z$, hence $(\alpha_i^{-1}s_i)_i$ converges to $u + \alpha^{-1}z$, so in particular $\|u + \alpha^{-1}z\| = 1$.

Since $\lim_{i\to+\infty} \langle u^*, u + \alpha_i^{-1}s_i\rangle = 2$ by (28), we have

$$2 = \langle u^*, 2u + \alpha^{-1}z\rangle \leq \|2u + \alpha^{-1}z\| \leq \liminf_{i\to+\infty} \|u + \alpha_i^{-1}s_i\| \leq 2,$$

and hence $\|2u + \alpha^{-1}z\| = 2$. Taking into account the equalities $\|u + \alpha^{-1}z\| = 1$ and $\|u + (u + \alpha^{-1}z)\| = 2$, the strict convexity of $\|\cdot\|$ together with the equality $\|u\| = 1$ guarantee that $u + \alpha^{-1}z = u$, thus $z = 0$. Consequently, $\lim_{i\to+\infty} \|\alpha_i^{-1}s_i - u\| = 0$, which is in contradiction with (17). □

## 6 Attainment sets of the Klee envelope

Our aim in this section is to investigate the connection between the attainment sets $Q_{\lambda,p}f(x)$ and $Q_{\lambda,p}\big(\overline{\text{co}}\, f\big)(x)$ for an appropriate $x \in X$. Let us start with the following example which shows that the inclusion $Q_{\lambda,p}f(x) \subset Q_{\lambda,p}\big(\overline{\text{co}}\, f\big)(x)$ may be strict.

*Example 1* Consider the lower semicontinuous function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 2 & \text{if } x = 2 \\ +\infty & \text{otherwise.} \end{cases}$$

For $\lambda = 1$ and $p = 1$, we have

$$\overline{co}\, f(x) = \begin{cases} x & \text{if } x \in [1, 2] \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\kappa_{1,1}(\overline{\text{co}}\, f)(1) = -1, \quad Q_{1,1}(\overline{\text{co}}\, f)(1) = [1, 2], \quad Q_{1,1}f(1) = \{1, 2\}.$$

We recall that the reals $\lambda$ and $p$ *are taken as* $\lambda > 0$ and $p \geq 1$.

**Theorem 2** *Let* $(X, \|\cdot\|)$ *be a normed space and let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous function whose domain* dom $f$ *is bounded. Then for every* $x \in$ dom $\kappa_{\lambda,p}$co $f$ *such that* dom $f - x$ *satisfies NSLUC property we have*

$$d \in Q_{\lambda,p}(\overline{\text{co}}\, f)(x) \implies d \in \text{co}(Q_{\lambda,p}f(x) \cap [x + \text{span}(d - x)]),$$

*and hence*

$$Q_{\lambda,p}f(x) \subset Q_{\lambda,p}(\overline{\text{co}}\, f)(x) \subset \text{co}\, Q_{\lambda,p}f(x).$$

*Moreover, if $f$ is constant on its domain then $Q_{\lambda,p}(\overline{\text{co}}\, f)(x) = Q_{\lambda,p} f(x)$.*

*Proof* Let $x \in \operatorname{dom} \kappa_{\lambda,p} f$, so $x \in \operatorname{dom} \kappa_{\lambda,p}(\overline{\text{co}}\, f)$ by Proposition 7. Pick $d \in Q_{\lambda,p}(\overline{\text{co}}\, f)(x)$. Without loss of generality we may assume that $\lambda = 1$.

*Case 1:* If $d = x$, then since the equality $\inf\limits_{y\in X} \overline{\text{co}} f(y) = \inf\limits_{y\in X} f(y)$ is obvious, by Proposition 7, we get

$$\begin{aligned}\kappa_{\lambda,p} f(d) &= \kappa_{\lambda,p}(\overline{\text{co}}\, f)(d) = -\overline{\text{co}}\, f(d)\\ &= \sup_{y\in X} -\overline{\text{co}}\, f(y) = \sup_{y\in X} -f(y)\\ &= -\inf_{y\in X} f(y)\end{aligned}$$

because

$$-\overline{\text{co}}\, f(d) = \kappa_{\lambda,p}(\overline{\text{co}}\, f)(d) \geq \sup_{y\in X}(-\overline{\text{co}}\, f)(y) \geq -\overline{\text{co}}\, f(d).$$

Taking a sequence $(y_n)_n$ such that $\lim\limits_{n\to+\infty} f(y_n) = \inf\limits_{y\in X} f(y) = \overline{\text{co}} f(d)$ and noting that

$$-f(y_n) \leq \|y_n - d\|^p - f(y_n) \leq \kappa_{\lambda,p} f(d) = -\overline{\text{co}}\, f(d)$$

it follows that $\lim\limits_{n\to+\infty} y_n = d$. So using the lower semicontinuity of $f$, we get

$$f(d) \leq \liminf_{n\to+\infty} f(y_n) = \overline{\text{co}}\, f(d) \leq f(d)$$

and this implies that $\kappa_{\lambda,p} f(d) = -f(d)$, and consequently $d \in Q_{\lambda,p} f(x)$.

*Case 2:* If $d \neq x$, then without loss of generality, we may assume that $\|d - x\| = 1$. Since $d \in Q_{\lambda,p}(\overline{\text{co}}\, f)(x)$, then

$$\kappa_{\lambda,p} f(x) = \kappa_{\lambda,p}(\overline{\text{co}}\, f)(x) = \|d - x\|^p - \overline{\text{co}} f(d).$$

For each $n \in \mathbb{N}$ there exist $m_n \in \mathbb{N}$, $t_1^n, \ldots, t_{m_n}^n \in ]0, 1]$, with $\sum_{i=1}^{m_n} t_i^n = 1$, and $y_1^n, \ldots, y_{m_n}^n \in \operatorname{dom} f$ such that

$$\lim_{n\to+\infty} \sum_{i=1}^{m_n} t_i^n y_i^n = d, \quad \overline{\text{co}} f(d) = \lim_{n\to+\infty} \sum_{i=1}^{m_n} t_i^n f(y_i^n). \tag{31}$$

Note that the sequence $(y_1^n, \ldots, y_{m_n}^n)_n$ is bounded according to the boundedness of $\operatorname{dom} f$. On the other hand, as for all $n \in \mathbb{N}$ and $i = 1, \ldots, m_n$,

$$\|y_i^n - x\|^p - f(y_i^n) \leq \|d - x\|^p - \overline{\text{co}} f(d),$$

we also have

$$\sum_{i=1}^{m_n} t_i^n [\|y_i^n - x\|^p - f(y_i^n)] \leq \|d - x\|^p - \overline{\text{co}} f(d),$$

and hence

$$\left\| \sum_{i=1}^{m_n} t_i^n y_i^n - x \right\|^p \leq \sum_{i=1}^{m_n} t_i^n \|y_i^n - x\|^p \leq \|d - x\|^p - \overline{\text{co}}\, f(d) + \sum_{i=1}^{m_n} t_i^n f(y_i^n),$$

which combined with (31) entails

$$\lim_{n \to +\infty} \sum_{i=1}^{m_n} t_i^n \|y_i^n - x\|^p = \|d - x\|^p. \tag{32}$$

Pick $u^* \in \partial \| \cdot \|(d - x)$. For each $n \in \mathbb{N}$, the inequality

$$\|y_i^n - x\| \geq |\langle u^*, y_i^n - x \rangle| \quad \text{with } \; i = 1, \ldots, m_n$$

along with the convexity of $| \cdot |^p$ ensures that

$$\sum_{i=1}^{m_n} t_i^n \|y_i^n - x\|^p \geq \sum_{i=1}^{m_n} t_i^n |\langle u^*, y_i^n - x \rangle|^p \geq \left| \left\langle u^*, \sum_{i=1}^{m_n} t_i^n y_i^n - x \right\rangle \right|^p,$$

hence

$$\lim_{n \to +\infty} \sum_{i=1}^{m_n} t_i^n |\langle u^*, y_i^n - x \rangle|^p = \|d - x\|^p = 1. \tag{33}$$

Let $\mu > 0$ be arbitrary and consider, for each $n \in \mathbb{N}$, the following sets

$$\begin{aligned} I_\mu^n &= \{i \in \{1, \ldots, m_n\} : \|y_i^n - x - \langle u^*, y_i^n - x \rangle (d - x)\| \geq \mu\}, \\ A_\mu^n &:= \{i \in \{1, \ldots, m_n\} : \|y_i^n - x - \langle u^*, y_i^n - x \rangle (d - x)\| \leq \mu \quad \text{and} \\ &\qquad \|y_i^n - x\|^p - f(y_i^n) + \mu \leq \kappa_{\lambda, p} f(x)\} \end{aligned}$$

and

$$B_\mu^n := \{i \in \{1, \ldots, m_n\} \backslash (I_\mu^n \cup A_\mu^n) : \langle u^*, y_i^n - x \rangle \leq 1 - \mu\}.$$

We have $\limsup\limits_{n \to +\infty} \sum_{i \in A_\mu^n} t_i^n = 0$, because

$$
\begin{aligned}
\kappa_{\lambda,p}f(x) &= \lim_{n\to+\infty}\left(\sum_{i\in A_\mu^n} t_i^n[\|y_i^n - x\|^p - f(y_i^n)] + \sum_{i\notin A_\mu^n} t_i^n[\|y_i^n - x\|^p - f(y_i^n)]\right) \\
&\le \kappa_{\lambda,p}f(x) - \mu \limsup_{n\to+\infty}\sum_{i\in A_\mu^n} t_i^n \le \kappa_{\lambda,p}f(x).
\end{aligned}
$$

We claim that $\limsup_{n\to+\infty}\sum_{i\in I_\mu^n} t_i^n = 0$. Suppose the contrary, that is,

$$
\limsup_{n\to+\infty}\sum_{i\in I_\mu^n} t_i^n = r > 0.
$$

Extracting subsequence, we may assume that $\lim_{n\to+\infty}\sum_{i\in I_\mu^n} t_i^n = r$. Clearly, the set $P := \{n \in \mathbb{N} : I_\mu^n \neq \emptyset\}$ is infinite (keep in mind that in the definition of $r$ the limit is involved) and the set $S' := \{y_i^n - x : n \in P, i \in I_\mu^n\}$ is a bounded subset of dom $f - x$ and by the definition of $I_\mu^n$ it fulfills the condition (17) with $S :=$ dom $f - x$. By (18) (applied with $S =$ dom $f - x$), there exists $\beta := \beta(\mu) > 0$ (not depending on $n$) such that for all $n \in P$ and $i \in I_\mu^n$,

$$
\begin{aligned}
\|y_i^n - x\| &\ge |\langle u^*, y_i^n - x\rangle| + \beta\|y_i^n - x - \langle u^*, y_i^n - x\rangle(d - x)\| \\
&\ge |\langle u^*, y_i^n - x\rangle| + \beta\mu.
\end{aligned}
$$

Fix any $n \in P$. Since

$$
\begin{aligned}
&\sum_{i=1}^{m_n} t_i^n[\|y_i^n - x\|^p - f(y_i^n)] \\
&= -\sum_{i=1}^{m_n} t_i^n f(y_i^n) + \sum_{i\notin I_\mu^n} t_i^n\|y_i^n - x\|^p + \sum_{i\in I_\mu^n} t_i^n\|y_i^n - x\|^p \\
&\ge -\sum_{i=1}^{m_n} t_i^n f(y_i^n) + \sum_{i\notin I_\mu^n} t_i^n|\langle u^*, y_i^n - x\rangle|^p + \sum_{i\in I_\mu^n} t_i^n\|y_i^n - x\|^p \\
&\ge -\sum_{i=1}^{m_n} t_i^n f(y_i^n) + \sum_{i\notin I_\mu^n} t_i^n|\langle u^*, y_i^n - x\rangle|^p + \sum_{i\in I_\mu^n} t_i^n(|\langle u^*, y_i^n - x\rangle| + \beta\mu)^p,
\end{aligned}
$$

we have

$$
\begin{aligned}
&\sum_{i=1}^{m_n} t_i^n[\|y_i^n - x\|^p - f(y_i^n)] \\
&\ge -\sum_{i=1}^{m_n} t_i^n f(y_i^n) + \sum_{i\notin I_\mu^n} t_i^n|\langle u^*, y_i^n - x\rangle|^p + \sum_{i\in I_\mu^n} t_i^n|\langle u^*, y_i^n - x\rangle|^p + \sum_{i\in I_\mu^n} t_i^n(\beta\mu)^p
\end{aligned}
$$

$$= \sum_{i=1}^{m_n} t_i^n [|\langle u^*, y_i^n - x\rangle|^p - f(y_i^n)] + \sum_{i \in I_\mu^n} t_i^n (\beta\mu)^p.$$

Using (33) and (31) and passing to the limit as $P \ni n \to \infty$, we get

$$\kappa_{\lambda,p}(\overline{\text{co}}\, f)(x) \geq \kappa_{\lambda,p}(\overline{\text{co}}\, f)(x) + r(\beta\mu)^p$$

and this contradiction establishes our claim.

If for all $n \in \mathbb{N}$, $B_\mu^n = \{1, \ldots, m_n\} \backslash (I_\mu^n \cup A_\mu^n)$, then $\lim_{n\to+\infty} \sum_{i \in B_\mu^n} t_i^n = 1$ and, by the definition of $B_\mu^n$ and the fact $\lim_{n\to+\infty} \sum_{i\in B_\mu^n} t_i^n y_i^n = d$ [see (31)], we have the following contradiction $1 = \|d - x\| = \langle u^*, d - x\rangle \leq 1 - \mu$. Then $B_\mu^n \neq \{1, \ldots, m_n\} \backslash (I_\mu^n \cup A_\mu^n)$ for some $n \in \mathbb{N}$.

Thus we can choose an increasing sequence $(n(\mu))_\mu$ (we take $\mu$ in a discrete set such that $\mu \to 0^+$), with $i(\mu) \in \{1, \ldots, m_{n(\mu)}\} \backslash (I_\mu^{n(\mu)} \cup A_\mu^{n(\mu)} \cup B_\mu^{n(\mu)})$, so

$$\kappa_{\lambda,p} f(x) = \|d - x\|^p - \overline{\text{co}} f(d) \leq \|y_{i(\mu)}^{n(\mu)} - x\|^p - f(y_{i(\mu)}^{n(\mu)}) + \mu;\ \langle u^*, y_{i(\mu)}^{n(\mu)} - x\rangle \geq 1 - \mu. \tag{34}$$

Thus the sequence $(y_{i(\mu)}^{n(\mu)})_\mu$ is bounded (since dom $f$ is bounded) and, by the definition of $I_\mu^{n(\mu)}$,

$$\lim_{\mu\to 0^+} d(y_{i(\mu)}^{n(\mu)}, x + \text{ span } (d - x)) = 0 \quad \text{and} \quad \liminf_{\mu\to 0^+} \langle u^*, y_{i(\mu)}^{n(\mu)} - x\rangle \geq 1.$$

Without loss of generality we may suppose that

$$\lim_{\mu\to 0^+} y_{i(\mu)}^{n(\mu)} = \bar{y} \in x + \text{ span } (d - x)$$

[because of the boundedness of $(y_{i(\mu)}^{n(\mu)})_\mu$ and the fact that span$(d - x)$ is finite-dimensional]. The lower semicontinuity of $f$ and the relation (34) ensure that

$$\kappa_{\lambda,p} f(x) = \|d - x\|^p - \overline{\text{co}} f(d) \leq \|\bar{y} - x\|^p - f(\bar{y}) \leq \kappa_{\lambda,p} f(x), \quad \text{and } \langle u^*, \bar{y} - x\rangle \geq 1$$

then $\bar{y} \in Q_{\lambda,p} f(x)$. Since

$$\bar{y} \in x + \text{span}\,(d - x),\ \langle u^*, \bar{y} - x\rangle \geq 1 \quad \text{and} \quad \langle u^*, d - x\rangle = 1,$$

there exists $s_1 \geq 1$ such that

$$\bar{y} - x = s_1(d - x). \tag{35}$$

Now, we consider the sets

$$C_\mu^n := \{i \in \{1, \ldots, m_n\} \backslash (I_\mu^n \cup A_\mu^n) : \langle u^*, y_i^n - x\rangle \geq 1 + \mu\}.$$

Repeating the same reasoning with $C_\mu^n$ instead of $B_\mu^n$ provides the existence of $\bar{z} \in Q_{\lambda,p} f(x)$ and $s_2 \le 1$ such that

$$\bar{z} - x = s_2(d - x). \tag{36}$$

Combining relations (35) and (36), we get

$$d \in \mathrm{co}(Q_{\lambda,p} f(x) \cap [x + \text{ span } (d - x)]).$$

Now suppose that $f$ is constant on its domain. If $d = x$, by the case 1 we have $d \in Q_{\lambda,p} f(x)$. So, let us suppose $d \neq x$ and show that $\bar{y} = d$, where $\bar{y}$ is as obtained in case 2. Since $\overline{\mathrm{co}} f(\bar{y}) \le f(\bar{y})$ and

$$\begin{aligned} \|\bar{y} - x\|^p - f(\bar{y}) = \kappa_{\lambda,p} f(x) &= \|d - x\|^p - \overline{\mathrm{co}}\, f(d) \\ &= \kappa_{\lambda,p}(\overline{\mathrm{co}}\, f)(x) \\ &\ge \|\bar{y} - x\|^p - \overline{\mathrm{co}}\, f(\bar{y}), \end{aligned}$$

it follows that $\overline{\mathrm{co}}\, f(\bar{y}) = f(\bar{y}) = \overline{\mathrm{co}} f(d)$ and hence $\|\bar{y} - x\| = \|d - x\|$. As $\bar{y} \in x + \text{ span } (d - x)$ and $\|\bar{y} - x\| = \|d - x\|$, there exists $\gamma \in \{-1, 1\}$ such that $\bar{y} - x = \gamma(d - x)$. Since $i(\mu) \notin B_\mu^{n(\mu)}$, then $\gamma \ge 0$ and hence $\bar{y} = d$ and the proof is completed. ◻

From Proposition 7 we know that $\mathrm{dom}\, \kappa_{\lambda,p} f = \mathrm{dom}\, \kappa_{\lambda,p}(\overline{\mathrm{co}}\, f)$. Consequently, Propositions 10, 11, 12, 13 and 14 combined with Theorem 2 guarantee the following:

**Theorem 3** *Let $(X, \|\cdot\|)$ be a normed space and let $f : X \to \cup\mathbb{R}\{+\infty\}$ be a proper lower semicontinuous function whose domain* dom $f$ *is bounded. Under anyone of the assumptions (1)–(5), one has the inclusion*

$$Q_{\lambda,p}(\overline{\mathrm{co}} f)(x) \subset \mathrm{co}\big(Q_{\lambda,p} f(x)\big) \quad \textit{for all } x \in X.$$

(1) *The norm of $(X, \|\cdot\|)$ has the LUR property;*
(2) *The space $(X, \|\cdot\|)$ is a fully k-convex Banach space;*
(3) *The norm of $(X, \|\cdot\|)$ is strictly convex and has the Kadec-Klee property, and* dom $f$ *is relatively weakly sequentially ball-compact;*
(4) *The space $(X, \|\cdot\|)$ is a Banach space whose norm has the Kadec-Klee property and the dual norm is Gâteaux differentiable off the origin;*
(5) *The norm of $X$ is strictly convex and* dom $f$ *is relatively ball-compact.*

*If in addition $f$ is constant on its domain, then one has*

$$Q_{\lambda,p}(\overline{\mathrm{co}} f)(x) = Q_{\lambda,p} f(x) \quad \textit{for all } x \in X.$$

As a direct consequence of Theorem 2 we get.

**Corollary 7** *Let* $(X, \|\cdot\|)$ *be a normed space. For any bounded closed set* $S$ *of* $X$ *satisfying the NSLUC property, one has the equality*

$$Q_S(x) = Q_{\overline{\text{co}}\, S}(x) \quad \textit{for all } x \in X.$$

Continuing with the case of sets, other properties hold true. Let us establish the following one which follows ideas in the proof of [15, Theorem 2].

**Proposition 15** *Let* $(X, \|\cdot\|)$ *be a normed space fulfilling the Mazur Intersection Property. Then, for any bounded set* $S$ *such that* Dom $Q_S$ *is dense in* $X$*, one has the equality*

$$\overline{\text{co}}\,(\text{Rge}\, Q_S) = \overline{\text{co}}\, S.$$

*Proof* Fix any $x \notin \overline{\text{co}}\,(\text{Rge}\, Q_S)$. By the Mazur Intersection Property, there are $x_0 \in X$ and a real $r > 0$ such that

$$x \notin B[x_0, r] \supset \overline{\text{co}}\,(\text{Rge}\, Q_S).$$

We can choose some real $\varepsilon > 0$ such that $\|x - x_0\| > r + \varepsilon$. According to the density assumption, choose $\bar{x} \in \text{Dom}\, Q_S$ with $\|\bar{x} - x_0\| < \varepsilon/2$. Taking some $\bar{y} \in Q_S(\bar{x})$, we have $\bar{y} \in B[x_0, r]$. Then for any $y \in S$, we have

$$\begin{aligned} \|y - x_0\| &\le \|y - \bar{x}\| + \|\bar{x} - x_0\| \le \|\bar{y} - \bar{x}\| + \|\bar{x} - x_0\| \\ &\le \|\bar{y} - x_0\| + 2\|\bar{x} - x_0\| < r + \varepsilon. \end{aligned}$$

This entails $S \subset B[x_0, r + \varepsilon]$, hence $\overline{\text{co}}\, S \subset B[x_0, r + \varepsilon]$, which ensures $x \notin \overline{\text{co}}\, S$ since $\|x - x_0\| > r + \varepsilon$. Consequently, $\overline{\text{co}}\,(\text{Rge}\, Q_S) \supset \overline{\text{co}}\, S$, and this inclusion is an equality as asserted, since the reverse inclusion is obvious. □

Lau's theorem [27, Theorem 2.3] says, for any weakly compact set $S$ of a Banach space $(X, \|\cdot\|)$, that Dom $Q_S$ contains a dense $G_\delta$ set of $X$. Consequently, the equality in Proposition 15 is valid for every weakly compact set of a Banach space satisfying the Mazur intersection property. This contains [15, Theorem 2]. A result showing that Dom $Q_S$ contains a dense $G_\delta$ set of $X^*$, whenever $S \subset X^*$ is weakly* compact, can be found in [12, Proposition 3].

## 7 The Klee envelope: an approach with a distance from a set

The aim of this section is to show that subdifferential properties of the Klee envelope $\kappa_{\lambda,1} f$ can be investigated through the distance function, namely we want to show that

$$\kappa_{\lambda,1} f(x) + \frac{1}{\lambda} d(x, W_\alpha) = m_\alpha \quad \forall x \in \text{cl}\big(X \backslash W_\alpha\big), \tag{37}$$

where $\alpha > 0$, $m_\alpha = \alpha + \inf\limits_{y \in X} \kappa_{\lambda,1} f(y)$ and

$$W_\alpha = \{y \in X : \kappa_{\lambda,1} f(y) \ge m_\alpha\}.$$

Whenever $\kappa_{\lambda,1} f$ is finite at some point $\bar{x}$, according to the coercivity [see (7)] and the Lipschitz continuity of $\kappa_{\lambda,1} f$, the set $W_\alpha$ is nonempty and closed. The finiteness of $\kappa_{\lambda,1} f$ at $\bar{x}$ also ensures that $f$ is finite at some $\bar{y}$, so writing, for all $x \in X$,

$$\kappa_{\lambda,1} f(x) = \sup_{y \in X} \left( \frac{1}{\lambda} \|x - y\| - f(y) \right) \geq -f(\bar{y}),$$

we see that $\kappa_{\lambda,1} f$ is bounded from below. Then $m_\alpha > \inf_{y \in Y} \kappa_{\lambda,1} f$, hence the set $\{x \in X : \kappa_{\lambda,1} f(x) < m_\alpha\}$ is nonempty. This nonemptiness combined with the continuity and convexity of $\kappa_{\lambda,1} f$ easily entails that

$$\text{cl}\,\{x \in X : \kappa_{\lambda,1} f(x) < m_\alpha\} = \{x \in X : \kappa_{\lambda,1} f(x) \leq m_\alpha\} \tag{38}$$

and

$$\text{int}\,\{x \in X : \kappa_{\lambda,1} f(x) \leq m_\alpha\} = \{x \in X : \kappa_{\lambda,1} f(x) < m_\alpha\}. \tag{39}$$

Moreover, if $f$ is the indicator function of a non-singleton set $S$ for which $Q_{\lambda,1} f(x)$ is a singleton for all $x \in X$, we show that the set $P_{W_\alpha}(x)$ of nearest points of $x$ in $W_\alpha$ is a singleton for every $\alpha > \kappa_{\lambda,1} f(x)$; see Theorem 5. In other words, if $S$ (not singleton) is such a set that for every $x$ the set $Q_S(x)$ is a singleton, then we can construct an open bounded convex nonempty set $U \subset X$ such the set $X \backslash U$ is Chebyshev and $d(\cdot, X \backslash U) + \Delta_S(\cdot)$ is a constant function on $U$. Observe that $X \backslash U$ can not be a convex set. Thus the Klee question, that is, the problem of singleton property of sets with unique farthest points (see Problem 6 in [23]) turns out to be a question on the convexity of Chebyshev sets, see Problem 5 in [23] and also the Goebel-Schöneberg problem [21], whenever $X$ is a Hilbert space. Namely, if we have the convexity of Chebyshev sets, then only singletons have unique farthest points, we refer to [6,7,10,14,20,21,23,25,38,42] and their references for several results concerning convexity of Chebyshev set in Hilbert spaces.

**Theorem 4** *Let $(X, \|\cdot\|)$ be a normed space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function. If $\kappa_{\lambda,1} f$ is finite at some point, then relation (37) holds true.*

*Proof* We saw above that the finitness of $\kappa_{\lambda,1} f$ at some point implies that $m_0 := \inf_{u \in X} \kappa_{\lambda,1} f(u)$ is well defined in $\mathbb{R}$. Let us fix any $x \in \text{int}\, D_\alpha$ with $\kappa_{\lambda,1} f(x) > -f(x)$, where [see (38)]

$$D_\alpha := \text{cl}\left(X \backslash W_\alpha\right) = \{u \in X : \kappa_{\lambda,1} f(u) \leq m_\alpha\}.$$

Take a sequence $(y_n)_n$ of elements of $X$ such that

$$\kappa_{\lambda,1} f(x) < \frac{1}{n} + \frac{1}{\lambda} \|y_n - x\| - f(y_n),$$

and observe that $y_n \neq x$ for $n$ large enough, say $n \geq n_0$, since $\kappa_{\lambda,1} f(x) > -f(x)$. Fix any integer $n \geq n_0$. Since $\kappa_{\lambda,1} f(u) \to +\infty$ as $\|u\| \to \infty$ by (7) and $\kappa_{\lambda,1} f(x) < m_\alpha$

by (39), we can choose by the continuity of $\kappa_{\lambda,1} f$ and the intermediate value theorem some $z_n = x + t_n(x - y_n)$ with $t_n > 0$ such that $\kappa_{\lambda,1} f(z_n) = m_\alpha$.

Choose some $x_n^* \in \partial\|\cdot\|(x - y_n)$ and note that $\langle x_n^*, x - y_n\rangle = \|x - y_n\|$ since $y_n \neq x$. Then we have

$$m_\alpha - \kappa_{\lambda,1} f(x) + \frac{1}{n} = \kappa_{\lambda,1} f(z_n) - \kappa_{\lambda,1} f(x) + \frac{1}{n}$$
$$\geq \frac{1}{\lambda}\|z_n - y_n\| - f(y_n) + \left(-\frac{1}{\lambda}\|x - y_n\| + f(y_n)\right),$$

which gives

$$m_\alpha - \kappa_{\lambda,1} f(x) + \frac{1}{n} \geq \frac{1}{\lambda}\big(\|z_n - y_n\| - \|x - y_n\|\big)$$
$$\geq \frac{1}{\lambda}\langle x_n^*, z_n - x\rangle = \frac{1}{\lambda} t_n\|x - y_n\|$$
$$= \frac{1}{\lambda}\|x - z_n\| \geq \frac{1}{\lambda} d(x, W_\alpha).$$

Passing to the limit as $n \to \infty$, we get

$$m_\alpha \geq \kappa_{\lambda,1} f(x) + \frac{1}{\lambda} d(x, W_\alpha). \tag{40}$$

On the other hand, since $\kappa_{\lambda,1} f$ is Lipschitz with constant $1/\lambda$, we also have

$$m_\alpha \leq \kappa_{\lambda,1} f(y) \leq \kappa_{\lambda,1} f(x) + \frac{1}{\lambda}\|x - y\|, \quad \forall y \in W_\alpha,$$

and consequently $m_\alpha \leq \kappa_{\lambda,1} f(x) + \frac{1}{\lambda} d(x, W_\alpha)$. It ensues that

$$m_\alpha = \kappa_{\lambda,1} f(x) + \frac{1}{\lambda} d(x, W_\alpha), \quad \text{for all } x \in \operatorname{int} D_\alpha \text{ with } \kappa_{\lambda,1} f(x) > -f(x).$$

In order to finish the proof, it is enough to show that the set

$$\{x \in \operatorname{int} D_\alpha : \kappa_{\lambda,1} f(x) > -f(x)\}$$

is dense in int $D_\alpha$. Assume the contrary, that is, there are $y_0 \in \operatorname{int} D_\alpha$ and $r_0 > 0$ such that

$$B(y_0, r_0) \subset \{x \in \operatorname{int} D_\alpha : \kappa_{\lambda,1} f(x) = -f(x)\}.$$

This implies that for every $y \in B(y_0, r_0)$

$$-f(y) = \sup_{z \in X}\left(\frac{1}{\lambda}\|z - y\| - f(z)\right) \geq \sup_{z \in X}(-f(z)) \geq -f(y).$$

It ensues that $f$ is constant over $B(y_0, r)$, or equivalently

$$f(y) = f(y_0) \quad \text{for all } y \in B(y_0, r_0).$$

It follows that

$$-f(y_0) = \kappa_{\lambda,1} f(y_0) \geq \sup_{y \in B(y_0,r_0)} \left( \frac{1}{\lambda} \|y_0 - y\| - f(y_0) \right) = \frac{r_0}{\lambda} - f(y_0),$$

which is a contradiction. Thus the set $\{x \in \text{ int } D_\alpha : \kappa_{\lambda,1} f(x) > -f(x)\}$ is dense in int $D_\alpha$, hence also in $D_\alpha$ according to (38) and (39). Since the function $x \mapsto \kappa_{\lambda,1} f(x) + \frac{1}{\lambda} d(x, W_\alpha)$ is continuous then (37) holds true and the proof is completed. □

*Remark 7* It follows from Theorem 4, taking $\alpha > 0$ large enough, the differentiability or subdifferentialbility of $\kappa_{\lambda,1} f$ at point $x \in X$ can be inferred from differentiability or subdifferentiability of the opposite of the distance function to the complement of bounded convex set. □

*Remark 8* Theorem 4 can be stated in a more general setting as follows : Let $g : X \to \mathbb{R}$ be a Lipschitz continuous convex function, with Lipschitz constant equal to 1, satisfying the following assumptions:

1. $\lim_{\|x\| \to +\infty} g(x) = +\infty$;
2. There exists a dense set $G$ of $X$ such that $\partial_{\frac{1}{n}} g(x) \cap \mathbb{S}_{X^*} \neq \emptyset$, for all $x \in G$ and $n \in \mathbb{N}$, where

$$\partial_\varepsilon g(x) := \{x^* \in X^* : \langle x^*, u - x \rangle + g(x) \leq g(u) + \varepsilon, \quad \forall u \in X\}$$

denotes the approximate $\varepsilon$-subdifferential of the convex function $g$ at $x$.

Then relation (37) holds true with $g$ instead of $\kappa_{\lambda,1} f(x)$. □

There is a partial connexion between the set of farthest points in a set $S$ and the set of nearest points in $S$ to points outside $S$.

**Proposition 16** *Let $(X, \|\cdot\|)$ be a normed space and $S$ be a nonempty closed bounded subset of $X$. Then*

$$\text{Rge } Q_S \subset P_S(X \backslash S).$$

*Proof* We may suppose that $S$ is not a singleton, since otherwise the inclusion is trivial. Let $y \in \text{Rge } Q_S$, so $y \in S$. There exists $x \in X$ such that $\|x - y\| = \Delta_S(x) > 0$. For $x' := 2y - x$, we see that $\|x' - x\| = 2\Delta_S(x)$, so in particular $x' \in X \backslash S$ since $\|x' - x\| > \Delta_S(x)$. Further, for every $u \in S$ we have $\|x - u\| \leq \Delta_S(x)$ according to the definition of $\Delta_S(x)$, hence

$$\|x' - u\| = \|2y - x - u\| = \|2(y - x) - (u - x)\|$$

$$\begin{aligned}&\geq 2\|y-x\|-\|u-x\|\\&\geq 2\Delta_S(x)-\Delta_S(x)\\&=\Delta_S(x)=\|x-y\|=\|x'-y\|.\end{aligned}$$

This guarantees that $y$ is a nearest point in $S$ of $x' \in X\backslash S$, so $y \in P_S(X\backslash S)$ as desired. □

The problem of possible convexity of a Chebyshev set and the question if a set with the unique farthest point property is itself a singleton are not solved in a Hilbert setting (see Problems 5 and 6 in [23]). In order to show that the question related to the unique farthest point property is a question of possible convexity of Chebyshev sets, we restrict ourselves to the Hilbert space setting. So, through the rest of this section we shall assume that $X$ is a Hilbert space, although the subsequent result can be obtained in a more general set up.

By the main result in the previous section (Theorem 2), we may consider only the case when the set with the unique farthest point property is convex and closed.

**Theorem 5** *Let $S$ be a closed convex subset of a (real) Hilbert space $(X, \|\cdot\|)$ such that for every $x \in X$ the set*

$$Q_S(x) = \{s \in S : \|s-x\| = \sup_{s'\in S}\|s'-x\|\}$$

*is a singleton. If $S$ is not a singleton, then there exists a nonempty bounded convex open set $U \subset X$ such that*

$$d(u, X\backslash U) + \Delta_S(u) = 1 + \inf_{x\in X}\Delta_S(x), \quad \forall u \in U$$

*and the set $W := X\backslash U$ is a Chebyshev set, that is, for every $u \in U$ there exists exactly one $w \in W$ such that*

$$d(u, W) = \|u - w\|.$$

*Proof* Theorem 4 with $f$ as the indicator function of $S$ (so, $\kappa_{1,1} f = \Delta_S$) asserts that for

$$W = \{w \in X : \Delta_S(w) \geq 1 + \inf_{x\in X}\Delta_S(x)\} \quad \text{and} \quad U := X\backslash W,$$

we have

$$d(u, W) + \Delta_S(u) = 1 + \inf_{x\in X}\Delta_S(x), \quad \forall u \in U. \tag{41}$$

So, we only need to show that $W$ is a Chebyshev set. Fix any $u \in U$ and take $s = Q_S(u)$, that is, (because $S$ is not a singleton)

$$\forall q \in S, q \neq s,\ \Delta_S(u) = \sup_{s'\in S}\|s'-u\| = \|u-s\| > \|u-q\|,$$

and this allows us to say that $u \neq s$. Since $U$ is an open bounded convex set with $u \in U$, clearly $(u + \text{cone}(u - s)) \cap \text{bdry}\, U \neq \emptyset$, hence we can take $w$ in this intersection, or equivalently $w \in W \cap \text{cl}U \cap \{u + \text{cone}(u - s)\}$. Observe that

$$\|w - u\| \geq \Delta_S(w) - \Delta_S(u) \geq \|w - s\| - \|u - s\| = \|w - u\|,$$

and hence $\Delta_S(w) - \Delta_S(u) = \|w - u\|$. On the other hand, since $w \in \text{bdry}\, U$ we also have $\Delta_S(w) = 1 + \inf_{x \in X} \Delta_S(x)$, so (41) gives $\Delta_S(w) = d(u, W) + \Delta_S(u)$. It results that

$$\Delta_S(w) - \Delta_S(u) = \|w - u\| \geq d(u, W) = \Delta_S(w) - \Delta_S(u),$$

thus

$$d(u, W) = \|w - u\|.$$

It remains to show that $w$ is the unique nearest point in $W$ of $u$. Suppose that there exists $w_1 \in W$, with $w_1 \neq w$, and $d(u, W) = \|u - w_1\|$. Note that $w_1 \in W \cap \text{cl}\, U$. Take any $x \in ]u, w_1[$. It follows from Proposition 4.1 and Theorem 4.2 in [25] (see also Theorem 4.1 in [40]) that the function $d(\cdot, W)$ is Fréchet differentiable at $x$, with derivative $Dd(\cdot, W)(x) = \frac{u - w_1}{\|u - w_1\|}$. Since $]u, w_1[\subset U$, by (41) the function $\Delta_S = \kappa_{1,1} f$ is also Fréchet differentiable at $x$, with derivative

$$D\Delta_S(x) = -\frac{u - w_1}{\|u - w_1\|}, \quad \forall x \in ]u, w_1[.$$

Using the closedness of the graph of the set-valued mapping $z \mapsto \partial\Delta_S(z)$, we get $\frac{w_1 - u}{\|u - w_1\|} \in \partial\Delta_S(u)$. Then, for every $n \in \mathbb{N}$, we have

$$\Delta_S(u + n^{-1}(w_1 - u)) - \Delta_S(u) \geq \left\langle \frac{w_1 - u}{\|w_1 - u\|}, n^{-1}(w_1 - u) \right\rangle = n^{-1}\|w_1 - u\|. \quad (42)$$

Put $b_n = u + n^{-1}(w_1 - u)$ and $b_n^* := \big(b_n - Q_S(b_n)\big)/\|b_n - Q_S(b_n)\|$. Then $b_n^* \in \partial\Delta_S(b_n)$ by Proposition 3, and hence by (42)

$$\langle b_n^*, b_n - u\rangle \geq \Delta_S(b_n) - \Delta_S(u) \geq \frac{1}{n}\|w_1 - u\|,$$

which entails

$$\langle b_n^*, w_1 - u\rangle \geq \|w_1 - u\|.$$

Since $X$ is a Hilbert space, it results that $b_n^* = \frac{w_1 - u}{\|w_1 - u\|}$ or equivalently $Q_S(b_n) = b_n + \frac{\Delta_S(b_n)}{d(u,W)}(u - w_1)$, because $\Delta_S(b_n) = \|b_n - Q_S(b_n)\|$ and $\|w_1 - u\| = d(u, W)$. Now using the continuity of $\Delta_S$ and the fact that $\lim_{n \to +\infty} b_n = u$, we deduce

$$\lim_{n \to +\infty} Q_S(u + n^{-1}(w_1 - u)) = u + \frac{\Delta_S(u)}{d(u, W)}(u - w_1).$$

Since $Q_S(u)$ is a singleton and $S$ is closed, we get

$$u + \frac{\Delta_S(u)}{d(u, W)}(u - w) = Q_S(u) = u + \frac{\Delta_S(u)}{d(u, W)}(u - w_1),$$

where the first equality is due to the definition of $w$ [i.e., $w - u = \gamma(u - Q_S(u))$] and $\frac{\gamma\, \Delta_S(u)}{d(u,W)} = 1$, a contradiction since $w \neq w_1$. □

*Remark 9* It is an obvious observation from Theorem 4 that if $S$ is a singleton, then any subset $W$ of $X$ satisfying (37) for $\kappa_{\lambda,1} f(x) := \Delta_S(x)$ can not be a Chebyshev set. □

*Remark 10* If the set $W$ constructed in Theorem 5 is convex then both functions $d(\cdot, W)$ and $\Delta_S(\cdot)$ are Fréchet differentiable on $U$, so $S$ is singleton. □

Using the fact that locally compact Chebyshev sets are convex, the following well-known result is a direct consequence of Theorem 5.

**Corollary 8** *Let $S$ be a closed convex subset of the Euclidean space $\mathbb{R}^n$ such that for every $x \in X$ the set*

$$Q_S(x) = \left\{ s \in S : \|s - x\| = \sup_{s' \in S'} \|s' - x\| \right\}$$

*is a singleton. Then $S$ is a singleton.*

**Acknowledgments** We thank both referees for their comments which allowed us to improve the presentation of the paper.

## References

1. Asplund, E.: Farthest points in reflexive locally uniformly rotund Banach spaces. Israel J. Math. **4**, 213–216 (1966)
2. Asplund, E.: Sets with unique farthest points. Israel J. Math. **5**, 201–209 (1967)
3. Bandyopadhyay, P.: The Mazur Intersection property for familes of closed bounded convex sets in Banach spaces. Colloq. Math. **63**, 45–56 (1992)
4. Bandyopadhyay, P., Dutta, S.: Farthest points and the farthest distance map. Bull. Aust. Math. Soc. **71**, 425–433 (2005)
5. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. CMS Books in Mathematics. Springer, Berlin (2011)
6. Bauschke, H.H., Macklem, S., Wang, X.: Chebyshev Sets, Klee Sets, and Chebyshev Centers with Respect to Bregman Distances: Recent Results and Open Problems, 1–21 in Fixed-Point Algorithms for Inverse Problems in Sciences and Engineering. Springer Optimization and Its Applications. Springer, New York (2011)
7. Borwein, J., Fitzpatrick, S.: Existence of nearest points in Banach spaces. Can. J. Math. **4**, 702–720 (1989)
8. Chen, Dongjian, Lin, Bor-Luh: Ball separation properties in Banach spaces. Rocky Mt. J. Math. **28**, 835–873 (1998)
9. Cibulka, R., Fabian, M.: Attainment and (sub) differentiability of the supremal convolution of a function and square of the norm. J. Math. Anal. Appl. **393**, 632–643 (2012)
10. Deutsch, F.: Best Approximation in Inner Product Spaces. Springer, Berlin (2001)

11. Deville, R., Godefroy, G., Zizler, V.: Smoothness and renormings in Banach spaces. Longman scientific and technical (1993)
12. Deville, R., Zizler, V.: Farthest points in $w^*$-compact sets. Bull. Aust. Math. Soc. **38**, 433–439 (1985)
13. Diestel, J.: Geometry of Banach Spaces-Selected Topics. Springer, Berlin (1975)
14. Dutta, S.: Generalized subdifferential of the distance function. Proc. Am. Math. Soc. **133**, 2949–2955 (2005)
15. Edelstein, M.: Farthest points in uniformly convex Banach spaces. Israel J. Math. **4**, 171–176 (1966)
16. Edelstein, M., Lewis, J.: On exposed and farthest points in normed linear spaces. J. Aust. Math. Soc. **12**, 301–308 (1971)
17. Fabian, M., Habala, P., Hájek, P., Montesinos, V., Zizler, V.: Banach Space Theory. CMS Books in Mathrmatics. Springer, New York (2011)
18. Fan, K., Glicksberg, I.: Fully convex normed linear spaces. Proc. Natl. Acad. Sci. USA **41**, 947–953 (1955)
19. Fan, K., Glicksberg, I.: Some geometric properties of the sphere in a normed linear space. Duke Math. J. **25**, 553–568 (1958)
20. Fitzpatrick, S.: Metric projections and the differentiability of distance functions. Bull. Aust. Math. Soc. **22**, 291–312 (1980)
21. Goebel, K., Schöneberg, R.: Moons, bridges, birds $\cdots$ and nonexpansive mappings in Hilbert space. Bull. Aust. Math. Soc. **17**, 463–466 (1977)
22. Hiriart-Urruty, J.-B.: La conjecture des points les plus éloignés revisitée. Annales des Sciences Mathématiques du Québec **29**, 197–214 (2005)
23. Hiriart-Urruty, J.-B.: Potpourri of conjectures and open questions in nonlinear analysis and optimization. SIAM Rev. **49**, 255–273 (2007)
24. Holmes, R.B.: Geometrical Functional Analysis and Applications. Springer, New York (1975)
25. Jourani, A., Thibault, L., Zagrodny, D.: Differential properties of the Moreau envelope. J. Funct. Anal. **266**, 1185–1237 (2014)
26. Klee, V.: Convexity of Chebyshev sets. Math. Ann. **142**, 292–304 (1961)
27. Lau, K.S.: Farthest points in weakly compact sets. Israel J. Math. **22**, 168–176 (1975)
28. Mazur, S.: Über schwache Konvergenz in den Raümen ($L^P$). Stud. Math. **4**, 128–133 (1933)
29. Montesinos, V., Zizler, P., Zizler, V.: Some remarks on farthest points. Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales, Serie A, Matematicas **105**, 119–131 (2011)
30. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation. I: Basic Theory, Grundlehren Series (Fundamental Principles of Mathematical Sciences), vol. 330, p. 584. Springer, Berlin (2006)
31. Moreau, J.J.: Proximité et dualité dans un espace hilbertien. Bull. Soc. Math. France **93**, 273–299 (1965)
32. Moreau, J.J.: Inf-convolution, sous-additivité, convexité des fonctions numériques. J. Math. Pures Appl. **49**, 109–154 (1970)
33. Moreau, J.J.: Fonctionnelles Convexes, Collège de France, 1966, 2nd edn. Tor Vergata University, Roma (2003)
34. Motzkin, T.S., Straus, E.G., Valentine, F.A.: The number of farthest points. Pac. J. Math. **3**, 221–232 (1953)
35. Polak, T., Sims, B.: A Banach space which is fully $2-$rotund but not locally uniformly rotund. Can. Math. Bull. **26**, 118–120 (1983)
36. Schaefer, H.H.: Topological Vector Spaces. Springe, New York (1971). Third Printing Corrected
37. Schirotzek, W.: Nonsmooth Analysis. Springer, Berlin (2007)
38. Wang, X.: On Chebyshev functions and Klee functions. J. Math. Anal. Appl. **368**, 293–310 (2010)
39. Whestphal, U., Schwartz, T.: Farthest points and monotone operators. Bull. Autral. Math. Soc. **58**, 75–92 (1998)
40. Wu, Z., Ye, J.J.: Equivalence among various derivatives and subdifferentials of the distance function. J. Math. Anal. Appl. **282**, 629–647 (2003)
41. Zagrodny, D.: The cancellation law for inf-convolution of convex functions. Stud. Math. **110**, 271–282 (1994)
42. Zagrodny, D.: On the best approximation in real Hilbert spaces. Set-Valued Var. Anal. (submitted)
43. Zizler, V.: On some extremal problems in Banach spaces. Math. Scand. **32**, 214–224 (1973)