

Solutions of the fractional Allen–Cahn equation which are invariant under screw motion

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ABSTRACT

We establish existence and non-existence results for entire solutions to the fractional Allen–Cahn equation in \mathbb{R}^3 , which vanish on helicoids and are invariant under screw motion. In addition, we prove that helicoids are surfaces with vanishing non-local mean curvature.

1. Introduction

In this paper, we establish existence results for a class of entire solutions to the fractional Allen–Cahn equation

$$(-\Delta)^\alpha u + F'(u) = 0 \quad \text{in } \mathbb{R}^n, \quad (1.1)$$

where F is a double-well potential; that is, it satisfies the following properties:

- (1) $t \mapsto F(t)$ is an even, positive function of class $C^{2,\gamma}$, with $\gamma > \max\{0, 1 - 2\alpha\}$;
- (2) $F(t) \geq F(\pm 1)$ and equality holds if and only if $t = \pm 1$.

Moreover, we assume also that

$$F''(0) < 0 \quad \text{and} \quad F''(0)t \leq F'(t) \quad \text{for every } t \geq 0. \quad (1.2)$$

A classical example of such potential is $F(t) = \frac{1}{4}(1 - t^2)^2$.

The fractional Laplacian $(-\Delta)^\alpha$ is a pseudodifferential operator with symbol $|\xi|^{2\alpha}$ (where ξ is the variable in the frequency space, $\alpha \in (0, 1)$), and it can also be defined in the following way:

$$(-\Delta)^\alpha u(x) := C_{n,\alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2\alpha}} dy, \quad (1.3)$$

where P.V. stands for the Cauchy principal value and $C_{n,\alpha}$ denotes a constant depending only on n and α .

In previous years, there has been much interest in the study of solutions to the fractional Allen–Cahn equation. In [8–10, 21], the existence of layer-type solutions, that is solutions monotone in one direction with limits ± 1 at $\pm\infty$, has been established, while in [14], one of the authors proved existence for saddle-shaped solutions, which are solutions that vanish on the Simons cone

$$\mathcal{C} = \{(x, \xi) \in \mathbb{R}^m \times \mathbb{R}^m : |x| = |\xi|\};$$

they are odd with respect to \mathcal{C} and even with respect to the coordinate axis. In all these works, the proof of existence relies on a variational argument, which makes use of the symmetries

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of the problem. Also, in this paper, we are interested in solutions of (1.1) which satisfy some symmetry properties, and the technique we use relies on the variational structure of (1.1).

More precisely, we establish existence and non-existence results for solutions which vanish on helicoids and are invariant under screw motion (see (1.4) and (1.5) for precise definitions). For the classical Allen–Cahn equation, analogue results are contained in a work by Del Pino, *et al.* [18].

The interest in the study of solutions that vanish on helicoids comes from the fact that helicoids are surfaces with zero mean curvature.

There is a very strict connection between the Allen–Cahn equation and the classical theory of minimal surfaces. The classical result by Modica and Mortola [20] establishes that the energy functional associated to the classical Allen–Cahn equation, after a suitable rescaling, Γ -converges to the Perimeter functional.

In the fractional setting, an analogue Γ -convergence-type result has been established in [1, 19] for powers $\frac{1}{2} \leq \alpha < 1$, and in [22] for any power $0 < \alpha < 1$: after a suitable rescaling, the energy functional associated to the fractional Allen–Cahn equation Γ -converges to the classical perimeter if $\frac{1}{2} \leq \alpha < 1$ and to the *non-local perimeter* if $0 < \alpha < \frac{1}{2}$.

This deep relation between the theory of minimal surfaces and the theory of phase transitions is also one of the main motivations to the well-known Conjecture of De Giorgi about one-dimensional symmetry of bounded monotone solutions to the Allen–Cahn equation. In the fractional setting, this problem has been studied in several works (see [4, 5, 9, 10]).

The notion of non-local perimeter has been introduced by Caffarelli *et al.* [12], where existence, regularity results and a monotonicity formula for non-local minimal surfaces have been established. Similarly to the case of the classical perimeter, performing the first variation of the non-local perimeter functional, one can define the notion of *non-local mean curvature* (see (1.10)).

In this paper, we focus our attention both on the partial differential equation and on the geometric problem. Indeed, in Theorems 1.1 and 1.2, we establish existence and non-existence results for solutions to (1.1) (in the case of space dimension $n = 3$), which vanish on helicoids and are invariant under screw motion. Moreover, in Theorem 1.3, we prove that helicoids have zero non-local mean curvature.

We recall now the definition of helicoid and screw motion. We will work in dimension $n = 3$. Given $\lambda > 0$, the *helicoid* H_λ is the minimal surface which can be parametrized in the following way:

$$\mathbb{R} \times \mathbb{R} \ni (t, \theta) \longmapsto \left(te^{i\theta}, \frac{\lambda}{\pi} \theta \right) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3. \quad (1.4)$$

The screw motion of parameter λ acting on $\mathbb{C} \times \mathbb{R}$ is given by

$$\sigma_\lambda^\beta(z, s) = \left(e^{i\beta} z, s + \frac{\lambda}{\pi} \beta \right). \quad (1.5)$$

Obviously H_λ is invariant under the action of σ_λ^β for every $\beta \in \mathbb{R}$.

Our first main result is the construction of a non-trivial entire solution to (1.1) in dimension 3 which vanishes on H_λ , provided λ is chosen sufficiently large. We define

$$\lambda_* := \frac{\pi}{(-F''(0))^{1/2\alpha}}. \quad (1.6)$$

THEOREM 1.1. *Let $n = 3$. Assume that F is a double-well potential satisfying (1.2) and that $\lambda > \lambda_*$.*

Then, there exists a solution of the fractional Allen–Cahn equation (1.1) whose zero set is equal to H_λ . This solution is invariant under the screw motion of parameter λ , that is,

$$u \circ \sigma_\lambda^\beta = u,$$

for every $\beta \in \mathbb{R}$.

In the following result, we also prove that Theorem 1.1 is, in some sense, sharp.

THEOREM 1.2. *Let $n = 3$. Assume that F is a double-well potential satisfying (1.2) and that $\lambda \leq \lambda_*$.*

Then, there are no non-trivial bounded solutions of (1.1) that vanish on the helicoid H_λ and are invariant under the screw motion of parameter λ .

To prove the previous results, we will realize the non-local problem (1.1) as a local problem in \mathbb{R}_+^{n+1} with a nonlinear Neumann condition on $\partial\mathbb{R}_+^{n+1} = \mathbb{R}^n$ (the so-called Caffarelli–Silvestre extension [13]).

More precisely, if $u = u(x)$ is a function defined on \mathbb{R}^n , then we consider its s -harmonic extension $v = v(x, y)$ in $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, +\infty)$. It is well known (see [10, 13]) that u is a solution of (1.1) if and only if v satisfies

$$\begin{cases} \operatorname{div}(y^{1-2\alpha}\nabla v) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ -\frac{1}{c_\alpha} \lim_{y \rightarrow 0} y^{1-2\alpha} \partial_y v = -F'(v) & \text{on } \mathbb{R}^n = \partial\mathbb{R}_+^{n+1}, \end{cases} \tag{1.7}$$

where c_α is the constant given in (2.3).

The energy associated to problem (1.7) is

$$E(v) = \frac{1}{2c_\alpha} \int_0^{+\infty} \int y^{1-2\alpha} |\nabla v|^2 \, dx \, dy + \int F(v(x, 0)) \, dx. \tag{1.8}$$

Both proofs of Theorems 1.1 and 1.2 follow the ideas contained in [18] for the local case.

Using the symmetries of the problem, the proof of existence for solutions to the problem (1.7) above is based on an existence result for an analogous problem in the two-dimensional cylinder $[0, \lambda] \times (0, +\infty)$ with zero-Dirichlet boundary conditions on the lateral boundary $\partial[0, \lambda] \times (0, +\infty)$. Existence and non-existence results for this two-dimensional problem will be studied in Section 3 (see Lemma 3.1). Here we emphasize that the length λ of the interval coincides with the parameter of the helicoid H_λ .

Another important tool in the proof of the existence result Theorem 1.1 will be an energy estimate for minimizers, which will ensure that the limit of a minimizing sequence does not identically vanish. In this step, we follow the technique used in [14], based on a suitable choice of a cut-off function.

On the other hand, for the non-existence result, a crucial ingredient will be an exponential decay in the y -variable of the extended solution v of (1.7), under our symmetry assumption (see Proposition 5.1).

We recall now the notions of non-local minimal surface and of non-local mean curvature. Non-local minimal surfaces were introduced in [12] as boundaries of measurable sets E whose characteristic function χ_E minimizes an H^α -norm. More precisely, for any $0 < \alpha < \frac{1}{2}$, the non-local α -perimeter functional of a set E in an open set $\Omega \subset \mathbb{R}^n$ is given by

$$\operatorname{Per}_{2\alpha}(E, \Omega) := L(E \cap \Omega, \mathbb{R}^n \setminus E) + L(E \setminus \Omega, \Omega \setminus E), \tag{1.9}$$

where, for two disjoint measurable sets A and B , $L(A, B)$ denotes the quantity

$$L(A, B) := \int_A \int_B \frac{1}{|x - \bar{x}|^{n+2\alpha}} dx d\bar{x}.$$

A set E is said to be α -minimal in Ω if

$$\text{Per}_{2\alpha}(E, \Omega) \leq \text{Per}_{2\alpha}(F, \Omega)$$

for any measurable set F with $E \Delta F \subset \subset \Omega$. Note that in the literature (see, for example, [12]) the fractional s -perimeter is defined for any $s \in (0, 1)$ and corresponds to (1.9) for $s = 2\alpha$. Here, we prefer to keep this notation to be consistent with the fractional power of the Laplacian.

Analogously to the classical theory of minimal surfaces, performing the first variation of the non-local perimeter functional, we end up with the notion of *non-local mean curvature*. More precisely, the Euler–Lagrange equation for $\text{Per}_{2\alpha}(E, \mathbb{R}^n)$ (on the boundary of E) is given by

$$\mathcal{H}_{\partial E}^{2\alpha}(x) := \int_{\mathbb{R}^n} \frac{\chi_E(\bar{x}) - \chi_{E^C}(\bar{x})}{|x - \bar{x}|^{n+2\alpha}} d\bar{x} = 0 \quad \text{for any } x \in \partial E, \quad (1.10)$$

where $E^C := \mathbb{R}^n \setminus E$ and $\mathcal{H}_{\partial E}^{2\alpha}$ denotes the non-local mean curvature of the boundary of E (we write NMC for short).

Recently, there has been much interest in the study of surfaces with vanishing or constant non-local mean curvature. In [17], two of the authors and J. Wei provide examples of surfaces with zero NMC. More precisely, they establish existence of non-local minimal Lawson cones (for any $0 < \alpha < \frac{1}{2}$) and proved their stability in dimension 7 for α small. Moreover, for $\alpha \rightarrow \frac{1}{2}$, they constructed the non-local analogue of catenoids. Concerning the study of surfaces with constant NMC, in [7, 15] the analogue of the Alexandrov Theorem, characterizing spheres as the only closed embedded hypersurfaces in \mathbb{R}^n with constant mean curvature, has been established. In [6, 7], the existence of Delaunay-type surfaces is established, while in [16] periodic and cylindrical symmetric hypersurfaces, which minimize a certain fractional perimeter under a volume constraint, are considered.

In our last result, we provide a new example of surface with zero NMC. Using the fact that the (double) helicoid H_λ can be seen as the boundary of a set E_+ (as will be made precise in (6.1)), we prove that helicoids, which have zero (classical) mean curvature, also have zero NMC.

THEOREM 1.3. *For any $\lambda > 0$ and for any $0 < \alpha < \frac{1}{2}$, we have that*

$$\mathcal{H}_{H_\lambda}^{2\alpha} \equiv 0.$$

The paper is organized as follows.

(1) In Section 2, we recall some preliminaries on the fractional Laplacian in a bounded domain with zero-Dirichlet boundary condition.

(2) In Section 3, we establish existence and non-existence results for a fractional Dirichlet problem in the one-dimensional case.

(3) In Section 4, we prove our existence result Theorem 1.1.

(4) In Section 5, we give the proof of Theorem 1.2.

(5) In Section 6, we prove Theorem 1.3.

2. Preliminaries

As mentioned in Section 1, owing to the symmetries of our problem, the proof of our existence result Theorem 1.1 is based on an existence result for the following two-dimensional problem

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla v(s, y)) = 0 & \text{for } (s, y) \in [0, \lambda] \times (0, +\infty), \\ v(s, y) = 0 & \text{for } (s, y) \in \partial[0, \lambda] \times (0, +\infty), \\ -\frac{1}{c_\alpha} \lim_{y \rightarrow 0} y^{1-2\alpha} \partial_y v = -F'(v) & \text{on } [0, \lambda] \times \{y = 0\}, \end{cases}$$

which we will establish in Lemma (3.1) below.

As we will clarify later on in this section, this is the extended problem associated to a fractional one-dimensional problem with zero-Dirichlet boundary condition in the interval $[0, \lambda]$. Hence, in particular, the existence result of Lemma 3.1 implies an existence result for solutions of a fractional Dirichlet problem on an interval $[0, \lambda]$.

For the sake of clarity and completeness, in this section we collect some well-known facts about the fractional Laplacian in a bounded domain (see [3, 11]) with zero-Dirichlet boundary condition. The results contained in this section will be used later on in Section 3 and in Section 5, where we will use in particular the exponential decay (2.12).

Let Ω be a sufficiently regular (say Lipschitz) domain in \mathbb{R}^n . We denote by A_α the fractional power of the Laplacian in Ω with zero-Dirichlet boundary condition on $\partial\Omega$.

To define A_α , let us consider $\{\mu_k, \zeta_k\}_{k=1}^\infty$ the eigenvalues and corresponding eigenfunctions of the Laplacian $-\Delta$ in Ω with zero-Dirichlet boundary condition:

$$\begin{cases} -\Delta \zeta_k = \mu_k \zeta_k & \text{in } \Omega, \\ \zeta_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $u = \sum_{k=1}^\infty a_k \zeta_k$; then we define

$$A_\alpha u = \sum_{k=1}^\infty a_k \mu_k^\alpha \zeta_k. \tag{2.1}$$

We stress that the operator A_α defined above is different from the operator $(-\Delta)^\alpha$ defined in (1.3), as shown in [23], where the two operators have been compared with particular reference to their spectrum.

Let now \mathcal{C}_Ω be the cylinder $\mathcal{C}_\Omega = \Omega \times (0, \infty)$ and $\partial_L \mathcal{C}_\Omega = \partial\Omega \times (0, \infty)$ be its lateral boundary. Following [3, 11], we can consider the extension operator with zero-Dirichlet boundary condition on all the $\partial_L \mathcal{C}_\Omega$.

DEFINITION 2.1. We define the α -harmonic extension $v = \mathcal{E}_\alpha(u)$ in \mathcal{C}_Ω of a function u defined in Ω and vanishing on $\partial\Omega$ as the solution of the problem

$$\begin{cases} \operatorname{div}(y^{1-2\alpha}\nabla v) = 0 & \text{in } \mathcal{C}_\Omega, \\ v = 0 & \text{on } \partial_L \mathcal{C}, \\ v = u & \text{on } \Omega \times \{y = 0\}. \end{cases} \tag{2.2}$$

It is well known that (see [3, 11])

$$A_\alpha u(x) = -\frac{1}{c_\alpha} \lim_{y \rightarrow 0} y^{1-2\alpha} \partial_y v(x, y),$$

where

$$c_\alpha = \frac{2^{1-2\alpha}\Gamma(1-\alpha)}{\Gamma(\alpha)}. \tag{2.3}$$

We recall now the explicit expression (see [3, Lemma 3.4]) for $\mathcal{E}_\alpha(u)$ in terms of the spectral decomposition (2.1).

LEMMA 2.2 ([3, Lemma 3.4]). *Let $\{\mu_k, \zeta_k\}$ be, as before, the eigenvalues and eigenfunctions of $-\Delta$ in Ω (with zero-Dirichlet boundary condition). Let $u = \sum_{k=1}^\infty a_k \zeta_k$ be such that $\sum_{k=1}^\infty a_k \mu_k^\alpha < \infty$. Then, the α -harmonic extension of u is given by*

$$v(x, y) = \mathcal{E}_\alpha(u)(x, y) = \sum_{k=1}^\infty a_k \zeta_k(x) \varphi(\mu_k^{1/2} y),$$

where φ is a solution of the problem

$$\begin{cases} \varphi'' + \frac{1-2\alpha}{y} \varphi' - \varphi = 0 & \text{for } y > 0, \\ -\lim_{y \rightarrow 0} y^{1-2\alpha} \varphi'(y) = c_\alpha, \\ \varphi(0) = 1. \end{cases} \tag{2.4}$$

The solution φ coincides with the solution of the following problem:

$$\varphi'' + \frac{1-2\alpha}{y} \varphi' - \varphi = 0, \quad \varphi(0) = 1, \quad \lim_{y \rightarrow \infty} \varphi(y) = 0,$$

and minimizes the functional

$$\int_0^\infty y^{1-2\alpha} (|\varphi(y)|^2 + |\varphi'(y)|^2) dy.$$

Moreover, it is a combination of Bessel functions, as shown in the following lemma.

LEMMA 2.3 ([2, Lemma 2.2]). *The solution of the ordinary differential equation*

$$\varphi'' + \frac{1-2\alpha}{y} \varphi' - \varphi = 0 \tag{2.5}$$

may be written as $\varphi(y) = y^\alpha \psi(y)$, where ψ solves the well-known Bessel equation

$$y^2 \psi'' + y \psi' - (y^2 + \alpha^2) \psi = 0. \tag{2.6}$$

In addition, (2.6) has two linearly independent solutions, I_α, Z_α , which are the modified Bessel functions; their asymptotic behaviour is given precisely by

$$\begin{aligned} I_\alpha(y) &\sim \frac{1}{\Gamma(\alpha+1)} \left(\frac{y}{2}\right)^\alpha \left(1 + \frac{y^2}{4(\alpha+1)} + \frac{y^4}{32(\alpha+1)(\alpha+2)} + \dots\right), \\ Z_\alpha(y) &\sim \frac{\Gamma(\alpha)}{2} \left(\frac{2}{y}\right)^\alpha \left(1 + \frac{y^2}{4(1-\alpha)} + \frac{y^4}{32(1-\alpha)(2-\alpha)} + \dots\right) \\ &\quad + \frac{\Gamma(-\alpha)}{2^\alpha} \left(\frac{y}{2}\right)^\alpha \left(1 + \frac{y^2}{4(\alpha+1)} + \frac{y^4}{32(\alpha+1)(\alpha+2)} + \dots\right), \end{aligned} \tag{2.7}$$

for $y \rightarrow 0^+$, $\alpha \notin \mathbb{Z}$. And when $y \rightarrow +\infty$,

$$\begin{aligned} I_\alpha(y) &\sim \frac{1}{\sqrt{2\pi y}} e^y \left(1 - \frac{4\alpha^2 - 1}{8y} + \frac{(4y^2 - 1)(4y^2 - 9)}{2!(8y)^2} + \dots \right), \\ Z_\alpha(y) &\sim \sqrt{\frac{\pi}{2y}} e^{-y} \left(1 - \frac{4\alpha^2 - 1}{8y} + \frac{(4y^2 - 1)(4y^2 - 9)}{2!(8y)^2} + \dots \right). \end{aligned} \tag{2.8}$$

In the sequel, we will use both types of solutions given in Lemma 2.3 (one is growing exponentially as $y \rightarrow \infty$, one is decaying exponentially to 0). Up to a normalization constant chosen in such a way that $\varphi(0) = 1$, we set

$$\varphi_1(y) := y^\alpha I_\alpha(y) \quad \text{and} \quad \varphi_2(y) := y^\alpha Z_\alpha(y). \tag{2.9}$$

REMARK 2.4. We observe that if φ satisfies (2.5), then the function $\bar{\varphi}(y) := \varphi(\mu y)$ is a solution of

$$\partial_y(y^{1-2\alpha} \partial_y \bar{\varphi}) = y^{1-2\alpha} \mu^2 \bar{\varphi}.$$

REMARK 2.5. As said before, φ_2 is the solution of (2.4) and, in particular, it satisfies

$$\lim_{y \rightarrow 0} \varphi_2(y) = 1, \tag{2.10}$$

$$-\lim_{y \rightarrow 0} y^{1-2\alpha} \partial_y \varphi_2(y) = c_\alpha, \tag{2.11}$$

where c_α is defined in (2.3).

Moreover, by (2.8) we have

$$\varphi_2(y) \sim y^{\alpha-1/2} e^{-y} \quad \text{as } y \rightarrow \infty. \tag{2.12}$$

3. The one-dimensional solution

In this section, we study the existence for solutions of the following problem:

$$\begin{cases} \operatorname{div}(y^{1-2\alpha} \nabla v) = 0 & \text{in } [0, \lambda] \times (0, +\infty), \\ v = 0 & \text{in } \partial[0, \lambda] \times (0, +\infty), \\ -\frac{1}{c_\alpha} \lim_{y \rightarrow 0} y^{1-2\alpha} \partial_y v = -F'(v) & \text{on } [0, \lambda] \times \{0\}. \end{cases} \tag{3.1}$$

The energy functional associated to problem (3.1) is given by

$$E_0(v) := \frac{1}{c_\alpha} \int_0^{+\infty} \int_0^\lambda \frac{1}{2} y^{1-2\alpha} |\nabla v(s, y)|^2 ds dy + \int_0^\lambda F(v(s, 0)) ds. \tag{3.2}$$

We consider the energy functional E_0 in the class of functions

$$H_{0,\ell}^1([0, \lambda] \times (0, +\infty), y^{1-2\alpha}) := \{v : [0, \lambda] \times (0, +\infty) \rightarrow \mathbb{R} \mid v = 0 \text{ on } \partial[0, \lambda] \times (0, +\infty), y^{1-2\alpha}(|v|^2 + |\nabla v|^2) \in L^1([0, \lambda] \times (0, +\infty))\}.$$

As explained in Section 2 (see [11, 24]), if v solves (3.1), then its trace u on $[0, \lambda]$ is a solution of the fractional Dirichlet problem

$$\begin{cases} A_\alpha u + F'(u) = 0 & \text{in } [0, \lambda], \\ u(0) = u(\lambda) = 0. \end{cases} \tag{3.3}$$

In the following lemma, we give a sufficient and necessary condition on the parameter λ for the existence of non-trivial solutions to problem (3.1).

LEMMA 3.1. *Let λ_* be defined as in (1.6) and assume that $\lambda > \lambda_*$ is fixed. Then, there exists a non-trivial positive weak solution of (3.1), which is a minimizer of E_0 in the space $H_{0,\ell}^1([0, \lambda] \times (0, +\infty), y^{1-2\alpha})$. Assume that $\lambda \leq \lambda_*$; then there are no positive weak solutions of (3.1) and the trivial solution 0 is the unique minimizer of E_0 .*

Proof. We start by proving the existence of a minimizer for E_0 in the class $H_{0,\ell}^1([0, \lambda] \times (0, +\infty), y^{1-2\alpha})$. Following [14, proof of Theorem 1.6] and [9, proof of Lemma 4.1], we introduce a new potential \tilde{F} with the following properties:

- (1) $\tilde{F} = F$ in $[-1, 1]$;
- (2) $\tilde{F} > 0$ in $\mathbb{R} \setminus [-1, 1]$;
- (3) \tilde{F} is even;
- (4) \tilde{F} has linear growth at infinity.

We consider the new energy functional

$$\tilde{E}_0(v) = \frac{1}{c_0} \int_0^{+\infty} \int_0^\lambda \frac{1}{2} y^{1-2\alpha} |\nabla v(s, y)|^2 ds dy + \int_0^\lambda \tilde{F}(v(s, 0)) ds.$$

Note that every minimizer w of \tilde{E}_0 in $H_{0,\ell}^1([0, \lambda] \times (0, +\infty), y^{1-2\alpha})$ such that $-1 \leq w \leq 1$ is also a minimizer of E_0 in the set

$$\{v \in H_{0,\ell}^1([0, \lambda] \times (0, +\infty), y^{1-2\alpha}) : -1 \leq v \leq 1\}.$$

We show now that \tilde{E}_0 admits a minimizer in $H_{0,\ell}^1([0, \lambda] \times (0, +\infty), y^{1-2\alpha})$. Indeed, by the properties of \tilde{F} , it follows that \tilde{E}_0 is well-defined, bounded below and coercive in $H_{0,\ell}^1([0, \lambda] \times (0, +\infty), y^{1-2\alpha})$.

By a standard variational argument, and using compactness of the inclusion $H_{0,\ell}^1([0, \lambda] \times (0, +\infty), y^{1-2\alpha}) \subset \tilde{L}^2([0, \lambda])$ (which was proved in [24, Section 4]), taking a minimizing sequence $v_k \in H_{0,\ell}^1([0, \lambda] \times (0, +\infty), y^{1-2\alpha})$ and a subsequence convergent in $\tilde{L}^2([0, \lambda])$, we conclude the existence of a minimizer v_0 for \tilde{E}_0 . Note moreover that, without loss of generality, we may assume that $0 \leq v_0 \leq 1$ because, if not, we can replace the minimizing sequence v_k with the sequence $\min\{|v_k|, 1\}$. Indeed, it is also minimizing because \tilde{F} is even and $\tilde{F} \geq \tilde{F}(1)$.

Now we want to prove that if $\lambda > \lambda_*$, then 0 cannot be a minimizer. To this aim, we consider the function

$$w(s, y) := \sin\left(\frac{\pi}{\lambda} s\right) \varphi_2\left(\frac{\pi}{\lambda} y\right),$$

where φ_2 is defined as in (2.9). By Remark 2.4 and by (2.10) and (2.11) w satisfies the problem

$$\begin{cases} \operatorname{div}(y^{1-2\alpha} \nabla w) = 0 & \text{in } [0, \lambda] \times (0, +\infty), \\ w = 0 & \text{on } \partial[0, \lambda] \times (0, +\infty), \\ -\frac{1}{c_\alpha} \lim_{y \rightarrow 0} y^{1-2\alpha} \partial_y w = \left(\frac{\pi}{\lambda}\right)^{2\alpha} w & \text{on } [0, \lambda] \times \{y = 0\}. \end{cases} \tag{3.4}$$

We use a small multiple of w as a test function to prove that 0 is not a minimizer when $\lambda > \lambda_*$. First of all, we observe that

$$E_0(0) = \lambda F(0).$$

On the other hand, using Taylor expansion for F , we have that

$$E_0(\varepsilon w) = \lambda F(0) + \frac{\varepsilon^2}{2c_\alpha} \int_0^{+\infty} \int_0^\lambda y^{1-2\alpha} |\nabla w(s, y)|^2 ds dy + \frac{\varepsilon^2}{2} \int_0^\lambda F''(0) w(s, 0)^2 ds + \mathcal{O}(\varepsilon^4).$$

We first observe that

$$\int_0^\lambda F''(0) w(s, 0)^2 ds = F''(0) \int_0^\lambda \sin^2\left(\frac{\pi}{\lambda} s\right) ds = \frac{\lambda}{2} F''(0). \tag{3.5}$$

To compute the Dirichlet energy, we use the change of variable $\bar{y} = \frac{\pi}{\lambda} y$ and we integrate by parts in \bar{y} to get

$$\begin{aligned} & \frac{1}{2c_\alpha} \int_0^{+\infty} \int_0^\lambda y^{1-2\alpha} |\nabla w(s, y)|^2 ds dy \\ &= \frac{1}{2c_\alpha} \int_0^{+\infty} \int_0^\lambda y^{1-2\alpha} \left(\frac{\pi}{\lambda}\right)^2 \left[\cos^2\left(\frac{\pi}{\lambda} s\right) \varphi_2^2\left(\frac{\pi}{\lambda} y\right) + \sin^2\left(\frac{\pi}{\lambda} s\right) \varphi_2^2\left(\frac{\pi}{\lambda} y\right) \right] ds dy \\ &= \frac{1}{c_\alpha} \frac{\lambda}{4} \left(\frac{\pi}{\lambda}\right)^{2\alpha} \int_0^{+\infty} \bar{y}^{1-2\alpha} [\varphi_2^2(\bar{y}) + \dot{\varphi}_2^2(\bar{y})] d\bar{y} \\ &= \frac{1}{c_\alpha} \frac{\lambda}{4} \left(\frac{\pi}{\lambda}\right)^{2\alpha} \int_0^{+\infty} (\bar{y}^{1-2\alpha} \varphi_2^2(\bar{y}) - \varphi_2(\bar{y}) \partial_{\bar{y}}(\bar{y}^{1-2\alpha} \partial_{\bar{y}} \varphi_2(\bar{y}))) d\bar{y} \\ &\quad - \frac{1}{c_\alpha} \frac{\lambda}{4} \left(\frac{\pi}{\lambda}\right)^{2\alpha} \lim_{\bar{y} \rightarrow 0} \bar{y}^{1-2\alpha} \varphi_2(\bar{y}) \dot{\varphi}_2(\bar{y}) \\ &= \frac{\lambda}{4} \left(\frac{\pi}{\lambda}\right)^{2\alpha}, \end{aligned} \tag{3.6}$$

where in the last equality we have used that φ_2 satisfies (2.5) and the asymptotic behaviours (2.10) and (2.11).

Therefore, combining together (3.5) and (3.6), we deduce that

$$E_0(\varepsilon w) = \lambda F(0) + \frac{\varepsilon^2}{4} \lambda \left(\left(\frac{\pi}{\lambda}\right)^{2\alpha} + F''(0) \right) + \mathcal{O}(\varepsilon^4) < E_0(0),$$

for ε small enough, provided that $\lambda > \lambda_*$. This concludes the proof of the first part of the statement, since we get a non-trivial minimizer for E_0 , which can be chosen to be positive by standard arguments.

To prove the non-existence of positive solutions for $\lambda \leq \lambda_*$, we multiply the first equation of (3.1) by w and we integrate by parts to get

$$\begin{aligned} 0 &= \frac{1}{2c_\alpha} \int_0^{+\infty} \int_0^\lambda \operatorname{div}(y^{1-2\alpha} \nabla v(s, y)) w(s, y) ds dy \\ &= -\frac{1}{2c_\alpha} \int_0^{+\infty} \int_0^\lambda y^{1-2\alpha} \nabla v(s, y) \cdot \nabla w(s, y) ds dy \\ &\quad - \int_0^\lambda \lim_{y \rightarrow 0} y^{1-2\alpha} \partial_y v(s, y) w(s, 0) ds \\ &= \frac{1}{2c_\alpha} \int_0^\lambda \lim_{y \rightarrow 0} y^{1-2\alpha} \partial_y w(s, y) v(s, y) ds + \int_0^\lambda F'(v) w(s, 0) ds \\ &= \frac{1}{2} \int_0^\lambda w(s, 0) \left(F'(v) + \left(\frac{\pi}{\lambda}\right)^{2\alpha} v(s, 0) \right) ds, \end{aligned}$$

where in the last two equalities we have used that w satisfies (3.4). This concludes the proof of the lemma; indeed, by assumption $F'(v) \geq F''(0)v$ for any $v \geq 0$, and therefore we must have $v(s, 0) \equiv 0$ when $\lambda \leq \lambda_*$, which implies $v \equiv 0$ by uniqueness of solutions to the extended problem (2.2). \square

For $\lambda > \lambda_*$, we will denote by v_0 the non-trivial minimizer of E_0 , whose existence is established in Theorem 3.1. From what we have seen above, we have the inequality

$$E_0(v_0) < \lambda F(0). \tag{3.7}$$

As a consequence of Lemma 3.1, we deduce the following corollary.

COROLLARY 3.2. *There exists a non-trivial positive solution of the fractional Dirichlet problem (3.3) on $[0, \lambda]$ if and only if $\lambda > \lambda_*$, where λ_* is defined as in (1.6).*

4. The existence result for $\lambda > \lambda_*$

This section is devoted to the proof of Theorem 1.1.

In the following, we will use cylindrical coordinates $(r, \theta, s, y) \in [0, +\infty) \times \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}^+$ to parametrize $\mathbb{R}^3 \times \mathbb{R}^+$. In order to find a solution of (1.1) which is invariant under the screw motion σ_λ^β , we will look for a solution v of the extended problem (1.7) which is invariant under the transformation, which for simplicity of notation we still denote by σ_λ^β , acting on $\mathbb{R}^3 \times \mathbb{R}^+$ in the natural way

$$\sigma_\lambda^\beta(z, s, y) = \left(e^{i\beta}z, s + \frac{\lambda}{\pi}\beta, y \right).$$

More precisely, we require that

$$v(r, \theta, s, y) = v\left(r, \theta + \beta, s + \frac{\lambda}{\pi}\beta, y\right) \quad \text{for every } \beta \in \mathbb{R},$$

which implies that

$$\begin{aligned} v(r, \theta, s, y) &= v(r, \theta, s + 2\lambda, y) \quad \text{and} \\ v(r, \theta, s, y) &= v\left(r, 0, s - \frac{\lambda}{\pi}\theta, y\right). \end{aligned} \tag{4.1}$$

Moreover, we assume that

$$v(r, \theta, -s, y) = -v(r, \theta, s, y).$$

Using these invariances, it is clear that in order to construct the solution v , it is enough to construct it for $\theta = 0$; that is, we just need to know

$$V(r, s, y) = v(r, 0, s, y) \quad \text{for } (r, s, y) \in [0, +\infty) \times [0, \lambda] \times \mathbb{R}^+.$$

Indeed, once we have $V(r, s, y)$ defined on $[0, +\infty) \times [0, \lambda] \times \mathbb{R}^+$, it is enough to reflect it oddly with respect to $\{s = 0\}$ and $\{s = \lambda\}$; in this way we obtain a function $V(r, s, \lambda)$ defined on all $[0, +\infty) \times \mathbb{R} \times \mathbb{R}^+$. Finally, we use (4.1) to define v in terms of V . Of course, by construction, we have that if V is positive in $(0, +\infty) \times (0, \lambda) \times \mathbb{R}^+$ and vanishes on $\partial([0, +\infty) \times [0, \lambda]) \times \mathbb{R}^+$, then the zero-level set of v is $H_\lambda \times \mathbb{R}^+$, and therefore the zero-level set of its trace u on $\{y = 0\}$, that is a solution of (1.1), is exactly the helicoid H_λ .

We need now to write problem (1.7) and the energy (1.8) in the cylindrical coordinates introduced above for functions invariant under screw motion. Problem (1.7) becomes

$$\begin{cases} V_{rr} + \frac{1}{r}V_r + \left(1 + \frac{\lambda^2}{\pi^2 r^2}\right)V_{ss} + V_{yy} + \frac{1-2\alpha}{y}V_y = 0 & \text{in } [0, +\infty) \times [0, \lambda] \times \mathbb{R}^+, \\ V = 0 & \text{on } \{s = 0\} \cup \{s = \lambda\}, \\ -\frac{1}{c_\alpha} \lim_{y \rightarrow 0} y^{1-2\alpha} \partial_y V = -F'(V) & \text{on } \{y = 0\}. \end{cases} \tag{4.2}$$

In what follows, we will use the following notations:

$$S_R := [0, R] \times [0, \lambda] \quad \text{and} \quad C_{R,L} := S_R \times [0, L].$$

We define the following subsets of $\partial C_{R,L}$:

$$\begin{aligned} \partial^+ C_{R,L} &:= \overline{\partial C_{R,L} \cap \{y > 0\}}, \\ \partial^0 C_{R,L} &:= \partial C_{R,L} \setminus \partial^+ C_{R,L}. \end{aligned}$$

The energy associated to problem (4.2) in the cylinder $C_{R,L}$ is given by

$$\begin{aligned} E(V, C_{R,L}) &= \frac{1}{2c_\alpha} \iint_{C_{R,L}} y^{1-2\alpha} \left(|V_r|^2 + \left(1 + \frac{\lambda^2}{\pi^2 r^2}\right) |V_s|^2 + |V_y|^2 \right) r \, dr \, ds \, dy \\ &\quad + \int_{\partial^0 C_{R,L}} F(V) r \, dr \, ds. \end{aligned}$$

Let $H^1(C_{R,L}, y^{1-2\alpha})$ denote the weighted Sobolev space

$$H^1(C_{R,L}, y^{1-2\alpha}) = \{v : C_{R,L} \rightarrow \mathbb{R} \mid y^{1-2\alpha}(v^2 + |\nabla v|^2) \in L^1_r(C_{R,L})\},$$

where L^1_r denotes the space $L^1(C_{R,L})$ with respect to the measure $r \, dr \, ds \, dy$, and let $\tilde{H}^1_0(C_{R,L}, y^{1-2\alpha})$ be the space

$$\tilde{H}^1_0(C_{R,L}, y^{1-2\alpha}) = \{v \in H^1(C_{R,L}, y^{1-2\alpha}) \mid |v| \leq 1, v \equiv 0 \text{ on } \partial^+ C_{R,L}\}.$$

We recall that (see the proof of Lemma 4.1 in [9]) the inclusion

$$\tilde{H}^1_0(C_{R,L}, y^{1-2\alpha}) \subset\subset L^2(\partial^0 C_{R,L}) \tag{4.3}$$

is compact.

We are now ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Similarly to the proof of Lemma (3.1), we introduce the functional $\tilde{E}(\cdot, C_{R,L})$ obtained by just replacing the potential F with the function \tilde{F} (which is the same one in the proof of Lemma (3.1)). We observe as before that every minimizer W of \tilde{E} in $\tilde{H}^1_0(C_{R,L}, y^{1-2\alpha})$, such that $|W| \leq 1$, is also a minimizer for E in the class $\{V \in H^1_0(C_{R,L}, y^{1-2\alpha}) : |V| \leq 1\}$. By a standard variational argument and using compactness of the inclusion (4.3), taking a minimizing sequence $\{V_{R,L}^k\} \in \tilde{H}^1_0(C_{R,L}, y^{1-2\alpha})$ and a subsequence convergent in $L^2(\partial^0 C_{R,L})$, we conclude that $\tilde{E}(\cdot, C_{R,L})$ admits an absolute minimizer $V_{R,L}$ in $\tilde{H}^1_0(C_{R,L}, y^{1-2\alpha})$. Note moreover that, without loss of generality, we may assume that $0 \leq V_{R,L} \leq 1$ because, if not, we can replace the minimizing sequence $V_{R,L}^k$ with the sequence $\min\{|V_{R,L}^k|, 1\}$. Indeed, it is also minimizing because \tilde{F} is even and $\tilde{F} \geq \tilde{F}(1)$.

It is easy to check that $V_{R,L}$ is a solution of (4.2) in $C_{R,L}$ (with zero-Dirichlet boundary condition on $\partial^+ C_{R,L}$). Arguing as in the [14, proof of Theorem 1.3], one can prove that $V_{R,L}$ extends to a solution of

$$\begin{cases} \operatorname{div}(y^{1-2\alpha}\nabla v) = 0 & \text{in } C_{R,L}, \\ v \equiv 0 & \text{on } \partial^+ C_{R,L}, \\ -\frac{1}{c_\alpha} \lim_{y \rightarrow 0} y^{1-2\alpha} \partial_y v = -F'(v) & \text{on } \partial^0 C_{R,L}, \end{cases}$$

where, for simplicity, we keep the notation $C_{R,L}$ to denote the corresponding cylinder in \mathbb{R}_+^4 , that is,

$$C_{R,L} = \{(x, y) = (r, \theta, s, y) \in \mathbb{R}_+^4 : 0 < r < R, 0 < y < L\},$$

and the subsets of its boundary

$$\partial^+ C_{R,L} = \overline{\partial C_{R,L} \cap \mathbb{R}_+^4}, \quad \partial^0 C_{R,L} := \partial C_{R,L} \setminus \partial^+ C_{R,L}.$$

Some care is needed to show that it is a solution close to $\{r = 0\}$, and we refer the reader to [14] for details. We now wish to pass to the limit in R and L , and obtain a solution in all of $[0, \infty) \times [0, \lambda] \times \mathbb{R}^+$. Let $S > 0$, $L' > 0$ and consider the family $\{V_{R,L}\}$ of solutions in $[0, S + 2] \times [0, \lambda] \times [0, L' + 2]$, with $R > S + 2$ and $L > L' + 2$. Since $|V_{R,L}| \leq 1$, regularity results proved in [8, Proposition 4.6] give a uniform $C^\gamma([0, S] \times [0, \lambda] \times [0, L'])$ bound for $V_{R,L}$, for some $\gamma \in (0, 1)$ (uniform with respect to R and L).

Choose now $L = R^b$, with $\frac{1}{2} < b < 1$ (this choice will be used later to prove that the solution that we construct is not identically zero). By the Ascoli–Arzelà Theorem, a subsequence of $\{V_{R,R^b}\}$ converges in $C^\gamma([0, S] \times [0, \lambda] \times [0, S^b])$ to a solution in $([0, S] \times [0, \lambda] \times [0, S^b])$. Taking $S = 1, 2, 3, \dots$ and making a Cantor diagonal argument, we obtain a sequence V_{R_j, R_j^b} converging in $C_{loc}^\gamma([0, \infty) \times [0, \lambda] \times \mathbb{R}^+)$ to a solution $V \in C_{loc}^\gamma([0, \infty) \times [0, \lambda] \times \mathbb{R}^+)$. By construction, we have found a solution V with $|V| \leq 1$.

It only remains to prove that V is positive.

We start by proving that V is not identically zero. In order to do that, following the argument in [14], we establish an energy estimate for V using a comparison argument, based on the minimality of V_{R,R^b} in the set C_{R,R^b} .

Suppose by contradiction that $\lambda > \lambda_*$ and that $V \equiv 0$. Then, given $R > 0$, the minimizing sequence V_{R,R^b} converges uniformly to 0 on C_{R,R^b} when $R \rightarrow +\infty$. The energy of 0 in C_{R,R^b} is clearly

$$E(0, C_{R,R^b}) = \frac{\lambda}{2} F(0) R^2.$$

Therefore, for any function W vanishing on $\partial^+ C_{R,R^b}$, we have that

$$E(W, C_{R,R^b}) \geq E(0, C_{R,R^b}) = \frac{\lambda}{2} F(0) R^2. \tag{4.4}$$

We build now a suitable competitor W and we arrive to a contradiction with (4.4). First, we define two smooth cut-off functions η and ξ as follows:

$$\eta(r) := \begin{cases} 0 & \text{for } r \in [0, \frac{1}{2}] \cup [R - \frac{1}{2}, R], \\ 1 & \text{for } r \in [1, R - 1], \end{cases}$$

and

$$\xi(y) = \begin{cases} 1 & \text{if } 0 < y \leq R^b - R^a, \\ \frac{\log R^b - \log y}{\log R^b - \log (R^b - R^a)} & \text{if } R^b - R^a < y \leq R^b. \end{cases}$$

We set

$$W(r, s, y) := \eta(r)\xi(y)v_0(s, y),$$

where v_0 is the minimizer of E_0 (see (3.2)), whose existence is established in Lemma 3.1 and which satisfies (3.7).

We compute now the energy of W . First, we observe that the potential energy is estimated by

$$\int_0^\lambda \int_0^R F(W(r, s, 0))r \, dr \, ds \leq \frac{R^2}{2} \int_0^\lambda F(v_0(s, 0)) \, ds + \mathcal{O}(R). \tag{4.5}$$

We estimate now the Dirichlet energy. Since $|\eta|, |\xi|, |v_0| \leq 1$, we have that

$$\begin{aligned} & \frac{1}{2c_\alpha} \iint_{C_{R,R^b}} y^{1-2\alpha} |\nabla W|^2 \, dx \, dy \\ & \leq \frac{1}{2c_\alpha} \iint_{C_{R,R^b}} y^{1-2\alpha} (|\nabla \eta|^2 + |\nabla v_0|^2 + |\dot{\xi}|^2) \, dx \, dy. \end{aligned} \tag{4.6}$$

This, together with (4.5), implies

$$E(W, C_{R,R^b}) \leq \frac{R^2}{2} E_0(v_0) + \frac{1}{2c_\alpha} \iint_{C_{R,R^b}} y^{1-2\alpha} (|\nabla \eta|^2 + |\dot{\xi}|^2) \, dx \, dy + \mathcal{O}(R).$$

We estimate now the second term on the right-hand side above. In the following, C_α will denote positive, possibly different, constants depending only on α . Using the definition of η , we have that

$$\iint_{C_{R,R^b}} y^{1-2\alpha} |\nabla \eta|^2 \, dx \, dy \leq C_\alpha \lambda R \int_0^{R^b} y^{1-2\alpha} \, dy = C_\alpha \lambda R^{1+2b(1-\alpha)}. \tag{4.7}$$

On the other hand, by the definition of ξ and using polar coordinates, we have

$$\begin{aligned} & \int_0^{R^b} \int_0^\lambda \int_0^R y^{1-2\alpha} |\dot{\xi}(y)|^2 r \, dr \, ds \, dy \leq C_\alpha \lambda R^2 \frac{1}{(\log R^b / (R^b - R^a))^2} \int_{R^b - R^a}^{R^b} \frac{y^{1-2\alpha}}{y^2} \, dy \\ & \leq C_\alpha \lambda R^2 \frac{1}{-\log(1 - R^{a-b})^2} \left[\frac{1}{R^{2\alpha b} - R^{2\alpha a}} - \frac{1}{R^{2\alpha b}} \right] \\ & \leq C_\alpha \lambda R^2 \cdot R^{2(b-a)} \cdot R^{-2\alpha b} = C_\alpha \lambda R^{2+2b(1-\alpha)-2a}. \end{aligned} \tag{4.8}$$

Therefore, plugging (4.5), (4.7) and (4.8) into (4.6), we deduce

$$E(W, C_{R,R^b}) \leq C_\alpha \lambda R^{1+2b(1-\alpha)} + C_\alpha \lambda R^{2+2b(1-\alpha)-2a} + \frac{R^2}{2} E_0(v_0) + \mathcal{O}(R). \tag{4.9}$$

Now we choose $\frac{1}{2} < a < b < 1/2(1 - \alpha)$. With this choice of b and a , there exists $\varepsilon = \varepsilon(b)$ such that

$$E(W, C_{R,R^b}) \leq C_\alpha \lambda R^{2-\varepsilon} + \frac{R^2}{2} E_0(v_0) + \mathcal{O}(R^{2-\varepsilon}).$$

This, together with (4.4), implies

$$\frac{1}{2} F(0) R^2 \leq \frac{R^2}{2} E_0(v_0) + \mathcal{O}(R).$$

This gives a contradiction, since for $\lambda > \lambda_*$ we have that

$$E_0(v_0) < \lambda F(0),$$

which was established in (3.7).

This implies that V is not identically zero. Since by construction $V \geq 0$, we conclude that V is strictly positive using the strong maximum principle. \square

5. The non-existence result

In this section, we give the proof of our non-existence result Theorem 1.2.

In order to do this, we need to establish exponential decay in the y -variable of a bounded solution v of (1.7), which vanishes on $H_\lambda \times \mathbb{R}^+$ and is invariant under screw motion. We stress that, in general, exponential decay in the y variable for bounded solutions in all the half-space of problem (1.7) is not true, as one can see in the particular case of the one-dimensional Peierls–Nabarro problem:

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}_+^2, \\ -\partial_y v(x, 0) = \frac{1}{\pi} \sin(\pi u) & \text{on } \mathbb{R}, \end{cases}$$

for which an explicit solution is given by $v(x, y) = 2/\pi \arctan(x/(1+y))$ (see [10] and references therein). On the other hand, as shown in Lemmas 2.2 and 2.3, the solution of problem (2.2) in a cylinder $\mathcal{C}_\Omega = \Omega \times \mathbb{R}^+$, with Ω bounded, with zero-Dirichlet condition on the lateral boundary $\partial\Omega \times \mathbb{R}^+$ decays exponentially in y . In our situation, we are able to prove exponential decay, owing to the symmetry of the problem and to the invariances of the solution v .

PROPOSITION 5.1. *Suppose that v is a bounded solution of*

$$\begin{cases} \operatorname{div}(y^{1-2\alpha} \nabla v) = 0 & \text{in } \mathbb{R}_+^4, \\ -\frac{1}{c_\alpha} \lim_{y \rightarrow 0} y^{1-2\alpha} \partial_y v(x, y) = -F'(v), \end{cases}$$

which vanishes on $H_\lambda \times \mathbb{R}^+$ and is invariant under screw motion.

Then, there exist a positive constant K such that

$$|v(x, y)| \leq K \varphi_2(y) \sim y^{\alpha-1/2} e^{-y} \quad \text{as } y \rightarrow \infty. \tag{5.1}$$

Proof. Using polar coordinates and the invariance of v , as done in Section 4, we have that the function $V(r, s, y) = v(r, 0, s, y)$ satisfies

$$\begin{cases} V_{rr} + \frac{1}{r} V_r + \left(1 + \frac{\lambda^2}{\pi^2 r^2}\right) V_{ss} + V_{yy} + \frac{1-2\alpha}{y} V_y = 0 & \text{in } [0, +\infty) \times [0, \lambda] \times \mathbb{R}^+, \\ V = 0 & \text{on } s \in \{0, \lambda\}, \\ -\frac{1}{c_\alpha} \lim_{y \rightarrow 0} y^{1-2\alpha} \partial_y v(x, y) = -F'(v). \end{cases} \tag{5.2}$$

We define now the following function, which provides an upper barrier for V . For $C > e^2$, K large enough to be chosen later and $\varepsilon > 0$, we set

$$w^\varepsilon(r, s, y) := K \sin\left(\frac{\pi}{\lambda} s\right) \cdot \left[\varphi_2\left(\frac{\pi}{\lambda} y\right) + \varepsilon \varphi_1\left(\frac{\pi}{\lambda} y\right) + \varepsilon(e^{(1/2)(\pi/\lambda)r} + C e^{-(1/2)(\pi/\lambda)r})\right], \tag{5.3}$$

where φ_1 and φ_2 were defined in (2.9). To conclude the proof, it is enough to prove that, for any $\varepsilon > 0$ (sufficiently small),

$$V \leq w^\varepsilon. \tag{5.4}$$

Then, the conclusion follows by sending $\varepsilon \rightarrow 0$.

We start by showing that w_ε satisfies

$$w_{rr}^\varepsilon + \frac{1}{r}w_r^\varepsilon + \left(1 + \frac{\lambda^2}{\pi^2 r^2}\right)w_{ss}^\varepsilon + w_{yy}^\varepsilon + \frac{1-2\alpha}{y}w_y^\varepsilon \leq 0. \tag{5.5}$$

By a direct computation, we have that

$$\begin{aligned} & \frac{1}{K} \left[w_{rr}^\varepsilon + \frac{1}{r}w_r^\varepsilon + \left(1 + \frac{\lambda^2}{\pi^2 r^2}\right)w_{ss}^\varepsilon + w_{yy}^\varepsilon + \frac{1-2\alpha}{y}w_y^\varepsilon \right] \\ &= -\frac{3}{4}\varepsilon \left(\frac{\pi}{\lambda}\right)^2 \sin\left(\frac{\pi}{\lambda}s\right) \left(e^{(1/2)(\pi/\lambda)r} + C e^{-(1/2)(\pi/\lambda)r}\right) \\ & \quad + \frac{1}{2r}\varepsilon \frac{\pi}{\lambda} \sin\left(\frac{\pi}{\lambda}s\right) \left(e^{(1/2)(\pi/\lambda)r} - C e^{-(1/2)(\pi/\lambda)r}\right) - \frac{1}{r^2}w^\varepsilon \\ &\leq -\frac{3}{4}\varepsilon \left(\frac{\pi}{\lambda}\right)^2 \sin\left(\frac{\pi}{\lambda}s\right) \left(e^{(1/2)(\pi/\lambda)r} + C e^{-(1/2)(\pi/\lambda)r}\right) \\ & \quad + \frac{1}{2r}\varepsilon \frac{\pi}{\lambda} \sin\left(\frac{\pi}{\lambda}s\right) \left(e^{(1/2)(\pi/\lambda)r} - C e^{-(1/2)(\pi/\lambda)r}\right) = A_1 + A_2. \end{aligned} \tag{5.6}$$

We study now separately the cases $r \geq \lambda/\pi$ and $r < \lambda/\pi$.

If $r \geq \lambda/\pi$, then the last term in (5.6) is bounded above by

$$\begin{aligned} A_2 &\leq \frac{1}{2}\varepsilon \left(\frac{\pi}{\lambda}\right)^2 \sin\left(\frac{\pi}{\lambda}s\right) \left(e^{(1/2)(\pi/\lambda)r} - C e^{-(1/2)(\pi/\lambda)r}\right) \\ &\leq \frac{1}{2}\varepsilon \left(\frac{\pi}{\lambda}\right)^2 \sin\left(\frac{\pi}{\lambda}s\right) \left(e^{(1/2)(\pi/\lambda)r} + C e^{-(1/2)(\pi/\lambda)r}\right). \end{aligned}$$

This, combined with (5.6) implies (5.5) when $r \geq \lambda/\pi$.

When $r < \lambda/\pi$, we immediately deduce that

$$e^{(1/2)(\pi/\lambda)r} - C e^{-(1/2)(\pi/\lambda)r} \leq 0,$$

since we have chosen $C > e^2$ and therefore (5.5) holds. It remains to prove that $w^\varepsilon \geq V$ on $\partial([0, +\infty) \times [0, \lambda] \times \mathbb{R}^+)$. On the set $\{s = 0\} \cup \{s = \lambda\}$, this is easy, since by definition $w^\varepsilon = 0 = V$, while on $\{r = 0\}$ we have $w^\varepsilon \geq 0 = V$. When $y \rightarrow \infty$ and $r \rightarrow \infty$, it is also true. Indeed, V is bounded and, for any fixed ε , w^ε can be made arbitrarily large for y and r sufficiently large. To conclude, we just have to prove that $w^\varepsilon \geq V$ on $\{y = 0\}$. On this part of the boundary, by definition of w^ε , we have that, for any $\varepsilon > 0$,

$$w^\varepsilon(r, s, 0) \geq K \sin\left(\frac{\pi}{\lambda}s\right),$$

since $\varphi_1(0) = \varphi_2(0) = 1$. By [8, Proposition 4.6], we know that v has bounded x -gradient, in particular $v_s = V_s$ is bounded. Therefore, since $V = 0$ on $\{s = 0\} \cup \{s = \lambda\}$, we deduce that there exists a constant \tilde{C} such that

$$|V(r, s, 0)| \leq \tilde{C} \min\{s, \lambda - s\}.$$

To conclude, we observe that it is possible to choose K sufficiently large independently of ε , so that

$$w^\varepsilon(r, s, 0) \geq K \sin\left(\frac{\pi}{\lambda}s\right) \geq \tilde{C} \min\{s, \lambda - s\} \geq |V(r, s, 0)|. \tag{5.7}$$

We remark that K can be chosen independently of ε since it only needs to satisfy the second inequality in (5.7) above, in which ε does not appear anymore. We have proved that w^ε is an upper barrier for V , and therefore (5.4) holds. This concludes the proof of the proposition. \square

We can now prove our non-existence result.

Proof of Theorem 1.2. We start by observing that, by uniqueness of solutions to problem

$$\begin{cases} \operatorname{div}(y^{1-2\alpha}\nabla v) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ v(x, 0) = u(x) \end{cases}$$

and since the operator $\operatorname{div}(y^{1-2\alpha}\nabla)$ is invariant under the screw motion σ_λ^β , in order to prove Theorem 1.2, it is enough to prove the corresponding non-existence result for the extended problem (1.7). We write $V(r, s, y) = v(r, 0, s, y)$ and we consider problem (4.2) written in cylindrical coordinates. Let η be a cut-off function only depending on r such that

$$\eta(r) = \begin{cases} 1 & r \leq R \\ 0 & r \geq 2R \end{cases} \quad \text{and} \quad |\nabla\eta| \leq \frac{C}{R}.$$

We multiply (4.2) by $V\eta^2$ and we integrate by parts to obtain

$$\begin{aligned} & \int_0^{+\infty} \int y^{1-2\alpha} \left(|V_r|^2 + \left(1 + \frac{\lambda^2}{\pi^2 r^2} \right) |V_s|^2 + |V_y|^2 \right) \eta^2 r \, dr \, ds \, dy \\ & + \int F'(V)V\eta^2 r \, dr \, ds = 2 \int_0^{+\infty} \int y^{1-2\alpha} V V_r \eta \eta_r r \, dr \, ds \, dy, \end{aligned} \tag{5.8}$$

where the domain of integration is $[0, \infty) \times [0, \lambda]$ where it is not explicitly written. Now, we use the assumption $F'(t)t \geq F''(0)t^2$ for any $t \in \mathbb{R}$ to get

$$\begin{aligned} & \int_0^\infty \int_0^\lambda y^{1-2\alpha} (|V_s|^2 + |V_y|^2) \, ds \, dy + \int_0^\lambda F'(V)V \, ds \\ & \geq \int_0^\infty \int_0^\lambda y^{1-2\alpha} (|V_s|^2 + |V_y|^2) \, ds \, dy + \int_0^\lambda F''(0)V^2 \, ds. \end{aligned}$$

Now we want to use the stability of V to prove that the quantity above is non-negative. Since V is not compactly supported in y , in order to do that, we use a truncation argument in the y -variable. More precisely, given $L > 1$, let ψ be a cut-off function depending only on y such that $\psi(y) = 1$ for $0 < y < L - 1$ and $\psi(y) = 0$ for $y > L$. We set $V^L := V\psi$, so that $V_s^L = V_s\psi$ and $V_y^L = V_y\psi + V\psi'$.

Using V^L as a test function in the stability inequality and the fact that 0 is the unique absolute minimizer for the one-dimensional problem when $\lambda \leq \lambda_*$, we have

$$\begin{aligned} 0 & \leq \int_0^\infty \int_0^\lambda y^{1-2\alpha} (|V_s^L|^2 + |V_y^L|^2) \, ds \, dy + \int_0^\lambda F''(0)(V^L)^2 \, ds \\ & \leq \int_0^L y^{1-2\alpha} (|V_s|^2 + |V_y|^2) \psi^2 \, ds \, dy + \int_{L-1}^L \int_0^\lambda (\psi'^2 V^2 + 2\psi\psi'VV_y) + \int_0^\lambda F''(0)(V^L)^2 \, ds. \end{aligned}$$

Using now that $\psi \leq 1$, ψ' and V_y are bounded for $y \in (L - 1, L)$ and the exponential decay for V established in Proposition (5.1), passing to the limit as $L \rightarrow \infty$, we get

$$\int_0^\infty \int_0^\lambda y^{1-2\alpha} (|V_s|^2 + |V_y|^2) \, ds \, dy + \int_0^\lambda F''(0)V^2 \, ds \geq 0.$$

Therefore, from (5.8) we deduce

$$\int_0^\infty \int y^{1-2\alpha} |V_r|^2 \eta^2 r \, dr \, ds \, dy \leq 2 \int_0^\infty \int y^{1-2\alpha} V V_r \eta \eta_r r \, dr \, ds \, dy.$$

Using Cauchy–Schwarz inequality on the right-hand side, we have

$$\begin{aligned} \int_0^\infty \int y^{1-2\alpha} |V_r|^2 \eta^2 r \, dr \, ds \, dy &\leq \left(\int_0^\infty \int_{r \in [R, 2R]} y^{1-2\alpha} |V_r|^2 \eta^2 r \, dr \, ds \, dy \right)^{1/2} \\ &\cdot \left(\int_0^\infty \int_{r \in [R, 2R]} y^{1-2\alpha} |V|^2 |\eta_r|^2 r \, dr \, ds \, dy \right)^{1/2}. \end{aligned} \tag{5.9}$$

Now, using that $|\nabla \eta| \leq C/R$ and the exponential decay (5.1) of V in the variable y , we deduce that the second integral on the right-hand side is bounded, which implies that the integral

$$\int_0^\infty \int_{r \in [R, 2R]} y^{1-2\alpha} |V_r|^2 \eta^2 r \, dr \, ds \, dy$$

is bounded independently of R . Letting R tend to infinity, we conclude that

$$\int_0^\infty \int y^{1-2\alpha} |V_r|^2 r \, dr \, ds \, dy \leq C.$$

In particular, there exists a sequence $R_i \rightarrow \infty$ for which

$$\lim_{i \rightarrow \infty} \int_0^\infty \int_{r \in [R_i, 2R_i]} y^{1-2\alpha} |V_r|^2 \eta^2 r \, dr \, ds \, dy = 0.$$

This, together with (5.9), implies that

$$\int_0^\infty \int y^{1-2\alpha} |V_r|^2 r \, dr \, ds \, dy \leq 0,$$

which concludes the proof. □

6. Proof of Theorem 1.3

In this last section, we prove that helicoids have vanishing non-local mean curvature.

Proof of Theorem 1.3. We have to show that

$$\mathcal{H}_{H_\lambda}^{2\alpha}(x_0) = 0$$

for all $x_0 \in H_\lambda$. We can assume for this calculation that $x_0 = (t_0, 0, 0) = (t_0 e^{i \cdot 0}, 0)$, where $t_0 > 0$.

We can write

$$\mathbb{R}^3 \setminus H_\lambda = E_+ \cup E_-,$$

where

$$E_+ = \left\{ \left(t e^{i\theta}, \frac{\lambda}{\pi}(\theta + z) \right) \mid t > 0, \theta \in \mathbb{R}, z \in (0, \pi) \right\}, \tag{6.1}$$

$$E_- = \left\{ \left(t e^{i\theta}, \frac{\lambda}{\pi}(\theta + z) \right) \mid t < 0, \theta \in \mathbb{R}, z \in (0, \pi) \right\} \tag{6.2}$$

are the two connected components of $\mathbb{R}^3 \setminus H_\lambda$. We observe that the double helicoid H_λ is the boundary of E_+ and that $E_+^C = E_-$ (up to a measure zero set). Then, the non-local mean

curvature of $H_\lambda = \partial E_+$ is given by

$$\mathcal{H}_{H_\lambda}^{2\alpha}(x_0) = \int_{\mathbb{R}^3} \frac{\chi_{E_+}(x) - \chi_{E_-}(x)}{|x - x_0|^{3+2\alpha}} dx.$$

Consider the transformation

$$f\left(te^{i\theta}, \frac{\lambda}{\pi}(\theta + z)\right) = \left(te^{-i\theta}, -\frac{\lambda}{\pi}(\theta + z)\right)$$

defined for $(te^{i\theta}, (\lambda/\pi)(\theta + z)) \in E_+$. Let us verify that $(te^{-i\theta}, -(\lambda/\pi)(\theta + z)) \in E_-$ if $(te^{i\theta}, (\lambda/\pi)(\theta + z)) \in E_+$. Indeed,

$$\left(te^{-i\theta}, -\frac{\lambda}{\pi}(\theta + z)\right) = \left(-te^{i(-\pi-\theta)}, \frac{\lambda}{\pi}(-\pi - \theta + (\pi - z))\right) \in E_-$$

if $t > 0$ and $z \in (0, \pi)$. Moreover, f is a bijection from E_+ onto E_- , which preserves volume. Moreover, writing $x = (te^{i\theta}, (\lambda/\pi)(\theta + z)) \in E_+$, $x^* = f(x) = (te^{-i\theta}, -(\lambda/\pi)(\theta + z)) \in E_-$ and $x_0 = (t_0, 0, 0)$, we have

$$\begin{aligned} |x - x_0|^2 &= (t \cos(\theta) - t_0)^2 + t^2 \sin^2(\theta) + \frac{\lambda^2}{\pi^2}(\theta + z)^2 \\ &= |x^* - x_0|^2. \end{aligned}$$

Therefore, changing x to $x^* = f(x)$ in the second integral below,

$$\begin{aligned} \mathcal{H}_{H_\lambda}^{2\alpha}(x_0) &= \lim_{r \rightarrow 0} \int_{\{|x-x_0| \geq r\}} \frac{\chi_{E_+}(x) - \chi_{E_-}(x)}{|x - x_0|^{3+2\alpha}} dx \\ &= \lim_{r \rightarrow 0} \left(\int_{\{|x-x_0| \geq r\} \cap E_+} \frac{1}{|x - x_0|^{3+2\alpha}} dx - \int_{\{|x-x_0| \geq r\} \cap E_-} \frac{1}{|x - x_0|^{3+2\alpha}} dx \right) \\ &= \lim_{r \rightarrow 0} \left(\int_{\{|x-x_0| \geq r\} \cap E_+} \frac{1}{|x - x_0|^{3+2\alpha}} dx - \int_{\{|x-x_0| \geq r\} \cap E_+} \frac{1}{|x^* - x_0|^{3+2\alpha}} dx^* \right) \\ &= 0. \end{aligned} \quad \square$$

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