SOLUTIONS OF SOME BOUNDARY VALUE PROBLEMS FOR A CLASS OF CONSTITUTIVE RELATIONS FOR NON-LINEAR ELASTIC BODIES THAT IS NOT GREEN ELASTIC

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Summary

Several boundary value problems are solved for a new class of constitutive equation, where the left Cauchy–Green stretch tensor is given as a non-linear function of the Cauchy stress tensor. Some constitutive inequalities and restrictions are proposed as well.

1. Introduction

Some new constitutive relations for elastic bodies have been presented in the literature, which cannot be classified as neither Green nor Cauchy elastic bodies, see for example, (**1**–**8**). One of such relatively new constitutive relations corresponds to an implicit relation between the Cauchy stress tensor $\mathbf{T}$ and the left Cauchy–Green stretch tensor $\mathbf{B}$, which is of the form $\mathfrak{F}(\mathbf{T}, \mathbf{B}) = \mathbf{0}$ (see (**2**)). Some interesting subclasses of the above implicit model are the classical constitutive equation for a Cauchy elastic body $\mathbf{T} = \mathfrak{g}(\mathbf{B})$, the new constitutive equation $\mathbf{B} = \mathfrak{h}(\mathbf{T})$ and its subclass $\boldsymbol{\varepsilon} = \mathfrak{f}(\mathbf{T})$, where $\boldsymbol{\varepsilon}$ is the linearised strain tensor. This last constitutive equation has been analysed in several recent works (**6**, **7**, **9**–**20**), and as mentioned, for example, in (**8**), it has different potential applications in the mathematical modelling of the mechanical behaviour of rock and concrete (**21**, **22**), fracture mechanics (**13**, **15**, **20**), and in the modelling of some gum metals (**23**–**26**). Regarding the more general implicit relation $\mathfrak{F}(\mathbf{T}, \mathbf{B}) = \mathbf{0}$, in (**27**) some possible applications in biomechanics are discussed.

About the constitutive equation $\mathbf{B} = \mathfrak{h}(\mathbf{T})$, a few works have appeared in the literature analysing that equation (see **28–30**), presenting the solutions of some boundary value problems considering time dependent and also time-independent deformations. In (**30**) a short note was presented about some restrictions for such constitutive equations, such that the body is actually elastic, in the sense of no dissipating work into heat for any process.

Some possible applications for these new constitutive relations $\mathbf{B} = \mathfrak{h}(\mathbf{T})$ could be found in fracture mechanics (for bodies that can show large elastic deformations), in the study of residual stresses, for problems where from the experimental point of view we can control the stresses applied on a sample rather than the deformation[1] and in the study of incompressible and nearly incompressible bodies.

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[1] See the comments in §6 of (**31**) and the references cited therein.

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In the case of the study of problems considering stress concentration, in (**12**, **13**, **15**) interesting results have been obtained for the subclass $\boldsymbol{\varepsilon} = \mathfrak{f}(\mathbf{T})$, where in particular it has been shown that for some special expressions for $\mathfrak{f}$ the magnitude of the strains is limited, independently of how large the stresses can be. One hypothesis is that something similar could be obtained for bodies considering large elastic deformations, for some special expressions for $\mathfrak{h}$ in $\mathbf{B} = \mathfrak{h}(\mathbf{T})$. Regarding the possible application of $\mathbf{B} = \mathfrak{h}(\mathbf{T})$ in the study of residually stressed bodies, we can see Equation 20 of (**32**), where the residual stress is incorporated as a variable for the energy function for a Green elastic material, that is, in (**32**) they have something like $\mathbf{T} = \mathfrak{g}(\mathbf{B}, \mathbf{T}_R)$, where $\mathbf{T}_R$ are the residual stresses. Something similar but simpler could be proposed by considering $\mathbf{B} = \mathfrak{h}(\mathbf{T})$, where the residual stresses $\mathbf{T}_R$ could be defined (for a given reference configuration) as the stresses for which there is no external traction, and which satisfy the equilibrium equation and the relation $\mathbf{I} = \mathfrak{h}(\mathbf{T}_R)$ (see, for example, (**33**)). About the modelling of the behaviour of incompressible bodies, from (**34**) we can see that such constraints can be incorporated in a rather direct way when using $\mathbf{B} = \mathfrak{h}(\mathbf{T})$, since the function $\mathfrak{h}$ should simply satisfy the equation $\det[\mathfrak{h}(\mathbf{T})] = 1$ for any $\mathbf{T}$, and the assumption that 'any deformation compatible with the constraint does not do work' (used in the classical theory of elasticity) is not necessary here. The same could be done in the case of nearly incompressible bodies, which could be defined as bodies for which $\det[\mathfrak{h}(\mathbf{T})] = 1 + r(\mathbf{T})$, where $|r(\mathbf{T})| \sim O(\delta)$, $\delta \ll 1$.

The implicit constitutive theory developed by Rajagopal and his co-workers, and the subclasses derived from it that have been mentioned before, should be considered as new theoretical tools for people interested in the modelling of the behaviour of elastic bodies. They would not replace the classical theory of elasticity entirely, since there could be some problems for which this new theory $\mathbf{B} = \mathfrak{h}(\mathbf{T})$ could be more convenient, and some other problems where it is well known the effectiveness of the classical theory of non-linear elasticity (which is based on the assumption $\mathbf{T} = \mathfrak{g}(\mathbf{B})$) for the modelling of elastic bodies. In order to decide which of the theories (the classical or the new) is better for a given problem, we first need to study these new constitutive equations as much as we can from the theoretical and the experimental point of view. That is the main aim of this work (devoted in this case only to some theoretical aspects), where we study the constitutive equation $\mathbf{B} = \mathfrak{h}(\mathbf{T})$ in more detail, assuming that the function $\mathfrak{h}$ can be expressed as the derivative of a scalar function $\Pi = \Pi(\mathbf{T})$ in the stress tensor (see Section 2). Thereafter for the special case, where Π is an isotropic function, we express Π in terms of the principal stresses, and some constitutive restrictions and inequalities are proposed. For a specific expression for Π, several boundary value problems are solved considering homogeneous distributions for the stress and the strains (see Section 3), and also some in-homogeneous distributions for the stresses and strains (see Section 4). In Section 5 some final remarks are given.

2. Basic equations

2.1 *Kinematics and equation of equilibrium*

The elastic body is denoted $\mathscr{B}$ and an arbitrary point belonging to the body is denoted as X. The position of such point in the reference configuration is denoted $\mathbf{X}$ and $\mathbf{X} = \boldsymbol{\kappa}_r(X)$. The reference configuration is denoted $\kappa_r(\mathscr{B})$. The position of the same point at a time t in the current configuration is denoted $\mathbf{x}$, and it is assumed that there exists a one-to-one function $\boldsymbol{\chi}(\mathbf{X}, t)$ such that $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. The current configuration is denoted $\kappa_t(\mathscr{B})$. The deformation gradient and the left Cauchy–Green stretch tensor are defined as

$$\mathbf{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \tag{1}$$

where it assumed that $J = \det \mathbf{F} > 0$. More details about the kinematics of deforming bodies can be found, for example, in (**35**, **36**).

In the present work no time effects will be considered. If $\mathbf{T}$ is used to denote the Cauchy stress tensor, the equation of equilibrium reads

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \mathbf{0}, \tag{2}$$

where ρ is the density of the body, div is the divergence operator and $\mathbf{b}$ represents the body forces in the current configuration.

2.2 *Constitutive relations*

Following the discussion presented in the Introduction, let us consider the following implicit constitutive relation for elastic bodies (see Equation (3.1) of (**2**)):

$$\mathfrak{F}(\mathbf{T}, \mathbf{B}) = \mathbf{0}, \tag{3}$$

which in the case that $\mathfrak{F}$ is isotropic it becomes (see **37**, **38**)

$$\begin{aligned} \alpha_0 \mathbf{I} + \alpha_1 \mathbf{T} + \alpha_2 \mathbf{T}^2 + \alpha_3 \mathbf{B} + \alpha_4 \mathbf{B}^2 + \alpha_5 (\mathbf{TB} + \mathbf{BT}) + \alpha_6 (\mathbf{T}^2 \mathbf{B} + \mathbf{B} \mathbf{T}^2) \\ + \alpha_7 (\mathbf{B}^2 \mathbf{T} + \mathbf{T} \mathbf{B}^2) + \alpha_8 (\mathbf{T}^2 \mathbf{B}^2 + \mathbf{B}^2 \mathbf{T}^2) = \mathbf{0}, \end{aligned} \tag{4}$$

where the scalar functions α_i, $i = 0, 1, 2, ..., 8$ depend on the following invariants

$$I_1 = \operatorname{tr}(\mathbf{T}), \quad I_2 = \frac{1}{2} \operatorname{tr}(\mathbf{T}^2), \quad I_3 = \frac{1}{3} \operatorname{tr}(\mathbf{T}^3), \tag{5}$$

$$I_4 = \operatorname{tr}(\mathbf{B}), \quad I_5 = \frac{1}{2} \operatorname{tr}(\mathbf{B}^2), \quad I_6 = \frac{1}{3} \operatorname{tr}(\mathbf{B}^3), \tag{6}$$

$$I_7 = \operatorname{tr}(\mathbf{TB}), \quad I_8 = \operatorname{tr}(\mathbf{T}^2 \mathbf{B}) \quad I_9 = \operatorname{tr}(\mathbf{T} \mathbf{B}^2), \quad I_{10} = \operatorname{tr}(\mathbf{T}^2 \mathbf{B}^2). \tag{7}$$

In the particular case that $\alpha_j = 0, j = 4, 5, 6, 7, 8$ and that α_i, $i = 0, 1, 2, 3$ do not depend on $\mathbf{B}$ (with $\alpha_3 \neq 0$), from (4) we obtain

$$\mathbf{B} = \mathfrak{h}(\mathbf{T}) = \beta_0 \mathbf{I} + \beta_1 \mathbf{T} + \beta_2 \mathbf{T}^2, \tag{8}$$

where the scalar functions β_0, β_1 and β_2 depend on the invariants I_1, I_2 and I_3 from (5).

In (4), (8) and in the expressions for the constitutive equations to be presented later on, we work with a dimensionless stress tensor $\frac{1}{\sigma_o}\mathbf{T}$, where σ_o is a characteristic value for the stress. We do not use a different notation for that stress.

In (**30**) a discussion was presented on restrictions on $\mathfrak{h}(\mathbf{T})$ such that the body is actually elastic, in the sense of not dissipating mechanical work into heat for any motion. It was found that if one assumes the existence of a scalar function $\Pi = \Pi(I_1, I_2, I_3)$ such that $\mathbf{B} = \frac{\partial \Pi}{\partial \mathbf{T}}$, it is easier to impose such conditions on Π, which we do not discuss here, such that the body is elastic and (8) becomes

$$\mathbf{B} = \Pi_1 \mathbf{I} + \Pi_2 \mathbf{T} + \Pi_3 \mathbf{T}^2, \tag{9}$$

where $\Pi_i = \frac{\partial \Pi}{\partial I_i}$, $i = 1, 2, 3$. The scalar function Π does not have a direct physical meaning, but it is related to the elastic energy stored by the body as shown in (**30**).

If λ_i, $i = 1, 2, 3$ are used to denote the principal values of $\mathbf{F}$ and σ_i, $i = 1, 2, 3$ denote the principal stresses, then from (9) it is easy to obtain the alternative representation

$$\lambda_i^2 = \frac{\partial \Pi}{\partial \sigma_i}, \tag{10}$$

where $\Pi(\mathbf{T}) = \Pi(\sigma_1, \sigma_2, \sigma_3) = \Pi(\sigma_2, \sigma_1, \sigma_3) = \Pi(\sigma_1, \sigma_3, \sigma_2)$. In this work the simpler form (10) will be used in the subsequent sections.

Since $\lambda_i^2 > 0$, $i = 1, 2, 3$ we have the restriction (see Section 4.1 in (**30**), Section 2.4 of (**28**) and Section 2.2 of (**29**)):

$$\frac{\partial \Pi}{\partial \sigma_i} > 0, \quad i = 1, 2, 3. \tag{11}$$

If in the reference configuration the body is unstressed and unstrained, we have the condition

$$\frac{\partial \Pi}{\partial \sigma_i}(0, 0, 0) = 1, \quad i = 1, 2, 3. \tag{12}$$

An additional restriction is presented now. If $\sigma_a > \sigma_b$ then we propose the constitutive inequality

$$\frac{\partial \Pi}{\partial \sigma_a} > \frac{\partial \Pi}{\partial \sigma_b}, \tag{13}$$

which implies that $\lambda_a^2 > \lambda_b^2$. The physical meaning of (13) is that we are requiring that the largest stretch is produced in the same direction of the largest principal stress. This restriction (13) can be re-written as

$$(\sigma_a - \sigma_b)\left(\frac{\partial \Pi}{\partial \sigma_a} - \frac{\partial \Pi}{\partial \sigma_b}\right) > 0,$$

which is assumed to be valid if $\sigma_a \neq \sigma_b$, $a, b = 1, 2, 3$, and if $\sigma_a = \sigma_b$ we further assume that $\frac{\partial \Pi}{\partial \sigma_a} = \frac{\partial \Pi}{\partial \sigma_b}$. The above restriction can be presented as $(\sigma_a - \sigma_b)(\lambda_a^2 - \lambda_b^2) > 0$, and since $\lambda_i > 0$ the above inequality would be equivalent to $(\sigma_a - \sigma_b)(\lambda_a - \lambda_b) > 0$, which is the well-known Baker–Ericksen inequality[2] (see (**39**) and Section 51 of (**46**)). It is interesting to mention that the Baker–Ericksen and the empirical inequalities (see Section 51 of (**46**)) have been used to study boundary value problems, considering the classical Cauchy isotropic elastic body $\mathbf{T} = \mathfrak{g}(\mathbf{B})$, where the stress distribution $\mathbf{T}$ is given as data, and $\mathbf{B}$ must be obtained by inverting $\mathbf{T} = \mathfrak{g}(\mathbf{B})$, see for example, (**31**, **40–42**). In the next section, more comments are given about the comparison of the results presented in those works with the results shown in this communication.

The inequalities presented in (11), (12) and (13), and some additional expressions to be proposed in Section 3, impose restrictions on the function Π which are not easy to satisfy. In particular (11) preclude the use of some simple expressions for Π, as explained in more detail at the beginning of Section 3.2.

[2] Actually, the Baker–Ericksen inequality is presented as $(\sigma_a - \sigma_b)(\lambda_a - \lambda_b) > 0$ if $\lambda_a \neq \lambda_b$, that is, the difference with our case is the interchange of the stretches with the principal stresses as the independent variables.

2.3 *On the boundary value problem and the semi-inverse method*

In the classical theory of non-linear elasticity considering a Cauchy elastic body $\mathbf{T} = \mathfrak{g}(\mathbf{B})$, the procedure used to solve some boundary value problems is based on assuming some simplified expressions for the deformation field[3] $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, which is used to calculate $\mathbf{B}$, from where we obtain the components of the stress tensor from $\mathbf{T} = \mathfrak{g}(\mathbf{B})$, which are replaced in (2). In general, the idea is to reduce the original non-linear partial differential equations into simpler partial or ordinary differential equations, which can be solved exactly or numerically.

In our case we are interested in studying constitutive equations of the form $\mathbf{B} = \mathfrak{h}(\mathbf{T})$. In the linearised theory of elasticity, when expressing the linearised strain tensor in terms of the stresses, the classical procedure to solve some boundary value problems is to express the stress tensor in terms of a stress potential, such that (2) would be satisfied automatically. Thereafter, the strains in terms of such stress potential are replaced into the compatibility equation, from where the biharmonic equation is obtained (see, for example, (**43**)). In the present case, considering (8)–(10), such a procedure would not be practical, due to the highly non-linear structure of the compatibility equations for large elastic deformations (see, for example, Section 34 of (**36**)), and also due to the in general non-linear nature of the functions $\mathfrak{h}(\mathbf{T})$. Therefore, in the present work we adopt the method used, for example, in (**19**, **44**) for the case of the linearised strain tensor given as a non-linear function of the Cauchy stress tensor $\boldsymbol{\varepsilon} = \mathfrak{f}(\mathbf{T})$. That procedure is based on assuming a simplified expression for the stress tensor $\mathbf{T}$, and in parallel on assuming as well a possible expression for the deformation field $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ that such a stress tensor may produce. Thereafter, we solve in parallel (2) and (8), that is, we look for $\mathbf{T}$ and $\mathbf{x}$ such that

$$\operatorname{div}\mathbf{T} + \rho\mathbf{b} = \mathbf{0}, \qquad \frac{\partial \mathbf{x}}{\partial \mathbf{X}}\frac{\partial \mathbf{x}}{\partial \mathbf{X}}^{\mathrm{T}} = \mathfrak{h}(\mathbf{T}) \tag{14}$$

are satisfied simultaneously. It is necessary to mention that when the deformation field $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ is prescribed, there is no need to consider the compatibility equations as discussed in (**45**). In (14) for a fully three-dimensional problem we have three equilibrium equations plus the six components from the constitutive equations, therefore, in total we have nine equations. Such nine equations must be solved to obtain the six independent components of the Cauchy stress tensor, plus the three components of the deformation vector $\mathbf{x}$, that is, we have nine equations for nine unknowns.

3. A specific expression for the constitutive relation. Solutions of some boundary value problems considering homogeneous deformations and stresses

In this and in the following sections we choose the following particular form for the function Π

$$\begin{aligned}\Pi(\sigma_1, \sigma_2, \sigma_3) = \sigma_1 + \sigma_2 + \sigma_3 + \mathfrak{h}_1(\sigma_1) + \mathfrak{h}_1(\sigma_2) + \mathfrak{h}_1(\sigma_3) + \mathfrak{h}_2(\sigma_1)(\sigma_2 + \sigma_3)\\ \mathfrak{h}_2(\sigma_2)(\sigma_1 + \sigma_3) + \mathfrak{h}_2(\sigma_3)(\sigma_1 + \sigma_2) + \mathfrak{h}_3\left(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right),\end{aligned} \tag{15}$$

where $\mathfrak{h}_1$, $\mathfrak{h}_2$ and $\mathfrak{h}_3$ are single valued functions.

From (15) the restriction (12) is met if

$$\mathfrak{h}_1'(0) = 0, \quad \mathfrak{h}_3'(0) = 0, \quad \mathfrak{h}_2(0) = 0, \tag{16}$$

[3] This is the idea of the the semi-inverse method, see, for example, (**46**).

where we use the notation $\mathfrak{h}'(x)=\frac{d\mathfrak{h}}{dx}$. Regarding the inequality (13) in the case of (15) it becomes

$$\mathfrak{h}_1'(\sigma_1)-\mathfrak{h}_1'(\sigma_2)+\mathfrak{h}_2'(\sigma_1)\sigma_2-\mathfrak{h}_2'(\sigma_2)\sigma_1+\mathfrak{h}_2(\sigma_2)-\mathfrak{h}_2(\sigma_1)>0, \quad \text{if} \quad \sigma_1>\sigma_2>0. \tag{17}$$

These restrictions hold for the rest of the article.

The specific expression for Π shown in (15) can be explained as follow: The first part $\sigma_1+\sigma_2+\sigma_3$ is such that when derived and when we consider $\sigma_i=0$ we have that $\lambda_j=1$ (see (12) and (16)). Regarding $\mathfrak{h}_1(\sigma_1)+\mathfrak{h}_1(\sigma_2)+\mathfrak{h}_1(\sigma_3)$, this can be seen as the counterpart for our new constitutive theory of the model by Vanalis and Landel (see Equation (29) of (**47**)). About the part $\mathfrak{h}_2(\sigma_1)(\sigma_2+\sigma_3)+\mathfrak{h}_2(\sigma_2)(\sigma_1+\sigma_3)+\mathfrak{h}_2(\sigma_3)(\sigma_1+\sigma_2)$, this would be used in order to model the lateral expansion-compression that in the linearised theory of elasticity is captured with the Poisson ratio, that is, it would be the non-linear counterpart of that simple model. Finally, $\mathfrak{h}_3\left(\frac{\sigma_1+\sigma_2+\sigma_3}{3}\right)$ would be used to model in a separate manner the effect of the spherical part of the stress $\frac{\sigma_1+\sigma_2+\sigma_3}{3}$ in the behaviour of the body.

3.1 *Homogeneous deformation and stress distributions*

In this section, we study some problems where the stresses and the components of the deformation gradient are constant. If the Cauchy stress tensor does not depend on the position $\mathbf{X}$, then if we neglect the body forces the equilibrium equations (2) are satisfied automatically. If we assume expressions for $\mathbf{x}=\boldsymbol{\chi}(\mathbf{X})$ for which $\frac{\partial\mathbf{x}}{\partial\mathbf{X}}$ does not depend on the position $\mathbf{X}$, the relations (8), (10) can be used to find such $\frac{\partial\mathbf{x}}{\partial\mathbf{X}}$ and $\mathbf{x}=\boldsymbol{\chi}(\mathbf{X})$ in terms of the stresses. Some of these problems have been already studied in (**30**).

3.1.1 Traction of a cylinder.

Let us consider the cylinder described in the reference configuration as

$$0\le R\le R_o, \quad 0\le\Theta\le 2\pi, \quad 0\le Z\le L. \tag{18}$$

It is assumed that this cylinder deforms under the influence of the uniform stress tensor

$$\mathbf{T}=\sigma_z\mathbf{e}_z\otimes\mathbf{e}_z, \tag{19}$$

where σ_z is constant. In this case the principal stresses are $\sigma_1=\sigma_z$, $\sigma_2=\sigma_3=0$.

Let us suppose that the cylinder deforms as

$$r=\lambda_r R, \quad \theta=\Theta, \quad z=\lambda_z Z, \tag{20}$$

where λ_r, λ_z are constants.

For this and the following boundary value problems we may have more than one solution, but in this article we do not study such possible non-uniqueness.

We have $\lambda_1=\lambda_z$, $\lambda_2=\lambda_3=\lambda_r$. From (10) we obtain

$$\lambda_z^2=\frac{\partial\Pi}{\partial\sigma_1}(\sigma_z,0,0), \quad \lambda_r^2=\frac{\partial\Pi}{\partial\sigma_2}(\sigma_z,0,0), \tag{21}$$

which in the case of considering (15) become

$$\lambda_z^2 = 1 + \mathfrak{h}_1'(\sigma_z) + \frac{1}{3}\mathfrak{h}_3'\left(\frac{\sigma_z}{3}\right), \tag{22}$$

$$\lambda_r^2 = 1 + \mathfrak{h}_2(\sigma_z) + \mathfrak{h}_2'(0)\sigma_z + \frac{1}{3}\mathfrak{h}_3'\left(\frac{\sigma_z}{3}\right). \tag{23}$$

From the physical point of view for this problem we can request that the function Π satisfies the inequalities

$$\sigma_z > 0 \quad \Rightarrow \quad \lambda_z^2 > 1, \quad 0 < \lambda_r^2 < 1, \tag{24}$$

and

$$\sigma_z < 0 \quad \Rightarrow \quad \lambda_r^2 > 1, \quad 0 < \lambda_z^2 < 1. \tag{25}$$

The physical meanings of these inequalities is that for a cylinder deforming only due to the effect of a uniform axial stress distribution, we expect that under traction the cylinder is stretched (and in the radial direction it is compressed), whereas the application of a compressive axial stress should produce an axial compression and a radial expansion.

The restriction $(25)_2$ can be complemented recognising that in compression, when the stress σ_z is very large, λ_z should approach zero from above (remembering that $\lambda_i > 0$ always), therefore we add the restriction

$$\sigma_z \to -\infty \quad \Rightarrow \quad \lambda_z \to 0. \tag{26}$$

For λ_r we can also propose a similar restriction, in this case when we have traction. If the stress is positive, independently of the behaviour of λ_z, we can expect that for the cylinder there is a limit for λ_r to become small[4], that is:

$$\sigma_z \to \infty \quad \Rightarrow \quad \lambda_r \to 0. \tag{27}$$

From (22), (23) the two restrictions (24), (25) become

$$\sigma_z > 0 \quad \Rightarrow \quad \mathfrak{h}_1'(\sigma_z) + \frac{1}{3}\mathfrak{h}_3'\left(\frac{\sigma_z}{3}\right) > 0, \quad -1 < \mathfrak{h}_2(\sigma_z) + \mathfrak{h}_2'(0)\sigma_z + \frac{1}{3}\mathfrak{h}_3'\left(\frac{\sigma_z}{3}\right) < 0, \tag{28}$$

$$\sigma_z < 0 \quad \Rightarrow \quad -1 < \mathfrak{h}_1'(\sigma_z) + \frac{1}{3}\mathfrak{h}_3'\left(\frac{\sigma_z}{3}\right) < 0, \quad \mathfrak{h}_2(\sigma_z) + \mathfrak{h}_2'(0)\sigma_z + \frac{1}{3}\mathfrak{h}_3'\left(\frac{\sigma_z}{3}\right) > 0. \tag{29}$$

It is interesting to compare the results presented above with the study of Batra (**41**) and Marzano (**42**), where the same problem of calculating the expression for **B** was considered, for the case **T** is given as in (19), but for an isotropic Cauchy elastic body $\mathbf{T} = \mathfrak{g}(\mathbf{B})$.

3.1.2 Shear of a slab. Let us consider the slab defined in the reference configuration as

$$-\frac{L_i}{2} \leq X_i \leq \frac{L_i}{2}, \quad i = 1, 2, 3. \tag{30}$$

[4] We could have materials for which there exists a, b with $0 \leq a < 1, 0 \leq b < 1$ such that for this problem $\sigma_z \to -\infty \Rightarrow \lambda_z \to a$ and $\sigma_z \to \infty \Rightarrow \lambda_r \to b$, but for simplicity in this work we do not consider such more general cases.

This slab is under the stress distribution

$$\mathbf{T} = \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \tag{31}$$

where τ is constant. The principal stresses are

$$\sigma_1 = \tau, \quad \sigma_2 = -\tau, \quad \sigma_3 = 0. \tag{32}$$

We assume that the slab deforms as

$$x_1 = \lambda_a X_1 + \kappa X_2, \quad x_2 = \lambda_b X_2, \quad x_3 = \lambda_c X_3, \tag{33}$$

where λ_a, λ_b, λ_c and κ are constants. The tensor $\mathbf{B}$ is of the form

$$\mathbf{B} = (\lambda_a^2 + \kappa^2)\mathbf{e}_1 \otimes \mathbf{e}_1 + \kappa\lambda_b(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + \lambda_b^2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_c^2 \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{34}$$

The eigenvectors of $\mathbf{T}$ and $\mathbf{B}$ should be the same if Π is an isotropic function (see (10)), and it is easy to see that that is the case if

$$\lambda_b^2 = \lambda_a^2 + \kappa^2. \tag{35}$$

Considering the above relation from (34) we obtain the principal stretches

$$\lambda_1 = \sqrt{\lambda_a^2 + \kappa^2 + \kappa\sqrt{\lambda_a^2 + \kappa^2}}, \tag{36}$$

$$\lambda_2 = \sqrt{\lambda_a^2 + \kappa^2 - \kappa\sqrt{\lambda_a^2 + \kappa^2}}, \tag{37}$$

$$\lambda_3 = \lambda_c. \tag{38}$$

From (10) considering (32) we have

$$\lambda_1^2 = \frac{\partial \Pi}{\partial \sigma_1}(\tau, -\tau, 0), \quad \lambda_2^2 = \frac{\partial \Pi}{\partial \sigma_2}(\tau, -\tau, 0), \quad \lambda_3^2 = \frac{\partial \Pi}{\partial \sigma_3}(\tau, -\tau, 0). \tag{39}$$

and from (36)–(38) we obtain

$$\lambda_1^2 = 1 + \mathfrak{h}_1'(\tau) - \mathfrak{h}_2'(\tau)\tau + \mathfrak{h}_2(-\tau), \tag{40}$$

$$\lambda_2^2 = 1 + \mathfrak{h}_1'(-\tau) + \mathfrak{h}_2'(-\tau)\tau + \mathfrak{h}_2(\tau), \tag{41}$$

$$\lambda_3^2 = 1 + \mathfrak{h}_2(\tau) + \mathfrak{h}_2(-\tau). \tag{42}$$

The results presented here can be compared with the results obtained, for example, by Destrade *et al*. (**31**) (see also (**40**)), where they considered the problem of a slab deforming due to a shear stress of the form (31) for an isotropic Green elastic solid. It is interesting to compare, for example, (33) with Equation (3.11) of that paper.

Unlike the case of the cylinder in traction/compression presented in the previous section, in the case of the shear of a slab is not simple to propose some restrictions similar to (24)–(27), therefore here we refrain from doing so.

3.1.3 A slab under a spherical stress. Consider the same slab described in (30), now deforming under the influence of the spherical stress

$$\mathbf{T} = \sigma_S \sum_{i=1}^{3} \mathbf{e}_i \otimes \mathbf{e}_i, \tag{43}$$

from where we have $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_S$.

It is assumed that this slab deforms as

$$x_i = \lambda X_i, \quad i = 1, 2, 3, \tag{44}$$

where λ is constant. From (44) we have that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ and from (10) we obtain

$$\lambda^2 = \frac{\partial \Pi}{\partial \sigma_1}(\sigma_S, \sigma_S, \sigma_S). \tag{45}$$

In this problem it is possible to propose the following inequalities or restrictions

$$\sigma_S > 0 \quad \Rightarrow \quad \lambda^2 > 1, \quad \sigma_S < 0 \quad \Rightarrow \quad 0 < \lambda^2 < 1. \tag{46}$$

The physical meaning of (46) is that for a compressible slab under a spherical stress, we expect a volumetric expansion if such stress is positive, and a compression if that stress is negative. Compare this with the P-C (Pressure-Compression) inequality presented in Section 51 of (**46**), which is of the form $(\sigma_S^* - \sigma_S)(\lambda^* - \lambda) > 0$ if $\sigma_S^* \neq \sigma_S$.

As in the case of the problem about the traction/compression of a cylinder presented in Section 3.1.1, here we can also propose something similar to (26) as:

$$\sigma_S \to -\infty \quad \Rightarrow \quad \lambda \to 0, \tag{47}$$

that is, we are saying that the slab becomes a point only when the spherical stress (in compression) would go to infinite.

From (15) we have

$$\lambda^2 = 1 + \mathfrak{h}_1'(\sigma_S) + 2\mathfrak{h}_2'(\sigma_S)\sigma_S + 2\mathfrak{h}_2(\sigma_S) + \frac{1}{3}\mathfrak{h}_3'(\sigma_S). \tag{48}$$

and from (46) and (48) we obtain

$$\sigma_S > 0 \quad \Rightarrow \quad \mathfrak{h}_1'(\sigma_S) + 2\mathfrak{h}_2'(\sigma_S)\sigma_S + 2\mathfrak{h}_2(\sigma_S) + \frac{1}{3}\mathfrak{h}_3'(\sigma_S) > 0, \tag{49}$$

$$\sigma_S < 0 \quad \Rightarrow \quad -1 < \mathfrak{h}_1'(\sigma_S) + 2\mathfrak{h}_2'(\sigma_S)\sigma_S + 2\mathfrak{h}_2(\sigma_S) + \frac{1}{3}\mathfrak{h}_3'(\sigma_S) < 0. \tag{50}$$

3.1.4 Biaxial stress on a plate. For the same slab described in (30), let us suppose that it deforms under the presence of the uniform stress

$$\mathbf{T} = \sigma_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_2 \mathbf{e}_2 \otimes \mathbf{e}_2. \tag{51}$$

This plate is supposed to deform as

$$x_i = \lambda_i X_i, \quad \text{no sum in } i, \quad i = 1, 2, 3. \tag{52}$$

Using this and (51) from (10) we obtain

$$\lambda_1^2 = \frac{\partial \Pi}{\partial \sigma_1}(\sigma_1, \sigma_2, 0), \quad \lambda_2^2 = \frac{\partial \Pi}{\partial \sigma_2}(\sigma_1, \sigma_2, 0), \quad \lambda_3^2 = \frac{\partial \Pi}{\partial \sigma_3}(\sigma_1, \sigma_2, 0), \tag{53}$$

and from (15) we have

$$\lambda_1^2 = 1 + \mathfrak{h}_1'(\sigma_1) + \mathfrak{h}_2'(\sigma_1)\sigma_2 + \mathfrak{h}_2(\sigma_2) + \frac{1}{3}\mathfrak{h}_3'\left(\frac{\sigma_1 + \sigma_2}{3}\right), \tag{54}$$

$$\lambda_2^2 = 1 + \mathfrak{h}_1'(\sigma_2) + \mathfrak{h}_2'(\sigma_2)\sigma_1 + \mathfrak{h}_2(\sigma_2) + \frac{1}{3}\mathfrak{h}_3'\left(\frac{\sigma_1 + \sigma_2}{3}\right), \tag{55}$$

$$\lambda_3^2 = 1 + \mathfrak{h}_2'(0)(\sigma_1 + \sigma_2) + \mathfrak{h}_2(\sigma_1) + \mathfrak{h}_2(\sigma_2) + \frac{1}{3}\mathfrak{h}_3'\left(\frac{\sigma_1 + \sigma_2}{3}\right). \tag{56}$$

3.2 *A particular expression for the constitutive relation*

In (15) we have three single valued functions to be found from experiments. Such functions could be found considering, for example, experimental results for the traction/compression of a cylindrical bar (the problem described in Section 3.1.1), and the volumetric expansion/compression of a slab presented in Section 3.1.3. If $\lambda_z(\sigma_z)$, $\lambda_r(\sigma_z)$ and $\lambda(\sigma_S)$ would be known from experiments, we could find $\mathfrak{h}_i(x)$, $i = 1, 2, 3$ by solving the ordinary differential equations (22), (23) and (48).

Considering that these new constitutive theories have been proposed very recently, and that it is still necessary to study for which specific materials such new theories provide a better choice than the classical Green elastic body, in this article we propose some prototype expressions for $\mathfrak{h}_i$, $i = 1, 2, 3$ instead looking for such functions from experiments. Such expressions must satisfy the different restrictions on Π or $\mathfrak{h}_i$, $i = 1, 2, 3$ mentioned in the previous sections. The restrictions (26) and (27) mean that for a cylinder in compression, we can expect that independently of the specific material being modelled, λ_z becomes constant and close to zero for $\sigma_z < 0$ if $|\sigma_z|$ is large enough, whereas the same happens for λ_r, but now in the case $\sigma_z > 0$ (traction). In the case of the restriction (47) for a slab in compression under a spherical stress, we can also expect that λ becomes constant and close to zero for σ_S large enough. These and the other restrictions on $\mathfrak{h}_i$, $i = 1, 2, 3$, especially $\lambda_j > 0$, are strong and not easy to be satisfied. In the present communication we use the following expressions for the functions $\mathfrak{h}_1$, $\mathfrak{h}_2$ and $\mathfrak{h}_3$:

$$\mathfrak{h}_1(x) = a_1\left[x \arctan\left(\frac{x}{b_1}\right) - \frac{1}{2}b_1 \ln\left(b_1^2 + x^2\right)\right] + c_1\left[x \arctan\left(\frac{x}{d_1}\right) - \frac{1}{2}d_1 \ln\left(d_1^2 + x^2\right)\right], \tag{57}$$

$$\mathfrak{h}_2(x) = a_2 \arctan\left(\frac{x}{b_2}\right) + c_2 \arctan\left(\frac{x}{d_2}\right), \tag{58}$$

$$\mathfrak{h}_3(x) = a_3\left[x \arctan\left(\frac{x}{b_3}\right) - \frac{1}{2}b_3 \ln\left(b_3^2 + x^2\right)\right] + c_3\left[x \arctan\left(\frac{x}{d_3}\right) - \frac{1}{2}d_3 \ln\left(d_3^2 + x^2\right)\right], \tag{59}$$

Table 1 Values for the constants used in (57)–(59)

a_1	b_1	c_1	d_1	a_2	b_2	c_2	d_2	a_3	b_3	c_3	d_3
1	1	−0.520	1	0.433	1	−0.520	1	0.433	1	-0.371	1

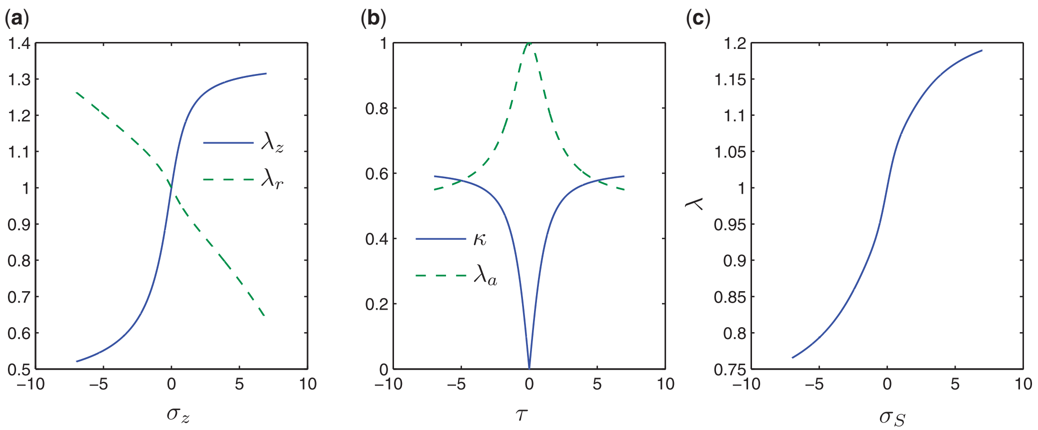

Fig. 1 (a) Results for λ_z and λ_r in terms of σ_z for the problem of uniform traction of a cylinder. (b) Results for κ and λ_a for the shear of a slab. (c) Results for a slab under a spherical stress

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2$ and c_3 are constants. From (57)–(59) we obtain

$$\mathfrak{h}_1'(x) = a_1 \arctan\left(\frac{x}{b_1}\right) + c_1 \arctan\left(\frac{x}{d_1}\right), \tag{60}$$

$$\mathfrak{h}_2'(x) = \frac{a_2}{b_2\left(1+\frac{x^2}{b_2^2}\right)} + \frac{c_2}{d_2\left(1+\frac{x^2}{d_2^2}\right)}, \tag{61}$$

$$\mathfrak{h}_3'(x) = a_3 \arctan\left(\frac{x}{b_3}\right) + c_3 \arctan\left(\frac{x}{d_3}\right). \tag{62}$$

In Table 1 some numerical values for the different constants are given.

In Fig. 1(a) results are presented for the traction (compression) of a cylinder, where we have λ_z and λ_r as functions of σ_z considering (22), (23) and (57)–(59). It is possible to see that the results satisfy (11) for the range of values considered for σ_z. In Fig. 1(b) results are presented for the shear of a slab (see (40)–(42)), in particular for λ_a^2 and κ as functions of τ. From (57)–(59) and (42) it is easy to see that for that particular expression for Π we have $\lambda_3^2 = 1$ for any τ. In Fig. 1(c) results are presented for λ as a function of σ_S for a slab under spherical stress considering (48), (57)–(59). Finally, in Fig. 2 we have a contour plot for $\mathfrak{h}_1'(\sigma_1) - \mathfrak{h}_1'(\sigma_2) + \mathfrak{h}_2'(\sigma_1)\sigma_2 - \mathfrak{h}_2'(\sigma_2)\sigma_1 + \mathfrak{h}_2(\sigma_2) - \mathfrak{h}_2(\sigma_1)$, for $\sigma_1 > \sigma_2 > 0$. It is possible to see that the inequality (17) is satisfied for Π given by (15), (57)–(59).

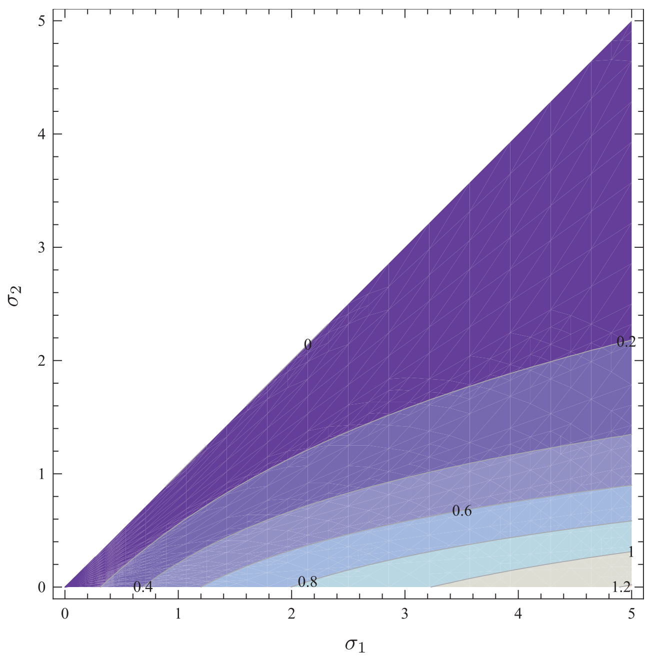

Fig. 2 Contour plot for $\mathfrak{h}_1'(\sigma_1) - \mathfrak{h}_1'(\sigma_2) + \mathfrak{h}_2'(\sigma_1)\sigma_2 - \mathfrak{h}_2'(\sigma_2)\sigma_1 + \mathfrak{h}_2(\sigma_2) - \mathfrak{h}_2(\sigma_1)$ for the problem of biaxial traction of a plate, where $\sigma_1 > \sigma_2 > 0$

4. Non-homogeneous deformations

In this section some boundary value problems are solved, wherein we have non-homogeneous distributions for the stresses and strains, considering the particular expression for Π given in Sections 3 and 3.2. Some of these problems have been studied already by Rajagopal and Saravanan considering (8) (see **28**, **29**), using different expressions for the function $\mathfrak{h}(\mathbf{T})$.

The problems to be studied in this section correspond to the inflation of a sphere, the flexure of a slab and the closing, radial expansion and stretching of an opened cylindrical tube. In the classical theory of non-linear elasticity when we assume $\mathbf{T} = \mathfrak{g}(\mathbf{B})$, exact solutions have been obtained for the above problems (see, for example, Section 57 of (**46**)). In that context the first problem has been treated by Green and Shield (**48**), whereas the problem of the flexure of a slab was considered originally by Rivlin (**49**) and thereafter by Ericksen (**50**). The problem of the opened cylindrical tube was studied originally by Rivlin (**51**) (see footnote 1, pp. 189 of (**46**) for a detailed account about that problem).

4.1 *Inflation of a sphere*

Let us consider the sphere defined in the reference configuration as

$$R_{\mathrm{i}} \leq R \leq R_{\mathrm{o}}, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq \Phi \leq \pi. \tag{63}$$

This sphere is assumed to deform under the influence of a Cauchy stress tensor field of the form

$$\mathbf{T} = \sigma_r(r)\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_\theta(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_\phi(r)\mathbf{e}_\phi \otimes \mathbf{e}_\phi. \tag{64}$$

It is assumed that $\sigma_\theta(r) = \sigma_\phi(r)$.

Under the influence of this stress tensor we suppose that the sphere deforms as

$$r = f(R), \quad \theta = \Theta, \quad \phi = \Phi, \tag{65}$$

from where we obtain

$$\mathbf{B} = (f')^2\mathbf{e}_r \otimes \mathbf{e}_r + \left(\frac{f}{R}\right)^2 \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \left(\frac{f}{R}\right)^2 \mathbf{e}_\phi \otimes \mathbf{e}_\phi, \tag{66}$$

where we use the notation $f' = \frac{df}{dR}$.

For this problem we have $\sigma_1 = \sigma_r, \sigma_2 = \sigma_3 = \sigma_\theta$ and $\lambda_1 = f'(R), \lambda_2 = \lambda_3 = \frac{f(R)}{R}$. The components σ_r, σ_θ and the function f must satisfy the equilibrium equation (2) and the constitutive relation (10):

$$\frac{d\sigma_r}{dr} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0 \quad \Leftrightarrow \quad \frac{d\sigma_r}{dR} + \frac{2f'}{f}(\sigma_r - \sigma_\theta) = 0, \tag{67}$$

$$(f')^2 = \frac{\partial \Pi}{\partial \sigma_r}, \quad \left(\frac{f}{R}\right)^2 = \frac{\partial \Pi}{\partial \sigma_\theta}, \tag{68}$$

where Π is given in (15), (57)–(59). These three equations must be solved to find the functions $\sigma_r(R)$, $\sigma_\theta(R)$ and $f(R)$.

For the particular expression for Π given in (15), (57)–(59) it has not been possible to find exact solutions for $(67)_2$ and (68), therefore, such equations are solved numerically using the finite element method and the program Comsol 3.4 (**52**). To do that, let us rewrite the equations in the following manner. From $(68)_2$ we have

$$f(R) = R\sqrt{\frac{\partial \Pi}{\partial \sigma_\theta}}, \tag{69}$$

therefore, in $(68)_1$ we obtain

$$\frac{d}{dR}\left(R\sqrt{\frac{\partial \Pi}{\partial \sigma_\theta}}\right) = \sqrt{\frac{\partial \Pi}{\partial \sigma_r}}. \tag{70}$$

Using (69) and $f' = \sqrt{\frac{\partial \Pi}{\partial \sigma_r}}$ (see $(68)_1$), $(67)_2$ becomes

$$-\frac{d\sigma_r}{dR} = 2\frac{\sqrt{\frac{\partial \Pi}{\partial \sigma_r}}}{R\sqrt{\frac{\partial \Pi}{\partial \sigma_\theta}}}(\sigma_r - \sigma_\theta). \tag{71}$$

We have eliminated the function $f(R)$ from the list of unknowns, and (70), (71) can be used to obtain $\sigma_r(R)$ and $\sigma_\theta(R)$.

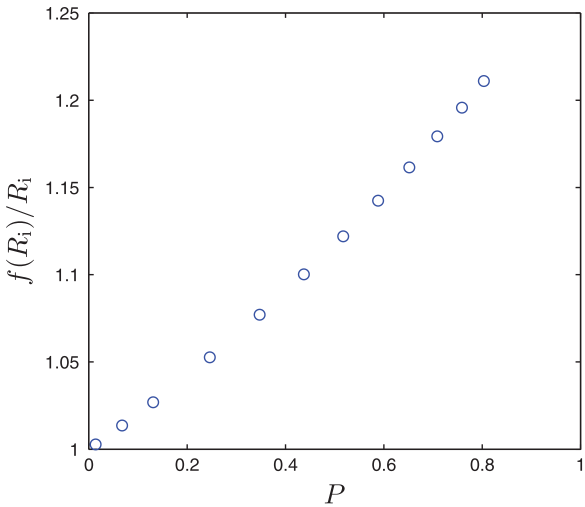

Fig. 3 Dimensionless radial expansion of the inner surface of the sphere as a function of the pressure

To solve (70), (71) using the finite element method we assume there exist functions $\varsigma_r(R)$ and $\varsigma_\theta(R)$ such that

$$\sigma_r(R) = \frac{d\varsigma_r}{dR}, \quad \sigma_\theta(R) = \frac{d\varsigma_\theta}{dR}. \tag{72}$$

Regarding the boundary conditions, let us assume that for the inner surface of the sphere we apply a pressure P, whereas we assume a stress free condition for the outer surface, that is:

$$\sigma_r(R_\mathrm{i}) = -P, \quad \sigma_r(R_\mathrm{o}) = 0. \tag{73}$$

The boundary condition $(73)_2$ is satisfied directly, but $(73)_1$ is satisfied indirectly by assuming

$$\varsigma_\theta(R_\mathrm{i}) = \varsigma_{\theta_\mathrm{i}}, \tag{74}$$

where $\varsigma_{\theta_\mathrm{i}}$ is a given value for ς_θ. Additionally we assume

$$\varsigma_\theta(R_\mathrm{o}) = 0, \quad \varsigma_r(R_\mathrm{i}) = 0. \tag{75}$$

In Figs. 3 and 4 some results are presented for the particular case $R_\mathrm{i} = 0.2$m and $R_\mathrm{o} = 0.4$m. In Fig. 3 we see the behaviour of $f(R_\mathrm{i})/R_\mathrm{i}$ (the dimensionless expansion of the inner surface of the sphere) as a function of P. Notice that P and the different components of the stress tensor are dimensionless, where we have divided them by σ_o (see paragraph after (8)).

In Fig. 4 for the case[5] $P \approx 0.803$ results for $\sigma_r(R)$, $\sigma_\theta(R)$, $\frac{f(R)}{R_\mathrm{i}}$ and $J = \lambda_1\lambda_2^2$ are presented.

[5] The particular value for P, which was used to obtain the results presented in Fig. 4, was the maximum pressure that it was possible to apply without having problems of convergence for the finite element code.

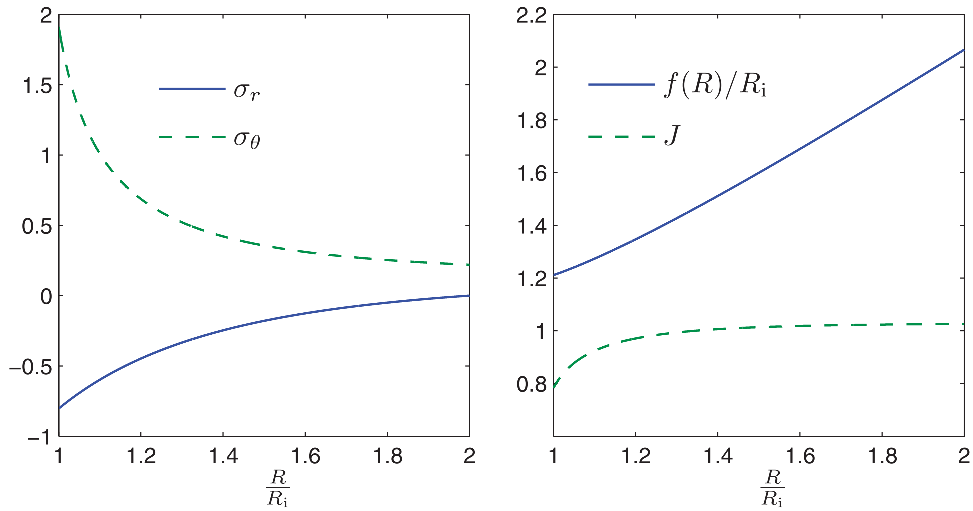

Fig. 4 On the left, distributions for the components of the stress. On the right, behaviour of $\frac{f(R)}{R_i}$ and J (in both cases $P = 0.803$)

4.2 *Flexure of a slab*

Consider the slab defined in the reference configuration as

$$0 \le X \le H, \quad -\frac{L_Y}{2} \le Y \le \frac{L_Y}{2}, \quad -\frac{L_Z}{2} \le Z \le \frac{L_Z}{2}, \tag{76}$$

where we use the notation X, Y, Z for X_i, $i = 1, 2, 3$, respectively. This slab is assumed to deform under the application of a stress tensor of the form (in the current configuration)

$$\mathbf{T} = \sigma_r(r)\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_\theta(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_z(r)\mathbf{e}_z \otimes \mathbf{e}_z. \tag{77}$$

We further assume that the slab deforms as

$$r = f(X), \quad \theta = kY, \quad z = \lambda Z, \tag{78}$$

where k, λ are constants and we impose the restriction $f(X) > 0$, $0 \le X \le H$. For this problem the left Cauchy–Green stretch tensor is given as

$$\mathbf{B} = (f')^2\mathbf{e}_r \otimes \mathbf{e}_r + (fk)^2\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda^2\mathbf{e}_z \otimes \mathbf{e}_z, \tag{79}$$

where $f' = \frac{df}{dX}$. We take $\sigma_1 = \sigma_r$, $\sigma_2 = \sigma_\theta$, $\sigma_3 = \sigma_z$, $\lambda_1 = f'(X)$, $\lambda_2 = kf(X)$ and $\lambda_3 = \lambda$.

From the equations of equilibrium and the constitutive relations (10) we obtain the following four equations, which can be solved to find $r = f(X)$, $\sigma_r = \sigma_r(r) = \sigma_r(X)$, $\sigma_\theta = \sigma_\theta(r) = \sigma_\theta(X)$ and

$\sigma_z = \sigma_z(r) = \sigma_z(X)$:

$$\frac{d\sigma_r}{dr} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0 \quad \Leftrightarrow \quad \frac{d\sigma_r}{dX} + \frac{f'}{f}(\sigma_r - \sigma_\theta) = 0, \tag{80}$$

$$(f')^2 = \frac{\partial \Pi}{\partial \sigma_r}, \quad (fk)^2 = \frac{\partial \Pi}{\partial \sigma_\theta}, \quad \lambda^2 = \frac{\partial \Pi}{\partial \sigma_z}. \tag{81}$$

These equations are also solved using the finite element method as in the previous section. In order to do so, we rewrite these equations in the following way. From $(81)_2$ we have

$$f(X) = \frac{1}{k}\sqrt{\frac{\partial \Pi}{\partial \sigma_\theta}}. \tag{82}$$

Using this in $(81)_1$ we obtain

$$\frac{d}{dX}\left(\sqrt{\frac{\partial \Pi}{\partial \sigma_\theta}}\right) = k\sqrt{\frac{\partial \Pi}{\partial \sigma_r}}. \tag{83}$$

From $(81)_1$ we have $f'(X) = \sqrt{\frac{\partial \Pi}{\partial \sigma_r}}$ and using (82) in $(80)_2$ we obtain

$$-\frac{d\sigma_r}{dX} = k\frac{\sqrt{\frac{\partial \Pi}{\partial \sigma_r}}}{\sqrt{\frac{\partial \Pi}{\partial \sigma_\theta}}}(\sigma_r - \sigma_\theta). \tag{84}$$

Finally, $(81)_3$, which could be considered as an algebraic equation that can be used to find, for example, σ_z, is converted into a differential equation by taking its derivative in X, becoming

$$0 = \frac{d}{dX}\left(\frac{\partial \Pi}{\partial \sigma_z}\right). \tag{85}$$

As in Section 4.1, in order to solve (83)–(85) with the finite element method, we assume the existence of functions $\varsigma_r(X)$, $\varsigma_\theta(X)$ and $\varsigma_z(X)$ such that

$$\sigma_r(X) = \frac{d\varsigma_r}{dX}, \quad \sigma_\theta(X) = \frac{d\varsigma_\theta}{dX}, \quad \sigma_z(X) = \frac{d\varsigma_z}{dX}. \tag{86}$$

Regarding the boundary conditions, we assume that the upper and lower surfaces of the slab are free of traction, that is:

$$\sigma_r(0) = 0, \quad \sigma_r(H) = 0. \tag{87}$$

Regarding $(81)_3$, since we are solving it considering the alternative representation (85), we impose the condition

$$\frac{\partial \Pi}{\partial \sigma_z}(X = 0) = \lambda^2. \tag{88}$$

Two additional boundary conditions are

$$\varsigma_\theta(H) = 0, \quad \varsigma_z(H) = 0. \tag{89}$$

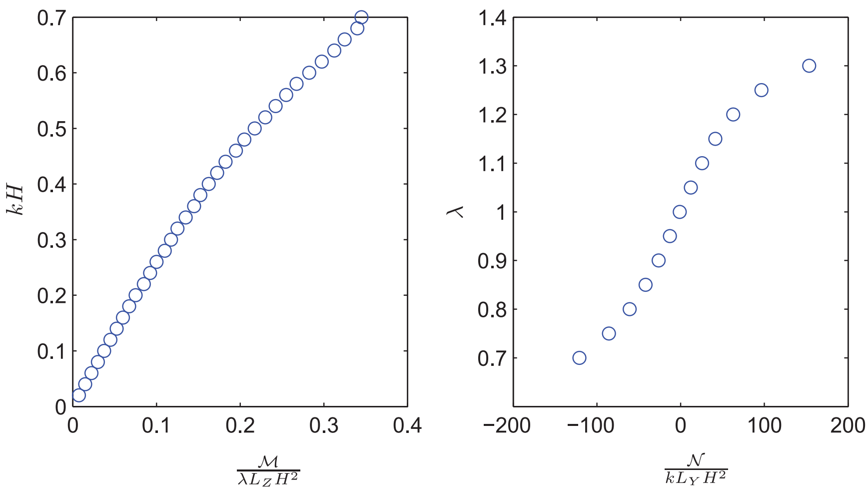

Fig. 5 (a) The results for kH in terms of the dimensionless total bending moment. (b) The results for the axial stretching λ as a function of the dimensionless total axial force

The boundary condition $(87)_2$ is imposed directly, but regarding $(87)_1$ we use instead

$$\varsigma_\theta(0) = \varsigma_{\theta_\mathrm{i}}, \tag{90}$$

where $\varsigma_{\theta_\mathrm{i}}$ is a given value for that constant such that $(87)_1$ is satisfied.

In Figs 5 and 6 some results are presented for the particular case $H = 0.2\mathrm{m}$. In Fig. 5(a) results are presented for kH (see (78)) in terms of the dimensionless total bending moment $\frac{\mathcal{M}}{\lambda L_Z H^2}$, where $\mathcal{M} = \int_{\mathscr{S}_\theta} \sigma_\theta (r - r_\mathrm{i})\, da$ and $\mathscr{S}_\theta$ is the surface defined in the current configuration as $\theta = \frac{kL_Y}{2}$. We have that

$$\frac{\mathcal{M}}{\lambda L_Z H^2} = \frac{1}{H^2} \int_{r_\mathrm{i}}^{r_\mathrm{o}} \sigma_\theta(r)(r - r_\mathrm{i})\, dr, \tag{91}$$

where r_i and r_o are the inner and outer radii of the bent slab, that is, $r_\mathrm{i} = f(0)$, $r_\mathrm{o} = f(H)$. In order to obtain such results it was assumed that $\lambda = 1$. It is necessary to indicate that the condition $(87)_1$ was satisfied indirectly by considering (90) using the bisection method. In Fig. 5(b) results are presented for λ the axial stretch (see (78)) as a function of the dimensionless total axial force $\frac{\mathcal{N}}{kL_Y H^2}$, where $\mathcal{N} = \int_{\mathscr{S}_z} \sigma_z\, da$ and $\mathscr{S}_z$ is the surface defined (in the current configuration) as $z = \frac{\lambda L_Z}{2}$. We have

$$\frac{\mathcal{N}}{kL_Y H^2} = \frac{1}{H^2} \int_{r_\mathrm{i}}^{r_\mathrm{o}} \sigma_z(r) r\, dr. \tag{92}$$

For those results (Fig. 5(b)) we assumed that $k = 2\frac{1}{\mathrm{m}}$.

Finally, in Fig. 6 results for $\sigma_r(X)$, $\sigma_\theta(X)$, $\sigma_z(X)$, $f(X)/H$ and $J = \lambda_1\lambda_2\lambda_3$ are presented for the case $\lambda = 1$, $k = 3.5\frac{1}{\mathrm{m}}$.

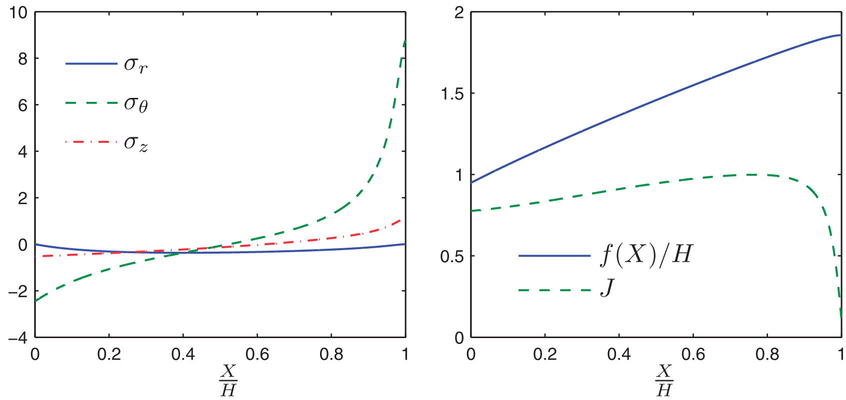

Fig. 6 Distributions for the components of the stress tensor, $f(X)/H$ and J for the case $\lambda = 1$ and $k = 3.5\frac{1}{\mathrm{m}}$

4.3 *Closing, radial expansion and stretching of a cylindrical tube*

In this last problem we consider the cylindrical and opened tube defined in the reference configuration as

$$R_\mathrm{i} \leq R \leq R_\mathrm{o}, \quad 0 \leq \Theta \leq 2\pi - \alpha, \quad 0 \leq Z \leq L, \tag{93}$$

where α is the initial opening angle. This tube is assumed to be deformed under the action of a stress tensor field of the form

$$\mathbf{T} = \sigma_r(r)\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_\theta(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_z(r)\mathbf{e}_z \otimes \mathbf{e}_z. \tag{94}$$

This stress field is supposed to deform the tube as

$$r = f(R), \quad \theta = k\Theta, \quad z = \lambda Z, \tag{95}$$

where $k = \frac{2\pi}{2\pi - \alpha}$ and λ is a positive constant. Under this deformation the opened tube is closed. From (95) the left Cauchy–Green deformation tensor becomes

$$\mathbf{B} = (f')^2\mathbf{e}_r \otimes \mathbf{e}_r + \left(\frac{fk}{R}\right)^2 \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda^2\mathbf{e}_z \otimes \mathbf{e}_z, \tag{96}$$

where $f' = \frac{df}{dR}$.

From the equations of equilibrium and the constitutive relations, considering the above expressions for the stress tensor and the left Cauchy–Green stretch tensor, we obtain the following four equations that can be used to find $r = f(R)$, $\sigma_r = \sigma_r(r) = \sigma_r(R)$, $\sigma_\theta = \sigma_\theta(r) = \sigma_\theta(R)$ and $\sigma_z = \sigma_z(r) = \sigma_z(R)$:

$$\frac{d\sigma_r}{dr} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0 \quad \Leftrightarrow \quad \frac{d\sigma_r}{dR} + \frac{f'}{f}(\sigma_r - \sigma_\theta) = 0, \tag{97}$$

$$(f')^2 = \frac{\partial \Pi}{\partial \sigma_r}, \quad \left(\frac{fk}{R}\right)^2 = \frac{\partial \Pi}{\partial \sigma_\theta}, \quad \lambda^2 = \frac{\partial \Pi}{\partial \sigma_z}, \tag{98}$$

where we have taken $\sigma_1 = \sigma_r$, $\sigma_2 = \sigma_\theta$, $\sigma_3 = \sigma_z$, $\lambda_1 = f'(R)$, $\lambda_2 = \frac{f(R)k}{R}$ and $\lambda_3 = \lambda$.

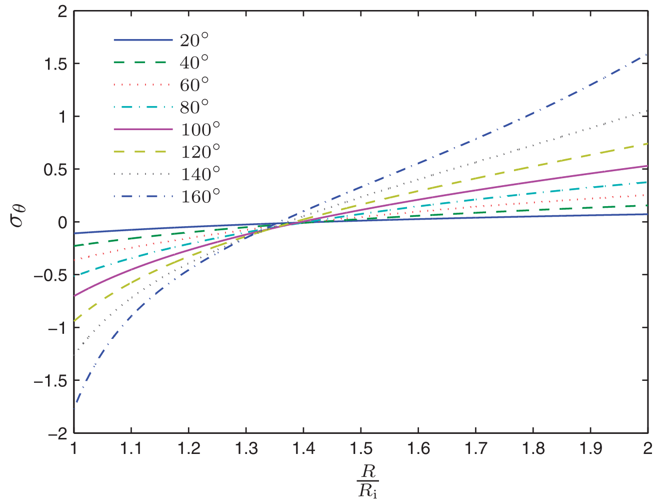

Fig. 7 Distribution for the circumferential stress considering different opening angles (in degrees), assuming $\lambda = 1$ and $\sigma_r(R_\mathrm{i}) = 0$

These equations are solved using the finite element method, the procedure to solve these equations is the same as outlined in the previous two Sections 4.1 and 4.2, and so we do not repeat that here. Regarding the boundary conditions, we assume

$$\sigma_r(R_\mathrm{i}) = -P, \quad \sigma_r(R_\mathrm{o}) = 0. \tag{99}$$

An interesting case to study is when $P = 0$, where the stresses and strains appear only due to the closing of the tube.

In Figs. 7–9 some results are presented for a tube where $R_\mathrm{i} = 0.2\mathrm{m}$, $R_\mathrm{o} = 0.4\mathrm{m}$. In Fig. 7, the distribution of circumferential stress is presented for different values of α (in degrees), assuming $\lambda = 1$ and $\sigma_r(R_\mathrm{i}) = 0$ (no load on the inner surface of the tube).

In Fig. 8(a) we have results for λ in terms of the total dimensionless axial force $\frac{\mathcal{N}}{2\pi R_\mathrm{i}^2} = \frac{1}{R_\mathrm{i}^2}\int_{r_\mathrm{i}}^{r_\mathrm{o}} \sigma_z(r) r\, dr$, where $\mathcal{N} = \int_{\mathscr{S}_z} \sigma_z\, da$ and the surface $\mathscr{S}_z$ is defined by $z = \lambda L$. Such results where obtained assuming $\alpha = 120°$ and $\sigma_r(R_\mathrm{i}) = 0$. In Fig. 8(b) results are shown for the dimensionless inner radial expansion $f(R_\mathrm{i})/R_\mathrm{i}$ as a function of P (the inner pressure), assuming $\alpha = 120°$ and $\lambda = 1$.

Finally, in Fig. 9 results are presented for the different components of the stress, $f(R)/R_\mathrm{i}$ and $J = \lambda_1\lambda_2\lambda_3$, for the case $\lambda = 1$, $\alpha = 160°$ and $\sigma_r(R_\mathrm{i}) = 0$.

5. Final remarks

In this work, we have solved several boundary value problems for a relatively new class of constitutive relations, where the left Cauchy–Green stretch tensor is given as a non-linear function of the Cauchy stress tensor. Considering that so far it has not been possible to find exact solutions for the case of non-homogeneous distributions for the stresses and strains, we needed to propose a particular expression

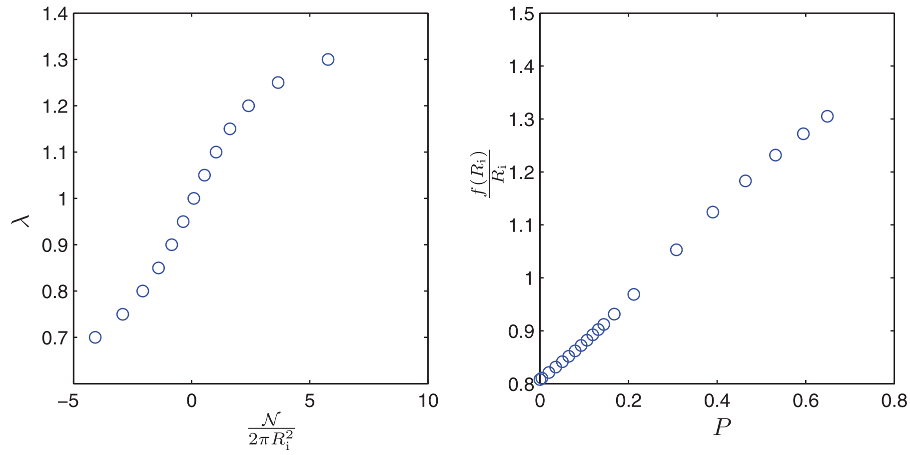

Fig. 8 (a) Axial stretch λ as a function of the dimensionless total axial force $\frac{\mathcal{N}}{2\pi R_{\mathrm{i}}^2}$, assuming $\alpha = 120°$ and $\sigma_r(R_{\mathrm{i}}) = 0$. (b) Radial expansion $\frac{f(R_{\mathrm{i}})}{R_{\mathrm{i}}}$ at $R = R_{\mathrm{i}}$ as a function of the applied pressure P, assuming $\alpha = 120°$ and $\lambda = 1$

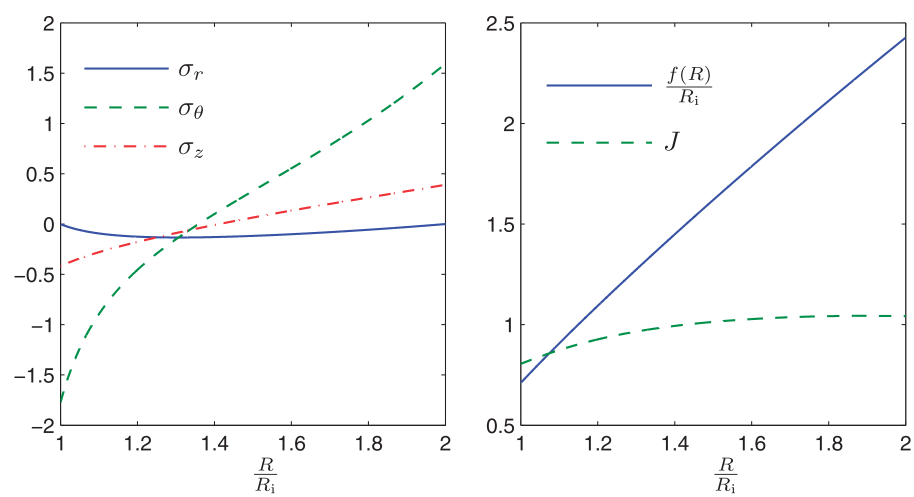

Fig. 9 Components of the stress tensor, $f(R)/R_{\mathrm{i}}$ and $J = \lambda_1\lambda_2\lambda_3$ for the case $\lambda = 1$, $\alpha = 160°$ and $\sigma_r(R_{\mathrm{i}}) = 0$

for the constitutive equation (see (15), (57)–(59)), which was used to solve three boundary value problems with the finite element method. That expression for Π is such that we have limiting strain behaviour for the different simple boundary value problems considering homogeneous distributions of stresses and strains, such as the extension of a cylinder (see Fig. 1(a)), the shear of a slab (see Fig. 1(b)) and the expansion-compression of a slab under a spherical stress distribution (see

Fig. 1(c)). From such results presented in Section 3 we see that the constitutive inequalities or restrictions proposed in Section 2.2 are satisfied. Regarding the results obtained in Section 4, we can observe that the distribution for the stresses are qualitatively similar to what it is observed for similar problems, considering the classical Green elastic body.

As indicated in the Introduction, this relatively new class of constitutive equation $\lambda_i^2 = \frac{\partial \Pi}{\partial \sigma_i}$ could be useful for problems where we have stress concentration and the strains are limited, and in the modelling of residually stressed bodies. To be able to asses the possible usefulness of this new theory for such problems, it is necessary to study that constitutive equation as much as possible from the theoretical and also the experimental point of view, and the present article aims to fill that gap (in this case addressing only some theoretical aspects), presenting some simple constitutive inequalities, exploring some possible expressions for the constitutive relations, and also showing a method to solve some simple boundary value problems.

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