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► **To cite this version:**

Jorge San Martin, Takéo Takahashi, Marius Tucsnak. An optimal control approach to ciliary locomotion. Mathematical Control and Related Fields, AIMS, 2016. <hal-01062663>

**HAL Id: hal-01062663**

**<https://hal.archives-ouvertes.fr/hal-01062663>**

Submitted on 10 Sep 2014

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# An optimal control approach to ciliary locomotion\*

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## Abstract

We consider a class of low Reynolds number swimmers, of prolate spheroidal shape, which can be seen as simplified models of ciliated microorganisms. Within this model, the form of the swimmer does not change, the propelling mechanism consisting in tangential displacements of the material points of swimmer's boundary. Using explicit formulas for the solution of the Stokes equations at the exterior of a translating prolate spheroid the governing equations reduce to a system of ODE's with the control acting in some of its coefficients (bilinear control system). The main theoretical result asserts the exact controllability of the prolate spheroidal swimmer. In the same geometrical situation, we define a concept of efficiency which reduces to the classical one in the case of a spherical swimmer and we consider the optimal control problem of maximizing this efficiency during a stroke. Moreover, we analyse the sensitivity of this efficiency with respect to the eccentricity of the considered spheroid. We provide semi-explicit formulas for the Stokes equations at the exterior of a prolate spheroid, with an arbitrary tangential velocity imposed on the fluid-solid interface. Finally, we use numerical optimization tools to investigate the dependence of the efficiency on the number of inputs and on the eccentricity of the spheroid. The "best" numerical result obtained yields an efficiency of 30.66% with 13 scalar inputs. In the limiting case of a sphere our best numerically obtained efficiency is of 30.4%, whereas the best computed efficiency previously reported in the literature is of 22%.

## 1 Introduction and statement of the main results

Ciliates are swimming microorganisms which exploit the bending of a large number of small and densely packed organelles, termed cilia, in order to propel themselves in a viscous fluid. In this work we consider an envelope model for such ciliary locomotion, where the dynamics of the individual cilia are replaced by time periodic tangential displacements of the points on the boundary of the microorganism. We refer to the works of Taylor [18], Blake [4], Childress [5], Ishikawa, Simmonds and Pedley [10] or to the recent review paper of Lauga and Powers [12] for a detailed description of this model. The aim of this work is to study the locomotion mechanism of these organisms, combining classical tools of low Reynolds number flow theory with the modern techniques of optimal control.

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\*The last two authors acknowledge the support of the French National Research Agency (ANR) via the grant 11-BS03-0002 HAMECMOPSYs.

The main novelty we bring is the analysis of a non spherical case, which is closer to the elongated form of most ciliated microorganisms than the spherical one previously considered in the literature.

Our main theoretical result gives the controllability of the considered system (roughly speaking, this means that for every initial position  $P_1$  and every final position  $P_2$  there exists a time-periodic motion of the cilia steering the organism from  $P_1$  to  $P_2$ ). We also consider the optimal control problem, with state and control constraints. This problem is to maximize the efficiency of the swimmer. Tackling this optimal control problem with a reasonable computational cost requires semi-explicit formulas for the solutions of the Stokes system in the exterior of a prolate spheroid, with boundary conditions which correspond either to a translation of the spheroid or to an arbitrary tangential velocity field on the surface. The first type of boundary conditions have been tackled in classical reference such as Happel and Brenner [9] whereas tackling the tangential boundary conditions requires very technical calculations involving Gegenbauer functions. Moreover, the study of the problem for spheroids close to a sphere requires a careful asymptotic analysis of these special functions. Our results can be seen as extensions of those from the spherical case obtained, from a control theoretic viewpoint, in San Martín, Takahashi and Tucsnak [15], Sigalotti and Vivalda [17] and of those obtained, from an optimization perspective, in Lauga and Michelin [14]. Related results, namely considering swimmers with shapes which change during the stroke, have been obtained in Shapere and Wilczek [16], Alouges, DeSimone and Lefebvre [2, 3], Lohéac, Scheid and Tucsnak [13].

To give the precise statement of our results we need some notation from spheroidal geometry. We first recall that a spheroid is a revolution ellipsoid. In Cartesian coordinates  $(x_1, x_2, x_3)$ , the equation of a spheroid with  $Ox_3$  as the symmetry axis is

$$\frac{x_1^2 + x_2^2}{a_1^2} + \frac{x_3^2}{a_3^2} = 1. \quad (1.1)$$

The spheroid is called prolate if  $a_1 < a_3$  and oblate if  $a_1 > a_3$ . To study the Stokes equations at the exterior of a prolate spheroid, it is convenient to use prolate spheroidal coordinates, denoted by  $(\eta, \theta, \varphi)$ . Following, for instance, Dassios et al. [6] or [9], these coordinates are related to the usual Cartesian ones by

$$x_1 = c \sinh \eta \sin \theta \cos \varphi, \quad x_2 = c \sinh \eta \sin \theta \sin \varphi, \quad x_3 = c \cosh \eta \cos \theta, \quad (1.2)$$

where the *focal distance*  $c$  is a fixed positive number and

$$0 \leq \eta < \infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi.$$

The level lines  $\theta = \theta_0, \eta > 0, \varphi = 0$  and  $\theta \in [0, \pi], \eta = \eta_0, \varphi = 0$  are described in Figure 1, for various values of  $\theta_0, \eta_0$  and for a fixed value of the focal distance  $c$ .

To each  $\eta_0 \in (0, \infty)$  there corresponds the prolate spheroid of equation  $\eta = \eta_0$ . This equation writes, in Cartesian coordinates, in the form (1.1), with

$$a_1 = c \sinh \eta_0, \quad a_3 = c \cosh \eta_0. \quad (1.3)$$

Figure 2 represents a family of spheroids for various values of  $c$  and of  $\eta_0$  such that  $a_3 = c \cosh \eta_0 = 1$ . In the limit case  $c \rightarrow 0$  (so that  $\eta_0 \rightarrow \infty$ ), we recover the unit sphere of  $\mathbb{R}^3$ .

Assume that at  $t = 0$  the swimmer occupies the closed bounded set  $S_0$  delimited by the spheroid, centered at the origin, of equation  $\eta = \eta_0$ , for some  $\eta_0 > 0$ . For  $t > 0$  we denote by  $S(t)$  the closed set occupied by the swimmer at instant  $t$ . Within Blake's envelope model, the propelling mechanism of ciliates consists of time periodic tangential displacements of the points on the swimmer's boundary. For the sake of simplicity, we assume that these displacements are azimuthally symmetric with respect to the axis  $Ox_3$  of the unit vector  $e_3$ . This implies, in particular, that at each instant  $t$ , the position of the center of mass of the swimmer is  $h(t)e_3$ , where  $h$  a real valued function, and that the domain

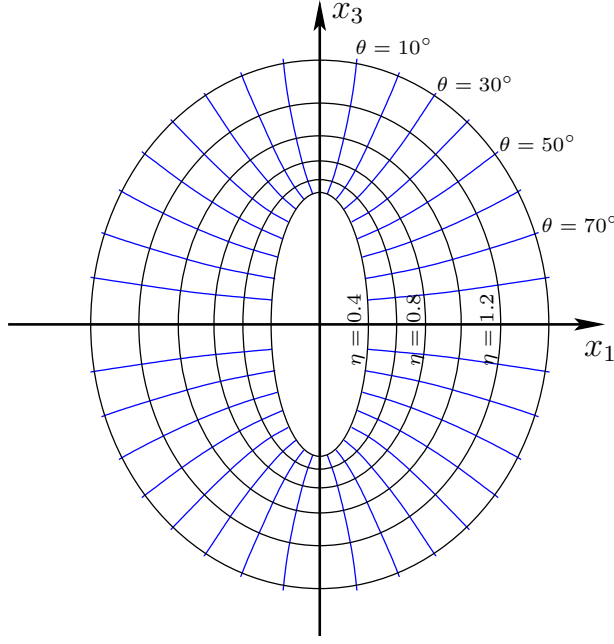


Figure 1: Level lines for prolate spheroidal coordinates

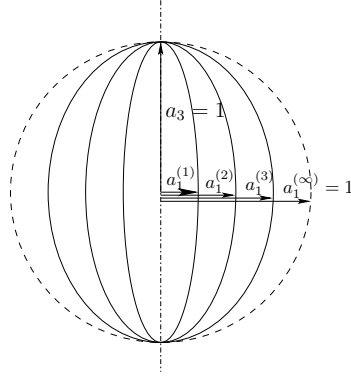


Figure 2: A family of prolate spheroids converging to the unit sphere

$S(t)$  occupied by the swimmer at instant  $t$  is simply obtained by a translation of the vector  $h(t)e_3$  of  $S_0$ , i.e., we have

$$S(t) = h(t)e_3 + S_0 \quad (t \geq 0).$$

Let  $\tilde{v}$  (respectively by  $\tilde{p}$ ) be the Eulerian velocity (respectively the pressure) field in the fluid at instant  $t$ . These fields are defined, for each instant  $t$ , in the time dependent domain  $\mathbb{R}^3 \setminus S(t)$ . For the remaining part of this work, we choose to work, instead of  $\tilde{v}$  and  $\tilde{p}$ , with the fields defined by

$$v(x, t) = \tilde{v}(x + h(t)e_3, t), \quad p(x, t) = \tilde{p}(x + h(t)e_3, t) \quad (x \in \overline{\mathbb{R}^3 \setminus S_0}, t \geq 0). \quad (1.4)$$

This allows us to take the space variable  $x$  in the fixed domain  $\overline{\mathbb{R}^3 \setminus S_0}$ .

The model that we consider for the displacements of the boundary points of  $S_0$  is the following: for each  $t \geq 0$ , the point  $x \in \partial S_0$ , of prolate spheroidal coordinates  $(\eta_0, \xi, \varphi)$ , is displaced to a point of  $\partial S_t$  whose prolate spheroidal coordinates are  $(\eta_0, \theta, \varphi)$ , where

$$\theta = \chi(\xi, t) \quad (\xi \in [0, \pi], t \geq 0), \quad (1.5)$$

and where  $\chi(\cdot, t)$  is a diffeomorphism from  $[0, \pi]$  onto itself.

In what follows, we suppose that the function  $\chi$  in (1.5) can be written as

$$\theta = \chi(\xi, t) = \xi + \sum_{i=1}^N \alpha_i(t) g_i(\xi) \quad (\xi \in [0, \pi], t \geq 0), \quad (1.6)$$

where  $N$  is a positive integer and the given functions  $(g_i)_{1 \leq i \leq N}$  are supposed to be smooth, with  $g_i(0) = g_i(\pi) = 0$  for every  $i \in \{1, \dots, N\}$ . Consequently, the input functions at the disposal of the swimmer are supposed to be  $(\alpha_i)_{1 \leq i \leq N}$ , where  $N$  is a positive integer. Moreover, we assume, in order to ensure the fact that  $\chi(\cdot, t)$  is a diffeomorphism, that

$$\sum_{i=1}^N \alpha_i(t) g'_i(\xi) > -1 \quad (t \geq 0). \quad (1.7)$$

Denoting by  $x_1(t)$ ,  $x_2(t)$  and  $x_3(t)$  the Cartesian coordinates corresponding to the motion defined by (1.6), we have, using (1.2), that

$$x_1(t) = c \sinh \eta_0 \sin \chi(\xi, t) \cos \varphi, \quad x_2(t) = c \sinh \eta_0 \sin \chi(\xi, t) \sin \varphi, \quad x_3(t) = c \cosh \eta_0 \cos \chi(\xi, t).$$

Differentiating the above formula with respect to  $t$ , the velocity field  $v$  defined in (1.4) of a point on  $\partial S_0$  is given by

$$v(\eta_0, \theta, \varphi, t) = \dot{h}(t) e_3 + c \sqrt{\cosh^2 \eta_0 - \cos^2 \theta} \frac{\partial \chi}{\partial t}(\chi^{-1}(\theta, t), t) e_\theta \quad (\theta \in [0, \pi], \varphi \in [0, 2\pi], t \geq 0), \quad (1.8)$$

where, for each point  $P$  of coordinates  $(\eta_0, \theta_0, \varphi_0)$  of the spheroid,  $e_\theta(\eta_0, \theta_0, \varphi_0)$  is the unit tangent vector at  $P$  to the curve of parametric equation  $\lambda \mapsto (\eta_0, \theta_0 + \lambda, \varphi_0)$ .

The full system describing the coupled motion of the swimmer and of the surrounding fluid writes

$$-\mu \Delta v + \nabla p = 0, \quad \operatorname{div} v = 0 \quad (\eta > \eta_0, \theta \in [0, \pi], \varphi \in [0, 2\pi], t \geq 0), \quad (1.9)$$

$$\lim_{\eta \rightarrow \infty} v(\eta, \theta, \varphi, t) = 0 \quad (\theta \in [0, \pi], \varphi \in [0, 2\pi], t \geq 0), \quad (1.10)$$

$$v(\eta_0, \theta, \varphi, t) = \dot{h} e_3 + c \sqrt{\cosh^2 \eta_0 - \cos^2 \theta} \left( \sum_{i=1}^N \beta_i g_i(\chi^{-1}(\theta, t)) \right) e_\theta \quad (\theta \in [0, \pi], \varphi \in [0, 2\pi], t \geq 0), \quad (1.11)$$

$$\dot{\alpha}_i(t) = \beta_i(t) \quad (i \in \{1, \dots, N\}, t \geq 0), \quad (1.12)$$

$$\int_{\partial S_0} \sigma(v, p) n \, d\Gamma = 0 \quad (t \geq 0), \quad (1.13)$$

$$h(0) = 0, \quad \alpha(0) = 0. \quad (1.14)$$

where the function  $\chi$  is defined by (1.6) and  $\alpha$  stands for the vector valued function  $(\alpha_i)_{1 \leq i \leq N}$ . In the above system, we have denoted by  $\mu$  the viscosity coefficient of the fluid, and we have assumed that the mass center of the swimmer is at the origin for  $t = 0$ . We denote by  $\sigma(v, p)$  the field of Cauchy stress tensors in the fluid defined by

$$\sigma(v, p) = -p \operatorname{Id}_3 + \mu [(\nabla v) + (\nabla v)^*]. \quad (1.15)$$

Moreover, as in the remaining part of this work, we have denoted by  $n$  the unit normal vector field to  $\partial S_0$  directed to the interior of  $S_0$ .

**Remark 1.1.** Note that, in the system (1.9)-(1.14), we have used the low Reynolds number approximation, in which all the inertia terms are neglected. Moreover, the equilibrium condition for the angular momentum is automatically satisfied due to the azimuthal symmetry of the problem. Since we want to control the final value of the functions  $(\alpha_i)$  (which should vanish at the end of the motion), it is more convenient to use the derivatives  $(\beta_i)$  of these functions with respect to time as inputs of our system. This fact motivates the occurrence of (1.12) as one of the governing equations.

As we shall see in Section 3, equations (1.9)-(1.14) uniquely determine  $v$ ,  $p$ ,  $(\alpha_i)_{1 \leq i \leq N}$  and  $h$  from the knowledge of the input functions  $(\beta_i)_{1 \leq i \leq N}$ .

We are now in a position to state our main controllability result.

**Theorem 1.2.** Assume that  $N \geq 2$  and that there exist  $i, j \in \{1, \dots, N\}$ ,  $i \neq j$  such that

$$\int_0^\pi [g'_j(\xi)g_i(\xi) - g'_i(\xi)g_j(\xi)] \sin^2 \xi \, d\xi \neq 0,$$

where the functions  $(g_i)$  have been introduced in (1.6).

Then for every  $h_0 \in \mathbb{R}$  and  $\varepsilon > 0$ , there exist  $T > 0$  and  $\beta \in C^\infty([0, T], \mathbb{R}^N)$  such that the solution  $(v, p, h, \alpha)$  of (1.9)-(1.14) satisfies:

1.  $h(0) = 0$ ,  $\alpha(0) = 0$ ,
2.  $h(T) = h_0$  and  $\alpha(T) = 0$ ,
3.  $|\beta(t)| \leq 1$  and  $|\alpha(t)| \leq \varepsilon$  for every  $t \in [0, T]$ .

In the above theorem and in all what follows, we denote by  $|\cdot|$  the Euclidean norm of  $\mathbb{R}^k$ ,  $k \geq 0$ . Note that, for  $\varepsilon$  small enough the third condition in the above theorem implies condition (1.6).

An important part of this work is devoted to the maximization of swimmer's efficiency, which is classically defined (see, for instance, [4]) as the ratio between the average power that an external force would spend to translate the system rigidly at the same average speed and the average power expended by the swimmer during a stroke starting and ending at the undeformed shape.

More precisely, let us consider the solution  $(v^{(0)}, p^{(0)})$  of

$$-\mu \Delta v^{(0)} + \nabla p^{(0)} = 0 \text{ in } \mathbb{R}^3 \setminus S_0, \quad \operatorname{div} v^{(0)} = 0 \text{ in } \mathbb{R}^3 \setminus S_0, \quad (1.16)$$

$$\lim_{|x| \rightarrow \infty} v^{(0)} = 0, \quad v^{(0)} = e_3 \text{ on } \partial S_0, \quad (1.17)$$

and assume that  $v$ ,  $p$ ,  $h$ ,  $(\alpha_i)_{1 \leq i \leq N}$  is the solution of (1.9)-(1.14) associated with  $\beta$ . Then the efficiency  $\operatorname{eff}$  of a motion driving the mass center from the origin to the final position  $h_0 e_3$ , with  $h_0 \neq 0$ , in time  $T$  is given by

$$\operatorname{eff}(\beta; c, \eta_0) = \frac{\int_0^T \int_{\partial S_0} \sigma(v^{(0)}, p^{(0)}) n \cdot \left(\frac{h_0}{T}\right)^2 e_3 \, d\Gamma \, dt}{\int_0^T \int_{\partial S_0} \sigma(v, p) n \cdot v \, d\Gamma \, dt}, \quad (1.18)$$

To derive the formula for the numerator in the right hand side of (1.18) we have used the fact that the mean velocity necessary to swim from the origin to  $h_0 e_3$  in time  $T$  is  $\frac{h_0}{T} e_3$ , so that the work needed to translate the prolate spheroid from the origin to a position centered in  $h_0 e_3$  in time  $T$  is given by

$$W_0 = \int_0^T \int_{\partial S_0} \sigma\left(\frac{h_0}{T} v^{(0)}, \frac{h_0}{T} p^{(0)}\right) n \cdot \frac{h_0}{T} e_3 \, d\Gamma \, dt = \int_0^T \int_{\partial S_0} \sigma(v^{(0)}, p^{(0)}) n \cdot \left(\frac{h_0}{T}\right)^2 e_3 \, d\Gamma \, dt. \quad (1.19)$$

As suggested by the above notation, if we fix the final position  $h_0$  and the final time  $T$ , this efficiency depends on the input  $\beta$  and on the geometric parameters  $c$  and  $\eta_0$ . If the geometry is fixed and if there is no risk of confusion, we simply denote the efficiency defined in (1.18) by  $\text{eff}(\beta)$

An important problem is to maximize the efficiency  $\text{eff}$  with respect to the input  $\beta$  which varies in a set of *admissible controls*. In the present case, given  $T > 0$  and  $h_0 \in \mathbb{R}$ , we choose the set  $\mathcal{U}_{h_0, T}$  of admissible controls to be the subset of all the functions in  $\beta \in L^\infty((0, \infty), \mathbb{R}^N)$  satisfying condition 2 in Theorem 1.2, together with (1.6). The result in Theorem 1.2 can be rephrased to say that for every  $h_0$  there exists  $T > 0$  such that  $\mathcal{U}_{h_0, T} \neq \emptyset$ .

We investigate in Section 6 the influence of the focal distance  $c$  on swimmers efficiency. More precisely, we consider the family of spheroids in Figure 2 for which  $a_3 = 1$ . For these spheroids we have, according to (1.3) that

$$\eta_0 = \cosh^{-1} \left( \frac{1}{c} \right),$$

so that the efficiency defined above depends only on  $\beta$  and on  $c$ . However, obtaining a simple formula for this efficiency defined in (1.18) seems much more complicated than in the spherical case. Therefore, for the above mentioned family of spheroids, we introduce a modified efficiency concept, which can be expressed as function of  $\beta$  and  $e = c^2$  only, denoted by  $\text{eff}_K(\beta; e)$ . This efficiency coincides when  $e \rightarrow 0$  (so that the family in Figure 2 tends to the unit sphere) to the swimming efficiency of a spherical ciliate (called a *squirmers*). Using this definition, we show that the efficiency of the swimmer increases when the sphere deforms to a prolate spheroid. Finally, in Section 7 we propose a methodology to solve the above mentioned optimal control problem numerically and we provide some of the results of our simulations. One of the main difficulty we had to tackle is the presence of constraints on the state and on the control. The maximum efficiency obtained numerically is of 30.66%, in the case in which 13 scalar input functions are used. In the limiting case of a sphere our best numerically obtained efficiency is of 30.4%, whereas the best computed efficiency previously reported in the literature, see [14], is of 22%.

## 2 The Stokes problem at the exterior of a prolate spheroidal obstacle in translation

In this section, we first recall some known results on the exterior Stokes problem for a fluid filling the exterior of a prolate spheroidal obstacle and then we derive an explicit formula for the shear stress on the surface of the prolate spheroid in the case of an obstacle translating with constant velocity.

Consider the Stokes system, with Dirichlet boundary conditions, posed in  $\Omega = \mathbb{R}^3 \setminus S_0$ , where  $S_0$  is the closed bounded set delimited by a prolate spheroid.

We consider the classical homogeneous Sobolev space

$$D^{1,2}(\Omega) = \{v \in L^1_{loc}(\Omega), \nabla v \in L^2(\Omega)\},$$

endowed with the norm

$$|v|_{1,2} = \|\nabla v\|_{L^2(\Omega)} \quad (v \in D^{1,2}(\Omega)).$$

The following well-posedness result for the exterior Stokes system is well-known (see, for instance, Galdi, [7, chap. 5, Theorem 2.1, p. 251]).

**Theorem 2.1.** *Let  $V \in H^{\frac{1}{2}}(\partial\Omega)$ . Then there exists a unique solution  $(v, p) \in D^{1,2}(\Omega) \times L^2(\Omega)$  of:*

$$\begin{cases} -\mu\Delta v + \nabla p = 0, & \text{in } \Omega, \\ \operatorname{div} v = 0, & \text{in } \Omega, \\ v = V, & \text{on } \partial S_0, \\ \lim_{r \rightarrow +\infty} \int_0^\pi \int_0^{2\pi} |v(r, \theta, \varphi)| \sin \theta \, d\varphi \, d\theta = 0. \end{cases} \quad (2.1)$$

Moreover, there exists  $c(\Omega) > 0$  such that:

$$\|v\|_{1,2} + \|p\|_{L^2(\Omega)} \leq c(\Omega) \|V\|_{H^{\frac{1}{2}}(\partial S_0)} \quad (V \in H^{\frac{1}{2}}(\partial S_0)).$$

Finally, if  $V \in C^\infty(\partial S_0)$  then  $(u, p) \in C^\infty(\Omega)$ .

In the above theorem  $H^{\frac{1}{2}}(\partial S_0)$  is, as usual, the fractional order Sobolev space as defined, for instance, in Grisvard [8, Ch.1]. Moreover, the theorem holds for every  $S_0$  which is the closure of an open bounded subset of  $\mathbb{R}^3$  with smooth boundary.

In the case of an obstacle  $S_0$  which is the closure of the interior of a prolate spheroid, more detailed information on the solution  $(v, p)$  of (2.1) can be obtained by using special functions.

To this aim, following Dassios et al. [6] and [9], we modify the prolate spheroidal coordinates which have been introduced in (1.2) and we set

$$\zeta = \cos \theta, \quad \tau = \cosh \eta, \quad (2.2)$$

so that we have

$$-1 \leq \zeta \leq 1, \quad 1 \leq \tau < \infty.$$

With this modification, the change of coordinates (1.2) becomes

$$x_1 = c\sqrt{\tau^2 - 1}\sqrt{1 - \zeta^2} \cos \varphi, \quad x_2 = c\sqrt{\tau^2 - 1}\sqrt{1 - \zeta^2} \sin \varphi, \quad x_3 = c\tau\zeta, \quad (2.3)$$

with

$$-1 \leq \zeta \leq 1, \quad 1 \leq \tau < \infty, \quad 0 \leq \varphi \leq 2\pi. \quad (2.4)$$

Writing  $x = (x_1, x_2, x_3)$  with  $x_i$  given by (2.3), the natural basis corresponding to the above coordinate system is given by

$$g_\zeta = \frac{\partial x}{\partial \zeta}, \quad g_\tau = \frac{\partial x}{\partial \tau}, \quad g_\varphi = \frac{\partial x}{\partial \varphi}.$$

Moreover, the Lamé coefficients (also called scale factors) corresponding to the change of coordinates (2.3) are given by

$$h_\zeta = \left\| \frac{\partial x}{\partial \zeta} \right\| = c\sqrt{\frac{\tau^2 - \zeta^2}{1 - \zeta^2}}, \quad h_\tau = \left\| \frac{\partial x}{\partial \tau} \right\| = c\sqrt{\frac{\tau^2 - \zeta^2}{\tau^2 - 1}}, \quad h_\varphi = \left\| \frac{\partial x}{\partial \varphi} \right\| = c\sqrt{\tau^2 - 1}\sqrt{1 - \zeta^2}. \quad (2.5)$$

It can be easily checked that  $\{g_\zeta, g_\tau, g_\varphi\}$  form an orthogonal basis in  $\mathbb{R}^3$ . Writing

$$e_\zeta = \frac{1}{h_\zeta} g_\zeta, \quad e_\tau = \frac{1}{h_\tau} g_\tau, \quad e_\varphi = \frac{1}{h_\varphi} g_\varphi,$$

we obtain a right-handed orthonormal basis  $\{e_\zeta, e_\tau, e_\varphi\}$  expressed by

$$e_\zeta = -\frac{\zeta\sqrt{\tau^2 - 1}}{\sqrt{\tau^2 - \zeta^2}} \cos \varphi e_1 - \frac{\zeta\sqrt{\tau^2 - 1}}{\sqrt{\tau^2 - \zeta^2}} \sin \varphi e_2 + \frac{\tau\sqrt{1 - \zeta^2}}{\sqrt{\tau^2 - \zeta^2}} e_3, \quad (2.6)$$



$$e_\tau = \frac{\tau\sqrt{1-\zeta^2}}{\sqrt{\tau^2-\zeta^2}} \cos\varphi e_1 + \frac{\tau\sqrt{1-\zeta^2}}{\sqrt{\tau^2-\zeta^2}} \sin\varphi e_2 + \frac{\zeta\sqrt{\tau^2-1}}{\sqrt{\tau^2-\zeta^2}} e_3, \quad (2.7)$$

$$e_\varphi = -(\sin\varphi) e_1 + (\cos\varphi) e_2.$$

Note that the prolate spheroid equation (1.1) can be rewritten as

$$\frac{x_1^2 + x_2^2}{c^2(\tau_0^2 - 1)} + \frac{x_3^2}{c^2\tau_0^2} = 1, \quad (2.8)$$

where  $\tau_0 > 1$  is fixed.

A simple calculation shows that the surface element of the prolate spheroid obtained by taking  $\tau = \tau_0$  in (2.3) is given by

$$d\Gamma = c^2 \sqrt{(\tau_0^2 - 1)(\tau_0^2 - \zeta^2)} d\zeta d\varphi. \quad (2.9)$$

In this work we are interested in solutions of (2.1) which are axisymmetrical with respect to  $Ox_3$ . This means, in particular that the velocity field  $v$  writes

$$v = v(\zeta, \tau) = v_\zeta(\zeta, \tau)e_\zeta + v_\tau(\zeta, \tau)e_\tau,$$

where  $e_\zeta$  and  $e_\tau$  have been introduced in (2.6) and (2.7) and the pressure satisfies  $p = p(\zeta, \tau)$ . It is not difficult to check that the  $(\tau, \zeta)$  and  $(\zeta, \tau)$  components of the gradient of  $v$  are given by

$$(\nabla v)_{\tau, \zeta} = \frac{1}{h_\tau} \frac{\partial v_\zeta}{\partial \tau} - \frac{1}{h_\tau h_\zeta} \frac{\partial h_\tau}{\partial \zeta} v_\tau, \quad (\nabla v)_{\zeta, \tau} = \frac{1}{h_\zeta} \frac{\partial v_\tau}{\partial \zeta} - \frac{1}{h_\tau h_\zeta} \frac{\partial h_\zeta}{\partial \tau} v_\zeta,$$

where the Lamé coefficients  $h_\tau$  and  $h_\zeta$  are given by (2.5). Inserting the expression of  $h_\tau$  and  $h_\zeta$  from (2.5) in the last formula, we obtain

$$(\nabla v)_{\tau, \zeta} = \frac{1}{c} \sqrt{\frac{\tau^2 - 1}{\tau^2 - \zeta^2}} \frac{\partial v_\zeta}{\partial \tau} + \frac{\zeta \sqrt{1 - \zeta^2}}{c(\tau^2 - \zeta^2)^{\frac{3}{2}}} v_\tau, \quad (\nabla v)_{\zeta, \tau} = \frac{1}{c} \sqrt{\frac{1 - \zeta^2}{\tau^2 - \zeta^2}} \frac{\partial v_\tau}{\partial \zeta} - \frac{\tau \sqrt{\tau^2 - 1}}{c(\tau^2 - \zeta^2)^{\frac{3}{2}}} v_\zeta. \quad (2.10)$$

It follows that the component  $\sigma_{\tau, \zeta} := (\sigma e_\tau) \cdot e_\zeta$  of the corresponding stress tensor is given by

$$\begin{aligned} \sigma_{\tau, \zeta}(\tau, \zeta) &= \frac{\mu}{c\sqrt{\tau^2 - \zeta^2}} \left[ \sqrt{1 - \zeta^2} \frac{\partial v_\tau}{\partial \zeta}(\zeta, \tau) + \sqrt{\tau^2 - 1} \frac{\partial v_\zeta}{\partial \tau}(\zeta, \tau) \right] \\ &+ \frac{\mu}{c(\tau^2 - \zeta^2)^{\frac{3}{2}}} \left[ \zeta \sqrt{1 - \zeta^2} v_\tau(\zeta, \tau) - \tau \sqrt{\tau^2 - 1} v_\zeta(\zeta, \tau) \right] \quad (\tau \geq \tau_0, -1 \leq \zeta \leq 1, 0 \leq \varphi < 2\pi). \end{aligned} \quad (2.11)$$

The axisymmetrical case is characterized (see [9]) by the existence of a stream function  $\psi$ . According to [6] this function is related to the velocity field  $v$  by

$$v_\zeta(\zeta, \tau) = -\frac{1}{c^2 \sqrt{(\tau^2 - \zeta^2)(1 - \zeta^2)}} \frac{\partial \psi}{\partial \tau}(\zeta, \tau). \quad (2.12)$$

$$v_\tau(\zeta, \tau) = \frac{1}{c^2 \sqrt{\tau^2 - \zeta^2} \sqrt{\tau^2 - 1}} \frac{\partial \psi}{\partial \zeta}(\zeta, \tau), \quad (2.13)$$

Moreover, the stream function  $\psi$  lies in a kernel of a relatively simple differential operator. More precisely, consider the differential operator  $E^2$  which associates to each smooth function  $f$  of variables  $\zeta$  and  $\tau$  the function  $E^2 f$  defined by

$$E^2 f = \frac{1}{c^2(\tau^2 - \zeta^2)} \left[ (1 - \zeta^2) \frac{\partial^2 f}{\partial \zeta^2} + (\tau^2 - 1) \frac{\partial^2 f}{\partial \tau^2} \right]. \quad (2.14)$$

Then (see, for instance, [9, p.104]) the fact that  $(v, p)$  satisfies the Stokes system is equivalent to the fact that  $\psi$  satisfies the equation

$$E^4\psi = 0, \quad (2.15)$$

where  $E^4$  is the square of the operator  $E^2$  introduced in (2.14).

In the case of a prolate spheroid translating with a constant velocity equal to  $e_3$  we have the following result.

**Proposition 2.2.** *Assume that  $S_0$  is the closure of the interior of the prolate spheroid in  $\mathbb{R}^3$  defined by equation (2.3). Then a stream function  $\psi^{(0)}$  corresponding to the unique solution  $(v^{(0)}, p^{(0)}) \in D^{1,2}(\Omega) \times L^2(\Omega)$  of (1.16)–(1.17) is given, using prolate spheroidal coordinates, by*

$$\psi^{(0)}(\tau, \zeta, \varphi) = \frac{2c^2 [(\tau_0^2 + 1)H_2(\tau) - \tau]}{(\tau_0^2 + 1) \coth^{-1}(\tau_0) - \tau} G_2(\zeta), \quad (2.16)$$

where  $G_2$  and  $H_2$  are the second order Gegenbauer functions defined in Appendix A. Moreover, the corresponding stress tensor  $\sigma^{(0)}$  satisfies, on  $\partial S_0$ ,

$$\sigma_{\zeta, \tau}^{(0)}(\zeta, \tau_0) = -\frac{2\mu\tau_0}{c\sqrt{\tau_0^2 - 1} [(\tau_0^2 + 1) \coth^{-1} \tau_0 - \tau_0]} \frac{\sqrt{1 - \zeta^2}}{\tau_0^2 - \zeta^2} \quad (\zeta \in [-1, 1]). \quad (2.17)$$

Finally, the force exerted by the solid on the fluid is

$$\int_{\partial S_0} \sigma^{(0)} n \, d\Gamma = \frac{8\pi\mu c}{(\tau_0^2 + 1) \coth^{-1} \tau_0 - \tau_0} e_3. \quad (2.18)$$

*Proof.* We refer to [9, Section 4.30] for the formula of the stream function (2.16) and for the formula (2.18). More precisely, it is proved in [9, Section 4.30] that

$$\psi^{(0)}(\zeta, \tau) = -\frac{c^2}{2}(\tau^2 - 1)(1 - \zeta^2) \frac{\frac{\tau_0^2 + 1}{\tau_0^2 - 1} \coth^{-1} \tau - \frac{\tau}{\tau^2 - 1}}{\frac{\tau_0^2 + 1}{\tau_0^2 - 1} \coth^{-1} \tau_0 - \frac{\tau_0}{\tau_0^2 - 1}} \quad (\tau \geq \tau_0, \zeta \in [-1, 1]). \quad (2.19)$$

satisfies (2.15) and that the velocity  $v^{(0)}$  associated to  $\psi^{(0)}$  satisfies  $v^{(0)}(\zeta, \tau_0) = e_3$ . A simple calculation and (A.19), (A.25), show that the function  $\psi^{(0)}$  can be written as (2.16)

Therefore, we only prove here (2.17). To do this, we use formula (2.11).

First note that, using (2.7) and (2.6), we have

$$v^{(0)}(\zeta, \tau_0) = e_3 = \frac{\tau_0 \sqrt{1 - \zeta^2}}{\sqrt{\tau_0^2 - \zeta^2}} e_\zeta + \frac{\zeta \sqrt{\tau_0^2 - 1}}{\sqrt{\tau_0^2 - \zeta^2}} e_\tau \quad (-1 < \zeta < 1), \quad (2.20)$$

By inserting (2.20) in the last term in the right hand side of (2.11) we obtain

$$\begin{aligned} \frac{\mu}{c(\tau_0^2 - \zeta^2)^{\frac{3}{2}}} \left( \zeta \sqrt{1 - \zeta^2} v_\tau^{(0)} - \tau_0 \sqrt{\tau_0^2 - 1} v_\zeta^{(0)} \right) &= \frac{\mu}{c(\tau_0^2 - \zeta^2)^{\frac{3}{2}}} \sqrt{\frac{(\tau_0^2 - 1)(1 - \zeta^2)}{\tau_0^2 - \zeta^2}} (\zeta^2 - \tau_0^2) \\ &= -\frac{\mu \sqrt{(\tau_0^2 - 1)(1 - \zeta^2)}}{c(\tau_0^2 - \zeta^2)} \quad (-1 \leq \zeta \leq 1). \end{aligned} \quad (2.21)$$

Concerning the first term in the right hand side of (2.11), we first tackle the part which can be evaluated directly using (2.20), i.e., we write

$$\begin{aligned} \frac{\mu \sqrt{1 - \zeta^2}}{c\sqrt{\tau_0^2 - \zeta^2}} \frac{\partial v_\tau^{(0)}}{\partial \zeta}(\zeta, \tau_0) &= \frac{\mu \sqrt{1 - \zeta^2}}{c\sqrt{\tau_0^2 - \zeta^2}} \left[ \frac{\sqrt{\tau_0^2 - 1}}{\sqrt{\tau_0^2 - \zeta^2}} + \frac{\zeta^2 \sqrt{\tau_0^2 - 1}}{(\tau_0^2 - \zeta^2)^{\frac{3}{2}}} \right] \\ &= \frac{\mu \sqrt{(\tau_0^2 - 1)(1 - \zeta^2)}}{c(\tau_0^2 - \zeta^2)} \left( 1 + \frac{\zeta^2}{\tau_0^2 - \zeta^2} \right) \quad (-1 \leq \zeta \leq 1). \end{aligned} \quad (2.22)$$

Finally, we note that (2.12) implies that

$$\begin{aligned} & \frac{\mu\sqrt{\tau_0^2-1}}{c\sqrt{\tau_0^2-\zeta^2}} \frac{\partial v_\zeta^{(0)}}{\partial \tau}(\zeta, \tau_0) \\ &= -\frac{\mu\sqrt{\tau_0^2-1}}{c^3\sqrt{(\tau_0^2-\zeta^2)(1-\zeta^2)}} \left( \frac{1}{\sqrt{\tau_0^2-\zeta^2}} \frac{\partial^2 \psi^{(0)}}{\partial \tau^2}(\zeta, \tau_0) - \frac{\tau_0}{(\tau_0^2-\zeta^2)^{\frac{3}{2}}} \frac{\partial \psi^{(0)}}{\partial \tau}(\zeta, \tau_0) \right) \quad (-1 \leq \zeta \leq 1), \end{aligned} \quad (2.23)$$

where  $\psi^{(0)}$  is given by (2.16). Using next (2.16), together with the fact (following from (A.28)) that

$$H_2''(\tau) = \frac{2H_2(\tau)}{\tau^2-1} \quad (\tau \in \mathbb{R}),$$

we obtain

$$\frac{\partial^2 \psi^{(0)}}{\partial \tau^2}(\zeta, \tau_0) = -c^2(1-\zeta^2) \frac{(\tau_0^2+1)[(\tau_0^2-1)\coth^{-1}\tau_0 - \tau_0]}{(\tau_0^2-1)[(\tau_0^2+1)\coth^{-1}\tau_0 - \tau_0]} \quad (-1 \leq \zeta \leq 1).$$

On the other hand, (2.12) and (2.20) imply that

$$\frac{\partial \psi^{(0)}}{\partial \tau}(\zeta, \tau_0) = c^2\sqrt{\tau_0^2-\zeta^2}\sqrt{\tau_0^2-1}v_\zeta^{(0)}(\zeta, \tau_0) = -c^2\tau_0(1-\zeta^2) \quad (-1 \leq \zeta \leq 1).$$

Using the last two formulas in (2.23), it follows that

$$\frac{\partial v_\zeta^{(0)}}{\partial \tau}(\zeta, \tau_0) = -\frac{\tau_0^2\sqrt{1-\zeta^2}}{(\tau_0^2-\zeta^2)^{\frac{3}{2}}} + \frac{(\coth^{-1}\tau_0 - \frac{\tau_0}{\tau_0^2-1})\sqrt{1-\zeta^2}}{(\coth^{-1}\tau_0 - \frac{\tau_0}{\tau_0^2+1})\sqrt{\tau_0^2-\zeta^2}} \quad (-1 \leq \zeta \leq 1).$$

The above formula, (2.21), (2.22) and (2.11) imply, after some calculation, the conclusion (2.17).  $\square$

**Remark 2.3.** *By using (1.19) and (2.18), it follows that the work needed to translate the prolate spheroid from the origin to a position centered in  $(0, 0, h_0)$  in time  $T$  is given by*

$$W_0 = \int_0^T \int_{\partial S_0} \sigma(v^{(0)}, p^{(0)})n \cdot \left(\frac{h_0}{T}\right)^2 e_3 \, d\Gamma \, dt = \frac{8\pi\mu ch_0^2}{T[(\tau_0^2+1)\coth^{-1}\tau_0 - \tau_0]}.$$

### 3 The controllability result

The aim of this section is to prove Theorem 1.2. As a first step, we show that equations (1.9)-(1.14) can be written in a form which allows the application of some classical control theoretical tools.

**Proposition 3.1.** *Let  $\beta = (\beta_i)_{1 \leq i \leq N}$  in  $L^\infty([0, \infty), \mathbb{R}^N)$  and  $(g_i)_{1 \leq i \leq N}$  in  $C^\infty[0, \pi] \cap H_0^1(0, \pi)$  be such that*

$$\sum_{i=1}^N g_i(\xi) \int_0^t \beta_i(s) \, ds > -1 \quad (\xi \in [0, \pi], t \geq 0). \quad (3.1)$$

*Then equations (1.9)-(1.14) admit a unique solution  $(v, p, h, \alpha)$ . More precisely, the function  $h$  is obtained by solving the initial value problem*

$$\dot{h}(t) = \sum_{i=1}^N \beta_i(t) F_i(\alpha_1(t), \dots, \alpha_N(t)), \quad (3.2)$$

$$\dot{\alpha}_i(t) = \beta_i(t) \quad (i \in \{1, \dots, N\}), \quad (3.3)$$

$$h(0) = 0, \quad \alpha(0) = 0, \quad (3.4)$$

where

$$F_i(\alpha_1, \dots, \alpha_N) = c\tau_0 \int_0^\pi g_i'(\xi) \mathcal{F}(\chi(\xi, t)) d\xi \quad (i \in \{1, \dots, N\}), \quad (3.5)$$

$$\mathcal{F}(x) = \frac{1}{8} (\sin(2x) - 2x) \quad (x \in \mathbb{R}), \quad (3.6)$$

and  $\chi$  has been defined in (1.6). The solution  $(v, p)$  can be decomposed as

$$v = \dot{h}(t)v^{(0)} + v^{(1)}, \quad p = \dot{h}(t)p^{(0)} + p^{(1)}, \quad (3.7)$$

where  $(v^{(0)}, p^{(0)})$  is the solution of the Stokes system (1.16)–(1.17) whereas  $(v^{(1)}, p^{(1)})$  is the solution of the Stokes system

$$-\mu\Delta v^{(1)} + \nabla p^{(1)} = 0 \text{ in } \mathbb{R}^3 \setminus S_0, \quad \operatorname{div} v^{(1)} = 0 \text{ in } \mathbb{R}^3 \setminus S_0, \quad (3.8)$$

$$\lim_{|x| \rightarrow \infty} v^{(1)} = 0, \quad v^{(1)} = V \text{ on } \partial S_0, \quad (3.9)$$

with

$$V(\zeta, t) = -c\sqrt{\tau_0^2 - \zeta^2} \left( \sum_{i=1}^N \beta_i g_i(\chi^{-1}(\arccos \zeta, t)) \right) e_\zeta \quad (\zeta \in [-1, 1], \quad t \geq 0). \quad (3.10)$$

*Proof.* For  $t \geq 0$ , we set  $\alpha(t) = \int_0^t \beta(s) ds$ . Using (3.1), it follows that  $\chi$  defined in (1.6) is one to one from  $[0, \pi]$  onto  $[0, \pi]$ . Let us fix, for a moment,  $h \in W_{\text{loc}}^{1, \infty}(0, \infty)$ . One can see that if we define  $(v, p)$  by (3.7), (3.8)–(3.9), (3.10), then  $(v, p)$  satisfies (1.9)–(1.11). Therefore,  $(v, p, h, \alpha)$  is solution of (1.9)–(1.14) if and only if (1.13) holds. This equation can be written as

$$\dot{h}(t) \int_{\partial S_0} \sigma(v^{(0)}, p^{(0)}) n d\Gamma + \int_{\partial S_0} \sigma(v^{(1)}, p^{(1)}) n d\Gamma = 0 \quad (t \geq 0). \quad (3.11)$$

At this point it is useful to note, using a simple integration by parts, that we have the *reciprocity relation*

$$\int_{\partial S_0} \sigma(v^{(0)}, p^{(0)}) n \cdot V d\Gamma = \int_{\partial S_0} \sigma(v^{(1)}, p^{(1)}) n \cdot e_3 d\Gamma$$

where  $V$  is given in (3.10). Taking the inner product of (3.11) with  $e_3$ , using the above reciprocity relation and (2.18), formula (3.11) is equivalent to

$$\dot{h}(t) = -\frac{(\tau_0^2 + 1) \coth^{-1} \tau_0 - \tau_0}{8\pi\mu c} \int_{\partial S} \sigma(v^{(0)}, p^{(0)}) n \cdot V(\zeta, t) d\Gamma \quad (t \geq 0).$$

Combining the above formula with (2.9), (2.17) and with  $n = -e_\tau$  implies

$$\dot{h}(t) = \frac{c\tau_0}{2} \sum_{i=1}^N \beta_i \int_{-1}^1 g_i(\chi^{-1}(\arccos \zeta, t)) \sqrt{1 - \zeta^2} d\zeta \quad (t \geq 0). \quad (3.12)$$

Relations (3.2), (3.5) and (3.6) follow now by making the change of variables  $\xi = \chi^{-1}(\arccos \zeta, t)$ , followed by an integration by parts, in the above formula.  $\square$

**Remark 3.2.** Using the Gegenbauer functions introduced in Appendix A and the relation (3.12) obtained in the above proof, the equation for  $h$  can be written in a useful alternative form. Indeed, since the Gegenbauer functions of the first kind  $(G_n)_{n \geq 2}$  (defined by formulae (A.17), (A.18) in Appendix 1) form a basis in  $L^2[-1, 1]$ , we can write

$$-\sqrt{1 - \zeta^2} \sum_{i=1}^N \beta_i g_i(\chi^{-1}(\arccos \zeta, t), t) = \sum_{n \geq 2} A_n(t) G_n(\zeta) \quad (\zeta \in [-1, 1]), \quad (3.13)$$

where  $(A_n)_{n \geq 1}$  is a sequence in  $C([0, \infty), l^2)$ . Plugging (3.13) into (3.12), and using (A.19), we obtain

$$\dot{h}(t) = -c\tau_0 \int_{-1}^1 \sum_{n \geq 2} A_n(t) G_n(\zeta) G_2(\zeta) \frac{1}{1-x^2} d\zeta.$$

The above equation and (A.21) imply that the equation governing the motion of the mass center of the swimmer simply writes

$$\dot{h}(t) = -\frac{c\tau_0 A_2(t)}{3} \quad (t \geq 0). \quad (3.14)$$

Since we reduced the governing equations to a finite dimensional bilinear control system, we are now in a position to use a well known result due to Chow. We recall this result below and we refer, for instance, to [19, chap. 5, Proposition 5.14, p. 89] or [11] for more information on this theorem.

**Theorem 3.3.** *Let  $m, n \in \mathbb{N}^*$  and let  $(f_i)_{i=1, \dots, n}$  be  $C^\infty$  vector fields on  $\mathbb{R}^n$ . Consider the control system, of state trajectory  $X$ ,*

$$\dot{X} = \sum_{i=1}^m u_i f_i(X), \quad (3.15)$$

with input function  $u = (u_i)_{i=1, \dots, m} \in C^\infty([0, +\infty[, B_r)$ , where  $r > 0$  and  $B_r$  is the closed ball of radius  $r$  and centered at the origin in  $\mathbb{R}^m$ . Let  $\mathcal{O}$  an open and connected set of  $\mathbb{R}^n$  and assume that

$$\text{Lie}_X\{f_1, \dots, f_m\} = \mathbb{R}^n \quad (X \in \mathcal{O}).$$

Then the system (3.15) is controllable, i.e., for every  $X_0, X_1 \in \mathcal{O}$  there exist  $T > 0$  and  $u \in C^\infty([0, T], B_r)$  such that  $X(0) = X_0$  and  $X(T) = X_1$  and  $X(t) \in \mathcal{O}$  for every  $t \in [0, T]$ .

In the above theorem, the Lie-bracket of two  $N$  dimensional smooth vector fields  $f_1, f_2$  is a new vector field defined by

$$[f_1, f_2](X) = Df_2(X)(f_1(X)) - Df_1(X)(f_2(X)) \quad (X \in \mathbb{R}^N). \quad (3.16)$$

We recall that a Lie-algebra is a space closed for the Lie-bracket  $[\cdot, \cdot]$  and that  $(\text{Lie}\{f_1, \dots, f_m\}, [\cdot, \cdot])$  is the smallest Lie-algebra containing  $\{f_1, \dots, f_m\}$ . Moreover,  $\text{Lie}_X\{f_1, \dots, f_m\}$  is the subspace of  $\mathbb{R}^n$  spanned by all the values in  $X$  of the vector fields in  $\text{Lie}\{f_1, \dots, f_m\}$ .

We are now in a position to prove Theorem 1.2.

*Proof of Theorem 1.2.* It clearly suffices to prove the result for  $N = 2$  and

$$\int_0^\pi [g_2'(\xi)g_1(\xi) - g_1'(\xi)g_2(\xi)] \sin^2 \xi d\xi \neq 0. \quad (3.17)$$

We first remark that, for  $N = 2$ , equations (3.2)-(3.3) can be written in the form (3.15), provided that we set  $X = \begin{bmatrix} h \\ \alpha \end{bmatrix}$  and

$$f_1(X) = \begin{bmatrix} F_1(\alpha) \\ 1 \\ 0 \end{bmatrix}, \quad f_2(X) = \begin{bmatrix} F_2(\alpha) \\ 0 \\ 1 \end{bmatrix}, \quad (3.18)$$

where the functions  $F_1$  and  $F_2$  have been defined in (3.5) and (3.6). It is easily checked that

$$Df_k(X) = \begin{bmatrix} 0 & \frac{\partial F_k}{\partial \alpha_1}(\alpha) & \frac{\partial F_k}{\partial \alpha_2}(\alpha) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (k \in \{1, 2\}, \quad X \in \mathbb{R}^3).$$

The above formula, (3.16) and (3.18) yield that

$$[f_1, f_2](X) = \begin{bmatrix} \frac{\partial F_2}{\partial \alpha_1}(\alpha) - \frac{\partial F_1}{\partial \alpha_2}(\alpha) \\ 0 \\ 0 \end{bmatrix} \quad (X \in \mathbb{R}^3). \quad (3.19)$$

On the other hand, going back to (3.5) and (3.6) and using the fact that  $\mathcal{F}'(x) = -\frac{\sin^2 x}{2}$ , we obtain that

$$\frac{\partial F_i}{\partial \alpha_j}(0) = -\frac{c\tau_0}{2} \int_0^\pi g'_i(\xi)g_j(\xi) \sin^2(\xi) \, d\xi \quad (i, j \in \{1, 2\}).$$

The above formula and (3.19) imply that

$$[f_1, f_2] \begin{bmatrix} h \\ 0 \end{bmatrix} = -\frac{c\tau_0}{2} \begin{bmatrix} \int_0^\pi [g'_2(\xi)g_1(\xi) - g'_1(\xi)g_2(\xi)] \sin^2(\xi) \, d\xi \\ 0 \\ 0 \end{bmatrix} \quad (h \in \mathbb{R}).$$

Using (3.17) and (3.18) it follows that if  $X = \begin{bmatrix} h \\ 0 \end{bmatrix}$  with  $h \in \mathbb{R}$  then  $f_1(X)$ ,  $f_2(X)$  and  $[f_1, f_2](X)$  span  $\mathbb{R}^3$ . By continuity, there exists  $\tilde{\varepsilon} \in (0, \varepsilon]$  such that the same property holds for  $X = \begin{bmatrix} h \\ \alpha \end{bmatrix}$ , with  $|\alpha| < \tilde{\varepsilon}$ . We can thus conclude the proof by using Theorem 3.3 with  $\mathcal{O} = \mathbb{R} \times \{\alpha \in \mathbb{R}^2 ; |\alpha| < \tilde{\varepsilon}\}$ .  $\square$

## 4 The Stokes problem at the exterior of a prolate spheroid with a tangential velocity on the boundary

In order to tackle the second term in the right hand side of the boundary condition (1.11) we investigate in this section the problem (2.1), with  $S_0$  being the prolate spheroid of equations (2.3) and with the boundary velocity  $V$  is azimuthally symmetric with respect to  $e_3$  and tangential to  $\partial S_0$ . Unlike in the case of a spherical swimmer, studied in [4], we have no longer a separation of variables (see [6]). Therefore there is no simple way of expressing the solution of (2.1) in function of the coefficients occurring in the decomposition of  $V$  in an appropriate basis. This leads us to consider an approximation of the exact solution involving several tridiagonal matrices.

To describe this approximation we use the prolate spheroidal coordinates introduced in (2.2), (2.3), (2.4) to write the boundary condition (1.11) as

$$v(\zeta, \tau_0, \varphi, t) = \dot{h}e_3 - c\sqrt{\tau_0^2 - \zeta^2} \left( \sum_{i=1}^N \beta_i g_i(\chi^{-1}(\arccos \zeta, t)) \right) e_\zeta.$$

Section 2 was devoted to solving (2.1) with  $V = e_3$  and the main aim of this section is to provide an approximate solution of (2.1) with

$$V = -u(\zeta)e_\zeta \quad \zeta \in [-1, 1], \quad (4.1)$$

where  $u$  is a smooth function.

We first give, following [6], an exact formula for a class of fields  $(v, p)$  satisfying the Stokes equation outside  $S_0$  and with the normal component of  $v$  vanishing on  $\partial S_0$ . We next show that for every small enough value of the the geometric parameter  $e = c^2$  we can choose one of the fields  $(v, p)$  in the above mentioned family such that the tangential trace of  $v$  on  $\partial S_0$  has, up to an error term of order  $O(e)$ , is equal to  $u$  in (4.1).

Let us consider the functions  $\psi_n$  defined by

$$\begin{aligned} \psi_n(\tau, \zeta) = & \frac{\alpha_n}{2(2n-3)} \left[ H_n(\tau_0) \left( \frac{H_n(\tau)}{H_n(\tau_0)} - \frac{H_{n-2}(\tau)}{H_{n-2}(\tau_0)} \right) G_{n-2}(\zeta) \right. \\ & \left. + H_{n-2}(\tau_0) \left( \frac{H_{n-2}(\tau)}{H_{n-2}(\tau_0)} - \frac{H_n(\tau)}{H_n(\tau_0)} \right) G_n(\zeta) \right] \\ & - \frac{\beta_n}{2(2n+1)} \left[ H_{n+2}(\tau_0) \left( \frac{H_{n+2}(\tau)}{H_{n+2}(\tau_0)} - \frac{H_n(\tau)}{H_n(\tau_0)} \right) G_n(\zeta) \right. \\ & \left. + H_n(\tau_0) \left( \frac{H_n(\tau)}{H_n(\tau_0)} - \frac{H_{n+2}(\tau)}{H_{n+2}(\tau_0)} \right) G_{n+2}(\zeta) \right] \quad (\tau \geq \tau_0, \quad \zeta \in [-1, 1]), \quad (4.2) \end{aligned}$$

for  $n \geq 4$  and

$$\begin{aligned} \psi_n(\tau, \zeta) = & \frac{\alpha_n}{2(2n-3)} \left[ H_{n-2}(\tau_0) \left( \frac{H_{n-2}(\tau)}{H_{n-2}(\tau_0)} - \frac{H_n(\tau)}{H_n(\tau_0)} \right) G_n(\zeta) \right] \\ & - \frac{\beta_n}{2(2n+1)} \left[ H_{n+2}(\tau_0) \left( \frac{H_{n+2}(\tau)}{H_{n+2}(\tau_0)} - \frac{H_n(\tau)}{H_n(\tau_0)} \right) G_n(\zeta) \right. \\ & \left. + H_n(\tau_0) \left( \frac{H_n(\tau)}{H_n(\tau_0)} - \frac{H_{n+2}(\tau)}{H_{n+2}(\tau_0)} \right) G_{n+2}(\zeta) \right] \quad (\tau \geq \tau_0, \quad \zeta \in [-1, 1]), \quad (4.3) \end{aligned}$$

for  $n = 2, 3$ , where  $H_n$  are the Gegenbauer functions of the second kind defined by (A.23)–(A.24), where  $G_n$  are the Gegenbauer functions of the first kind defined by (A.17)–(A.18), and where  $\alpha_n$  and  $\beta_n$  are coefficients related to  $G_n$  and  $H_n$  that are defined by (A.35)–(A.36).

**Lemma 4.1.** *Assume  $n \geq 2$  and consider  $\psi_n$  defined by (4.2) with  $v_n$  associated with  $\psi_n$  through (2.12)–(2.13). Then there exists  $p_n$  such that  $(v_n, p_n)$  satisfies (1.9)–(1.10). Moreover,*

$$v_n(\zeta, \tau_0) = -\frac{1}{c^2 \sqrt{(\tau_0^2 - \zeta^2)(1 - \zeta^2)}} [a_n G_{n-2}(\zeta) + b_n G_n(\zeta) + c_n G_{n+2}(\zeta)] e_\zeta \quad (\zeta \in [-1, 1]), \quad (4.4)$$

with

$$a_2 = a_3 = 0 \quad \text{and for } n \geq 4, \quad a_n = -\frac{\alpha_n}{2(2n-3)} H_n(\tau_0) \left( \frac{Q_{n-1}(\tau_0)}{H_n(\tau_0)} - \frac{Q_{n-3}(\tau_0)}{H_{n-2}(\tau_0)} \right), \quad (4.5)$$

$$b_n = -\frac{\alpha_n}{2(2n-3)} H_{n-2}(\tau_0) \left( \frac{\widehat{Q}_{n-3}(\tau_0)}{H_{n-2}(\tau_0)} - \frac{Q_{n-1}(\tau_0)}{H_n(\tau_0)} \right) + \frac{\beta_n}{2(2n+1)} H_{n+2}(\tau_0) \left( \frac{Q_{n+1}(\tau_0)}{H_{n+2}(\tau_0)} - \frac{Q_{n-1}(\tau_0)}{H_n(\tau_0)} \right), \quad (4.6)$$

$$c_n = \frac{\beta_n}{2(2n+1)} H_n(\tau_0) \left( \frac{Q_{n-1}(\tau_0)}{H_n(\tau_0)} - \frac{Q_{n+1}(\tau_0)}{H_{n+2}(\tau_0)} \right). \quad (4.7)$$

Here  $Q_n$  are the Legendre functions of the second kind defined by (A.7)–(A.8), and  $\widehat{Q}_n = Q_n$  for  $n \geq 1$ ,  $\widehat{Q}_0 = 0$ ,  $\widehat{Q}_{-1} = 1$ .

*Proof.* Let us first prove that  $\psi_n$  defined by (4.2) satisfies (2.15). In order to do this, we use (A.22) and (A.28) to deduce that

$$c^2(\tau^2 - \zeta^2) E^2(H_n(\tau) G_m(\zeta)) = (n-m)(m+n-1) H_n(\tau) G_m(\zeta). \quad (4.8)$$

In particular, for  $n \geq 2$ ,

$$c^2(\tau^2 - \zeta^2) E^2(H_n(\tau) G_n(\zeta)) = 0, \quad (4.9)$$

$$c^2(\tau^2 - \zeta^2) E^2(H_{n+2}(\tau) G_n(\zeta)) = 2(2n+1) H_{n+2}(\tau) G_n(\zeta), \quad (4.10)$$

$$c^2(\tau^2 - \zeta^2) E^2(H_n(\tau) G_{n+2}(\zeta)) = -2(2n+1) H_n(\tau) G_{n+2}(\zeta), \quad (4.11)$$

Using (4.9), we obtain, for  $n \geq 4$ ,

$$\begin{aligned} c^2(\tau^2 - \zeta^2)E^2(\psi_n) &= \frac{\alpha_n}{2(2n-3)} [E^2 H_n(\tau)G_{n-2}(\zeta) + E^2 H_{n-2}(\tau)G_n(\zeta)] \\ &\quad - \frac{\beta_n}{2(2n+1)} [E^2 H_{n+2}(\tau)G_n(\zeta) + E^2 H_n(\tau)G_{n+2}(\zeta)] \quad (\tau \geq \tau_0, \quad \zeta \in [-1, 1]). \end{aligned}$$

Then, combining the above equality with (4.10) and (4.11) we obtain

$$\begin{aligned} c^2(\tau^2 - \zeta^2)E^2(\psi_n) &= \alpha_n [H_n(\tau)G_{n-2}(\zeta) - H_{n-2}(\tau)G_n(\zeta)] \\ &\quad - \beta_n [H_{n+2}(\tau)G_n(\zeta) - H_n(\tau)G_{n+2}(\zeta)] \quad (\tau \geq \tau_0, \quad \zeta \in [-1, 1]). \end{aligned}$$

The recurrence relations (A.38) and (A.39) transform the above relation into

$$c^2(\tau^2 - \zeta^2)E^2(\psi_n) = (\zeta^2 - \tau^2)H_n(\tau)G_n(\zeta) \quad (\tau \geq \tau_0, \quad \zeta \in [-1, 1]),$$

from which we deduce (2.15) (using again (4.9)).

For  $n = 2, 3$ , similar calculations give the same result (we use (A.37) instead of (A.38)).

It remains to show (4.4). Since  $\psi_n$  is constant on  $S_0 = \{\tau = \tau_0\}$ , we deduce from (2.12), (2.13) that

$$v_n(\zeta, \tau_0) = -\frac{1}{c^2 \sqrt{(\tau_0^2 - \zeta^2)(1 - \zeta^2)}} \frac{\partial \psi_n}{\partial \tau}(\zeta, \tau_0) e_\zeta.$$

Then for  $n \geq 4$ , we use (4.2) to deduce

$$\begin{aligned} a_n &= \frac{\alpha_n}{2(2n-3)} H_n(\tau_0) \left( \frac{H'_n(\tau_0)}{H_n(\tau_0)} - \frac{H'_{n-2}(\tau_0)}{H_{n-2}(\tau_0)} \right), \\ b_n &= \frac{\alpha_n}{2(2n-3)} H_{n-2}(\tau_0) \left( \frac{H'_{n-2}(\tau_0)}{H_{n-2}(\tau_0)} - \frac{H'_n(\tau_0)}{H_n(\tau_0)} \right) - \frac{\beta_n}{2(2n+1)} H_{n+2}(\tau_0) \left( \frac{H'_{n+2}(\tau_0)}{H_{n+2}(\tau_0)} - \frac{H'_n(\tau_0)}{H_n(\tau_0)} \right), \\ c_n &= -\frac{\beta_n}{2(2n+1)} H_n(\tau_0) \left( \frac{H'_n(\tau_0)}{H_n(\tau_0)} - \frac{H'_{n+2}(\tau_0)}{H_{n+2}(\tau_0)} \right). \end{aligned}$$

Combining (A.27) (since  $n-2 \geq 2$ ) and the three above equations, we deduce (4.5)-(4.7).

For the particular case  $n = 2, 3$ , we get directly  $a_n = 0$  and we also use (A.27) with

$$H'_n = -\widehat{Q}_{n-1}$$

for  $n = 0, 1$ . □

If the shape of the prolate is close to a ball, one can obtain asymptotic formulas of the above results. More precisely, in what follows, we set

$$e := c^2,$$

and we assume  $c\tau_0 = 1$  so that

$$e = \frac{1}{\tau_0^2}.$$

We next study the above formulas and their consequences when  $e \rightarrow 0$ . We first normalize the functions  $\psi_n$  defined in (4.2), (4.3) by setting

$$\widetilde{\psi}_n = \frac{(2n-3)}{K_n e^{(n-3)/2}} \psi_n, \tag{4.12}$$



with  $K_n$  defined by (A.33) and we define the velocity  $\widetilde{v}_n$  associated to  $\widetilde{\psi}_n$  through (2.12)–(2.13).

Let us define

$$C_n(e) = \frac{1}{2} \left( \frac{(n+1)\widetilde{Q}_{n+1}\left(\frac{1}{\sqrt{e}}\right)}{\widetilde{H}_{n+2}\left(\frac{1}{\sqrt{e}}\right)} - \frac{(n-1)\widetilde{Q}_{n-1}\left(\frac{1}{\sqrt{e}}\right)}{\widetilde{H}_n\left(\frac{1}{\sqrt{e}}\right)} \right) \quad (n \geq 0), \quad (4.13)$$

with  $\widetilde{Q}_n$  and  $\widetilde{H}_n$  defined by (A.14), (A.29) and with  $\widetilde{Q}_{-1} = 1$ .

**Lemma 4.2.** *Assume  $v_n$  and  $\psi_n$  are defined as in Lemma 4.1. Then*

$$\widetilde{v}_n(\zeta, \tau_0) = -\frac{1}{\sqrt{(1-e\zeta^2)(1-\zeta^2)}} \left[ \widetilde{a}_n G_{n-2}(\zeta) + \widetilde{b}_n G_n(\zeta) + \widetilde{c}_n G_{n+2}(\zeta) \right] e_\zeta, \quad (4.14)$$

with

$$\widetilde{a}_n = -e\alpha_n \widetilde{H}_n(1/\sqrt{e}) C_{n-2}(e), \quad (4.15)$$

$$\widetilde{b}_n = \widetilde{H}_{n-2}(1/\sqrt{e}) C_{n-2}(e) + e^2 \frac{\beta_n \alpha_{n+2} (2n-3)}{(2n+1)} \widetilde{H}_{n+2}(1/\sqrt{e}) C_n(e), \quad (4.16)$$

$$\widetilde{c}_n = -e \frac{\beta_n (2n-3)}{(2n+1)} \widetilde{H}_n(1/\sqrt{e}) C_n(e). \quad (4.17)$$

*Proof.* Using (A.29), (A.31), (A.14), we obtain that

$$\frac{Q_{n-1}(\tau_0)}{H_n(\tau_0)} = \sqrt{e} \frac{(n-1)\widetilde{Q}_{n-1}(1/\sqrt{e})}{\widetilde{H}_n(1/\sqrt{e})}.$$

Then using the definition (4.13) of  $C_n$  and (A.29), we deduce that  $a_n, b_n, c_n$  defined by (4.5)–(4.7) satisfy for  $n \geq 4$

$$a_n = -\frac{\alpha_n}{2n-3} K_n e^{n/2} \widetilde{H}_n(1/\sqrt{e}) C_{n-2}(e), \quad (4.18)$$

and for  $n \geq 2$

$$b_n = \frac{\alpha_n}{(2n-3)} K_{n-2} e^{(n-2)/2} \widetilde{H}_{n-2}(1/\sqrt{e}) C_{n-2}(e) + \frac{\beta_n}{(2n+1)} K_{n+2} e^{(n+2)/2} \widetilde{H}_{n+2}(1/\sqrt{e}) C_n(e), \quad (4.19)$$

$$c_n = -\frac{\beta_n}{(2n+1)} K_n e^{n/2} \widetilde{H}_n(1/\sqrt{e}) C_n(e). \quad (4.20)$$

By combining (A.39) with (A.29)–(A.32) it follows that

$$K_n = \alpha_n K_{n-2} \quad (n \geq 2). \quad (4.21)$$

Using the above equation, we deduce that

$$b_n = \frac{K_n}{(2n-3)} e^{(n-2)/2} \widetilde{H}_{n-2}(1/\sqrt{e}) C_{n-2}(e) + \frac{\beta_n \alpha_{n+2}}{(2n+1)} K_n e^{(n+2)/2} \widetilde{H}_{n+2}(1/\sqrt{e}) C_n(e). \quad (4.22)$$

Let us note that (4.22) holds true in particular for  $n = 2, 3$  with the definition of  $\widetilde{Q}_n$  and  $\widetilde{H}_n$  for  $n = 0, -1$ .

In particular, we deduce from (4.12) that

$$\begin{aligned} \widetilde{v}_n(\zeta, \tau_0) &= -\frac{1}{\sqrt{(1-e\zeta^2)(1-\zeta^2)}} \frac{1}{\sqrt{e}} \frac{(2n-3)}{K_n e^{(n-3)/2}} [a_n G_{n-2}(\zeta) + b_n G_n(\zeta) + c_n G_{n+2}(\zeta)] e_\zeta \\ &= -\frac{1}{\sqrt{(1-e\zeta^2)(1-\zeta^2)}} \left[ \widetilde{a}_n G_{n-2}(\zeta) + \widetilde{b}_n G_n(\zeta) + \widetilde{c}_n G_{n+2}(\zeta) \right] e_\zeta, \end{aligned}$$

with (4.15)–(4.17). □

Now, we want to consider a linear combination of the  $\tilde{v}_n$  in order to approximate the boundary condition (1.11). More precisely, we will choose  $(B_n)_{n \in \{2, \dots, K\}}$  so that

$$\sum_{n=2}^K B_n \tilde{v}_n$$

is close to velocity produced by the cilia motions, that is

$$-c\sqrt{\tau_0^2 - \zeta^2} \left( \sum_{i=1}^N \beta_i g_i(\chi^{-1}(\arccos \zeta, t)) \right) e_\zeta = -\sqrt{1 - e\zeta^2} \left( \sum_{i=1}^N \beta_i g_i(\chi^{-1}(\arccos \zeta, t)) \right) e_\zeta.$$

In order to do this, we use that  $(G_n)_{n \geq 2}$  is an orthonormal basis of  $L^2(-1, 1)$  (for the weight  $w(x) = 1/(1-x^2)$ ) to write

$$\sqrt{1 - \zeta^2} \left( \sum_{i=1}^N \beta_i g_i(\chi^{-1}(\arccos(\zeta), t)) \right) = \sum_{n=2}^{\infty} A_n G_n(\zeta). \quad (4.23)$$

With this decomposition, the velocity produced by the cilia motions can be written as

$$-c\sqrt{\tau_0^2 - \zeta^2} \left( \sum_{i=1}^N \beta_i g_i(\chi^{-1}(\arccos \zeta, t)) \right) e_\zeta = -\frac{\sqrt{1 - e\zeta^2}}{\sqrt{1 - \zeta^2}} \sum_{n=2}^{\infty} A_n G_n(\zeta) e_\zeta. \quad (4.24)$$

More precisely, one has the following result (whose proof is classical and omitted).

**Lemma 4.3.** *Assume (4.23). Then the coefficients  $A_n$  are given by*

$$A_n = \frac{n(n-1)(2n-1)}{2} \sum_{i=1}^N \beta_i \int_0^\pi g_i(\xi) G_n(\cos(\chi(\xi, t))) \frac{\partial \chi}{\partial \xi}(\xi, t) d\xi \quad (4.25)$$

Let us fix for all what follows

$$K \in \mathbb{N}, \quad K \geq 4. \quad (4.26)$$

Let us define the matrices  $L$  and  $M$  as

$$M_{n,n}(e) = 1 - e\gamma_n \quad (2 \leq n \leq K), \quad (4.27)$$

$$M_{n,n+2}(e) = -e\alpha_{n+2} \quad (2 \leq n \leq K-2), \quad (4.28)$$

$$M_{n,n-2}(e) = -e\beta_{n-2} \quad (4 \leq n \leq K), \quad (4.29)$$

$$M_{n,m}(e) = 0 \quad (|m-n| \notin \{0, 2\}) \quad (4.30)$$

and

$$L_{n,n-2}(e) = -e \frac{2n-7}{2n-3} \beta_{n-2} \tilde{H}_{n-2} \left( \frac{1}{\sqrt{e}} \right) C_{n-2}(e) \quad (4 \leq n \leq K), \quad (4.31)$$

$$L_{n,n+2}(e) = -e\alpha_{n+2} \tilde{H}_{n+2} \left( \frac{1}{\sqrt{e}} \right) C_n(e) \quad (2 \leq n \leq K-2), \quad (4.32)$$

$$L_{n,n}(e) = \tilde{H}_{n-2} \left( \frac{1}{\sqrt{e}} \right) C_{n-2}(e) + e^2 \frac{2n-3}{2n+1} \beta_n \alpha_{n+2} \tilde{H}_{n+2} \left( \frac{1}{\sqrt{e}} \right) C_n(e) \quad (2 \leq n \leq K), \quad (4.33)$$

$$L_{n,m}(e) = 0 \quad (|m-n| \notin \{0, 2\}). \quad (4.34)$$

We notice that  $L(e)$  goes to the identity matrix  $I_K$  as  $e \rightarrow 0$  so that  $L(e)$  is invertible for  $e$  small enough.

The result below provides a relatively simple approximation (for small values of  $e$ ) of the stream function associated to the boundary value problem (2.1).

**Proposition 4.4.** Assume  $\beta \in \mathbb{R}^N$  and let consider  $A = (A_n)_{n=2,\dots,K}$  defined by (4.25). Suppose  $B = (B_n)_{n=2,\dots,K}$  is given by

$$B = L(e)^{-1}M(e)A. \quad (4.35)$$

Then

$$\psi^{(1)} = \sum_{n=2}^K B_n \tilde{\psi}_n$$

is a stream function corresponding to a solution  $(v^{(1)}, p^{(1)})$  of

$$-\mu \Delta v^{(1)} + \nabla p^{(1)} = 0 \text{ in } \mathbb{R}^3 \setminus S_0, \quad \operatorname{div} v^{(1)} = 0 \text{ in } \mathbb{R}^3 \setminus S_0, \quad (4.36)$$

$$\lim_{|x| \rightarrow \infty} v^{(1)} = 0, \quad v^{(1)} = -\frac{\sqrt{1-e\zeta^2}}{\sqrt{1-\zeta^2}} \left( \sum_{n=2}^K A_n G_n(\zeta) + R_K(e) \right) e_\zeta \text{ on } \partial S_0, \quad (4.37)$$

where the remaining term  $R_K(e)$  is such that

$$R_K(e) \in O(e), \quad R_K(e) \in \operatorname{span}\{G_{K+1}, G_{K+2}\}. \quad (4.38)$$

*Proof.* Using (A.37) and (A.38), we deduce that for  $K \geq 4$ ,

$$\zeta^2 \sum_{n=2}^K A_n G_n(\zeta) = \sum_{n=2}^K A_n (\hat{\alpha}_n G_{n-2}(\zeta) + \gamma_n G_n(\zeta) + \beta_n G_{n+2}(\zeta))$$

where  $\hat{\alpha}_n = \alpha_n$  for  $n \geq 4$  and  $\hat{\alpha}_2 = \hat{\alpha}_3 = 0$ . The above relation can be transformed as

$$\zeta^2 \sum_{n=2}^K A_n G_n(\zeta) = \sum_{n=2}^{K-2} A_{n+2} \alpha_{n+2} G_n(\zeta) + \sum_{n=2}^K A_n \gamma_n G_n(\zeta) + \sum_{n=4}^{K+2} A_{n-2} \beta_{n-2} G_n(\zeta). \quad (4.39)$$

In particular, using the definition (4.27)–(4.30) of  $M$ , we deduce

$$\begin{aligned} & \frac{\sqrt{1-e\zeta^2}}{\sqrt{1-\zeta^2}} \sum_{n=2}^K A_n G_n(\zeta) \\ &= \frac{1}{\sqrt{(1-e\zeta^2)(1-\zeta^2)}} \left( \sum_{n=2}^K (MA)_n G_n(\zeta) - eA_{K-1}\beta_{K-1}G_{K+1}(\zeta) - eA_K\beta_K G_{K+2}(\zeta) \right). \end{aligned} \quad (4.40)$$

On the other hand, applying Lemma 4.2, we have on  $\partial S_0$

$$\sum_{n=2}^K B_n \tilde{v}_n = -\frac{1}{\sqrt{(1-e\zeta^2)(1-\zeta^2)}} \sum_{n=2}^K B_n \left[ \tilde{a}_n G_{n-2}(\zeta) + \tilde{b}_n G_n(\zeta) + \tilde{c}_n G_{n+2}(\zeta) \right] e_\zeta,$$

and thus

$$\sum_{n=2}^K B_n \tilde{v}_n = -\frac{1}{\sqrt{(1-e\zeta^2)(1-\zeta^2)}} \left( \sum_{n=2}^K (LB)_n G_n(\zeta) + B_{K-1}\tilde{c}_{K-1}G_{K+1}(\zeta) + B_K\tilde{c}_K G_{K+2}(\zeta) \right).$$

We conclude by using (4.17).  $\square$

## 5 The modified swimming efficiency

In this section, we define and study a notion of efficiency for the prolate. Let us recall that

$$\eta_0 = \cosh^{-1} \left( \frac{1}{c} \right). \quad (5.1)$$

We also recall (see Section 1) that the motion of the points on the swimmers surface is defined, using prolate spheroidal coordinates, by

$$\theta = \chi(\xi, t) = \xi + \sum_{i=1}^N \alpha_i(t) g_i(\xi) \quad (\xi \in [0, \pi], t \geq 0), \quad (5.2)$$

and that the input functions of the considered system are the functions

$$\beta_i(t) = \dot{\alpha}_i(t) \quad (i \in \{1, \dots, N\}).$$

Given the function  $(\beta_i)_{1 \leq i \leq N}$  and an eccentricity  $e$ , the corresponding efficiency  $\text{eff} \left( \beta; \sqrt{e}, \cosh^{-1} \left( \frac{1}{\sqrt{e}} \right) \right)$  is defined by (1.18). For  $K \in \mathbb{N}$  we approximate the efficiency defined in (1.18) by the function  $\begin{bmatrix} \beta \\ e \end{bmatrix} \mapsto \text{eff}_K(\beta, e)$  by

$$\text{eff}_K(\beta; e) = \frac{\int_0^T \int_{\partial S_0} \sigma(v^{(0)}, p^{(0)})_n \cdot \left( \frac{h_0}{T} \right)^2 e_3 \, d\Gamma \, dt}{\int_0^T \int_{\partial S_0} \sigma(\dot{h}(t)v^{(0)} + v^{(1)}, \dot{h}(t)p^{(0)} + p^{(1)})_n \cdot (\dot{h}(t)v^{(0)} + v_K^{(1)}) \, d\Gamma \, dt}, \quad (5.3)$$

where

- $(A_n)_{2 \leq n \leq K}$  is defined from  $\beta$  by (4.25);
- $(v^{(0)}, p^{(0)})$  is the solution of (1.16)–(1.17);
- $(v^{(1)}, p^{(1)})$  is the solution of (4.36)–(4.37) considered in Proposition 4.4;
- $v_K^{(1)} := -\frac{\sqrt{1-e\zeta^2}}{\sqrt{1-\zeta^2}} \sum_{n=2}^K A_n G_n(\zeta) e_\zeta$  is a truncation of the trace of  $v^{(1)}$  on  $\partial S_0$ ;
- $h$  is defined by (3.11).

The approximate character of  $\text{eff}_K$  is due to the replacement of the last occurrence of  $v^{(1)}$  by its truncation  $v_K^{(1)}$  in the definition (1.18) of  $\text{eff} \left( \beta; \sqrt{e}, \cosh^{-1} \left( \frac{1}{\sqrt{e}} \right) \right)$ . This allows us to express  $\text{eff}_K$  using the coefficients  $(A_n)_{2 \leq n \leq K}$  defined by (4.25). To accomplish this goal we first prove the following result.

**Proposition 5.1.** *Let  $(v^{(1)}, p^{(1)})$  be the solution of (4.36)–(4.37) considered in Proposition 4.4 and let  $v_K^{(1)}$  be defined as above. Then*

$$\int_{\partial S_0} \sigma(v^{(1)}, p^{(1)})_n \cdot v_K^{(1)} \, d\Gamma = 4\pi\mu \sum_{n=2}^K \frac{A_n f_n + 2(1-e)A_n^2}{n(n-1)(2n-1)} \quad (5.4)$$

where  $f = MDB$ , where  $B$  is defined by (4.35), where  $M$  is the matrix defined by (4.27)–(4.30) and where

$$D_{mn}(e) = \tilde{H}_n(1/\sqrt{e})(2n-3)\delta_{mn} \quad (m \geq 2, n \geq 2). \quad (5.5)$$

*Proof.* Using (2.11) and the fact that  $v_\tau^{(1)} = 0$  on  $\partial S_0$  it follows that

$$\sigma_{\tau,\zeta}^{(1)}(\tau_0, \zeta, \varphi) = \frac{\mu\sqrt{\tau_0^2-1}}{c\sqrt{\tau_0^2-\zeta^2}} \frac{\partial v_\zeta^{(1)}}{\partial \tau}(\tau_0, \zeta) - \frac{\mu\tau_0\sqrt{\tau_0^2-1}}{c(\tau_0^2-\zeta^2)^{\frac{3}{2}}} v_\zeta^{(1)}(\tau_0, \zeta) \quad (-1 \leq \zeta \leq 1). \quad (5.6)$$

On the other hand, using (2.12) and (5.1), it follows that

$$\sigma_{\tau,\zeta}^{(1)}(\tau_0, \zeta) = -\frac{\mu\sqrt{1-e}}{e(1-e\zeta^2)\sqrt{1-\zeta^2}} \frac{\partial^2 \psi^{(1)}}{\partial \tau^2}(\tau_0, \zeta) - \frac{2\mu\sqrt{1-e}}{(1-e\zeta^2)^{\frac{3}{2}}} v_\zeta^{(1)}(\tau_0, \zeta) \quad (-1 \leq \zeta \leq 1). \quad (5.7)$$

Using (2.9) and the fact that  $n = -e_\tau$ , it follows from the above relation that

$$\begin{aligned} \int_{\partial S_0} \sigma(v^{(1)}, p^{(1)})n \cdot v_K^{(1)} d\Gamma &= -2\pi\mu \frac{1-e}{e} \int_{-1}^1 \frac{1}{1-\zeta^2} \frac{\partial^2 \psi}{\partial \tau^2}(\tau_0, \zeta) \left( \sum_{n=2}^K A_n G_n(\zeta) \right) d\zeta \\ &\quad - 4\pi\mu(1-e) \int_{-1}^1 \frac{v_\zeta^{(1)}(\tau_0, \zeta) \left( \sum_{n=2}^K A_n G_n(\zeta) \right)}{\sqrt{1-e\zeta^2}\sqrt{1-\zeta^2}} d\zeta. \end{aligned} \quad (5.8)$$

For the second integral in the right hand side of (5.8), we use (A.21) and (4.37), (4.38) to get

$$-4\pi\mu(1-e) \int_{-1}^1 \frac{v_\zeta^{(1)}(\tau_0, \zeta) \left( \sum_{n=2}^K A_n G_n(\zeta) \right)}{\sqrt{1-e\zeta^2}\sqrt{1-\zeta^2}} d\zeta = 8\pi\mu(1-e) \sum_{n=2}^K \frac{A_n^2}{n(n-1)(2n-1)}. \quad (5.9)$$

For the first integral in the right hand side of (5.8), we differentiate (4.2) and (4.3) two times with respect to  $\tau$  and we use (A.28):

$$\begin{aligned} \frac{\partial^2 \psi_n}{\partial \tau^2}(\zeta, \tau_0) &= \frac{1}{\tau_0^2-1} \left( \alpha_n H_n(\tau_0) G_{n-2}(\zeta) - \alpha_n H_{n-2}(\tau_0) G_n(\zeta) - \beta_n H_{n+2}(\tau_0) G_n(\zeta) \right. \\ &\quad \left. + \beta_n H_n(\tau_0) G_{n+2}(\zeta) \right) \end{aligned} \quad (5.10)$$

for  $n \geq 4$  and

$$\frac{\partial^2 \psi_n}{\partial \tau^2}(\zeta, \tau_0) = \frac{1}{\tau_0^2-1} \left( -\alpha_n H_{n-2}(\tau_0) G_n(\zeta) - \beta_n H_{n+2}(\tau_0) G_n(\zeta) + \beta_n H_n(\tau_0) G_{n+2}(\zeta) \right) \quad (5.11)$$

for  $n = 2, 3$ . Inserting (A.39) into (5.10) and (5.11) yields that  $\tilde{\psi}_n$  defined by (4.12) satisfies

$$\frac{\partial^2 \tilde{\psi}_n}{\partial \tau^2}(\zeta, \tau_0) = e \frac{2n-3}{1-e} \tilde{H}_n(1/\sqrt{e}) \left( \alpha_n e G_{n-2}(\zeta) + (e\gamma_n - 1) G_n(\zeta) + e\beta_n G_{n+2}(\zeta) \right) \quad (5.12)$$

for  $n \geq 4$  and

$$\frac{\partial^2 \tilde{\psi}_n}{\partial \tau^2}(\zeta, \tau_0) = e \frac{2n-3}{1-e} \tilde{H}_n(1/\sqrt{e}) \left( (e\gamma_n - 1) G_n(\zeta) + e\beta_n G_{n+2}(\zeta) \right) \quad (5.13)$$

for  $n = 2, 3$ . We deduce from (5.12) and (5.13) that

$$\frac{\partial^2 \psi^{(1)}}{\partial \tau^2}(\tau_0, \zeta) = -\frac{e}{1-e} (MDB)_n G_n(\zeta) + \frac{e^2}{1-e} \sum_{n=K-1}^K (2n-3) \tilde{H}_n(1/\sqrt{e}) B_n \beta_n G_{n+2}(\zeta). \quad (5.14)$$

Combining the last formula with (A.21), we obtain that

$$-2\pi\mu \frac{1-e}{e} \int_{-1}^1 \frac{1}{1-\zeta^2} \frac{\partial^2 \psi}{\partial \tau^2}(\tau_0, \zeta) \left( \sum_{n=2}^K A_n G_n(\zeta) \right) d\zeta = 4\pi\mu \sum_{n=2}^K \frac{A_n f_n}{n(n-1)(2n-1)}.$$

Inserting the last formula and (5.9) in (5.8) we obtain the conclusion.  $\square$

The result below provides an useful expression for the efficiency defined in (5.3).

**Proposition 5.2.** *Let  $(\beta_i)_{1 \leq i \leq N}$  be a family of functions in  $L^\infty(0, \infty)$ . Then the swimmer's efficiency, defined by (5.3), is given by:*

$$\text{eff}_K(\beta; e) = \frac{\left( \int_0^T A_2 dt \right)^2}{T \int_0^T \left( -A_2^2 + \frac{9[(1+e)\tanh^{-1}(\sqrt{e}) - \sqrt{e}]}{2e^{\frac{3}{2}}} \sum_{n=2}^K \frac{A_n f_n + 2(1-e)A_n^2}{n(n-1)(2n-1)} \right) dt}. \quad (5.15)$$

*Proof.* We first note that, applying (2.18), the numerator of the right hand side of (1.18) writes

$$\int_0^T \int_{\partial S_0} \sigma(v^{(0)}, p^{(0)})_n \cdot \left( \frac{h_0}{T} \right)^2 e_3 d\Gamma dt = \frac{1}{T} \frac{8\pi\mu c h_0^2}{(\tau_0^2 + 1) \coth^{-1} \tau_0 - \tau_0}.$$

On the other hand, from (3.14) it follows that

$$h_0 = -\frac{c\tau_0}{3} \int_0^T A_2(s) ds.$$

The last two formulas imply that

$$\int_0^T \int_{\partial S_0} \sigma(v^{(0)}, p^{(0)})_n \cdot \left( \frac{h_0}{T} \right)^2 e_3 d\Gamma dt = \frac{1}{9T} \frac{8\pi\mu c^3 \tau_0^2}{(\tau_0^2 + 1) \coth^{-1} \tau_0 - \tau_0} \left[ \int_0^T A_2(s) ds \right]^2. \quad (5.16)$$

To compute the denominator of the right hand side of (1.18) we first note that

$$\int_{\partial S_0} \sigma(\dot{h}v^{(0)} + v^{(1)}, \dot{h}p^{(0)} + p^{(1)})_n \cdot \dot{h}v^{(0)} d\Gamma = \dot{h}e_3 \cdot \left( \int_{\partial S_0} \sigma(\dot{h}v^{(0)} + v^{(1)}, \dot{h}p^{(0)} + p^{(1)})_n d\Gamma \right) = 0 \quad (5.17)$$

from (1.13).

On the other hand, by linearity,

$$\begin{aligned} \int_{\partial S_0} \sigma(\dot{h}v^{(0)} + v^{(1)}, \dot{h}p^{(0)} + p^{(1)})_n \cdot v_K^{(1)} d\Gamma &= \dot{h} \int_{\partial S_0} \sigma(v^{(0)}, p^{(0)})_n \cdot v_K^{(1)} d\Gamma \\ &\quad + \int_{\partial S_0} \sigma(v^{(1)}, p^{(1)})_n \cdot v_K^{(1)} d\Gamma. \end{aligned} \quad (5.18)$$

Using (2.9), that  $n = -e_\tau$ , (2.17), (3.14) and Proposition 5.1, it follows that

$$\begin{aligned} \int_{\partial S_0} \sigma(\dot{h}v^{(0)} + v^{(1)}, \dot{h}p^{(0)} + p^{(1)})_n \cdot v_K^{(1)} d\Gamma &= -\frac{8\pi\mu c^3 \tau_0^2 A_2^2}{9[(1+\tau_0^2)\coth^{-1} \tau_0 - \tau_0]} \\ &\quad + 4\pi\mu \sum_{n=2}^K \frac{A_n f_n + 2(1-e)A_n^2}{n(n-1)(2n-1)}. \end{aligned} \quad (5.19)$$

The last formula, (5.16) and (5.3) clearly imply that

$$\text{eff}_K(\beta; e) = \frac{\frac{8\pi\mu c^3 \tau_0^2 \left(\int_0^T A_2 dt\right)^2}{9T((\tau_0^2 + 1) \coth^{-1} \tau_0 - \tau_0)}}{\int_0^T \left( \frac{-8\pi\mu c^3 \tau_0^2 A_2^2}{9[(1 + \tau_0^2) \coth^{-1} \tau_0 - \tau_0]} + 4\pi\mu \sum_{n=2}^K \frac{A_n f_n + 2\tau_0(\tau_0^2 - 1)c^3 A_n^2}{n(n-1)(2n-1)} \right) dt},$$

which yields the conclusion.  $\square$

## 6 Sensitivity with respect to the focal distance

In this section, we study the behavior of the function  $\text{eff}_K$  defined in (5.3) when  $e \rightarrow 0$ . This means that we consider a family of prolate spheroids like those in Figure 2, i.e., with

$$e = c^2 \rightarrow 0, \quad \cosh \eta_0 = \tau_0 = \frac{1}{c}.$$

An important role in the proofs below will be played by the behavior when  $e \rightarrow 0$  of the matrices  $M$ ,  $L$  and  $D$  defined by (4.27)–(4.30), by (4.31)–(4.34) and by (5.5). Let us define the matrices  $M^{(1)}$ ,  $D^{(0)}$ ,  $D^{(1)}$  and  $L^{(1)}$  by

$$M_{nn}^{(1)} = -\gamma_n, \quad M_{n,n+2}^{(1)} = -\alpha_{n+2}, \quad M_{n+2,n}^{(1)} = -\beta_n \quad (n \geq 2), \quad (6.1)$$

$$M_{n,m}^{(1)} = 0 \quad (n \geq 2, |m-n| \notin \{0, 2\}), \quad (6.2)$$

$$D_{nm}^{(0)} = (2n-3)\delta_{nm}, \quad D_{n,m}^{(1)} = \frac{n(n-1)(2n-3)}{2(2n+1)}\delta_{nm} \quad (n, m \geq 2), \quad (6.3)$$

$$L_{nn}^{(1)} = \frac{n(n-1)}{2(2n+1)}, \quad L_{n,n+2}^{(1)} = -\alpha_{n+2}, \quad L_{n+2,n}^{(1)} = -\frac{2n-3}{2n+1}\beta_n \quad (n \geq 2), \quad (6.4)$$

$$L_{n,m}^{(1)} = 0 \quad (n \geq 2, |m-n| \notin \{0, 2\}). \quad (6.5)$$

**Lemma 6.1.** *With the above notation, we have the following estimates as  $e \rightarrow 0$ :*

$$M(e) = I + eM^{(1)}, \quad (6.6)$$

$$D(e) = D^{(0)} + eD^{(1)} + O(e^2), \quad (6.7)$$

$$L(e) = I + eL^{(1)} + O(e^2). \quad (6.8)$$

We are now in a position to state and to prove our result of dependence of  $\text{eff}_K$  with respect to  $e$ .

**Theorem 6.2.** *The efficiency  $\text{eff}_K$  defined by (5.3) satisfies*

$$\text{eff}_K(e) = \frac{\left(\int_0^T A_2 dt\right)^2}{T \int_0^T \left(2A_2^2 + 6 \sum_{n=3}^K \frac{A_n^2}{n(n-1)}\right) dt} \left(1 + e \text{eff}_K^{(1)} + O(e^2)\right),$$

with  $\text{eff}_K^{(1)} > 0$ .

**Remark 6.3.** *Let us emphasize that the quantity*

$$\frac{\left(\int_0^T A_2 dt\right)^2}{T \int_0^T \left(2A_2^2 + 6 \sum_{n \geq 3} \frac{A_n^2}{n(n-1)}\right) dt}$$

*is the efficiency of the spherical swimmer. More precisely, if  $S_0$  is the sphere centered at the origin and of radius 1, then one can consider the same problem as for the prolate:*

$$-\mu \Delta v + \nabla p = 0 \text{ in } \mathbb{R}^3 \setminus S_0, \quad \operatorname{div} v = 0 \text{ in } \mathbb{R}^3 \setminus S_0, \quad (6.9)$$

$$\lim_{|x| \rightarrow \infty} v = 0, \quad v = \dot{e}_3 + \left( \sum_{i=1}^N \beta_i g_i(\chi^{-1}(\theta, t)) \right) e_\theta \text{ on } \partial S_0, \quad (6.10)$$

$$\dot{\alpha}_i(t) = \beta_i(t) \quad (i \in \{1, \dots, N\}, t \geq 0), \quad (6.11)$$

$$\int_{\partial S_0} \sigma(v, p) n d\Gamma = 0 \quad (t \geq 0), \quad (6.12)$$

$$h(0) = 0, \quad \alpha(0) = 0, \quad (6.13)$$

$$\theta = \chi(\xi, t) = \xi + \sum_{i=1}^N \alpha_i(t) g_i(\xi) \quad (\xi \in [0, \pi], t \geq 0). \quad (6.14)$$

*Then, one can decompose*

$$\sum_{i=1}^N \beta_i g_i(\chi^{-1}(\theta, t)) = \frac{1}{\sin \theta} \sum_{n=2}^{\infty} A_n(t) G_n(\cos \theta) \quad (6.15)$$

*and in that case one can obtain (see, for instance, [14]) that the efficiency defined by (1.18) satisfies*

$$\operatorname{eff}(\beta) = \frac{\left(\int_0^T A_2 dt\right)^2}{T \int_0^T \left(2A_2^2 + 6 \sum_{n \geq 3} \frac{A_n^2}{n(n-1)}\right) dt}. \quad (6.16)$$

*Let us note that in [14], the tangential velocity is written by using the Legendre function of first kind  $\theta \mapsto P_n^1(\cos(\theta))$  ( $n \geq 1$ ), but that the obtained formulas can be seen as a particular case of the corresponding results for the prolate spheroidal case (see (A.5)). Indeed, it suffices to note that, using (A.20) and (A.22), we have*

$$P_n^1(\cos(\theta)) = \frac{n(n+1)}{\sin(\theta)} G_{n+1}(\cos(\theta)) \quad (n \geq 1).$$

*Finally, it is easy to check that the efficiency in (6.16) is less than 1/2 (see [14]). Obtaining such a theoretical upper bound is an interesting open question in the prolate spheroidal case. From the numerical point of view, see Section 7, getting close to the theoretical upper bound seems difficult because of the singular behavior of the optimal stroke (we refer again to [14]).*

*Proof of Theorem 6.2.* Assume  $\beta \in L^\infty((0, T); \mathbb{R}^N)$ . Then  $\alpha$  is given by (1.12) and  $\chi$  by (1.6). As a consequence, we note from (4.25) that the functions  $(A_n)_{n \geq 2}$ , do not depend on  $e$



We next can check by simple calculation that

$$\frac{9 \left[ (1+e) \tanh^{-1}(\sqrt{e}) - \sqrt{e} \right]}{2e^{\frac{3}{2}}} = 6 + \frac{12}{5}e + O(e^2). \quad (6.17)$$

Recalling that  $f = MDB$  (Proposition 5.1) and that  $B = L^{-1}MA$  (see (4.35)), we obtain that

$$f = MDL^{-1}MA.$$

The above relation and (6.6), (6.7), (6.8) yield

$$f = D^{(0)}A + eEA + O(e^2), \quad (6.18)$$

with  $E = M^{(1)}D^{(0)} + D^{(1)} - D^{(0)}L^{(1)} + D^{(0)}M^{(1)}$ .

After some calculations using (6.1), (6.2), (6.3), (6.4) and (6.5), together with (A.35) and (A.36), we obtain

$$E_{nn} = -2\frac{2n^2 - 2n - 3}{2n + 1}, \quad E_{n,n+2} = -\frac{n(n-1)}{2n+3}, \quad E_{n+2,n} = -\frac{(n+1)(n+2)}{2n-1} \quad (n \geq 2), \quad (6.19)$$

$$E_{n,m} = 0 \quad (n \geq 2, |m-n| \notin \{0, 2\}). \quad (6.20)$$

Combining (5.15) with (6.17) and (6.18), we deduce

$$\text{eff}_K(\beta; e) = \frac{\left( \int_0^T A_2 dt \right)^2}{T \int_0^T \left( 2A_2^2 + 6 \sum_{n \geq 3} \frac{A_n^2}{n(n-1)} \right) dt} \left( 1 + \text{eff}_K^{(1)}(\beta) + O(e^2) \right),$$

with

$$\text{eff}_K^{(1)}(\beta) = -\frac{\int_0^T \left( \sum_{n \geq 2} \frac{12}{5} \frac{A_n^2}{n(n-1)} + 6 \frac{A_n(EA)_n - 2A_n}{n(n-1)(2n-1)} \right) dt}{\int_0^T \left( 2A_2^2 + 6 \sum_{n \geq 3} \frac{A_n^2}{n(n-1)} \right) dt}.$$

Some calculation gives

$$\text{eff}_K^{(1)}(\beta) = \frac{\frac{12}{5} \int_0^T \sum_{n=2}^K \frac{A_n^2(6n^2-9)}{n(n-1)(2n-1)(2n+1)} + \sum_{n=2}^{K-2} \frac{5A_n A_{n+2}}{(2n-1)(2n+3)} dt}{\int_0^T \left( 2A_2^2 + 6 \sum_{n \geq 3} \frac{A_n^2}{n(n-1)} \right) dt}. \quad (6.21)$$

In order to conclude, we show now that there exists  $c > 0$  such that for any  $(A_n) \in \mathbb{R}^{K-2}$ ,

$$\sum_{n=2}^K \frac{A_n^2(6n^2-9)}{n(n-1)(2n-1)(2n+1)} + \sum_{n=2}^{K-2} \frac{5A_n A_{n+2}}{(2n-1)(2n+3)} \geq c \sum_{n=2}^K \frac{A_n^2}{(2n-1)^2}. \quad (6.22)$$

To prove (6.22), one can, for instance, perform the change of variables  $\widetilde{A}_n := A_n/(2n-1)$  so that

$$\begin{aligned} \sum_{n=2}^K \frac{A_n^2(6n^2-9)}{n(n-1)(2n-1)(2n+1)} + \sum_{n=2}^{K-2} \frac{5A_n A_{n+2}}{(2n-1)(2n+3)} \\ = \sum_{n=2}^K \frac{\widetilde{A}_n^2(6n^2-9)(2n-1)}{n(n-1)(2n+1)} + \sum_{n=2}^{K-2} 5\widetilde{A}_n \widetilde{A}_{n+2}. \end{aligned} \quad (6.23)$$

Some calculation gives that the sequence  $\kappa_n := \frac{(6n^2-9)(2n-1)}{n(n-1)(2n+1)}$  is such that

$$\begin{aligned} \kappa_2 &> \frac{5}{2}, & \kappa_3 &> \frac{5}{2}, \\ \kappa_n &> 5 & (n &\geq 4). \end{aligned}$$

We deduce from these properties that the quadratic form

$$(\widetilde{A}_n) \mapsto \sum_{n=2}^K \frac{\widetilde{A}_n^2 (6n^2 - 9)(2n - 1)}{n(n-1)(2n+1)} + \sum_{n=2}^{K-2} 5\widetilde{A}_n \widetilde{A}_{n+2}$$

is positive definite. This property, (6.22) and (6.23) imply the result.  $\square$

*Proof of Lemma 6.1.* Relation (6.6) is only a consequence of the definition (4.27)–(4.30) of  $M$ .

Using Lemma A.1, we have

$$\widetilde{H}_n \left( \frac{1}{\sqrt{e}} \right) = 1 + \frac{(n-1)n}{2(2n+1)}e + O(e^2). \quad (6.24)$$

The above relation and (5.5) imply (6.7).

In order to obtain (6.8), we need first several preliminary results. Combining (A.14), (A.27), (A.29), and (A.31), we deduce

$$\widetilde{Q}_n(x) = \widetilde{H}_{n+1}(x) - \frac{x}{n} \widetilde{H}'_{n+1}(x) \quad (x > 1).$$

The above relation and (4.13) yield

$$C_n(e) = 1 - \frac{1}{2\sqrt{e}} \frac{\widetilde{H}'_{n+2}(1/\sqrt{e})}{\widetilde{H}_{n+2}(1/\sqrt{e})} + \frac{1}{2\sqrt{e}} \frac{\widetilde{H}'_n(1/\sqrt{e})}{\widetilde{H}_n(1/\sqrt{e})} \quad (n \geq 2).$$

We deduce from this equality and Lemma A.1 that

$$C_n(e) = \left( 1 + \frac{(2n^2 + 6n + 1)}{(2n + 1)(2n + 5)}e + O(e^2) \right) \quad (n \geq 2). \quad (6.25)$$

One can check that the same formula holds for  $n = 0$  and  $n = 1$ .

Combining (6.25) and (6.24), we obtain that  $L$  satisfies (6.8).  $\square$

## 7 Numerical computation of the prolate spheroidal swimmer efficiency

We first recall the main steps to obtain the efficiency  $\text{eff}_K$  defined in (5.3). We fix  $g_i$ ,  $i = 1, \dots, N$  such that the hypothesis of Theorem 1.2 holds true.

Then, for  $\beta \in L^\infty(0, T; \mathbb{R}^N)$ , we compute

$$\alpha(t) := \int_0^t \beta(s) ds, \quad \chi(\xi, t) = \xi + \sum_{i=1}^N \alpha_i(t) g_i(\xi) \quad (\xi \in [0, \pi], t \geq 0).$$

Then we use formula (4.25)

$$A_n(t) = \frac{n(n-1)(2n-1)}{2} \sum_{i=1}^N \beta_i(t) \int_0^\pi g_i(\xi) G_n(\cos(\chi(\xi, t))) \frac{\partial \chi}{\partial \xi}(\xi, t) d\xi$$

that gives in particular (see (3.14))

$$h(t) = -\frac{1}{3} \int_0^t A_2(s) ds.$$

Then we compute  $L$ ,  $M$  and  $D$  defined by (4.27)–(4.30), by (4.31)–(4.34) and by (5.5) and

$$f = MDL^{-1}MA.$$

Using (5.15), we can finally compute the approximate efficiency

$$\text{eff}_K(\beta; e) = \frac{\left( \int_0^T A_2 dt \right)^2}{T \int_0^T \left( -A_2^2 + \frac{9[(1+e)\tanh^{-1}(\sqrt{e}) - \sqrt{e}]}{2e^{\frac{3}{2}}} \sum_{n=2}^K \frac{A_n f_n + 2(1-e)A_n^2}{n(n-1)(2n-1)} \right) dt}.$$

One of the difficulties in computing the efficiency of the prolate spheroidal swimmer is to evaluate formulas containing  $H_n(1/\sqrt{e})$  which, for small  $e$ , may lead to important numerical errors. Therefore, for small  $e$ , we use the asymptotic formula (derived in Theorem 6.2),

$$\text{eff}_K(\beta; e) \approx \frac{\left( \int_0^T A_2 dt \right)^2}{T \int_0^T \left( 2A_2^2 + 6 \sum_{n \geq 3} \frac{A_n^2}{n(n-1)} \right) dt} \left( 1 + e \text{eff}_K^{(1)}(\beta) \right),$$

with

$$\text{eff}_K^{(1)}(\beta) = -\frac{\int_0^T \left( \sum_{n \geq 2} \frac{12}{5} \frac{A_n^2}{n(n-1)} + 6 \frac{A_n(EA)_n - 2A_n}{n(n-1)(2n-1)} \right) dt}{\int_0^T \left( 2A_2^2 + 6 \sum_{n \geq 3} \frac{A_n^2}{n(n-1)} \right) dt}$$

and with  $E$  given by (6.19)–(6.20).

To evaluate  $\text{eff}_K$ , for given  $\beta$  and  $e$ , we discretize the time interval  $[0, T]$  and the angle interval  $[0, \pi]$  by using uniform meshes of size  $N_T$  and  $N_\xi$ . With this full discretization the original optimal control problem, of input function  $\beta$ , reduces to an optimal control problem where the unknown is a matrix of size  $N \times N_T$ . The constraints in this problem are the injectivity of  $\chi$  and the periodicity of  $\alpha$  (i.e.  $\alpha(T) = 0$ ). This maximization is performed using the IPOPT (Interior Point Optimizer) package.

We take

$$g_i(\xi) = \sin(i\xi), i \geq 1.$$

Table 1 contains the results of numerical computations of the optimal efficiency for various values of the number of scalar controls and with the discretization parameters  $K = 50$ ,  $N_T = 250$ ,  $N_\xi = 500$ . The above values of  $K$ ,  $N_T$  and  $N_\xi$  are large enough (for the considered values of  $N$ ) to make the results stable (not more than 0.05% of modification of the efficiency) with respect to coherent augmentations of these parameters. In the above statement the term ‘‘coherent augmentations’’ signifies that increasing the number  $K$  of basis functions implies (due to high frequency oscillations) a significant augmentation of  $N_\xi$ .

In Figure 3 we describe the evolution of  $h$  with respect to time and the evolution of the cilia (of  $\chi$ ) at several times:  $t = 0$ ,  $t = T/16$ ,  $t = T/8$  et  $t = 3T/16$ .

	$e = 0$	$e = 1.0 \cdot 10^{-3}$	$e = 5.0 \cdot 10^{-3}$	$e = 1.0 \cdot 10^{-2}$	$e = 1.5 \cdot 10^{-2}$
$N = 2$	7.91%	7.92%	7.94%	7.96%	7.99%
$N = 5$	21.02%	21.04%	21.09%	21.15%	21.22%
$N = 13$	30.4%	30.41%	30.49%	30.57%	30.66%

Table 1: Optimal values of  $\text{eff}_K$  obtained for different values of  $N$  and  $e$ .

$N = 2$	$\max  \chi - \text{id}  = 58.43^\circ$
$N = 5$	$\max  \chi - \text{id}  = 102.89^\circ$
$N = 13$	$\max  \chi - \text{id}  = 104^\circ$

Table 2: Maximum displacement for the optimal  $\text{eff}_K$  for different values of  $N$ .

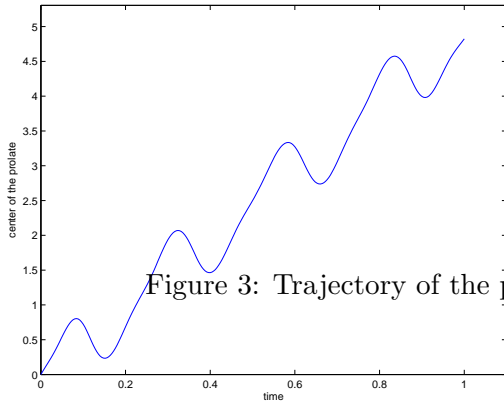


Figure 3: Trajectory of the prolate and evolution of the cilia ( $N = 2$ )

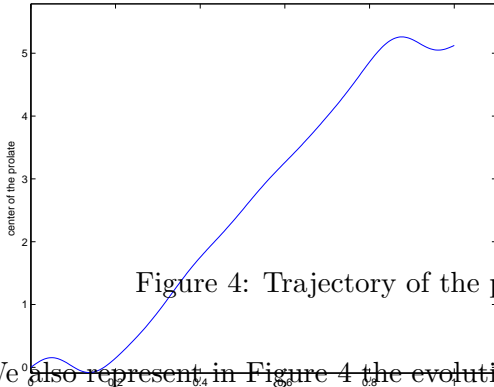
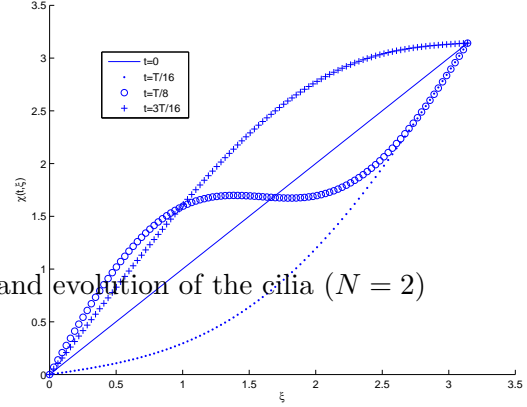
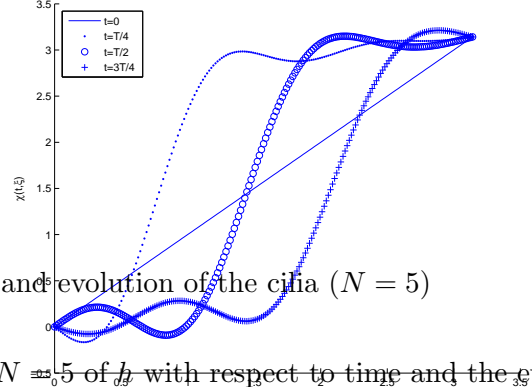


Figure 4: Trajectory of the prolate and evolution of the cilia ( $N = 5$ )



We also represent in Figure 4 the evolution for  $N = 5$  of  $h$  with respect to time and the evolution of the cilia (of  $\chi$ ) at several times:  $t = 0$ ,  $t = T/4$ ,  $t = T/2$  et  $t = 3T/4$ . The same graphics are drawn in Figure 5 for  $N = 13$ .

Our first conclusion is that, in the spherical case, we obtain numerically an efficiency of 30.4%,

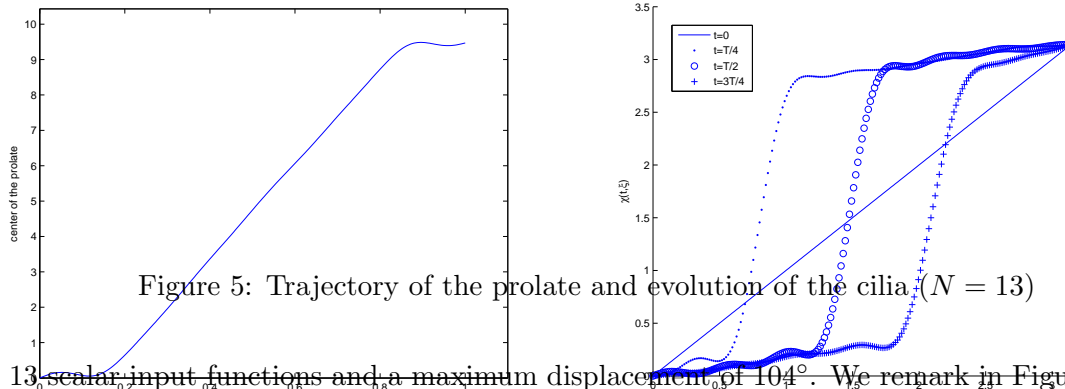


Figure 5: Trajectory of the prolate and evolution of the cilia ( $N = 13$ )

with 13 scalar input functions and a maximum displacement of  $104^\circ$ . We remark in Figure 5, that the obtained displacement field has similar features with the “shock structure”<sup>5</sup> obtained in [14] (where the maximal obtained efficiency is 22% with a maximal displacement  $52.6^\circ$ ). We believe that obtaining numerically, in the spherical case, an efficiency closer to the theoretical bound of 50% requires to increase the number of modes  $N$  and, consequently, refining the discretization. Accomplishing this program by using the full-discretization and the standard optimization algorithms proposed in this paper seems a difficult computational issue.

For the prolate spheroidal ciliate we have shown, both analytically and numerically, that the optimal efficiency increases by small augmentations of the eccentricity  $e$ . It would be interesting to perform similar computations for larger values of  $e$ . From a theoretical view-point, the main difficulty in this case would be to establish that the matrix  $L$  in (4.31)-(4.34) is invertible, whereas computationally one should use “exact” formulas instead of the asymptotic ones in order to evaluate the Gegenbauer functions.

Finally, let us mention that, according to the left pictures in Figures 3, 4, 5 the position of the mass center of the ciliate goes backward at a certain stage of the stroke. We think that this feature should appear independently of the number of modes (although in a less obvious manner, as we can see in Figure 5) and that a mathematical proof of this assertion is an interesting challenge.

## A Some background on Legendre and Gegenbauer functions

We gather in this section some definitions and properties of functions of Legendre and Gegenbauer type. For details on this rich topic, including the proofs of the results we state here, we refer, for instance, to Abramowitz and Stegun [1] or Whittaker and Watson [20].

We first recall some classical special functions. An important role will be played by the Legendre polynomials  $(P_n)_{n \geq 0}$  which are defined by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n] \quad (n \geq 0, x \in \mathbb{R}). \quad (\text{A.1})$$

We have, in particular,

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3x^2 - 1}{2} \quad (x \in \mathbb{R}). \quad (\text{A.2})$$

It is well-known that the Legendre polynomials satisfy the differential equation

$$((1-x^2)P'_n(x))' = -n(n+1)P_n(x) \quad (\text{A.3})$$

and that they satisfy the orthogonality conditions

$$\int_{-1}^1 P_n(x)P_m(x) dx = \frac{2}{2m+1} \delta_{nm} \quad (n, m \geq 1). \quad (\text{A.4})$$

We also need the associated function of the first kind, denoted by  $(P_n^1)_{n \geq 1}$  which are defined by

$$P_n^1(x) = \sqrt{1-x^2}P'_n(x) \quad (n \geq 1, x \in [-1, 1]). \quad (\text{A.5})$$

According to classical results, the functions

$$\theta \mapsto P_n^1(\cos \theta) \quad (n \geq 1),$$

form a complete set in  $L^2[0, \pi]$  and they satisfy the orthogonality conditions

$$\int_0^\pi P_n^1(\cos \theta)P_m^1(\cos \theta) \sin \theta d\theta = \frac{2m(m+1)}{2m+1} \delta_{nm} \quad (n, m \geq 1). \quad (\text{A.6})$$

The Legendre functions of the second kind  $(Q_n)_{n \geq 0}$  can be defined by

$$Q_0(x) = \coth^{-1} x, \quad Q_1(x) = x \coth^{-1} x - 1, \quad (\text{A.7})$$

where

$$\coth^{-1} x = \frac{1}{2} \log \frac{x+1}{x-1} \quad (x \in (1, \infty)),$$

and then recursively by

$$Q_{n+1}(x) = \frac{(2n+1)xQ_n(x) - nQ_{n-1}(x)}{n+1} \quad (n \geq 1). \quad (\text{A.8})$$

In particular, we have

$$Q_2(x) = P_2(x) \coth^{-1} x - \frac{3x}{2}, \quad (\text{A.9})$$

$$Q_3(x) = P_3(x) \coth^{-1} x - \frac{5}{2}x^2 + \frac{2}{3}, \quad (\text{A.10})$$

$$Q_4(x) = P_4(x) \coth^{-1} x - \frac{35}{8}x^3 + \frac{55}{24}x. \quad (\text{A.11})$$

The Legendre functions of the second kind satisfy the recurrence relation

$$(2n+1)Q_n = \frac{d}{dx} (Q_{n+1} - Q_{n-1}) \quad (\text{A.12})$$

and the differential equation

$$\frac{d}{dx} ((1-x^2)Q'_n) = -n(n+1)Q_n. \quad (\text{A.13})$$

The behavior of  $Q_n(x)$  when  $x \rightarrow \infty$  plays an important role in our calculation and it is given by (see, for instance, [20, Section 15.31])

$$Q_n(x) = \frac{M_n}{x^{n+1}} \tilde{Q}_n(x). \quad (\text{A.14})$$

where

$$M_n = \frac{1}{2^n} \int_0^1 (1-t^2)^n dt \quad (n \geq 1), \quad (\text{A.15})$$

and

$$\tilde{Q}_n(x) = 1 + \mathcal{O}\left(\frac{1}{x^2}\right) \quad (x \rightarrow \infty). \quad (\text{A.16})$$

Within this work, the functions  $(\tilde{Q}_n)$  will be called *normalized Legendre functions of the second kind*. The Gegenbauer functions of the first kind  $(G_n)_{n \geq 0}$  are defined by

$$G_0(x) = 1, \quad G_1(x) = -x \quad (x \in \mathbb{R}), \quad (\text{A.17})$$

$$G_n(x) = \frac{P_{n-2}(x) - P_n(x)}{2n-1} \quad (n \geq 2, x \in \mathbb{R}). \quad (\text{A.18})$$

In particular, using the last formula and (A.2), we have

$$G_2(x) = \frac{1-x^2}{2} \quad (x \in \mathbb{R}). \quad (\text{A.19})$$

Among the useful properties of these functions, we recall the formulas

$$G'_n(x) = -P_{n-1}(x) \quad (n \geq 1, x \in \mathbb{R}), \quad (\text{A.20})$$

$$\int_{-1}^1 \frac{G_n(x)G_m(x)}{1-x^2} dx = \frac{2}{n(n-1)(2n-1)} \delta_{nm} \quad (n, m \geq 2), \quad (\text{A.21})$$

$\delta_{nm}$  being the Kronecker delta. In particular, the Gegenbauer polynomials  $(G_n)_{n \geq 2}$  form an orthogonal basis of  $L^2[-1, 1]$  for the scalar product associated to the weight  $w(x) = 1/(1-x^2)$ . Moreover the Gegenbauer functions of the first kind satisfy the differential equation

$$(x^2 - 1)G''_n(x) = n(n-1)G_n(x). \quad (\text{A.22})$$

The Gegenbauer functions of the second kind  $(H_n)_{n \geq 0}$  are defined by

$$H_0(x) = -x, \quad H_1(x) = -1 \quad (x \in \mathbb{R}), \quad (\text{A.23})$$

$$H_n(x) = \frac{Q_{n-2}(x) - Q_n(x)}{2n-1} \quad (n \geq 2, x > 1). \quad (\text{A.24})$$

In particular,

$$H_2(x) = \frac{1-x^2}{2}Q_0(x) + \frac{x}{2} \quad (x > 1). \quad (\text{A.25})$$

By combining (A.24) and (A.8) it follows that

$$H_n(x) = \frac{1}{n}(Q_{n-2}(x) - xQ_{n-1}(x)). \quad (\text{A.26})$$

Among the useful properties of the Gegenbauer functions of the second kind we note that

$$H'_n(x) = -Q_{n-1}(x) \quad (n \geq 2), \quad (\text{A.27})$$

and that  $H_n$  satisfies the differential equation

$$H''_n(x) = \frac{n(n-1)}{x^2-1}H_n(x) \quad (n \geq 0). \quad (\text{A.28})$$

Using formula (A.26) is very inaccurate for large values of  $x$ . In this case we need a formula taking explicitly in consideration the behavior of  $H_n$  when  $x \rightarrow \infty$ . To reach this aim, we combine (A.14) and (A.27) to obtain that

$$H_n(x) = \frac{K_n}{x^{n-1}} \tilde{H}_n(x) \quad (n \geq 0). \quad (\text{A.29})$$

where

$$K_0 = K_1 = -1. \quad (\text{A.30})$$

$$K_n = \frac{M_{n-1}}{n-1} \quad (n \geq 2), \quad (\text{A.31})$$

with  $(M_n)$  defined in (A.15) and

$$\tilde{H}_n(x) = 1 + \mathcal{O}\left(\frac{1}{x^2}\right) \quad (x \rightarrow \infty). \quad (\text{A.32})$$

The functions  $(\tilde{H}_n)$  are called *normalized Gegenbauer functions of the second kind*. Note that

$$K_2 = 1/3 \quad K_{n+1} = \frac{n-1}{2n+1} K_n \quad (n \geq 2) \quad (\text{A.33})$$

It is useful in the context of this work to have a more detailed description of the behavior of  $\tilde{H}_n$  when  $x \rightarrow \infty$ .

**Lemma A.1.** *The normalized Gegenbauer functions of the second kind  $(\tilde{H}_n)_{n \geq 2}$  satisfy*

$$\tilde{H}_n(x) = \sum_{k=0}^{\infty} a_{n,k} x^{-2k} \quad (x > 1),$$

where, for each  $n \geq 2$ , the sequence  $(a_{n,k})_{k \geq 0}$  is defined by  $a_{n,0} = 1$ , together with

$$a_{n,k+1} = \frac{(n+2k-1)(n+2k)}{2(k+1)(2n+2k+1)} a_{n,k} \quad (k \geq 0). \quad (\text{A.34})$$

*Proof.* We know (see, for instance [20, Section 15.31]) that  $Q_n(x)$  admits, for each  $n \geq 0$  an expansion as a power-series in  $x^{-1}$  of the form

$$Q_n(x) = \frac{1}{x^{n+1}} \sum_{k \geq 0} b_{n,k} x^{-2k} \quad (x > 1).$$

Combining the last formula with (A.14), it follows that we have the expansion

$$Q_n(x) = \frac{M_n}{x^{n+1}} \left( 1 + \sum_{k \geq 1} c_{n,k} x^{-2k} \right) \quad (x > 1).$$

The last formula, combined with (A.24) and (A.31), implies that we have the expansion

$$H_n(x) = \frac{K_n}{x^{n-1}} \left( 1 + \sum_{k \geq 1} a_{n,k} x^{-2k} \right) \quad (x > 1).$$

Using the fact that  $H_n$  satisfies the differential equation (A.28) we can see, after some calculation, that  $(a_{n,k})$  satisfy (A.34). Finally, the conclusion is obtained by simply using (A.29).  $\square$



An important role in this work is played by the recurrence relation satisfied by  $(G_n)$  and  $(H_n)$ . These formulas involve the sequences  $(\alpha_n)_{n \geq 2}$ ,  $(\beta_n)_{n \geq 2}$  and  $(\gamma_n)_{n \geq 2}$  defined by

$$\alpha_2 = -\frac{1}{3}, \quad \alpha_3 = -\frac{1}{15}, \quad \alpha_n = \frac{(n-3)(n-2)}{(2n-3)(2n-1)} \quad (n \geq 4), \quad (\text{A.35})$$

$$\beta_n = \frac{(n+1)(n+2)}{(2n-1)(2n+1)}, \quad \gamma_n = \frac{2n^2 - 2n - 3}{(2n+1)(2n-3)} \quad (n \geq 2). \quad (\text{A.36})$$

With the above notation, the recurrence relation for  $G_n$  are

$$x^2 G_n(x) = \gamma_n G_n(x) + \beta_n G_{n+2}(x) \quad (n = 2, 3), \quad (\text{A.37})$$

$$x^2 G_n(x) = \alpha_n G_{n-2}(x) + \gamma_n G_n(x) + \beta_n G_{n+2}(x) \quad (n \geq 4), \quad (\text{A.38})$$

whereas the recurrence relation for  $(H_n)$  writes

$$x^2 H_n(x) = \alpha_n H_{n-2}(x) + \gamma_n H_n(x) + \beta_n H_{n+2}(x) \quad (n \geq 2). \quad (\text{A.39})$$

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