SPECTRAL (ISOTROPIC) MANIFOLDS AND THEIR DIMENSION

By

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Abstract. A set of $n \times n$ symmetric matrices whose ordered vector of eigenvalues belongs to a fixed set in $\mathbf{R}^n$ is called spectral or isotropic. In this paper, we establish that every locally symmetric C^k submanifold $\mathcal{M}$ of $\mathbf{R}^n$ gives rise to a C^k spectral manifold for $k \in \{2, 3, \ldots, \infty, \omega\}$. An explicit formula for the dimension of the spectral manifold in terms of the dimension and the intrinsic properties of $\mathcal{M}$ is derived. This work builds upon the results of Sylvester and Šilhavý and uses characteristic properties of locally symmetric submanifolds established in recent works by the authors.

1 Introduction

Let $\mathbf{R}^n_{\geq} = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_1 \geq x_2 \geq \cdots \geq x_n\}$. Denoting by $\mathbf{S}^n$ the euclidean space of $n \times n$ symmetric matrices with inner product $\langle X, Y \rangle = \mathrm{tr}\,(XY)$, we consider the **spectral mapping** λ, i.e., the function from $\mathbf{S}^n$ to $\mathbf{R}^n$ which associates to $X \in \mathbf{S}^n$ the vector $\lambda(X) \in \mathbf{R}^n_{\geq}$ of its eigenvalues; i.e., for $X \in \mathbf{S}^n$, $\lambda(X) = (\lambda_1(X), \ldots, \lambda_n(X))$, where $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$ are the eigenvalues of X counted with multiplicity. The objects of study in this paper are the spectral sets of $\mathbf{S}^n$. A subset $M \subset \mathbf{S}^n$ is a **spectral set** if $U^\top X U \in M$ for all $X \in M$ and $U \in \mathbf{O}^n$, where $\mathbf{O}^n$ is the group of $n \times n$ orthogonal matrices. In other words, if a matrix X lies in a spectral set $M \subset \mathbf{S}^n$, then so does its orbit under the natural action of the group $\mathbf{O}^n$

$$\mathbf{O}^n.X := \{U^\top X U : U \in \mathbf{O}^n\}.$$

The spectral sets are determined entirely by their eigenvalues and can be defined equivalently by

$$\lambda^{-1}(M) := \{X \in \mathbf{S}^n : \lambda(X) \in M\} \quad \text{for some } M \subset \mathbf{R}^n.$$

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For example, if M is the euclidean unit ball $B(0,1)$ of $\mathbf{R}^n$, then $\lambda^{-1}(M)$ is the euclidean unit ball of $\mathbf{S}^n$ as well. A spectral set can be written as a union of orbits

$$\lambda^{-1}(M) = \bigcup_{x \in M} \mathbf{O}^n.\mathrm{Diag}(x), \tag{1.1}$$

where $\mathrm{Diag}(x)$ denotes the diagonal matrix with the vector $x \in \mathbf{R}^n$ on the main diagonal. Notice that each orbit is an analytic submanifold of $\mathbf{S}^n$; see Example 2.7 for details.

In this context, a general question arises. Which properties of M remain true for the corresponding spectral set $\lambda^{-1}(M)$? In the sequel, we say that a property obeys the **transfer principle** if it holds for $\lambda^{-1}(M)$ whenever it holds for M. The spectral mapping λ has nice geometrical properties, but it may behave very badly as far as, for example, differentiability is concerned. This imposes intrinsic difficulties for the formulation of a generic transfer principle.

Invariance properties of M under permutations often correct such bad behavior and allow us to deduce transfer properties between the sets M and $\lambda^{-1}(M)$. A permutation σ on n elements acts on $\mathbf{R}^n$ by $\sigma(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots x_{\sigma(n)})$. A set $M \subset \mathbf{R}^n$ is **symmetric** if $\sigma M = M$ for all σ. Thus, if $M \subset \mathbf{R}^n$ is symmetric, properties such as closedness and convexity are transferred between M and $\lambda^{-1}(M)$; namely, M is closed (respectively, convex [14], prox-regular [5]) if and only if $\lambda^{-1}(M)$ is closed (respectively, convex, prox-regular). The next result is another interesting example of such a transfer.

Proposition 1.1 (Transferring algebraicity). *Let $\mathcal{M} \subset \mathbf{R}^n$ be a symmetric algebraic variety. Then, $\lambda^{-1}(\mathcal{M})$ is an algebraic variety of $\mathbf{S}^n$.*

Proof. Let p be a polynomial equation of $\mathcal{M}$, i.e., $p(x) = 0$ if and only if $x \in \mathcal{M}$. Define the symmetric polynomial $q(x) := \sum_\sigma p^2(\sigma x)$. Observe that q is again a polynomial equation of $\mathcal{M}$ and $q(\lambda(X))$ is an equation of $\lambda^{-1}(\mathcal{M})$. We just have to prove that $q \circ \lambda$ is a polynomial in the entries of X. It is known that q can be written as a polynomial of the elementary symmetric polynomials $p_1, p_2, \dots, p_n$. Each $p_j(\lambda(X))$, up to a sign, is a coefficient of the characteristic polynomial of X and thus is a polynomial in X. □

Concurrently, similar transfer properties hold for **spectral functions**, functions $F : \mathbf{S}^n \to \mathbf{R}^n$ which are constant on the orbits $\mathbf{O}^n.X$ or, equivalently, functions F that can be written as $F = f \circ \lambda$ with $f : \mathbf{R}^n \to \mathbf{R}$ symmetric, i.e., invariant under any permutation of entries of x. Since f is symmetric, closedness and convexity are transferred between f and F; see [14] for details. More surprisingly, differentiability properties are also transferred; see [1], [20], [19], [13], [15], and

[18]. As established recently in [5], the same occurs for the variational property of prox-regularity, for the definition of which we refer to [17].

In this work, we study the transfer of differentiable structure of a submanifold $\mathcal{M}$ of $\mathbf{R}^n$ to the corresponding spectral set. This gives rise to an orbit-closed set $\lambda^{-1}(\mathcal{M})$ of $\mathbf{S}^n$, which, in case it is a manifold, is called the **spectral manifold**. Such spectral manifolds often appear in engineering sciences, often as constraints in feasibility problems (e.g., in the design of tight frames [21] in image processing or in the design of low-rank controllers [16] in control). However, given a manifold $\mathcal{M}$, the answer to the question of whether the spectral set $\lambda^{-1}(\mathcal{M})$ is a manifold of $\mathbf{S}^n$ is not immediate. Indeed, a careful glance at (1.1) reveals that $\mathbf{O}^n.\mathrm{Diag}(x)$ has a natural (quotient) manifold structure; the question there is how the different strata combine as x moves along $\mathcal{M}$.

For functions, transferring local properties such as differentiability requires some symmetry, albeit not with respect to all permutations. Many properties still hold under **local symmetry**, i.e., invariance under permutations that preserve balls centered at the point of interest. We make this notion more precise in Subsection 2.1.

The main goal here is to prove that local smoothness of $\mathcal{M}$ is transferred to the spectral set $\lambda^{-1}(\mathcal{M})$ whenever $\mathcal{M}$ is locally symmetric. More precisely, our aims here are

- to prove that every connected C^k locally symmetric manifold $\mathcal{M}$ of $\mathbf{R}^n$ is "lifted" to a connected C^k manifold $\lambda^{-1}(\mathcal{M})$ of $\mathbf{S}^n$, for $k \in \{2, 3, \ldots, \infty, \omega\}$ (where C^ω stands for real analytic);
- to derive a formula for the dimension of $\lambda^{-1}(\mathcal{M})$ in terms of the dimension of $\mathcal{M}$ and some characteristic properties of $\mathcal{M}$.

We achieved these aims in 2009 for the cases $k = 2$, $k = \infty$, and $k = \omega$ through a long technical proof in the unpublished technical note [6]. Here, we provide a shorter, tractable version of the aforementioned proof, which moreover encompasses all cases $k \in \{2, 3, \ldots, \infty, \omega\}$. Notation and arguments have been simplified, and additional comments providing extra intuition have been added. We use the results from [20] and [19] and the properties of locally symmetric submanifolds of [7].

The particular case of the lift of a C^∞ manifold is recovered in a very recent work [4] through an indirect technique based on metric projections. However, this technique is specific for the case $k = \infty$ and does not provide any information on the dimension of the spectral manifold $\lambda^{-1}(\mathcal{M})$.

The main result of the current manuscript is Theorem 3.20, which proves that the lift of a locally symmetric C^k submanifold of $\mathbf{R}^n$ is a C^k manifold of $\mathbf{S}^n$, and provides a formula for its dimension. We obtain this result using extensively differential properties of spectral functions as well as structural properties of locally symmetric manifolds. Roughly speaking, given a manifold $\mathcal{M}$ which is locally symmetric around $x \in \mathcal{M}$, the proof splits in the following two steps:

Step 1. exhibiting a simple locally symmetric affine manifold $\mathcal{D}$ (see (3.8)) which will be used as the domain for a locally symmetric local equation for the manifold $\mathcal{M}$ around x (Lemma 3.10);

Step 2. showing that $\lambda^{-1}(\mathcal{D})$ is an analytic manifold (Theorem 3.15) and using it as a domain to build a local equation of $\lambda^{-1}(\mathcal{M})$ (cf. (3.12)), in order to establish that this latter spectral set is a manifold (Theorem 3.20). Let us remark, however, that $\lambda^{-1}(\mathcal{D})$ is not an affine manifold in general; see comments at the end of Section 3.2.

2 Locally symmetric functions and manifolds

This section does not contain any new results, but introduces relevant background notation and revises material established in [7] (and previously in [6], though in a less elaborated form) concerning the structure of a locally symmetric submanifold $\mathcal{M}$ of $\mathbf{R}^n$. A key notion is that of a characteristic partition (see Section 2.3), as well as the existence of a locally symmetric reduced tangential parametrization (Theorem 2.17).

2.1 Notation and definitions. A **partition** P of a finite set N is a collection of non-empty, pairwise disjoint subsets of N whose union is N. The elements of a partition are sometimes called **blocks**. The partition $\{\{i\} : i \in N\}$ is denoted by id_N. The set of all partitions of N is denoted by Π_N. The symbol $\mathbf{R}^N$ denotes the set of all functions from N to $\mathbf{R}$. Set $\mathbb{N}_n := \{1, \dots, n\}$. When $N = \mathbb{N}_n$, we simply write Π_n, for Π_N, id_n for id_N, and $\mathbf{R}^n$ for $\mathbf{R}^N$. The partition induced by $x \in \mathbf{R}^N$, denoted by P_x, is defined by the indexes of the equal coordinates of x. More precisely, for $i, j \in N$, i, j are in the same subset of P_x if and only if $x_i = x_j$. Given two partitions P and P' of Π_N, we say that P' is a **refinement** of P, written $P \preceq P'$, if every set in P is a (disjoint) union of sets from P'. Given a partition P of Π_N, define the subset Δ_P of $\mathbf{R}^N$ by

$$\Delta_P := \{x \in \mathbf{R}^N : \; P_x = P\}. \tag{2.1}$$

Obviously, Δ_P is an affine manifold, which is not connected in general. (By affine manifold, we mean an open subset of an affine subspace of a vector space.) The collection $\{\Delta_P : P \in \Pi_N\}$ is an affine stratification of $\mathbf{R}^N$ i.e., a finite decomposition of $\mathbf{R}^N$ into affine manifolds (strata) that fit together in a regular way. For each $x \in \mathbf{R}^N$, there exists a $\delta > 0$ such that the ball $B(x, \delta)$ intersects only strata Δ_P with $P \succeq P_x$; see [7, Section 2.2]. If the partition $P \in \Pi_N$ is given by $P = \{I_1, \ldots, I_m\}$, then the orthogonal and bi-orthogonal spaces of Δ_P have the expressions

$$\Delta_P^{\perp} = \Big\{ x \in \mathbf{R}^N : \sum_{j \in I_i} x_j = 0 \text{ for all } i \in \mathbb{N}_m \Big\}, \tag{2.2}$$

$$\Delta_P^{\perp\perp} = \{x \in \mathbf{R}^N : x_i = x_j \text{ for all } i, j \in I_k,\ k \in \mathbb{N}_m\}, \tag{2.3}$$

respectively. Note that $\Delta_P^{\perp\perp} = \overline{\Delta_P}$, where the latter set is the closure of Δ_P. Also, $\Delta_P^{\perp\perp} = \bigcup_{P' \preceq P} \Delta_{P'}$.

The group Σ^n of permutations over $\mathbb{N}_n$ has a natural action on $\mathbf{R}^n$ and Π_n defined for $x \in \mathbf{R}^n$ by $\sigma x := (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$, and for a partition $P = \{I_1, \ldots, I_m\}$ by $\sigma P := \big\{\{\sigma(i) : i \in I_k\} : k = 1, \ldots, m\big\}$. For a vector $x \in \mathbf{R}^n$ and a partition $P \in \Pi_n$, define the subgroups of permutations

$$\Sigma_P^n := \{\sigma \in \Sigma^n : \sigma P = P\} \quad \text{and} \quad \Sigma_x^n := \{\sigma \in \Sigma^n : \sigma x = x\}.$$

Note that $\Sigma_{P_x}^n = \Sigma_x^n$ for all $x \in \mathbf{R}^n$.

Definition 2.1. A map f defined on $\mathbf{R}^n$ is called **symmetric** if $f(\sigma y) = f(y)$ for all $y \in \mathbf{R}^n$ and all $\sigma \in \Sigma^n$. The function f is called **locally symmetric** at $x \in \mathbf{R}^n$ if there exists an open ball B centered at x and

$$f(\sigma y) = f(y) \quad \text{for all } y \in B \text{ and all } \sigma \in \Sigma_x^n.$$

Locally symmetric functions are those which are symmetric on an open ball centered at x under all permutations of entries of x that preserve this ball; see [7, Section 3.1]. The above property is exactly the invariance property required on f which allows the transfer of its differentiability properties to the spectral function $f \circ \lambda$; see Theorem 2.2, below. For a proof, we refer to [20] and [22]. (Note that although the main result in [20] is stated for symmetric functions f, the supporting results are stated in locally symmetric language, and the argument remains unchanged in this case.) In the sequel, given a vector $x \in \mathbf{R}^n$, $\operatorname{Diag} x$ denotes the diagonal matrix with the vector x on the main diagonal, and $\operatorname{diag}: \mathbf{S}^n \to \mathbf{R}^n$ denotes its adjoint operator, defined by $\operatorname{diag}(X) := (x_{11}, \ldots, x_{nn})$ for any matrix $X = (x_{ij})_{i,j} \in \mathbf{S}^n$.

Theorem 2.2 (Derivatives of spectral functions)**.** *Let $f : \mathbf{R}^n \to \mathbf{R}$ be locally symmetric at $x \in \mathbf{R}^n_{\geq}$, and $k \in \{1, 2, \ldots, \infty, \omega\}$. The function $F : \mathbf{S}^n \to \mathbf{R}$ defined* *by* $F = f \circ \lambda$ *is* C^k *in a neighborhood of* $X \in \lambda^{-1}(x)$ *if and only if* f *is* C^k *in a neighborhood of* x*. In that case,*

$$\nabla F(X) = U^\top (\operatorname{Diag} \nabla f(\lambda(X)))U,$$

where U *is an orthogonal matrix such that* $X = U^\top (\operatorname{Diag} \lambda(X))U$*. Equivalently,*

$$\nabla F(X)[H] = \nabla f(\lambda(X)))[\operatorname{diag}(UHU^\top)] \tag{2.4}$$

for every direction $H \in \mathbf{S}^n$*.*

The differentiability of spectral functions will be intensively used in the sequel. Before giving the definition of spectral manifolds and locally symmetric manifolds, let us first recall the definition of a submanifold.

Definition 2.3. A nonempty set $\mathcal{M} \subset \mathbf{R}^n$ is a C^k **submanifold** of dimension d (with $d \in \{0, \dots, n\}$ and $k \in \mathbb{N} \cup \{\omega\}$) if for every $x \in \mathcal{M}$, there exist a neighborhood $U \subset \mathbf{R}^n$ of x and a C^k function $\varphi\colon U \to \mathbf{R}^{n-d}$ with Jacobian matrix $J\varphi(x)$ of full rank such that for all $y \in U$, $y \in \mathcal{M}$ if and only if $\varphi(y) = 0$. The map φ is called a **local equation of** $\mathcal{M}$ **around** x.

Definition 2.4. A set $S \subseteq \mathbf{R}^n$ is **strongly locally symmetric** if $\sigma S = S$ for all $x \in S$ and all $\sigma \in \Sigma^n_x$. The set S is **locally symmetric** if for every $x \in S$, there exists $\delta > 0$ such that $S \cap B(x, \delta)$ is a strongly locally symmetric set.

In other words, S is locally symmetric if for every $x \in S$, there exists $\delta > 0$ such that

$$\sigma(S \cap B(x, \delta)) = S \cap B(x, \delta) \text{ for all } y \in S \cap B(x, \delta) \text{ and } \sigma \in \Sigma^n_y. \tag{2.5}$$

Observe that if S satisfies (2.5), then for $\rho \leq \delta$, $S \cap B(x, \rho)$ is a strongly locally symmetric set as well.

Example 2.5. Obviously, $\mathbf{R}^n$ is (strongly locally) symmetric. It is also easily seen from the definition that any stratum Δ_P is a strongly locally symmetric affine manifold. If $x \in \Delta_P$ and the ball $B(x, \delta)$ is so small that it intersects only strata $\Delta_{P'}$ with $P' \succeq P$, then $B(x, \delta)$ is strongly locally symmetric.

2.2 Locally symmetric manifolds.

In this subsection, we recall the formal definition of a locally symmetric manifold (submanifold of $\mathbf{R}^n$) from [7] and illustrate this notion by means of characteristic examples.

Definition 2.6. A subset $\mathcal{M}$ of $\mathbf{R}^n$ is said to be a **(strongly) locally symmetric manifold** if $\mathcal{M}$ is a connected submanifold of $\mathbf{R}^n$ without boundary, a (strongly) locally symmetric set, and satisfies $\mathcal{M} \cap \mathbf{R}^n_{\geq} \neq \varnothing$.

The above definition includes the technical assumption $\mathcal{M} \cap \mathbf{R}^n_{\geq} \neq \varnothing$, since the entries of the eigenvalue vector $\lambda(X)$ are non-increasing (by definition of λ). This assumption is not restrictive, since we can always reorder the orthogonal basis of $\mathbf{R}^n$ to satisfy this property.

Our aim is to show that $\lambda^{-1}(\mathcal{M} \cap \mathbf{R}^n_{\geq})$ is a manifold, an objective ultimately realized in Section 3 by Theorem 3.20. First we sketch two simple approaches that could be adopted in order to prove this result and illustrate the difficulties that appear.

Consider the expression (1.1) for the spectral set $\lambda^{-1}(\mathcal{M})$. Although each orbit $\mathbf{O}^n.\mathrm{Diag}(x)$ is well known to be an analytic manifold (see Example 2.7 below), there is no straightforward approach for showing that the union (1.1) is also a smooth manifold. Our strategy, developed in Section 3, uses crucial properties of locally symmetric manifolds derived in [7, Section 5], namely, the existence of a locally symmetric tangential parametrization in an appropriately reduced ambient space. This is used to provide a locally symmetric local equation defined in a reduced ambient space (see Section 3.2), which, in turn, is used to exhibit an explicit smooth local equation for the spectral manifold $\lambda^{-1}(\mathcal{M})$; see Sections 3.3–3.4.

Example 2.7 (The case $\mathcal{M} = \{x\}$.)**.** Recall that the **stabilizer $\mathbf{O}^n_X$** of a matrix $X \in \mathbf{S}^n$ under the action of the orthogonal group $\mathbf{O}^n$ is defined by

$$\mathbf{O}^n_X := \{U \in \mathbf{O}^n : U^\top X U = X\}.$$

For $x \in \mathbf{R}^n_{\geq}$, we have an exact description of the stabilizer $\mathbf{O}^n_{\mathrm{Diag}(x)}$ of the matrix $\mathrm{Diag}(x)$. Indeed, with the partition $P_x = \{I_1, \ldots, I_m\}$, $U \in \mathbf{O}^n_{\mathrm{Diag}(x)}$ is a block-diagonal matrix made up of matrices $U_i \in \mathbf{O}^{|I_i|}$. Conversely, every such block-diagonal matrix clearly belongs to $\mathbf{O}^n_{\mathrm{Diag}(x)}$. In other words, we have the identification $\mathbf{O}^n_{\mathrm{Diag}(x)} \simeq \mathbf{O}^{|I_1|} \times \cdots \times \mathbf{O}^{|I_m|}$. Since $\mathbf{O}^p$ is a manifold of dimension $p(p-1)/2$, $\mathbf{O}^n_{\mathrm{Diag}(x)}$ is a manifold of dimension $\sum_{i=1}^m |I_i|(|I_i|-1)/2$. It is well-known that the orbit $\mathbf{O}^n.\mathrm{Diag}(x)$ is diffeomorphic to the quotient manifold $\mathbf{O}^n/\mathbf{O}^n_{\mathrm{Diag}(x)}$. Thus, $\mathbf{O}^n.\mathrm{Diag}(x)$ is a submanifold of $\mathbf{S}^n$ of dimension

$$\begin{aligned}\dim \mathbf{O}^n.\mathrm{Diag}(x) = \dim \mathbf{O}^n - \dim \mathbf{O}^n_{\mathrm{Diag}(x)} &= \frac{n(n-1)}{2} - \sum_{i=1}^m \frac{|I_i|(|I_i|-1)}{2} \\ &= \frac{n^2 - \sum_{i=1}^m |I_i|^2}{2} = \sum_{1 \leq i < j \leq m} |I_i||I_j|,\end{aligned}$$

where we have twice used the fact that $n = \sum_{i=1}^m |I_i|$.

Let us now explain how a natural approach to show that $\lambda^{-1}(\mathcal{M})$ is a manifold using local equations, would fail. Assume that the manifold $\mathcal{M}$ of dimension $d \in \{0, 1, \ldots, n\}$ is described by a smooth equation $\varphi\colon \mathbf{R}^n \to \mathbf{R}^{n-d}$ around the point $x \in \mathcal{M} \cap \mathbf{R}^n_{\geq}$. This gives a function $\varphi \circ \lambda$, whose zeros characterize $\lambda^{-1}(\mathcal{M})$ around $X \in \lambda^{-1}(\mathcal{M})$; i.e., for all $Y \in \mathbf{S}^n$ around X,

$$Y \in \lambda^{-1}(\mathcal{M}) \iff \lambda(Y) \in \mathcal{M} \iff \varphi(\lambda(Y)) = 0. \tag{2.6}$$

However, we cannot guarantee that $\Phi := \varphi \circ \lambda$ is smooth, unless φ is locally symmetric (in which case, Theorem 2.2 applies). However, as shown in the next easy example (cf. [7, Example 5.5] and also [6, Example 3.8]), local equations $\varphi\colon \mathbf{R}^n \to \mathbf{R}$ of a locally symmetric submanifold of $\mathbf{R}^n$ can fail to be locally symmetric.

Example 2.8 (A symmetric manifold without symmetric equations)**.** Consider the symmetric (affine) submanifold of $\mathbf{R}^2$ of dimension 1

$$\mathcal{M} = \{(x, y) \in \mathbf{R}^2 : x = y\} \;=\; \Delta((12)).$$

The associated spectral set

$$\lambda^{-1}(\mathcal{M}) = \{A \in \mathbf{S}^2 : \lambda_1(A) = \lambda_2(A)\} = \{\alpha I_2 :\ \alpha \in \mathbf{R}\}$$

is a submanifold of $\mathbf{S}^2$ around $I_2 = \lambda^{-1}(1, 1)$. However, although $\lambda^{-1}(\mathcal{M})$ is a (spectral) 1-dimensional submanifold of $\mathbf{S}^2$, it cannot be described by a local equation that is a composition of λ with a symmetric local equation $\varphi\colon \mathbf{R}^2 \to \mathbf{R}$ of $\mathcal{M}$ around $(1, 1)$. Indeed, assume to the contrary that such a local equation of $\mathcal{M}$ exists, i.e., there exists a smooth symmetric function $\varphi\colon \mathbf{R}^2 \to \mathbf{R}$ with surjective derivative $\nabla\varphi(1, 1)$ which satisfies $\varphi(x, y) = 0$ if and only if $x = y$. Consider now the two smooth paths $c_1 : t \mapsto (t, t)$ and $c_2 : t \mapsto (t, 2 - t)$. Since $\varphi \circ c_1(t) = 0$,

$$\nabla\varphi(1, 1)(1, 1) = 0. \tag{2.7}$$

On the other hand, since $c_2'(1) = (1, -1)$ is normal to $\mathcal{M}$ at $(1, 1)$ and since φ is symmetric, the smooth function $t \mapsto (\varphi \circ c_2)(t)$ has a critical point at $t = 1$. Thus

$$0 = (\varphi \circ c_2)'(1) = \nabla\varphi(1, 1)(1, -1). \tag{2.8}$$

But (2.7) and (2.8) imply that $\nabla\varphi(1, 1) = (0, 0)$, which is a contradiction.

We close this section by observing that the property of local symmetry introduced in Definition 2.4 is necessary and, in a sense, minimal. In any case, as revealed by the following examples, it cannot easily be relaxed.

Example 2.9 (A manifold without symmetry)**.** Consider the manifold $\mathcal{N} = \{(t, 0) : t \in (-1, 1)\} \subset \mathbf{R}^2$. We have an explicit expression for $\lambda^{-1}(\mathcal{N})$:

$$\lambda^{-1}(\mathcal{N}) = \left\{ \begin{bmatrix} t\cos^2\theta & t(\sin 2\theta)/2 \\ t(\sin 2\theta)/2 & t\sin^2\theta \end{bmatrix}, \begin{bmatrix} -t\sin^2\theta & t(\sin 2\theta)/2 \\ t(\sin 2\theta)/2 & -t\cos^2\theta \end{bmatrix} : t \geq 0 \right\}.$$

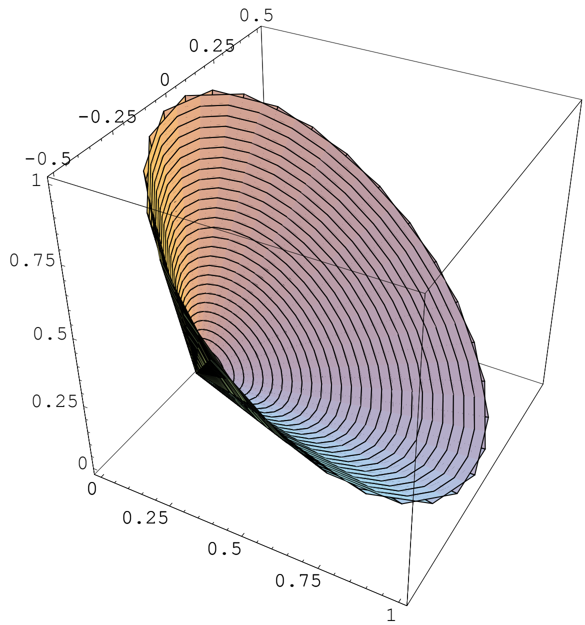

Figure 1. A spectral subset of $\mathbf{S}^2$ represented in $\mathbf{R}^3$.

It can be proved that this lifted set is not a submanifold of S^2, since it has a sharp point at the zero matrix, as suggested by Figure 1.

Example 2.10 (A manifold without enough symmetry)**.** Consider the set $\mathcal{N} = \{(t, 0, -t) : t \in (-\epsilon, \epsilon)\} \subset \mathbf{R}^3$, and let $x = (0, 0, 0) \in \mathcal{N}$. Then, $\Delta_{P_x} = \{(\alpha, \alpha, \alpha) : \alpha \in \mathbf{R}\}$, and $\mathcal{N}$ is a smooth submanifold of $\mathbf{R}^3$ which is symmetric with respect to the affine set Δ_{P_x}. However, it is not locally symmetric. Indeed, it can be easily seen that the set $\lambda^{-1}(\mathcal{N})$ is not a submanifold of $\mathbf{S}^3$ around the zero matrix.

2.3 Properties of locally symmetric manifolds. In this subsection, we collect definitions and results from [7] that are needed in the present work. Note first that if $x \in \mathcal{M} \cap \mathbf{R}^n_{\geq}$, then every set in P_x contains consecutive integers.

Definition 2.11. Consider partitions $P, P' \in \Pi_n$.

- The partition P' is said to be **much smaller** than P, written $P' \prec\prec P$, whenever $P' \prec P$ and there exists some set in P' which is formed by merging at least two sets from P, at least one of which contains at least two elements.
- When $P' \prec P$ but P' is not much smaller than P, we write $P' \prec\sim P$. In other words, if $P' \prec P$ but P' is not much smaller than P, then every set in P' that is not in P is formed by uniting one-element sets from P.

Suppose that $\mathcal{M}$ is a locally symmetric manifold. Among the partitions $P \in \Pi_n$ such that $\mathcal{M} \cap \mathbf{R}^n_{\geq} \cap \Delta_P \neq \varnothing$, there is a unique maximal partition P_* called the **characteristic partition** of $\mathcal{M}$. The characteristic partition describes the strata that may intersect $\mathcal{M}$:

$$\mathcal{M} \subseteq \Delta_{P_*} \cup \Big(\bigcup_{P \prec\sim P_*} \Delta_P \Big) \subseteq \Delta_{P_*}^{\perp\perp}. \tag{2.9}$$

Formula (2.9) implies that every set in P_* contains consecutive integers, and the tangent space $T_{\mathcal{M}}(x)$ of $\mathcal{M}$ at x satisfies

$$T_{\mathcal{M}}(x) \subset \Delta_{P_*}^{\perp\perp}. \tag{2.10}$$

The stratum Δ_{P_*} is dense in $\mathcal{M}$: for every $x \in \mathcal{M}$ and $\delta > 0$,

$$\mathcal{M} \cap \Delta_{P_*} \cap B(x, \delta) \neq \varnothing. \tag{2.11}$$

Define

$$\begin{aligned} \mathbb{N}_n^1 &:= \text{ the union of all sets in } P_* \text{ with exactly one element, and} \\ \mathbb{N}_n^2 &:= \text{ the union of all sets in } P_* \text{ with more than one elements.} \end{aligned}$$

Clearly, $\mathbb{N}_n$ is the disjoint union of $\mathbb{N}_n^1$ and $\mathbb{N}_n^2$. (Either $\mathbb{N}_n^1$ or $\mathbb{N}_n^2$ may possibly be empty.)

Definition 2.12. The characteristic partition P_* of $\mathcal{M}$ yields a **canonical split** of $\mathbf{R}^n$ as a direct sum of the spaces $\mathbf{R}^{\mathbb{N}_n^1}$ and $\mathbf{R}^{\mathbb{N}_n^2}$ as follows. Each vector $x \in \mathbf{R}^n$ is represented as

$$x = x^F \otimes x^M, \tag{2.12}$$

where

- $x^F \in \mathbf{R}^{\mathbb{N}_n^1}$ is the subvector of x obtained by collecting the coordinates that have indices in one-element sets of P_*, preserving their relative order;
- $x^M \in \mathbf{R}^{\mathbb{N}_n^2}$ is the subvector of x obtained by collecting the remaining coordinates, again preserving their order.

It is readily seen that the canonical split is both a linear and a reversible operation. Reversibility means that given $x^F \in \mathbf{R}^{\mathbb{N}_n^1}$ and $x^M \in \mathbf{R}^{\mathbb{N}_n^2}$, there exists a unique $x^F \otimes x^M \in \mathbf{R}^n$ such that

$$(x^F \otimes x^M)^F = x^F \quad \text{and} \quad (x^F \otimes x^M)^M = x^M.$$

In the particular case $P_* = \mathrm{id}_n$, $x = x^F$ for all $x \in \mathbf{R}^n$.

Definition 2.13. $P \in \Pi_n$ is called **P_*-decomposable** if $P \succeq P_\circ$ for some $P_\circ \prec\sim P_*$.

Note that a P_*-decomposable partition P has the property that if a set in P contains elements from $\mathbb{N}_n^1$, then it cannot contain elements from $\mathbb{N}_n^2$. According to (2.9), if $x \in \mathcal{M}$, P_x is P_*-decomposable; moreover, every $P \succeq P_x$ is P_*-decomposable.

Definition 2.14. For a P_*-decomposable partition P, define the partitions $P^F \in \Pi_{\mathbb{N}_n^1}$ and $P^M \in \Pi_{\mathbb{N}_n^2}$ as follows:

- P^F contains those sets of P that contain only elements from $\mathbb{N}_n^1$;
- P^M contains the remaining sets of P (those containing only elements from $\mathbb{N}_n^2$).

The disjoint union $P = P^F \cup P^M$ is called the **P_*-decomposition of** P.

For example, applying the P_*-decomposition to P_* yields $P_*^F = \mathrm{id}_{\mathbb{N}_n^1}$. Notice that the P_*-decomposition cannot be applied to partitions P that are much smaller than P_*, since these partitions may have sets containing elements from both $\mathbb{N}_n^1$ and $\mathbb{N}_n^2$. We now summarize the properties of P_* and the P_*-decomposition that are needed later.

Proposition 2.15. (i) *$P \prec\sim P_*$ if and only if $P^F \prec \mathrm{id}_{\mathbb{N}_n^1}$ and $P^M = P_*^M$;*
(ii) *If $x \in \mathcal{M}$ and $P_x \preceq P$, then $P_x^F \preceq P^F \preceq \mathrm{id}_{\mathbb{N}_n^1}$ and $P_*^M = P_x^M \preceq P^M$.*

If $P \in \Pi_n$ is P_*-decomposable, then the partitions $P^F \in \Pi_{\mathbb{N}_n^1}$ and $P^M \in \Pi_{\mathbb{N}_n^2}$ define strata in $\mathbf{R}^{\mathbb{N}_n^1}$ and $\mathbf{R}^{\mathbb{N}_n^2}$, respectively; see (2.1), (2.2), and (2.3). A glance at (2.2) and (2.3) reveals the following relations:

$$\Delta_P^{\perp\perp} = \Delta_{P^F}^{\perp\perp} \otimes \Delta_{P^M}^{\perp\perp} \quad \text{and} \quad \Delta_P^{\perp} = \Delta_{P^F}^{\perp} \otimes \Delta_{P^M}^{\perp}. \tag{2.13}$$

As above, for $x \in \mathcal{M}$, let $T_{\mathcal{M}}(x)$ be the tangent space of $\mathcal{M}$ at x, and let $N_{\mathcal{M}}(x)$ be the normal space of $\mathcal{M}$ at x. The local symmetry of $\mathcal{M}$ implies that these spaces are invariant under all permutations $\sigma \in \Sigma_x^n$. For all $x \in \mathcal{M}$,

$$T_{\mathcal{M}}(x) = (T_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp\perp}) \oplus (T_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp}), \text{ and} \tag{2.14}$$

$$N_{\mathcal{M}}(x) = (N_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp\perp}) \oplus (N_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp}). \tag{2.15}$$

It has been established in [7, Section 5.1] that for all $x \in \mathcal{M}$,

$$\text{if } w \in T_{\mathcal{M}}(x) \text{ then } w^M \in \Delta^{\perp\perp}_{P_x^M}; \text{ and} \tag{2.16}$$

$$\text{if } v \in N_{\mathcal{M}}(x), \text{ then } v^F \in \Delta^{\perp\perp}_{P_x^F}. \tag{2.17}$$

The following lemma complements the structural property (2.16).

Lemma 2.16 ([7, Lemma 6.1])**.** *For every $x \in \mathcal{M}$ and $\epsilon > 0$, there exists $w \in T_{\mathcal{M}}(x) \cap B(0, \epsilon)$ such that in the vector $w^F \in \mathbf{R}^{\mathbb{N}_n^1}$, every subvector w_I^F has distinct coordinates for every set I in the partition P_x^F.*

In the rest of this section, we briefly recall a local equation, the tangential parametrization, for a submanifold of $\mathbf{R}^n$, specialized to our context of a locally symmetric manifold $\mathcal{M}$. Let $\pi_T : \mathbf{R}^n \to T_{\mathcal{M}}(x)$ denote the orthogonal projection onto the tangent space at x, and let π_N be the orthogonal projection onto the normal space $N_{\mathcal{M}}(x)$. Let $\bar{\pi}_T : \mathbf{R}^n \to x + T_{\mathcal{M}}(x)$ be the projection onto the affine space $x + T_{\mathcal{M}}(x)$; and, similarly, let $\bar{\pi}_N$ denote the projection of $\mathbf{R}^n$ onto $x + N_{\mathcal{M}}(x)$. Note that for all $y \in \mathbf{R}^n$ sufficiently close to x,

$$\bar{\pi}_T(y) + \bar{\pi}_N(y) = x + y. \tag{2.18}$$

The local symmetry of $\mathcal{M}$ implies the existence of $\delta > 0$ such that

$$\sigma\bar{\pi}_T(y) = \bar{\pi}_T(\sigma y) \quad \text{and} \quad \sigma\bar{\pi}_N(y) = \bar{\pi}_N(\sigma y) \tag{2.19}$$

for all $y \in B(x, \delta)$ and all $\sigma \in \Sigma_x^n$. Choosing smaller $\delta > 0$, if necessary, we can ensure that the following conditions hold.

A1. The restriction $\bar{\pi}_T : \mathcal{M} \cap B(x, \delta) \to x + T_{\mathcal{M}}(x)$ is a diffeomorphism onto its image.

A2. The ball $B(x, \delta)$ intersects only strata Δ_P with $P \succeq P_x$.

Under the above conditions, $\mathcal{M} \cap B(x, \delta)$ is a strongly locally symmetric manifold; see (2.5). In addition, there exists a smooth map

$$\phi \colon (x + T_{\mathcal{M}}(x)) \cap B(x, \delta) \to N_{\mathcal{M}}(x) \tag{2.20}$$

such that

$$\mathcal{M} \cap B(x, \delta) = \{y + \phi(y) \in \mathbf{R}^n : y \in (x + T_{\mathcal{M}}(x)) \cap B(x, \delta)\}. \tag{2.21}$$

The map ϕ measures the difference between the manifold and its tangent space. Clearly, $\phi \equiv 0$ if $\mathcal{M}$ is an affine manifold around x. Note that, technically speaking, the domain of the map ϕ is the (strongly symmetric) open set $\bar{\pi}_T(\mathcal{M} \cap B(x, \delta))$, which may be a proper subset of $(x+T_{\mathcal{M}}(x))\cap B(x,\delta)$. To keep the paper readable, we do not introduce the more precise (but also more complicated notation) of a rectangular neighborhood around x.

We call the map $\psi\colon (x+T_{\mathcal{M}}(x))\cap B(x,\delta)\to \mathcal{M}\cap B(x,\delta)$ defined by

$$\psi(y) = y+\phi(y) \tag{2.22}$$

the **tangential parametrization of** $\mathcal{M}$ **around** x. This function is smooth, one-to-one and onto, with a full rank Jacobian matrix $J\psi(x)$: it is a local diffeomorphism at x; and its inverse is $\overline{\pi}_T$, i.e., locally $\overline{\pi}_T(\psi(y)) = y$.

We can now state the main result of [7], noting that the proof of (2.23) utilizes the fundamental relation (2.17) established in [7, Theorem 5.1].

Theorem 2.17 ([7, Theorem 5.1].)**.** *For $x\in\mathcal{M}$, the function ϕ in* (2.22) *satisfies*

$$\phi(x)\in N_{\mathcal{M}}(x)\cap \Delta_{P_x}^{\perp\perp}. \tag{2.23}$$

Moreover, for all $y\in(x+T_{\mathcal{M}}(x))\cap B(x,\delta)$ and for all $\sigma\in\Sigma_x^n$,

$$\psi(\sigma y) = \sigma\psi(y) \tag{2.24}$$

and

$$\phi(\sigma y) = \sigma\phi(y) = \phi(y). \tag{2.25}$$

3 Spectral manifolds

We begin this section with an example of the special case in which the (locally symmetric) manifold $\mathcal{M}$ is (a relatively open subset of) a stratum Δ_P. In this case, basic algebraic arguments allow us to conclude directly that $\lambda^{-1}(\mathcal{M})$ is a smooth manifold.

Example 3.1 (Lift of stratum Δ_P)**.** Suppose that a manifold $\mathcal{M}$ is (a relatively open subset of) a stratum Δ_P and intersects $\mathbf{R}^n_{\geq}$. We show directly that the spectral set $\lambda^{-1}(\mathcal{M})$ is an analytic (fiber) manifold, using basic arguments exposed in Example 2.7. As stated there, the orbit $\mathbf{O}^n_{\mathrm{Diag}(x)}$ is a submanifold of $\mathbf{S}^n$ of dimension $\sum_{1\leq i<j\leq m}|I_i||I_j|$, where $P=\{I_1,\dots,I_m\}$. The key is to observe that, in our case, for all $x\in\mathcal{M}$, $\mathbf{O}^n_{\mathrm{Diag}(x)}\simeq \mathbf{O}^{|I_1|}\times\cdots\times\mathbf{O}^{|I_m|}$ and $P_x=P$. Then, all the orbits $\mathbf{O}^n.\mathrm{Diag}(x)$ are manifolds diffeomorphic to $\mathbf{O}^n/\mathbf{O}^n_{\mathrm{Diag}(\bar{x})}$ (fibers), hence of the same dimension. We deduce that $\lambda^{-1}(\mathcal{M})$ is a submanifold of $\mathbf{S}^n$ diffeomorphic to the direct product $\mathcal{M}\times\big(\mathbf{O}^n/\mathbf{O}^n_{\mathrm{Diag}(\bar{x})}\big)$, with dimension

$$\dim\lambda^{-1}(\mathcal{M}) = d+\sum_{1\leq i<j\leq m}|I_i||I_j|. \tag{3.1}$$

The proof of the general situation ($\mathcal{M}$ an arbitrary locally symmetric manifold) is a (non-trivial) generalization of the above arguments. The strategy is more precisely explained in Section 3.2. First, in Section 3.1, we treat the special case $P_* = \mathrm{id}_n$. In this case, the proof that a locally symmetric manifold lifts smoothly in $\mathbf{S}^n$ goes through without extra technicalities, illustrating the main ideas. Moreover, the main result in the special case $P_* = \mathrm{id}_n$ is a needed step in the general case.

We require the following definition.

Definition 3.2 (Ordered partition)**.** A partition $P = \{I_1, \ldots, I_m\}$ of $\mathbb{N}_n$ is called **ordered** if whenever $1 \le i < j \le m$, the smallest element in I_i is (strictly) smaller than the smallest element in I_j.

For example, the ordered version of the partition $\{\{4\}, \{3, 2\}, \{1, 5\}\}$ of $\mathbb{N}_5$ is $\{\{1, 5\}, \{2, 3\}, \{4\}\}$. Consider an ordered partition $P = \{I_1, \ldots, I_m\}$ of $\mathbb{N}_n$. Consider the space $\mathbf{S}^n_P$ of all block-diagonal symmetric matrices in which the ℓ-th block is of size $|I_\ell|$, and denote by $\mathbf{O}^n_P$ the subgroup of block-diagonal orthogonal matrices in which the ℓ-th block is of size $|I_\ell|$. Denote by $X_P = \mathrm{Diag}(X_1, \ldots, X_m)$ an element of $\mathbf{S}^n_P$, where $X_\ell \in \mathbf{S}^{|I_\ell|}$. For $X_P \in \mathbf{S}^n_P$, we define

$$\lambda_P(X_P) := \big(\lambda(X_1), \ldots, \lambda(X_m)\big) \in \mathbf{R}^n.$$

Note the difference between $\lambda_P(X_P)$ and $\lambda(X_P)$: the coordinates of $\lambda_P(X_P)$ are ordered within each block, whereas those of $\lambda(X_P)$ are ordered globally. For technical reasons, we need a slight modification of Theorem 2.2 (Derivatives of spectral functions) to cover the case of spectral functions of the type $f \circ \lambda_P$ on $\mathbf{S}^n_P$.

Lemma 3.3. *Suppose that $f : \mathbf{R}^n \to \mathbf{R}$ is locally symmetric at $x \in \mathbf{R}^n_\geq$. For sufficiently small $\delta > 0$, the function $F : \mathbf{S}^n_{P_x} \to \mathbf{R}$ defined by $F = f \circ \lambda_{P_x}$ is C^k on $\lambda^{-1}_{P_x}(B(x, \delta))$ if and only if f is C^k on $B(x, \delta)$. The Jacobian of $f \circ \lambda_{P_x}$ at $X \in \lambda^{-1}_{P_x}(B(x, \delta))$ applied to $H \in \mathbf{S}^n_{P_x}$ is*

$$J(f \circ \lambda_{P_x})(X)[H] = J(f \circ \lambda)(X)[H].$$

Here $k \in \{1, 2, 3, \ldots, \infty, \omega\}$.

Proof. Let $P_x = \{I_1, \ldots, I_m\}$ and $X = \mathrm{Diag}(X_1, \ldots, X_m) \in \lambda^{-1}_{P_x}(B(x, \delta))$. Suppose $B(x, \delta)$ intersects only strata Δ_P with $P \succeq P_x$. The fact that $x \in \mathbf{R}^n_\geq$ implies that $\lambda_{\min}(X_\ell) > \lambda_{\max}(X_{\ell+1})$ for $1 \le \ell \le m - 1$. Hence $\lambda_{P_x}(X) = \lambda(X)$, and the claim follows from Theorem 2.2. $\square$

3.1 Lift into $\mathbf{S}^n$ (case $P_* = \mathrm{id}_n$). In this subsection, we consider the case $P_* = \mathrm{id}_n$. This condition implies that $P^F = P$ for any P_*-decomposable partition P, and $v^F = v$ for any $v \in \mathbf{R}^n$. Thus, Property (2.17) simplifies to

$$N_{\mathcal{M}}(x) \subseteq \Delta_{P_x}^{\perp\perp} \quad \text{for all } x \in \mathcal{M}. \tag{3.2}$$

The goal here is to establish, under conditions (A1)–(A2), that if $x \in \mathcal{M} \cap \mathbf{R}^n_{\geq}$, then $\lambda_{P_x}^{-1}(\mathcal{M} \cap B(x,\delta))$ is a submanifold of $\mathbf{S}^n_{P_x}$, and to calculate its dimension. This is an intermediate step on the way to proving that $\lambda^{-1}(\mathcal{M})$ is a submanifold of $\mathbf{S}^n$. The advantage of treating first the special case $P_* = \mathrm{id}_n$ is twofold: on the one hand, the results in this subsection are needed later; and, on the other, the succession of arguments in the general case is similar to the that for the special case.

Using (3.2), we can easily exhibit a locally symmetric local equation of $\mathcal{M}$. Thus, we fix $x \in \mathcal{M} \cap \mathbf{R}^n_{\geq}$ and recall the definitions of the projections $\overline{\pi}_T$ and $\overline{\pi}_N$.

Lemma 3.4. *If $P_* = \mathrm{id}_n$, then $\overline{\pi}_N$ is locally symmetric at x.*

Proof. Take $y \in B(x,\delta')$, where $\delta > \delta' > 0$ is so small that $B(x,\delta)$ intersects only strata Δ_P for $P \succeq P_x$. Without loss of generality, assume that there exists $z \in \mathcal{M} \cap B(x,\delta)$ such that $\overline{\pi}_N(y) = \overline{\pi}_N(z)$. The fact that $z \in \Delta_{P_z}^{\perp\perp}$ together with (3.2), applied to z, gives that $z + N_{\mathcal{M}}(z) \subseteq \Delta_{P_z}^{\perp\perp}$. Therefore, $\overline{\pi}_N(z) \in \Delta_{P_z}^{\perp\perp}$; and, consequently, for all $\sigma \in \Sigma^n_x$, $\overline{\pi}_N(\sigma y) = \sigma\overline{\pi}_N(y) = \sigma\overline{\pi}_N(z) = \overline{\pi}_N(z) = \overline{\pi}_N(y)$. This means that $\overline{\pi}_N$ is locally symmetric at x. ☐

Recall also the definition of ϕ given by (2.20) and the conditions on the ball $B(x,\delta)$ there. Define the function $\overline{\phi} : B(x,\delta) \to N_{\mathcal{M}}(x)$ by

$$y \mapsto x + \phi(\overline{\pi}_T(y)) - \overline{\pi}_N(y). \tag{3.3}$$

Lemma 3.5 (Existence of a locally symmetric local equation in the case $P_* = \mathrm{id}_n$)**.** *The function $\overline{\phi}$ defined by* (3.3) *is a local equation of $\mathcal{M}$ around $x \in \mathcal{M}$ which is locally symmetric. In other words,*

$$\overline{\phi}(\sigma y) = \sigma\overline{\phi}(y) = \overline{\phi}(y) \quad \textit{for all } y \in B(x,\delta) \textit{ and all } \sigma \in \Sigma^n_x.$$

Proof. Successive use of (2.18) and (2.21) gives, for $y \in B(x,\delta)$, the equivalence of following statements:

(i) $\overline{\phi}(y) = 0$;

(ii) $\overline{\pi}_N(y) = x + \phi(\overline{\pi}_T(y))$;

(iii) $y = \overline{\pi}_T(y) + \phi(\overline{\pi}_T(y))$;

(iv) $y \in \mathcal{M} \cap B(x,\delta)$.

The Jacobian mapping $J\overline{\phi}(y)$ is a linear map from $\mathbf{R}^n$ to $N_{\mathcal{M}}(x)$, which, when applied to any direction h, yields $J\overline{\phi}(y)[h] = J\phi(\overline{\pi}_T(y))[\pi_T(h)] - \pi_N(h)$. Clearly, $J\overline{\phi}(x)[h] = -h$ for $h \in N_{\mathcal{M}}(x)$, which shows that the Jacobian is onto and hence of full rank. Thus, $\overline{\phi}$ is a local equation of $\mathcal{M}$ around x. In view of Theorem 2.17 and the symmetries of the projections, $(\phi \circ \overline{\pi}_T)(\sigma y) = (\phi \circ \overline{\pi}_T)(y)$ for all $\sigma \in \Sigma^n_x$ and $y \in B(x, \delta)$. This implies that $\sigma^{-1}\overline{\phi}(\sigma y) = \sigma^{-1}(x + (\phi \circ \overline{\pi}_T)(y) - \sigma\overline{\pi}_N(y)) = \overline{\phi}(y)$. Since $\overline{\phi}(y) \in N_{\mathcal{M}}(x) \subset \Delta^{\perp\perp}_{P_x}$, we obtain the second claimed equality $\sigma\overline{\phi}(y) = \overline{\phi}(y)$. $\square$

Next, consider the map $\overline{\Phi} : \lambda^{-1}_{P_x}(B(x, \delta)) \to N_{\mathcal{M}}(x)$ defined by

$$X \mapsto (\overline{\phi} \circ \lambda_{P_x})(X) = x + \phi(\overline{\pi}_T(\lambda_{P_x}(X))) - \overline{\pi}_N(\lambda_{P_x}(X)). \tag{3.4}$$

Since $\overline{\phi}$ is a local equation of $\mathcal{M}$ around x, we deduce that for $X \in \mathbf{S}^n_{P_x}$,

$$X \in \lambda^{-1}_{P_x}(\mathcal{M} \cap B(x, \delta)) \iff \lambda_{P_x}(X) \in \mathcal{M} \cap B(x, \delta) \iff \overline{\Phi}(X) = 0. \tag{3.5}$$

Thus, in order to prove that $\overline{\Phi}$ is a local equation for $\lambda^{-1}_{P_x}(\mathcal{M} \cap B(x, \delta))$, it remains to establish that $\overline{\Phi}$ is C^k-differentiable and that its Jacobian $J\overline{\Phi}$ has full rank at $X \in \lambda^{-1}_{P_x}(x)$. This is accomplished in Theorem 3.7. First we need the following lemma.

Lemma 3.6. *The function $\overline{\pi}_N \circ \lambda_{P_x}$ is analytic on $\lambda^{-1}_{P_x}(B(x, \delta))$. Moreover, at each $X \in \lambda^{-1}_{P_x}(B(x, \delta))$ and every direction $H \in \mathbf{S}^n_{P_x}$,*

$$J(\overline{\pi}_N \circ \lambda_{P_x})(X)[H] = \pi_N(\operatorname{diag}(UHU^\top)),$$

where $U \in \mathbf{O}^n_{P_x}$ is such that $X = U^\top\big(\operatorname{Diag}\lambda_{P_x}(X)\big)U$.

Proof. By Lemma 3.4, $\overline{\pi}_N$ is locally symmetric at x. By Lemma 3.3, $\overline{\pi}_N \circ \lambda_{P_x}$ is analytic on $\lambda^{-1}_{P_x}(B(x, \delta))$. Its Jacobian at $X \in \lambda^{-1}_{P_x}(B(x, \delta))$ in the direction $H \in \mathbf{S}^n_{P_x}$ is

$$\begin{aligned} J(\overline{\pi}_N \circ \lambda_{P_x})(X)[H] &= J(\overline{\pi}_N \circ \lambda)(X)[H] = J\overline{\pi}_N(\lambda(X))[\operatorname{diag}(UHU^\top)] \\ &= \pi_N(\operatorname{diag}(UHU^\top)). \end{aligned}$$

The second equality follows by (2.4). $\square$

Theorem 3.7 (Main result (case $P_* = \mathrm{id}_n$))**.** *Let $\mathcal{M}$ be a locally symmetric C^k submanifold of $\mathbf{R}^n$ of dimension d. Suppose $P_* = \mathrm{id}_n$, fix $x \in \mathcal{M} \cap \mathbf{R}^n_{\geq}$ and let $\delta > 0$ be such that conditions* (A1)–(A2) *hold. Then $\lambda^{-1}(\mathcal{M} \cap B(x, \delta))$ is a C^k submanifold of $\mathbf{S}^n$ with codimension $n - d$. Here $k \in \{2, 3, \ldots, \infty, \omega\}$.*

Proof. By Theorem 2.17 and (2.19), the function $\phi \circ \bar{\pi}_T$ is locally symmetric at x. Therefore, Lemma 3.3 yields that $\phi \circ \bar{\pi}_T \circ \lambda_{P_x}$ is C^k on $\lambda_{P_x}^{-1}(B(x,\delta)) = \lambda^{-1}(B(x,\delta))$. Combining this with Lemma 3.6, we deduce that the function $\bar{\Phi}$ defined by (3.4) is C^k on $\lambda^{-1}(B(x,\delta))$.

Let us now show that the Jacobian $J\bar{\Phi}$ has full rank at $X \in \lambda^{-1}(B(x,\delta))$. First, the gradient of the i-th coordinate function $\phi_i \circ \bar{\pi}_T$ at x applied to the direction h is

$$\nabla(\phi_i \circ \bar{\pi}_T)(x)[h] = \nabla\phi_i(\bar{\pi}_T(x))[\pi_T(h)].$$

Second, Lemma 3.3 and Theorem 2.2 give that the gradient of $\phi_i \circ \bar{\pi}_T \circ \lambda$ at X in the direction $H \in \mathbf{S}^n$ is

$$\nabla(\phi_i \circ \bar{\pi}_T \circ \lambda)(X)[H] = \nabla\phi_i(\bar{\pi}_T(\lambda(X)))[\pi_T(\operatorname{diag}(UHU^\top))],$$

where $U \in \mathbf{O}^n$ satisfies $X = U^\top(\operatorname{Diag}\lambda(X))U$. Combining this with Lemma 3.6 we obtain the following expression for the derivative of the map $\bar{\Phi}$:

$$J\bar{\Phi}(X)[H] = J\phi(\bar{\pi}_T(\lambda(X)))[\pi_T(\operatorname{diag}(UHU^\top))] - \pi_N(\operatorname{diag}(UHU^\top)).$$

Notice that for each $h \in N_{\mathcal{M}}(x)$, defining $H := U^\top(\operatorname{Diag} h)U \in \mathbf{S}^n$, we have $J\bar{\Phi}(X)[H] = -h$, which shows that the linear map $J\bar{\Phi}(X)\colon \mathbf{S}^n \to N_{\mathcal{M}}(x)$ is onto and thus has full rank. In view of (3.5), $\bar{\Phi}$ is a local equation of $\mathcal{M}$ around X.

Since $d = \dim(\mathcal{M}) = \dim(T_{\mathcal{M}}(x))$ and $\dim(N_{\mathcal{M}}(x)) = n - d$, and since $\bar{\phi}$ and $\bar{\Phi}$ are local equations of the manifolds $\mathcal{M}$ and $\lambda^{-1}(\mathcal{M} \cap B(x,\delta))$, respectively, these manifolds have the same codimension $n - d$. □

3.2 Reduction of the ambient space (general case). We now consider a manifold $\mathcal{M}$ with general characteristic partition P_* and $\delta > 0$ such that conditions (A1)–(A2) hold. Using (2.21) and (2.23), we obtain the inclusion

$$\mathcal{M} \cap B(x,\delta) \subset \left(x + T_{\mathcal{M}}(x) \oplus \left(N_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp\perp}\right)\right) \cap B(x,\delta).$$

To define a local equation of $\mathcal{M}$ in the appropriate space, we introduce the reduced tangent and normal spaces:

$$T_{\mathcal{M}}^{\mathrm{red}}(x) := T_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp} \quad \text{and} \quad N_{\mathcal{M}}^{\mathrm{red}}(x) := N_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp\perp}. \tag{3.6}$$

Note that these spaces are invariant under permutations $\sigma \in \Sigma_x^n$. For later use, when calculating the dimension of spectral manifolds, we write

$$n^{\mathrm{red}} := \dim N_{\mathcal{M}}^{\mathrm{red}}(x). \tag{3.7}$$

We now describe the set on which the local equation of $\lambda^{-1}(\mathcal{M})$ is to be defined. Let $x = x^F \otimes x^M$ be the canonical split of x in $\mathbf{R}^n$. Naturally, $B(x^F, \delta_1)$ denotes the open ball in $\mathbf{R}^{\mathbb{N}_n^1}$ centered at x^F with radius δ_1, and $B(x^M, \delta_2)$ denotes the open ball in $\mathbf{R}^{\mathbb{N}_n^2}$ centered at x^M with radius δ_2. Define the rectangular neighborhood of x

$$B(x, \delta_1, \delta_2) := B(x^F, \delta_1) \otimes B(x^M, \delta_2).$$

Choose δ_1, δ_2 such that $B(x, \delta_1, \delta_2) \subset B(x, \delta)$. By conditions (A1)–(A2) and Proposition 2.15 (ii), $B(x^F, \delta_1)$ intersects only strata $\Delta_{P^F} \subset \mathbf{R}^{\mathbb{N}_n^1}$ for $P^F \succeq P_x^F$, and similarly for the ball $B(x^M, \delta_2)$. Thus, $B(x, \delta_1, \delta_2)$ is invariant under permutations $\sigma \in \Sigma_x^n$.

The key element in our next development is the set

$$\mathcal{D} := \big(x + T_{\mathcal{M}}(x) \oplus N_{\mathcal{M}}^{\mathrm{red}}(x)\big) \cap B(x, \delta_1, \delta_2), \tag{3.8}$$

which plays the role of a new ambient space. Indeed, $\mathcal{D}$ is an affine manifold of $\mathbf{R}^n$ and, as it turns out, is the domain of a symmetric local equation of $\mathcal{M}$. We gather properties of $\mathcal{D}$ in the next proposition.

Proposition 3.8. *In the situation above,*

$$T_{\mathcal{M}}(x) \oplus N_{\mathcal{M}}^{\mathrm{red}}(x) \;=\; T_{\mathcal{M}}^{\mathrm{red}}(x) \oplus \Delta_{P_x}^{\perp\perp}. \tag{3.9}$$

Hence

$$\mathcal{D} \;=\; \big(x + \big(T_{\mathcal{M}}^{\mathrm{red}}(x) \oplus \Delta_{P_x}^{\perp\perp}\big)\big) \;\cap\; B(x, \delta_1, \delta_2).$$

Moreover, $\mathcal{D}$ is invariant under all permutations $\sigma \in \Sigma_x^n$, and so is a locally symmetric set.

Proof. Since $T_{\mathcal{M}}(x)$ and $N_{\mathcal{M}}(x)$ are orthogonal complements, applying successively (3.6) and (2.14), we have

$$\begin{aligned} T_{\mathcal{M}}(x) \oplus N_{\mathcal{M}}^{\mathrm{red}}(x) &= T_{\mathcal{M}}(x) \oplus \big(N_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp\perp}\big) \\ &= (T_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp}) \oplus (T_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp\perp}) \oplus (N_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp\perp}) \\ &= \big(T_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp}\big) \oplus \Delta_{P_x}^{\perp\perp} \\ &= T_{\mathcal{M}}^{\mathrm{red}}(x) \oplus \Delta_{P_x}^{\perp\perp}, \end{aligned}$$

which yields (3.9) since $x \in \Delta_{P_x}^{\perp\perp}$ and $0 \in T_{\mathcal{M}}^{\mathrm{red}}(x)$. The invariance of $\mathcal{D}$ follows from the invariance of each set in the intersection. □

Let $\bar{\pi}_N^{\mathrm{red}}$ and π_N^{red} be the projections onto $x + N_{\mathcal{M}}^{\mathrm{red}}(x)$ and $N_{\mathcal{M}}^{\mathrm{red}}(x)$, respectively. Note that

$$x + y \;=\; \bar{\pi}_T(y) + \bar{\pi}_N^{\mathrm{red}}(y) \text{ for all } y \in x + T_{\mathcal{M}}(x) \oplus N_{\mathcal{M}}^{\mathrm{red}}(x). \tag{3.10}$$

The next result is the analogue of Lemma 3.4.

Lemma 3.9. *The projection $\overline{\pi}_N^{\mathrm{red}}$ is locally symmetric at x.*

Proof. Projecting onto $x + N_{\mathcal{M}}^{\mathrm{red}}(x)$ can be accomplished in two steps: first projecting onto $x + \Delta_{P_x}^{\perp\perp}$ and then onto $x + N_{\mathcal{M}}^{\mathrm{red}}(x)$. Now, the projection onto $x + \Delta_{P_x}^{\perp\perp}$ is given by $y \mapsto x + \frac{1}{|\Sigma_x^n|}\sum_{\sigma\in\Sigma_x^n}\sigma y$ [7, Lemma 2.9]. Since it is locally symmetric at x, the result follows. □

Similarly to (3.3), we define the map $\overline{\phi} : \mathcal{D} \to N_{\mathcal{M}}^{\mathrm{red}}(x)$ by

$$y \mapsto x + \phi(\overline{\pi}_T(y)) - \overline{\pi}_N^{\mathrm{red}}(y). \tag{3.11}$$

The next lemma is an analogue of Lemma 3.5.

Lemma 3.10. *The map $\overline{\phi}$ is well-defined, locally symmetric, and is a local equation of $\mathcal{M}$ around x.*

Proof. The set $\mathcal{D}$ is chosen so that ϕ is well-defined. Thanks to (2.23) and the fact that $x - \overline{\pi}_N^{\mathrm{red}}(y) \in N_{\mathcal{M}}^{\mathrm{red}}(x)$, the range of $\overline{\phi}$ is in $N_{\mathcal{M}}^{\mathrm{red}}(x)$. The remainder of the proof follows closely that of Lemma 3.5. In view of (3.10), (2.21) and Theorem 2.17, we see that for all $y \in \mathcal{D}$, the following statements are equivalent:

(i) $\overline{\phi}(y) = 0$;
(ii) $\overline{\pi}_N^{\mathrm{red}}(y) = x + \phi(\overline{\pi}_T(y))$;
(iii) $y = \overline{\pi}_T(y) + \phi(\overline{\pi}_T(y))$;
(iv) $y \in \mathcal{M} \cap B(x, \delta)$.

The Jacobian of $\overline{\phi}$ at y is the linear map from $T_{\mathcal{M}}(x) \oplus N_{\mathcal{M}}^{\mathrm{red}}(x)$ to $N_{\mathcal{M}}^{\mathrm{red}}(x)$ given by

$$J\overline{\phi}(y)[h] = J\phi(\overline{\pi}_T(y))[\pi_T(h)] - \pi_N^{\mathrm{red}}(h).$$

Clearly, for $h \in N_{\mathcal{M}}^{\mathrm{red}}(x)$, $J\overline{\phi}(x)[h] = -h$, which shows that the Jacobian $J\overline{\phi}$ at x is onto and has full rank. Thus, $\overline{\phi}$ is a local equation of $\mathcal{M}$ around x. Finally, Theorem 2.17 shows that for each $\sigma \in \Sigma_x^n$ and $y \in \mathcal{D}$, $\phi(\overline{\pi}_T(\sigma y)) = \phi(\sigma\overline{\pi}_T(y)) = \phi(\overline{\pi}_T(y))$. This, together with Lemma 3.9, gives the local symmetry of $\overline{\phi}$. □

We introduce the spectral function $\overline{\Phi} : \lambda^{-1}(\mathcal{D}) \to N_{\mathcal{M}}^{\mathrm{red}}(x)$ associated with $\overline{\phi}$, defined by

$$X \mapsto (\overline{\phi} \circ \lambda)(X) = x + \phi(\overline{\pi}_T(\lambda(X))) - \overline{\pi}_N^{\mathrm{red}}(\lambda(X)). \tag{3.12}$$

By construction, the zeros of $\overline{\Phi}$ characterize $\mathcal{M}$, since

$$X \in \lambda^{-1}(\mathcal{M} \cap B(x,\delta)) \iff \lambda(X) \in \mathcal{M} \cap B(x,\delta) \iff \overline{\Phi}(X) = 0. \tag{3.13}$$

At this stage, let us compare (3.12) with (3.4) and the particular treatment in Section 3.1. In Section 3.1, we had $N_{\mathcal{M}}(x) \subseteq \Delta_{P_x}^{\perp\perp}$, yielding $N_{\mathcal{M}}^{\mathrm{red}}(x) = N_{\mathcal{M}}(x)$ and thus $\mathcal{D} = B(x, \delta_1, \delta_2)$, which is an open subset of $\mathbf{R}^n$. Unfortunately, in the general case, there is an extra difficulty, stemming from the fact that $\mathcal{D}$ is not open in $\mathbf{R}^n$. Consequently, $\overline{\Phi}$ is defined on a subset $\lambda^{-1}(\mathcal{D})$ of $\mathbf{S}^n$, of lower dimension. To work around this diffuculty, we proceed as follows.

1 *Transfer of local approximation.* We show that the set $\lambda^{-1}(\mathcal{D})$ is an analytic manifold locally around $X \in \lambda^{-1}(x)$, and calculate its dimension.
2 *Transfer of local equation.* We show that the function $\overline{\Phi}$ defined on $\lambda^{-1}(\mathcal{D})$ is differentiable with derivative at X of full rank (as a linear map on the tangent space of $\lambda^{-1}(\mathcal{D})$).

3.3 Transfer of the local approximation. The goal of this subsection is to show that locally around $X \in \lambda^{-1}(x)$ the set $\lambda^{-1}(\mathcal{D})$ is an analytic submanifold of $\mathbf{S}^n$. We do this in two steps. The first step consists of showing that the F-part and the M-part of $\mathcal{D}$ give rise to analytic submanifolds in the spaces $\mathbf{S}^{|\mathbb{N}_n^1|}_{P_x^F}$ and $\mathbf{S}^{|\mathbb{N}_n^2|}_{P_x^M}$ respectively. In the second step, we show that 'intertwining' the two parts preserves this property in the space $\mathbf{S}^n$. Suppose the partition $P_x = P_x^F \cup P_x^M$ is made up of the sets

$$P_x^F = \{I_1, \dots, I_\kappa\} \quad \text{and} \quad P_x^M = \{I_{\kappa+1}, \dots, I_{\kappa+m}\}. \tag{3.14}$$

Lemma 3.11. *The affine manifold $\mathcal{D}$ can be decomposed as*

$$\mathcal{D} = \{y^F \otimes y^M : y^F \in \mathcal{D}^F,\ y^M \in \mathcal{D}^M\},$$

where $\mathcal{D}^F$ and $\mathcal{D}^M$ are affine manifolds defined by

$$\begin{aligned}\mathcal{D}^F &:= \left([T^{\mathrm{red}}_{\mathcal{M}}(x)]^F \oplus \Delta_{P_x^F}\right) \cap B(x^F, \delta_1) \text{ and}\\ \mathcal{D}^M &:= \Delta_{P_x^M} \cap B(x^M, \delta_2)\end{aligned}$$

and $[T^{\mathrm{red}}_{\mathcal{M}}(x)]^F$ is the F-part of the reduced space $T^{\mathrm{red}}_{\mathcal{M}}(x)$. The sets $\mathcal{D}^F$ and $\mathcal{D}^M$ are locally symmetric, and $\dim \mathcal{D}^F = d + n^{\mathrm{red}} - m$.

Proof. Recalling the definition of $T^{\mathrm{red}}_{\mathcal{M}}(\bar{x})$ and using (2.16) and the right part of (2.13), one sees that $y^M = 0$ for every $y = y^F \otimes y^M \in T^{\mathrm{red}}_{\mathcal{M}}(x)$. By the left part of (2.13) with $P = P_x$, combined with Proposition 3.8,

$$\begin{aligned}\mathcal{D} &= \{y^F \otimes y^M : y^F \in \left([T^{\mathrm{red}}_{\mathcal{M}}(x)]^F \oplus \Delta^{\perp\perp}_{P_x^F}\right) \cap B(x^F, \delta_1),\ \ y^M \in \Delta^{\perp\perp}_{P_x^M} \cap B(x^M, \delta_2)\}\\ &= \{y^F \otimes y^M : y^F \in \left([T^{\mathrm{red}}_{\mathcal{M}}(x)]^F \oplus \Delta_{P_x^F}\right) \cap B(x^F, \delta_1),\ \ y^M \in \Delta_{P_x^M} \cap B(x^M, \delta_2)\},\end{aligned}$$

where we have used the facts that the ball $B(x^F, \delta_1)$ intersects only strata Δ_{P^F} with $P^F \succeq P_x^F$ and similarly for the ball $B(x^M, \delta_2)$. The desired expressions for $\mathcal{D}^F$ and $\mathcal{D}^M$ follow.

By Proposition 3.8, the set $\mathcal{D}$ is invariant under all permutations in Σ^n_x. Thus, by Proposition 2.15(ii), being the F-part and the M-part of $\mathcal{D}$ respectively, the sets $\mathcal{D}^F$ and $\mathcal{D}^M$ are invariant with respect to the permutations preserving P^F_x and P^M_x, respectively. We now compute the dimension of $\mathcal{D}^F$. Proposition 3.8 yields

$$x + T_{\mathcal{M}}(x) \oplus N^{\mathrm{red}}_{\mathcal{M}}(x) = T^{\mathrm{red}}_{\mathcal{M}}(x) \oplus \Delta^{\perp\perp}_{P_x} = \left([T^{\mathrm{red}}_{\mathcal{M}}(x)]^F \oplus \Delta^{\perp\perp}_{P^F_x}\right) \otimes \left(\{0\} \oplus \Delta^{\perp\perp}_{P^M_x}\right),$$

where the zero vector is of dimension $|\mathbb{N}^2_n|$. Thus, using (3.9), (3.7), and the fact that $\dim \Delta^{\perp\perp}_{P^M_x} = m$, we get $d + n^{\mathrm{red}} = \dim \mathcal{D}^F + \dim \Delta^{\perp\perp}_{P^M_x}$, completing the proof.□

In the next two lemmas, we show that the two parts of $\mathcal{D}$ lift to manifolds $\lambda^{-1}_{P^M_x}(\mathcal{D}^M)$ and $\lambda^{-1}_{P^F_x}(\mathcal{D}^F)$. Let us start with the easier case, concerning the M-part.

Lemma 3.12. *The set $\mathcal{S}^M := \lambda^{-1}_{P^M_x}(\mathcal{D}^M)$ is an analytic submanifold in $\mathbf{S}^{|\mathbb{N}^2_n|}_{P^M_x}$ with codimension $\sum_{i=1}^m \big(|I_{\kappa+i}|(|I_{\kappa+i}|+1)\big)/2 - m$.*

Proof. Vectors in $\Delta_{P^M_x}$ have equal coordinates within each block $I_{\kappa+i}$. Each block lifts to a multiple of the identity matrix (of appropriate dimension). Since the lifting $\lambda^{-1}_{P^M_x}$ is block-wise, $\mathcal{S}^M$ is a direct product of multiples of identity matrices. Hence it is an analytic submanifold with dimension m. □

Lemma 3.13. *The set $\mathcal{S}^F := \lambda^{-1}_{P^F_x}(\mathcal{D}^F)$ is an analytic submanifold in $\mathbf{S}^{|\mathbb{N}^1_n|}_{P^F_x}$ with codimension $|\mathbb{N}^1_n| - (d + n^{\mathrm{red}} - m)$.*

Proof. By Lemma 3.11, $\mathcal{D}^F$ is a locally symmetric affine submanifold of $\mathbf{R}^{\mathbb{N}^1_n}$. Our aim is to show that the characteristic partition of $\mathcal{D}^F$ is $\mathrm{id}_{\mathbb{N}^1_n}$. Then, applying Theorem 3.7 to $\mathcal{D}^F$ shows that $\mathcal{S}^F$ is an analytic submanifold of codimension $|\mathbb{N}^1_n| - (d + n^{\mathrm{red}} - m)$.

To this end, fix $\epsilon > 0$, and let $\omega \in T_{\mathcal{M}}(\overline{x}) \cap B(0, \epsilon)$ have the properties stated in Lemma 2.16, i.e., $\omega^F \in \mathbf{R}^{\mathbb{N}^1_n}$ is such that each subvector $\omega^F_{I_i}$ has distinct coordinates for all $i \in \mathbb{N}_\kappa$. By (2.14), there exists a unique representation $\omega = \omega_\perp + \omega_{\perp\perp}$,where $\omega_\perp \in T^{\mathrm{red}}_{\mathcal{M}}(x)$ and $\omega_{\perp\perp} \in T_{\mathcal{M}}(x) \cap \Delta^{\perp\perp}_{P_x}$. Taking the F-parts, we have $\omega^F = \omega^F_\perp + \omega^F_{\perp\perp}$ with $\omega^F_\perp \in [T^{\mathrm{red}}_{\mathcal{M}}(\overline{x})]^F$ and $\omega^F_{\perp\perp} \in \Delta^{\perp\perp}_{P^F_x}$. Recall that $P^F_x = \{I_1, \dots, I_\kappa\}$, and write $\omega^F_\perp = \omega^F - \omega^F_{\perp\perp}$. Since the subvector $\omega^F_{I_i}$ has distinct coordinates, whereas $(\omega^F_{\perp\perp})_{I_i}$ has equal coordinates, the subvector $(\omega^F_\perp)_{I_i}$ has distinct coordinates for all $i \in \mathbb{N}_\kappa$.

Consider now $\mathcal{D}^F$. Fix $x^F \in \Delta_{P^F_x} \cap B(x^F, \delta_1)$. Taking ω close to 0 ensures that $\omega^F_\perp$ is close to 0, all of the coordinates of $\omega^F_\perp + x^F$ are distinct, and $\omega^F_\perp + x^F \in \mathcal{D}^F$. Thus $\mathcal{D}^F \cap \Delta_{\mathrm{id}_{\mathbb{N}^1_n}} \neq \varnothing$, and the characteristic partition of the affine manifold $\mathcal{D}^F$ is $\mathrm{id}_{\mathbb{N}^1_n}$, as asserted. □

We show that $\lambda^{-1}(\mathcal{D})$, the intended domain of the local equation of $\lambda^{-1}(\mathcal{M})$, is an analytic manifold by merging the results of the two preceding lemmas using the following technical result.

Proposition 3.14 (Local canonical split of $\mathbf{S}^n$ induced by P_x)**.** *For each $x \in \mathcal{M} \cap \mathbf{R}^n_{\geq}$, there exist an open neighborhood $W \subset \mathbf{S}^n$ of $X \in \lambda^{-1}(x)$ and two analytic maps $\Theta^F : W \to \mathbf{S}^{\mathbb{N}_n^1}_{P_x^F}$ and $\Theta^M : W \to \mathbf{S}^{\mathbb{N}_n^2}_{P_x^M}$ such that*

(i) $\lambda(Y) = \lambda_{P_x^F}(\Theta^F(Y)) \otimes \lambda_{P_x^M}(\Theta^M(Y))$ *for all $Y \in W$;*

(ii) *the Jacobians of the analytic maps Θ^F and Θ^M have full rank at X.*

Proof. To each set in the partition $P_x = \{I_1, \ldots, I_m\}$, we apply the following classical result on eigenvalues [2, Example 3.98]. Let $X \in \mathbf{S}^n$ have eigenvalues

$$\lambda_1(X) \geq \cdots \geq \lambda_{k-1}(X) > \lambda_k(X) = \cdots = \lambda_{k+r-1}(X) > \lambda_{k+r}(X) \geq \cdots \geq \lambda_n(X);$$

then there exist an open neighborhood $W \subset \mathbf{S}^n$ of X and an analytic map $\Theta\colon W \to \mathbf{S}^r$ such that

(i) $\{\lambda_k(Y), \ldots, \lambda_{k+r-1}(Y)\} = \{\lambda_1(\Theta(Y)), \ldots, \lambda_r(\Theta(Y))\}$ for all $Y \in W$, and

(ii) the Jacobian of Θ has full rank at X.

Recall now that each I_ℓ contains consecutive integers and assume, without loss of generality, that P_x is an ordered partition, i.e., for all $1 \leq \ell_1 < \ell_2 \leq m$, $i \in I_{\ell_1}$, $j \in I_{\ell_2}$ implies $i < j$. In other words, $\lambda_i(X) > \lambda_j(X)$. Apply the above result to each I_ℓ to get an open neighborhood $W_\ell \subset \mathbf{S}^n$ of X and an analytic map $\Theta_\ell\colon W_\ell \to \mathbf{S}^{|I_\ell|}$ having full rank Jacobian. Set $W = \bigcap_{\ell=1}^m W_\ell$ and put the F-pieces and the M-pieces together; i.e., restricting each Θ_ℓ to W, define the direct products

$$\Theta^F := \times\{\Theta_\ell : I_\ell \in P_x^F\} \quad \text{and} \quad \Theta^M := \times\{\Theta_\ell : I_\ell \in P_x^M\}.$$

The order of multiples in the direct products follows the order of the sets I_ℓ in P_x. Then the functions defined above have the desired properties. □

Theorem 3.15. *The set $\lambda^{-1}(\mathcal{D})$ is an analytic submanifold of $\mathbf{S}^n$ around $X \in \lambda^{-1}(x)$ with dimension*

$$\dim \lambda^{-1}(\mathcal{D}) = \frac{n(n+1)}{2} + d + n^{\mathrm{red}} - |\mathbb{N}_n^1| - \sum_{i=1}^{m} \frac{|I_{\kappa+i}|(|I_{\kappa+i}|+1)}{2}. \tag{3.15}$$

Proof. By Proposition 3.14, there exist a neighborhood $W \subset \mathbf{S}^n$ of X and analytic maps Θ^F and Θ^M such that

$$\lambda(Y) = \lambda_{P_x^F}(\Theta^F(Y)) \otimes \lambda_{P_x^F}(\Theta^M(Y)) \text{ for all } Y \in W. \tag{3.16}$$

Set $X^F := \Theta^F(X) \in \mathbf{S}^{|\mathbb{N}_n^1|}_{P_x^F}$ and $X^M := \Theta^M(X) \in \mathbf{S}^{|\mathbb{N}_n^2|}_{P_x^M}$. Then (3.16) gives $x = \lambda(X) = \lambda_{P_x^F}(X^F) \otimes \lambda_{P_x^M}(X^M)$, and hence $x^F = \lambda_{P_x^F}(X^F)$ and $x^M = \lambda_{P_x^M}(X^M)$, from which we conclude that $X^F \in \mathcal{S}^F$ and $X^M \in \mathcal{S}^M$ (recall Lemmas 3.12 and 3.13). Consider the respective codimensions

$$s_1 := \text{co-dim}\,\mathcal{S}^F = |\mathbb{N}_n^1| - (d + n^{\text{red}} - m), \text{ and} \tag{3.17}$$

$$s_2 := \text{co-dim}\,\mathcal{S}^M = \sum_{i=1}^{m} \frac{|I_{\kappa+i}|(|I_{\kappa+i}|+1)}{2} - m. \tag{3.18}$$

Since Θ^F and Θ^M have Jacobians of full rank at X, they are open around it. By shrinking W if necessary, we may assume that there exist analytic maps

$$\Psi^F : \Theta^F(W) \to \mathbf{R}^{s_1} \quad \text{and} \quad \Psi^M : \Theta^M(W) \to \mathbf{R}^{s_2},$$

with Jacobians having full rank at X^F and X^M, respectively, such that

$$\Psi^F(Y) = 0 \Leftrightarrow Y \in \mathcal{S}^F \cap \Theta^F(W) \quad \text{and} \quad \Psi^M(Y) = 0 \Leftrightarrow Y \in \mathcal{S}^M \cap \Theta^M(W).$$

We now define a local equation $\Psi : W \to \mathbf{R}^{s_1} \times \mathbf{R}^{s_2}$ for $\lambda^{-1}(\mathcal{D})$ around X by

$$X \mapsto (\Psi^F \circ \Theta^F)(X) \times (\Psi^M \circ \Theta^M)(X).$$

Indeed, using (3.16), we have

$$\Psi(Y) = 0 \iff \lambda(Y) = \lambda_{P_x^F}(\Theta^F(Y)) \otimes \lambda_{P_x^M}(\Theta^M(Y)) \in \mathcal{D} \iff Y \in \lambda^{-1}(\mathcal{D})$$

for all $Y \in W$. That the Jacobian of Ψ has full rank at X follows from the chain rule and the fact that the Jacobians $J\Theta^F(X)$, $J\Theta^M(X)$, $J\Psi^F(X^F)$, and $J\Psi^M(X^M)$ are all of full rank. Thus Ψ is an analytic local equation of $\lambda^{-1}(\mathcal{D})$ around X, from which it follows that $\lambda^{-1}(\mathcal{D})$ is a submanifold of $\mathbf{S}^n$ around X. Using (3.17) and (3.18), we compute its dimension as follows:

$$\begin{aligned} \dim \lambda^{-1}(\mathcal{D}) &= \dim \mathbf{S}^n - \left(\text{co-dim}\,\mathcal{S}^F + \text{co-dim}\,\mathcal{S}^M\right) \\ &= \frac{n(n+1)}{2} + d + n^{\text{red}} - |\mathbb{N}_n^1| - \sum_{i=1}^{m} \frac{|I_{\kappa+i}|(|I_{\kappa+i}|+1)}{2}. \end{aligned}$$ □

Theorem 3.15 is an important intermediate result used in Subsection 3.4 below, which contains the final step of the proof. In the following particular case, Theorem 3.15 actually gives us the result directly.

Example 3.16 (Lift of strata)**.** For a partition $P^\circ \in \Pi_n$, the set

$$\mathcal{M} := \Delta_{P^\circ} \cup \Big(\bigcup_{P \prec\sim P^\circ} \Delta_P \Big)$$

is a locally symmetric manifold with characteristic permutation P° [7, Remark 4.6]. Suppose, in addition, that the sets in P° contain consecutive integers. Then $\mathcal{M} \cap \mathbf{R}^n_{\geq} \neq \varnothing$. For each $x \in \mathcal{M} \cap \mathbf{R}^n_{\geq}$, $N^{\mathrm{red}}_{\mathcal{M}}(x) = \{0\}$, i.e.,$n^{\mathrm{red}} = 0$. This means that the affine manifolds $\mathcal{M}$ and $\mathcal{D}$ coincide locally around x; see (3.8). In this case, Theorem 3.15 shows directly that $\lambda^{-1}(\mathcal{M})$ is a manifold in $\mathbf{S}^n$ with dimension given by (3.15).

At first glance, it may appear that the dimension depends on the particular choice of x. However, since $P_x = P_\circ$ or $P_x \prec\sim P_\circ$, Proposition 2.15(ii) implies that $P^M_x = P^M_\circ =: \{I^\circ_{\kappa+1}, \dots, I^\circ_{\kappa+m}\}$. Let $P^F_\circ = \{I^\circ_1, \dots, I^\circ_\kappa\}$. Since $n = |\mathbb{N}^1_n| + |\mathbb{N}^2_n| = \sum_{i=1}^{\kappa} |I^\circ_i| + \sum_{i=\kappa+1}^{\kappa+m} |I^\circ_i|$, one can verify that (3.15) becomes

$$\dim \lambda^{-1}(\mathcal{M}) = d + \sum_{1 \leq i < j \leq \kappa+m} |I^\circ_i||I^\circ_j|.$$

Thus, by (3.1), $\dim \lambda^{-1}(\mathcal{M}) = \dim \lambda^{-1}(\Delta_{P_\circ})$. This is a particular case of the more general formula (3.21), below. □

In Example 3.16, the manifold $\mathcal{M}$ has a trivial reduced normal space. The following remark sheds more light on this aspect.

Remark 3.17. Let $\mathcal{M}$ be a locally symmetric manifold with characteristic partition P_*, and let $x \in \mathcal{M} \cap \mathbf{R}^n_{\geq}$. Then, by (2.21) and (2.23), it can be seen that

$$N^{\mathrm{red}}_M(x) = \{0\} \iff \mathcal{M} \cap B(x,\delta) = (x + T_{\mathcal{M}}(x)) \cap B(x,\delta) \ \text{ for some } \delta > 0.$$

Inclusion (2.9) shows that $x \in \Delta^{\perp\perp}_{P_*}$, which, together with (2.10), implies that $(x + T_{\mathcal{M}}(x)) \subset \Delta^{\perp\perp}_{P_*}$. Thus

$$N^{\mathrm{red}}_M(x) = \{0\} \iff \mathcal{M} \cap B(x,\delta) = \Delta^{\perp\perp}_{P_*} \cap B(x,\delta) \ \text{ for some } \delta > 0.$$

3.4 Transfer of local equations, proof of the main result. This subsection contains the final step of our argument. We show that (3.12) is indeed a local equation of $\mathcal{M}$ around $X \in \lambda^{-1}(x)$.

Lemma 3.18. *The map $\overline{\Phi}$ defined in* (3.12) *is of class C^k at $X \in \lambda^{-1}(x)$. Denote the differential of $\overline{\Phi}$ at X by $D\overline{\Phi}(X)\colon T_{\lambda^{-1}(\mathcal{D})}(X) \to N^{\mathrm{red}}_{\mathcal{M}}(x)$,*

For every direction $H \in T_{\lambda^{-1}(\mathcal{D})}(X)$,

$$D\overline{\Phi}(X)\,[H] = D\phi\,(\overline{\pi}_T(\lambda(X)))\,[\pi_T(\mathrm{diag}(U\,H\,U^\top))] - \pi^{\mathrm{red}}_N(\mathrm{diag}(U\,H\,U^\top)), \tag{3.19}$$

where $U \in \mathbf{O}^n$ is such that $X = U^\top(\mathrm{Diag}\,\lambda(X))U$.

Proof. We deduce from Theorem 2.17 that for all $\sigma \in \Sigma_x^n$ and $y \in \mathcal{D}$,

$$(\phi \circ \bar{\pi}_T)(\sigma y) = (\phi \circ \bar{\pi}_T)(y). \tag{3.20}$$

In addition, the gradient of the i-th coordinate function $\phi_i \circ \bar{\pi}_T$ at x applied to any direction $h \in T_{\mathcal{D}}(x) = T_{\mathcal{M}}^{\mathrm{red}}(x) \oplus \Delta_{P_x}^{\perp\perp}$ (see Proposition 3.8) yields

$$\nabla(\phi_i \circ \bar{\pi}_T)(x)[h] = \nabla\phi_i(\bar{\pi}_T(x))[\pi_T(h)].$$

Thus, by Theorem 2.2, we obtain for $H \in T_{\lambda^{-1}(D)}(X)$

$$\nabla(\phi_i \circ \bar{\pi}_T \circ \lambda)(X)[H] = \nabla\phi_i\,(\bar{\pi}_T(\lambda(X)))\,[\pi_T(\mathrm{diag}\,(UHU^\top))] \quad \text{for } i \in \mathbb{N}_n,$$

where $U \in \mathbf{O}^n$ is such that $X = U^\top(\mathrm{Diag}\,\lambda(X))U$. By Lemma 3.9, the projection $\bar{\pi}_N^{\mathrm{red}}$ is locally symmetric at x. Thus, by (2.4),

$$J(\bar{\pi}_N^{\mathrm{red}} \circ \lambda)(X)[H] = J\bar{\pi}_N^{\mathrm{red}}(\lambda(X))[\mathrm{diag}(UHU^\top)] = \pi_N^{\mathrm{red}}(\mathrm{diag}\,(UHU^\top)),$$

from which (3.19) follows. □

Next, we show that the differential of $\bar{\Phi}$ at X is of full rank. We accomplish this without actually computing the tangent space of the manifold $\lambda^{-1}(\mathcal{D})$ at X. Instead, we show that the tangent space is sufficiently rich to guarantee surjectivity.

Lemma 3.19. *The differential of $\bar{\Phi}$ at X*

$$D\bar{\Phi}(X)\colon T_{\lambda^{-1}(\mathcal{D})}(X) \to N_{\mathcal{M}}^{\mathrm{red}}(x)$$

is onto, and hence has a full rank.

Proof. Let $U \in \mathbf{O}^n$ be such that $X = U^\top(\mathrm{Diag}\,\lambda(X))\,U$. The tangent space of $\mathbf{O}^n$ at U is $\{UA : A$ is an $n \times n$ skew-symmetric matrix$\}$. Thus, for each $n \times n$ skew symmetric matrix A, there exists an analytic curve $t \mapsto U(t) \in \mathbf{O}^n$ such that $U(0) = U$ and $\dot{U}(0) := \frac{d}{dt}U(0) = UA$. Now fix $h \in N_{\mathcal{M}}^{\mathrm{red}}(x)$ and consider the curve $t \mapsto U(t)^\top(\mathrm{Diag}\,(x + th))U(t)$. For all values of t sufficiently close to 0, this curve lies in $\lambda^{-1}(\mathcal{D})$, because $x+th$ lies in $\mathcal{D}$. Introduce the vector x_t whose entries are those of $x + th$ reordered in decreasing way. Since $N_{\mathcal{M}}^{\mathrm{red}}(x)$ is invariant under all permutations $\sigma \in \Sigma_x^n$, $x_t \in x + N_{\mathcal{M}}^{\mathrm{red}}(x)$, for t sufficiently close to zero. The derivative of this curve at $t = 0$ (i.e., a tangent vector in $T_{\lambda^{-1}(\mathcal{D})}(X)$) is

$$\begin{aligned} H &:= \dot{U}(0)^\top(\mathrm{Diag}\,x)U(0) + U(0)^\top(\mathrm{Diag}\,h)U(0) + U(0)^\top(\mathrm{Diag}\,x)\dot{U}(0) \\ &= -AU^\top(\mathrm{Diag}\,x)U + U^\top(\mathrm{Diag}\,h)U + U^\top(\mathrm{Diag}\,x)UA, \end{aligned}$$

where we have used the fact that $A^\top = -A$. Substituting the above expression for H into (3.19) and using the facts that $UU^\top = U^\top U = I$ and that $UAU^\top(\mathrm{Diag}\,x)$ and $(\mathrm{Diag}\,x)UAU^\top$ have the same diagonal, we obtain $D\bar{\Phi}(X)[H] = -h$, which shows that $D\bar{\Phi}(X)$ is surjective onto $N_{\mathcal{M}}^{\mathrm{red}}(x)$. □

Theorem 3.20 (The Main Result)**.** *Let $\mathcal{M}$ be a locally symmetric C^k submanifold ($k \in \{2, 3, \dots, \infty, \omega\}$) of $\mathbf{R}^n$ of dimension d. Then $\lambda^{-1}(\mathcal{M})$ is a C^k submanifold of $\mathbf{S}^n$ of dimension*

$$\dim \lambda^{-1}(\mathcal{M}) = d + \sum_{1 \le i < j \le m^*} |I_i^*||I_j^*|, \tag{3.21}$$

where $P_* = \{I_1^*, \dots, I_{m^*}^*\}$ is the characteristic partition of $\mathcal{M}$.

Proof. Fix $x \in \mathcal{M} \cap \mathbf{R}^n_{\ge}$ and $X \in \lambda^{-1}(x)$, and consider the spectral function $\bar{\Phi}$ introduced in (3.12). Equation (3.13) shows that $\bar{\Phi}$ is a local equation of $\mathcal{M}$. Lemmas 3.18 and 3.19 prove that $\bar{\Phi}$ is a C^k local equation of $\lambda^{-1}(\mathcal{M})$ around X. Thus $\lambda^{-1}(\mathcal{M})$ is a C^k submanifold of $\mathbf{S}^n$ around X. Moreover, the dimension of $\lambda^{-1}(\mathcal{M})$ is $\dim \lambda^{-1}(\mathcal{M}) = \dim \lambda^{-1}(\mathcal{D}) - \dim(N_{\mathcal{M}}^{\mathrm{red}}(x))$.

Suppose that (3.14) holds. By Proposition 2.15(ii), $P_x^M = P_*^M$. So suppose that $I_{\kappa+i} = I^*_{m^*-m+i}$ for all $i = 1, \dots, m$. Recall that $n = |\mathbb{N}_n^1| + |\mathbb{N}_n^2|$ and that $\sum_{i=1}^m |I_{\kappa+i}| = n - |\mathbb{N}_n^1|$. Using (3.6) and Theorem 3.15, we get

$$\begin{aligned}
\dim \lambda^{-1}(\mathcal{M}) &= d + \frac{n(n+1)}{2} - |\mathbb{N}_n^1| - \sum_{i=1}^m \frac{|I_{\kappa+i}|(|I_{\kappa+i}|+1)}{2} \\
&= d + \frac{n^2}{2} - \frac{|\mathbb{N}_n^1|}{2} - \sum_{i=1}^m \frac{|I_{\kappa+i}|^2}{2} \\
&= d + \frac{n^2}{2} - \frac{|\mathbb{N}_n^1|}{2} - \frac{1}{2}\Big(\sum_{i=1}^m |I^*_{m^*-m+i}|\Big)^2 + \sum_{1\le i<j\le m} |I^*_{m^*-m+i}||I^*_{m^*-m+j}| \\
&= d + \frac{|\mathbb{N}_n^1|(|\mathbb{N}_n^1|-1)}{2} + |\mathbb{N}_n^1|(n - |\mathbb{N}_n^1|) + \sum_{1\le i<j\le m} |I^*_{m^*-m+i}||I^*_{m^*-m+j}| \\
&= d + \sum_{1\le i<j\le m^*} |I_i^*|\,|I_j^*|.
\end{aligned}$$

The last equality comes from the fact that all the sets in $P_*^F = \{I_1^*, \dots, I^*_{m^*-m}\}$ have size one. □

Observe that the dimension (3.21) of $\lambda^{-1}(\mathcal{M})$ depends only on the dimension of the underlying manifold $\mathcal{M}$ and its characteristic permutation P_*. This is not the case with the dimension (3.15) of $\lambda^{-1}(\mathcal{D})$, which also depends on the partition P_x (via n^{red}, κ and m).

Remark 3.21. The case of a locally symmetric C^1 manifold $\mathcal{M}$ is compromised by [7, Lemma 3.13] (Determination of isometries), which uses the intrinsic Riemannian structure of $\mathcal{M}$, thus requiring C^2 regularity. Lemma 3.13 from [7] was used there to obtain the reduction of the ambient space for the tangential parametrization of $\mathcal{M}$, which is one of the main ingredients in establishing Theorem 2.17.

Let us now give a few applications of the main result. It is known that the set of all symmetric matrices in $\mathbf{S}^n$ of rank k is an analytic manifold; see, for example, [9, Proposition 1.14, p.133]. This also follows from our main result, together with the formula for its dimension.

Example 3.22 (Matrices of constant rank)**.** Let

$$\mathcal{M} = \{x \in \mathbf{R}^n : x \text{ has exactly } n-k \text{ zeros}\}.$$

Then $\lambda^{-1}(\mathcal{M}) = \{A \in S^n : \operatorname{rank} A = k\}$. Fix a subset J of $\{1, 2, \ldots, n\}$ with $n-k$ consecutive elements, and let $\mathcal{M}' := \{x \in \mathbf{R}^n : x_i = 0, i \in J\}$, which is a connected component of $\mathcal{M}$. Then $\dim \mathcal{M}' = k$, and the characteristic partition of $\mathcal{M}'$ is $P_* = \{i : i \in J\} \cup \{\{i\} : i \notin J\}$. By Theorem 3.20, $\lambda^{-1}(\mathcal{M})$ is an analytic submanifold of $\mathbf{S}^n$ with dimension

$$\dim \lambda^{-1}(\mathcal{M}) = \dim \lambda^{-1}(\mathcal{M}') = k + \frac{k(k-1)}{2} + k(n-k) = \frac{k(2n-k+1)}{2}.$$

In particular, the dimension of rank-one matrices ($k = 1$) is n, whereas the dimension of the invertible matrices ($k = n$) is $\binom{n}{2}$.

Example 3.23 (The unit shpere)**.** The unit sphere

$$\mathcal{M} := \{x \in \mathbf{R}^n : x_1^2 + \cdots + x_n^2 = 1\} \subset \mathbf{R}^n$$

is a symmetric analytic manifold of dimension $n-1$ and characteristic partition $P_* = \{\{i\} : i = 1, \ldots, n\}$. By Theorem 3.20, $\lambda^{-1}(\mathcal{M})$ is an analytic submanifold of $\mathbf{S}^n$ with dimension

$$\dim \lambda^{-1}(\mathcal{M}) = (n-1) + \binom{n}{2} = \binom{n+1}{2} - 1.$$

Indeed, it is easy to see that

$$\lambda^{-1}(\mathcal{M}) = \{A \in \mathbf{S}^n : \|\lambda(A)\| = 1\} = \{A \in \mathbf{S}^n : \|A\| = 1\}.$$

i.e.,, $\lambda^{-1}(\mathcal{M})$ is the unit sphere in $\mathbf{S}^n$.

Remark 3.24 (The case $|\mathbb{N}_n^1| \in \{0, 1\}$)**.** If $\mathcal{M}$ is a connected, submanifold of $\mathbf{R}^n$ of dimension d such that $|\mathbb{N}_n^1| \in \{0, 1\}$, then $\mathcal{M} \subset \Delta_{P_*}$. The same arguments as in Example 3.1 allow us to conclude that $\lambda^{-1}(\mathcal{M})$ is a spectral manifold of dimension given by (3.1).

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