Mathematical
Methods in the
Applied Sciences

Minimization of the ground state of the mixture of two conducting materials in a small contrast regime

Carlos Conca[a], Marc Dambrine[b*†], Rajesh Mahadevan[c] and Duver Quintero[d]

Communicated by H. Ammari

We consider the problem of distributing two conducting materials with a prescribed volume ratio in a given domain so as to minimize the first eigenvalue of an elliptic operator with Dirichlet conditions. The gap between the two conductivities is assumed to be small (low contrast regime). For any geometrical configuration of the mixture, we provide a complete asymptotic expansion of the first eigenvalue. We then consider a relaxation approach to minimize the second-order approximation with respect to the mixture. We present numerical simulations in dimensions two and three to illustrate optimal distributions and the advantage of using a second-order method. Copyright © 2016 John Wiley & Sons, Ltd.

Keywords: shape optimization; eigenvalue problem; homogenization

1. Introduction

Problems of minimizing the ground state of composite materials appear frequently and are of interest in applications. We refer to Henrot [1], Cox and McLaughlin [2, 3], Cox and Lipton[4], and included references. In this article, we consider the following problem. Given a domain Ω and a subdomain B and two nonnegative numbers α and β, we define the ground state $\lambda(B)$ of the mixture as the infimum of the λ such that there exists $0 \neq u$ such that

$$-\operatorname{div}\left((\alpha + (\beta - \alpha)\chi_B)\nabla u\right) = \lambda u \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega. \tag{1.1}$$

In other words, $\lambda(B)$ is the smallest eigenvalue of the operator $-\operatorname{div}\left((\alpha + (\beta - \alpha)\chi_B)\nabla.\right)$ on $H_0^1(\Omega)$. We are then interested in minimizing $\lambda(B)$ with respect to B among the subdomains of Ω of given volume. In general, it is well known that this problem is not well posed: the infimum is not usually reached at a given B, and we have to consider a relaxed version corresponding to a situation of homogenization [4].

Nevertheless, when Ω is a ball, the infimum is reached on a radially symmetric domain B^* [5],[6]. In the recent years, much attention has been put on the determination of the corresponding B^*. First, Conca *et al.* conjectured in [7] that the global minimizer B^* in Ω should be a concentric ball of the prescribed volume. The conjecture was motivated by the situation in dimension one and by numerical simulations. Then, Dambrine and Kateb reinforced the conjecture by an order two sensitivity analysis in [8] by proving that the concentric ball of prescribed volume is a local strict minimizer of $\lambda(B)$.

However, Conca *et. al.* proved in [9] that the conjecture is false. Their strategy was the following. They consider the case of small contrast, that is to say, α and β such that the difference of both conductivities is small: $\beta = \alpha(1 + \varepsilon)$ and provide the first-order asymptotic expansion $\lambda_1(B)$ of $\lambda(B)$ with respect to the small parameter ε for any admissible domain $B \subset \Omega$. Then, they minimize the new objective functional $\lambda_1(B)$ with respect to B and observe that the minimizer B_1 of this approximation is not always the concentric ball of prescribed volume. Finally, thanks to a precise estimate of the remainder in the approximation, they prove that $\lambda(B_1) < \lambda(B^*)$.

Finally, Laurain proved in [10] that the global minimum of the first eigenvalue in low contrast regime is either a centered ball or the union of a centered ball and of a centered ring touching the boundary, depending on the prescribed volume ratio between the two

a Department of Engineering Mathematics, Center for Mathematical Modelling, Universidad de Chile, Santiago, Chile
b Department of Mathematics, University of Pau, Pau, France
c Departamento de Matematica, Universidad de Concepciòn, Concepciòn, Chile
d Department of Engineering Mathematics, Center for Mathematical Modelling, Universidad de Chile, Santiago, Chile
* Correspondence to: Marc Dambrine, Department of Engineering Mathematics, Center for Mathematical Modelling, Universidad de Chile, Santiago, Chile.
† E-mail: marc.dambrine@univ-pau.fr

materials. Thus, the small contrast case is well understood when the domain is a ball. Notice that in the large contrast case and the case of some volume defects, the asymptotic expansion can be found in [11, 12].

We aim in this work to make a precise analysis of the small contrast case in general domains. In Section 2, to begin with, we characterize completely the full asymptotic expansion of $\lambda(B)$ with respect to the small parameter ε. Subsequently, we obtain a second-order approximation $\lambda_2(B)$ of $\lambda(B)$ with uniform estimates for the remainder, uniform with respect to B. This means that minimizers for the second-order approximation $\lambda_2(B)$ are approximate minimizers for the original objective functional $\lambda(B)$. With this motivation, in Section 3, we study the problem of minimizing λ_2. Unlike the first-order approximation $\lambda_1(B)$, the minimization problem for $\lambda_2(B)$ is not, *a priori*, well posed and thus, qualitatively, resembles more closely the minimization problem for $\lambda(B)$. A relaxed formulation for the minimization problem for $\lambda_2(B)$ is obtained using *H*-measures. It can be seen that the relaxed problem for $\lambda_2(B)$ has a much more simple aspect compared with the relaxed problem for $\lambda(B)$ obtained in Cox and Lipton [4]. Then, in Section 4, the optimality conditions for the relaxed problem for $\lambda_2(B)$ are obtained and the minimization problem is studied numerically using a descent algorithm. Finally, we present a numerical comparison of optimal solution obtained for the first and second order.

2. Asymptotic expansion of the first eigenvalue with respect to the contrast

We consider the low contrast regime, that is to say, α and β such that the difference of both conductivities is small: $\beta = \alpha(1 + \varepsilon)$. We shall denote the first eigenvalue in the problem (1.1) by $\lambda_\varepsilon(B)$ for a given distribution B of the material with conductivity β and a given value of the contrast parameter $\varepsilon > 0$.

The existence of an asymptotic development for $\lambda_\varepsilon(B)$, for given B, is classical from perturbation theory of simple eigenvalues. By the Krein–Rutman theorem, the first eigenvalue $\lambda_\varepsilon(B)$ in (1.1) is simple. The corresponding normalized eigenfunction, with unit L^2 norm and taken to be nonnegative, will be denoted by $u_\varepsilon(B)$. So, by classical results from perturbation theory (see, for instance, Theorem 3, Chapter 2.5 of Rellich [13]), for a given B, the map $\varepsilon \mapsto (\lambda_\varepsilon, u_\varepsilon)$ is analytic in $(\mathbb{R}, H_0^1(\Omega))$. Therefore, there are sequences (λ_i) of real numbers and (u_i) of functions in $H_0^1(\Omega)$ such that

$$\lambda_\varepsilon = \sum_{i=0}^\infty \lambda_i \varepsilon^i \text{ and } u_\varepsilon = \sum_{i=0}^\infty u_i \varepsilon^i. \tag{2.1}$$

As a consequence, there are constants $C_n(B)$ such that

$$\left| \lambda_\varepsilon - \sum_{i=0}^n \lambda_i \varepsilon^i \right| \le C_n(B)\varepsilon^{n+1} \text{ and } \left\| u_\varepsilon - \sum_{i=0}^n u_i \varepsilon^i \right\|_{H_0^1} \le C_n(B)\varepsilon^{n+1}.$$

In this section, we will first identify the coefficients λ_i, u_i then prove that the constants $C_n(B)$ can be taken uniform in B. This will serve in obtaining an approximate model problem for the eigenvalue minimization problem.

2.1. Computation of the coefficients in (2.1)

The terms in the asymptotic expansions in (2.1) may be identified, formally, by injecting the expansions in the equations defining $(\lambda_\varepsilon, u_\varepsilon)$, that is,

$$-\text{div}\left(\alpha(1 + \chi_B \varepsilon)\nabla\left(\sum_{i=0}^\infty u_i \varepsilon^i\right)\right) = \left(\sum_{i=0}^\infty \lambda_i \varepsilon^i\right)\left(\sum_{i=0}^\infty u_i \varepsilon^i\right) \text{ in } \Omega,$$

$$\sum_{i=0}^\infty u_i \varepsilon^i = 0 \text{ on } \partial\Omega,$$

$$\int_\Omega \left(\sum_{i=0}^\infty u_i \varepsilon^i\right)^2 = 1.$$

and we obtain, then, the following relationships by identifying the coefficients of the same order in the previous power series.

$$\begin{cases} -\alpha\Delta u_0 - \lambda_0 u_0 = 0 \text{ in } \Omega \\ \qquad\qquad u_0 = 0 \text{ on } \partial\Omega \quad \forall i \ge 0, \\ \int_\Omega u_0^2 = 1. \end{cases} \tag{2.2}$$

$$\begin{cases} -\alpha\Delta u_i - \lambda_0 u_i = \text{div}\,(\alpha\chi_B \nabla u_{i-1}) + \sum_{k=1}^i \lambda_k u_{i-k} \text{ in } \Omega \quad \forall i \ge 1, \\ \qquad\qquad u_i = 0 \text{ on } \partial\Omega \quad \forall i \ge 0, \\ \sum_{k=0}^i \int_\Omega u_k u_{i-k} = 0 \quad \forall i \ge 1. \end{cases} \tag{2.3}$$

It is possible to rigorously justify the relations by using the expansions (2.1) in the weak formulation of the partial differential equation in (1.1). We then have an iterative procedure to compute the pair (λ_i, u_i).

The case: $i = 0$**.** By definition, one has

$$-\alpha\Delta u_0 - \lambda_0 u_0 = 0 \text{ in } \Omega \tag{2.4}$$

$$u_0 = 0 \text{ on } \partial\Omega. \tag{2.5}$$

Hence, the couple (λ_0, u_0) is an eigenpair of $-\alpha\Delta$ with homogeneous Dirichlet boundary condition. Clearly, $u_0 \geq 0$ in Ω because $u_\varepsilon \to u_0$ as $\varepsilon \to 0$, and the eigenmodes u_ε are nonnegative. Now, by the Krein–Rutman theorem, because all eigenmodes change sign except those associated with the first eigenvalue, we obtain that λ_0 is the ground state of $-\alpha\Delta$ with Dirichlet boundary condition and u_0 is the positive eigenmode with L²-norm 1.

Now assume that, for a given i, we have knowledge of all the λ_k, u_k for $k < i$. We now then treat

The case $k = i$**.** We know that u_i satisfies the equation

$$-\alpha\Delta u_i - \lambda_0 u_i = \text{div}\,(\alpha\chi_B\nabla u_{i-1}) + \sum_{k=1}^{i}\lambda_k u_{i-k} \text{ in } \Omega, \tag{2.6}$$

$$u_i = 0 \text{ on } \partial\Omega,$$

Notice that the right-hand side has the unknown quantity λ_i. We shall first obtain an expression for λ_i in terms of λ_k's and u_k's for $k < i$, which have been assumed to be calculated previously. The compatibility condition, the Fredholm alternative for Equation (2.6), imposes the orthogonality of the right-hand side of the former equation to the kernel of $-\alpha\Delta - \lambda_0 I$ with Dirichlet boundary condition, which is spanned by u_0

$$\int_\Omega\left(\text{div}\,(\alpha\chi_B\nabla u_{i-1}) + \sum_{k=1}^{i}\lambda_k u_{i-k}\right)u_0 = 0.$$

This gives the expression for the eigenvalue λ_i

$$\lambda_i = \int_B\alpha\nabla u_{i-1}\cdot\nabla u_0 - \sum_{k=2}^{i-1}\int_\Omega\lambda_{i-k}u_0 u_k \tag{2.7}$$

taking into account the fact that the L^2 norm of u_0 is 1 and u_0 and u_1 are orthogonal. In the sequel, whenever there is a sum whose upper limit is less than the lower limit, we shall adopt the convention that the sum is 0.

Now, to end, we note that u_i is not completely determined by Equation (2.6), but only up to the kernel of $-\alpha\Delta - \lambda_0 I$. For $i = 0$, the nonnegativity of u_0 and the normalization condition (the third relation in (2.2)) determined uniquely the u_0. For general i, having determined uniquely the u_k for $k < i$, the term u_i is determined uniquely using the normalization condition (the third relation in (2.3), which can be written as

$$\int_\Omega u_i u_0 = -\frac{1}{2}\sum_{k=1}^{i-1}\int_\Omega u_k u_{i-k}. \tag{2.8}$$

and should be understood as the orthogonality relation $\int_\Omega u_i u_0 = 0$ when $i = 1$.

2.2. Uniform estimate of the remainders

We seek to estimate the remainder in the expansions (2.1), uniformly in B. Our main results in this section are the following estimates.

Proposition 2.1
There exists a constant C, independent of B, such that

$$|\lambda_\varepsilon - (\lambda_0 + \varepsilon\lambda_1)| \leq \sqrt{\frac{\lambda_0}{\alpha}}C\varepsilon^2. \tag{2.9}$$

Proposition 2.2
There is a constant $C > 0$ independent of B such that

$$|\lambda_\varepsilon - (\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2)| \leq 2\,C\varepsilon^3\sqrt{\frac{\lambda_0}{\alpha}}. \tag{2.10}$$

The main tool we use for the estimation of the remainders is the notion of $h - quasimode$ with $h = \mathcal{O}(\varepsilon^k)$, for $k = 1, 2$ in the sequel. The notion of quasimode is defined as follows.

Definition 2.3
Let A be a self-adjoint operator on a Hilbert space H with domain $D(A)$. For a fixed $h > 0$, a pair $(\lambda, u) \in \mathbb{R} \times D(A) \setminus \{0\}$ is called an h-quasimode if we have

Math. Meth. Appl. Sci. **2016**, 39 3549–3564

3551

$$\|(A - \lambda)u\|_H \le h\|u\|_H.$$

The interest of such a definition relies on the following fact: if (λ, u) is an *h-quasimode* of A, then the distance from λ to the spectrum of A is less than h and the distance between u and certain eigenspaces of A can be estimated (See Lemma 2-2 in [14]). We will prove that our truncated power series expansions are quasimodes in the Hilbert space $H^{-1}(\Omega)$.

2.2.1. Remainder of order one.   The first step is to prove a uniform bound in B of $\|u\|_{H^1(\Omega)}$.

Lemma 2.4
There exists C, which is independent of B, such that

$$\|u_1\|_{H_0^1(\Omega)} \le C \text{ and } |\lambda_\varepsilon - \lambda_0| \le C\varepsilon. \tag{2.11}$$

Proof of Lemma 2.4
By using (2.7), with $i = 1$, we have the following expression and uniform bounds for $\lambda_1(B)$

$$\lambda_1 = \int_B \alpha |\nabla u_0|^2 \le \alpha \int_\Omega |\nabla u_0|^2 = \lambda_0. \tag{2.12}$$

By (2.3), for $i = 1$, u_1 satisfies the following:

$$-\alpha \Delta u_1 - \lambda_0 u_1 = \text{div} (\alpha \chi_B \nabla u_0) + \lambda_1 u_0 \text{ in } \Omega, \tag{2.13}$$

$$u_1 = 0 \text{ on } \partial\Omega, \tag{2.14}$$

$$\int_\Omega u_0 u_1 = 0. \tag{2.15}$$

After multiplying the first relation by u_1 and integrating over Ω, by integration by parts, we obtain

$$\int_\Omega \alpha |\nabla u_1|^2 - \lambda_0 \int_\Omega u_1^2 = - \int_B \alpha \nabla u_0 \cdot \nabla u_1.$$

By the characterization of the spectrum of an elliptic self-adjoint operator using the Rayleigh's quotient, we know that for all v in $H_0^1(\Omega)$ orthogonal to the first eigenfunction u_0, it holds that

$$\lambda^1 \int_\Omega v^2 \le \alpha \int_\Omega |\nabla v|^2, \tag{2.16}$$

where $\lambda^1 > \lambda_0$ is the second eigenvalue of $-\alpha \Delta$ in $H_0^1(\Omega)$. We have used the superscript here to distinguish the second eigenvalue λ^1 from λ_1 which appears in the second term of the expansion (2.1). Because u_1 is orthogonal to u_0, it follows using (2.16) that

$$\alpha \left(1 - \frac{\lambda_0}{\lambda^1}\right) \int_\Omega |\nabla u_1|^2 \le \int_\Omega \alpha |\nabla u_1|^2 - \lambda_0 \int_\Omega u_1^2 \le \alpha \|u_0\|_{H_0^1(\Omega)} \|u_1\|_{H_0^1(\Omega)} \tag{2.17}$$

where at the end, we have used (2.13) and followed it by a simple estimation. We have obtained the upper bound for u_1. Finally, using the variational characterization of the first eigenvalue for elliptic self-adjoint operators, we obtain

$$\lambda_0 = \int_\Omega \alpha |\nabla u_0|^2 \le \int_\Omega \alpha |\nabla u_\varepsilon|^2 \le \int_\Omega \alpha(1 + \chi_B \varepsilon)|\nabla u_\varepsilon|^2 = \lambda_\varepsilon$$

$$\le \int_\Omega \alpha(1 + \chi_B \varepsilon)|\nabla u_0|^2 \le (1 + \varepsilon) \int_\Omega \alpha |\nabla u_0|^2 = (1 + \varepsilon)\lambda_0$$

which allows us to conclude that $|\lambda_\varepsilon - \lambda_0| \le C\varepsilon$. □

To use the quasimode strategy, we compute

$$
\begin{aligned}
-\text{div} &(\alpha(1 + \chi_B \varepsilon)\nabla(u_0 + \varepsilon u_1)) - (\lambda_0 + \varepsilon\lambda_1)(u_0 + \varepsilon u_1) \\
&= -\alpha\Delta u_0 - \lambda_0 u_0 + \varepsilon \left(-\alpha\Delta u_1 - \lambda_0 u_1 - \lambda_1 u_0 - \text{div} (\alpha \chi_B \nabla u_0)\right) \\
&\quad + \varepsilon^2 \left(-\lambda_1 u_1 - \text{div} (\alpha \chi_B \nabla u_1)\right) \\
&= \varepsilon^2 \left(-\lambda_1 u_1 - \text{div} (\alpha \chi_B \nabla u_1)\right)
\end{aligned} \tag{2.18}
$$

where we have used (2.4) and (2.6) with $i = 1$.

Proof of Proposition 2.1
We need a uniform bound on the normalized right-hand side: $\lambda_1 u_1 + \text{div}\,(\alpha \chi_B \nabla u_1)$. Obviously, this term is only defined in $H^{-1}(\Omega)$; hence, we have to make the estimation in the $H^{-1}(\Omega)$ norm. To that end, we use a test function $\varphi \in H_0^1(\Omega)$ and compute the duality product:

$$\langle -\text{div}\,(\alpha \chi_B \nabla u_1), \varphi \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = \int_\Omega \alpha \chi_B \nabla u_1 \cdot \nabla \varphi = \int_B \alpha \nabla u_1 \cdot \nabla \varphi$$
$$\leq \alpha \|u_1\|_{H_0^1(\Omega)} \|\varphi\|_{H_0^1(\Omega)}.$$

This proves that

$$\|\text{div}\,\alpha(\chi_B \nabla u_1)\|_{H^{-1}(\Omega)} \leq \alpha \|u_1\|_{H_0^1(\Omega)}. \tag{2.19}$$

And

$$\langle \lambda_1 u_1, \varphi \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = \int_\Omega \lambda_1 u_1 \varphi \leq \lambda_1 \|u_1\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)}$$
$$\leq \lambda_1 \|u_1\|_{H_0^1(\Omega)} \|\varphi\|_{H_0^1(\Omega)} \leq C \|\varphi\|_{H_0^1(\Omega)}$$

using the estimation (2.12) and the fact that u_1 is bounded independently of B proved in Lemma 2.4. This gives

$$\|\lambda_1 u_1\|_{H^{-1}(\Omega)} \leq C. \tag{2.20}$$

Hence, we obtain from (2.18), using (2.19) and (2.20) that there exists a constant C independent of B such that

$$\| -\text{div}\,(\alpha(1 + \chi_B \varepsilon) \nabla(u_0 + \varepsilon u_1)) - (\lambda_0 + \varepsilon \lambda_1)(u_0 + \varepsilon u_1) \|_{H^{-1}(\Omega)} \leq C \varepsilon^2 \tag{2.21}$$

Moreover, using $u_0 \in H_0^1$ as test function in the definition of the H^{-1}-norm of $u_0 + \varepsilon u_1$, we obtain

$$\|u_0 + \varepsilon u_1\|_{H^{-1}(\Omega)} = \sup_{\varphi \in H_0^1(\Omega)} \frac{\langle u_0 + \varepsilon u_1, \varphi \rangle_{H^{-1}, H_0^1}}{\|\varphi\|_{H_0^1(\Omega)}} = \sup_{\varphi \in H_0^1(\Omega)} \frac{\int_\Omega (u_0 + \varepsilon u_1)\varphi}{\|\varphi\|_{H_0^1(\Omega)}}$$
$$\geq \frac{\int_\Omega (u_0 + \varepsilon u_1)u_0}{\|u_0\|_{H_0^1(\Omega)}} = \frac{\int_\Omega u_0^2}{\left(\int_\Omega |\nabla u_0|^2\right)^{\frac{1}{2}}} = \sqrt{\frac{\alpha}{\lambda_0}}. \tag{2.22}$$

Hence, by (2.21) and (2.22), we obtain

$$\| -\text{div}\,(\alpha(1 + \chi_B \varepsilon)\nabla(u_0 + \varepsilon u_1)) - (\lambda_0 + \varepsilon \lambda_1)(u_0 + \varepsilon u_1)\|_{H^{-1}(\Omega)} \leq \sqrt{\frac{\lambda_0}{\alpha}}\, C \varepsilon^2 \|u_0 + \varepsilon u_1\|_{H^{-1}(\Omega)}$$

As a consequence of the theory of quasimode, there is an element of the spectrum of the self-adjoint operator $-\text{div}(\alpha(1 + \chi_B \varepsilon)\nabla \cdot)$ in $H^{-1}(\Omega)$ at distance at most $\sqrt{\frac{\lambda_0}{\alpha}}\, C \varepsilon^2$ from $\lambda_0 + \varepsilon \lambda_1$. To finish, we need to argue that this element of the spectrum is λ_ε, the first eigenvalue of $-\text{div}\,(\alpha(1 + \chi_B \varepsilon)\nabla \cdot)$. If these were higher eigenvalues, then as $\varepsilon \to 0$, they would tend to a higher eigenvalue of the operator $-\alpha \Delta$. But this would lead to a contradiction, because this sequence is within a distance $O(\varepsilon^2)$ from the sequence $\lambda_0 + \varepsilon \lambda_1$ which tends to λ_0, the first eigenvalue of $-\alpha \Delta$, which is simple. $\qquad \square$

2.2.2. Remainder of order two. We first prove a uniform upper bound for λ_2 and u_2.

Lemma 2.5
There exists C, which is independent of B, such that

$$\|u_2\|_{H_0^1(\Omega)} \leq C \text{ and } \lambda_2 \leq C. \tag{2.23}$$

Proof of Lemma 2.5
First, notice that by (2.7) applied with $i = 2$, we obtain

$$\lambda_2 = \int_B \alpha \nabla u_0 \cdot \nabla u_1 \leq \alpha \|u_0\|_{H_0^1(\Omega)} \|u_1\|_{H_0^1(\Omega)} \leq C \tag{2.24}$$

where C is independent of B by the estimate (2.11). In a second step, we search a uniform estimate for u_2. To that end, we follow the strategy already used to estimate u_1. The main change is that u_2 is not orthogonal to u_0, so the adaptation is not straightforward. To overcome the difficulty, we introduce the combination $u_2 + au_0$ where

Math. Meth. Appl. Sci. **2016**, 39 3549–3564

3553

$$a = -\int_\Omega u_2 u_0$$

is chosen such that $u_2 + au_0$ is $L^2(\Omega)$-orthogonal to u_0.

By (2.8) for $i = 2$, we have

$$\int_\Omega u_2 u_0 = -\frac{1}{2}\int_\Omega u_1^2 \tag{2.25}$$

which gives

$$a = \frac{1}{2}\int_\Omega u_1^2 \leq \frac{1}{2}\|u_1\|_{H^1(\Omega)}^2 \leq C \tag{2.26}$$

with C independent of B (by (2.11)). We now estimate $u_2 + au_0$. For this, we multiply Equation (2.4) by a and add it to Equation (2.6) to obtain

$$-\alpha\Delta(u_2 + au_0) - \lambda_0(u_2 + au_0) = \text{div}\,(\alpha\chi_B\nabla u_1) + \lambda_1 u_1 + \lambda_2 u_0, \quad \text{in } \Omega$$
$$u_2 + au_0 = 0 \quad \text{on } \partial\Omega$$

Using $u_2 + au_0$ as a test function, it follows that

$$\begin{aligned}
\alpha\int_\Omega |\nabla(u_2 + au_0)|^2 &- \lambda_0\int_\Omega (u_2 + au_0)^2 \\
&= -\int_B \alpha\nabla u_1 \cdot \nabla(u_2 + au_0) + \int_\Omega \lambda_1 u_1(u_2 + au_0) + \int_\Omega \lambda_2 u_0(u_2 + au_0) \\
&\leq \left(\alpha\|u_1\|_{H_0^1(\Omega)} + \lambda_1\|u_1\|_{H_0^1(\Omega)} + |\lambda_2|\,\|u_0\|_{H_0^1(\Omega)}\right)\|u_2 + au_0\|_{H_0^1(\Omega)} \\
&\leq C\|u_2 + au_0\|_{H_0^1(\Omega)}
\end{aligned} \tag{2.27}$$

where C is independent of B, by estimates (2.11), (2.12), and (2.24). Because $u_2 + au_0$ is orthogonal to u_0, similarly as in the estimation (2.17), we conclude that $u_2 + au_0$ is bounded in $H_0^1(\Omega)$ uniformly in B. Therefore,

$$\|u_2\|_{H_0^1(\Omega)} \leq C + a\|u_0\|_{H_0^1(\Omega)} \leq C'$$

with C' independent of B by estimate (2.26). □

Proof of Proposition 2.2

We compute

$$\begin{aligned}
&-\text{div}\,\left(\alpha(1 + \chi_B\varepsilon)\nabla(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)\right) - (\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2) \\
&= -\alpha\Delta u_0 - \lambda_0 u_0 + \varepsilon\left(-\alpha\Delta u_1 - \lambda_0 u_1 - \lambda_1 u_0 - \text{div}\,(\alpha\chi_B\nabla u_0)\right) \\
&\quad + \varepsilon^2\left(-\alpha\Delta u_2 - \lambda_0 u_2 - \lambda_1 u_1 - \lambda_2 u_0 - \text{div}\,(\alpha\chi_B\nabla u_1)\right) \\
&\quad + \varepsilon^3\left(-\lambda_1 u_2 - \lambda_2 u_1 - \text{div}\,(\alpha\chi_B\nabla u_2)\right) + \varepsilon^4(\lambda_2 u_2) \\
&= \varepsilon^3\left(-\lambda_1 u_2 - \lambda_2 u_1 - \text{div}\,(\alpha\chi_B\nabla u_2)\right) + \varepsilon^4(\lambda_2 u_2)
\end{aligned} \tag{2.28}$$

using Equations (2.4) and (2.6) for $i = 1, 2$. Then, as

$$\|-\text{div}\,(\alpha\chi_B\nabla u_2)\|_{H^{-1}(\Omega)} \leq \alpha\,\|u_2\|_{H_0^1(\Omega)},$$

it follows from Equation (2.28) and estimates (2.11), (2.12), and (2.23), that for $\varepsilon \ll 1$,

$$\begin{aligned}
&\|\text{div}\,\left(\alpha(1 + \chi_B\varepsilon)\nabla(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)\right) + (\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)\|_{H^{-1}(\Omega)} \\
&\leq \left((\alpha + \lambda_1)\|u_2\|_{H_0^1(\Omega)} + |\lambda_2|\,\|u_1\|_{H_0^1(\Omega)}\right)\varepsilon^3 + (|\lambda_2|\,\|u_2\|_{H_0^1(\Omega)})\varepsilon^4 \\
&\leq C_1\varepsilon^3 + C_2\varepsilon^4 \leq C\varepsilon^3,
\end{aligned} \tag{2.29}$$

Moreover, one has

$$\begin{aligned}
\|u_0 + \varepsilon u_1 + \varepsilon^2 u_2\|_{H^{-1}(\Omega)} &= \sup_{\varphi\in H_0^1(\Omega)} \frac{\int_\Omega (u_0 + \varepsilon u_1 + \varepsilon^2 u_2)\varphi}{\|\varphi\|_{H_0^1(\Omega)}} \geq \frac{\int_\Omega (u_0 + \varepsilon u_1 + \varepsilon^2 u_2)u_0}{\|u_0\|_{H_0^1(\Omega)}} \\
&= \frac{\int_\Omega u_0^2 + \varepsilon^2\int_\Omega u_0 u_2}{\|u_0\|_{H_0^1(\Omega)}}.
\end{aligned}$$

Then, using relation (2.25), we obtain

$$\|u_0 + \varepsilon u_1 + \varepsilon^2 u_2\|_{H^{-1}(\Omega)} \geq \frac{1 - \frac{\varepsilon^2}{2}\int_\Omega u_1^2}{\|u_0\|_{H_0^1(\Omega)}} \geq \frac{1 - \frac{\varepsilon^2}{2}C^2}{\|u_0\|_{H_0^1(\Omega)}},$$

because u_1 is bounded in $H_0^1(\Omega)$ and consequently, in $L^2(\Omega)$ as shown in (2.11). For $\varepsilon < \frac{1}{C}$, we obtain

$$\|u_0 + \varepsilon u_1 + \varepsilon^2 u_2\|_{H^{-1}(\Omega)} \geq \frac{1}{2\|u_0\|_{H_0^1(\Omega)}} = \frac{1}{2}\sqrt{\frac{\alpha}{\lambda_0}} \tag{2.30}$$

By (2.29) and (2.30), we then have for $\varepsilon < 1/C$ small enough

$$\|-\text{div}\left(\alpha(1+\chi_B\varepsilon)\nabla(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)\right) - (\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)\|_{H^{-1}(\Omega)}$$
$$\leq 2C\varepsilon^3\sqrt{\frac{\lambda_0}{\alpha}}\|u_0 + \varepsilon u_1 + \varepsilon^2 u_2\|_{H^{-1}(\Omega)}. \tag{2.31}$$

By the quasimode argument, there is an element of the spectrum of $-\text{div}\left(\alpha(1+\chi_B\varepsilon)\nabla\cdot\right)$ in $H^{-1}(\Omega)$ whose distance from $\lambda_0+\varepsilon\lambda_1+\varepsilon^2\lambda_2$ is at most $2C\varepsilon^3\sqrt{\frac{\lambda_0}{\alpha}}$. By similar arguments as those at the end of Proposition 2.1, one concludes that such an element is precisely λ_ε, the first eigenvalue of $-\text{div}(\alpha(1+\chi_B\varepsilon)\nabla\cdot)$. □

3. Minimization of the second-order approximation of $\lambda(B)$

Although our main interest is to minimize the ground state λ_ε with respect to the set B, given $\varepsilon > 0$, the general feeling is that the optimization problem is not well posed. A relaxed problem, which is not so simple to describe, was obtained in Cox and Lipton [4]. In order to understand the nature of the problem for small contrasts, Conca *et al.* used a first-order approximation [9]. Indeed, after proving a slightly weaker estimate as compared with Proposition 2.1 using a more ad hoc method of estimation, they conclude that

$$\left|\inf_B \lambda_\varepsilon(B) - \lambda_0 - \varepsilon\inf_B \lambda_1(B)\right| \leq C\varepsilon^{\frac{3}{2}}. \tag{3.1}$$

This permits to obtain approximate minimizers for the eigenvalue functional λ_ε by minimizing, instead, the functional $\lambda_0 + \varepsilon\lambda_1$. This is a well-posed problem, and because the original problem may not be well posed, it may fail to capture some of the features of the original minimization problem. With this motivation, we go further and do a second-order approximation. Indeed, Proposition 2.2 allows us to conclude that

$$\left|\inf_B \lambda_\varepsilon(B) - \inf_B(\lambda_0 + \varepsilon\lambda_1(B) + \varepsilon^2\lambda_2(B))\right| \leq C\varepsilon^3. \tag{3.2}$$

Thus, we can obtain approximate minimizers for the functional λ_ε, for given $\varepsilon > 0$ small enough, by minimizing the functional $\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2$, which is a second-order approximation of λ_ε. We then study the following problem:

$$\text{minimize } \{\lambda_0 + \varepsilon\lambda_1(B) + \varepsilon^2\lambda_2(B)\,;\, B \subseteq \Omega, |B| = m\}, \quad 0 < m < |\Omega|, \; m \text{ fixed}$$

or equivalently

$$\text{minimize } \{\lambda_1(B) + \varepsilon\lambda_2(B)\,;\, B \subseteq \Omega, |B| = m\},$$

because λ_0 is independent of B and $\varepsilon > 0$ is fixed. From the expressions for $\lambda_1(B), \lambda_2(B)$ computed in the previous section, we finally consider the problem

$$\text{minimize } F(\chi) := \alpha\int_\Omega \chi(\nabla u_0 + \varepsilon\nabla v(\chi)) \cdot \nabla u_0$$

over the class of admissible domains represented by their characteristic functions

$$\mathcal{U}_{ad} := \{\chi\,;\, \chi = \chi_B, B \subseteq \Omega, |B| = m\} \subseteq L^\infty(\Omega),$$

and $v = v(\chi) \in H_0^1(\Omega)$ satisfies

$$-\alpha\Delta v - \lambda_0 v = \lambda_1(\chi)u_0 + \text{div}\,(\alpha\chi\nabla u_0), \tag{3.3}$$

$$\lambda_1(\chi) := \int_\Omega \alpha\chi|\nabla u_0|^2, \tag{3.4}$$

$$v \perp u_0 \text{ in } L^2(\Omega).$$

3.1. Relaxation of the minimization problem

The functional F is lower-semicontinuous for the weak-∗ topology on $L^\infty(\Omega)$, being quadratic with respect to χ, but the admissible set $\mathcal{U}_{ad}$ is not closed for this topology. In order to have a well-posed minimization problem, we need to work on the closure $\overline{\mathcal{U}_{ad}}$ and calculate the *lower semicontinuous envelope* of F with respect to the weak-∗ topology on $L^\infty(\Omega)$.

$$\bar{F}(\theta) := \inf\{\liminf F(\chi_n) \ : \ \chi_n \rightharpoonup \theta \text{ in } L^\infty(\Omega)^*\}, \ \theta \in \overline{\mathcal{U}_{ad}},$$

where

$$\overline{\mathcal{U}_{ad}} = \overline{\mathcal{U}_{ad}}^{L^\infty(\Omega)^*} = \{\theta \in L^\infty(\Omega) ; 0 \leq \theta \leq 1, \int_\Omega \theta = m\}.$$

We shall follow the general procedure to compute $\bar{F}$ and obtain the following theorem.

Theorem 3.1
For any $\theta \in \overline{\mathcal{U}_{ad}}$, we have

$$\bar{F}(\theta) = \alpha \int_\Omega \theta \left[\nabla u_0 + \varepsilon \nabla v_\infty(\theta)\right].\nabla u_0 - \varepsilon\theta(1-\theta)|\nabla u_0|^2,$$

where $v_\infty(\theta) \in H_0^1(\Omega)$ is solution of

$$-\alpha\Delta v - \lambda_0 v = \lambda_1(\theta)u_0 + \text{div}\,(\alpha\theta\nabla u_0), \tag{3.5}$$

$$\lambda_1(\theta) := \int_\Omega \alpha\theta|\nabla u_0|^2, \tag{3.6}$$

$$v \perp u_0 \text{ in } L^2(\Omega).$$

The proof of Theorem 3.1 will use some results on *H*-measures. This tool was introduced by Gérard [15] and Tartar [16] to understand the obstruction to compactness via a matrix of complex-valued Radon measures $(\mu_{ij}(x, xi))_{1\leq i,j\leq p}$ on $\mathbb{R}^N \times \mathbb{S}^{N-1}$ on the space-frequency domain associated with weakly convergent sequences. We refer to the two previous references for a complete presentation of *H*-measures and to [17] for their applications in small contrast homogenization. We will need the two following results (Theorem 2-2 and Lemma 2-3 in [17]).

Theorem 3.2 ([17])
Let u_ε be a sequence which weakly converges to 0 in $L^2(\mathbb{R}^N)^p$. There exists a subsequence and an *H*-measure μ such that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} q(u_\varepsilon).\bar{u}_\varepsilon = \int_{\mathbb{R}^N}\int_{\mathbb{S}^{N-1}}\sum_{i,j=1}^p q_{ij}(x,\xi)\mu_{ij}(dx,d\xi)$$

for any polyhomogeneous pseudo-differential operator q of degree 0 with symbol $(q_{ij}(x,\xi))$.

We shall also use the following lemma due to Kohn and Tartar that deals with the special case of sequences of characteristic functions.

Lemma 2.6 ([17])
Let χ_ε be a sequence of characteristic functions that weakly-∗ converges to some θ in $L^\infty(\Omega, [0, 1])$. Then the corresponding *H*-measure μ for the sequence $(\chi_\varepsilon - \theta)$ is necessarily of the type

$$\mu(dx, d\xi) = \theta(x)(1 - \theta(x))\nu(dx, d\xi),$$

where, for a given x, the measure $\nu(dx, d\xi)$ is a probability measure with respect to ξ.

Conversely, for any such probability measure $\nu \in \mathcal{P}(\Omega, \mathbb{S}^{N-1})$, there exists a sequence χ_ε of characteristic functions which weakly-∗ converges to $\theta \in L^\infty(\Omega, [0, 1])$ such that $\theta(1-\theta)\nu$ is the *H*-measure of $(\chi_\varepsilon - \theta)$.

Proof of Theorem 3.1
Let $\theta \in \overline{\mathcal{U}_{ad}}$. Let $\{\chi_n\}$ be a sequence in $\mathcal{U}_{ad}$ such that

$$\chi_n \overset{\star}{\rightharpoonup} \theta \in \overline{\mathcal{U}_{ad}}. \tag{3.7}$$

We then analyze the limit of

$$F(\chi_n) = \alpha \underbrace{\int_\Omega \chi_n|\nabla u_0|^2}_{A_n} + \alpha\,\varepsilon \underbrace{\int_\Omega \chi_n\nabla v_n \cdot \nabla u_0}_{B_n},$$

with $v_n := v(\chi_n) \in H_0^1(\Omega)$ such that

$$-\alpha\Delta v_n - \lambda_0 v_n = \lambda_1(\chi_n)u_0 + \text{div}\,(\alpha\chi_n\nabla u_0), \tag{3.8}$$

$$\lambda_1(\chi_n) = \int_\Omega \alpha\chi_n|\nabla u_0|^2,$$

$$v_n \perp u_0 \text{ in } L^2(\Omega). \tag{3.9}$$

Step 1: Passing to the limit in A_n is easy. By the convergence (3.7), we have

$$A_n = \lambda_1(\chi_n) \longrightarrow \alpha \int_\Omega \theta \, |\nabla u_0|^2 = \lambda_1(\theta). \tag{3.10}$$

Step 2: Now we study the limit of the sequence v_n. By (3.9), we know that

$$\left(1 - \frac{\lambda_0}{\lambda^1}\right) \int_\Omega |\nabla v_n|^2 \le C,$$

using a similar estimation as (2.17). Then $\|v_n\|_{H_0^1}^2 \le C$ and hence,

$$v_n \rightharpoonup v_\infty = v_\infty(\theta) \quad \text{weak-}H_0^1(\Omega)$$

up to a subsequence. Because $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$,

$$v_n \longrightarrow v_\infty \quad \text{in } L^2(\Omega)$$

up to a subsequence. Therefore, we can pass to variational limit from (3.8) to obtain

$$-\alpha \Delta v_\infty - \lambda_0 v_\infty = \lambda_1(\theta) u_0 + \text{div} \, (\alpha \, \theta \nabla u_0). \tag{3.11}$$

Moreover, passing to the limit from (3.9), we have

$$v_\infty \perp u_0 \text{ in } L^2(\Omega),$$

accordingly, because $\|u_0\|_{L^2} = 1$, $v_\infty = v_\infty(\theta)$ is uniquely defined in (3.11) and v_∞ depends (linearly) only on θ and not on the convergent subsequence of $\{v_n\}$.

Step 3: The main difficulty is to pass to the limit in B_n, which is quadratic with respect to χ_n. First, we can rewrite B_n as

$$B_n = \int_\Omega \chi_n \nabla w_n \cdot \nabla u_0 + \int_\Omega \chi_n \nabla z_n \cdot \nabla u_0, \tag{3.12}$$

$w_n, z_n \in H_0^1(\Omega)$ such that

$$-\alpha \Delta w_n = \lambda_0 v_n + \lambda_1(\chi_n) u_0, \tag{3.13}$$

$$-\Delta z_n = \text{div} \, (\chi_n \nabla u_0). \tag{3.14}$$

On the one hand, because

$$\lambda_0 v_n + \lambda_1(\chi_n) u_0 \longrightarrow \lambda_0 v_\infty + \lambda_1(\theta) u_0 \quad \text{in } L^2(\Omega),$$

(3.13) implies

$$w_n \longrightarrow w \quad \text{in } H_0^1(\Omega),$$

where $w \in H_0^1(\Omega)$ satisfies the equation

$$-\alpha \Delta w = \lambda_0 v_\infty + \lambda_1(\theta) u_0$$

and, in consequence,

$$\int_\Omega \chi_n \nabla w_n \cdot \nabla u_0 \longrightarrow \int_\Omega \theta \, \nabla w \cdot \nabla u_0. \tag{3.15}$$

The difficulty is now to calculate the limit in the second term of B_n in (3.12). We observe that div $\chi_n \nabla u_0 \rightharpoonup$ div $\theta \nabla u_0$ weakly in $H^{-1}(\Omega)$, and because $(-\Delta)^{-1}$ is a isomorphism from $H^{-1}(\Omega)$ into $H_0^1(\Omega)$, we obtain L^2-weak convergence of ∇z_n. However, this is not enough for passing to the limit in the second term of B_n because, in the product $\chi_n \nabla z_n$, both sequences χ_n and ∇z_n only converge weakly. For handling this convergence problem, we use the results on *H*-convergence stated before.

Step 4: For simplicity if Ω is $\mathbb{R}^n$, in view of Theorem 3.2 and Lemma 2.6, the limit of the second term in (3.12) becomes

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \chi_n \nabla z_n \cdot \nabla u_0 = \int_{\mathbb{R}^N} \theta \, Q(\theta) \cdot \nabla u_0 - \int_{\mathbb{R}^N} \theta(1-\theta) M \nabla u_0 \cdot \nabla u_0,$$

where the pseudo-differential operator Q is defined in Lemma A1 (in the Appendix), and its symbol has been calculated therein and

$$M = \int_{\mathbb{S}^{N-1}} \xi \otimes \xi \, \nu(x, \text{d}\xi),$$

$\nu = \nu(x, \xi)$ is a probability measure with respect to ξ that depends on the sequence $\{\chi_n\}$ and $Q(\theta) = \nabla z$ with $z \in H_0^1(\Omega)$ verifies the equation

$$-\Delta z = \text{div} \, (\theta \, \nabla u_0).$$

Math. Meth. Appl. Sci. **2016**, 39 3549–3564

3557

Step 5: But we need to work on Ω bounded. To that end, we use a localization procedure. This argument proceeds as follows. Let (ζ_k) be a sequence of smooth compactly supported functions in $C_0^\infty(\mathbb{R}^N)$ such that supp $\zeta_k \subset \Omega$ for all k and ζ_k converges to 1 strongly in $L^2(\Omega)$. Then the second term on the right-hand side of (3.12) can be written as

$$\int_\Omega \chi_n \nabla z_n \cdot \nabla u_0 = \int_{\mathbb{R}^N} \zeta_k \chi_n \nabla z_n \cdot \nabla u_0 + \int_{\mathbb{R}^N} (1 - \zeta_k)\chi_n \nabla z_n \cdot \nabla u_0. \tag{3.16}$$

Note that the last term in (3.16) converges to 0 uniformly with respect to n when k tends to infinity because z_n is bounded in $H^1(\Omega)$. We now fix k and consider another smooth compactly supported function $\psi_k \in C_0^\infty$ such that $\psi_k \equiv 1$ inside the support of ζ_k. The first term on the right-hand side of (3.16) is thus equal to

$$\int_\Omega \zeta_k(\psi_k \chi_n)\nabla(\psi_k z_n) \cdot \nabla u_0. \tag{3.17}$$

Rewriting the equation (3.14) in $\mathbb{R}^N$ as

$$-\Delta(\psi_k z_n) - \Delta((1 - \psi_k)z_n) = \text{div }(\psi_k \chi_n \nabla u_0) + \text{div }((1 - \psi_k)\chi_n \nabla u_0),$$

we can show that the function $\psi_k z_n$ is the sum of $\tilde{z}_n, \check{z}_n$ on the support of ζ_k being $\tilde{z}_n, \check{z}_n$ solutions of the following equations in the whole space $\mathbb{R}^N$

$$-\Delta \tilde{z}_n = \text{div }\psi_k \chi_n \nabla u_0 \quad \text{in } \mathbb{R}^N,$$
$$\Delta \check{z}_n = \text{div }z_n \nabla \psi_k + \nabla \psi_k \cdot (\chi_n \nabla u_0 + \nabla z_n) \quad \text{in } \mathbb{R}^N.$$

We then notice that

$$\text{div }\psi_k \chi_n \nabla u_0 \rightharpoonup \text{div }\psi_k \theta \nabla u_0 \quad \text{weakly in } H^{-1}(\mathbb{R}^N) \text{ and}$$

$$\text{div }z_n \nabla \psi_k + \nabla \psi_k \cdot (\chi_n \nabla u_0 + \nabla z_n) \to \text{div }z \nabla \psi_k + \nabla \psi_k \cdot (\theta \nabla u_0 + \nabla z) \quad \text{strongly in } H^{-1}(\mathbb{R}^N)$$

because this last term clearly converges weak-$L^2(\Omega)$. Using the fact that $(-\Delta)^{-1}$ is an isomorphism from $H^{-1}(\mathbb{R}^N)$ into $H^1(\mathbb{R}^N)$, we thus have

$$\tilde{z}_n \rightharpoonup \tilde{z} \quad \text{weakly in } H^1(\mathbb{R}^N)$$

and

$$\check{z}_n \to \check{z} \quad \text{strongly in } H^1(\mathbb{R}^N)$$

where $\tilde{z}, \check{z}$ verify

$$-\Delta \tilde{z} = \text{div }\psi_k \theta \nabla u_0 \quad \text{in } \mathbb{R}^N,$$
$$\Delta \check{z} = \text{div }z \nabla \psi_k + \nabla \psi_k \cdot (\theta \nabla u_0 + \nabla z) \quad \text{in } \mathbb{R}^N.$$

Obviously $z = \tilde{z} + \check{z}$ on the support of ζ_k.

Now noting that the integral (3.17) has close relationship with the formulation of the H-measures, we see that as in the whole space case, $\nabla \tilde{z}_n$ depends linearly on $(\psi_k \chi_n)$ through the pseudo-differential operator Q of symbol (A2). Therefore, applying Theorem 2 of [15], we conclude that the limit of the first term on the right-hand side of (3.16) is equal to

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \zeta_k(\psi_k \chi_n)\nabla(\check{z}_n + \tilde{z}_n) \cdot \nabla u_0 = \int_{\mathbb{R}^N} \zeta_k(\psi_k \theta)\nabla \check{z} \cdot \nabla u_0 + \lim_{n\to\infty} \int_{\mathbb{R}^N} \zeta_k(\psi_k \chi_n)\nabla \tilde{z}_n \cdot \nabla u_0$$

$$= \int_{\mathbb{R}^N} \zeta_k(\psi_k \theta)\nabla \check{z} \cdot \nabla u_0 + \int_{\mathbb{R}^N} \zeta_k(\psi_k \theta)\nabla \tilde{z} \cdot \nabla u_0 - \int_{\mathbb{R}^N} \zeta_k \psi_k \theta(1 - \theta)M\nabla u_0 \cdot \nabla u_0$$

$$= \int_\Omega \zeta_k \theta \nabla z \cdot \nabla u_0 - \int_\Omega \zeta_k \theta(1 - \theta)M\nabla u_0 \cdot \nabla u_0.$$

Finally, making k tends to ∞, we obtain the desired bounded domain case.

We go back to the calculation of the limit in (3.12). Indeed, gathering the limit (3.15) and limit calculated in the previous discussion, it follows that

$$\lim_{n\to\infty} B_n = \int_\Omega \theta \, \nabla v_\infty(\theta) \cdot \nabla u_0 - \int_\Omega \theta(1 - \theta) \int_{\mathbb{S}^{N-1}} (\xi \cdot \nabla u_0)^2 \, \nu(\text{d}x, \text{d}\xi). \tag{3.18}$$

From (3.10) and (3.18), finally, one has

$$\lim_{n\to\infty} F(\chi_n) = \lim_{n\to\infty} A_n + \varepsilon \lim_{n\to\infty} B_n$$

$$= \frac{1}{\alpha}\lambda_1(\theta) + \varepsilon \int_\Omega \theta \, \nabla v_\infty(\theta) \cdot \nabla u_0 - \varepsilon \int_\Omega \theta(1 - \theta) \int_{\mathbb{S}^{N-1}} (\xi \cdot \nabla u_0)^2 \, \nu(\text{d}x, \text{d}\xi).$$

Step 6: Now we calculate

$$\bar{F}(\theta) = \inf_\nu \lim F(\chi_n).$$

To that end, we notice that

$$\int_{\mathbb{S}^{N-1}} (\xi \cdot \nabla u_0)^2 \, \nu(\mathrm{d}x, \mathrm{d}\xi) \le |\nabla u_0|^2(x) \quad a.e. \ x \in \Omega,$$

because ν is a probability measure with respect to ξ *a.e.* $x \in \Omega$. Moreover, this value is reached when we take the Dirac measure $\delta_{\nabla u_0(x)}$, that is, when

$$\nu(x, \xi) = \delta_{\xi_x} \mathrm{d}x, \quad \xi_x = \nabla u_0(x) / \|\nabla u_0(x)\|.$$

From the converse part of Lemma 2.3 in [17], the minimum for

$$\inf_{\nu} \lim F(\chi_n)$$

is also achieved. So, finally, we can conclude

$$\bar{F}(\theta) = \int_{\Omega} \theta \, (\nabla u_0 + \varepsilon \nabla v_{\infty}(\theta)) \cdot \nabla u_0 - \varepsilon \int_{\Omega} \theta \, (1 - \theta) \, |\nabla u_0|^2. \tag{3.19}$$

Recall that $v_{\infty} = v_{\infty}(\theta)$ depends linearly on θ. □

3.2. Optimality conditions for the relaxed problem.

The relaxed functional $\bar{F}$ achieves its minimum of $\overline{\mathcal{U}_{ad}}$ because it is lower-semicontinuous, and the constraint set is compact for the weak-$*$ topology. We first investigate the differentiability properties of $\bar{F}$ in order to obtain optimality conditions for a minimizer of $\bar{F}$ on the compact convex set $\overline{\mathcal{U}_{ad}}$.

Proposition 3.4
The functional $\bar{F}$ is Fréchet differentiable of every order, and we have the following expressions for the Gateaux derivatives of the first and second order

$$\bar{F}'(\theta)\varphi = \int_{\Omega} [2\varepsilon(\nabla v_{\infty}(\theta) + \theta \nabla u_0) + (1 - \varepsilon)\nabla u_0] \cdot \nabla u_0 \, \varphi. \tag{3.20}$$

and

$$\bar{F}''(\theta)(\varphi, \varphi) = 2\varepsilon \int_{\Omega} (\nabla v_{\infty}(\varphi) + \varphi \nabla u_0) \cdot \nabla u_0 \, \varphi. \tag{3.21}$$

Proof
The linearity of the application $\theta \mapsto v_{\infty}(\theta)$ and the expression for $\bar{F}$ show clearly that it is quadratic with respect to θ. So, the Fréchet derivatives exist. In order to calculate the first-order derivative, we rewrite (3.19) as

$$\bar{F}(\theta) = \varepsilon \int_{\Omega} \theta \nabla v_{\infty}(\theta) \cdot \nabla u_0 + \varepsilon \int_{\Omega} \theta^2 |\nabla u_0|^2 + (1 - \varepsilon) \int_{\Omega} \theta |\nabla u_0|^2.$$

But, using $v_{\infty}(\theta)$ as test function in (3.11), we obtain

$$\bar{F}(\theta) = -\varepsilon \int_{\Omega} |\nabla v_{\infty}(\theta)|^2 - \frac{\lambda_0}{\alpha} v_{\infty}^2(\theta) + \varepsilon \int_{\Omega} \theta^2 |\nabla u_0|^2 + (1 - \varepsilon) \int_{\Omega} \theta |\nabla u_0|^2.$$

A simple calculation gives us

$$\bar{F}'(\theta)\varphi = -2\varepsilon \int_{\Omega} \nabla v_{\infty}(\theta) \cdot \nabla v_{\infty}(\varphi) - \frac{\lambda_0}{\alpha} v_{\infty}(\theta) v_{\infty}(\varphi) + 2\varepsilon \int_{\Omega} \theta |\nabla u_0|^2 \varphi + (1 - \varepsilon) \int_{\Omega} |\nabla u_0|^2 \varphi.$$

We now notice that $v_{\infty}(\varphi)$ satisfies (3.11). Then, again taking $v_{\infty}(\theta)$ as test function, we can explicitly write the aforementioned expression in terms of φ to obtain (3.20) then (3.21). □

We wish to investigate the critical points for the constrained minimization problem of minimizing $\bar{F}$ over $\overline{\mathcal{U}_{ad}}$. To that end, we use the Lagrange's multipliers method with the constraint

$$C(\theta) := \int_{\Omega} \theta = m, \quad \theta \in \overline{\mathcal{U}_{ad}} \text{ hence } C'(\theta)\varphi = \int_{\Omega} \varphi.$$

Therefore, the critical points satisfy the Euler–Lagrange equation: for all admissible φ

$$[\bar{F}'(\theta) + \Lambda C'(\theta)]\varphi = 0$$

for some $\Lambda \in \mathbb{R}$; that is,

Copyright © 2016 John Wiley & Sons, Ltd.

Math. Meth. Appl. Sci. **2016**, 39 3549–3564

3559

$$\int_\Omega [2\varepsilon(\nabla v_\infty(\theta) + \theta \nabla u_0) + (1 - \varepsilon)\nabla u_0] \cdot \nabla u_0 \, \varphi + \Lambda \int_\Omega \varphi = 0 \quad \forall \varphi.$$

Consequently, the density of $\overline{\mathcal{U}_{ad}}$ in $L^2(\Omega)$ implies

$$2\varepsilon \nabla v_\infty(\theta) \cdot \nabla u_0 + (2\varepsilon\theta + 1 - \varepsilon)|\nabla u_0|^2 = \Lambda \quad \text{on } \Omega.$$

Proposition 3.5
If θ^* is optimal in the relaxed formulation, then there is real Λ such that

$$2\varepsilon \nabla v_\infty(\theta) \cdot \nabla u_0 + (2\varepsilon\theta + 1 - \varepsilon)|\nabla u_0|^2 = \Lambda \quad \text{in } \Omega.$$

Integrating over Ω and considering u_0 as test function in (3.11), we obtain the following consequence

$$\int_\Omega \nabla v_\infty(\theta) \cdot \nabla u_0 = 0 \text{ and } 2\varepsilon \int_\Omega \theta |\nabla u_0|^2 + \frac{1-\varepsilon}{\alpha} \lambda_0 = \Lambda |\Omega|.$$

4. Numerical illustrations

Let us emphasize that the original problem of minimizing the leading eigenvalue with respect to the inclusion B is not well posed; usually, it admits no solution.

In this section, we shall illustrate the behavior of the solution of the approximated problem through numerical simulations. To that end, we place ourselves under assumption of low contrast regime, that is, $\beta = \alpha(1 + \varepsilon)$ for small ε. In the following examples, we will consider $\varepsilon = 0.1$ and $\varepsilon = 10^{-6}$.

We use an optimization algorithm to minimize $\bar{F}$: we have implemented a gradient-based steepest descend numerical algorithm for the local proportion θ. At each step of the optimization algorithm, we update the local proportion with a step $\rho_i > 0$ by

$$\theta_i = \min(1, \max(0, \tilde{\theta}_i)) \text{ with } \tilde{\theta}_i = \theta_{i-1} - \rho_i \bar{F}'(\theta_{i-1}) + \Lambda_i$$

where Λ_i is the Lagrange multiplier for the volume constraint. The Lagrange multipliers Λ_i are approximated at each iteration by simple dichotomy in order to obtain the constraint $\int_\Omega \theta_i = m$ corresponding to a fixed proportion.

The optimization procedure is coupled with finite elements approximations of the boundary values problems needed to compute both $\bar{F}$ and its derivative $\bar{F}'$. To calculate the eigenpair (λ_0, u_0) and all the states $v_{i,\infty}$, we use $\mathbb{P}_2$ finite elements, while the local proportions θ_i have been discretized with $\mathbb{P}_1$.

We will present examples in dimensions two and three. The computations have been made with the FEM library FreeFem++ [18]. The subsequent figures show the local proportion of the material with higher conductivity. We do a comparative analysis in dimensions two and three for square and cube cases, respectively, confirming the mentioned properties in [9] with respect to the distribution of the material with higher conductivity that depends on the shape of the domain Ω. The volume always refers to the percentage of volume occupied by the higher conductivity material.

4.1. The square and the cube.

The computations are made on the unit square $[0, 1]^2$ with a regular mesh of 80,000 triangles. For a very small value of ε, here 10^{-6}, we have obtained the optimal designs displayed into Figure 1 for different volume proportions. The dark red region corresponds to B and material β, and the local proportion is then 1. The blue region corresponds to material α, and the local proportion is then 0.

The numerically computed optimal region B contains neighborhoods around corners, and the center always is also included. Similar results were obtained by Conca *et al.* [9] with a first-order approximation only. Nevertheless, the local proportion is very often either 0 or 1. Let us now consider the same cases with a much larger parameter ε. In Figure 2, we present the results obtained with $\varepsilon = 0.1$. We observe that the mixture is much more important: the interest of higher-order approximation appear,s as one can expect, for reasonably large value of the small parameter ε. For very small values of ε, like 10^{-6}, the first-order model should already provide a very good approximation.

(a) $m/|\Omega| = 0.2$ (b) $m/|\Omega| = 0.4$ (c) $m/|\Omega| = 0.7$ (d) $m/|\Omega| = 0.9$
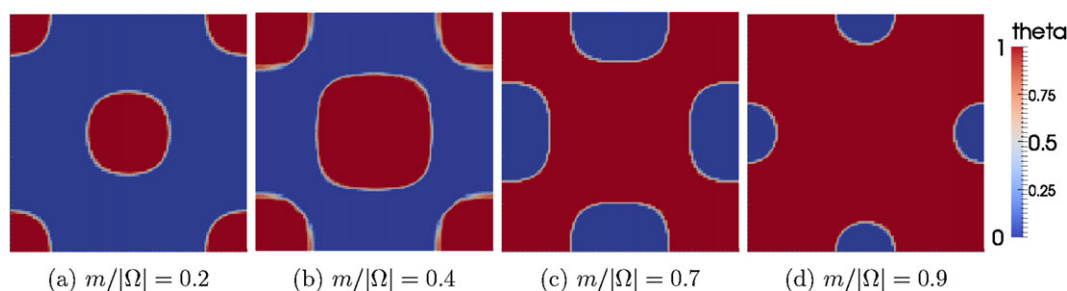
Figure 1. Nearly optimal distribution B in the square case for $\varepsilon = 10^{-6}$.

Let us now present in Figure 3 the simulations on the unit cube $[0,1]^3$. For visualization, we have remove the phase where $\theta = 0$. Because the computation have been made on a laptop, the resolution is coarser in these simulations in dimension three, and we kept the same numbers of degree of freedom.
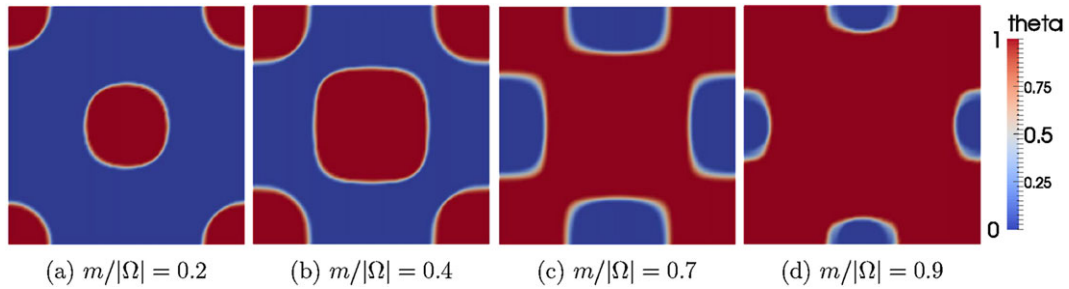

(a) $m/|\Omega| = 0.2$ (b) $m/|\Omega| = 0.4$ (c) $m/|\Omega| = 0.7$ (d) $m/|\Omega| = 0.9$

Figure 2. Nearly optimal distribution *B* in the square case for $\varepsilon = 0.1$.

(a) $m/|\Omega| = 0.125$ (b) $m/|\Omega| = 0.25$ (c) $m/|\Omega| = 0.375$ (d) $m/|\Omega| = 0.5$
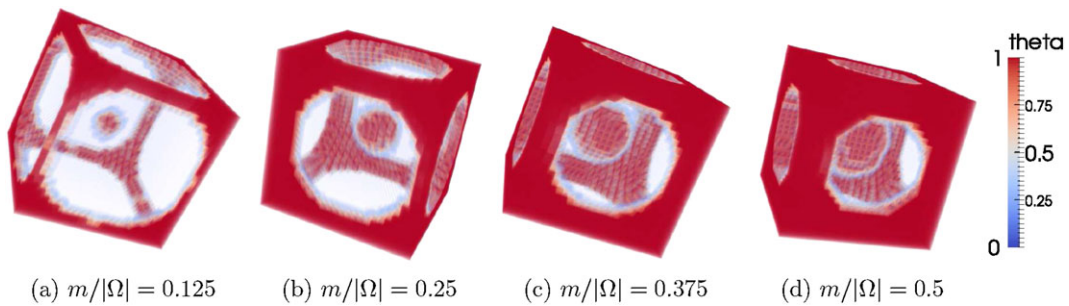
Figure 3. Nearly optimal distribution *B* in the cube case for $\varepsilon = 10^{-6}$.

(a) $m/|\Omega| = 0.2$ (b) $m/|\Omega| = 0.4$ (c) $m/|\Omega| = 0.7$ (d) $m/|\Omega| = 0.9$

(e) $m/|\Omega| = 0.2$ (f) $m/|\Omega| = 0.4$ (g) $m/|\Omega| = 0.7$ (h) $m/|\Omega| = 0.9$

(i) $m/|\Omega| = 0.2$ (j) $m/|\Omega| = 0.4$ (k) $m/|\Omega| = 0.7$ (l) $m/|\Omega| = 0.9$
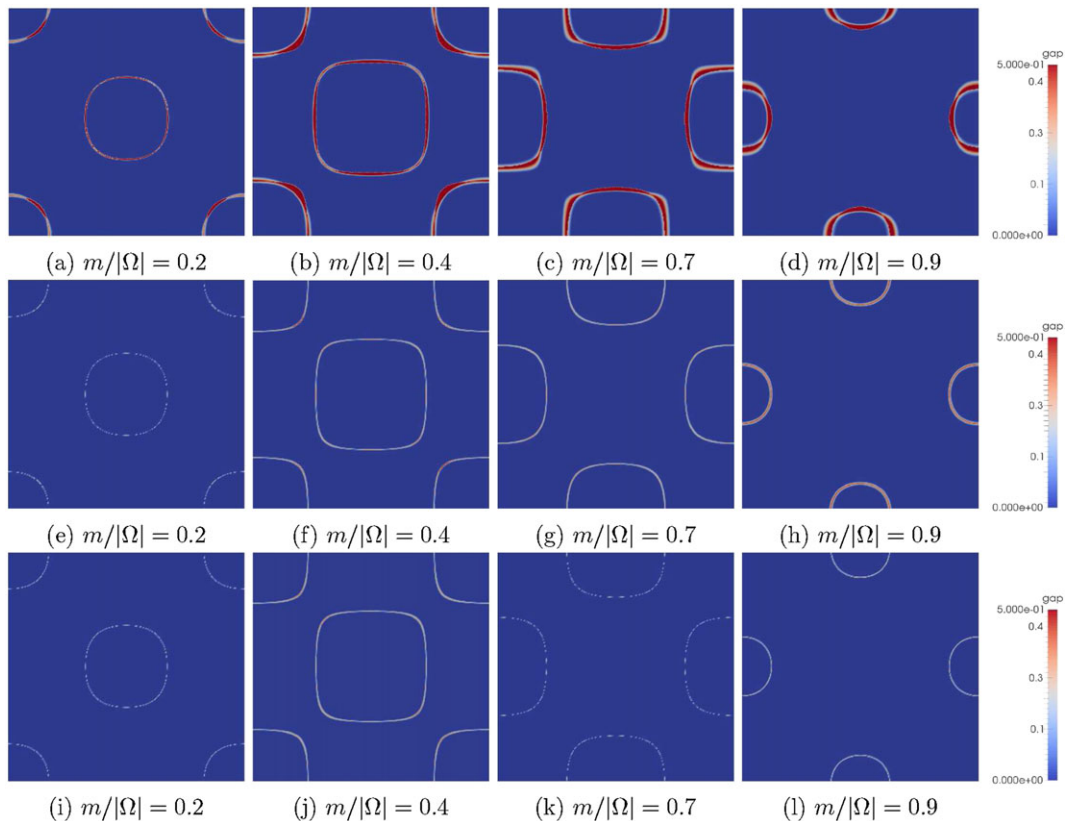
Figure 4. Absolute value of the gap between optimal design for the first-order and second-order models. The parameter ε takes the value 10^{-1} on the first line, 5.10^{-3} on the second line and 10^{-6} on the third line.

Copyright © 2016 John Wiley & Sons, Ltd.

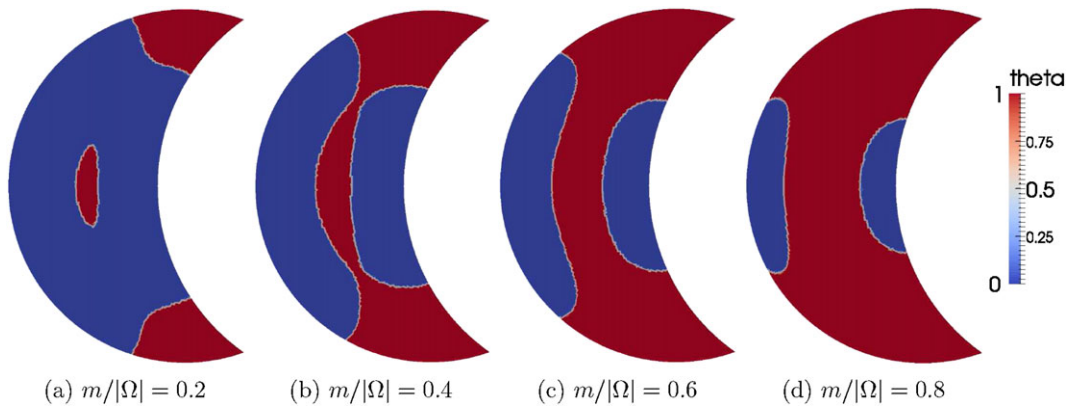*Math. Meth. Appl. Sci.* **2016**, 39 3549–3564

3561

Figure 5. Nearly optimal distribution B in a crescent for $\varepsilon = 10^{-6}$.
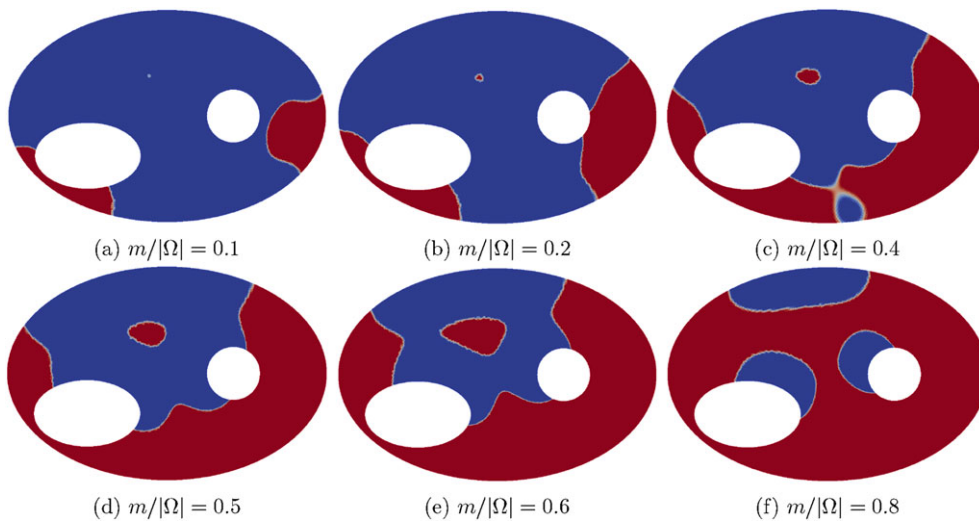

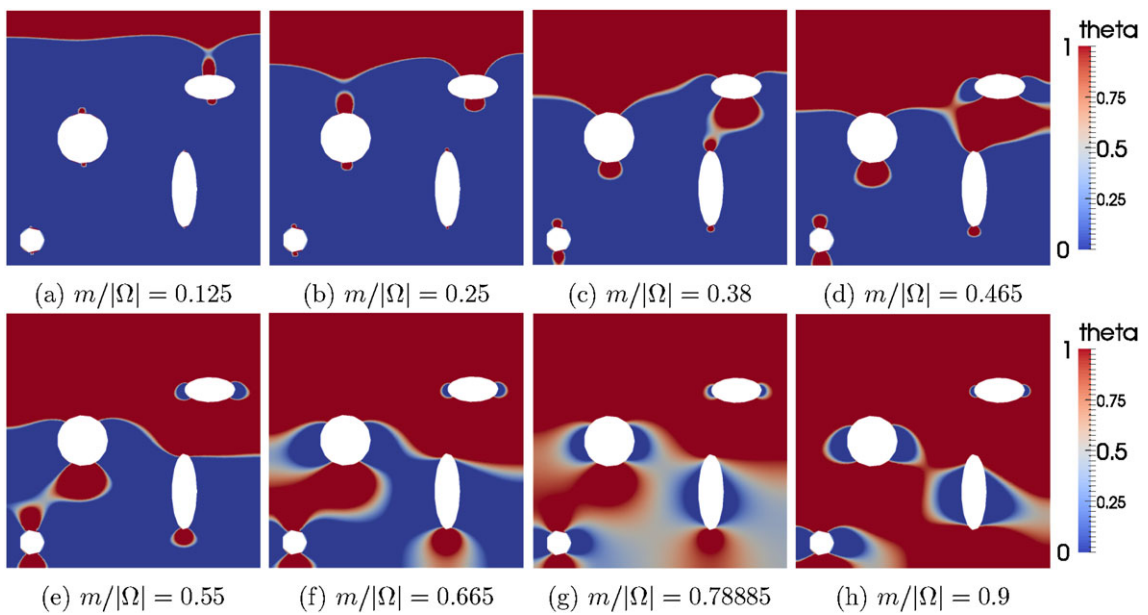**Figure 6.** Nearly optimal distribution B in a perforated ellipse for $\varepsilon = 10^{-6}$.

Figure 7. Optimal design for second-order model for various fractions. The parameter ε takes the value 10^{-1}.

3562

Copyright © 2016 John Wiley & Sons, Ltd.

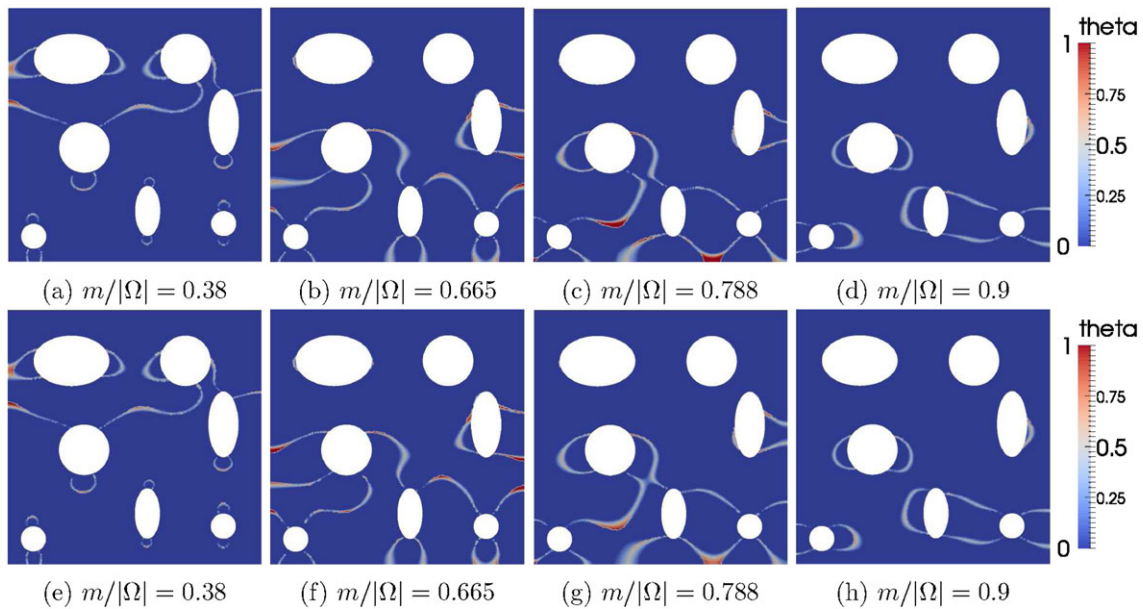*Math. Meth. Appl. Sci.* **2016**, 39 3549–3564

Figure 8. Absolute value of the gap between optimal design for the first-order and second-order models. The parameter ε takes the value 10^{-1} on the first line and 10^{-3} on the second line.

4.2. Numerical comparison of the first-order and second-order model.

Because the original problem does not have in general a solution, it makes little sense to test how good are the optimal domains obtained by the first-order approximation method [5] and by the second-order method. Nevertheless, one may wonder if the optimal solution obtained by the first-order and second-order methods really differ. Even if the computed value of the ground state only varies between the first-order and second-order model, at the second order in ε (so that it can hardly be seen), we claim that the optimal design really differs in general. Of course, it depends on the range of the small parameter ε. In order to illustrate and defend this claim, we have plotted in Figure 4 the absolute value of the difference between the characteristic function of the optimal domain computed by the method of [9] (based on the first-order approximation) and the optimal density computed with the method presented in this work. In order to catch more precisely this gap, we use a refined mesh made of 180,000 triangles and $\mathbb{P}_2$ finite elements.

It appears that the second-order method really differs for rather large values of ε and brings a real gain in decreasing the leading eigenvalue by generating mixture in a transition zone between the two phases as it is expected. The size of the mesh appears in the last line when $\varepsilon = 10^{-6}$, and it corresponds to the width of the light curved lines. On the contrary, the mixing zone has the size of several element when ε take larger values.

4.3. Others domains

For the sake of completeness, we present computations in other plane domains for the comparison with [9]: a crescent in Figure 5 and a perforated ellipse in Figure 6.

Let us emphasize that in the last case, even for $\varepsilon = 10^{-6}$, we observe clearly in Figure 6(c) a small area where θ takes values strictly between 0 and 1 where we see the effect of the modeling with a second-order approximation. This can be explained by the fact that the first eigenmode is more oscillating in a more complex geometrical configuration.

In order to enlighten this observation, let us consider a perforated square. The need of relaxation appears clearly in Figures 7 and 8 for various configurations of perforated squares. The results illustrated through Figures 7 and 8 for domains with many inclusions provide a convincing case for the use of the second-order approximation. The used meshes involve around 110,000 triangles.

Appendix

Lemma A1
For $\theta \in L^\infty(R^N)$, if $z \in H_0^1(R^N)$ solves

$$-\Delta z = \text{div} (\theta \nabla u_0) \quad \text{in } \mathbb{R}^N, \tag{A1}$$

then $\theta \mapsto Q(\theta) := \nabla z$ defines a pseudo-differential operator with symbol

$$q(x, \xi) = -\frac{\xi \cdot \nabla u_0(x)}{|\xi|^2} \xi. \tag{A2}$$

Note that q is homogenous of degree 0 in ξ.

Math. Meth. Appl. Sci. **2016**, 39 3549–3564

3563

Heuristic Proof of Lemma A1

We first consider the whole space case in order to use Fourier calculus. Indeed, denoting by $\widehat{}$ the Fourier transform and starting from Equation (3.14), formally we freeze ∇_{u_0} we can calculate as follows

$$(-\Delta z)\widehat{}(\xi) = (\mathrm{div}\,(\theta \nabla u_0))\widehat{}(\xi)$$

$$-(-|\xi|^2\widehat{z}) = -i\xi \cdot \nabla u_0(x)\widehat{\theta}$$

$$\widehat{z} = -\frac{i\xi \cdot \nabla u_0(x)}{|\xi|^2}\widehat{\theta},$$

which gives

$$\widehat{\nabla z}(\xi) = -i\xi\widehat{z}(\xi) = -i\xi\left(-\frac{i\xi \cdot \nabla u_0(x)}{|\xi|^2}\widehat{\theta}\right) = \underbrace{\left(-\frac{\xi \cdot \nabla u_0(x)}{|\xi|^2}\xi\right)}_{q(x,\xi)}\widehat{\theta}.$$

For a rigorous derivation, we refer to [16]   □

Acknowledgements

References

1. Henrot A. *Extremum problems for eigenvalues of elliptic operators*, Frontiers in Mathematics: Birkhäuser Verlag, Basel, 2006.
2. Cox S, McLaughlin JR. Extremal eigenvalue problems for composite membranes i. *Applied Mathematics and Optimization* 1990; **22**(2):153–167.
3. Cox S, McLaughlin JR. Extremal eigenvalue problems for composite membranes ii. *Applied Mathematics and Optimization* 1990; **22**(2):169–187.
4. Cox S, Lipton R. Extremal eigenvalue problems for two-phase conductors. *Archive in Rational Mechanics and Analysis* 1996; **136**(2):101–117.
5. Alvino A, Trombetti G, Lions PL. On optimization problems with prescribed rearrangements. *Nonlinear Analysis* 1989; **13**(2):185–220.
6. Conca C, Mahadevan R, Sanz L. An extremal eigenvalue problem for a two-phase conductor in a ball. *Applied Mathematics and Optimization* 2009; **60**(2):173–184.
7. Conca C, Mahadevan R, Sanz L. Shape derivative for a two-phase eigenvalue problem and optimal configurations in a ball. *CANUM 2008, volume 27 of ESAIM Proceedings*, EDP Sci., Les Ulis, 2009, 311–321.
8. Dambrine M, Kateb D. On the shape sensitivity of the first Dirichlet eigenvalue for two-phase problems. *Applied Mathematics and Optimization* 2011; **63**(1):45–74.
9. Conca C, Laurain A, Mahadevan R. Minimization of the ground state for two phase conductors in low contrast regime. *SIAM Journal on Applied Mathematics* 2012; **72**(4):1238–1259.
10. Laurain A. Global minimizer of the ground state for two phase conductors in low contrast regime. *ESAIM: Control, Optimisation and Calculus of Variations* 2014; **20**(4):362–388.
11. Ammari H, Kang H, Lim M, Zribi H. Layer potential techniques in spectral analysis. Part I: complete asymptotic expansions for eigenvalues of the Laplacian in domains with small inclusions. *Transaction of the American Mathematical Society* 2010; **362**(6):2901–2922.
12. Ammari H, Kang H, Lee H. *Layer potential techniques in spectral analysis, volume 153 of Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2009.
13. Rellich F. *Perturbation theory of eigenvalue problems*, Assisted by J. Berkowitz. With a preface by Jacob T. Schwartz. Gordon and Breach Science Publishers: New York, 1969.
14. Dauge M, Djurdjevic I, Faou E, Rössle A. Eigenmode asymptotics in thin elastic plates. *Journal des Mathématiques Pures et Appliquées* 1999; **78**(9): 925–964.
15. Gérard P. Microlocal defect measures. *Communications in Partial Differential Equations* 1991; **16**(11):1761–1794.
16. Tartar L. *H*-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations. *Proceeding of the Royal Society Edinburgh Section A* 1990; **115**(3-4):193–230.
17. Allaire G, Gutiérrez S. Optimal design in small amplitude homogenization. *ESAIM Mathematical Modeling and Numerical Analysis* 2007; **41**(3): 543–574.
18. Hecht F. New development in Freefem++. *Journal of Numerical Mathematics* 2012; **20**(3-4):251–265.