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Weaker conditions for subdifferential calculus of convex functions [☆]



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ARTICLE INFO

Article history:

Received 21 June 2014

Accepted 21 May 2016

Available online 1 June 2016

Communicated by H. Brezis

MSC:

26B05

26J25

49H05

Keywords:

Convex functions

Fenchel subdifferential

Subdifferential calculus rules

Convex infinite-dimensional

optimization

ABSTRACT

In this paper we establish new rules for the calculus of the subdifferential mapping of the sum of two convex functions. Our results are established under conditions which are at an intermediate level of generality among those leading to the Hiriart-Urruty and Phelps formula (Hiriart-Urruty and Phelps, 1993 [15]), involving the approximate subdifferential, and the stronger assumption used in the well-known Moreau–Rockafellar formula (Rockafellar 1970, [23]; Moreau 1966, [20]), which only uses the exact subdifferential. We give an application to derive asymptotic optimality conditions for convex optimization.

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[☆] Research partially supported by CONICYT projects No. 1151003 and No. 1150909, by Math-Amsud program No. 13MATH-01 2013, by MINECO of Spain and FEDER of EU grants No. MTM2014-59179-C2-1-P and No. MTM2011-29064-C03-02, and by the Australian Research Council project No. DP160100854.

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1. Introduction

In the sixties, J.J. Moreau and R.T. Rockafellar gave the pioneering formula for the Fenchel subdifferential of the sum of two convex functions $f, g : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$:

$$\partial(f + g) = \partial f + \partial g. \quad (1)$$

When X is finite-dimensional the condition for the validity of formula (1) is [23, Theorem 23.8]

$$\text{ri}(\text{dom } f) \cap \text{ri}(\text{dom } g) \neq \emptyset. \quad (2)$$

It can be easily shown that this condition is equivalent to

$$\text{dom } f \cap \text{ri}(\text{dom } g) \neq \emptyset \text{ AND } \text{ri}(\text{dom } f) \cap \text{dom } g \neq \emptyset. \quad (3)$$

In the infinite-dimensional setting, when X is a (real) separated locally convex space, the condition for (1) is ([19], [20], [22, p. 47])

$$\text{dom } f \cap \text{cont } g \neq \emptyset \text{ OR } \text{cont } f \cap \text{dom } g \neq \emptyset,$$

where $\text{cont } h$ is the set of continuity points of h from $\text{dom } h$. In particular, when X is a Banach space and f is a proper lower semicontinuous (lsc, for short) convex function, it is known that $\text{cont } f = \text{int}(\text{dom } f)$ and so the condition above reads

$$\text{dom } f \cap \text{int}(\text{dom } g) \neq \emptyset \text{ OR } \text{int}(\text{dom } f) \cap \text{dom } g \neq \emptyset. \quad (4)$$

Observe that the conditions above can be weakened when one or both functions are polyhedral. For instance, according to [23, Theorem 23.8], (1) also follows when X is finite-dimensional, if (2) is replaced by

$$g \text{ is polyhedral AND } \text{dom } g \cap \text{ri}(\text{dom } f) \neq \emptyset.$$

Compared with (2), condition (4) is obviously a (nonsymmetric) one-sided relation. Hence, the two conditions (3) and (4) are not equivalent in general. This kind of asymmetry arises when passing from the finite to the infinite-dimensional setting. The approach to this issue constitutes one of the main contributions of this paper.

The consequences of calculus rules for the subdifferential are undoubtedly of the greatest importance in convex analysis and in the whole theory of convex optimization, giving rise to standard constraint qualifications like Slater, Abadie, and many others (e.g. [24] and references therein).

Conditions (3) and (4) lead to formula (1), which involves only the exact subdifferential of the corresponding functions. However, under the only assumption of lower

semicontinuity of f and g , we have the following Hiriart-Urruty and Phelps formula [15] (which uses the approximate subdifferential rather than the exact one), stating that for every $x \in X$

$$\partial(f + g)(x) = \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon f(x) + \partial_\varepsilon g(x)) \tag{5}$$

(the closure in the last expression is with respect to any topology compatible with the duality pair X and its dual). See also [16] for other related calculus rules using the ε -subdifferential approach. It is worth recalling that the first calculus rule which requires no condition is the Brøndsted formula [7] given for the maximum of two convex functions (see also [13] and [14] for an extension of this result).

Rule (5) is also a powerful tool in optimization theory, and also in algorithmic approaches based on the use of approximate subdifferentials of the corresponding data functions. The subtle condition relative to the lower semicontinuity of f and g seems to be restrictive in some situations; indeed, it is not explicitly evoked when conditions (2) or (4) are used. However, it is implicit there because both of these conditions imply the following relationship between the closed hulls of the involved functions (see, e.g., [23, p. 146] and [14, Lemma 15])

$$\overline{f + g} = \overline{f} + \overline{g}, \tag{6}$$

that is obviously verified when f and g are lsc. Moreover, it is shown in Proposition 2 that (5) holds under this last general condition.

Our main purpose in this paper is to derive new formulas for the subdifferential of the sum under conditions which are at an intermediate level of generality among those leading to the Hiriart-Urruty and Phelps formula (5), which holds for lsc proper convex functions, and the Moreau–Rockafellar formula (1), which indirectly requires the convex functions to satisfy the closure condition (6). This is, for instance, the case when only a half (one part) of (2) is used; that is, the domains of the involved functions overlap *quasi-sufficiently*. In particular, our main result, Theorem 12, establishes that, under (6) (or any other condition guaranteeing the validity of (5)), the asymmetric assumption

$$\text{dom } f \cap \text{ri}(\text{dom } g) \neq \emptyset \text{ and } g|_{\text{aff}(\text{dom } g)} \text{ is continuous on } \text{ri}(\text{dom } g), \tag{7}$$

provides the following asymmetric calculus rule:

$$\partial(f + g)(x) = \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon f(x) + \partial g(x)), \text{ for every } x \in X. \tag{8}$$

This formula is in fact intermediate between the classical results (1) and (5) since it uses the approximate subdifferential of one of the two functions and the exact subdifferential of the other one. This second function is called *qualified function*, and it is the one whose relative interior is involved in condition (7). A particular situation occurs when

the subdifferential of the qualified function is empty at the reference point x ; in that case, the exact calculus rule for the sum (1) obviously follows from (8).

Formula (8) is also valid for a large family of functions for which the convex cone

$$\mathbb{R}_+(\text{epi } g - (x, g(x))) \text{ is closed}$$

(condition (i) in Theorem 12). This is the case of infinite-dimensional polyhedral functions, whose analysis is meaningful in our approach and is carried out in Corollaries 14 and 20.

When functions f and g are both qualified, that is (2) holds, together with the following symmetric condition

$$f|_{\text{aff}(\text{dom } f)} \text{ and } g|_{\text{aff}(\text{dom } g)} \text{ are respectively continuous on } \text{ri}(\text{dom } f) \text{ and } \text{ri}(\text{dom } g), \tag{9}$$

it is proved in Theorem 15 that

$$\partial(f + g)(x) = \text{cl}(\partial f(x) + \partial g(x)), \text{ for every } x \in X.$$

In addition, if one of the subdifferential sets of f or g at x is locally compact, then this last formula reduces to the exact rule (1); this is our Corollary 19.

In this respect, our hypothesis (9) constitutes the infinite-dimensional counterpart, which is needed, together with (2), to extend the aforementioned result of Rockafellar (Theorem 23.8 in [23]) to the setting of locally convex spaces. Obviously, (9) is automatically satisfied when $X = \mathbb{R}^n$, in which case it is equivalent to the fact that $\mathbb{R}_+(\text{dom } f - \text{dom } g)$ is a linear subspace. When X is a Banach space, the condition

$$\mathbb{R}_+(\text{dom } f - \text{dom } g) \text{ is a closed linear subspace,}$$

together with the lower semicontinuity of f and g , constitutes the so-called Attouch–Brézis condition [2], extended by Zălinescu [28, Theorem 2.8.7] to the case in which X is a Fréchet space. In this way, our assumption can be seen as a counterpart of the Attouch–Brézis condition for *general* locally convex spaces. Other conditions guaranteeing the fulfillment of the exact formula (1) can be found in [3–5,8,9,21,25,27], among others.

Regarding our assumption (7), a legitimate criticism could be addressed at a first glance to the use of the relative interior out of the finite-dimensional framework. Indeed, the relative interior of convex sets is intrinsically connected to the finite-dimensional setting since it is always nonempty in this framework, but could be empty in infinite-dimensional spaces. However, our analysis goes beyond this difficulty by using the family of functions g_L given as the sum of g and the indicator functions I_L of finite-dimensional subspaces L of X intersecting $\text{dom } g$. In this way, we guarantee that

$$\text{dom } f \cap \text{ri}(\text{dom } g_L) \neq \emptyset.$$

Moreover, the continuity of $g|_{\text{aff}(\text{dom } g)}$ on $\text{ri}(\text{dom } g_L)$ is automatically satisfied as the last set is finite-dimensional. Thus, we arrive at [Corollary 13](#) to the following alternative characterization of the subdifferential of the sum:

$$\partial(f + g)(x) = \bigcap_{\substack{L \in \mathcal{F}(x) \\ \varepsilon > 0}} \text{cl}(\partial_\varepsilon f(x) + \partial g_L(x)), \text{ for every } x \in X,$$

where the intersection is taken over $\varepsilon > 0$ and a certain family $\mathcal{F}(x)$, defined in [\(37\)](#), whose members are finite-dimensional subspaces intersecting $\text{dom } g$. The last expression then gives another route for the subdifferential of the sum, filtering the approximate subdifferential of f on the one hand and the exact subdifferential of an augmented function g_L of g on the other hand. Moreover, by repeating this argument on f we arrive at [Corollary 18](#) to the following formula

$$\partial(f + g)(x) = \bigcap_{L \in \mathcal{G}(x)} \text{cl}(\partial f_L(x) + \partial g_L(x)), \text{ for every } x \in X,$$

where $\mathcal{G}(x)$ is defined in [\(47\)](#). This formula then can be viewed as an alternative for [\(5\)](#), which uses only the exact subdifferential of the augmented functions f_L and g_L of f and g , respectively.

Another interesting situation, not covered by the classical calculus rules, occurs when f and g are lsc convex proper functions respectively defined on X and \mathbb{R}^n , $A : X \rightarrow \mathbb{R}^n$ is a continuous linear mapping with continuous adjoint A^* , and

$$A(\text{dom } f) \cap \text{ri}(\text{dom } g) \neq \emptyset.$$

Because g is defined in \mathbb{R}^n , $\text{ri}(\text{dom } g)$ is given in the usual sense. Then, [\(8\)](#), when applied to an appropriate choice of functions, leads us to the following characterization (see [Corollary 23](#)):

$$\partial(f + g \circ A)(x) = \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon f(x) + A^* \partial g(Ax)).$$

In the last section, the above results are applied to derive new optimality conditions for convex programming problems of the form

$$\begin{aligned} (P) \quad & \text{Min} \quad f(x) \\ & \text{s.t.} \quad f_t(x) \leq 0, \quad t \in T, \\ & \quad \quad x \in C, \end{aligned}$$

where T is an arbitrary infinite index set, $C \subset X$ is a nonempty closed convex subset, and $f, f_t : X \rightarrow \mathbb{R} \cup \{+\infty\}$, $t \in T$, are lsc proper convex functions. As a consequence of our conditions, the resulting optimality conditions will involve the exact subdifferential of the constraint functions.

The summary of the paper is as follows. Section 2 sets the notation and gathers the preliminary results. Our main results in this paper are given in section 3. In particular, Theorems 12 and 15 provide the desired calculus rules, although most of the previous results in this section have their own interest in infinite-dimensional convex analysis. Finally, section 4 addresses the objective of deriving necessary and sufficient asymptotic optimality conditions for convex infinite-dimensional optimization problems.

2. Notation and preliminary results

In this paper X is a real (separated) locally convex space, its topological dual space being denoted by X^* and endowed, unless otherwise stated, with the w^* -topology. The spaces X and X^* are paired in duality by means of the canonical bilinear form $(x, x^*) \in X \times X^* \mapsto \langle x^*, x \rangle := x^*(x)$. The zero vector in both spaces is denoted by θ , and the convex neighborhoods of θ are called θ -neighborhoods. We shall adopt the convention that $\infty + (-\infty) = \infty$.

If A, B are sets in X (or in X^*), and $\Lambda \subset \mathbb{R}$ is nonempty, we define

$$A + B := \{x + y \mid x \in A, y \in B\} \text{ and } \Lambda A := \{\lambda x \mid \lambda \in \Lambda, x \in A\}, \tag{10}$$

with the conventions $\Lambda\emptyset = A + \emptyset = \emptyset$, $\Lambda x := \Lambda\{x\}$, $\lambda A := \{\lambda\}A$ and $x + A := \{x\} + A$. By $\text{co } A$, $\text{cone } A$ (or $\mathbb{R}_+ A$), and $\text{aff } A$, we denote the *convex hull*, the *conic hull*, and the *affine hull* of A , respectively. Moreover, $\text{int } A$ is the *interior* of A , and $\text{cl } A$ and \overline{A} are indistinctly used for denoting the *closure* of A (w^* -closure if $A \subset X^*$). Thus, $\overline{\text{co}}A := \text{cl}(\text{co } A)$, $\overline{\text{aff}}A := \text{cl}(\text{aff } A)$, etc. By $\text{ri } A$ we represent the interior of A in the topology relative to $\text{aff } A$ if $\text{aff } A$ is closed, and the empty set otherwise [28, p. 15]. We also associate with A the (*negative*) *dual cone* of A defined by

$$A^\circ := \{x^* \in X^* \mid \langle x^*, x \rangle \leq 0 \text{ for all } x \in A\}.$$

If the set A is convex and closed, we define the recession cone of A as

$$A_\infty := \{v \in X \mid x + tv \in A \text{ for all } t \geq 0\},$$

where x is any point of A . If A is convex and $\varepsilon \geq 0$, we define

$$N_A^\varepsilon(x) := \begin{cases} \{x^* \in X^* \mid \langle x^*, y - x \rangle \leq \varepsilon \text{ for all } y \in A\}, & \text{if } x \in A, \\ \emptyset, & \text{if } x \notin A, \end{cases}$$

with $N_\emptyset^\varepsilon \equiv \emptyset$. If $\varepsilon = 0$ we omit the reference to ε and write $N_A(x)$, which is the usual *normal cone* of A at x .

The following property is used very often: If A is convex,

$$\lambda \text{ri } A + (1 - \lambda) \text{cl } A \subset \text{ri } A, \text{ for every } \lambda \in]0, 1]. \tag{11}$$

As a consequence of (11), if A and B are convex sets in X such that $\text{ri}(A) \cap \text{ri}(B) \neq \emptyset$, then we have

$$\overline{A \cap B} = \overline{A} \cap \overline{B} = \overline{\text{ri}(A) \cap \text{ri}(B)} = \overline{\text{ri}(A) \cap \text{ri}(B)}. \tag{12}$$

In particular, if $B = X$, this property yields

$$\text{ri}(A) \neq \emptyset \implies \overline{A} = \overline{\text{ri}(A)}.$$

We say that a function $h : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is *proper* if its *effective domain*, $\text{dom } h := \{x \in X \mid h(x) < +\infty\}$, is nonempty and h never takes the value $-\infty$; and it is *convex* (*lower semicontinuous* or *lsc*, for short) if its *epigraph*, $\text{epi } h := \{(x, \lambda) \in X \times \mathbb{R} \mid h(x) \leq \lambda\}$, is convex (closed). The *restriction* of h to the set A is denoted by $h|_A$. The *lsc envelope* of h is the function \bar{h} such that $\text{epi}(\bar{h}) = \text{cl}(\text{epi } h)$. Equivalently, we have

$$\bar{h}(x) = \liminf_{y \rightarrow x} h(y) \quad \text{for all } x \in X.$$

We shall denote by $\Gamma_0(X)$ the family of lsc proper convex functions defined on X .

The *Fenchel conjugate* of a proper function h is the function $h^* : X^* \rightarrow \mathbb{R} \cup \{\pm\infty\}$ defined by

$$h^*(x^*) := \sup_{x \in X} \{\langle x^*, x \rangle - h(x)\}.$$

The *indicator* and the *support* functions of $A \subset X$ are, respectively, defined as

$$I_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{if } x \notin A, \end{cases}$$

$$\sigma_A(x^*) := I_A^*(x^*) = \sup\{\langle x^*, x \rangle \mid x \in A\}, \text{ for } x^* \in X^*,$$

with $\sigma_\emptyset \equiv -\infty$.

Suppose now that h is convex. If $\varepsilon \geq 0$, the ε -*subdifferential* of h at a point x where h is finite is the w^* -closed convex set given by

$$\partial_\varepsilon h(x) := \{x^* \in X^* \mid h(y) - h(x) \geq \langle x^*, y - x \rangle - \varepsilon \text{ for all } y \in X\}.$$

If $h(x) \notin \mathbb{R}$, or if $h(x)$ is not defined (of the form $-\infty + \infty$), then we set $\partial_\varepsilon h(x) := \emptyset$. In particular, for $\varepsilon = 0$ we get the *Fenchel subdifferential* of h at x , $\partial h(x) := \partial_0 h(x)$. The following implications are very useful (e.g. [19])

$$\partial h(x) \neq \emptyset \implies h(x) = \bar{h}(x) \implies \partial_\varepsilon h(x) = \partial_\varepsilon \bar{h}(x). \tag{13}$$

Moreover, when $\partial_\varepsilon h(x) \neq \emptyset$ we have that

$$h(x) \leq \bar{h}(x) + \varepsilon. \tag{14}$$

The ε -directional derivative of h at x where it is finite is the function $h'_\varepsilon(x; \cdot) : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ defined by

$$h'_\varepsilon(x; v) = \inf_{t>0} \frac{h(x + tv) - h(x) + \varepsilon}{t}.$$

For $\varepsilon = 0$ we simply write $h'(x; \cdot) := h'_0(x; \cdot)$. If, in addition to being convex, h is lsc, for $x \in X$ where h is finite we have the following important relations:

$$h'_\varepsilon(x; \cdot) = \sigma_{\partial_\varepsilon h(x)} \text{ for every } \varepsilon > 0, \tag{15}$$

$$h'(x; \cdot) = \sigma_{\partial h(x)} \text{ when } \partial h(x) \neq \emptyset \text{ and } h'(x; \cdot) \text{ is lsc,} \tag{16}$$

$$\text{dom } h'_\varepsilon(x; \cdot) = \text{cone}(\text{dom } h - x) \text{ for every } \varepsilon \geq 0. \tag{17}$$

The following lemma will be used in the paper.

Lemma 1. *Let us consider two convex functions $f, g : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ such that the equality in (6) holds at a given $x \in X$; i.e., $\overline{(f + g)}(x) = \bar{f}(x) + \bar{g}(x)$. If $\partial_\varepsilon(f + g)(x) \neq \emptyset$ for some $\varepsilon \geq 0$, then the functions $f, \bar{f}, g,$ and \bar{g} are all proper and satisfy*

$$f(x) \leq \bar{f}(x) + \varepsilon, g(x) \leq \bar{g}(x) + \varepsilon.$$

Proof. The assumption $\partial_\varepsilon(f + g)(x) \neq \emptyset$ entails that $f(x)$ and $g(x) \in \mathbb{R}$. Moreover, the inclusion $\partial_\varepsilon(f + g)(x) \subset \partial_\varepsilon(\overline{(f + g)})(x)$ implies $\overline{(f + g)}(x) \in \mathbb{R}$ so that $\bar{f}(x), \bar{g}(x) \in \mathbb{R}$ (by (6) at x). Thus, \bar{f} and \bar{g} are both proper. Because of the relations $f \geq \bar{f}$ and $g \geq \bar{g}$, we see that f and g are also proper. Finally, from (14) and equality in (6) at x we have that

$$f(x) + g(x) \leq \bar{f}(x) + \bar{g}(x) + \varepsilon.$$

This relation together with $\bar{f}(x) \leq f(x)$ and $\bar{g}(x) \leq g(x)$ give us the desired inequalities. \square

The following simple example shows the necessity of condition (6) in Lemma 1.

Example 1. Consider the convex functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$\begin{aligned} f(x) &:= +\infty \text{ if } x < 0; 0 \text{ if } x = 0; +\infty \text{ if } x > 0, \\ g(x) &:= -\infty \text{ if } x < 0; 0 \text{ if } x = 0; +\infty \text{ if } x > 0. \end{aligned}$$

Then, we easily check that $\partial(f + g)(0) \neq \emptyset$ but the equality in (6) does not hold at 0.

Let us give, for the sake of completeness, a direct and easy proof of formula (5) when the convex functions f and g satisfy the closure condition (6).

Proposition 2. Formula (5) holds for every couple of convex functions $f, g : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ satisfying (6).

Proof. If $\partial(f + g)(x) \neq \emptyset$, by Lemma 1 above, the functions f, g, \bar{f}, \bar{g} are all proper and we have that $\bar{f}(x) = f(x) \in \mathbb{R}$ and $\bar{g}(x) = g(x) \in \mathbb{R}$. By (13) and (6),

$$\partial(f + g)(x) = \partial(\overline{f + g})(x) = \partial(\bar{f} + \bar{g})(x).$$

Since $\sigma_{\partial(\bar{f} + \bar{g})(x)}$ is the closure of the function $(\bar{f} + \bar{g})'(x; \cdot)$, we have for all $\varepsilon > 0$,

$$\begin{aligned} \sigma_{\partial(f + g)(x)} &= \sigma_{\partial(\bar{f} + \bar{g})(x)} \leq (\bar{f} + \bar{g})'(x; \cdot) = \bar{f}'(x; \cdot) + \bar{g}'(x; \cdot) \\ &\leq \bar{f}'_{\varepsilon}(x; \cdot) + \bar{g}'_{\varepsilon}(x; \cdot) = \sigma_{\partial_{\varepsilon}\bar{f}(x)} + \sigma_{\partial_{\varepsilon}\bar{g}(x)} = \sigma_{\partial_{\varepsilon}f(x)} + \sigma_{\partial_{\varepsilon}g(x)}. \end{aligned}$$

Hence, $\partial(f + g)(x) \subset \bigcap_{\varepsilon > 0} \text{cl}(\partial_{\varepsilon}f(x) + \partial_{\varepsilon}g(x))$ and, so, (5) follows due to the straightforwardness of the opposite inclusion. \square

Finally, we recall that if $f, g \in \Gamma_0(X)$ and $\varepsilon \geq 0$, then we have, in addition to (5),

$$\partial_{\varepsilon}(f + g)(x) = \bigcap_{\eta > 0} \text{cl} \left(\bigcup_{\varepsilon_1 + \varepsilon_2 = \varepsilon + \eta} \partial_{\varepsilon_1}f(x) + \partial_{\varepsilon_2}g(x) \right) \tag{18}$$

(e.g. [15]). It is worth noting that (18) also holds on the domain of the subdifferential mapping $\partial(f + g)$, when f and g are proper and convex, and satisfy condition (6) rather than being lsc.

Proposition 3. Let $f, g : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ be convex functions satisfying (6). Then, (18) holds at every $x \in X$ where $\partial(f + g)(x) \neq \emptyset$.

Proof. If $\partial(f + g)(x) \neq \emptyset$, then $\overline{(f + g)}(x) = (f + g)(x)$ and so, invoking the current assumption (6),

$$f(x) = \bar{f}(x), g(x) = \bar{g}(x), \tag{19}$$

$$\begin{aligned} \partial_{\varepsilon}(f + g)(x) &= \partial_{\varepsilon}(\overline{f + g})(x) = \partial_{\varepsilon}(\bar{f} + \bar{g})(x), \partial_{\varepsilon}f(x) = \partial_{\varepsilon}\bar{f}(x), \text{ and} \\ \partial_{\varepsilon}g(x) &= \partial_{\varepsilon}\bar{g}(x). \end{aligned} \tag{20}$$

Hence, (18) follows by applying the Hiriart-Urruty and Phelps formula [15] to the lsc proper (recall Lemma 1) convex functions \bar{f} and \bar{g} . \square

3. The sum rules

In this section, we prove some new sum rules for the subdifferential; they constitute the core part of this paper. We begin by establishing the following proposition, which

gives rise to some specialization of the formula in (8) for the case in which g is an indicator function.

Proposition 4. *Let $C \subset X$ and $\{A_\varepsilon\}_{\varepsilon>0}$ be a family of convex subsets of X^* nonincreasing as ε goes to 0. Fix $x \in C$ and assume*

either (i) $\mathbb{R}_+(C - x)$ is closed

or (ii) there exists $\varepsilon_0 > 0$ such that $\text{ri}(C - x) \cap \text{dom } \sigma_{A_{\varepsilon_0}} \neq \emptyset$.

Then,

$$\bigcap_{\varepsilon>0} \text{cl} \{A_\varepsilon + N_C^\varepsilon(x)\} = \bigcap_{\varepsilon>0} \text{cl} \{A_\varepsilon + N_C(x)\}. \tag{21}$$

Proof. We denote by B the left-hand-side set in (21). It is clear that

$$\bigcap_{\varepsilon>0} \text{cl} \{A_\varepsilon + N_C(x)\} \subset B,$$

as $N_C(x) \subset N_C^\varepsilon(x)$, and only the opposite inclusion needs to be proved. We suppose that all the subsets A_ε are nonempty (as $\emptyset + N_C(x) = \emptyset$). To this aim, it is enough to prove that $B \subset \text{cl} \{A_\varepsilon + N_C(x)\}$ for all $\varepsilon \in]0, \varepsilon_0[$ with $\varepsilon_0 > 0$ (because A_ε is nonincreasing as $\varepsilon \downarrow 0$). Since $B \subset \text{cl} \{A_\varepsilon + N_C(x)\}$ is equivalent to

$$\sigma_B(u) \leq \sigma_{A_\varepsilon + N_C(x)}(u) \text{ for all } u \in X, \tag{22}$$

we shall distinguish the following cases relative to the position of u :

(a) If $u \notin \overline{\text{cone}(C - x)} = (N_C(x))^\circ$, then

$$\sigma_{A_\varepsilon + N_C(x)}(u) = +\infty,$$

and, so, (22) trivially holds.

(b) If $u = y - x$, with $y \in C$, then for a given $\varepsilon > 0$ and every $\delta \in]0, \varepsilon[$ we write

$$\sigma_{A_\delta + N_C^\delta(x)}(u) \leq \sigma_{A_\delta}(u) + \delta \leq \sigma_{A_\varepsilon}(u) + \delta \leq \sigma_{A_\varepsilon + N_C(x)}(u) + \delta, \tag{23}$$

which in turn leads us to

$$\sigma_B(u) \leq \inf_{\delta \in]0, \varepsilon[} (\sigma_{A_\varepsilon + N_C(x)}(u) + \delta) = \sigma_{A_\varepsilon + N_C(x)}(u),$$

that is, (22) follows.

(c) If $u \in \text{cone}(C - x)$, (22) follows from (b) due to the positive homogeneity of the support function $\sigma_{A_\varepsilon + N_C(x)}$.

At this step, the proof of (22) is finished under assumption (i).

Let us assume (ii). We only need to investigate the case (d) below.

(d) If $u \in \overline{\text{cone}}(C - x) \setminus \text{cone}(C - x)$, we pick $u_0 \in \text{ri}(C - x) \cap \text{dom } \sigma_{A_{\varepsilon_0}}$. Then, for any $\lambda \in]0, 1[$, by (11) we have that $u_\lambda := \lambda u + (1 - \lambda)u_0 \in \text{ri}(\text{cone}(C - x)) \subset \text{cone}(C - x)$. Hence, because of the lower semicontinuity and convexity of σ_B , inequality (23) applied to $u_\lambda \in \text{cone}(C - x)$, the inequality $\sigma_{N_C(x)}(u_0) \leq 0$, and the inclusion $\text{dom } \sigma_{A_{\varepsilon_0}} \subset \text{dom } \sigma_{A_\varepsilon}$, for any $\varepsilon \in]0, \varepsilon_0[$

$$\begin{aligned} \sigma_B(u) &\leq \liminf_{\lambda \rightarrow 1} \sigma_B(u_\lambda) \\ &\leq \liminf_{\lambda \rightarrow 1} \sigma_{A_\varepsilon + N_C(x)}(u_\lambda) \text{ (by (c))} \\ &\leq \liminf_{\lambda \rightarrow 1} \{ \lambda \sigma_{A_\varepsilon + N_C(x)}(u) + (1 - \lambda) \sigma_{A_\varepsilon + N_C(x)}(u_0) \} \\ &\leq \sigma_{A_\varepsilon + N_C(x)}(u) + \limsup_{\lambda \uparrow 1} (1 - \lambda) \sigma_{A_\varepsilon}(u_0) \\ &= \sigma_{A_\varepsilon + N_C(x)}(u). \end{aligned}$$

This finishes the proof of the proposition. \square

The following corollary provides the announced specialization of the formula in (5) for the case in which g is an indicator function.

Corollary 5. *Let $f \in \Gamma_0(X)$ and $C \subset X$ be a closed convex set. If for a given $x \in C$ either $\mathbb{R}_+(C - x)$ is closed or $\text{ri}(C - x) \cap \mathbb{R}_+(\text{dom } f - x) \neq \emptyset$, then*

$$\partial(f + I_C)(x) = \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon f(x) + N_C(x)).$$

Proof. By (15) and (17), the second assumption is equivalent to

$$\text{ri}(C - x) \cap \text{dom } \sigma_{\partial_\varepsilon f(x)} \neq \emptyset \text{ for all } \varepsilon > 0.$$

Now we apply formula (5) (f and I_C are convex, proper, and lsc) and (21) to get the desired conclusion. \square

Lemma 6. *The following statements are equivalent for any couple of proper convex functions f and g defined on X :*

- (i) $\text{dom } f \cap \text{ri}(\text{dom } g) \neq \emptyset$ and $g|_{\text{aff}(\text{dom } g)}$ is continuous on $\text{ri}(\text{dom } g)$;
- (ii) $((\text{dom } f - x) \times \mathbb{R}) \cap (\text{ri}(\text{epi } g - (x, g(x)))) \neq \emptyset$ for all $x \in \text{dom } g$;
- (iii) $((\text{dom } f - x) \times \mathbb{R}) \cap (\text{ri}(\text{epi } g - (x, g(x)))) \neq \emptyset$ for some $x \in \text{dom } g$.

Proof. (i) \Rightarrow (ii). If $x_0 \in \text{ri}(\text{dom } g) \cap \text{dom } f$ and $g|_{\text{aff}(\text{dom } g)}$ is continuous on $\text{ri}(\text{dom } g)$, then it is obvious that $(x_0, g(x_0) + 1) \in \text{ri}(\text{epi } g)$. Now, for $x \in \text{dom } g$ one has

$$(x_0, g(x_0) + 1) - (x, g(x)) \in \text{ri}(\text{epi } g) - (x, g(x)) = \text{ri}(\text{epi } g - (x, g(x))).$$

At the same time, $(x_0 - x, g(x_0) - g(x) + 1) \in (\text{dom } f - x) \times \mathbb{R}$.

(ii) \Rightarrow (iii). Obvious (g is proper).

(iii) \Rightarrow (i). Take $(z, \mu) \in (\text{ri}(\text{epi } g - (x_0, g(x_0)))) \cap ((\text{dom } f - x_0) \times \mathbb{R})$ for $x_0 \in \text{dom } g$. From one side, $(z, \mu) + (x_0, g(x_0)) \in \text{ri}(\text{epi } g)$ and this entails $g|_{\text{aff}(\text{dom } g)}$ is continuous at $z + x_0 \in \text{ri}(\text{dom } g)$; hence, continuous on $\text{ri}(\text{dom } g)$ (see, e.g., [28, Corollary 2.2.10]). From the other side, relation $(z, \mu) \in (\text{dom } f - x_0) \times \mathbb{R}$ entails $z + x_0 \in \text{dom } f$ so that $z + x_0 \in \text{ri}(\text{dom } g) \cap \text{dom } f$. \square

The following proposition provides some properties of functions f such that $\mathbb{R}_+(\text{epi } f - (x, f(x)))$ is closed for some $x \in \text{dom } f$. This family is shown in Lemma 8 below to include all *polyhedral functions*; that is, those (convex and lsc) functions whose epigraphs are polyhedral sets. However, there are nonpolyhedral functions which satisfy (24); for instance, locally positive homogeneous functions. Other related properties are the so-called *local polyhedrality* [1] and *exactness of tangent approximations* [18].

Proposition 7. *Let f be a proper convex function defined on X . If $x \in X$ is such that $f(x) \in \mathbb{R}$ and $\mathbb{R}_+(\text{epi } f - (x, f(x)))$ is closed, then \bar{f} is proper and we have that*

$$\text{epi } f'(x; \cdot) = \mathbb{R}_+(\text{epi } f - (x, f(x))). \tag{24}$$

As a consequence of that, $\partial f(x) \neq \emptyset$, $f'(x; \cdot) = (\bar{f})'(x; \cdot) = \sigma_{\partial f(x)}$, and f is lsc at x .

Proof. Let us first show that (24) holds. It is clear that $\text{epi } f - (x, f(x)) \subset \text{epi } f'(x; \cdot)$ and, since $f'(x; \cdot)$ is sublinear, $\mathbb{R}_+(\text{epi } f - (x, f(x))) \subset \text{epi } f'(x; \cdot)$. Conversely, if $(u, \lambda) \in \text{epi } f'(x; \cdot)$, then for any given $\delta > 0$ there exists $t > 0$ such that $t^{-1}(f(x + tu) - f(x)) < \lambda + \delta$; that is,

$$(u, \lambda + \delta) \in t^{-1}(\text{epi } f - (x, f(x))) \subset \mathbb{R}_+(\text{epi } f - (x, f(x))). \tag{25}$$

Thus, by the current assumption we conclude

$$\text{epi } f'(x; \cdot) = \mathbb{R}_+(\text{epi } f - (x, f(x))),$$

so that $f'(x; \cdot)$ is lsc and proper (recall that a lsc convex function finite at a point never takes the value $-\infty$). Consequently, $\partial f(x) \neq \emptyset$ and $f'(x; \cdot) = \sigma_{\partial f(x)}$ so that $\bar{f}(x) = f(x)$ and \bar{f} is proper. Moreover, since

$$\mathbb{R}_+(\text{epi } \bar{f} - (x, f(x))) = \mathbb{R}_+(\overline{\text{epi } f} - (x, f(x))) = \mathbb{R}_+(\text{epi } f - (x, f(x))), \tag{26}$$

we infer that $f'(x; \cdot) = (\bar{f})'(x; \cdot) = \sigma_{\partial \bar{f}(x)} = \sigma_{\partial f(x)}$. \square

The following lemma is addressed to prove the aforementioned assertion that polyhedral functions satisfy condition (24).

Lemma 8. *Given a polyhedral set $A \subset X$, for every $x \in A$ the set $\mathbb{R}_+(A - x)$ is closed.*

Proof. We write set A as

$$A := \{z \in X \mid \langle a_j, z \rangle \leq b_j, \quad j \in \overline{1, n}\}, \tag{27}$$

and denote $J_x := \{j \in \overline{1, n} \mid \langle a_j, x \rangle = b_j\}$ for $x \in A$. If $J_x = \emptyset$, then $x \in \text{int}(A)$ and, so, $\mathbb{R}_+(A - x) = X$. Otherwise, if $J_x \neq \emptyset$, we prove that

$$\mathbb{R}_+(A - x) = \{z \in X \mid \langle a_j, z \rangle \leq 0, \text{ for } j \in J_x\}.$$

The inclusion ‘ \subset ’ is clear. To verify the other inclusion we take z in the right-hand side and choose an $\alpha > 0$ such that

$$\langle a_j, \alpha z + x \rangle \leq b_j \text{ for all } j \in \overline{1, n} \setminus J_x.$$

Then, since for all $j \in J_x$

$$\langle a_j, \alpha z + x \rangle = \langle a_j, \alpha z \rangle + b_j \leq b_j,$$

we deduce that $\alpha z + x \in A$ and, so, $z \in \alpha^{-1}(A - x) \subset \mathbb{R}_+(A - x)$. \square

The following lemma shows a stability aspect of the relation in (24).

Lemma 9. *Let f, g be two convex functions on X , and let $x \in X$ such that $f(x)$ and $g(x)$ are finite. If the sets*

$$\mathbb{R}_+(\text{epi } f - (x, f(x))) \text{ and } \mathbb{R}_+(\text{epi } g - (x, g(x))) \text{ are closed,}$$

then so is the set $\mathbb{R}_+(\text{epi}(f + g) - (x, f(x) + g(x)))$.

Proof. Let nets $(\alpha_i)_{i \in I} \subset \mathbb{R}_+$ and $(x_i, \lambda_i)_{i \in I} \subset \text{epi}(f + g)$ be such that $\alpha_i((x_i, \lambda_i) - (x, f(x) + g(x)))$ converges to some $(u, \mu) \in X \times \mathbb{R}$; i.e. the net $\alpha_i(x_i - x)$ converges to u and $\alpha_i(\lambda_i - f(x) - g(x))$ converges to μ . If $y^* \in \partial f(x)$ and $z^* \in \partial g(x)$ (due to Proposition 7, $\partial f(x)$ and $\partial g(x)$ are both nonempty), by taking into account that $f(x_i) + g(x_i) \leq \lambda_i$ we can easily prove that

$$\langle y^*, \alpha_i(x_i - x) \rangle \leq \alpha_i(f(x_i) - f(x)) \leq \alpha_i(\lambda_i - f(x) - g(x)) - \langle z^*, \alpha_i(x_i - x) \rangle,$$

and so, for every $\varepsilon > 0$, the net $\alpha_i(f(x_i) - f(x))$ is eventually contained in the interval $[\langle y^*, u \rangle - \varepsilon, \mu - \langle z^*, u \rangle + \varepsilon]$.

A similar argument yields the conclusion that, for every $\varepsilon > 0$, the net $\alpha_i(g(x_i) - g(x))$ is eventually contained in the interval $[\langle z^*, u \rangle - \varepsilon, \mu - \langle y^*, u \rangle + \varepsilon]$. Therefore, we may suppose that the (eventually bounded) nets $(\alpha_i(f(x_i) - f(x)))_{i \in I}$ and $(\alpha_i(g(x_i) - g(x)))_{i \in I}$ converge to some $\mu_1, \mu_2 \geq 0$, respectively. On the other hand, we have for every $i \in I$,

$$\alpha_i(f(x_i) - f(x)) + \alpha_i(g(x_i) - g(x)) \leq \alpha_i(\lambda_i - f(x) - g(x))$$

entailing

$$\mu_1 + \mu_2 \leq \mu. \tag{28}$$

In other words, we have that

$$\alpha_i((x_i, f(x_i)) - (x, f(x))) \rightarrow (u, \mu_1) \text{ and } \alpha_i((x_i, g(x_i)) - (x, g(x))) \rightarrow (u, \mu_2).$$

Thus, by the current assumption, there are $\gamma_1, \gamma_2 \geq 0$ such that $(u, \mu_1) \in \gamma_1(\text{epi } f - (x, f(x)))$ and $(u, \mu_2) \in \gamma_2(\text{epi } g - (x, g(x)))$. More specifically, since $\text{epi } f - (x, f(x))$ and $\text{epi } g - (x, g(x))$ are convex sets and contain $(\theta, 0)$ we may assume that $\gamma_1 = \gamma_2 \geq 1$; that is, denoting $\gamma := \gamma_1 = \gamma_2$,

$$(x, f(x)) + \gamma^{-1}(u, \mu_1) \in \text{epi } f \text{ and } (x, g(x)) + \gamma^{-1}(u, \mu_2) \in \text{epi } g.$$

Hence, by (28),

$$\begin{aligned} f(x + \gamma^{-1}u) + g(x + \gamma^{-1}u) &\leq f(x) + \gamma^{-1}\mu_1 + g(x) + \gamma^{-1}\mu_2 \\ &\leq f(x) + g(x) + \gamma^{-1}\mu. \end{aligned}$$

Therefore, $(x + \gamma^{-1}u, f(x) + g(x) + \gamma^{-1}\mu) \in \text{epi}(f + g)$ and, so, $(u, \mu) \in \gamma(\text{epi}(f + g) - (x, f(x) + g(x))) \subset \mathbb{R}_+(\text{epi}(f + g) - (x, f(x) + g(x)))$. \square

The following result completes Proposition 2 by providing other sufficient conditions for the validity of the subdifferential formula in (5).

Proposition 10. *Consider two convex functions $f, g : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ and $x \in X$ such that $\partial(f + g)(x) \neq \emptyset$. Then the functions f, g, \bar{f} , and \bar{g} are proper, and formula (5) holds at x ; that is,*

$$\partial(f + g)(x) = \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon f(x) + \partial_\varepsilon g(x)),$$

provided that one of the following conditions holds:

- (i) The equality $\overline{f + g} = \bar{f} + g$ holds on X and the cone $\mathbb{R}_+(\text{epi } g - (x, g(x)))$ is closed.
- (ii) The cones $\mathbb{R}_+(\text{epi } f - (x, f(x)))$ and $\mathbb{R}_+(\text{epi } g - (x, g(x)))$ are closed.
- (iii) The cone $\mathbb{R}_+(\text{epi } f - (x, f(x)))$ is closed, $\text{dom } f \cap \text{ri}(\text{dom } g) \neq \emptyset$, and the function $g|_{\text{aff}(\text{dom } g)}$ is continuous on $\text{ri}(\text{dom } g)$.

Proof. Due to Proposition 7, conditions (i) and (ii) imply that $(\overline{f + g})(x) = \bar{f}(x) + g(x) = \bar{f}(x) + \bar{g}(x)$. Thus, Lemma 1 ensures that the functions f, g, \bar{f} , and \bar{g} are all proper. Also by Proposition 7, since $(\overline{f + g})(x) = (f + g)(x) \in \mathbb{R}$, condition (iii) implies that $(\overline{f + g})(x) = \bar{f}(x) + g(x) \in \mathbb{R}$. Take $x_0 \in \text{dom } f \cap \text{ri}(\text{dom } g) \neq \emptyset$, $\lambda \in (0, 1)$, and denote $x_\lambda := \lambda x_0 + (1 - \lambda)x$; hence, $x_\lambda \in \text{dom } f \cap \text{ri}(\text{dom } g)$. Then, by lsc and convexity of $\overline{f + g}$, together with the continuity assumption on g ,

$$\begin{aligned} \bar{f}(x) + g(x) &= (\overline{f + g})(x) \leq \liminf_{\lambda \downarrow 0} (\overline{f + g})(x_\lambda) \leq \liminf_{\lambda \downarrow 0} (f + g)(x_\lambda) \\ &= \liminf_{\lambda \downarrow 0} f(x_\lambda) + \bar{g}(x_\lambda) \leq \liminf_{\lambda \downarrow 0} \lambda(f + \bar{g})(x_0) + (1 - \lambda)(f + \bar{g})(x) = f(x) + \bar{g}(x), \end{aligned}$$

and so $\bar{g}(x) = g(x) \in \mathbb{R}$. This shows that $(\overline{f + g})(x) = \bar{f}(x) + g(x) = \bar{f}(x) + \bar{g}(x)$, and so, again by Lemma 1, it follows that the functions f, g, \bar{f} , and \bar{g} are also proper under condition (iii). This establishes the first statement of the proposition.

To continue the proof we proceed by investigating each one of the current conditions:

- (i) By [28, Theorem 2.4.1(ii)] and the condition $\overline{f + g} = \bar{f} + g$ we obtain that

$$\partial(f + g)(x) = \partial(\overline{f + g})(x) = \partial(\bar{f} + g)(x);$$

hence, $(f + g)(x) = (\bar{f} + g)(x)$ so that $\bar{f}(x) = f(x) \in \mathbb{R}$ (recall that g is lsc at x due to Proposition 7). Thus, for every $\varepsilon > 0$ we write (thanks to [28, Theorem 2.4.11] and Proposition 7),

$$\begin{aligned} \sigma_{\partial(f+g)(x)} &= \sigma_{\partial(\bar{f}+g)(x)} \leq (\bar{f} + g)'(x; \cdot) \\ &= \bar{f}'(x; \cdot) + (\bar{g})'(x; \cdot) \\ &\leq (\bar{f})'_\varepsilon(x; \cdot) + (\bar{g})'_\varepsilon(x; \cdot) \\ &\leq \sigma_{\partial_\varepsilon \bar{f}(x)} + \sigma_{\partial_\varepsilon \bar{g}(x)} = \sigma_{\partial_\varepsilon f(x)} + \sigma_{\partial_\varepsilon g(x)}. \end{aligned}$$

Hence, $\partial(f + g)(x) \subset \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon f(x) + \partial_\varepsilon g(x))$ and, so, (5) follows due to the straightforwardness of the opposite inclusion.

- (ii) By using again Proposition 7 we write, for every $\varepsilon > 0$,

$$\begin{aligned} \sigma_{\partial(f+g)(x)} &\leq (f + g)'(x; \cdot) \\ &= f'(x; \cdot) + g'(x; \cdot) \\ &= (\bar{f})'(x; \cdot) + (\bar{g})'(x; \cdot) \\ &\leq \sigma_{\partial_\varepsilon \bar{f}(x)} + \sigma_{\partial_\varepsilon \bar{g}(x)} = \sigma_{\partial_\varepsilon f(x)} + \sigma_{\partial_\varepsilon g(x)}. \end{aligned}$$

Thus, we conclude as in the paragraph above.

(iii) We take $v \in \text{dom } f \cap \overline{\text{dom } g}$ and denote $v_\lambda := v + \lambda(x_0 - v)$, $\lambda \in (0, 1)$, where as above $x_0 \in \text{dom } f \cap \text{ri}(\text{dom } g)$. So, $v_\lambda \in \text{dom } f \cap \text{ri}(\text{dom } g)$ and $x + t(v_\lambda - x) \in \text{ri}(\text{dom } g)$ for every $\lambda, t \in (0, 1)$. Then, since $\bar{g}(x + t(v_\lambda - x)) = g(x + t(v_\lambda - x))$ by the current assumption, and $g(x) = \bar{g}(x)$ as shown in the beginning of the proof, we get

$$\begin{aligned} g'(x; v_\lambda - x) &= \inf_{0 < t < 1} \frac{g(x + t(v_\lambda - x)) - g(x)}{t} \\ &= \inf_{0 < t < 1} \frac{\bar{g}(x + t(v_\lambda - x)) - \bar{g}(x)}{t} = \bar{g}'(x; v_\lambda - x). \end{aligned}$$

Consequently, invoking Proposition 7, for every $\varepsilon > 0$ we write (\bar{f} and \bar{g} are proper and satisfy $f(x) = \bar{f}(x)$, $g(x) = \bar{g}(x)$; see the beginning of the proof)

$$\begin{aligned} \sigma_{\partial(f+g)(x)}(v_\lambda - x) &\leq (f + g)'(x; v_\lambda - x) \\ &= f'(x; v_\lambda - x) + g'(x; v_\lambda - x) \\ &= (\bar{f})'(x; v_\lambda - x) + \bar{g}'(x; v_\lambda - x) \\ &\leq (\bar{f})'_\varepsilon(x; v_\lambda - x) + \bar{g}'_\varepsilon(x; v_\lambda - x) \\ &= \sigma_{\partial_\varepsilon \bar{f}(x)}(v_\lambda - x) + \sigma_{\partial_\varepsilon \bar{g}(x)}(v_\lambda - x) \\ &= \sigma_{\partial_\varepsilon f(x)}(v_\lambda - x) + \sigma_{\partial_\varepsilon g(x)}(v_\lambda - x) \\ &\leq \lambda(\sigma_{\partial_\varepsilon f(x)}(x_0 - x) + \sigma_{\partial_\varepsilon g(x)}(x_0 - x)) \\ &\quad + (1 - \lambda)(\sigma_{\partial_\varepsilon f(x)}(v - x) + \sigma_{\partial_\varepsilon g(x)}(v - x)). \end{aligned}$$

Hence, as $\lambda \downarrow 0$ we obtain that

$$\sigma_{\partial(f+g)(x)}(v - x) \leq \sigma_{\partial_\varepsilon f(x)}(v - x) + \sigma_{\partial_\varepsilon g(x)}(v - x),$$

showing that

$$\sigma_{\partial(f+g)(x)}(u) \leq \sigma_{\partial_\varepsilon f(x) + \partial_\varepsilon g(x)}(u) \quad \text{for all } u \in \mathbb{R}_+(\text{dom } f \cap \overline{\text{dom } g} - x). \tag{29}$$

Moreover, since for every $\varepsilon > 0$

$$\begin{aligned} \mathbb{R}_+(\text{dom } f - x) &= \text{dom } f'(x; \cdot) = \text{dom}(\bar{f})'(x; \cdot) \\ &= \text{dom}(\bar{f})'_\varepsilon(x; \cdot) = \text{dom } \sigma_{\partial_\varepsilon \bar{f}(x)} = \text{dom } \sigma_{\partial_\varepsilon f(x)}, \\ \mathbb{R}_+(\overline{\text{dom } g} - x) &\supset \text{dom}(\bar{g})'_\varepsilon(x; \cdot) = \text{dom } \sigma_{\partial_\varepsilon \bar{g}(x)} = \text{dom } \sigma_{\partial_\varepsilon g(x)}, \end{aligned}$$

it follows that

$$\begin{aligned} \text{dom } \sigma_{\partial_\varepsilon f(x) + \partial_\varepsilon g(x)} &= \text{dom } \sigma_{\partial_\varepsilon f(x)} \cap \text{dom } \sigma_{\partial_\varepsilon g(x)} \\ &\subset \mathbb{R}_+(\text{dom } f - x) \cap \mathbb{R}_+(\overline{\text{dom } g} - x) = \mathbb{R}_+(\text{dom } f \cap \overline{\text{dom } g} - x). \end{aligned}$$

This shows that inequality (29) obviously holds outside the set $\mathbb{R}_+(\text{dom } f \cap \text{dom } g - x)$. In other words, (29) holds for all u and we infer that $\partial(f+g)(x) \subset \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon f(x) + \partial_\varepsilon g(x))$.

The conclusion follows as in the paragraph above. \square

We shall also need the following lemma.

Lemma 11. *Let f and g be two proper convex functions defined on X . Then, (6) follows in each one of the following cases:*

(i) *$\text{ri}(\text{dom } f) \cap \text{ri}(\text{dom } g) \neq \emptyset$ and the functions $f_{|\text{aff}(\text{dom } f)}$ and $g_{|\text{aff}(\text{dom } g)}$ are continuous on $\text{ri}(\text{dom } f)$ and $\text{ri}(\text{dom } g)$, respectively.*

(ii) *$\text{dom } f \cap \text{ri}(\text{dom } g) \neq \emptyset$, the function $g_{|\text{aff}(\text{dom } g)}$ is continuous on $\text{ri}(\text{dom } g)$, and $\overline{f+g} = \bar{f} + g$.*

Proof. From the definition of the lsc envelope, it is clear that $\overline{f+g} \geq \bar{f} + \bar{g}$. Now, for a fixed $x \in X$, we have to prove

$$\overline{(f+g)}(x) \leq \bar{f}(x) + \bar{g}(x). \tag{30}$$

We may assume that x belongs to $\text{dom } \bar{f} \cap \text{dom } \bar{g} \subset \text{cl}(\text{dom } f) \cap \text{cl}(\text{dom } g)$.

(i) We choose $x_0 \in \text{ri}(\text{dom } f) \cap \text{ri}(\text{dom } g)$ and denote $x_\lambda := \lambda x + (1 - \lambda)x_0$, $\lambda \in]0, 1[$. Then, since $x_\lambda \in \text{ri}(\text{dom } f) \cap \text{ri}(\text{dom } g)$, by the continuity assumption we infer that

$$\bar{f}(x_\lambda) = f(x_\lambda) \text{ and } \bar{g}(x_\lambda) = g(x_\lambda)$$

(observe that $\overline{f_{|\text{aff}(\text{dom } f)}} = \bar{f}_{|\text{aff}(\text{dom } f)}$ and similarly for g). Consequently, using the convexity of f and g , for every $\lambda \in]0, 1[$ we write

$$\lambda(\bar{f}(x) + \bar{g}(x)) + (1 - \lambda)(\bar{f}(x_0) + \bar{g}(x_0)) \geq \bar{f}(x_\lambda) + \bar{g}(x_\lambda) = f(x_\lambda) + g(x_\lambda).$$

Thus, since $x_0 \in \text{dom } f \cap \text{dom } g \subset \text{dom } \bar{f} \cap \text{dom } \bar{g}$ and \bar{f}, \bar{g} are proper [28, Theorem 2.2.6(iv)], by taking limits for $\lambda \uparrow 1$ we obtain that

$$\begin{aligned} \overline{(f+g)}(x) &\leq \liminf_{\lambda \uparrow 1} (f+g)(x_\lambda) \\ &\leq \liminf_{\lambda \uparrow 1} \lambda(\bar{f} + \bar{g})(x) + (1 - \lambda)(\bar{f} + \bar{g})(x_0) = (\bar{f} + \bar{g})(x), \end{aligned}$$

proving the desired inequality.

(ii) We choose $x_0 \in \text{dom } f \cap \text{ri}(\text{dom } g)$ and denote x_λ as above so that $x_\lambda \in \text{dom } \bar{f} \cap \text{ri}(\text{dom } g)$ and $\bar{g}(x_\lambda) = g(x_\lambda)$. If \bar{f} is not proper, then $\bar{f}(x_\lambda) = -\infty$ for all $\lambda \in]0, 1[$ ([28, Theorem 2.2.6(iv)]) and, so, using the current assumption $\overline{f+g} = \bar{f} + g$,

$$\overline{(f + g)}(x) = \overline{\bar{f} + \bar{g}}(x) \leq \liminf_{\lambda \uparrow 1} \{\bar{f}(x_\lambda) + g(x_\lambda)\} = -\infty;$$

that is, (30) follows. Let us suppose now that \bar{f} is proper. Then, arguing as in the proof of (i) above, for every $\lambda \in]0, 1[$ we write

$$\bar{f}(x_\lambda) + g(x_\lambda) = \bar{f}(x_\lambda) + \bar{g}(x_\lambda) \leq \lambda(\bar{f}(x) + \bar{g}(x)) + (1 - \lambda)(\bar{f}(x_0) + \bar{g}(x_0)),$$

which leads us, as $\lambda \uparrow 1$, to

$$\overline{(f + g)}(x) \leq \liminf_{\lambda \uparrow 1} (\bar{f} + \bar{g})(x_\lambda) \leq \bar{f}(x) + \bar{g}(x),$$

which is the desired inequality. \square

Now we give the main result of this section, which uses the weaker condition requiring that the domains of the involved functions overlap quasi-sufficiently; that is, the domain of one of these functions meets the relative interior of the domain of the other one. The resulting formula involves the exact subdifferential of the qualified function (represented by the relative interior of its domain into the condition) and the approximate subdifferential of the other function. As we said in the introduction, this result uses conditions which are somewhat at an intermediate level of generality between the Moreau–Rockafellar’s rule (1) and the Hiriart-Urruty and Phelps formula (5).

Theorem 12. *Let f and g be two convex functions defined on X and satisfying $\overline{f + g} = \bar{f} + \bar{g}$. Given $x \in X$ such that $g(x) \in \mathbb{R}$, we assume*

- either (i) $\mathbb{R}_+(\text{epi } g - (x, g(x)))$ is closed*
- or (ii) $\text{dom } f \cap \text{ri}(\text{dom } g) \neq \emptyset$ and $g|_{\text{aff}(\text{dom } g)}$ is continuous on $\text{ri}(\text{dom } g)$.*

Then

$$\partial(f + g)(x) = \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon f(x) + \partial g(x)). \tag{31}$$

Proof. We may suppose that $x \in \text{dom } f$. Then, according to Proposition 10, condition (i) together with the assumption $\overline{f + g} = \bar{f} + \bar{g}$ guarantee that (5) holds at x ; that is,

$$\partial(f + g)(x) = \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon f(x) + \partial_\varepsilon g(x)). \tag{32}$$

In case (ii), in view of Lemma 11(ii) assumption $\overline{f + g} = \bar{f} + \bar{g}$ implies that $\overline{f + g} = \bar{f} + \bar{g}$ and, so, by Proposition 2 formula (32) also holds.

The inclusion “ \supset ” in (31) follows immediately from (32). To show the direct inclusion “ \subset ” we pick x^* in $\partial(f + g)(x)$. Then, using the relationship $\partial_\varepsilon g(x) \times \{-1\} \subset N_{\text{epi } g}^\varepsilon(x, g(x))$, from (32) we get

$$\begin{aligned} (x^*, -1) &\in \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon f(x) \times \{0\} + \partial_\varepsilon g(x) \times \{-1\}) \\ &\subset \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon f(x) \times \{0\} + N_{\text{epi } g}^\varepsilon(x, g(x))). \end{aligned} \tag{33}$$

We appeal now to [Proposition 4](#). If (i) holds, then it follows from the last inclusion that

$$(x^*, -1) \in \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon f(x) \times \{0\} + N_{\text{epi } g}(x, g(x))). \tag{34}$$

Otherwise, if (ii) holds, we have by [Lemma 6](#)

$$(\text{ri}(\text{epi } g - (x, g(x)))) \cap ((\text{dom } f - x) \times \mathbb{R}) \neq \emptyset,$$

and observing that

$$(\text{dom } f - x) \times \mathbb{R} \subset \text{dom } \sigma_{\partial_\varepsilon f(x)} \times \mathbb{R} = \text{dom } \sigma_{\partial_\varepsilon f(x) \times \{0\}},$$

we conclude

$$(\text{ri}(\text{epi } g - (x, g(x)))) \cap \text{dom } \sigma_{\partial_\varepsilon f(x) \times \{0\}} \neq \emptyset.$$

Therefore, [\(34\)](#) follows by applying once again [Proposition 4](#).

We claim that for all $\varepsilon > 0$ and all $u \in X$

$$\sigma_{\partial(f+g)(x)}(u) \leq \sigma_{\partial_\varepsilon f(x) + \partial g(x)}(u). \tag{35}$$

This will clearly ensure the inclusion ‘‘ \subset ’’ in [\(31\)](#). Fix $\varepsilon > 0$, $u \in X$, and take $\alpha < \sigma_{\partial(f+g)(x)}(u)$. Let $x^* \in \partial(f + g)(x)$ such that $\alpha < \langle x^*, u \rangle$. Then, using [\(34\)](#), we find nets $(y_i^*)_{i \in I} \subset \partial_\varepsilon f(x)$, $(z_i^*)_{i \in I} \subset X^*$, and $(\beta_i)_{i \in I} \subset \mathbb{R}_+$ such that $((z_i^*, -\beta_i))_{i \in I} \subset N_{\text{epi } g}(x, g(x))$, $\beta_i \rightarrow 1$, and

$$x^* = \lim_{i \in I} (y_i^* + z_i^*).$$

Since $\beta_i \rightarrow 1$ we can suppose that $\beta_i > 0$, and then $\beta_i^{-1} z_i^* \in \partial g(x)$. Then, writing

$$\sigma_{\partial_\varepsilon f(x)}(u) + \beta_i \sigma_{\partial g(x)}(u) \geq \langle y_i^*, u \rangle + \beta_i \langle \beta_i^{-1} z_i^*, u \rangle = \langle y_i^* + z_i^*, u \rangle,$$

and taking limits we get

$$\sigma_{\partial_\varepsilon f(x) + \partial g(x)}(u) = \sigma_{\partial_\varepsilon f(x)}(u) + \sigma_{\partial g(x)}(u) \geq \langle x^*, u \rangle > \alpha.$$

Hence, [\(35\)](#) follows when α approaches the value $\sigma_{\partial(f+g)(x)}(u)$. \square

Remark 1. The only purpose of using assumption

$$\overline{f + g} = \bar{f} + g \tag{36}$$

in Theorem 12 (coupled with one of conditions (i) and (ii)) is to guarantee the validity of formula (5) (see (32) in the proof of that theorem). Hence, according to Proposition 2, Theorem 12 is also valid if instead of (36) we use condition (6) or any one of the conditions in Proposition 10.

However, conditions (36) and (6) are not comparable in general; namely, on one hand, for the case $f = g$, condition (6) reads $2\bar{f} = \bar{f} + \bar{f}$, which is obviously always true, while the validity of (36) requires the lsc of f . On the other hand, (36) and (6) are equivalent whenever function g is lsc. Of course, our preference for condition (36) is justified by Lemma 11(ii), and also by its simple and asymmetric appearance, which does not require the calculation of the lsc hull of the qualified function g .

Following the discussion made in the introduction about the use of the relative interior in the infinite-dimensional setting, we give the following corollary in which the resulting formula of the subdifferential of the sum is given by means of the exact subdifferential of the qualified function.

We shall consider the functions

$$g_L := g + I_L, L \subset X,$$

and denote, for $x \in X$ such that $g(x) \in \mathbb{R}$,

$$\mathcal{F}(x) := \left\{ \begin{array}{l} x \in L = \text{aff}(L), \dim(L) < \infty, \text{ and} \\ \text{either } \text{dom } f \cap \text{ri}(L \cap \text{dom } g) \neq \emptyset \text{ or } \mathbb{R}_+(\text{epi } g_L - (x, g(x))) \text{ is closed} \end{array} \right\}. \tag{37}$$

Observe that $\mathcal{F}(x)$ always contains the set $\{x\}$. If $g(x) \notin \mathbb{R}$ we put $\mathcal{F}(x) := \emptyset$.

Corollary 13. *Let f and g be two lsc convex functions. Then for every $x \in X$ we have that*

$$\partial(f + g)(x) = \bigcap_{\substack{L \in \mathcal{F}(x) \\ \varepsilon > 0}} \text{cl}(\partial_\varepsilon f(x) + \partial g_L(x)),$$

with the convention that the right-hand side is empty when $\mathcal{F}(x) = \emptyset$. Consequently, we also have that

$$\partial(f + g)(x) = \bigcap_{\substack{L \in \mathcal{F}'(x) \\ \varepsilon > 0}} \text{cl}(\partial_\varepsilon f(x) + \partial g_L(x)),$$

where $\mathcal{F}'(x) := \{L \subset X \mid x \in L = \text{aff}(L), \dim(L) < \infty, \text{ and } \text{dom } f \cap \text{ri}(\text{dom } g_L) \neq \emptyset\}$

Proof. Fix x in $\text{dom } f$ such that $g(x) \in \mathbb{R}$. If $L \in \mathcal{F}(x)$ is given we easily verify that

$$\partial(f + g)(x) \subset \partial(f + g_L)(x).$$

But we have, since the lsc convex functions f and g_L satisfy the hypotheses of [Theorem 12](#),

$$\partial(f + g_L)(x) = \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon f(x) + \partial g_L(x)),$$

and so, by intersecting over L in $\mathcal{F}(x) \supset \mathcal{F}'(x)$,

$$\partial(f + g)(x) \subset \bigcap_{\substack{L \in \mathcal{F}(x) \\ \varepsilon > 0}} \partial(f + g_L)(x) \subset \bigcap_{\substack{L \in \mathcal{F}'(x) \\ \varepsilon > 0}} \partial(f + g_L)(x).$$

To establish the opposite inclusion, we pick x^* in the right-hand side. If $y \in \{\text{dom } f \cap \text{dom } g\} \setminus \{x\}$ is given, we consider the set $L := x + \mathbb{R}(y - x)$. Observe that the segment $]x, y[$ is included in $\text{dom } f \cap \text{ri}(L \cap \text{dom } g)$. Hence, $L \in \mathcal{F}'(x)$ and we have that $x^* \in \partial(f + g_L)(x)$; that is,

$$\langle x^*, y - x \rangle \leq (f + g_L)(y) - (f + g_L)(x) = (f + g)(y) - (f + g)(x).$$

This inequality being true when $y = x$ or y lies outside the set $\{\text{dom } f \cap \text{dom } g\} \setminus \{x\}$, we conclude that $x^* \in \partial(f + g)(x)$. \square

The following corollary is a special case of [Theorem 12](#).

Corollary 14. *Let f be a convex function and g be a polyhedral function both defined on X . If $\overline{f + g} = \bar{f} + g$, then for every $x \in X$*

$$\partial(f + g)(x) = \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon f(x) + \partial g(x)).$$

Proof. Due to [Lemma 8](#) the cone $\mathbb{R}_+(\text{epi } g - (x, g(x)))$ is closed for each $x \in \text{dom } f \cap \text{dom } g$. Thus, the conclusion follows by [Theorem 12\(i\)](#). \square

We give now another result, which constitutes an infinite-dimensional extension of [Theorem 23.8](#) in [\[23\]](#), and yields a *quasi-exact rule* for convex functions whose epigraphs overlap sufficiently.

Theorem 15. *Let f and g be two convex functions defined on X . Given $x \in X$ such that $f(x), g(x) \in \mathbb{R}$, we assume that one of the following conditions holds:*

- (i) $\mathbb{R}_+(\text{epi } f - (x, f(x)))$ is closed, $\text{dom } f \cap \text{ri}(\text{dom } g) \neq \emptyset$, and $g|_{\text{aff}(\text{dom } g)}$ is continuous on $\text{ri}(\text{dom } g)$,

(ii) $\mathbb{R}_+(\text{epi } f - (x, f(x)))$ and $\mathbb{R}_+(\text{epi } g - (x, g(x)))$ are closed,

(iii) $\text{ri}(\text{dom } f) \cap \text{ri}(\text{dom } g) \neq \emptyset$ and $f|_{\text{aff}(\text{dom } f)}$ and $g|_{\text{aff}(\text{dom } g)}$ are continuous on $\text{ri}(\text{dom } f)$ and $\text{ri}(\text{dom } g)$, respectively.

Then we have that

$$\partial(f + g)(x) = \text{cl}(\partial f(x) + \partial g(x)). \tag{38}$$

Proof. We shall prove that

$$\partial(f + g)(x) \subset \text{cl}(\partial f(x) + \partial g(x)), \tag{39}$$

since the converse inclusion in (38) is straightforward. We fix $x \in X$ such that $\partial(f+g)(x) \neq \emptyset$; hence, $x \in \text{dom } f \cap \text{dom } g$ and the function $\overline{f+g}$ is proper. Moreover, due to Proposition 10, each one of the conditions (i) and (ii) imply that f, g, \bar{f}, \bar{g} are proper. This is also the case under condition (iii). Indeed, take $x_0 \in \text{ri}(\text{dom } f) \cap \text{ri}(\text{dom } g)$. If, for instance, f is not proper, then $f(u) = -\infty$ for $u \in \text{ri}(\text{dom } f)$, whence $(f+g)(x_0) = -\infty$, and so $\overline{(f+g)}(x_0) = -\infty$. It follows that $\overline{(f+g)}(u) = -\infty$ for $u \in \text{dom}(\overline{f+g}) \supset \text{dom}(f+g)$. In particular, $(f+g)(x) = -\infty$, a contradiction. Consequently, f and g , and equivalently, \bar{f} and \bar{g} , are proper. On another hand, Proposition 10 yields the formula

$$\partial(f + g)(x) = \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon f(x) + \partial_\varepsilon g(x)). \tag{40}$$

We pick x^* in $\partial(f+g)(x)$. Then, using the relationship $\partial_\varepsilon f(x) \times \{-1\} \subset N_{\text{epi } f}^\varepsilon(x, f(x))$, from (40) we get

$$(x^*, -1) \in \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon g(x) \times \{0\} + N_{\text{epi } f}^\varepsilon(x, f(x))).$$

Hence, by Proposition 4, each one of the conditions (i) and (ii) implies that

$$(x^*, -1) \in \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon g(x) \times \{0\} + N_{\text{epi } f}(x, g(x))). \tag{41}$$

While, if (iii) holds, Lemma 6 guaranties that

$$(\text{ri}(\text{epi } f - (f(x)))) \cap ((\text{dom } g - x) \times \mathbb{R}) \neq \emptyset;$$

hence, since

$$(\text{dom } g - x) \times \mathbb{R} \subset \text{dom } \sigma_{\partial_\varepsilon g(x)} \times \mathbb{R} = \text{dom } \sigma_{\partial_\varepsilon g(x) \times \{0\}},$$

we conclude that

$$(\text{ri}(\text{epi } f - (x, f(x)))) \cap \text{dom } \sigma_{\partial_\varepsilon g(x) \times \{0\}} \neq \emptyset.$$

Therefore, thanks Proposition 4, formula (41) also holds under condition (iii). Now, as in the proof of Theorem 12 (when verifying (35)) we show that (41) yields the following inequality, for all $\varepsilon > 0$ and all $u \in X$,

$$\sigma_{\partial(f+g)(x)}(u) \leq \sigma_{\partial f(x)+\partial_\varepsilon g(x)}(u), \tag{42}$$

which in turn leads us to

$$x^* \in \partial(f + g)(x) = \bigcap_{\varepsilon > 0} \text{cl}(\partial f(x) + \partial_\varepsilon g(x)).$$

In order to remove ε from this expression, we proceed as above and write

$$(x^*, -1) \in \bigcap_{\varepsilon > 0} \text{cl}(\partial f(x) \times \{0\} + N_{\text{epi } g}^\varepsilon(x, g(x))).$$

Then, on the one hand, under condition (ii) Proposition 4 ensures that

$$(x^*, -1) \in \bigcap_{\varepsilon > 0} \text{cl}(\partial f(x) \times \{0\} + N_{\text{epi } g}(x, g(x))). \tag{43}$$

On the other hand, due to the relation

$$(\text{dom } f - x) \times \mathbb{R} \subset \text{dom } \sigma_{\partial f(x)} \times \mathbb{R} = \text{dom } \sigma_{\partial f(x) \times \{0\}},$$

according to Lemma 6 each one of the conditions (i) and (iii) leads us to

$$(\text{ri}(\text{epi } g - (x, g(x)))) \cap \text{dom } \sigma_{\partial f(x) \times \{0\}} \neq \emptyset.$$

Thus, (43) follows by appealing to Proposition 4. In this way, we have proved that each one of the conditions (i), (ii), and (iii) yields that

$$(x^*, -1) \in \text{cl}(\partial f(x) \times \{0\} + N_{\text{epi } g}(x, g(x))). \tag{44}$$

By repeating once again the proof of Theorem 12 (the part concerning (35), as we did with (42) above) we verify that, for every $\varepsilon > 0$ and $u \in X$,

$$\sigma_{\partial(f+g)(x)}(u) \leq \sigma_{\partial f(x)+\partial g(x)}(u). \tag{45}$$

Indeed, take $\alpha < \sigma_{\partial(f+g)(x)}(u)$ and choose $x^* \in \partial(f + g)(x)$ such that $\alpha < \langle x^*, u \rangle$. Then, according to (44), we find nets $(y_i^*)_{i \in I} \subset \partial f(x)$, $(z_i^*)_{i \in I} \subset X^*$, and $(\beta_i)_{i \in I} \subset \mathbb{R}_+$ such that $\beta_i^{-1} z_i^* \in \partial g(x)$, $\beta_i \rightarrow 1$, with $\beta_i > 0$, and

$$x^* = \lim_{i \in I} (y_i^* + z_i^*).$$

Then, taking the limit in the inequalities

$$\sigma_{\partial f(x)}(u) + \beta_i \sigma_{\partial g(x)}(u) \geq \langle y_i^*, u \rangle + \beta_i \langle \beta_i^{-1} z_i^*, u \rangle = \langle y_i^* + z_i^*, u \rangle,$$

we get

$$\sigma_{\partial f(x) + \partial g(x)}(u) \geq \langle x^*, u \rangle > \alpha.$$

Hence, (45) follows when α goes to $\sigma_{\partial(f+g)(x)}(u)$ and the inclusion $x^* \in \partial(f + g)(x) \subset \text{cl}(\partial f(x) + \partial g(x))$ follows. \square

The following result is immediate from Theorem 15.

Corollary 16. *Let C and D be two convex subsets of X such that*

$$\text{ri}(\text{dom } C) \cap \text{ri}(\text{dom } D) \neq \emptyset.$$

Then for every $x \in C \cap D$ we have that

$$N_{C \cap D}(x) = \text{cl}(N_C(x) + N_D(x)).$$

The previous results are applied in the following corollary to get calculus rules for the subdifferential of the composition of a convex function with a linear mapping. This result is also needed in Corollary 23 below.

Corollary 17. *Let us consider a continuous linear mapping $A : X \rightarrow Y$ with continuous adjoint A^* , where Y is another (real) separated locally convex space, and a convex function f defined on X . Given $x \in X$ such that $f(Ax) \in \mathbb{R}$, we assume*

- either (i) $\mathbb{R}_+(\text{epi } f - (Ax, f(Ax)))$ is closed,*
- or (ii) $A(X) \cap \text{ri}(\text{dom } f) \neq \emptyset$ and $f|_{\text{aff}(\text{dom } f)}$ is continuous relative to $\text{ri}(\text{dom } f)$.*

Then,

$$\partial(f \circ A)(x) = \text{cl}(A^* \partial f(Ax)).$$

Proof. We consider the convex functions $g, h : X \times Y \rightarrow \overline{\mathbb{R}}$ defined as

$$g(x, y) := f(y) \text{ and } h(x, y) := I_{\text{gph } A},$$

where $\text{gph } A$ denotes the graph of A . Then, on the one hand, by writing

$$\mathbb{R}_+(\text{epi } h - (x, Ax, 0)) = \mathbb{R}_+(\text{gph } A \times \mathbb{R}_+ - (x, Ax, 0)) = \text{gph } A \times \mathbb{R}_+,$$

it follows that $\mathbb{R}_+(\text{epi } h - (x, Ax, 0))$ is a closed set. On the other hand, we have that

$$\begin{aligned} \mathbb{R}_+(\text{epi } g - (x, Ax, f(Ax))) &= \mathbb{R}_+(X \times \text{epi } f - (x, Ax, f(Ax))) \\ &= X \times \mathbb{R}_+(\text{epi } f - (Ax, f(Ax))). \end{aligned}$$

Therefore, if (i) holds, then $\mathbb{R}_+(\text{epi } g - (x, Ax, f(Ax)))$ is closed and, so, by [Theorem 15\(i\)](#) we get

$$\partial(g + h)(x, Ax) = \text{cl}(\partial g(x, Ax) + \partial \mathbf{I}_{\text{gph } A}(x, Ax)) = \text{cl}(\{\theta\} \times \partial f(Ax) + \mathbf{N}_{\text{gph } A}(x, Ax))$$

and then

$$\begin{aligned} \partial(f \circ A)(x) \times \{\theta\} &= \partial(g + h)(x, Ax) \cap (X^* \times \{\theta\}) \\ &= \text{cl}\{(A^*v^*, y^* - v^*) \mid y^* \in \partial f(Ax), v^* \in Y^*\} \cap (X^* \times \{\theta\}). \end{aligned} \tag{46}$$

In other words, if x^* is given in $\partial(f \circ A)(x)$, then we find nets $(y_i^*) \subset \partial f(Ax)$ and $(v_i^*) \subset Y^*$ such that $x^* = \lim_i A^*v_i^*$ and $\lim_i (y_i^* - v_i^*) = \theta$. Since A^* is continuous, we deduce that $x^* = \lim_i A^*v_i^* = \lim_i A^*y_i^* \in \text{cl}(A^*\partial f(Ax))$. The opposite inclusion also easily follows from (46).

If (ii) holds, then it easily follows that $\text{dom } h \cap \text{ri}(\text{dom } g) \neq \emptyset$ and $g|_{\text{aff}(\text{dom } g)}$ is continuous relative to $\text{ri}(\text{dom } g)$. Thus, we conclude as in (i) above. \square

The following result is a continuation of [Corollary 13](#) in which we use the following notation, where $x \in X$ is such that $f(x), g(x) \in \mathbb{R}$,

$$\mathcal{G}(x) := \left\{ L \subset X \left| \begin{array}{l} L \text{ is a finite-dimensional affine manifold, } x \in L, \text{ and} \\ \text{either } \text{ri}(\text{dom } f_L) \cap \text{ri}(\text{dom } g_L) \neq \emptyset \\ \text{or } \text{dom } f_L \cap \text{ri}(\text{dom } g_L) \neq \emptyset \text{ and } \mathbb{R}_+(\text{epi } f_L - (x, f(x))) \text{ is closed} \\ \text{or } \mathbb{R}_+(\text{epi } f_L - (x, f(x))) \text{ and } \mathbb{R}_+(\text{epi } g_L - (x, g(x))) \text{ are closed} \end{array} \right. \right\} \tag{47}$$

(recall that $f_L = f + \mathbf{I}_L$ and $g_L = g + \mathbf{I}_L$). Observe that $\{x\} \in \mathcal{G}(x)$.

Corollary 18. *Let f and g be two proper convex functions both defined on X . Then for every $x \in X$ such that $f(x), g(x) \in \mathbb{R}$ we have*

$$\partial(f + g)(x) = \bigcap_{L \in \mathcal{G}(x)} \text{cl}(\partial f_L(x) + \partial g_L(x)).$$

Proof. According to [Theorem 15](#), the conclusion follows in view of the relationship

$$\partial(f + g)(x) = \bigcap_{L \in \mathcal{G}(x)} \partial(f_L + g_L)(x). \quad \square$$

Taking $\mathcal{G}'(x) := \{L \subset X \mid x \in L = \text{aff } L, \dim L < \infty, \text{ri}(\text{dom } f_L) \cap \text{ri}(\text{dom } g_L) \neq \emptyset\}$, the conclusion of [Corollary 18](#) is valid replacing $\mathcal{G}(x)$ by $\mathcal{G}'(x)$; in fact, it is true even with replacing $\mathcal{G}(x)$ by $\mathcal{G}''(x) := \{L \in \mathcal{G}'(x) \mid \dim L = 1\}$.

The closure in formula [\(38\)](#) of [Theorem 15](#) is removed in the next two corollaries.

Corollary 19. *Let functions f and g be the same as in [Theorem 15\(iii\)](#). If $x \in X$ is such that $\partial f(x)$ or $\partial g(x)$ is locally compact, then*

$$\partial(f + g)(x) = \partial f(x) + \partial g(x).$$

Proof. We may assume that $f(x), g(x) \in \mathbb{R}$. According to [Theorem 15\(iii\)](#), we have that $\partial(f + g)(x) = \text{cl}(\partial f(x) + \partial g(x))$. If $\partial f(x)$ or $\partial g(x)$ is empty, then we are done. Consequently, $\partial f(x)$ and $\partial g(x)$ as well as $\partial(f + g)(x)$ are all nonempty. Hence, by [Lemma 1](#) functions f and g are proper, and we have that $\bar{f}, \bar{g} \in \Gamma_0(X)$. Next (without loss of generality) we assume that $\theta \in \text{ri}(\text{dom } f) \cap \text{ri}(\text{dom } g)$ and let $U \subset X$ be a balanced and convex θ -neighborhood such that

$$U_1 := U \cap \text{aff}(\text{dom } f) \subset \text{dom } f, \quad U_2 := U \cap \text{aff}(\text{dom } g) \subset \text{dom } g.$$

We have that U_1 and U_2 are balanced and convex in $\text{aff}(\text{dom } f)$ and $\text{aff}(\text{dom } g)$, respectively. Take $v^* \in (\partial f(x))_\infty \cap (-\partial g(x))_\infty$. Thanks to [\[28, Theorem 2.4.1\(ii\) and Exercise 2.23\]](#) we can write

$$\begin{aligned} v^* \in (\partial \bar{f}(x))_\infty \cap (-\partial \bar{g}(x))_\infty &= N_{\text{dom } \bar{f}}(x) \cap (-N_{\text{dom } \bar{g}}(x)) \\ &= N_{\text{dom } f}(x) \cap (-N_{\text{dom } g}(x)). \end{aligned}$$

Then, for every $u_1 \in U_1$ and $u_2 \in U_2$ we obtain that

$$\langle v^*, u_1 - x \rangle \leq 0 \text{ and } \langle -v^*, u_2 - x \rangle \leq 0, \tag{48}$$

which by summing up gives us $\langle v^*, u_1 - u_2 \rangle \leq 0$. Moreover, since $-u_1 \in U_1$ and $-u_2 \in U_2$, we also have that

$$\langle v^*, u_2 - u_1 \rangle \leq 0 \text{ for all } u_1 \in U_1 \text{ and } u_2 \in U_2,$$

which shows that $v^* \in (\text{aff}(\text{dom } f - \text{dom } g))^\circ$ (U_1 and U_2 are absorbing in $\text{aff}(\text{dom } f)$ and $\text{aff}(\text{dom } g)$, respectively).

Thus, since the inclusion $(\text{aff}(\text{dom } f - \text{dom } g))^\circ \subset N_{\text{dom } f}(x) \cap (-N_{\text{dom } g}(x))$ always holds, we deduce that $N_{\text{dom } f}(x) \cap (-N_{\text{dom } g}(x)) = (\text{aff}(\text{dom } f - \text{dom } g))^\circ$. Therefore, the closedness of $\partial f(x) + \partial g(x)$ follows from Dieudonné’s [Theorem \[28, Theorem 1.1.8\]](#). \square

The following result is well-known in the finite-dimensional setting (e.g. [\[23, Theorem 23.8\]](#)). The infinite-dimensional case can also be reduced to the finite-dimensional setting, but here we give a proof based on [Theorem 15\(ii\)](#).

Corollary 20. *Let f and g be two polyhedral functions defined on X . Then for every $x \in X$ we have that*

$$\partial(f + g)(x) = \partial f(x) + \partial g(x).$$

Proof. We fix $x \in X$ such that $\partial(f + g)(x) \neq \emptyset$; hence, $f, g \in \Gamma_0(X)$ (by Lemma 1). Since the sets $\mathbb{R}_+(\text{epi } f - (x, f(x)))$ and $\mathbb{R}_+(\text{epi } g - (x, g(x)))$ are closed (see Lemma 8), according to Theorem 15(ii) we have that

$$\partial(f + g)(x) = \text{cl}(\partial f(x) + \partial g(x)). \tag{49}$$

Now, it can be shown (e.g. [6, Proposition 3.1.1]) that f (or similarly g) admits the following representation

$$f(x) = \max_{i=1, \dots, k} \{ \langle a_i, x \rangle - b_i \} + \mathbb{I}_{\{z \in X \mid \langle a_i, z \rangle \leq b_i, i=k+1, \dots, m\}}(x),$$

with $a_1, \dots, a_k, a_{k+1}, \dots, a_m \in X^*$ and $b_1, \dots, b_k, b_{k+1}, \dots, b_m \in \mathbb{R}$. Hence, using [28, Theorem 2.8.7(iii)] together with Valadier’s formula (e.g. [28, p. 136]), for every $x \in \text{dom } f$ the subdifferential of f at x is nonempty and is characterized by

$$\partial f(x) = \text{co}\{a_j \mid j \in I(x)\} + \mathbb{R}_+\{a_j \mid j \in J(x)\},$$

where $I(x) := \{j \in \{1, \dots, k\} \mid \langle a_j, x \rangle - b_j = \max_{i=1, \dots, k} \{ \langle a_i, x \rangle - b_i \} \}$ and $J(x) := \{j \in \{k + 1, \dots, m\} \mid \langle a_j, x \rangle = b_j\}$. This shows, again by Dieudonné’s Theorem, that the closure is superfluous in (49). \square

Below, Theorem 12 is extended to deal with finitely many functions. We will need the following proposition.

Proposition 21. *Let f and g be two proper convex functions defined on X . Suppose that $\text{ri}(\text{dom } f) \cap \text{ri}(\text{dom } g) \neq \emptyset$ and that the functions $f|_{\text{aff}(\text{dom } f)}$ and $g|_{\text{aff}(\text{dom } g)}$ are continuous on $\text{ri}(\text{dom } f)$ and $\text{ri}(\text{dom } g)$, respectively. Then*

$$\text{ri}(\text{dom}(f + g)) = \text{ri}(\text{dom } f) \cap \text{ri}(\text{dom } g) \tag{50}$$

and

$$(f + g)|_{\text{aff}(\text{dom}(f+g))} \text{ is continuous on } \text{ri}(\text{dom}(f + g)).$$

Proof. Let us first prove that given two convex sets A and B with $\text{ri}(A) \cap \text{ri}(B) \neq \emptyset$, we have that $\text{ri}(A) \cap \text{ri}(B) = \text{ri}(A \cap B)$. If $\theta \in \text{ri}(A) \cap \text{ri}(B)$, then $\text{aff } A = \text{span } A$ is closed and $\text{span } A = \mathbb{R}_+ A$; similarly for B . It follows that $\text{span } A \cap \text{span } B = \mathbb{R}_+ A \cap \mathbb{R}_+ B = \mathbb{R}_+(A \cap B) = \text{span}(A \cap B)$. Moreover, there exists a θ -neighborhood U such

that $U \cap \text{span } A \subset A$ and $U \cap \text{span } B \subset B$, whence $U \cap \text{span}(A \cap B) \subset A \cap B$; hence $\theta \in \text{ri}(A \cap B)$. Now if $x \in \text{ri}(A) \cap \text{ri}(B)$, then $\theta \in \text{ri}(A - x) \cap \text{ri}(B - x) = \text{ri}((A \cap B) - x)$ and so $x \in \text{ri}(A \cap B)$.

For the converse inclusion, take $x_0 \in \text{ri}(A) \cap \text{ri}(B)$; as above, we may assume that $x_0 = \theta$. Take $x \in \text{ri}(A \cap B)$. Then there exists $\alpha > 1$ such that $\alpha x \in A \cap B$. Since $\theta \in \text{ri}(A) \cap \text{ri}(B)$, $x = \alpha^{-1}(\alpha x) + (1 - \alpha^{-1})\theta \in \text{ri}(A) \cap \text{ri}(B)$. Hence $\text{ri}(A \cap B) = \text{ri}(A) \cap \text{ri}(B)$.

Now we go back to the proof of the proposition. Taking $A := \text{dom } f$ and $B := \text{dom } g$ we get (50). Moreover, since $f|_{\text{aff}(A)}$ is continuous on $\text{ri}(A)$ and $\text{aff}(A \cap B) = \text{aff}(A) \cap \text{aff}(B) \subset \text{aff}(A)$, it follows that $f|_{\text{aff}(A \cap B)}$ is continuous on $\text{ri}(A) \cap \text{aff}(A \cap B)$. Similarly, $g|_{\text{aff}(A \cap B)}$ is continuous on $\text{ri}(B) \cap \text{aff}(A \cap B)$, whence $(f + g)|_{\text{aff}(A \cap B)} = f|_{\text{aff}(A \cap B)} + g|_{\text{aff}(A \cap B)}$ is continuous on $\text{ri}(A) \cap \text{ri}(B) \cap \text{aff}(A \cap B) = \text{ri}(A \cap B) \cap \text{aff}(A \cap B) = \text{ri}(A \cap B)$. \square

Theorem 22. *Let $f_1, \dots, f_k, g_1, \dots, g_m : X \rightarrow \mathbb{R} \cup \{+\infty\}$, $k, m \geq 1$, be convex functions satisfying*

$$\overline{f_1 + \dots + f_k + g_1 + \dots + g_m} = \bar{f}_1 + \dots + \bar{f}_k + \bar{g}_1 + \dots + \bar{g}_m. \tag{51}$$

Given $x \in X$ where functions g_i are finite, we assume

either (i) $\mathbb{R}_+(\text{epi } g_i - (x, g_i(x)))$ is closed for $i = 1, \dots, m$,

or (ii) $\left(\bigcap_{i=1}^k \text{dom } f_i\right) \cap \left(\bigcap_{i=1}^m \text{ri}(\text{dom } g_i)\right) \neq \emptyset$ and $g_i|_{\text{aff}(\text{dom } g_i)}$ is continuous on $\text{ri}(\text{dom } g_i)$, for $i = 1, \dots, m$.

Then,

$$\begin{aligned} &\partial(f_1 + \dots + f_k + g_1 + \dots + g_m)(x) \\ &= \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon f_1(x) + \dots + \partial_\varepsilon f_k(x) + \partial g_1(x) + \dots + \partial g_m(x)). \end{aligned} \tag{52}$$

Proof. We use an inductive argument on the number of functions by considering the following relation:

(\mathcal{R}_n) : Every family of convex functions $f_1, \dots, f_k, g_1, \dots, g_m$, $k, m \geq 1, k + m \leq n$, satisfying (51) and either (i) or (ii), verifies (52) when

$$\partial(f_1 + \dots + f_k + g_1 + \dots + g_m)(x) \neq \emptyset.$$

Observe that (52) holds trivially when $\partial(f_1 + \dots + f_k + g_1 + \dots + g_m)(x) = \emptyset$ since that the inclusion “ \supset ” is always true.

On the other hand, following the same argument as in Lemma 1, we get that all the functions $f_1, \dots, f_k, g_1, \dots, g_m$ must be lsc at x , and that their closures $\bar{f}_1, \dots, \bar{f}_k, \bar{g}_1, \dots, \bar{g}_m$, must be proper. From the assumption on the closures we also have

$$\overline{\varphi + \psi} = \bar{\varphi} + \bar{\psi}, \tag{53}$$

for any fixed pair of functions $\{\varphi, \psi\}$ within the family $f_1, \dots, f_k, g_1, \dots, g_m$, and as a consequence of that, if, for instance, $\varphi = f_1$ and $\psi = f_k$, we deduce that

$$\overline{\varphi + \psi + f_2 + \dots + f_{k-1} + g_1 + \dots + g_m} = \overline{\varphi + \psi} + \overline{f_2 + \dots + f_{k-1}} + \overline{g_1 + \dots + g_m}, \tag{54}$$

and similarly when $\varphi = f_i, \psi = g_j$, or $\varphi = g_i, \psi = g_j$. Moreover, by (18) and the discussion following (18), for every $\varepsilon > 0$ we have that

$$\partial_\varepsilon(f + g)(x) \subset \text{cl}(\partial_{2\varepsilon}f(x) + \partial_{2\varepsilon}g(x)). \tag{55}$$

Now relation (\mathcal{R}_2) is true due to Theorems 12 and 15. Assume that (\mathcal{R}_n) is true and consider a family of functions $f_1, \dots, f_k, g_1, \dots, g_m, h$, with $k, m \geq 1$ and $k + m \leq n$, satisfying (51) and either (i) or (ii).

Let assume first that (i) holds. If the function h is such that $\mathbb{R}_+(\text{epi } h - (x, h(x)))$ is closed, then by Lemma 9 the set $\mathbb{R}_+(\text{epi}(g_m + h) - (x, g_m(x) + h(x)))$ is also closed. So, since (54) holds, by applying the induction argument to the family of functions $f_1, \dots, f_k, g_1, \dots, g_{m-1}, g_m + h$ we infer that

$$\partial \left(\sum_{i=1}^k f_i + \sum_{i=1}^m g_i + h \right) (x) = \bigcap_{\varepsilon > 0} \text{cl} \left(\sum_{i=1}^k \partial_\varepsilon f_i(x) + \sum_{i=1}^{m-1} \partial g_i(x) + \partial(g_m + h)(x) \right).$$

Hence, since $\partial(g_m + h)(x) = \text{cl}(\partial g_m(x) + \partial h(x))$, by Theorem 15, we infer that

$$\partial \left(\sum_{i=1}^k f_i + \sum_{i=1}^m g_i + h \right) (x) = \bigcap_{\varepsilon > 0} \text{cl} \left(\sum_{i=1}^k \partial_\varepsilon f_i(x) + \sum_{i=1}^m \partial g_i(x) + \partial h(x) \right). \tag{56}$$

Now, if $\mathbb{R}_+(\text{epi } h - (x, h(x)))$ is not closed, so that h is of type f , we apply the induction argument to the family $f_1, \dots, f_{k-1}, f_k + h, g_1, \dots, g_m$ to get

$$\partial \left((f_k + h) + \sum_{i=1}^{k-1} f_i + \sum_{i=1}^m g_i \right) (x) = \bigcap_{\varepsilon > 0} \text{cl} \left(\partial_\varepsilon(f_k + h)(x) + \sum_{i=1}^{k-1} \partial_\varepsilon f_i(x) + \sum_{i=1}^m \partial g_i(x) \right),$$

and, so, by (55) it follows that

$$\partial \left((f_k + h) + \sum_{i=1}^{k-1} f_i + \sum_{i=1}^m g_i \right) (x) = \bigcap_{\varepsilon > 0} \text{cl} \left(\partial_\varepsilon h(x) + \sum_{i=1}^k \partial_\varepsilon f_i(x) + \sum_{i=1}^m \partial g_i(x) \right). \tag{57}$$

Now we consider the case in which the family of functions $f_1, \dots, f_k, g_1, \dots, g_{m-1}, g_m, h$ satisfies condition (ii). If the function h is of type f ; that is, the current assumption relies merely on its domain, then we conclude as in (57). Otherwise, if h is of type g ; that is, the current assumption relies on the relative interior of its domain, by Proposition 21 we deduce that $\text{ri}(\text{dom}(g_m + h)) = \text{ri}(\text{dom } g_m) \cap \text{ri}(\text{dom } h) \neq \emptyset$ and

$(g_m + h)|_{\text{aff}(\text{dom } g_m \cap \text{dom } h)}$ is continuous on $\text{ri}(\text{dom } g_m \cap \text{dom } h)$.

Hence, by applying the induction argument to the family $f_1, \dots, f_k, g_1, \dots, g_{m-1}, g_m + h$ we get (using (54))

$$\partial \left(\sum_{i=1}^k f_i + \sum_{i=1}^m g_i + h \right) (x) = \bigcap_{\varepsilon > 0} \text{cl} \left(\sum_{i=1}^k \partial_\varepsilon f_i(x) + \sum_{i=1}^{m-1} \partial g_i(x) + \partial(g_m + h)(x) \right).$$

But, according to Theorem 15 we have that $\partial(g_m + h) = \text{cl}(\partial g_m(x) + \partial h(x))$, so that the last equality leads us to (56), and this shows that (\mathcal{R}_{n+1}) holds true. \square

The following corollary deals with a typical situation where finite and infinite settings are put together. We consider a continuous linear mapping $A : X \rightarrow \mathbb{R}^n$ with continuous adjoint A^* , and two convex functions f, g defined on \mathbb{R}^n and X , respectively. The Rockafellar’s usual condition [23, Theorem 23.8] for the subdifferential of the function $g + f \circ A$, adapted to the current setting in [5, Theorem 4.2], reads

$$\text{ri}(A \text{ dom } g) \cap \text{ri}(\text{dom } f) \neq \emptyset.$$

This condition has been translated in terms of the so-called quasi-relative interior as [5, Corollary 4.3]:

$$A(\text{qri}(\text{dom } g)) \cap \text{ri}(\text{dom } f) \neq \emptyset;$$

observe that $\text{ri}(A \text{ dom } g) = A(\text{qri}(\text{dom } g))$ provided that $\text{qri}(\text{dom } g)$ is nonempty. However, $\text{qri}(\text{dom } g)$ is always nonempty [5, Theorem 2.19] when the underlying space X is a separable Fréchet space (in particular, a separable Banach space) and C is cs-closed (see, also, [28, Proposition 1.2.9]).

The following result only requires the condition $A(\text{dom } g) \cap \text{ri}(\text{dom } f) \neq \emptyset$, and so it makes sense for any space X . The resulting formula uses the exact subdifferential of the qualified function g .

Corollary 23. *Let us consider a continuous linear mapping $A : X \rightarrow \mathbb{R}^n$, with continuous adjoint A^* , and two lsc convex functions f and g defined on \mathbb{R}^n and X , respectively. Assume that, for $x \in \text{dom } g$ such that $f(Ax) \in \mathbb{R}$,*

- either (i) $\mathbb{R}_+(\text{epi } f - (Ax, f(Ax)))$ is closed*
- or (ii) $A(\text{dom } g) \cap \text{ri}(\text{dom } f) \neq \emptyset$.*

Then,

$$\partial(g + f \circ A)(x) = \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon g(x) + A^* \partial f(Ax)).$$

In addition, if $\mathbb{R}_+(\text{epi } g - (x, g(x)))$ is closed, then the last formula reduces to

$$\partial(g + f \circ A)(x) = \text{cl}(\partial g(x) + A^* (\partial f(Ax))).$$

Proof. We suppose that $\partial(g + f \circ A)(x) \neq \emptyset$ so that, according to [Corollary 17](#),

$$\partial(f \circ A)(x) = \text{cl}(A^* \partial f(Ax)). \tag{58}$$

Hence, we only need to check that $f \circ A$ and g satisfy the conditions of [Theorem 12](#), with $f \circ A$ being the qualified function. First we assume that (i) holds, and take nets $(\alpha_i)_{i \in I} \subset \mathbb{R}_+$ and $(x_i, \lambda_i)_{i \in I} \subset \text{epi } f \circ A$ such that

$$\alpha_i((x_i, \lambda_i) - (x, f(Ax))) \rightarrow (u, \mu),$$

for some $(u, \mu) \in X \times \mathbb{R}$; hence, $\alpha_i((Ax_i, \lambda_i) - (Ax, f(Ax))) \rightarrow (Au, \mu)$ so that $(Au, \mu) \in \mathbb{R}_+(\text{epi } f - (Ax, f(Ax)))$, due to (i). This shows that the function $f \circ A$ satisfies condition (i) of [Theorem 12](#).

If (ii) holds we choose $x_0 \in \text{dom } g$ such that $y_0 = Ax_0 \in \text{ri}(\text{dom } f)$. Hence, there are some constant $m \geq 0$ and an θ -neighborhood $V \subset \mathbb{R}^n$ such that

$$f(y_0 + y) \leq m \text{ for all } y \in V \cap \text{aff}(\text{dom } f).$$

Let $U \subset X$ be a θ -neighborhood such that $AU \subset V$. Then, for every $z \in U \cap \text{aff}(\text{dom } f \circ A)$ it holds that

$$f \circ A(x_0 + z) = f(y_0 + Az) \leq m,$$

which implies that $f \circ A$ satisfies condition (ii) of [Theorem 12](#). Consequently, taking into account [\(58\)](#),

$$\begin{aligned} \partial(g + f \circ A)(x) &= \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon g(x) + \partial(f \circ A)(x)) \\ &= \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon g(x) + \text{cl}(A^* \partial f(Ax))) \\ &= \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon g(x) + A^* \partial f(Ax)), \end{aligned}$$

yielding the main conclusion. If, in addition, the set $\mathbb{R}_+(\text{epi } g - (x, g(x)))$ is closed, then we conclude by using [Theorem 15](#) in a similar way. \square

The following result establishes a rule with a sequential flavor in the spirit of [\[26\]](#) (see also [\[10\]](#) for other results). Here, we only use the exact subdifferential at the reference point of the qualified function (the one whose relative interior or epigraph is involved in the assumption), meanwhile the subdifferential of the other function is taken at nearby points. The result is stated in Banach spaces because it is the requirement

of the Brøndsted–Rockafeller theorem, which is needed in the proof; the reflexivity assumption comes to justify the use of sequences rather than nets. We denote by

$$\limsup_{z \rightsquigarrow x} (\partial f(z) + \partial g(x))$$

the set of elements $x^* \in X^*$ such that there are sequences $x_n \rightarrow x$ with $f(x_n) \rightarrow f(x)$, $x_n^* \in \partial f(x_n)$, and $y_n^* \in \partial g(x)$ such that $\langle x_n^*, x_n - x \rangle \rightarrow 0$ and

$$x_n^* + y_n^* \rightarrow x^*.$$

Corollary 24. *Assume that X is a reflexive Banach space, and let f, g be two lsc proper convex functions. Given $x \in \text{dom } f \cap \text{dom } g$, we assume*

either (i) $\mathbb{R}_+(\text{epi } g - (x, g(x)))$ is closed,

or (ii) $\text{dom } f \cap \text{ri}(\text{dom } g) \neq \emptyset$ and $g|_{\text{aff}(\text{dom } g)}$ is continuous on $\text{ri}(\text{dom } g)$.

Then,

$$\partial(f + g)(x) = \limsup_{z \rightsquigarrow x} (\partial f(z) + \partial g(x)).$$

Proof. Fix $x \in \text{dom } g \cap \text{dom } f$ and $x^* \in \partial(f + g)(x)$. Then, using Theorem 12, for each integer $n \geq 1$ there are $z_n^* \in \partial_{1/n^2} f(x)$ and $y_n^* \in \partial g(x)$ such that

$$x^* \in z_n^* + y_n^* + (1/n)\mathbb{B}^*,$$

where \mathbb{B}^* represents the closed unit ball for the (dual) norm in X^* . Now, appealing to Brøndsted–Rockafellar’s Theorem (e.g. [22]), we find $x_n \in x + (1/n)\mathbb{B}$ (here, \mathbb{B} is the closed unit ball in X) and $x_n^* \in \partial_{1/n} f(x_n)$ such that

$$|f(x_n) - f(x)| \leq \frac{1}{n}, \quad |\langle x_n^*, x_n - x \rangle| \leq \frac{1}{n}, \quad \text{and } x_n^* \in z_n^* + (1/n)\mathbb{B}^*.$$

Therefore, $x^* \in x_n^* + y_n^* + (2/n)\mathbb{B}^*$ and the conclusion follows. \square

4. Application to optimization

Let us consider next the *convex programming* problem:

$$\begin{aligned} (P) \quad & \text{Min} \quad f(x) \\ & \text{s.t.} \quad f_t(x) \leq 0, \quad t \in T, \\ & \quad \quad x \in C, \end{aligned} \tag{59}$$

where T is an arbitrary infinite index set, C is a nonempty closed convex subset of a (real) separated locally convex space X , and all the involved functions $f, f_t, t \in T$, belong to $\Gamma_0(X)$. We assume that the (convex) constraint system

$$\sigma := \{f_t(x) \leq 0, t \in T; x \in C\}, \tag{60}$$

is consistent; i.e. the *feasible set* of (P), which is denoted by F , is nonempty ($F \neq \emptyset$).

The convex infinite-dimensional version of *Farkas lemma* (see, for instance, [11, Theorem 4.1], extending [17, Theorem 3.2]) establishes that, given $u \in X^*$ and $\alpha \in \mathbb{R}$,

$$\langle u, x \rangle \leq \alpha \text{ is a consequence of } \sigma \text{ if and only if } (u, \alpha) \in \text{cl} K,$$

where

$$\begin{aligned} K &:= \text{cone co} \left\{ \bigcup_{t \in T} \text{epi } f_t^* \cup \text{epi } \sigma_C \right\} \\ &= \text{cone co} \left\{ \bigcup_{t \in T} \text{epi } f_t^* \right\} + \text{epi } \sigma_C, \end{aligned}$$

since $\text{epi } \sigma_C$ is a convex cone. The system σ is said to be *Farkas–Minkowski* if K is closed.

Next we provide optimality conditions for (P). Before, let us introduce some additional notation. We represent by $\mathbb{R}_+^{(T)}$ the cone of all the functions $\lambda : T \rightarrow \mathbb{R}_+$ such that $\lambda_t := \lambda(t)$ is zero except for finitely many t 's.

Theorem 25. *Given problem (P), let us assume that X is reflexive, that σ is Farkas–Minkowski and that $\text{ri}(F) \cap \text{dom } f \neq \emptyset$. Then, $\bar{x} \in F \cap \text{dom } f$ is a global minimizer of (P) if and only if for each fixed $\varepsilon > 0$ we have that, for every $\rho > 0$, there exists $\lambda^\rho \in \mathbb{R}_+^{(T)}$ such that the following condition holds:*

$$\theta \in \partial f_\varepsilon(\bar{x}) + \partial \left(\sum_{\{t \in T | \lambda_t^\rho > 0\}} \lambda_t^\rho f_t + I_C \right) (\bar{x}) + \rho \mathbb{B}^*, \tag{61}$$

where $f_t(\bar{x}) = 0$ for all $t \in T$ such that $\lambda_t^\rho > 0$.

Proof. The point $\bar{x} \in F \cap \text{dom } f$ is a minimizer of (P) if and only if

$$\theta \in \partial(f + I_F)(\bar{x}). \tag{62}$$

The assumption $\text{ri}(F) \cap (\text{dom } f) \neq \emptyset$ and Corollary 5 yield

$$\theta \in \bigcap_{\varepsilon > 0} \text{cl}(\partial f_\varepsilon(\bar{x}) + N_F(\bar{x})); \tag{63}$$

then, for each $\varepsilon > 0$ we have $\theta \in \text{cl}(\partial f_\varepsilon(\bar{x}) + N_F(\bar{x}))$. Since X is reflexive, $\text{cl}(\partial f_\varepsilon(\bar{x}) + N_F(\bar{x}))$ coincides with the closure of $\partial f_\varepsilon(\bar{x}) + N_F(\bar{x})$ for the topology of the (dual) norm in X^* and, so, for every $\rho > 0$,

$$\theta \in \partial f_\varepsilon(\bar{x}) + N_F(\bar{x}) + \rho \mathbb{B}^*.$$

The last relation entails the existence of $u^\rho \in N_F(\bar{x})$ such that $\theta \in \partial f_\varepsilon(\bar{x}) + u^\rho + \rho \mathbb{B}^*$. Moreover $u^\rho \in N_F(\bar{x})$ is equivalent to say that $\langle u^\rho, x \rangle \leq \langle u^\rho, \bar{x} \rangle$ is a consequence of σ . By Farkas lemma, and thanks to the fact that σ is FM, we have

$$(u^\rho, \langle u^\rho, \bar{x} \rangle) \in \text{cl } K = K = \text{cone co} \left\{ \bigcup_{t \in T} \text{epi } f_t^* \right\} + \text{epi } \sigma_C.$$

This yields (see [12, p. 130]) the existence of $\lambda^\rho \in \mathbb{R}_+^{(T)}$ such that

$$u^\rho \in \partial \left(\sum_{\{t \in T | \lambda_t^\rho > 0\}} \lambda_t^\rho f_t + I_C \right) (\bar{x}),$$

with $f_t(\bar{x}) = 0$ for all $t \in T$ such that $\lambda_t^\rho > 0$. The necessity is proved.

Conversely, if (61) holds then there exists $z^\rho \in X^*$ such that $\|z^\rho\|_* = 1$ and

$$\rho z^\rho \in \partial f_\varepsilon(\bar{x}) + \partial \left(\sum_{\{t \in T | \lambda_t^\rho > 0\}} \lambda_t^\rho f_t + I_C \right) (\bar{x}).$$

Thus,

$$f(x) + \sum_{\{t \in T | \lambda_t^\rho > 0\}} \lambda_t^\rho f_t(x) \geq f(\bar{x}) + \sum_{\{t \in T | \lambda_t^\rho > 0\}} \lambda_t^\rho f_t(\bar{x}) + \langle \rho z^\rho, x - \bar{x} \rangle - \varepsilon, \forall x \in C. \tag{64}$$

Since $f_t(\bar{x}) = 0$ for all $t \in T$ such that $\lambda_t^\rho > 0$, (64) implies

$$f(x) + \sum_{\{t \in T | \lambda_t^\rho > 0\}} \lambda_t^\rho f_t(x) - f(\bar{x}) \geq \langle \rho z^\rho, x - \bar{x} \rangle - \varepsilon, \quad \forall x \in C.$$

Then, for a fixed $x \in F$

$$f(x) \geq f(x) + \sum_{\{t \in T | \lambda_t^\rho > 0\}} \lambda_t^\rho f_t(x) \geq f(\bar{x}) + \langle \rho z^\rho, x - \bar{x} \rangle - \varepsilon,$$

and taking limits first for $\rho \rightarrow 0$ and after for $\varepsilon \rightarrow 0$, we have proved that \bar{x} is a global minimizer of (P) . \square

Remark. In [12] it was shown that if σ is Farkas–Minkowski and

$$\text{epi}(f^*) + K \text{ is a closed set,} \tag{65}$$

then $\bar{x} \in F \cap \text{dom } \partial f$ (e.g., $\partial f(\bar{x}) \neq \emptyset$) is a minimizer of (P) if and only if there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that the exact Karush–Kuhn–Tucker condition holds,

$$\theta \in \partial f(\bar{x}) + \partial \left(\sum_{\{t \in T | \lambda_t > 0\}} \lambda_t f_t + I_C \right) (\bar{x}) \quad (66)$$

and $f_t(\bar{x}) = 0$ for all $t \in T$ such that $\lambda_t > 0$. If, in addition, the f_t 's are continuous at some point of C , then (66) gives

$$\theta \in \partial f(\bar{x}) + \sum_{\{t \in T | \lambda_t > 0\}} \lambda_t \partial f_t(\bar{x}) + N_C(\bar{x})$$

and $f_t(\bar{x}) = 0$ for all $t \in T$ such that $\lambda_t > 0$. This constitutes an exact optimality condition for (P) involving only the exact subgradients, but under stronger assumptions, namely the closedness of the set in (65) and the nonemptiness of ∂f at \bar{x} .

Acknowledgments

We would like to thank the referee for his (her) careful reading and for providing valuable suggestions which definitely allowed us to improve the manuscript, to include some better proofs, and to avoid some incoherences.

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