ZAMM · Z. Angew. Math. Mech. **96**, No. 6, 721–732 (2016) / **DOI** 10.1002/zamm.201200181

Weak solutions of the transmission problem in anti-plane Cosserat elasticity

E. Atroshchenko[1,**] **and S. Potapenko**[2,*]

[1] Department of Mechanical Engineering, Faculty of Physical and Mathematical Sciences, University of Chile, Av. Beauchef 850, Santiago 8370448, Chile
[2] Department of Civil and Environmental Engineering, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

Received 22 September 2012, revised 28 September 2014, accepted 18 December 2014
Published online 4 September 2015

Key words Boundary integral equation method, inclusion problems, micropolar elasticity.

In this paper we prove the existence of weak solutions for the inclusion problem in anti-plane Cosserat elasticity in Sobolev space setting, and for the corresponding systems of boundary integral equations.

1 Introduction

The theory of Cosserat elasticity (also known as micropolar or asymmetric elasticity) was introduced by Cosserat brothers [1] and further developed by Eringen [2] to model the mechanical behavior of materials for which the microstructure is significant (see [3] for a review of works in this area and an extensive bibliography). The theory was intended to eliminate discrepancies between the classical elasticity and experiments since the classical elasticity failed to produce the acceptable results when the effects of material microstructure were known to significantly affect the body's overall deformation, for example, in the case of granular bodies with large molecules (e.g. polymers), human bones, porous and cellular solids or foams (see, for example, [4–6]). These cases are becoming increasingly important in the design and manufacture of modern day advanced materials as small-scale effects become paramount in the prediction of the overall mechanical behavior of these materials. Recent experiments [6] show that Cosserat theory gives more accurate results in comparison with the classical theory in modeling of many modern materials, such as fiber-reinforced composites, synthetic polymers or metallic foams.

In [7], the classical (Dirichlet, Neumann, mixed) boundary value problems of the three dimensional theory of elasticity were shown to be well-posed and subsequently solved in a rigorous setting using the boundary integral equation method. In a series of recent papers [8–12], the boundary integral equation method proposed in [7] has been extended for the rigorous treatment of the corresponding plane and antiplane problems of Cosserat elasticity and the boundary value problems arising in the theory of plates.

The inclusion problems have been receiving an increasing amount of attention in the literature [13–16]. This interest is caused by the rapid development and growing use of composite materials, since the interaction of fibers with the surrounding matrix can be modeled as an inclusion problem. The inclusion problem in micropolar elasticity is especially challenging due to the presence of independent micro-rotations. The solutions available in the literature are restricted by assumptions of a simplified shape of the inclusion or a form of the applied loading. For example, Cheng and He in [17, 18], Hansong and Gengkai in [19] considered a spherical, cylindrical and ellipsoidal inclusions, respectively, and derived the analytical expressions of the corresponding micropolar Eshelby tensors [20].

In [21, 22] the rigorous mathematical analysis of an inclusion problem in plane and anti-plane Cosserat elasticity, respectively, was presented. The boundary value problem for an inclusion of arbitrary shape with a homogeneously imperfect interface was formulated and showed to be well-posed, and subsequently reduced to the system of boundary integral equations with unique solutions. Despite the fact, that the inclusion problem was considered in a very general formulation, i.e. without restrictions for the shape of the inclusion and type of the boundary conditions, the analysis was done under assumption that the boundary of the inclusion is a C^2-curve. Therefore, a wide class of boundary shapes was excluded from consideration, such as for example, polygons and piecewise-defined curves. These cases are very important for applications, such as finite element method and boundary element method, but also very difficult for analysis due to the

* Corresponding author E-mail: spotapenko@uwaterloo.ca
** E-mail: eatroshchenko@ing.uchile.cl

uncertainty in the behavior of the solutions at the corners of the boundary, and a different approach should be engaged to accommodate these special points.

Chudinovich and Constanda in [23] suggested that the boundary value problems arising in the bending of plates be formulated in a weak (Sobolev) space setting. Later this approach was successfully used for various problems of plane and anti-plane Cosserat elasticity, [24–27].

In the present paper we introduce the weak formulation of the boundary value problem for an inclusion of arbitrary shape with a homogeneously imperfect interface in anti-plane Cosserat elasticity. We develop the modification of the boundary integral equation method suggested by Chudinovich and Constanda in [28] for the treatment of inclusions in plates, and reduce the problem to the systems of boundary integral equations and prove the existence and uniqueness of the corresponding solutions. Using the results presented herein those solutions can be represented in terms of modified integral potentials which further can be expanded into semi-analytic solutions using generalized Fourier series. Solutions in terms of generalized Fourier series can be easily employed for the treatment of inclusion problems arising in engineering applications in the cases when material microstructure is known to be significant. Such situations can occur in the consideration of elastic behaviour of fiber reinforced polymers, titanium implants in human bones, crllular solids and foams.

2 Preliminaries

In what follows Greek and Latin indices take the values 1, 2 and 1, 2, 3, respectively, the convention of summation over repeated indices is understood, $\mathcal{M}_{m\times n}$ is the space of $(m \times n)$- matrices, E_n is the identity element in $\mathcal{M}_{m\times n}$, a superscript T indicates matrix transposition and $(...)_{,\alpha} \equiv \partial(...)/\partial x_\alpha$. Also, if X is a space of scalar functions and v a matrix, $v \in X$ means that every component of v belongs to X.

Let S be a domain in $\mathbb{R}^2$ bounded by a piecewise $C^{0,1}$-curve ∂S, which consists of finitely many C^2-arcs, and occupied by a homogeneous and isotropic linearly elastic micropolar material with elastic constants λ, μ, κ, and γ. The state of plane micropolar strain is characterized by a displacement field $u(x') = (u_1(x'), u_2(x'), u_3(x'))^T$ and a microrotation field $\Phi(x') = (\phi_1(x'), \phi_2(x'), \phi_3(x'))^T$ of the form

$$\begin{aligned} &u_\alpha\left(x'\right) = 0, \qquad u_3\left(x'\right) = u_3\left(x\right), \\ &\phi_\alpha\left(x'\right) = \phi_\alpha\left(x\right), \qquad \phi_3(x') = 0, \end{aligned} \tag{1}$$

where $x' = (x_1, x_2, x_3)$ and $x = (x_1, x_2)$ are generic points in $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

In view of (1), in absence of body forces and couples, the equilibrium equations [2] are written in the form

$$L(\partial_x)u(x) = 0, \tag{2}$$

where ϕ_3 is denoted by u_3, $u = (u_1, u_2, u_3)^T$ and the matrix partial differential operator $L(\partial_x) = L(\partial/\partial x_1, \partial/\partial x_2)$ is defined by [8]

$$L(\xi_\alpha) = \begin{pmatrix} \gamma\Delta + (\alpha+\beta)\xi_1^2 - 2\kappa & (\alpha+\beta)\xi_1\xi_2 & \kappa\xi_2 \\ (\alpha+\beta)\xi_1\xi_2 & \gamma\Delta + (\alpha+\beta)\xi_2^2 - 2\kappa & -\kappa\xi_1 \\ -\kappa\xi_2 & \kappa\xi_1 & (\mu+\kappa)\Delta \end{pmatrix}, \tag{3}$$

where $\Delta = \xi_\alpha\xi_\alpha$.

We consider also the boundary stress operator $T(\partial_x) = T(\partial/\partial x_\alpha)$ defined by [8]

$$T(\xi_\alpha) = \begin{pmatrix} (\alpha+\beta+\gamma)\xi_1 n_1 + \gamma\xi_2 n_2 & \alpha\xi_2 n_1 + \beta\xi_1 n_2 & 0 \\ \alpha\xi_1 n_2 + \beta\xi_2 n_1 & (\alpha+\beta+\gamma)\xi_2 n_2 + \gamma\xi_1 n_1 & 0 \\ -\kappa n_2 & \kappa n_1 & (\mu+\kappa)\xi_\alpha n_\alpha \end{pmatrix}, \tag{4}$$

where $n = (n_1, n_2)^T$ is the unit outward normal to ∂S. We define the bilinear form $E(u, v)$ by

$$\begin{aligned} 2E(u,v) &= E_e(u,v) + E_r(u,v) \\ E_e(u,v) &= (\mu+\kappa)(u_{3,1}v_{3,1} + u_{3,2}v_{3,2}) + \kappa(u_{3,1}v_2 + v_{3,1}u_2 - u_{3,2}v_1 - v_{3,2}u_1 + 2(u_1v_1 + u_2v_2)) \\ E_r(u,v) &= (\alpha+\beta+\gamma)(u_{1,1}v_{1,1} + u_{2,2}v_{2,2}) + \alpha(u_{1,1}v_{2,2} + u_{2,2}v_{1,1}) \\ &\quad + \beta(u_{2,1}v_{1,2} + u_{1,2}v_{2,1}) + \gamma(u_{2,1}v_{2,1} + u_{1,2}v_{1,2}). \end{aligned} \tag{5}$$

The internal energy density is given by $E(u, u)$. Throughout what follows we assume that

$$2\mu + \kappa > 0, \qquad \alpha, \gamma, \kappa > 0, \qquad -\gamma < \beta < \gamma. \tag{6}$$

Conditions (6) guarantee the ellipticity of the system (2) and the positiveness of the internal energy density $E(u, u)$ [2]. In fact $E(u, u) = 0$ if and only if $u \in \mathcal{Z}$, where $\mathcal{Z}$ is the space of rigid displacements spanned by the columns of the matrix

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

Clearly, $Lz = 0$ in $\mathbb{R}^2$ and $Tz = 0$ on ∂S for any $z \in \mathcal{Z}$ and a generic rigid displacement $z \in \mathcal{Z}$ can be written as $z = Fk$, where $k \in \mathcal{M}_{3\times1}$ is constant and arbitrary.

Let $\mathcal{A}$ be the class of vectors $u \in \mathcal{M}_{3\times1}$ whose components in terms of polar coordinates, as $r = |x| \to \infty$, admit an asymptotic expansion of the form [8]:

$$\begin{aligned} u_1(r, \theta) &= r^{-2}\left(m_0 \sin 2\theta + m_1(1 - \cos 2\theta) + m_2\right) + O\left(r^{-3}\right), \\ u_2(r, \theta) &= r^{-2}\left(-m_0 \sin 2\theta - m_1(1 - \cos 2\theta) + m_3\right) + O\left(r^{-3}\right), \\ u_3(r, \theta) &= r^{-1}\left((m_3 - m_0)\cos\theta - (m_2 - m_1)\sin\theta\right) + O\left(r^{-2}\right), \end{aligned} \tag{8}$$

where $m_0, \ldots, m_3$ are arbitrary constants. We consider also the set

$$\mathcal{A}^* = \left\{u : u = \mathcal{F}k + s^{\mathcal{A}}\right\}, \tag{9}$$

where $k \in \mathcal{M}_{3\times1}$ is an arbitrary constant and $s^{\mathcal{A}} \in \mathcal{M}_{3\times1} \cap \mathcal{A}$. In view of (5), $\mathcal{A}$ and $\mathcal{A}^*$ are classes of finite energy functions.

The following theorems are known as Betti formulas and proved in [8].

Theorem 1. *(i) For any* $u, v \in C^2(S^+) \cap C^1(\bar{S}^+)$

$$\int_{S^+} v^T Lu\, dx = 2\int_{S^+} E(u, v)dx - \int_{\partial S} v^T Tu\, ds. \tag{10}$$

(ii) For any $u, v \in C^2(S^-) \cap C^1(\bar{S}^-) \cap \mathcal{A}^*$

$$\int_{S^-} v^T Lu\, dx = 2\int_{S^-} E(u, v)dx + \int_{\partial S} v^T Tu\, ds. \tag{11}$$

The matrix of fundamental solutions $D(x, y)$ of system (2) is given by [8]

$$D(x, y) = L^*(\partial_x)t(x, y), \tag{12}$$

where L^* is the adjoint of L,

$$t(x, y) = -\frac{a}{2\pi}\left\{\frac{1}{k_1^2 k_2^2}\ln r - \frac{1}{k_1^2\left(k_1^2 - k_2^2\right)}K_0(k_1 r) + \frac{1}{k_2^2\left(k_1^2 - k_2^2\right)}K_0(k_2 r)\right\}, \tag{13}$$

where $r = |x - y|$, K_0 is the modified Bessel function of order zero and constants a , k_1, k_2 are defined by

$$a^{-1} = \gamma(\mu + \kappa)(\alpha + \beta + \gamma), \quad k_1^2 = \frac{\kappa(\kappa + 2\mu)}{\gamma(\kappa + \mu)}, \quad k_2^2 = \frac{2\kappa}{\alpha + \beta + \gamma}. \tag{14}$$

In view of (8) and (9)

$$D(x, y) = (D(y, x))^T. \tag{15}$$

Along with $D(x, y)$ we consider the matrix of singular solutions $P(x, y)$ given by

$$P(x, y) = (T(\partial_y)D(y, x))^T. \tag{16}$$

It is easy to verify that the columns of $D(x, y)$ and $P(x, y)$ satisfy (2) at all $x \in \mathbb{R}^2, x \neq y$, and for any direction n independent of x.

We also recall Somigliana formulas as proved in [8].

Theorem 2. *(i) If* $u \in C^2(S^+) \cap C^1(\bar{S}^+)$ *is a solution of (2) in* S^+*, then*

$$\int_{\partial S}\left[D(x, y)T(\partial_y)u(y) - P(x, y)u(y)\right] ds_y = \begin{cases} u(x), & x \in S^+, \\ \frac{1}{2}u(x), & x \in \partial S, \\ 0, & x \in S^-. \end{cases} \tag{17}$$

(ii) If $u \in C^2(S^-) \cap C^1(\bar{S}^-) \cap \mathcal{A}$ is a solution of (2) in S^-, then

$$-\int_{\partial S} [D(x,y)T(\partial_y)u(y) - P(x,y)u(y)]\, ds_y = \begin{cases} 0, & x \in S^+, \\ \frac{1}{2}u(x), & x \in \partial S, \\ u(x), & x \in S^-. \end{cases} \tag{18}$$

3 Basic definitions of Sobolev spaces

Let S^+ be the bounded domain enclosed by ∂S and $S^- = \mathbb{R}^2 \backslash \bar{S}^+$. In what follows we use the same notation for spaces, norms and inner products of scalar and vector functions. The symbols $\|\cdot\|_{0;S}$ and $(\cdot,\cdot)_{0;S}$ denote the norm and inner product in $L^2(S)$. When $S = \mathbb{R}^2$ we use the notations $\|\cdot\|_0$ and $(\cdot,\cdot)_0$.

For any $m \in \mathbb{R}$, let $H_m(\mathbb{R}^2)$ be the standard real Sobolev space of three-component distributions, equipped with the norm

$$\|u\|_m^2 = \int_{\mathbb{R}^2} (1 + |\xi|^2)^m |\widetilde{u}(\xi)|^2 d\xi,$$

where $\widetilde{u}$ is the Fourier transform of u. In what follows we do not distinguish between equivalent norms and denote them by the same symbol; thus, the norm in $H_1(\mathbb{R}^2)$ can be defined by

$$\|u\|_1^2 = \|u\|_0^2 + \sum_{i=1}^{3} \|\nabla u_i\|_0^2.$$

The spaces $H_m(\mathbb{R}^2)$ and $H_{-m}(\mathbb{R}^2)$ are dual with respect to duality induced by $(\cdot,\cdot)_0$.

We introduce the space $L^2_\omega(\mathbb{R}^2)$ of all $u = (\overline{u}^T, u_3)^T$, where $\overline{u} = (u_1, u_2)^T$, such that

$$\|u\|_{0,\omega}^2 = \int_{\mathbb{R}^2} \left|\overline{u}(x)\right|^2 dx + \int_{\mathbb{R}^2} \frac{\left|u_3(x)\right|^2}{(1+|x|)^2(1+\ln^2(1+|x|))} dx < \infty.$$

We consider the bilinear form $a(u,v) = 2\int_{\mathbb{R}^2} E(u,v)dx$.

Let $H_{1,\omega}(\mathbb{R}^2)$ be the space of three-component distributions on $\mathbb{R}^2$ for which

$$\|u\|_{1,\omega}^2 = \|u\|_{0,\omega}^2 + a(u,u) < \infty,$$

The space $H_{-1,\omega}(\mathbb{R}^2)$ is dual to $H_{1,\omega}(\mathbb{R}^2)$ with respect to duality generated by $(\cdot,\cdot)_0$. The norm in $H_{-1,\omega}(\mathbb{R}^2)$ is denoted by $\|\cdot\|_{-1,\omega}$.

Let $\mathring{H}_m(S^+)$ be the subspace of $H_m(\mathbb{R}^2)$ consisting of all u such that $\operatorname{supp} u \subset \overline{S}^+$.$H_m(S^+)$ is the space of the restrictions to S^+ of all $u \in H_m(\mathbb{R}^2)$. Denoting by $\pi^\pm$ the operators of restrictions from $\mathbb{R}^2$ to $S^\pm$, respectively, we introduce the norm of $u \in H_m(S^+)$ by $\|u\|_{m;S^+} = \inf\limits_{v \in H_m(\mathbb{R}^2):\pi^+ v = u} \|v\|_m$. If $m = 1$, then the norms of $u \in \mathring{H}_1(S^+)$ and $u \in H_1(S^+)$ are equivalent to

$$\left\{ \|u\|_{0;S^+}^2 + \sum_{i=1}^{3} \int_{S^+} |\nabla u_i(x)|^2 dx \right\}^{1/2}.$$

The spaces $\mathring{H}_m(S^+)$ and $H_{-m}(S^+)$ are dual with respect to the duality induced by $(\cdot,\cdot)_{0;S^+}$.

Let $\mathring{H}_{1,\omega}(S^-)$ be the subspace of $H_{1,\omega}(\mathbb{R}^2)$ consisting of all u such that $\operatorname{supp} u \subset \overline{S}^-$. $H_{1,\omega}(S^-)$ is the space of the restrictions to S^- of all $u \in H_{1,\omega}(\mathbb{R}^2)$. The norm in $H_{1,\omega}(S^-)$ is defined by $\|u\|_{1,\omega;S^-} = \inf\limits_{v \in H_{1,\omega}(\mathbb{R}^2):\pi^- v = u} \|v\|_{1,\omega}$. It can be shown that the norm of $u \in H_{1,\omega}(S^-)$ is equivalent to

$$\left\{ \|u\|_{0,\omega;S^-}^2 + a_-(u,u) \right\}^{1/2},$$

where

$$\|u\|_{0,\omega;S^-}^2 = \int_{S^-} \left|\overline{u}(x)\right|^2 dx + \int_{S^-} \frac{\left|u_3(x)\right|^2}{(1+|x|)^2(1+\ln^2(1+|x|))} dx$$

and $a_\pm(u,v) = 2\int_{S^\pm} E(u,v)\, dx$.

The dual of $\overset{\circ}{H}_{1,\omega}(S^-)$ with respect to the duality generated by $(\cdot,\cdot)_{0;S^-}$ is the space $H_{-1,\omega}(S^-)$, with norm $\|\cdot\|_{-1,\omega;S^-}$; the dual of $H_{1,\omega}(S^-)$ is $\overset{\circ}{H}_{-1,\omega}(S^-)$, which can be identified with a subspace of $H_{-1,\omega}(\mathbb{R}^2)$.

Let $H_m(\partial S)$ be the standard Sobolev space of distributions on ∂S with norm $\|\cdot\|_{m;\partial S}$.$H_m(\partial S)$ and $H_{-m}(\partial S)$ are dual with respect to the duality generated by the inner product $(\cdot,\cdot)_{0;\partial S}$ in $L^2(\partial S)$.

We denote by γ^+ and γ^- the trace operators defined first on $C_0^\infty(S^\pm)$ and then extended by continuity to the surjections $\gamma^+ : H_1(S^+) \longrightarrow H_{1/2}(\partial S)$,$\gamma^- : H_{1,\omega}(S^-) \longrightarrow H_{1/2}(\partial S)$. This is possible because of the local equivalence of $H_{1,\omega}(S^-)$ and $H_1(S^-)$. We also consider a continuous extension operators $l^+ : H_{1/2}(\partial S) \longrightarrow H_1(S^+)$, $l^- : H_{1/2}(\partial S) \longrightarrow H_1(S^-)$, the latter, since norm in $H_1(S^-)$ is stronger than that in $H_{1,\omega}(S^-)$, can also be regarded as a continuous operator from $H_{1/2}(\partial S)$ into $H_{1,\omega}(S^-)$.

To proceed further we will need the following well-known fact from the functional analysis.

Theorem 3. *(Lax-Milgram Lemma) Let H be a Hilbert space and $b(u,v)$ be a bilinear functional defined for every ordinate pair $u, v \in H$, for which there exist two constants h and k such that:*

$$\left|b(u,v)\right| \leqslant h\,\|u\|\,\|v\|\,, \quad \|u\|^2 \leqslant k\left|b(u,u)\right| \qquad \forall u, v \in H,$$

in this case we say that $a(u,v)$ is coercive. Then however we assign the bounded linear functional $\mathcal{L}(v)$ on H there exists one and only one u such that

$$b(u,v) = \mathcal{L}(v), \ \ \forall v \in H, \ \ \|u\| \leqslant c\,\|\mathcal{L}\|_*\,,$$

where $\|\cdot\|_*$ is the norm on the dual H' of H.

The proof of this lemma can be found in [29].

4 Variational formulation of the transmission problem

We consider an infinite plane with a bounded inclusion occupying the domain $\overline{S}^+$ enclosed by ∂S with the unit outward normal $(n_1,n_2)^T$. The inclusion is homogeneous and isotropic with elastic constants $\mu_i, \alpha_i, \beta_i, \gamma_i$ and κ_i. The remainder of the plane, which lies in $\overline{S}^-$ is also homogeneous and isotropic with constants $\mu_e, \alpha_e, \beta_e, \gamma_e$ and κ_e. Let $L_i(\partial_x)$, $L_e(\partial_x)$ be the operator $L(\partial_x)$ with constants $\mu_i, \alpha_i, \beta_i, \gamma_i, \kappa_i$ and $\mu_e, \alpha_e, \beta_e, \gamma_e, \kappa_e$ respectively. The boundary stress operators $T_i(\partial_x)$, $T_e(\partial_x)$ and the bilinear forms $E_i(u,v)$, $E_e(u,v)$ are defined analogously. We assume both sets of constants satisfy conditions (6). The transmission problem for an inclusion with the perfect interface is formulated as follows.

Find $u = \{u_+, u_-\}$, $u_+ \in C^2(S^+) \cap C^1(\bar{S}^+)$ and $u_- \in C^2(S^-) \cap C^1(\bar{S}^-) \cap \mathcal{A}^*$ such that

$$L_i u_+ = q_+ \quad \text{in } S^+, \qquad L_e u_- = q_- \quad \text{in } S^-, \tag{19}$$

and

$$u_+ = u_-, \qquad T_i u_+ = T_e u_- \qquad \text{on } \partial S, \tag{20}$$

where q_+ and q_- are prescribed in S^+ and S^-, respectively. We write

$$a_{\nu\pm}(u,v) = 2\int\limits_{S^\pm} E_\nu(u,v), \qquad a_\nu(u,v) = 2\int\limits_{\mathbb{R}^2} E_\nu(u,v)dx, \quad \nu = i, e. \tag{21}$$

We introduce the space $H_{1,\omega}(\mathbb{R}^2)$ with norm $\|\cdot\|_{1,\omega}^2 = \|\cdot\|_{0,\omega}^2 + a_e(u,v)$, the space $\overset{\circ}{H}_{1,\omega}(\mathbb{R}^2) = \left\{u = \{u_+, u_-\} \in H_{1,\omega}(\mathbb{R}^2) : \gamma^+ u_+ = \gamma^- u_-\right\}$ and the spaces $H_{-1,\omega}(\mathbb{R}^2)$, $H_{m,\omega}(S^-)$, $\overset{\circ}{H}_{m,\omega}(S^-)$ (with $m = \pm 1$) as described above. For any $v \in C_0^\infty(\mathbb{R}^2)$ we write $v = \{v_+, v_-\}$ where $v_\pm = \pi^\pm v$. Multiplying the first equation of (19) by v^+ and the second equation of (19) by v^-, integrating over S^+ and S^-, respectively, and adding the results, we arrive at

$$(L_i u_+, v_+)_{0;S^+} + (L_e u_-, v_-)_{0;S^-} = (q_+, v_+)_{0;S^+} + (q_-, v_-)_{0;S^-}. \tag{22}$$

Using Betti formulas (10)-(11) Eq. (22) becomes

$$a_{i+}(u_+, v_+) + a_{e-}(u_-, v_-) = (q_+, v_+)_{0;S^+} + (q_-, v_-)_{0;S^-}. \tag{23}$$

We use the notations $q = \{q_+, q_-\}$ and $(q,v)_0 = (q_+, v_+)_{0;S^+} + (q_-, v_-)_{0;S^-}$. According to Eq. (23), we introduce the variational formulation of problem (19)–(20) as follows.

Find $u = \{u_+, u_-\} \in \overset{\circ}{H}_{1,\omega}(\mathbb{R}^2)$ such that

$$a_{i+}(u_+, v_+) + a_{e-}(u_-, v_-) = (q, v)_0 \qquad \forall v = \{v_+, v_-\} \in \overset{\circ}{H}_{1,\omega}(\mathbb{R}^2). \tag{24}$$

Taking $v = z \in \mathcal{Z}$ we deduce that (24) is solvable if and only if

$$(q, z)_0 = 0 \qquad \forall z \in \mathcal{Z}. \tag{25}$$

Let $\mathcal{H}_{-1,\omega}(\mathbb{R}^2)$ be the subspace of $H_{-1,\omega}(\mathbb{R}^2)$ consisting of all q satisfying (25).

5 Solvability of the transmission problem

Theorem 4. *For any $q \in \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$ the transmission problem (24) has a solution $u \in H_{1,\omega}(\mathbb{R}^2)$. Any two solutions of (24) differ by a rigid displacement $z \in \mathcal{Z}$ and there is a solution u_0 that satisfies the estimate*

$$\|u_0\|_{1,\omega} \le c\|q\|_{-1,\omega}, \tag{26}$$

where c is a positive constant.

Proof. We introduce the factor space $\mathbb{H}_{1,\omega}(\mathbb{R}^2) = \overset{\circ}{H}_{1,\omega}(\mathbb{R}^2)/\mathcal{Z}$, the bilinear form $A(U, V) = a_{i+}(u_+, v_+) + a_{e-}(u_-, v_-)$ defined on $\mathbb{H}_{1,\omega}(\mathbb{R}^2) \times \mathbb{H}_{1,\omega}(\mathbb{R}^2)$ and the linear functional $L(V) = (q, v)_0$ defined on $\mathbb{H}_{1,\omega}(\mathbb{R}^2)$, where u and v are arbitrary representatives of classes $U, V \in \mathbb{H}_{1,\omega}(\mathbb{R}^2)$, respectively. Due to the properties of rigid displacements, definitions of $A(U, V)$ and $L(V)$ are consistent. We define the norm on $\mathbb{H}_{1,\omega}(\mathbb{R}^2)$ by $\|U\|_{\mathbb{H}_{1,\omega}(\mathbb{R}^2)} = \inf_{u \in U} \|u\|_{1,\omega}$. We now consider the problem of finding $U \in \mathbb{H}_{1,\omega}(\mathbb{R}^2)$ such that

$$A(U, V) = L(V) \qquad \forall V \in \mathbb{H}_{1,\omega}(\mathbb{R}^2). \tag{27}$$

As it follows from definition of $A(U, V)$ and $L(V)$, for any $U, V \in \mathbb{H}_{1,\omega}(\mathbb{R}^2)$

$$A(U, V) \le c\|U\|_{\mathbb{H}_{1,\omega}(\mathbb{R}^2)}\|V\|_{\mathbb{H}_{1,\omega}(\mathbb{R}^2)}, \qquad |L(V)| \le c\|V\|_{\mathbb{H}_{1,\omega}(\mathbb{R}^2)}. \tag{28}$$

To prove that $A(U, V)$ is coercive on $\mathbb{H}_{1,\omega}(\mathbb{R}^2)$ we use the following results from [24]:

$$\begin{aligned} \|u_+\|^2_{1;S^+} &\leqslant c\left(a_{i+}(u_+, u_+) + \sum_{i=1}^{3}(\gamma^+ u_+, z^{(i)})^2_{0;\partial S}\right), \\ \|u_-\|^2_{1,\omega;S^-} &\leqslant c\left(a_{e-}(u_-, u_-) + \sum_{i=1}^{3}(\gamma^- u_-, z^{(i)})^2_{0;\partial S}\right), \end{aligned} \tag{29}$$

where $z^{(i)}$ is an $L^2(\partial S)$-orthonormal basis for $\mathcal{Z}$. We can choose $u \in U$ such that $(\gamma^\pm u_\pm, z)^2_{0;\partial S} = 0$ for all $z \in \mathcal{Z}$, then Eqs. (29) yield the estimate

$$\begin{aligned} \|U\|^2_{\mathbb{H}_{1,\omega}(\mathbb{R}^2)} &\le \|u\|^2_{1,\omega} = \|u_+\|^2_{1;S^+} + \|u_-\|^2_{1,\omega;S^-} \\ &\le \{a_{i+}(u_+, u_+) + a_{e-}(u_-, u_-)\} = cA(U, U). \end{aligned} \tag{30}$$

As it follows from Eqs. (28)–(30), $A(U, V)$ and $L(V)$ satisfy the conditions of Lax-Milgram Lemma, hence we conclude that there exists the unique solution $U \in \mathbb{H}_{1,\omega}(\mathbb{R}^2)$ of (27). This solution satisfies the estimate:

$$\|U\|_{\mathbb{H}_{1,\omega}(\mathbb{R}^2)} \le c\,\|q\|_{-1,\omega}. \tag{31}$$

Every $u \in U$ is a solution of (24). In view of definition of the norm in $\mathbb{H}_{1,\omega}(\mathbb{R}^2)$ there exists $u_0 \in U$ such that $\|U\|_{\mathbb{H}_{1,\omega}(\mathbb{R}^2)} = \|u_0\|_{1,\omega}$ and (26) follows from (31). □

Next we show that the problem (24) can be reduced to the analogous problem with homogeneous equilibrium equations and non-homogeneous boundary conditions.

We define two area potentials of density q by

$$(U_\nu q)(x) = \int_{\mathbb{R}^2} D_\nu(x, y)q(y)dy \qquad \alpha = i, e, \qquad x \in \mathbb{R}^2, \tag{32}$$

where D_i, D_e are matrices $D(x, y)$ constructed with $\mu_i, \alpha_i, \beta_i, \gamma_i, \kappa_i$ and $\mu_e, \alpha_e, \beta_e, \gamma_e, \kappa_e$, respectively. It has been shown in [30] that if $q \in \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$, then $U_\nu q \in H_{1,\omega}(\mathbb{R}^2)$ and

$$a_\nu(U_\pm q, v) = (q, v)_0 \qquad \forall v \in \overset{\circ}{H}_{1,\omega}(\mathbb{R}^2), \qquad \nu = i, e. \tag{33}$$

We represent the solution u of (24) in the form $u = w + Uq$, where $Uq = \{\pi^+ U_i q, \pi^- U_e q\}$ and $w = \{w_+, w_-\} \in H_{1,\omega}(\mathbb{R}^2)$, then w satisfies

$$a_{i+}(w_+, v_+) + a_{e-}(w_-, v_-) = (q, v)_0 - a_{i+}(\pi^+ U_i q, v_+) - a_{e-}(\pi^- U_e q, v_-) \quad \forall\, v \in \overset{\circ}{H}_{1,\omega}(\mathbb{R}^2) \tag{34}$$

and

$$\gamma^+ w_+ - \gamma^- w_- = -\gamma^+ \pi^+ U_i q + \gamma^- \pi^- U_e q. \tag{35}$$

Let us denote $f = -\gamma^+ \pi^+ U_i q + \gamma^- \pi^- U_e q \in H_{1/2}(\partial S)$. Next we show that the right hand side of (34) depends only on $v_0 = \gamma^+ v_+ = \gamma^- v_-$. Let us consider $v^{(1)}, v^{(2)} \in \overset{\circ}{H}_{1,\omega}(\mathbb{R}^2)$ such that $\gamma^+ v_+^{(1)} = \gamma^- v_-^{(1)} = \gamma^+ v_+^{(2)} = \gamma^- v_-^{(2)}$. Then, making use of the Betti formulae (10)–(11) we arrive at

$$\begin{aligned}&(q, v^{(1)} - v^{(2)})_0 - a_{i+}(\pi^+ U_i q, v_+^{(1)} - v_+^{(2)}) - a_{e-}(\pi^- U_e q, v_-^{(1)} - v_-^{(2)})\\ &\quad = (q, v^{(1)} - v^{(2)})_0 - (q^+, v_+^{(1)} - v_+^{(2)})_{0;S^+} - (q^-, v_-^{(1)} - v_-^{(2)})_{0;S^-} = 0.\end{aligned} \tag{36}$$

Hence we define the linear functional defined on $H_{1/2}(\partial S)$ by

$$L(v_0) = (q^+, l^+ v_0)_{0;S^+} + (q^-, l^- v_0)_{0;S^-} - a_{i+}(\pi^+ U_i q, l^+ v_0) - a_{e-}(\pi^- U_e q, l^- v_0). \tag{37}$$

The linearity of $L(v_0)$ follows from the properties of $a_i(\pi^+ U_i q, l^+ v_+)$ and $a_e(\pi^- U_e q, l^- v_-)$. The continuity follows from the following estimate.

$$\begin{aligned}&|L(v_0)|\\ &\quad = |(q^+, l^+ v_0)_{0;S^+} + (q^-, l^- v_0)_{0;S^-} - a_{i+}(\pi^+ U_i q, l^+ v_0) - a_{e-}(\pi^- U_e q^-, l^- v_0)|\\ &\quad \le c\left(\|q^+\|_{-1;S^+}\|l^+ v_0\|_{1;S^+} + \|q^-\|_{-1,\omega;S^-}\|l^- v_0\|_{1,\omega;S^-}\right.\\ &\qquad \left. + \|\pi^+ U_i q\|_{1;S^+}\|l^+ v_0\|_{1;S^+} + \|\pi^- U_e q\|_{1,\omega;S^-}\|l^- v_0\|_{1,\omega;S^-}\right)\\ &\quad \le c\|q^+\|_{-1,\omega}\|v_0\|_{1/2;\partial S}\end{aligned} \tag{38}$$

Hence, we have proved that $L(v_0)$ is a continuous linear functional on $H_{1/2}(\partial S)$, therefore there exists $g \in H_{-1/2}(\partial S)$ such that

$$L(v_0) = (g, v_0)_{0;\partial S} \qquad \forall v_0 \in H_{1/2}(\partial S) \tag{39}$$

and, as it follows from (38), $\|q\|_{-1/2,\partial S} \le c\|q\|_{-1,\omega}$.

According to Eqs. (34), (35) and (38), and (39) variational problem (24) is equivalent to finding $u \in H_{1,\omega}(\mathbb{R}^2)$ such that $\forall v = \{v_+, v_-\} \in \overset{\circ}{H}_{1,\omega}(\mathbb{R}^2) : \gamma^+ v_+ = \gamma^- v_- = v_0$

$$\begin{aligned}a_{i+}(u_+, v_+) + a_{e-}(u_-, v_-) &= (g, v_0)_{0;\partial S},\\ \gamma^+ u_+ - \gamma^- u_- &= f,\end{aligned} \tag{40}$$

where $f \in H_{1/2}(\partial S)$ and $g \in H_{-1/2}(\partial S)$ are prescribed on ∂S. As it follows from (37) and (39), if $q \in \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$ then $(g, z)_{0;\partial S} = 0$ for all $z \in \mathcal{Z}$.

Let us denote by $\mathcal{H}_{\pm 1/2}(\partial S)$, the subspaces of $H_{\pm 1/2}(\partial S)$, respectively of all g, such that

$$(g, z)_{0;\partial S} = 0 \qquad \forall z \in \mathcal{Z}. \tag{41}$$

6 The Poincaré - Steklov operators

Let $f \in H_{1/2}(\partial S)$ and $u_+ \in H_1(S^+)$ and $u_- \in H_{1,\omega}(S^-)$ be the solutions of the interior and the exterior Dirichlet problems with boundary data f for a plane with material constants $\mu_i, \alpha_i, \beta_i, \gamma_i$ and κ_i, namely

$$\begin{aligned}&a_{i+}(u_+, v_+) = 0 \quad \forall v_+ \in \overset{\circ}{H}_1(S^+), \quad \gamma^+ u_+ = f,\\ &a_{i-}(u_-, v_-) = 0 \quad \forall v_- \in \overset{\circ}{H}_{1,\omega}(S^-), \quad \gamma^- u_- = f.\end{aligned} \tag{42}$$

We consider an arbitrary $\phi \in H_{1/2}(\partial S)$ and let $v_+ = l^+\phi \in H_1(S^+)$ and $v_- = l^-\phi \in H_{1,\omega}(S^-)$. Using Riesz representation theorem [29], we can define the Poincaré-Steklov operators $\mathcal{T}_i^{\pm}$ by

$$(\mathcal{T}_i^+ f, \varphi)_{0;\partial S} = a_{i+}(u_+, v_+), \qquad (\mathcal{T}_i^- f, \varphi)_{0;\partial S} = -a_{i-}(u_-, v_-). \tag{43}$$

In order to demonstrate, that this definition doesn't depend on the choice of the extension operators $l^{\pm}$, let us consider another extensions w_+, w_- of ϕ. Then $w_+ - v_+ \in \overset{\circ}{H}_1(S^+)$ and $w_- - v_- \in \overset{\circ}{H}_{1,\omega}(S^-)$ and according to (42) $a_{i+}(u_+, w_+ - v_+) = 0$ and $a_{i-}(u_-, w_- - v_-) = 0$. Let us recall the properties of Poincaré-Steklov operators, discussed in details in [23].

Theorem 5. *(i) $\mathcal{T}_i^{\pm}$ are continuous from $H_{-1/2}(\partial S)$ to $H_{-1/2}(\partial S)$,*
(ii) $\mathcal{T}_i^{\pm}$ are self-adjoint in the sense that

$$(\mathcal{T}_i^{\pm} f, \phi)_{0;\partial S} = (f, \mathcal{T}_i^{\pm}\phi)_{0;\partial S} \qquad \forall f, \phi \in H_{1/2}(\partial S), \tag{44}$$

(iii) The kernels of $\mathcal{T}_i^{\pm}$ coincide with $\mathcal{Z}$,
(iv) The ranges of $\mathcal{T}_i^{\pm}$ coincide with $\mathcal{H}_{-1/2}(\partial S)$,
(v) The restrictions $\mathcal{N}_i^{\pm}$ of $\mathcal{T}_i^{\pm}$ to $\mathcal{H}_{1/2}(\partial S)$ are homeomorphisms from $\mathcal{H}_{1/2}(\partial S)$ to $\mathcal{H}_{-1/2}(\partial S)$.

The operators $\mathcal{T}_e^{\pm}, \mathcal{N}_e^{\pm}$ for a plane with constants μ_e, α_e, β_e, γ_e and κ_e are defined in the similar manner. Their properties are analogous to the properties of $\mathcal{T}_i^{\pm}, \mathcal{N}_i^{\pm}$. The following statement has been proved in [28].

Theorem 6. *$\mathcal{N} = \mathcal{N}_i^+ - \mathcal{N}_e^-$ is a homeomorphism from $\mathcal{H}_{1/2}(\partial S)$ to $\mathcal{H}_{-1/2}(\partial S)$.*

7 Elastic potentials

We introduce the single layer potential as

$$(V_\nu\varphi)(x) = \int_{\partial S} D_\nu(x, y)\varphi(y)ds(y), \qquad x \in \mathbb{R}^2 \tag{45}$$

and double layer potential as

$$(W_\nu\varphi)(x) = \int_{\partial S} P_\nu(x, y)\varphi(y)ds(y), \qquad x \in S^+ \cup S^- \tag{46}$$

where $\varphi \in \mathcal{M}_{3\times 1}$ and $\nu = i, e$. The properties of single and double layer integral potentials with smooth densities were discussed in [9]. Here we recall properties of $(V_\nu\varphi)(x)$ and $(W_\nu\varphi)(x)$ in Sobolev spaces [24].

Theorem 7. *If $\varphi \in \mathcal{H}_{-1/2}(\partial S)$ then*

(i) $V_\nu\varphi \in H_{1,\omega}(\mathbb{R}^2)$, $\|V_\nu\varphi\|_{1,\omega} \leq c\|\varphi\|_{-1/2;\partial S}$ and

$$\gamma^+\pi^+V_\nu\varphi = \gamma^-\pi^-V_\nu\varphi = (V_\nu\varphi)_0,$$

(ii) boundary operator $V_{\nu 0}$ of the single layer potential is defined by the direct value $(V_\nu\varphi)_0$ of $V_\nu\varphi$ on ∂S, i.e. $V_{\nu 0}\varphi = (V_\nu\varphi)_0$, this operator is continuous from $\mathcal{H}_{-1/2}(\partial S)$ to $H_{1/2}(\partial S)$.
(iii) the jump formula holds

$$(\mathcal{T}_\nu^+ - \mathcal{T}_\nu^-)V_{\nu 0}\varphi = \varphi. \tag{47}$$

Theorem 8. *If $\psi \in H_{1/2}(\partial S)$ then*

(i) $\pi^+W_\nu\psi \in H_1(S^+)$ and $\pi^-W_\nu\psi \in H_{1,\omega}(S^-)$ and

$$\|\pi^+W_\nu\psi\|_{1;S^+} + \|\pi^-W_\nu\psi\|_{1,\omega;S^-} \leq c\|\psi\|_{1/2;\partial S},$$

(ii) the operators $W_\nu^{\pm}$ of limiting values on ∂S of the double layer potential $W_\nu\varphi$ are defined by $W_\nu^{\pm}\psi = \gamma^{\pm}\pi^{\pm}W_\nu\psi$, and they are continuous from $H_{1/2}(\partial S)$ to $H_{1/2}(\partial S)$,
(iii) the jump formula holds

$$W_\nu^+\psi - W_\nu^-\psi = -\psi \qquad \text{and} \qquad \mathcal{T}_\nu^+W_\nu^+\psi = \mathcal{T}_\nu^-W_\nu^-\psi.$$

For $\varphi \in \mathcal{H}_{-1/2}(\partial S)$ we define a modified single layer potential and its corresponding boundary operator by

$$(\mathcal{V}_\nu\varphi)(x) = (V_\nu\varphi)(x) - \sum_{i=1}^{3}(V_{\nu 0}\varphi, z^{(i)})_{0;\partial S} z^{(i)}(x), \qquad x \in \mathbb{R}^2 \tag{48}$$

$$\mathcal{V}_{\nu 0}\varphi = V_{\nu 0}\varphi - \sum_{i=1}^{3}(V_{\nu 0}\varphi, z^{(i)})_{0;\partial S} z^{(i)}(x) \tag{49}$$

The operator $\mathcal{V}_\nu\varphi$ is continuous from $\mathcal{H}_{-1/2}(\partial S)$ to $H_{1,\omega}(\mathbb{R}^2)$, $\mathcal{V}_{\nu 0}\varphi$ is a homeomorphism from $\mathcal{H}_{-1/2}(\partial S)$ to $\mathcal{H}_{1/2}(\partial S)$. The following jump formulae hold

$$\gamma^+\pi^+\mathcal{V}_\nu\varphi = \gamma^-\pi^-\mathcal{V}_\nu\varphi = \mathcal{V}_{\nu 0}\varphi, \qquad (\mathcal{N}_\nu^+ - \mathcal{N}_\nu^-)\mathcal{V}_{\nu 0}\varphi = \varphi. \tag{50}$$

For $\psi \in \mathcal{H}_{1/2}(\partial S)$ we define a modified double layer potential and the operators of its limiting values by

$$(\mathcal{W}_\nu\psi)(x) = \begin{cases} (W_\nu\psi)(x) - \sum\limits_{i=1}^{3}(W_\nu^+\psi, z^{(i)})_{0;\partial S} z^{(i)}(x), & x \in S^+, \\ (W_\nu\psi)(x) - \sum\limits_{i=1}^{3}(W_\nu^-\psi, z^{(i)})_{0;\partial S} z^{(i)}(x), & x \in S^-, \end{cases} \tag{51}$$

$$\mathcal{W}_\nu^\pm\psi = \gamma^\pm\pi^\pm\mathcal{W}_\nu\psi. \tag{52}$$

The operators $\mathcal{W}_\nu^\pm\psi$ are homeomorphisms from $\mathcal{H}_{1/2}(\partial S)$ to $\in H_{1,\omega}(\mathbb{R}^2)$, $\pi^+\mathcal{W}_\nu\psi \in H_1(S^+)$, $\pi^-\mathcal{W}_\nu\psi \in H_{1,\omega}(S^-)$ and

$$\|\pi^+\mathcal{W}_\nu\psi\|_{1;S^+} + \|\pi^-\mathcal{W}_\nu\psi\|_{1,\omega;S^-} \le c\|\psi\|_{1/2;\partial S}. \tag{53}$$

The following jump formulae hold

$$\mathcal{W}_\nu^+\psi - \mathcal{W}_\nu^-\psi = -\psi, \qquad \mathcal{N}_\nu^+\mathcal{W}_\nu^+\psi = \mathcal{N}_\nu^-\mathcal{W}_\nu^-\psi. \tag{54}$$

Finally

$$\mathcal{W}_\nu^+ = \mathcal{V}_\nu\mathcal{N}_\nu^-, \qquad \mathcal{W}_\nu^- = \mathcal{V}_\nu\mathcal{N}_\nu^+. \tag{55}$$

The potentials $V_\nu\varphi$, $W_\nu\psi$, $\mathcal{V}_\nu\varphi$, $\mathcal{W}_\nu\psi$ with $\nu = i, e$ satisfy homogenous equilibrium equations with material constants $\mu_i, \alpha_i, \beta_i, \gamma_i, \kappa_i$ and $\mu_e, \alpha_e, \beta_e, \gamma_e, \kappa_e$, respectively.

8 Boundary integral equations

We seek a solution of (40) in the form $u = \{u_+, u_-\}$ such that

$$u_+ = \pi^+\mathcal{V}_i\varphi_+ + \sum_{i=1}^{3}(f, z^{(i)})_{0;\partial S} z^{(i)} + z, \qquad u_- = \pi^-\mathcal{V}_e\varphi_- + z, \tag{56}$$

where $\varphi_\pm \in \mathcal{H}_{-1/2}(\partial S)$ is the unknown density and $z \in \mathcal{Z}$ is arbitrary. This representation leads to the following system of boundary integral equations with singular and weakly singular kernels.

$$\mathcal{V}_{i0}\varphi_+ - \mathcal{V}_{e0}\varphi_- = \tilde{f}, \qquad \mathcal{N}_i^+\mathcal{V}_{i0}\varphi_+ - \mathcal{N}_e^-\mathcal{V}_{e0}\varphi_- = g, \tag{57}$$

where $\tilde{f} = f - \sum\limits_{i=1}^{3}(f, z^{(i)})_{0;\partial S} z^{(i)} \in \mathcal{H}_{1/2}(\partial S)$ and $\|\tilde{f}\|_{1/2;\partial S} \le c\,\|f\|_{1/2;\partial S}$.

Theorem 9. *For any $f \in H_{1/2}(\partial S)$ and $g \in \mathcal{H}_{-1/2}(\partial S)$ system (57) has a unique solution pair $\{\varphi_+, \varphi_-\} \in [\mathcal{H}_{-1/2}(\partial S)]^2$ and*

$$\|\varphi_\pm\|_{-1/2;\partial S} \le c\,(\|f\|_{1/2;\partial S} + \|g\|_{-1/2;\partial S}). \tag{58}$$

Subsequently, (56) is the solution of (40) for any $z \in \mathcal{Z}$ and there is a solution u_0 such that

$$\|u_0\|_{1,\omega} \le c\,(\|f\|_{1/2;\partial S} + \|g\|_{-1/2;\partial S}). \tag{59}$$

P r o o f . Applying $\mathcal{N}_i^+$ to the first equation of (57) we arrive at

$$\mathcal{N}_i^+\mathcal{V}_{i0}\varphi_+ - \mathcal{N}_i^+\mathcal{V}_{e0}\varphi_- = \mathcal{N}_i^+\tilde{f}. \tag{60}$$

Next, using the second equation of (57) together with (60) we arrive at

$$\begin{aligned}\varphi_- &= (\mathcal{V}_{e0})^{-1}\mathcal{N}^{-1}\left(g - \mathcal{N}_i^+\tilde{f}\right),\\ \varphi_+ &= (\mathcal{V}_{i0})^{-1}\left(\tilde{f} + \mathcal{N}^{-1}\left(g - \mathcal{N}_i^+\tilde{f}\right)\right).\end{aligned} \tag{61}$$

The estimate (58) follows from (61) and the properties of operators $\mathcal{V}_{e0}$, $\mathcal{V}_{i0}$, $\mathcal{N}$ and $\mathcal{N}_i^+$. The estimate (59) follows from (56) with $z = 0$ and (29). □

If we seek a solution of (40) in the form $u = \{u_+, u_-\}$ such that

$$u_+ = \pi^+\mathcal{W}_i\psi_+ + \sum_{i=1}^{3}(f, z^{(i)})_{0;\partial S}z^{(i)} + z, \qquad u_- = \pi^-\mathcal{W}_e\psi_- + z, \tag{62}$$

where $\psi_\pm \in \mathcal{H}_{1/2}(\partial S)$ is the unknown density and $z \in \mathcal{Z}$ is arbitrary, we arrive to the following system of boundary integral equations with singular and hyper-singular kernels.

$$\mathcal{W}_i^+\psi_+ - \mathcal{W}_e^-\psi_- = \tilde{f}, \qquad \mathcal{N}_i^+\mathcal{W}_i^+\psi_+ - \mathcal{N}_e^-\mathcal{W}_e^-\psi_- = g, \tag{63}$$

Theorem 10. *For any $f \in H_{1/2}(\partial S)$ and $g \in \mathcal{H}_{-1/2}(\partial S)$ system (63) has a unique solution pair $\{\psi_+, \psi_-\} \in [\mathcal{H}_{1/2}(\partial S)]^2$ and*

$$\|\psi_\pm\|_{1/2;\partial S} \le c\,(\|f\|_{1/2;\partial S} + \|g\|_{-1/2;\partial S}). \tag{64}$$

Subsequently, (62) is the solution of (40) for any $z \in \mathcal{Z}$ and there is a solution u_0 such that

$$\|u_0\|_{1,\omega} \le c\,(\|f\|_{1/2;\partial S} + \|g\|_{-1/2;\partial S}). \tag{65}$$

P r o o f . Applying $\mathcal{N}_i^+$ to the first equation of (63) we arrive at

$$\mathcal{N}_i^+\mathcal{W}_i^+\psi_+ - \mathcal{N}_i^+\mathcal{W}_e^-\psi_- = \mathcal{N}_i^+\tilde{f}. \tag{66}$$

Next, using the second equation of (63) together with (66) we arrive at

$$\begin{aligned}\psi_- &= (\mathcal{W}_e^-)^{-1}\mathcal{N}^{-1}\left(g - \mathcal{N}_i^+\tilde{f}\right),\\ \psi_+ &= (\mathcal{W}_i^+)^{-1}\left(\tilde{f} + \mathcal{N}^{-1}\left(g - \mathcal{N}_i^+\tilde{f}\right)\right).\end{aligned} \tag{67}$$

The estimate (64) follows from (67) and the properties of operators $\mathcal{W}_e^-$, $\mathcal{W}_i^+$, $\mathcal{N}$ and $\mathcal{N}_i^+$. The estimate (65) follows from (62) with $z = 0$ and (29). □

If we seek a solution of (40) in the form $u = \{u_+, u_-\}$ such that

$$u_+ = \pi^+\mathcal{V}_i\varphi_+ + \sum_{i=1}^{3}(f, z^{(i)})_{0;\partial S}z^{(i)} + z, \qquad u_- = \pi^-\mathcal{W}_e\psi_- + z, \tag{68}$$

where $\varphi_+ \in \mathcal{H}_{-1/2}(\partial S)$ and $\psi_- \in \mathcal{H}_{1/2}(\partial S)$ are the unknown densities and $z \in \mathcal{Z}$ is arbitrary, we arrive to the following system of boundary integral equations with weakly singular, singular, and hyper-singular kernels.

$$\mathcal{V}_{i0}\varphi_+ - \mathcal{W}_e^-\psi_- = \tilde{f}, \qquad \mathcal{N}_i^+\mathcal{V}_{i0}\varphi_+ - \mathcal{N}_e^-\mathcal{W}_e^-\psi_- = g, \tag{69}$$

Theorem 11. *For any $f \in H_{1/2}(\partial S)$ and $g \in \mathcal{H}_{-1/2}(\partial S)$ system (69) has a unique solution pair $\{\varphi_+, \psi_-\}$, $\varphi_+ \in \mathcal{H}_{-1/2}(\partial S)$, $\psi_- \in \mathcal{H}_{1/2}(\partial S)$ and*

$$\begin{aligned}\|\varphi_+\|_{1/2;\partial S} &\le c(\|f\|_{1/2;\partial S} + \|g\|_{-1/2;\partial S}),\\ \|\psi_-\|_{1/2;\partial S} &\le c(\|f\|_{1/2;\partial S} + \|g\|_{-1/2;\partial S}).\end{aligned} \tag{70}$$

Subsequently, (68) is the solution of (40) for any $z \in \mathcal{Z}$ and there is a solution u_0 such that

$$\|u_0\|_{1,\omega} \le c\,(\|f\|_{1/2;\partial S} + \|g\|_{-1/2;\partial S}). \tag{71}$$

P r o o f . Applying $\mathcal{N}_i^+$ to the first equation of (63) we arrive at

$$\mathcal{N}_i^+ \mathcal{V}_{i0} \varphi_+ - \mathcal{N}_i^+ \mathcal{W}_e^- \psi_- = \mathcal{N}_i^+ \tilde{f}. \tag{72}$$

Next, using the second equation of (69) together with (72) we arrive at

$$\psi_- = (\mathcal{W}_e^-)^{-1} \mathcal{N}^{-1} \left(g - \mathcal{N}_i^+ \tilde{f} \right),$$
$$\varphi_+ = (\mathcal{V}_{i0})^{-1} \left(\tilde{f} + \mathcal{N}^{-1} \left(g - \mathcal{N}_i^+ \tilde{f} \right) \right). \tag{73}$$

The estimate (70) follows from (73) and the properties of operators $\mathcal{W}_e^-$, $\mathcal{V}_{i0}$, $\mathcal{N}$ and $\mathcal{N}_i^+$. The estimate (71) follows from (68) with $z = 0$ and (29). □

Finally if we seek a solution of (40) in the form $u = \{u_+, u_-\}$ such that

$$u_+ = \pi^+ \mathcal{W}_i \psi_+ + \sum_{i=1}^{3} (f, z^{(i)})_{0;\partial S} z^{(i)} + z, \qquad u_- = \pi^- \mathcal{V}_e \varphi_- + z, \tag{74}$$

where $\varphi_- \in \mathcal{H}_{-1/2}(\partial S)$ and $\psi_+ \in \mathcal{H}_{1/2}(\partial S)$ are the unknown densities and $z \in \mathcal{Z}$ is arbitrary, we arrive to the following system of boundary integral equations with weakly singular, singular, and hyper-singular kernels.

$$\mathcal{W}_i^+ \psi_+ - \mathcal{V}_{i0} \varphi_- = \tilde{f}, \qquad \mathcal{N}_i^+ \mathcal{W}_i^+ \psi_+ - \mathcal{N}_e^- \mathcal{V}_{e0} \varphi_- = g, \tag{75}$$

Theorem 12. *For any $f \in H_{1/2}(\partial S)$ and $g \in \mathcal{H}_{-1/2}(\partial S)$ system (75) has a unique solution pair $\{\varphi_-, \psi_+\}$, $\varphi_- \in \mathcal{H}_{-1/2}(\partial S)$, $\psi_+ \in \mathcal{H}_{1/2}(\partial S)$ and estimates (70) and (71) hold.*

P r o o f . The proof of this theorem repeats the proof of theorem (11) with obvious changes. □

9 Conclusion

In this paper we have formulated the inclusion problem in anti-plane Cosserat elasticity in Sobolev spaces and established existence, uniqueness and continuous dependence on the data results for this problem. This is a necessary step to deal with such problem from the practical point of view, since it validates the subsequent application of numerical procedures such as finite element method or method of generalized Fourier series.

References

[1] E. Cosserat and F. Cosserat, Sur la theorie de l'elasticite, Ann. de l'Ecole Normale de Toulouse **10**(1), 1 (1896).
[2] A. C. Eringen, Linear theory of micropolar elasticity, J. Math. Mech. **15**, 909–923 (1966).
[3] W. Nowacki, Theory of asymmetric elasticity (Polish Scientific Publishers, Warsaw, 1986).
[4] R. Lakes, Experimental methods for study of Cosserat elastic solids and other generalized elastic continua, in Continuum Models for Materials with Microstructure (edited by H. B. Muhlhaus) (John Wiley and Sons, 1995), pp 1–13.
[5] R. S. Lakes, Experimental microelasticity of two porous solids, Int. J. Solids and Structures **22**, 55–63 (1986).
[6] H. C. Park and R. S. Lakes, Cosserat micromechanics of human bone: strain redistribution by a hydration-sensitive constituent, J. Biomechanics **19**, 385–397 (1986).
[7] V. D. Kupradze, et al., Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity (North-Holland, Amsterdam, 1979).
[8] P. Schiavone, Integral equation methods in plane asymmetric elasticity, J. Elasticity **43**, 31–43 (1996).
[9] S. Potapenko, P. Schiavone, and A. Mioduchowski, Anti-plane shear deformations in a linear theory of elasticity with microstructure, ZAMP **56**, 516–528 (2005).
[10] E. Shmoylova, S. Potapenko, and L. Rothenburg, Weak solutions of the interior boundary-value problems of plane Cosserat elasticity, ZAMP **57**, 506–522 (2006).
[11] C. Constanda, A mathematical analysis of bending of plates with transverse shear deformation (Longman Scientific & Technical, Harlow, 1990).
[12] D. Iesan, Existence theorems in the theory of micropolar elasticity, Int. J. Engng. Sci. **8**, 777–791 (1970).
[13] C. Q. Ru and P. Schiavone, A circular inclusion with circumferentially inhomogeneous interface in antiplane shear, Proc. R. Soc. Lond. A **453**, 2551–2572 (1997).
[14] H. Shen, P. Schiavone, C. Q. Ru, and A. Mioduchowski, Analysis of Internal Stress in an Elliptic Inclusion with Imperfect Interface in Plane Elasticity, Math. Mech. Solids **5**, 501–521 (2000).
[15] H. Shen, P. Schiavone, C. Q. Ru, and A. Mioduchowski, An elliptic inclusion with imperfect interface in anti-plane shear, Int. J. Solids and Structures **37**, 4557–4575 (2000).
[16] P. A. Martin, On the scattering of elastic waves by an elastic inclusion in two dimensions, Q. J. Mechanics Appl. Math. **43**, 275–291 (1990).

[17] Z.-Q. Cheng and L.-H. He, Micropolar elastic fields due to a spherical inclusion, Int. J. Eng. Sci. **33**, 389–397 (1995).
[18] Z.-Q. Cheng and L.-H. He, Micropolar elastic fields due to a circular cylindrical inclusion, Int. J. Eng. Sci. **35**, 659–668 (1997).
[19] H. Ma and G. Hu, Eshelby tensors for an ellipsoidal inclusion in a micropolar material, Int. J. Eng. Sci. **44**, 595–605 (2006).
[20] J. D. Eshelby, The determination of the elastic field of an ellipsoidal inclusion, and related problems, Proc. Roy. Soc. Lond. A **241**, 376–396 (1957).
[21] E. Atroshchenko and S. Potapenko, Existence and uniqueness theorems for transmission problem in plane Cosserat elasticity, Mathematics and Mechanics of Solids (accepted for publication), (2015)
[22] E. Atroshchenko and S. Potapenko, Existence and uniqueness theorems for transmission problem in anti-plane Cosserat elasticity, Archives of Mechanics (accepted for publication), (2015)
[23] I. Chudinovich and C. Constanda, Variational and Potential Methods in the Theory of Bending of Plates with Transverse Shear Deformation (Chapman and Hall/CRC, Boca Raton, Fl., 2000).
[24] E. Shmoylova, S. Potapenko, and A. Dorfmann, Weak solutions to anti-plane boundary value problems in a linear theory of elasticity with microstructure, Arch. Mech. **59**, 519–539 (2007).
[25] S. Potapenko and E. Shmoylova, Solvability of Weak Solutions to Anti-Plane Cosserat Elasticity by Means of Boundary Integral Equations, Math. Mech. Solids **15**(2), 209–228 (2010).
[26] E. Shmoylova, S. Potapenko, and L. Rothenburg, Weak solutions of the interior boundary value problems of plane Cosserat elasticity, ZAMP **57**, 506–522 (2006).
[27] E. Shmoylova, S. Potapenko, and A. Dorfmann, Weak solutions to fundamental boundary value problems in anti-plane elasticity with microstructure, Arch. Mech. **59**, 519–539 (2007a).
[28] I. Chudinovich and C. Constanda, The transmission problem in bending of plates with transverse shear deformation, IMA J. Appl. Math. **66**, 215–229 (2001).
[29] C. Miranda, Partial Differential Equations of Elliptic Type (Springer-Verlag, Berlin, 1970).
[30] I. Chudinovich and C. Constanda, Area potentials for elastic plates, Strathclyde Math. Res. Report **34**, (1997).

