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Additive representation of symmetric inverse M-matrices and potentials

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ABSTRACT

In this article we characterize the closed cones respectively generated by the symmetric inverse M-matrices and by the inverses of symmetric row diagonally dominant M-matrices. We show the latter has a finite number of extremal rays, while the former has infinitely many extremal rays. As a consequence we prove that every potential is the sum of ultrametric matrices.

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1. Introduction

Consider the cone $\mathcal{K}$ generated by the set of symmetric inverse M-matrices and the cone $\mathcal{KP}$ generated by potentials, that is, inverses of symmetric diagonally dominant M-matrices. We study these cones to understand the difference between inverse M-matrices and potentials. To our surprise, $\mathcal{KP}$ has only a finite number of extremal rays, that is, $\mathcal{KP}$ is a polyhedral cone, while $\mathcal{K}$ has infinitely many extremal rays.

The extremal rays of $\mathcal{K}$ are generated by the rank one matrices uu' where u is a nonnegative nonzero vector. This is shown in Theorem 2.1. On the other hand, the extremal rays of $\mathcal{KP}$ are the rank one matrices uu', where u is a $\{0,1\}$-valued nonzero vector. This is shown in Theorem 3.2. While the first result is simple to show, the second one is more involved and its proof relies on some properties of the adjoint of the symmetric polynomial matrix

$$M(n,(\mathbf{Y},\mathbf{Z})) = \begin{pmatrix} y_1+S_1 & -z_{12} & -z_{13} & \cdots & -z_{1n} \\ -z_{21} & y_2+S_2 & -z_{23} & \cdots & -z_{2n} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ -z_{n-1,1} & -z_{n-1,2} & \cdots & y_{n-1}+S_{n-1} & -z_{n-1,n} \\ -z_{n,1} & -z_{n,2} & \cdots & -z_{n,n-1} & y_n+S_n \end{pmatrix}, \tag{1.1}$$

where: $\mathbf{Y}=(y_1,\cdots,y_n)$, $\mathbf{Z}=(z_{ij}: i,j=1,\cdots n, i\neq j)$ with $z_{ij}=z_{ji}$ and $S_i=S_i(\mathbf{Z})=\sum\limits_{j\neq i} z_{ij}$. Sometimes we write $\mathbf{X}=(\mathbf{Y},\mathbf{Z})$ and $M(n,\mathbf{X})=M(n,(\mathbf{Y},\mathbf{Z}))$.

The important property of $V=adj(M)$ is the minimality of it, which simply says that V_{ij}, for $i\neq j$, is the intersection of the two polynomials V_{ii} and V_{jj} (see Definition 3.4). We also use results of Wang's algebra, which simplifies some of our computations.

In Appendix B, we include some historical remarks about the principal minors of $M(n,\mathbf{X})$ and its determinant.

2. Representation for inverse M-matrices

We fix $I=\{1,\cdots,n\}$. An M-matrix is a nonsingular matrix, whose off-diagonal elements are nonpositive and its inverse is a nonnegative matrix (every entry is nonnegative). Given a nonnegative matrix U, an important problem is to characterize (in terms of U) when it is the inverse of an M-matrix. In this direction, in the next result we study the cone generated by inverses of symmetric M-matrices.

Theorem 2.1. *Assume that U is the inverse of a symmetric M-matrix of order n. Then, there exist n nonnegative and linearly independent vectors $v_1,\cdots,v_n\in\mathbb{R}^n$, such that*

$$U=\sum_{k=1}^{n} v_k v_k'. \tag{2.1}$$

Proof. According to Theorem 11 and its Corollary 4 in [12], every symmetric inverse M-matrix U has a square root V, which is an inverse M-matrix. In particular V is a nonnegative matrix. V is also symmetric, because V^{-1} is a power series in U^{-1}. Hence $U = VV = VV' = \sum_{k=1}^{n} v_k\, v_k'$, where $V = [v_1 \cdots v_n]$. $\square$

Remark 2.1. The proof given above was suggested to us by the referee. Our original proof was much longer, but it supplies the following recursive algorithm to get (2.1). Decompose U, of order $n+1$, by blocks as

$$U = \begin{pmatrix} a & v' \\ v & T \end{pmatrix}$$

where $a > 0$, $v \geq 0 \in \mathbb{R}^n$ and T is a symmetric matrix of order n. It is well known that T is also a symmetric inverse M-matrix. On the other hand, U^{-1}, which is an M-matrix, is decomposed as

$$U^{-1} = \begin{pmatrix} \theta & -\gamma' \\ -\gamma & \Gamma \end{pmatrix},$$

where $\Gamma^{-1} = T - \frac{1}{a} vv'$ is also a symmetric inverse M-matrix, $v = \frac{1}{\theta} T\gamma \geq 0$ and $\theta = (a - v'T^{-1}v)^{-1} > 0$. Other useful relations are $\frac{1}{\theta}\gamma' v = v'T^{-1}v = a - \frac{1}{\theta}$.

A straightforward computation shows that $U = A \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} A'$, where

$$A = \begin{pmatrix} \frac{1}{\sqrt{\theta}} & \frac{1}{\theta}\gamma' \\ 0 & \mathbb{I} \end{pmatrix}$$

Now, suppose that T has a decomposition $T = \sum\limits_{i=1}^{n} z_i z_i'$, where $z_1, \cdots, z_n$ are linearly independent nonnegative vectors in $\mathbb{R}^n$. We extend these vectors to $\mathbb{R}^{n+1}$ by adding 0 as a first coordinate, that is, $w_i = (0, z_i')'$. We also add a new vector $w_{n+1} = (1, 0)' \in \mathbb{R}^{n+1}$. The vectors Aw_i $i = 1, \cdots, n+1$ are nonnegative, linearly independent and they provide a decomposition as in (2.1)

$$U = \sum_{i=1}^{n} Aw_i (Aw_i)' + Aw_{n+1}(Aw_{n+1})'.$$

In the next result we consider Hadamard products and Hadamard functions of matrices. As usual, given two matrices A, B of the same size $n \times m$, their Hadamard product $(A \odot B)$ is given by $(A \odot B)_{ij} = A_{ij} B_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq m$. On the other hand, given a function $\psi : \mathscr{D} \subseteq \mathbb{R} \to \mathbb{R}$ and a matrix A with entries in $\mathscr{D}$, we denote by $\psi(A)$ the matrix given by $(\psi(A))_{ij} = \psi(A_{ij})$, $1 \leq i \leq n$, $1 \leq j \leq m$. We call $\psi(A)$ the Hadamard function of A.

Recall that a nonnegative function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is an absolutely monotone function if all its derivatives $(\psi^{(k)} : k \geq 1)$ are nonnegative functions. Any absolutely monotone function defined in $\mathbb{R}_+$ has a power series expansion $\psi(x) = \sum_{k=0}^{\infty} \frac{\psi^{(k)}(0)}{k!} x^k$ (see [21] Theorem 3a). A subset of nonnegative matrices $\mathcal{B}$, is said to be closed under Hadamard absolutely monotone functions if $\psi(A) \in \mathcal{B}$ for every absolutely monotone function ψ and every $A \in \mathcal{B}$ (see [6,7] for some results on Hadamard products and functions of inverse M-matrices).

Corollary 2.2. *The closed convex cone $\mathcal{K}$ generated by the set of symmetric inverse M-matrices of order n satisfies*

(i) *$\mathcal{K}$ is closed under Hadamard products and Hadamard absolutely monotone functions;*
(ii) *the extremal rays are generated by the rank one matrices of the form uu' where u is a nonnegative nonzero vector in $\mathbb{R}^n$.*

Proof. (i) $\mathcal{K}$ is closed under Hadamard products because $uu' \odot vv' = (u \odot v)(u \odot v)'$. Hence, $\mathcal{K}$ is closed under Hadamard powers and therefore, it is also closed under Hadamard absolutely monotone functions.

Let us prove (ii). Assume u is a nonnegative nonzero vector and consider the matrix $U_\epsilon = uu' + \epsilon\mathbb{I}$, where $\epsilon > 0$. Its inverse $(uu' + \epsilon\mathbb{I})^{-1} = \frac{1}{\epsilon}\mathbb{I} - \frac{1}{\epsilon^2 + u'u} uu'$ is an M-matrix, proving that $uu' \in \mathcal{K}$.

From Theorem 2.1, we obtain that every extremal ray of $\mathcal{K}$ is generated by a rank one matrix uu', for some nonnegative nonzero vector u. To finish the result it is enough to show that every matrix uu', where u is a nonnegative nonzero vector, generates an extremal ray. Since every symmetric inverse M-matrix is positive semidefinite, we conclude that $\mathcal{K}$ is contained in the cone of semidefinite matrices, whose extremal rays are generated by the rank one matrices $vv', v \in \mathscr{R}^n$ (see [11], page 464). In particular, uu' generates an extremal ray in a larger cone and a fortiori it generates an extremal ray in $\mathcal{K}$. $\square$

Remark 2.2. We slightly modify representation (2.1) to get

$$U = \sum_{k=1}^{n} c_k \, u_k u_k',$$

where $c_k = \|v_k\|_\infty^2$ and $u_k = v_k / \|v_k\|_\infty$. We use the normalization to compare this representation with the one for potentials in the next section.

3. Representation for potentials

In this section we study the extremal rays for the closed convex cone generated by the set of symmetric potentials. For the sake of completeness we recall the notion of a potential matrix.

Definition 3.1. A nonnegative matrix U is said to be a **potential** if it is nonsingular and its inverse U^{-1} is a row diagonally dominant M-matrix, that is,

(i) $\forall i \neq j \ (U^{-1})_{ij} \leq 0$,
(ii) $\forall i \ \sum_j (U^{-1})_{ij} \geq 0$.

In this article we only consider symmetric potentials. A main results of this article is the following.

Theorem 3.2. *Let U be a symmetric potential of order n. Then, there exists a finite set of $\{0,1\}$-valued vectors $u_1, \cdots, u_t \in \mathbb{R}^n$ and positive constants $c_1, \cdots, c_t$ such that*

$$U = \sum_{s=1}^{t} c_s \ u_s u_s'. \tag{3.1}$$

Remark 3.1. To our knowledge Muir (see [17]) is the first in considering generic matrices of the form (3.1). He proved that for $n = 3$ and 4, their determinant is a sum of positive monomials on the variables (c_s) which appear with degree at most 1 (but the coefficients are not necessarily 1).

The way we prove this result requires the study of the inverse of a generic symmetric diagonally dominant M-matrix. For that reason, we will need to introduce some basic concepts and relations among polynomials in several variables. Given a polynomial $p(\mathbf{X})$ in the variables $\mathbf{X} = (x_1, \cdots, x_k)$ we consider its reduced form

$$p(\mathbf{X}) = \sum_{(m_1,\cdots,m_k)\in\mathbb{N}^k} p_{(m_1,\cdots,m_k)} \prod_{\ell=1}^{k} x_\ell^{m_\ell},$$

where only a finite number of coefficients are nonzero. Each term $p_{(m_1,\cdots,m_k)} \prod_{\ell=1}^{k} x_\ell^{m_\ell}$ for which $p_{(m_1,\cdots,m_k)} \neq 0$ is called a monomial of $p(\mathbf{X})$ and the number $p_{(m_1,\cdots,m_k)}$ the constant coefficient of this monomial. The degree of the monomial is $\sum_{\ell=1}^{k} m_\ell$, and m_ℓ is the degree of x_ℓ in this monomial.

Definition 3.3. Given two polynomials $p(\mathbf{X})$ and $q(\mathbf{X})$ in the variables $\mathbf{X} = (x_1, \cdots, x_k)$, we define the **intersection** of them as the polynomial $r(\mathbf{X})$ in the variables $\mathbf{X}$ such that

$$r_{(m_1,\cdots,m_k)} = \begin{cases} p_{(m_1,\cdots,m_k)} & \text{if } \ p_{(m_1,\cdots,m_k)} = q_{(m_1,\cdots,m_k)}, \\ 0 & \text{otherwise} \end{cases}.$$

We write $r = p \cap q$. We shall say that p is **smaller** than q if $p = p \cap q$. We denote this relation by $p \preccurlyeq q$. We shall say that a monomial r participates or belongs to a polynomial p if $r \preccurlyeq p$.

To clarify this concept, consider the example $(xy + xz + 2x + 3) \cap (xy + x + 2) = xy$.

Remark 3.2. If the polynomials p, q have coefficients in $\{0, 1\}$ then the intersection of them corresponds exactly to the Hadamard product of polynomials, which is the product obtained by multiplying the coefficients term by term, that is, $p \cap q = p \odot q$.

The following definition will play an important role.

Definition 3.4. Assume that $W = W(\mathbf{X})$ is a symmetric matrix whose entries are polynomials in $\mathbf{X}$.

(i) We say that W is **minimal** if for all $i \neq j$ it holds that

$$W_{ij}(\mathbf{X}) = W_{ii}(\mathbf{X}) \cap W_{jj}(\mathbf{X}).$$

(ii) Let $p[W](\mathbf{X}) = \bigcap_{i,j} W_{ij}(\mathbf{X})$ be the polynomial intersection over all the entries of the matrix. We call this polynomial the **minimum polynomial** of W. If W is minimal then $p[W](\mathbf{X}) = \bigcap_{i} W_{ii}(\mathbf{X})$ is the polynomial intersection over the diagonal of W.

The following theorem summarizes the known results about the inverse of $M(n, \mathbf{X})$. We write this theorem in terms of $V(n, \mathbf{X})$, the adjoint of $M(n, \mathbf{X})$. Its proof, as well as some extra properties, is contained in Section 4.

Theorem 3.5. *The determinant of $M(n, \mathbf{X})$ and the elements of $V(n, \mathbf{X}) = adj(M(n, \mathbf{X}))$ are polynomials in $\mathbf{X}$ such that*

(i) $\det(M(n, \mathbf{X}))$ *(respectively, the diagonal elements of V) is a sum of monomials of degree n (respectively, $n - 1$) whose nonzero constant coefficients are equal to 1 and the degree of each variable appearing in each monomial is* 1*;*

(ii) *for all $i \neq j$, we have $V_{ij}(n, \mathbf{X}) = V_{ii}(n, \mathbf{X}) \cap V_{jj}(n, \mathbf{X})$, that is, $V(n, \mathbf{X})$ is minimal. In particular, V_{ij} is also a sum of monomials of degree $n-1$, whose nonzero constant coefficients are equal to 1 and the degree of each variable appearing in each monomial is* 1.

Remark 3.3. Part (i) is a variant of a well-known theorem in graph theory and electrical networks theory, cf. [1,2,15,19]; a proof using Wang algebra appears in [3] and [10]. Part (ii) is less known; it is cited without a real proof in [10] and can be deduced from theorems on trees proved in [3] and [19] (see also Appendix B). We will provide a self-contained proof of both properties that uses some basic facts of finite potential theory for Markov processes (see Section 4.1).

Proof of Theorem 3.2. Take $N = U^{-1}$ and consider the polynomial matrix valued $M(n, \mathbf{X})$. We assign the values $\overline{\mathbf{x}}$ to the variables $\mathbf{X}$, given by

$$\overline{z}_{ij} = -N_{ij} \geq 0, \quad \text{for} \quad i \neq j$$

and for $i = 1, \cdots, n$

$$\overline{y}_i = N_{ii} - \sum_{j \neq i} N_{ij} \geq 0.$$

In this way, $N = M(n, \overline{\mathbf{x}})$.

We shall prove that the literal $V = V(n, \mathbf{X}) = \text{adj}(M(n, \mathbf{X}))$ has the desired decomposition and so U will also have the desired decomposition. The proof is based on a backward algorithm that starts from the matrix V and decreases towards the 0 matrix. At every step the corresponding matrix is minimal. In what follows, we use the pair (k, ℓ) where $k = n, \cdots, 1$ and $\ell = 1, \cdots, \binom{n}{k}$: k represents the size of subsets of $I = \{1, \cdots, n\}$ and ℓ represents a subset of size k.

We start with $V^{(n+1)} = V$. From Theorem 3.5 this matrix is minimal. We subtract from V its minimum polynomial $p[V](\mathbf{X})$, that is, $V^{(n)} = V^{(n+1)} - p[V](\mathbf{X})\ \mathbf{11}'$. It is clear that $0 \preccurlyeq V^{(n)} \preccurlyeq V^{(n+1)}$ and $V^{(n)}$ is a minimal matrix. Notice that $V^{(n+1)} - V^{(n)} = p[V](\mathbf{X})\ \mathbf{11}'$, so if $V^{(n)}$ is the zero matrix we stop and the result is proven (actually this can happens only if $n = 1$). Otherwise we continue.

One important observation is that the minimum polynomial of $V^{(n)}$ is $p[V^{(n)}](\mathbf{X}) = 0$. For $k = n - 1$ consider the $\binom{n}{k} = n$ subsets of I of size k. We note these sets by $\left(I_{k,\ell} :\ \ell \in \{1, \cdots, \binom{n}{k}\}\right)$. For each one of these subsets consider the principal submatrix of order k given by $V^{(k+1)}|_{I_{k,\ell}}$. Then, each of them is a minimal matrix (of order k) and for any $\ell \neq \ell'$

$$p[V^{(k+1)}|_{I_{k,\ell}}](\mathbf{X}) \cap p[V^{(k+1)}|_{I_{k,\ell'}}](\mathbf{X}) = 0.$$

This follows from the fact that $I_{k,\ell} \cup I_{k,\ell'} = I$ and that $p[V^{(k+1)}](\mathbf{X}) = 0$. Thus, we consider

$$V^{(k)} = V^{(k+1)} - \sum_{\ell=1}^{\binom{n}{k}} p[V^{(k+1)}|_{I_{k,\ell}}](\mathbf{X})\ \mathbf{1}_{I_{k,\ell}} \mathbf{1}'_{I_{k,\ell}}. \tag{3.2}$$

Here $\mathbf{1}_{I_{k,\ell}}$ is the $\{0, 1\}$-valued vector, with ones at the coordinates in $I_{k,\ell}$. The main properties to continue with the induction are

(i) $0 \preccurlyeq V^{(k)} \preccurlyeq V^{(k+1)}$;
(ii) $V^{(k)}$ is a minimal matrix;
(iii) 0 is the minimum polynomial for any principal submatrix of $V^{(k)}$ of order $m \geq k$.

So, assume the algorithm works for $n, n-1, \cdots, k+1$. We will show it also works for k (while $k \geq 1$). As before we take $I_{k,1}, \cdots, I_{k,\binom{n}{k}}$ the subsets of I, of size k, and the corresponding principal submatrices of order k, given by $V^{(k+1)}|_{I_{k,\ell}} : \ell = 1, \cdots, \binom{n}{k}$. Given two different $\ell \neq \ell'$, the set $J = I_{k,\ell} \cup I_{k,\ell'}$ has at least $k+1$ elements and then, by the induction hypothesis, the minimum polynomial associated to the submatrix of $V^{(k+1)}$ with index set J is 0. Equivalently, $p[V^{(k+1)}|_{I_{k,\ell}}](\mathbf{X}) \cap p[V^{(k+1)}_{I_{k,\ell'}}](\mathbf{X}) = 0$. Define $V^{(k)}$ as in (3.2). It is straightforward to check that (i), (ii), (iii) are satisfied for $V^{(k)}$.

In this way we reach $k = 1$, and $V^{(1)}$ satisfies the main restrictions (i), (ii), (iii). In particular it is a minimal matrix whose diagonal elements are 0. Thus, $V^{(1)} = 0$, proving that

$$V = \sum_{k=1}^{n} \sum_{\ell=1}^{\binom{n}{k}} p[V^{(k+1)}|_{I_{k,\ell}}](\mathbf{X})\ \mathbf{1}_{I_{k,\ell}} 1'_{I_{k,\ell}}. \tag{3.3}$$

We notice that U has the representation

$$U = \det(U) \sum_{k=1}^{n} \sum_{\ell=1}^{\binom{n}{k}} p[V^{(k+1)}|_{I_{k,\ell}}](\overline{\mathbf{x}})\ \mathbf{1}_{I_{k,\ell}} 1'_{I_{k,\ell}}. \quad \square$$

A symmetric nonnegative matrix U is said to be ultrametric, if it satisfies the set of inequalities

$$\forall i, j, k \quad U_{ij} \geq \min\{U_{ik}, U_{kj}\}$$

A nonsingular ultrametric matrix is a potential (see for example [4,5,9,14,18]) and therefore the sum of ultrametric matrices belongs to $\mathcal{KP}$.

Corollary 3.6. *The closed convex cone $\mathcal{KP}$ generated by the potentials of order n, satisfies*

(i) *$\mathcal{KP}$ is closed under Hadamard products and Hadamard absolutely monotone functions;*
(ii) *the extremal rays of $\mathcal{KP}$ are generated by the rank one matrices of the form uu' where u is a $\{0,1\}$-valued vector;*
(iii) *every matrix $V \in \mathcal{KP}$ is a finite sum of ultrametric matrices.*

Proof. If u is a $\{0,1\}$-valued vector, then $V = uu'$ is an ultrametric matrix. Indeed, if $V_{ik} = V_{kj} = 1$ then $u_i = u_j = u_k = 1$ and so $V_{ij} = 1$. The perturbed matrix $V_\epsilon = uu' + \epsilon\mathbb{I}$ is nonsingular and ultrametric, for every positive ϵ. Therefore, V_ϵ is a potential, showing that $uu' \in \mathcal{KP}$. This also shows (iii).

On the other hand, (i) and (ii) are shown as in Corollary 2.2. $\square$

In Appendix A we discuss in more details a relation between ultrametric matrices and $\mathcal{KP}$.

Since the cone $\mathcal{KP}$ is polyhedral, in principle, it is possible to determine in finite time if a matrix U is in $\mathcal{KP}$. The naive way to do this is to use $2^n - 1$ variables to describe this problem, which consists in

$$\min \quad \mathbf{1}'U\mathbf{1} - \mathbf{1}'\left[\sum_{a\in\mathcal{P}(I),\, a\neq\emptyset} w_a \mathbf{1}_a \mathbf{1}_a'\right]\mathbf{1}$$

$$s.t. \begin{cases} 0 \le w_a \le C \\ \sum_{a\in\mathcal{P}(I),\, a\neq\emptyset} w_a \mathbf{1}_a \mathbf{1}_a' \le U \end{cases}$$

We have taken $C = \max\{U\}$ and $\mathcal{P}(I)$ is the power set of I. The minimum is 0 if and only if U is the sum of potentials. We have been unable to find an efficient algorithm to check if U is in $\mathcal{KP}$.

The arguments below will require to fix some of the variables in $\mathbf{X}$ to 0. To do it in a precise way, we shall make explicit the variables that are free and as a consequence which are the variables that are fixed to be 0. Since we have distinguished between the diagonal variables $\mathbf{Y}$ and the off-diagonal variables $\mathbf{Z}$, we do the same with the free variables. The set of free variables will be indexed by two sets $\mathbb{F}_1 \subseteq I$ and a symmetric set $\mathbb{F}_2 \subseteq I \times I \setminus \{(i,i) : i \in I\}$. That is, for $i \in \mathbb{F}_1$, $(k,l) \in \mathbb{F}_2$ the variables y_i, z_{kl} are assumed free, and for $i \notin \mathbb{F}_1$, $(k,l) \notin \mathbb{F}_2$ we assume that y_i, z_{kl} are 0. By abuse of notation we shall denote $y_i \in \mathbb{F}$ (or $z_{kl} \in \mathbb{F}$) to mean that y_i is a free variable (respectively, z_{kl} is free). The complete set of free variables is denoted by $(\mathbf{Y},\mathbf{Z})_{\mathbb{F}}$.

For example take $n = 3$, $\mathbf{Y} = (a,b,c)$ and $\mathbf{Z}$ given by $z_{12} = z_{21} = x$, $z_{13} = z_{31} = y$, $z_{23} = z_{32} = z$. The matrix

$$N = \begin{pmatrix} a+x & -x & 0 \\ -x & x+z & -z \\ 0 & -z & z \end{pmatrix} \tag{3.4}$$

is given by $N = M(3,(\mathbf{Y},\mathbf{Z})_{\mathbb{F}})$ were $\mathbb{F}_1 = \{1\}$ and $\mathbb{F}_2 = \{(1,2),(2,1),(2,3),(3,2)\}$. Of course there is an abuse of notation in $M(3,(\mathbf{Y},\mathbf{Z})_{\mathbb{F}})$, because $(\mathbf{Y},\mathbf{Z})_{\mathbb{F}}$ is a reduced set of variables.

If $L \subseteq I$ we denote by $\mathbb{F}^L$ the set of free variables where $\mathbb{F}_1 = L$ and $\mathbb{F}_2 = (L \times L \cup L^c \times L^c) \setminus \{(i,i) : i \in I\}$, that is, we fix $y_k = z_{kl} = z_{lk} = 0$ for $l \in L$, $k \in L^c$.

The following proposition gives a relation between the minimum polynomial $p[V]$ and the diagonal elements of V. It also gives the minimum polynomials that appear in the additive representation given in (3.3). Its proof is given in Section 5.

Proposition 3.7. *For all i, we have $p[V](\mathbf{X}) = V_{ii}(n,(0,\mathbf{Z}))$. Moreover, for all $k = 1,\cdots,n$, $J \subseteq I$ of size k, and all $i \in J$ it holds that*

$$\begin{aligned} p[V^{(k+1)}|_J]((\mathbf{Y},\mathbf{Z})) &= V_{ii}(n,(\mathbf{Y},\mathbf{Z})_{\mathbb{F}^{J^c}}) \\ &= V_{11}(k,(0,\mathbf{Z}|_J))\det(M(n-k,(\mathbf{Y}_{J^c},\mathbf{Z}|_{J^c}))), \end{aligned} \tag{3.5}$$

where $(\mathbf{Y},\mathbf{Z})_{\mathbb{F}^{J^c}}$ means that $y_h = 0$, for $h \in J$, and $z_{h\ell} = z_{\ell h} = 0$ for $h \in J$, $\ell \in J^c$.

Remark 3.4. Notice that, after a suitable permutation, $M(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}^{J^c}})$ has the block form

$$\begin{pmatrix} M(k, (0, \mathbf{Z}|_J)) & 0 \\ 0 & M(n-k, (\mathbf{Y}_{J^c}, \mathbf{Z}|_{J^c})) \end{pmatrix}.$$

Hence, the last expression on (3.5) is simply the formula for the determinant of a matrix with block structure.

The next result is a consequence of Proposition 3.7, representation (3.3) for V and the fact that for all i, j we have $V_{ij} = V_{ii} \cap V_{jj}$ (see Theorem 3.5).

Theorem 3.8. *The following representation holds for V. For $i \in I$*

$$V_{ii}(n, (\mathbf{Y}, \mathbf{Z})) = \sum_{k=1}^{n} \sum_{\substack{J \,:\, i \in J \\ \#J = k}} V_{ii}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}^{J^c}}). \tag{3.6}$$

Moreover, different terms have no common monomials. Also we have the representation for the off-diagonal elements of V, for $i \neq j$

$$V_{ij}(n, (\mathbf{Y}, \mathbf{Z})) = \sum_{k=2}^{n} \sum_{\substack{J \,:\, i, j \in J \\ \#J = k}} V_{ii}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}^{J^c}}).$$

The main advantage of representation (3.6) over the one in (3.3), is that each term in the former can be computed directly without passing through the recurrence defining the sequence $(V^{(s)} : \; s = n+1, \cdots, 1)$.

It is straightforward to see that $V_{ii}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}^{J^c}}) = 0$ when $i \notin J$, because the block $J \times J$ of $M(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}^{J^c}})$ is singular (the row sums of this block are 0). Hence we have the following alternative expression of Theorem 3.8.

Corollary 3.9. *V has the following representation*

$$V(n, (\mathbf{Y}, \mathbf{Z})) = adj(M(n, (\mathbf{Y}, \mathbf{Z})) = \sum_{J \subseteq I, J \neq \emptyset} adj(M(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}^{J^c}})).$$

4. Some basic properties of $V(n, \mathbf{X})$, $\det(M(n, \mathbf{X}))$ and $p[V](\mathbf{X})$

In this section we derive properties for the polynomials $q_n(\mathbf{X}) = \det(M(n, \mathbf{X}))$ and $V(n, \mathbf{X})$, as well as the proof of Theorem 3.5. According to Theorem 7.1 in [19], q_n is a polynomial obtained as the sum of monomials of degree n, whose constant coefficients are 1 and all of them are constituted by the product of n different variables. That is, each variable participates in each monomial with degree at most 1.

Example 4.1. Consider the matrix of order 3

$$M(3,\mathbf{X}) = \begin{pmatrix} a+x+y & -x & -y \\ -x & b+x+z & -z \\ -y & -z & c+y+z \end{pmatrix}, \tag{4.1}$$

where $\mathbf{X} = (\mathbf{Y}, \mathbf{Z})$ and $\mathbf{Y} = (a, b, c)$, $\mathbf{Z} = (x, y, z)$. Then, the determinant of $M(3,\mathbf{X})$ is the polynomial that consists of 16 monomials given by

$$\begin{aligned} q_3(\mathbf{X}) &= abc + aby + abz + acx + acz + axy + axz + ayz \\ &\quad + bcx + bcy + bxy + bxz + byz + cxy + cxz + cyz. \end{aligned}$$

We notice that in this example, all the monomials are of degree 3, and all of them contain at least one of the variables a, b, c. This last observation follows from the fact that if $a = b = c = 0$ then the matrix is singular and then its determinant is 0. We also notice that there are $20 = \binom{6}{3}$ monomials of degree 3 and not all of them participate in the determinant. The ones that are excluded are xyz, abx, acy, bcz, which correspond to monomials given by certain cycles that we will explain later on (see Appendix B).

The adjoint of $M(3,\mathbf{X})$ is

$$\begin{aligned} V(3,\mathbf{X}) &= (xy + xz + zy)\mathbf{1}\mathbf{1}' \\ &\quad + \begin{pmatrix} bc+by+bz+xc+zc & xc & by \\ xc & ac+ay+az+xc+yc & az \\ by & az & ab+ax+az+xb+by \end{pmatrix}. \end{aligned}$$

The minimum polynomial of $V(3,\mathbf{X})$ is $p[V](\mathbf{X}) = xy + xz + zy$, which is obtained by evaluating $V(0,\mathbf{Z})$ and it corresponds to the sum of all the monomials of degree 2 in the variables x, y, z, which appear with degree at most 1.

The following lemma is the basis to prove these facts about q_n and $V(n,\mathbf{X})$. As usual, given $1 \le i \le n$, we denote by $\mathbf{Y}^{(i)}$ and $\mathbf{Z}^{(i)}$ the set of variables obtained by removing all variables with index i, that is, $\mathbf{Y}^{(i)} = (y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n)$ and $\mathbf{Z}^{(i)}$ is what in matrix analysis is denoted by $\mathbf{Z}^{(i,i)}$, which is the principal submatrix of order $n-1$ obtained by removing the ith row and column from $\mathbf{Z}$.

Lemma 4.1. *Assume that $n \ge 1$. Then,*

(i) *the sequence q_n satisfies the recursion*

$$q_n((\mathbf{Y},\mathbf{Z})) = y_1 q_{n-1}(\mathbf{Y}^{(1)} + (z_{12}, \cdots, z_{1n}), \mathbf{Z}^{(1)}) + q_n((0, y_2, \cdots, y_n), \mathbf{Z}), \tag{4.2}$$

where $q_0 = 1$;

(ii) *$q_n((\mathbf{Y},\mathbf{Z}))$ is a sum of monomials of degree n, all of which contain at least one variable in $\mathbf{Y}$;*

(iii) $V_{ii}(n, (\mathbf{Y}, \mathbf{Z})) = q_{n-1}(\mathbf{Y}^{(i)} + (z_{1,i}, \cdots, z_{i-1,i}, z_{i,i+1}, \cdots, z_{i,n}), \mathbf{Z}^{(i)})$. *In particular,*

$$q_n((\mathbf{Y}, \mathbf{Z})) = y_1 V_{11}(n, (\mathbf{Y}, \mathbf{Z})) + q_n((0, y_2, \cdots, y_n), \mathbf{Z}). \tag{4.3}$$

On the other hand, $V_{ii}(n, (\mathbf{Y}, \mathbf{Z}))$ *does not depend on* y_i.

(iv) $q_n((\mathbf{Y}, \mathbf{Z}))$ *is the sum of monomials of degree* n*, with constant coefficient 1. In each monomial, each variable appearing on it has degree* 1.

Proof. Parts (i), (ii), (iii) follow from standard formulae for the determinant of a matrix. The fact that q_n is a sum of monomials of degree n also follows from these formulae. So, the only thing left to prove is that each monomial contains at least one diagonal element. This follows from the fact that when $y_1 = \cdots = y_n = 0$ this determinant is 0. Alternatively, by iterating (4.2) we obtain

$$\begin{aligned} q_n(\mathbf{X}) &= y_1 V_{11}(n, ((0, y_2, \cdots, y_n), \mathbf{Z})) + y_2 V_{22}(n, ((0, 0, y_3, \cdots, y_n), \mathbf{Z})) \\ &\quad + q_n((0, 0, y_3, \cdots, y_n), \mathbf{Z}) \\ &= \sum_{i=1}^{n} y_i V_{ii}(n, ((0, \cdots, 0, y_{i+1}, \cdots, y_n), \mathbf{Z})) + q_n((0, \mathbf{Z})). \end{aligned}$$

Notice that $q_n((0, \mathbf{Z})) = 0$ because it is the determinant of a matrix whose row sums are 0. Then, we get

$$q_n(\mathbf{X}) = \sum_{i=1}^{n} y_i V_{ii}(n, (0, \cdots, 0, y_{i+1}, \cdots, y_n), \mathbf{Z}). \tag{4.4}$$

Part (iv) follows from formula (4.4) and induction. Indeed, for example

$$V_{ii}((n, (\mathbf{Y}, \mathbf{Z})) = q_{n-1}(\mathbf{Y}^{(i)} + (z_{1,i}, \cdots, z_{i-1,i}, z_{i,i+1}, \cdots, z_{i,n}), \mathbf{Z}^{(i)})$$

is a sum of monomials of degree $n - 1$ in which each variable has degree 1 and each monomial has constant factor 1. Since the variable y_i appears in no monomial, we conclude that $y_i V_{ii}(n, ((0, \cdots, 0, y_{i+1}, \cdots, y_n), \mathbf{Z}))$ is a sum of monomials of degree n with constant coefficient 1. On each monomial each variable that appears on it, has degree 1.

Observe that all the monomials in the expansion of $y_1 V_{11}(n, ((0, y_2, \cdots, y_n), \mathbf{Z}))$ contain the variable y_1, but no monomial in $y_2 V_{22}(n, ((0, 0, y_3, \cdots, y_n), \mathbf{Z}))$ contains y_1. This shows that both expansions do not contain a common monomial. The result is shown. □

We end this section with one of the most beautiful properties of $q_n(\mathbf{X})$. This determinant can be computed from the diagonal elements of $M(n, \mathbf{X})$ by using the algebra of Wang (see [10]). This is a commutative algebra on the sum of finite words with symbols in $\mathbf{X}$, with the restrictions

$$\forall x \in \mathbf{X} : \quad x \bigoplus x = 0 \ \text{ and } \ x \bigodot x = 0.$$

Obviously, every element of this algebra is a polynomial on $\mathbf{X}$, obtained as sum of monomials of degree at most the size of $\mathbf{X}$, where each variable has degree at most 1 and the constant coefficients of these monomials are 1.

A formula for the determinant is the following

$$\det(M(n,\mathbf{X})) = \bigodot_{i=1}^{n} M_{ii}(n,\mathbf{X}). \tag{4.5}$$

This is the Wang product of the diagonal elements of $M(n,\mathbf{X})$. Let see how this works for $n=3$. Consider

$$A = \begin{pmatrix} a+x+y & -x & -y \\ -x & b+x+z & -z \\ -y & -z & c+y+z \end{pmatrix}.$$

It is straightforward to show that $\det(A)$ contains exactly the same 16 monomials as the expansion of

$$\left(a \bigoplus x \bigoplus y\right) \bigodot \left(b \bigoplus x \bigoplus z\right) \bigodot \left(c \bigoplus y \bigoplus z\right)$$

This is an important ingredient to show Theorem 3.8 in Section 5.

4.1. Proof of Theorem 3.5

Part (i) of Theorem 3.5 is proven in Lemma 4.1. Now we turn to the proof of part (ii). In the next lemma we shall prove one of the inequalities of this claim.

Lemma 4.2. *For all $i \neq j$ we have $V_{ij} \preccurlyeq V_{ii} \cap V_{jj}$ and V_{ij} is a sum of monomials of degree $n-1$ with constant coefficients 1, where each variable has degree at most 1 in each monomial.*

Proof. We claim that $0 \le V_{ij}(\mathbf{X}) \le V_{ii}(\mathbf{X})$ as real functions. Indeed, consider any evaluation $\bar{x} \ge 0$ of $\mathbf{X}$. By using a perturbation argument we can assume that $M = M(n,\bar{x})$ is nonsingular and irreducible. On the other hand, M is a row diagonally dominant M-matrix and therefore its inverse $U = M^{-1}$ is a nonnegative matrix. Moreover, $\det(M)$ is positive, which shows that $V(\bar{x}) = \operatorname{adj}(M)$ is a nonnegative matrix, thus $0 \le V_{ij}(\bar{x})$.

The matrix M can be decomposed as $M = k(\mathbb{I} - P)$, where P is an irreducible substochastic matrix. The Markov chain $(\Xi_m : m \in \mathbb{N})$, taking values in $\{1, \cdots, n\}$ with transition kernel P, is a transient Markov chain with potential $W = kU = (\mathbb{I} - P)^{-1}$. Therefore,

$$W_{ij} = \mathbb{E}_i \left(\sum_{m \ge 0} \mathbb{1}_{\{j\}}(\Xi_m) \right)$$

is the expected number of visits to j, when the chain starts at i, until absorption.

If we denote by $\tau_j = \inf\{m \geq 0 : \Xi_m = j\}$ as the first time the chain visits j, then the strong Markov property shows

$$W_{ij} = \mathbb{P}_i(\tau_j < \infty) W_{jj}.$$

This property and the symmetry of M show that $V_{ij}(\bar{x}) \leq V_{ii}(\bar{x})$, $V_{ij}(\bar{x}) \leq V_{jj}(\bar{x})$, proving the claim.

Now, we shall prove that $0 \preccurlyeq V_{ij} \preccurlyeq V_{ii} \cap V_{jj}$. It is clear that V_{ij} is an algebraic cofactor of $M(n, \mathbf{X})$ and therefore is a sum of monomials of degree $n-1$, with integer coefficients. It is also clear that the diagonal variables $\mathbf{Y}$ have degree at most 1 on each of these monomials and each off-diagonal variable in $\mathbf{Z}$ has degree at most 2.

If one monomial r contains a variable z_{kl} of degree 2 then, the coefficient of this monomial has to be 0. Indeed, it is enough to consider an evaluation $\bar{x}$ for which all the variables outside r are 0, all the variables in r, except z_{kl} are 1 and $\bar{z}_{kl}$ is a large value. Then, the inequality

$$0 \leq V_{ij}(\bar{x}) \leq V_{ii}(\bar{x}),$$

is only possible if the coefficient of this monomial is 0 (recall that z_{kl} has at most degree 1 in every monomial in V_{ii}).

Using the same idea, we prove that all the monomials in V_{ij} have coefficients at least 1. Finally, if a monomial r is present in V_{ij} then its coefficient is 1 and r is present in V_{ii}. This finishes the proof. $\square$

Definition 4.3. Given the set of variables $\mathbf{X}$ and the set of free variables $\mathbb{F}$ we say that a monomial $r(\mathbf{X}) = c \prod_{\ell=1}^{k} x_{i_\ell}$ is **positive** if $c > 0$ and it contains only free variables. By extension, a polynomial p which is the sum of monomials with positive constant coefficients, is said to be positive if one of these monomials is positive.

A monomial with positive constant coefficient is positive if and only if takes positive values when the free variables are replaced by positive values. In particular the determinant of $M(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}})$ is positive if and only if it contains at least one positive monomial and this happens if and only if this determinant is positive (number) for some (any) particular positive value assigned to $(\mathbf{Y}, \mathbf{Z})_{\mathbb{F}}$.

Remark 4.1. The relation $\preccurlyeq$ and evaluation are not compatible in general. Take the example $p(x, y, z) = xy + yz$ and $q(x, y, z) = xy + yz - x$. Clearly $p \preccurlyeq q$ but if we take $y = 1$ then $p(x, 1, z) = x + z$ is not dominated by $q(x, 1, z) = z$. Nevertheless, the relation $\preccurlyeq$ and evaluation by 0 are compatible, that is, if $p(\mathbf{X}) \preccurlyeq q(\mathbf{X})$ and $\mathbb{F}$ is a set of free variables (the other variables are fixed to be 0), then $p(\mathbf{X}_{\mathbb{F}}) \preccurlyeq q(\mathbf{X}_{\mathbb{F}})$. Moreover, if q has only positive monomials then $p(\mathbf{X}_{\mathbb{F}}) \preccurlyeq q(\mathbf{X}_{\mathbb{F}})$ implies that $p(\mathbf{X}_{\mathbb{F}}) \leq q(\mathbf{X}_{\mathbb{F}})$ as functions on $\mathbb{R}_+^{\#\mathbb{F}}$.

We shall also need some basic results from potential theory for finite Markov chains, that we translate in the language of this article.

Definition 4.4. Given the set of free variables $\mathbb{F}$, two different points i, j are said to be $\mathbb{F}$-**connected** (or simply connected if the set of free variables is clear from the context) if there exists a finite path $i_0 = i, i_1, \cdots, i_p, i_{p+1} = j$ of different points in I such that for all $\ell = 0, \cdots, p$ it holds that $z_{i_\ell i_{\ell+1}} \in \mathbb{F}_2$.

Given $L \subseteq I$, the principal submatrix $A = M(n, (\mathbf{Y}, \mathbf{Z})_\mathbb{F})|_L$ is said to be $\mathbb{F}$-**irreducible** if any couple $i \neq j \in L$ are $\mathbb{F}$-connected by a path that remains in L.

Notice that i, j are $\mathbb{F}$-connected if and only if the monomial $\prod_{\ell=0}^{p} z_{i_\ell i_{\ell+1}}$ is positive for some path that joins i, j.

Example 4.2. The matrices N, $N|_{\{1,2\}}$ given in (3.4) are $\mathbb{F}$-irreducible, but the submatrix $A = N|_{\{1,3\}}$ is not.

Lemma 4.5. *Let $\mathbb{F}$ be a set of free variables.*

(i) *If $M(n, (\mathbf{Y}, \mathbf{Z})_\mathbb{F})$ is $\mathbb{F}$-irreducible, then $q_n((\mathbf{Y}, \mathbf{Z})_\mathbb{F}) = \det(M(n, (\mathbf{Y}, \mathbf{Z})_\mathbb{F}))$ is a positive polynomial if and only if $\mathbb{F}_1$ is not empty, that is there is at least one free diagonal variable;*

(ii) *If $i \neq j$ are not connected then $V_{ij}(n, (\mathbf{Y}, \mathbf{Z})_\mathbb{F}) = 0$. Conversely, if $q_n((\mathbf{Y}, \mathbf{Z})_\mathbb{F})$ is positive then $V_{ij}(n, (\mathbf{Y}, \mathbf{Z})_\mathbb{F}) = 0$ is a sufficient condition for i, j to be not connected.*

Proof. (i). Assume first that $\mathbb{F}_1$ is not empty. Assign any particular positive value to $(\mathbf{Y}, \mathbf{Z})_\mathbb{F}$ (recall that the variables outside $\mathbb{F}$ are 0). We denote this selection by $(\bar{y}, \bar{z})$ and by $\overline{M}$ the numerical matrix thus obtained. This matrix is a Z-matrix, which is row diagonally dominant and irreducible in the standard sense of matrices. The condition that $\mathbb{F}_1$ is not empty implies that it is strictly diagonally dominant at least at one row. This implies that $\overline{M}$ is a nonsingular M-matrix, so its determinant is a positive number. We conclude $q_n((\bar{y}, \bar{z})) > 0$ and therefore $q_n((\mathbf{Y}, \mathbf{Z})_\mathbb{F})$ is a positive polynomial.

Conversely, if $\mathbb{F}_1$ is empty then the row sums of $\overline{M}$ are 0, which implies that $q_n((\bar{y}, \bar{z})) = 0$ and therefore $q_n((\mathbf{Y}, \mathbf{Z})_\mathbb{F}) = 0$.

(ii). Notice that the property of being connected or not does not depend on the diagonal variables. Hence, if we enlarge the set of free variables to $\tilde{\mathbb{F}}$ by setting $\tilde{\mathbb{F}}_1 = I$ and $\tilde{\mathbb{F}}_2 = \mathbb{F}_2$ we will still have that i and j are not $\tilde{\mathbb{F}}$-connected. Assign positive values to the free variables in $\tilde{\mathbb{F}}$ and 0 to the others. As before we denote by $(\bar{y}, \bar{z})$ this selection. The matrix $N = M(n, (\bar{y}, \bar{z}))$ is a strictly row diagonally dominant M-matrix, and $-N$ is the generator of a Markov process on I. The fact that i, j are not $\tilde{\mathbb{F}}$-connected means that they belong to different irreducible classes for this Markov process and therefore we have $\mathrm{adj}(N)_{ij} = 0$, showing that the polynomial $\mathrm{adj}(M(n, (\mathbf{Y}, \mathbf{Z})_{\tilde{\mathbb{F}}}))_{ij} = 0$. On the other hand, $V_{ij} = \mathrm{adj}(M(n, (\mathbf{Y}, \mathbf{Z})_\mathbb{F}))_{ij} \preccurlyeq \mathrm{adj}(M(n, (\mathbf{Y}, \mathbf{Z})_{\tilde{\mathbb{F}}}))_{ij}$, because in the former there are fewer monomials than in the latter (exactly the ones that contain a variable in $I \setminus \mathbb{F}_1$). We have shown $V_{ij} = 0$.

Conversely, assume that $q_n((\mathbf{Y}, \mathbf{Z})_{\mathbb{F}})$ is positive and $V_{ij} = 0$. As before, we assign positive values to $(\mathbf{Y}, \mathbf{Z})_{\mathbb{F}}$, 0 to other variables, and we denote by $(\bar{y}, \bar{z})$ this selection. Recall that $M^{-1}(n, (\bar{y}, \bar{z}))_{ij}$ is the potential between i, j, which in this case is 0. This shows that i, j cannot be connected in the Markov process whose infinitesimal generator is $-M(n, (\bar{y}, \bar{z}))$ implying that i, j are not $\mathbb{F}$-connected. $\square$

Example 4.3. Let us see that we cannot remove the condition on the determinant in the second part of (ii) in the previous lemma. Take the 4×4 matrix

$$N = \begin{pmatrix} x & -x & 0 & 0 \\ -x & x+z & -z & 0 \\ 0 & -z & z & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly nodes 1 and 2 are connected but $V_{12} = 0$. Notice that in this example $\det(N) = 0$.

Proof of $V_{ij} = V_{ii} \cap V_{jj}$. This property is true for $n = 2$ and $n = 3$ so, in what follows, we assume that $n \geq 4$ and $i = 1$, $j = 2$. Since we will use different assignments to the variables $(\mathbf{Y}, \mathbf{Z})$ we fix $M = M(n, (\mathbf{Y}, \mathbf{Z}))$ and $V = V(n, (\mathbf{Y}, \mathbf{Z}))$.

Assume that $r = r((\mathbf{Y}, \mathbf{Z}))$ is a common monomial between V_{11} and V_{22}, which is not present in V_{12}. This monomial has degree $n - 1$ and all its variables have degree 1. For the moment we take $\mathbb{F}_1 = \{1\} \cup \{i : y_i \in r\}$ and $\mathbb{F}_2$ the symmetric set obtained from $\{(i, j) : z_{ij} \in r\}$. We recall that $y_1 \notin V_{11}$, which implies that $y_1 \notin r$ and therefore $V_{11}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}}) = r \preccurlyeq V_{22}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}})$. The same argument shows that $y_2 \notin r$ and so $2 \notin \mathbb{F}_1$. We set $N = M(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}})$ and $W = \mathrm{adj}(N)$. Notice that $W_{ij} = V_{ij}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}})$. According to (4.3) we have $y_1 V_{11} \preccurlyeq \det(M)$ and then $y_1 r \preccurlyeq \det(N)$, proving that $\det(N)$ is positive.

Since $V_{12} \preccurlyeq V_{11} \cap V_{22}$, we conclude that $W_{12} \preccurlyeq W_{11} \cap W_{22} = r$. On the other hand, $W_{12} \preccurlyeq V_{12}$, which implies that $W_{12} = 0$. Thus, according to Lemma 4.5 the nodes 1, 2 are not $\mathbb{F}$-connected. Consider the set L of nodes that are $\mathbb{F}$-connected to 1, that is, $L = \{j \neq 1 \in I : 1 \text{ and } j \text{ are } \mathbb{F}\text{-connected}\} \cup \{1\}$. Denote by $K = I \setminus L$. Since being $\mathbb{F}$-connected is a transitive and a symmetric relation we obtain that

$$\forall i \in L, j \in K \quad z_{ij} = 0.$$

A suitable permutation of rows and columns of N has the block structure

$$P = \begin{pmatrix} N_L & 0 \\ 0 & N_K \end{pmatrix}.$$

Now, we take $y_1 = 0$, which means that the set of free variables will be reduced to $\widetilde{\mathbb{F}}$, where $\widetilde{\mathbb{F}}_1 = \mathbb{F}_1 \setminus \{1\}$ and $\widetilde{\mathbb{F}}_2 = \mathbb{F}_2$. Since the property of being connected does not depend on the set of diagonal variables, we will still have that 1 is $\widetilde{\mathbb{F}}$-connected to L. In particular, if we denote by $\widetilde{N} = M(n, (\mathbf{Y}, \mathbf{Z})_{\widetilde{\mathbb{F}}})$, we have

$$\widetilde{P} = \begin{pmatrix} \widetilde{N}_L & 0 \\ 0 & \widetilde{N}_K \end{pmatrix}.$$

The determinant of this matrix is 0, because $\det(\widetilde{P}) = \det(M(n, (\mathbf{Y}, \mathbf{Z})_{\widetilde{\mathbb{F}}}))$ is formed by monomials of degree n, with the variables in each monomial having degree 1 and the only possible free variables are the ones in r, which is a monomial of degree $n-1$, that is, there are $n-1$ free variables. Hence $\det(\widetilde{P}) = 0$.

Letting $\widetilde{W} = \mathrm{adj}(\widetilde{N})$ we get that $r \preccurlyeq \widetilde{W}_{11} = \mathrm{adj}(\widetilde{N}_L)_{11} \det(\widetilde{N}_K)$, which implies that $\widetilde{W}_{11}$ is not zero. In particular, $\det(\widetilde{N}_K)$ is not 0. Similarly, $\det(\widetilde{N}_L)$ is not 0, which is a contradiction and the result is shown. □

5. Proof of Proposition 3.7

Recall that $\mathbb{F}^{J^c}$ is the set of free variables given by $\mathbb{F}_1^{J^c} = J^c$ and $\mathbb{F}_2^{J^c} = (J \times J \cup J^c \times J^c) \setminus \{(i,i) : i \in I\}$. Also recall that the intersection of two sets of free variables $\mathbb{F}$, $\mathbb{G}$ is just the set of free variables given by $(\mathbb{F} \cap \mathbb{G})_1 = \mathbb{F}_1 \cap \mathbb{G}_1$ and $(\mathbb{F} \cap \mathbb{G})_2 = \mathbb{F}_2 \cap \mathbb{G}_2$. The following lemma is quite useful.

Lemma 5.1. *Assume that $\mathbb{F}$, $\mathbb{G}$ are two sets of free variables, then for all i*

$$V_{ii}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F} \cap \mathbb{G}}) = V_{ii}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}}) \cap V_{ii}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{G}}).$$

In particular, if J, K are two different subsets of I, such that $1 \in J \cap K$, then

$$V_{11}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}^{J^c} \cap \mathbb{F}^{K^c}}) = V_{11}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}^{J^c}}) \cap V_{11}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}^{K^c}}) = 0.$$

Proof. It is clear that

$$V_{ii}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F} \cap \mathbb{G}}) \preccurlyeq V_{ii}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}}).$$

So, to prove the equality we need to show that any monomial r present in $V_{ii}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}}) \cap V_{ii}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{G}})$ is also present in $V_{ii}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F} \cap \mathbb{G}})$. This monomial r contains only variables in the intersection $\mathbb{F} \cap \mathbb{G}$ and therefore, it is clearly present in $V_{ii}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F} \cap \mathbb{G}})$.

For the second part of the lemma, we note that, after a permutation of rows and columns, the matrix $M(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}^{J^c} \cap \mathbb{F}^{K^c}})$ has the block structure

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{pmatrix},$$

where A is the matrix associated to the nonempty set of indexes $J \cap K$, which contains at least the index 1. The matrices B, C, D are associated to the sets $J \setminus K$, $K \setminus J$, $I \setminus (J \cup K)$, respectively. The hypothesis that J, K are different implies that at least one of the sets $J \setminus K$, $K \setminus J$ is not empty. Without loss of generality we assume that the first one is not empty. Since the free variables are $\mathbb{F}^{J^c} \cap \mathbb{F}^{K^c}$ we conclude the diagonal variables $\mathbf{Y}$ associated to B are all 0. Thus, the row sums of B are 0. This implies that $\det(B) = 0$ and then $V_{11}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}^{J^c} \cap \mathbb{F}^{K^c}}) = 0$. $\square$

Lemma 5.2. *Let r be a monomial present in $V_{11}(n, (\mathbf{Y}, \mathbf{Z}))$. Define $K = \{i : r \text{ is present in } V_{ii}(n, (\mathbf{Y}, \mathbf{Z}))\}$ and $J = I \setminus K$. Then, r is present in $V_{11}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}^{J^c}})$.*

Proof. We need to show that any of the variables z_{kj}, for $k \in K$, $j \in J$, is not present in r.

Assume for simplicity that $K = \{1, \cdots, \ell\}$, $J = \{\ell + 1, \cdots, n\}$. We shall prove that $z_{1\ell+1}$ does not participate in r.

On the one hand, since r participates in $V_{11}, \cdots, V_{\ell\ell}$ then r does not depend on $y_1, \cdots, y_\ell$ and so r must appear in the expansion of the determinant

$$\bigodot_{k=2}^{\ell} \left(\bigoplus_{t \neq k} z_{kt} \right) \bigodot_{j=\ell+1}^{n} \left(y_j \bigoplus_{s \neq j} z_{js} \right).$$

The only way $z_{1\ell+1}$ appears in r, is that r is present in the expansion of

$$\begin{aligned} &\bigodot_{k=2}^{\ell} \left(\bigoplus_{t \neq k} z_{kt} \right) \odot z_{1\ell+1} \bigodot_{j=\ell+2}^{n} \left(y_j \bigoplus_{s \neq j} z_{js} \right) \\ &= z_{1\ell+1} \odot \bigodot_{k=2}^{\ell} \left(\bigoplus_{t \neq k} z_{kt} \right) \bigodot_{j=\ell+2}^{n} \left(y_j \bigoplus_{s \neq j} z_{js} \right). \end{aligned}$$

Since the variable $z_{1\ell+1}$ does not appear in $\bigodot_{k=2}^{\ell} \left(\bigoplus_{t \neq k} z_{kt} \right) \bigodot_{j=\ell+2}^{n} \left(y_j \bigoplus_{s \neq j} z_{js} \right)$, we conclude r appears in the expansion of

$$\bigodot_{k=1}^{\ell} \left(\bigoplus_{t \neq k} z_{kt} \right) \bigodot_{j=\ell+2}^{n} \left(y_j \bigoplus_{s \neq j} z_{js} \right) \preccurlyeq V_{\ell+1,\ell+1}(n, (\mathbf{Y}, \mathbf{Z})).$$

This contradicts the maximally of K and the claim is shown. Therefore, r contains no variable z_{kj} for $k \in K$, $j \in J$ and no variable y_i for $i \in K$. So, r appears in $V_{11}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}^{J^c}})$ and the result is shown. $\square$

Notice that $V_{11}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}^{\emptyset}}) = V_{11}(n, (0, \mathbf{Z}))$ is the sum of monomials that do not depend on $\mathbf{Y}$ and we will show it is the minimum polynomial of V.

Proof of Proposition 3.7. Let us start with the representation of the minimum polynomial $p = p[V](\mathbf{X})$. This is the common polynomial in $V_{11}, \cdots, V_{nn}$. Since, for all i, the variable y_i is not present in V_{ii} it is clear that $p \preccurlyeq V_{ii}(n, (0, \mathbf{Z}))$. In order to show the desired representation we prove that

$$V_{11}(n, (0, \mathbf{Z})) = V_{22}(n, (0, \mathbf{Z})),$$

or equivalently $V_{11}(n, (0, \mathbf{Z})) \bigoplus V_{22}(n, (0, \mathbf{Z})) = 0$ in the Wang's algebra. This expression is given by

$$V_{11}(n, (0, \mathbf{Z})) \bigoplus V_{22}(n, (0, \mathbf{Z})) = \left(\bigoplus_{\ell \neq 1} z_{1\ell} \bigoplus_{k \neq 2} z_{2k} \right) \bigodot_{i=3}^{n} \left[\bigoplus_{j \neq i} z_{ij} \right].$$

Using that $z_{12} \bigoplus z_{12} = 0$ and collecting some terms we get

$$V_{11}(n, (0, \mathbf{Z})) \bigoplus V_{22}(n, (0, \mathbf{Z})) = \left(\bigoplus_{\ell=3}^{n} (z_{1\ell} \bigoplus z_{2\ell}) \right) \bigodot_{i=3}^{n} \left[\bigoplus_{j \neq i} z_{ij} \right].$$

This last expression corresponds to the expansion of the determinant of the matrix

$$\begin{pmatrix} \sum_{j=3}^{n} z_{1j} + z_{2j} & -(z_{13} + z_{23}) & -(z_{14} + z_{24}) & \cdots & -(z_{1n} + z_{2n}) \\ -(z_{13} + z_{23}) & \sum_{k=1, k \neq 3}^{n} z_{3k} & -z_{34} & \cdots & -z_{3n} \\ -(z_{14} + z_{24}) & -z_{34} & \sum_{k=1, k \neq 4}^{n} z_{4k} & \cdots & -z_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -(z_{1n} + z_{2n}) & -z_{3,n} & -z_{4,n} & \cdots & \sum_{k=1, k \neq n}^{n} z_{kn} \end{pmatrix},$$

which is zero because the row sums of this matrix are all 0.

Hence, $V_{11}(n, (0, \mathbf{Z})) = V_{22}(n, (0, \mathbf{Z}))$ and similarly $V_{11}(n, (0, \mathbf{Z})) = V_{ii}(n, (0, \mathbf{Z}))$ for all i, showing the desired representation for the minimum polynomial:

$$p[V]((\mathbf{Y}, \mathbf{Z})) = V_{11}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}^{\emptyset}}).$$

Let us compute the minimum polynomial for $V_{11} - p, \cdots, V_{n-1n-1} - p$, where $p = p[V](\mathbf{X})$. For that purpose consider the submatrix $N = M|_J$ where $J = \{1, \cdots, n-1\}$ and the Schur's complement

$$B = N - \frac{1}{y_n + \sum_{\ell=1}^{n-1} z_{\ell n}} (z_{1n}, \cdots, z_{n-1,n})'(z_{1n}, \cdots, z_{n-1,n}).$$

Recall that $B = B((\mathbf{R}, \mathbf{T})) = M(n-1, (\mathbf{R}, \mathbf{T}))$ where

$$\begin{aligned} r_i &= y_i + \tfrac{y_n z_{in}}{y_n + \sum_{\ell=1}^{n-1} z_{\ell n}}, \quad \text{for } i \in J, \\ t_{ij} &= z_{ij} + \tfrac{z_{in} z_{jn}}{y_n + \sum_{\ell=1}^{n-1} z_{\ell n}}, \quad \text{for } i \neq j \in J. \end{aligned} \tag{5.1}$$

Using that $(y_n + \sum_{\ell=1}^{n-1} z_{\ell n})\mathrm{adj}(B) = (\mathrm{adj}(M))|_J$, we conclude the minimal polynomial for $V|_J$ should be the minimum polynomial of $\mathrm{adj}(B)$ multiplied by $y_n + \sum_{\ell=1}^{n-1} z_{\ell n}$. So, the diagonal variables $r_1, \cdots, r_{n-1}$ are 0. This condition on the $\mathbf{Y}$, $\mathbf{Z}$ variables is that $y_1 = \cdots = y_{n-1} = 0$ together with one of the following two cases

$$\begin{array}{ll} (I) & y_n \text{ is free and } z_{1n} = \cdots = z_{n-1,n} = 0, \quad \text{or} \\ (II) & y_n = 0 \text{ and } z_{1n}, \cdots, z_{n-1,n} \text{ are free.} \end{array}$$

The main problem is that under these conditions the determinant of M is zero and then the relation $(y_n + \sum_{\ell=1}^{n-1} z_{\ell n})\mathrm{adj}(B) = (\mathrm{adj}(M))|_J$ may not hold. Under condition (II) we obtain the minimum polynomial of V and not the minimum polynomial of $V|_J$. So, condition (I) should give the right answer. Indeed, if we let $V^{(n)} = V - p[V]((\mathbf{Y}, \mathbf{Z}))$ we claim that

$$p[V^{(n)}|_J]((\mathbf{Y}, \mathbf{Z})) = V_{11}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}^{J^c}}), \tag{5.2}$$

where $\mathbb{F} = \mathbb{F}^{J^c}$ are the free variables obtained when $y_1 = \cdots = y_{n-1} = 0$ and $z_{1n} = \cdots = z_{n-1,n} = 0$.

Let us prove (5.2). As before, let us show that $V_{11}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}}) \bigoplus V_{22}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}})$ is 0. This polynomial is computed from

$$V_{11}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}}) \bigoplus V_{22}(n, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}}) = \left(\bigoplus_{\ell=3}^{n-1} (z_{1\ell} \bigoplus z_{2\ell}) \right) \bigodot_{i=3}^{n-1} \left[\bigoplus_{j \neq i, j \leq n-1} z_{ij} \right] \bigodot y_n.$$

This expression is the determinant of the matrix

$$\begin{pmatrix} \sum_{j=3}^{n-1} z_{1j} + z_{2j} & -(z_{13} + z_{23}) & -(z_{14} + z_{24}) & \cdots & -(z_{1,n-1} + z_{2,n-1}) & 0 \\ -(z_{13} + z_{23}) & \sum_{k=1, k \neq 3}^{n-1} z_{3k} & -z_{34} & \cdots & -z_{3,n-1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -(z_{1,n-1} + z_{2,n-1}) & -z_{3,n-1} & -z_{4,n-1} & \cdots & \sum_{k=1, k \neq n-1}^{n-1} z_{k,n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & y_n \end{pmatrix},$$

which is 0 because the submatrix given by the $n-1$ first rows and columns has 0 determinant (the corresponding row sum of this submatrix is 0).

Notice that $V_{11}(n,(\mathbf{Y},\mathbf{Z})_{\mathbb{F}})$ and $V_{11}(n,(\mathbf{0},\mathbf{Z}))$ cannot have common monomials because in the former all the terms contain the variable y_n and in the latter this variable is 0. This shows that for all $i \le n-1$

$$V_{ii}(n,(\mathbf{Y},\mathbf{Z})_{\mathbb{F}}) = V_{11}(n,(\mathbf{Y},\mathbf{Z})_{\mathbb{F}}) \preccurlyeq \bigcap_{j=1}^{n-1} \left(V_{jj}(n,(\mathbf{Y},\mathbf{Z})) - V_{jj}(n,(0,\mathbf{Z}))\right).$$

A monomial r that appears in the right hand side contains no variable from $y_1,\cdots,y_{n-1}$. Hence r appears in

$$\bigcap_{j=1}^{n-1} \left(V_{jj}(n,(0,y_n),\mathbf{Z})) - V_{jj}(n,(0,\mathbf{Z}))\right),$$

and therefore it must contain y_n. The only thing to show is r cannot contain $z_{1n},\cdots,$ $z_{n-1,n}$. This monomial r has to appear in the expansion of

$$\left(\bigodot_{i=2}^{n-1} \bigoplus_{j=1,j\neq i}^{n} z_{ij}\right) \bigodot y_n.$$

Hence, it does not contains z_{1n} and the representation (5.2) is shown.

Now, we prove by induction on $k=n,\cdots,1$ the following equality

$$\forall J \subseteq I,\ \#J=k, 1\in J \quad p[V^{(k+1)}|_J]((\mathbf{Y},\mathbf{Z})) = V_{11}(n,(\mathbf{Y},\mathbf{Z})_{\mathbb{F}^{J^c}}). \tag{5.3}$$

This has been proved for $k=n,n-1$ and now we show the inductive step. So, we assume the property holds for $n,n-1,\cdots,k+1$ and we show it holds for k. Thus, we take J of size k, such that $1\in J$. As in the case $k=n,n-1$ it is straightforward to show that for all $i\in J$

$$V_{ii}(n,(\mathbf{Y},\mathbf{Z})_{\mathbb{F}^{J^c}}) = V_{11}(n,(\mathbf{Y},\mathbf{Z})_{\mathbb{F}^{J^c}}).$$

Take now a monomial r present in $V_{11}(n,(\mathbf{Y},\mathbf{Z})_{\mathbb{F}^{J^c}})$. From the induction hypothesis, iteration (3.2) and Lemma 5.1 we conclude that r is present in

$$\begin{aligned} V_{11}^{(k+1)} &= V_{11}(n,(\mathbf{Y},\mathbf{Z})) - \sum_{m=k+2}^{n+1} \sum_{\substack{L\,:\,1\in L\\ \#L=m-1}} p[(V_m)|_L]((\mathbf{Y},\mathbf{Z})) \\ &= V_{11}(n,(\mathbf{Y},\mathbf{Z})) - \sum_{m=k+2}^{n+1} \sum_{\substack{L\,:\,1\in L\\ \#L=m-1}} V_{11}(n,(\mathbf{Y},\mathbf{Z})_{\mathbb{F}^{L^c}}). \end{aligned}$$

This fact allows us to conclude that r is present in $p[V^{(k+1)}|_J]((\mathbf{Y},\mathbf{Z}))$ and therefore

$$V_{11}(n,(\mathbf{Y},\mathbf{Z})_{\mathbb{F}^{J^c}}) \preccurlyeq p[V^{(k+1)}|_J]((\mathbf{Y},\mathbf{Z})).$$

To show the equality, consider a monomial r present in $p[V^{(k+1)}|_J]((\mathbf{Y},\mathbf{Z}))$. In particular r is present in $\bigcap_{i\in J} V_{ii}(n,(\mathbf{Y},\mathbf{Z}))$ and therefore r does not depend on $(y_i : i \in J)$.

If r is not present in $V_{11}(n,(\mathbf{Y},\mathbf{Z})_{\mathbb{F}^{J^c}})$ we deduce that r must contain a variable $z_{j\ell}$, for some $j \in J$, $\ell \in J^c$. As in the proof of Lemma 5.2, r must appear in $V_{\ell\ell}(n,(\mathbf{Y},\mathbf{Z}))$. Consider now the set $L = \{i \in I : r \text{ appears in } V_{ii}(n,(\mathbf{Y},\mathbf{Z}))\}$. Clearly $J \cup \{\ell\} \subseteq L$ and thus $m = \#L \geq k+1$. The maximality of L implies that r is not present in any of the minimum polynomials of $V|_K$ for any set K of cardinality larger than m.

The way the sequence $(V_s : s = n+1,\cdots,1)$ is constructed allows us to deduce that r is present on $(V_s)|_L$ for $s = n+1,\cdots,m+1$. The conclusion is that r is present in $p[(V_{m+1})|_L]((\mathbf{Y},\mathbf{Z}))$ and a fortiori it is not present in the entries of the matrix V_m (see formula (3.2)). Given that $(V_{k+1})_{11} \preccurlyeq (V_m)_{11}$ we arrive to a contradiction and therefore we have the equality

$$V_{11}(n,(\mathbf{Y},\mathbf{Z})_{\mathbb{F}^{J^c}}) = p[V^{(k+1)}|_J]((\mathbf{Y},\mathbf{Z})).$$

This finishes the proof of Proposition 3.7. $\square$

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Appendix A. Representation of $\mathcal{KP}$ using ultrametric matrices

Let us start with an example

Example A.1. (**Continuation of Example 4.1.**) Let us examine the representation of a potential given in Theorem 3.8 for $n = 3$. We have

$$M(3,\mathbf{X}) = \begin{pmatrix} a+x+y & -x & -y \\ -x & b+x+z & -z \\ -y & -z & c+y+z \end{pmatrix}.$$

The adjoint of $M(3,\mathbf{X})$ is again

$$V(3,\mathbf{X}) = (xy+xz+zy)\mathbf{1}\mathbf{1}' \\ + \begin{pmatrix} bc+by+bz+xc+zc & xc & by \\ xc & ac+ay+az+xc+yc & az \\ by & az & ab+ax+az+xb+by \end{pmatrix}.$$

This matrix can be expressed as a linear combination of the 7 matrices

$$E_0 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, E_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The decomposition is

$$\begin{aligned} V = {} & (xy + xz + zy)E_0 + xcE_1 + byE_2 + azE_3 + (bc + bz + zc)E_4 \\ & + (ac + ay + yc)E_5 + (ab + ax + xb)E_6. \end{aligned}$$

The term $(ac+ay+yc)E_5$ coincides with $\mathrm{adj}(M(3, (\mathbf{Y}, \mathbf{Z})_{\mathbb{F}})$, where $\mathbb{F} = \mathbb{F}^{\{2\}^c}$, as predicted by Corollary 3.9. Also, V can be expressed as a sum of 3 ultrametric matrices $U1$, $U2$, $U3$ given by

$$U1 = \tfrac{xy+xz+zy}{3}\mathbf{1}\mathbf{1}' + \begin{pmatrix} xc & xc & 0 \\ xc & xc & 0 \\ 0 & 0 & ab + ax + xb \end{pmatrix},$$

$$U2 = \tfrac{xy+xz+zy}{3}\mathbf{1}\mathbf{1}' + \begin{pmatrix} by & 0 & by \\ 0 & ac + ay + yc & 0 \\ by & 0 & by \end{pmatrix},$$

$$U3 = \tfrac{xy+xz+zy}{3}\mathbf{1}\mathbf{1}' + \begin{pmatrix} bc + bz + zc & 0 & 0 \\ 0 & az & az \\ 0 & az & az \end{pmatrix}.$$

We shall discuss the relation between $\mathcal{KP}$ and ultrametric matrices. As we have shown in Corollary 3.6 every matrix in $\mathcal{KP}$ is the sum of ultrametric matrices. Here we study in more details this relation and we construct another representation. For that purpose, it is important to use the following characterization of ultrametric matrices.

A conditional expectation $\mathbb{E}$ is a nonnegative symmetric matrix that satisfies $\mathbb{E}^2 = \mathbb{E}$, $\mathbb{E}\mathbf{1} = \mathbf{1}$, that is a nonnegative projection that preserves the constants. Every conditional expectation can be described in the following way. Consider a partition $\mathcal{R}$ of $I = \{1, \cdots, n\}$ given by its atoms $\mathcal{R} = \{A_1, \cdots, A_p\}$ and the corresponding cardinals $\#A_\ell$ for $\ell = 1, \cdots, p$. Then, the associated conditional expectation is

$$\mathbb{E} = \sum_{\ell=1}^{p} \frac{1}{\#A_\ell} \mathbf{1}_{A_\ell} \mathbf{1}'_{A_\ell}. \tag{A.1}$$

Given a conditional expectation $\mathbb{E}$, the partition that generates can be computed from the partition obtained from the invariant sets $\{A \subseteq I : \mathbb{E}\mathbf{1}_A = \mathbf{1}_A\}$. We denote by $J \in \mathbb{E}$ to mean that J is an atom of $\mathbb{E}$.

While $\mathbb{E}$ is singular (the only exception is when $\mathbb{E}$ is the identity), for any positive number $a > 0$ the matrix $\mathbb{E} + a\mathbb{I}$ is a potential. Moreover, (A.1) is a particular case of the representation (3.1).

To describe ultrametric matrices using conditional expectations, we need the notion of filtration, which is simply a chain of comparable conditional expectations: $\mathcal{F} = \mathbb{E}_1 < \mathbb{E}_2 < \cdots < \mathbb{E}_N$. The fact that they are comparable is the commutation relation $\mathbb{E}_r\mathbb{E}_s = \mathbb{E}_s\mathbb{E}_r = \mathbb{E}_{s\wedge r}$ for $s, r = 1, \cdots, N$.

We shall assume without loss of generality that $\mathcal{F}$ is maximal as a chain of projections. This implies that the projection $\mathbb{E}_{s+1} - \mathbb{E}_s$, for all s, projects over a space of dimension 1. The obvious conclusions are $N = n = \#I$, $\mathbb{E}_1 = \frac{1}{n}\mathbf{1}\mathbf{1}'$ is the projection over the constants and $\mathbb{E}_n = \mathbb{I}$. To every subset $J \subseteq I$ we associate the number

$$C(J) = \#\{(\mathbb{E}, \mathcal{F}) : J \in \mathbb{E} \in \mathcal{F}\},$$

that is, the number of times the atom J appears in all conditional expectations of all maximal filtrations. Clearly $C(J) \geq 1$. These numbers are not easy to compute and will be part of the decomposition we are searching.

The last ingredient we need is the following representation for every ultrametric matrix. The proof of this result can be found in [8]. To describe it for every vector z we denote by D_z the diagonal matrix associated to z.

Lemma A.1. *A nonnegative matrix U is ultrametric if and only if there exists a maximal filtration $\mathcal{F} = \mathbb{E}_1 < \mathbb{E}_2 < \cdots < \mathbb{E}_n$, a collection of nonnegative vectors $z_1, \cdots, z_n$ such that $\mathbb{E}_k(z_k) = z_k$ for all $k = 1, \cdots, n$ and*

$$U = \sum_{k=1}^{n} D_{z_k}\mathbb{E}_k.$$

Remark A.1. The condition $\mathbb{E}_k(z_k) = z_k$ is exactly the condition that $z_k \in \mathrm{Im}(\mathbb{E}_k)$, which is equivalent to the fact that z_k is constant on the atoms of $\mathbb{E}_k$. In the language of probability theory it is said that $z_1, \cdots, z_n$ is adapted to $\mathcal{F}$.

Now, we are in a position to construct a richer representation of a potential as a sum of ultrametric matrices. Consider a potential U with a representation like (3.1)

$$U = \sum_{i=1}^{t} c_i\, u_i u_i',$$

where $c_i \geq 0$ are constants and $u_i = \mathbf{1}_{H_i}$ is the $\{0, 1\}$-valued vector with support on $H_i \subseteq I$.

The index set for this decomposition is given by $(\mathbb{E}, \mathcal{F})$, where $\mathcal{F}$ is a maximal filtration and $\mathbb{E} \in \mathcal{F}$. Notice that

$$u_i u_i' = \frac{\#H_i}{C(H_i)} \sum_{(\mathbb{E},\mathcal{F}):H_i\in\mathbb{E}} D_{H_i}\mathbb{E}.$$

Therefore, we have

$$U = \sum_{i=1}^{t} c_i \frac{\#H_i}{C(H_i)} \sum_{(\mathbb{E},\mathcal{F}):H_i\in\mathbb{E}} D_{H_i}\mathbb{E} = \sum_{\mathcal{F}} \sum_{\mathbb{E}\in\mathcal{F}} D_{z(\mathbb{E},\mathcal{F})}\mathbb{E}, \tag{A.2}$$

where the vectors $z(\mathbb{E}, \mathcal{F})$ are obtained as

$$z(\mathbb{E},\mathcal{F}) = \sum_{i:\, H_i\in\mathbb{E}} c_i \frac{\#H_i}{C(H_i)} \mathbf{1}_{H_i}.$$

It is straightforward to show that $\mathbb{E}(z(\mathbb{E}, \mathcal{F})) = z(\mathbb{E}, \mathcal{F})$ and therefore (A.2) is a decomposition of U as a sum of ultrametric matrices.

Appendix B. Interpretation of $M(n, \mathbf{X})$ and $V(n, \mathbf{X})$ in graph theory

Let us start with the generic (or symbolic) **laplacian** symmetric matrix $\mathcal{L} = (\ell_{ij})$ of the complete graph K with $n+1$ vertices. Each edge (i, j) is labelled by z_{ij}, a "free variable", with $z_{ij} = z_{ji}$; by definition we have $\ell_{ij} = -z_{ij}$ for $i \neq j$ and $\ell_{ii} = \sum_{k\neq i} z_{ik}$. Of course $\mathcal{L}$ has zero row and column sums. In particular, it is not invertible and so it cannot be a generic M-matrix. Any principal submatrix of order n is like our $M(n, (\mathbf{Y}, \mathbf{Z}))$ except for the names of the variables. On the other hand, taking some free variables z_{ij} equal to 0, one gets the generic laplacian of any graph $\mathbb{G}$ with $n + 1$ vertices and, furthermore, taking the remaining variables equal to 1, one gets the classical laplacian of the graph. Similar remarks hold for the results given below.

The laplacian $\mathcal{L}$, or one of its principal submatrices of order n, is often called a **Kirchhoff matrix**, and a node-admittance (or conductance) matrix in circuit theory. Kirchhoff proved, at least implicitly, that all the cofactors (not only the principal ones) of $\mathcal{L}$ are equal. This result is quite surprising, but not difficult to prove directly. Nevertheless, the elucidation of the common value of these determinants is more involved.

We arrive to the very often called **matrix-tree theorem**, which can be found in the literature devoted to graph theory or circuit theory. This result is also called the **Kirchhoff's theorem**, or the **Maxwell's rule**. To explain it, let $\mathcal{T}$ be the set of spanning trees of the graph K and for each $T \in \mathcal{T}$ denote by $m(T)$ the product of the z_{ij} where (i, j) goes over the edges of T. Then, the matrix-tree theorem says that the common value of the preceding determinants (and so our $det(M(n, \mathbf{X}))$ is equal to $\sum_{T\in\mathcal{T}} m(T)$. Hence, there exists a natural bijection between the set of spanning trees and the set of monomials in the determinant.

Let us make some historical comments freely adapted from Moon (see [16] page 42). Actually Kirchhoff in 1847 (see [13]) gave, with proof, "only" an analogous of the matrix-tree theorem, since he was considering principally meshes and not nodes in circuit theory.

Sylvester in 1857 (see [20]), not aware of Kirchhoff's work, gave only the result without a proof as a rule to calculate the determinant. Borchardt, following [16], gave a proof in 1889. The third edition of the Maxwell's treatise [15] gave in the appendix of the chapter VI of part II (introduced by the editor J.J. Thomson) as a rule without proof and reference, looking at nodes and not at meshes like Kirchhoff. Finally, statements and proofs abounded in the XXth century.

The matrix-tree theorem gives a graphic interpretation of the cofactors of $\mathcal{L}$. We are going now to give such an interpretation for the cofactors of the principal submatrices of $\mathcal{L}$. So, to fix the ideas, we note M the matrix obtained by deleting the last row and last column of $\mathcal{L}$, and $V = (V_{ij})$ its adjoint (or adjugate) so that, except for some change of names of variables, we find again our original matrix V. On the other hand let us say that a subgraph of K is a 2-tree if it is a forest consisting of exactly two trees. Denote by $\mathbb{S}$ the set of all 2-trees spanning K and for each $S \in \mathbb{S}$ note $m(S)$ as before. Finally, for all i, j fixed, let $\mathbb{S}_{ij}$ be the subset of $\mathbb{S}$ whose elements contain vertices i and j in one of their components and the vertex $n+1$ in the other one. One has $V_{ij} = \sum_{S\in\mathbb{S}_{ij}} m(S)$. As an evident corollary, we have $V_{ij} = V_{ii} \cap V_{jj}$ as in Theorem 3.5 in Section 3, a formula we learned from [10] without a proof. This result is more recent and much less cited than the matrix-tree theorem. A proof can be found in [19] (attributed to Mayeda (1955)) or in [3].

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