SERIE DOCUMENTOS DE TRABAJO

FACULTAD ECONOMÍA Y NEGOCIOS UNIVERSIDAD DE CHILE

DEPARTAMENTO DE ECONOMÍA

SDT 362

ESSENTIAL EQUILIBRIA OF LARGE GAMES

Autores: Sofía Correa y Juan Pablo Torres-Martínez

Versión Original: Septiembre 2012
Versión Revisada: Diciembre 2012

Santiago, Diciembre de 2012

ESSENTIAL EQUILIBRIA OF LARGE GAMES

SOFIA CORREA AND JUAN PABLO TORRES-MARTÍNEZ

Abstract. We characterize essential stability of Cournot-Nash equilibria of generalized games with a continuum of players. As application, we rationalize the active participation of politically engaged individuals as the unique essential equilibrium in an electoral game with a continuum of Cournot-Nash equilibria.

Keywords. Essential equilibria - Essential sets and components - Large games
JEL Classification: C62, C72, C02.

1. Introduction

In this study we focus on essential stability of Cournot-Nash equilibria of large generalized games, analyzing how equilibrium allocations change when some characteristics of the game are perturbed. We allow for any kind of perturbation, provided that it can be defined through a continuous parametrization over a complete metric space of parameters.

In large generalized games both objective functions and admissible strategies may depend on the actions chosen by the players in the game. There are two types of players: (i) a continuum set of non-atomic players, with continuous objective functions and compact sets of actions; and (ii) a finite number of atomic players, with quasi-concave and continuous objective functions, and compact and convex sets of actions. Atomic players' actions may directly affect the decisions of other individuals, while decisions of non-atomic players impact others participants only through aggregate information. Indeed, the profiles of actions of non-atomic players are codified and aggregated, generating messages to other participants on the game. Under mild conditions on the characteristics of the generalized game, pure strategy Cournot-Nash equilibrium always exists (cf. Balder (1999, 2002), Riascos and Torres-Martínez (2012)).

Date: December, 2012.
Correa acknowledges financial support from Conicyt and University of Chile through graduate fellowships.

Sofía Correa
Chile's National Economic Prosecutor Office
Agustinas 853, 12th floor, Santiago, Chile.
e-mail: scorrea@fne.gob.cl

Juan Pablo Torres-Martínez
Department of Economics, University of Chile
Diagonal Paraguay 257, Santiago, Chile
e-mail: juan.torres@fen.uchile.cl.

In this context, it is natural to ask how equilibrium actions of atomic players and equilibrium messages induced by the decisions of non-atomic players—the pieces of information that fully determine the strategic behavior of players—change when the characteristics of the generalized game are perturbed. The focus is on essential stability: under which conditions Cournot-Nash equilibria of a (generalized) game can be approximated by equilibria of perturbed games.

We depart our analysis of essential stability assuming that each characteristic of a game can be perturbed, i.e., objective functions, actions sets, or correspondences of admissible strategies. Our first result ensures that for a dense residual subset of the space of generalized games, messages and atomic players' actions associated to Cournot-Nash equilibria are stable to perturbations (Theorem 1).[1] Also, we guarantee that uniqueness of equilibrium messages and actions for atomic players is a sufficient condition for stability. We also analyze the stability of subsets of equilibrium messages and actions, obtaining results analogous to those ensured in the literature for convex games with finitely many players: for any generalized game there are essential connected subsets of Cournot-Nash equilibria (Theorem 2).

These results of stability are extended to allow specific perturbations, that we capture through parametrizations of the set of generalized games. We prove that, when the set of parameters constitutes a complete metric space and the mapping associating parameters with generalized games is continuous, then stability results previously described still holds (Theorem 3) and essential sets varies continuously (Theorem 4). As byproduct, we obtain stability results for convex (generalized) games with finitely many players and, therefore, we extend results of the previous literature to allow for a great variety of admissible perturbations.

To obtain our results about essential stability, we prove that the compact-valued correspondence that associates generalized games with sets of equilibrium messages-actions, referred as Cournot-Nash correspondence, has closed graph. To guarantee this property, we use the fact that the set of non-atomic players has finite measure and their actions are transformed into finite-dimensional codes (which are integrated to obtain messages). Indeed, under these conditions, we can ensure the closed graph property of the Cournot-Nash correspondence applying the multidimensional Fatou's Lemma (cf. Hildenbrand (1974, Lemma 3, page 69)).

Essential stability of equilibrium messages and atomic player actions have relevant implications in applied game theory. Our results ensure that in models based in large generalized games, small errors in the estimation or calibration of some parameters does not necessarily affect player's decisions. In addition, essential stability can also be used as a refinement criteria in the presence of multiplicity of equilibria. We illustrate this last possibility through electoral games, with the aim to give a rationale for electoral participation of politically engaged individuals.

The rest of the paper is organized as follows: Section 2 is devoted to discuss the related literature. In Section 3 we describe the space of large generalized games. In Section 4 and 5 we analyze essential stability properties of Cournot-Nash equilibria. In Section 6 we apply our results to an electoral game. The proofs of our results are given in the Appendix.

[1] A subset of a metric space is residual if it contains the intersection of a countable family of dense and open sets.

2. Related Literature

The concept of essential stability has its origins in the mathematical analysis literature, where it was introduced as a natural property for fixed points of functions and correspondences. In a seminal paper, Fort (1950) introduces the concept of essential fixed point of a continuous function: a fixed point is essential if it can be approximated by fixed points of functions close to the original. In addition, a continuous function is essential if it has only essential fixed points. Considering the set of continuous functions from a compact metric space to itself, Fort (1950) ensures that the set of essential functions is dense. He also proves that continuous functions that have only one fixed point are essential. These concepts and properties have natural extensions to multivalued mappings, as shown by Jia-He (1962). However, not all mappings are essential and, therefore, it is natural to analyze the stability of subsets of fixed points. With this aim, Kinoshita (1952) introduces the concept of essential component of the set of fixed points of a function: a maximal connected set that is stable to perturbations on the characteristics of the function. He proves that any continuous mapping has at least one essential component. Jia-He (1963) and Yu and Yang (2004) extend these results to multivalued mappings. They prove that compact-valued upper hemicontinuous correspondences have at least one essential component, although fixed points of these correspondences may not be essential. These results are complemented by Yu, Yang, and Xiang (2005) who also analyze how essential components change when mappings are perturbed.

This literature motivates the study of equilibrium stability in games. Indeed, since in many non-cooperative games the set of Nash equilibria coincides with the set of fixed points of a correspondence, techniques described above allow to analyze how the equilibria of a game change when payoffs and action sets are perturbed. In this direction, essential stability of Nash equilibria for games with finitely many players is studied by Wu and Jia-He (1962), Yu (1999), Yu, Yang, and Xiang (2005), Zhou, Yu and Xiang (2007), Yu (2009), Carbonell-Nicolau (2010), and Scalzo (2012).

More precisely, Wu and Jia-He (1962) address the stability of the set of Nash equilibria for finite games. They ensure that any game can be approximated arbitrarily by a game whose equilibria are all essential. Yu (1999) formalizes and extends these results for convex games with a finite number of players, analyzing perturbations in payoffs, sets of actions, and correspondences of admissible strategies. Jia-He (1963), Yu, Yang, and Xiang (2005) and Yu (2009) supplement these results to analyze the existence of essential components of the set of Nash equilibria for games and generalized games. Zhou, Yu and Xiang (2007) study the notion of essential stability for mixed-strategy equilibria in games with compact sets of pure strategies and finitely many players. They also compare the concept of essential stability with strategic stability, a notion studied by Kohlberg and Mertens (1986), Hillas (1990), and Al-Najjar (1995). Recently, allowing for discontinuities on objective functions, Carbonell-Nicolau (2010) and Scalzo (2012) analyze the essential stability of Nash equilibria for games with finitely many players.

As we describe in the introduction, our goal is to contribute to this growing literature by addressing essential stability properties of Cournot-Nash equilibria in large generalized games. However, results of essential stability for games with finitely many players take advantage of the fact that the set of (pure strategy) equilibria is compact and non-empty. Actually, with these properties, to obtain some of the main results of essential stability it is sufficient to ensure that the correspondence that associates games with equilibrium allocations has closed graph. In our case, under mild conditions on the characteristics of the generalized game, a (pure strategy) Cournot-Nash equilibrium always exists, as was proved by Schmeidler (1973)—for the case of large games—and by Balder (1999, 2002)—for the case of generalized games. However, the set of pure strategy equilibria is not necessarily compact (see footnote 4). Therefore, the traditional analysis of essential stability can not be directly implemented in our context.

Nevertheless, associated to any Cournot-Nash equilibrium of a large generalized game there is a vector of messages (generated by the actions of non-atomic players) and a vector of optimal actions of atomic players. These vectors of messages-actions constitute all the relevant information that any player takes into account to make optimal decisions. Furthermore, when there is a compact set of non-atomic players with compact sets of actions, the set of equilibrium messages-actions coincides with the fixed points of a compact-graph correspondence and, therefore, it is a compact set too (see Riascos and Torres-Martínez (2012, Theorem 1)). Hence, we focus our analysis on the stability of equilibrium messages-actions to perturbations on the characteristics of the generalized game.

3. The Space $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$ of Generalized Games

We introduce large generalized games, as those studied by Riascos and Torres-Martínez (2012). Through our model some characteristics of the generalized game are fixed and summarized by a vector $(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$. The set of non-atomic players T_1 is a non-empty and compact subset of a metric space and there is a σ-algebra $\mathcal{A}$ such that, for some finite measure μ, $(T_1, \mathcal{A}, \mu)$ is a complete atomless measure space. The set of atomic players, denoted by T_2, is non-empty and finite. $\widehat{K}$ is a non-empty and compact metric space where non-atomic players' actions belongs. For any $t \in T_2$, the actions of atomic players belong to a non-empty and compact Frechet space $\widehat{K}_t$.[2] Finally, non-atomic players' action are codified by a function $H : T_1 \times \widehat{K} \to \mathbb{R}^m$, which is continuous with respect to the product topology induced by the metrics of T_1 and $\widehat{K}$.

In a game $\mathcal{G}((K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2})$ among players in $T_1 \cup T_2$, each $t \in T_1$ has associated a closed and non-empty action space $K_t \subseteq \widehat{K}$, while each $t \in T_2$ has a closed, convex and non-empty action space $K_t \subseteq \widehat{K}_t$. A profile of actions for players in T_1 is given by a function $f : T_1 \to \widehat{K}$ such that $f(t) \in K_t$, for any $t \in T_1$. Any vector $a = (a_t; t \in T_2) \in \prod_{t\in T_2} K_t$ constitutes a profile of actions for players in T_2. For each $i \in \{1, 2\}$, let $\mathcal{F}^i((K_t)_{t\in T_i})$ be the space of profiles of actions for agents in T_i. In addition, for any $t \in T_2$, let $\mathcal{F}^2_{-t}((K_j)_{j\in T_2\setminus\{t\}})$ be the set of vectors $a_{-t} \in \Pi_{j\in T_2\setminus\{t\}} K_j$.

Each participant considers aggregated information about the actions taken by players in T_1. Thus, if non-atomic players choose a profile of actions $f \in \mathcal{F}^1((K_t)_{t\in T_1})$, then the relevant characteristics of this actions are coded by the function H. Also, each player only take into account, for strategic purposes, aggregated information about these available characteristics through a message $m(f) := \int\limits_{T_1} H(t, f(t))d\mu$. For this reason, we concentrate our attention only on those action profiles for which

[2] That is, $\widehat{K}_t$ is a non-empty and compact metrizable locally convex topological vector space.

messages are well defined. That is, we consider profiles $f \in \mathcal{F}^1((K_t)_{t\in T_1})$ such that $H(\cdot, f(\cdot))$ is a measurable function from T_1 to $\mathbb{R}^m$.[3]

Therefore, the set of messages associated with action profiles of non-atomic players is given by $M((K_t)_{t\in T_1}) = \left\{ \int_{T_1} H(t, f(t))d\mu : \quad f \in \mathcal{F}^1((K_t)_{t\in T_1}) \;\wedge\; H(\cdot, f(\cdot)) \text{ is measurable} \right\}$.

Let $\widehat{M} = M((\widehat{K})_{t\in T_1})$, $\widehat{\mathcal{F}}^1 = \mathcal{F}^1((\widehat{K})_{t\in T_1})$, $\widehat{\mathcal{F}}^2 = \mathcal{F}^2((\widehat{K}_t)_{t\in T_2})$, and $\widehat{\mathcal{F}}^2_{-t} = \mathcal{F}^2((\widehat{K}_s)_{s\in T_2\setminus\{t\}})$. Messages and profiles of actions may restrict players admissible strategies. Indeed, the set of strategies available for a player $t \in T_1$ is determined by a continuous correspondence $\Gamma_t : \widehat{M} \times \widehat{\mathcal{F}}^2 \twoheadrightarrow K_t$ with non-empty and compact values. Analogously, the set of strategies that a player $t \in T_2$ can choose is determined by a continuous correspondence $\Gamma_t : \widehat{M} \times \widehat{\mathcal{F}}^2_{-t} \twoheadrightarrow K_t$ with non-empty, compact and convex values. We assume that, for any $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$, the correspondence that associates to any $t \in T_1$ the set of admissible strategies $\Gamma_t(m, a)$ is measurable.

Given a metric space S, let $\mathcal{U}(S)$ be the set of continuous functions $u : S \to \mathbb{R}$ endowed with the sup norm topology. We assume that each player $t \in T_1$ has a objective function $u_t \in \mathcal{U}(\widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2)$, while each atomic player $t \in T_2$ has a objective function $u_t \in \mathcal{U}(\widehat{M} \times \widehat{\mathcal{F}}^2)$ which is quasi-concave in its own strategy a_t (for convenience of notations, we refer to this subset of $\mathcal{U}(\widehat{M} \times \widehat{\mathcal{F}}^2)$ as $\mathcal{U}_t(\widehat{M} \times \widehat{\mathcal{F}}^2)$). The mapping $U : T_1 \to \mathcal{U}(\widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2)$ defined by $U(t) = u_t$ is measurable.

Definition 1. *A Cournot-Nash equilibrium of $\mathcal{G}((K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2})$ is given by action profiles $(f^*, a^*) \in \mathcal{F}^1((K_t)_{t\in T_1}) \times \mathcal{F}^2((K_t)_{t\in T_2})$ such that,*

$$u_t(f^*(t), m^*, a^*) \geq u_t(f(t), m^*, a^*), \quad \forall t \in T_1,\ \forall f(t) \in \Gamma_t(m^*, a^*),$$
$$u_t(m^*, a^*) \geq u_t(m^*, a_t, a^*_{-t}), \quad \forall t \in T_2,\ \forall a_t \in \Gamma_t(m^*, a^*_{-t}),$$

where the message $m^* := \int_{T_1} H(t, f^*(t))d\mu$.

Riascos and Torres-Martínez (2012, Theorem 1) ensure that, for any $\mathcal{G}((K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2})$ satisfying assumptions above the set of Cournot-Nash equilibria $\mathrm{CN}(\mathcal{G})$ is non-empty.

The Cournot-Nash Correspondence. We analyze stability properties of Cournot-Nash equilibria of a generalized game $\mathcal{G}((K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2})$ when parameters $(K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2}$ may change. To attempt this objective, we introduce the Cournot-Nash correspondence, which associates the parameters that define the generalized game $\mathcal{G}$ with the set of messages and actions $(m^*, a^*) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ such that, for some $f^* \in \widehat{\mathcal{F}}^1$, we have $m^* = m(f^*)$ and $(f^*, a^*) \in \mathrm{CN}(\mathcal{G})$.

Notice that, the set of Cournot-Nash equilibria of a large generalized game is not necessarily compact,[4] a property that was required by the previous literature of essential stability in games

[3]That is, for any Borelian set $E \subseteq \mathbb{R}^m$, $\{t \in T_1 : H(t, f(t)) \in E\}$ belongs to $\mathcal{A}$.

[4]For instance, consider an electoral game with a continuum of non-atomic players, $T_1 = [0, 1]$, which vote for a party in $\{a, b\}$. Let x_t be the action of player $t \in T_1$, and assume that his objective function, u_t, only takes into account the benefits that he receives for any party $\{v_t(a), v_t(b)\}$ weighted by the support that each party has in the population, i.e. $u_t \equiv v_t(a)\mu(\{s \in T_1 : x_s = a\}) + v_t(b)(1 - \mu(\{s \in T_1 : x_s = a\}))$, where μ denotes the Lebesgue measure in $[0, 1]$. That is, the utility level of a player $t \in T_1$ in unaffected by his own action and, therefore, any measurable profile $x : [0, 1] \to \{a, b\}$ constitutes a Nash equilibrium of the game. Hence, the set of Nash equilibria

with finitely many players. However, given any Cournot-Nash equilibrium $(f^*, a^*) \in \mathrm{CN}(\mathcal{G})$, the pair $(m(f^*), a^*)$ contains all the information that players require to take their decisions. Thus, we can concentrate our analysis of stability in the effects that perturbations on the characteristics of a game have on messages and actions of atomic players.[5]

Let $\mathbb{G} = \mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$ be the collection of generalized games that satisfies the hypotheses described in the previous section. The distance between two arbitrary elements of $\mathbb{G}$, namely $\mathcal{G}_1((K_t^1, \Gamma_t^1, u_t^1)_{t\in T_1\cup T_2})$ and $\mathcal{G}_2((K_t^2, \Gamma_t^2, u_t^2)_{t\in T_1\cup T_2})$, is defined by

$$\begin{aligned}
\rho(\mathcal{G}_1, \mathcal{G}_2) \;=\; & \max_{t\in T_1} \max_{(x,m,a)\in \widehat{K}\times\widehat{M}\times\widehat{\mathcal{F}}^2} |u_t^1(x,m,a) - u_t^2(x,m,a)| \\
& + \max_{t\in T_1} \max_{(m,a)\in \widehat{M}\times\widehat{\mathcal{F}}^2} d_H(\Gamma_t^1(m,a), \Gamma_t^2(m,a)) + \max_{t\in T_1} d_H(K_t^1, K_t^2) \\
& + \max_{t\in T_2} \max_{(m,x,a_{-t})\in \widehat{M}\times\widehat{K}_t\times\widehat{\mathcal{F}}^2_{-t}} |u_t^1(m,x,a_{-t}) - u_t^2(m,x,a_{-t})| \\
& + \max_{t\in T_2} \max_{(m,a_{-t})\in \widehat{M}\times\widehat{\mathcal{F}}^2_{-t}} d_{H,t}(\Gamma_t^1(m,a_{-t}), \Gamma_t^2(m,a_{-t})) + \max_{t\in T_2} d_{H,t}(K_t^1, K_t^2),
\end{aligned}$$

where d_H is the Hausdorff distance induced by the metric of $\widehat{K}$ over the collection of its non-empty and compact subsets, and for any $t \in T_2$, $d_{H,t}$ is the Hausdorff metric induced by the topological vector space $\widehat{K}_t$. Since $(T_1, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}))$ are compact sets, T_2 is finite, and $\widehat{M}$ is a non-empty and compact set (see Riascos and Torres-Martínez (2012, Theorem 1, Step 1)), the metric space $(\mathbb{G}, \rho)$ in complete (see detailed arguments in the Appendix).

For any $\mathcal{G}((K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2}) \in \mathbb{G}$, let $\Phi_{\mathcal{G}} : \widehat{M}\times\widehat{\mathcal{F}}^2 \twoheadrightarrow \widehat{M}\times\widehat{\mathcal{F}}^2$ be the set-valued mapping defined by $\Phi_{\mathcal{G}}(m,a) = \left(\Omega^{\mathcal{G}}(m,a), (B_t^{\mathcal{G}}(m,a_{-t}))_{t\in T_2}\right)$ where

$$\begin{aligned}
\Omega^{\mathcal{G}}(m,a) &= \int_{T_1} H(t, B_t^{\mathcal{G}}(m,a))d\mu; \\
B_t^{\mathcal{G}}(m,a) &= \operatorname*{argmax}_{x_t\in\Gamma_t(m,a)} u_t(x_t,m,a), \quad \forall t\in T_1; \\
B_t^{\mathcal{G}}(m,a_{-t}) &= \operatorname*{argmax}_{x_t\in\Gamma_t(m,a_{-t})} u_t(x_t,m,a_{-t}), \quad \forall t\in T_2.
\end{aligned}$$

It follows from Riascos and Torres-Martínez (2012, Theorem 1) that $\Phi_{\mathcal{G}}$ is upper hemicontinuous with non-empty, compact and convex values. Thus, by Kakutani's Fixed Point Theorem, the set of fixed points of $\Phi_{\mathcal{G}}$ is non-empty and compact. Furthermore, (f^*, a^*) is a Cournot-Nash equilibrium of $\mathcal{G}$ if and only if $(m^*, a^*) \in \widehat{M}\times\widehat{\mathcal{F}}^2$ is a fixed point of $\Phi_{\mathcal{G}}$, where $m^* = \int_{T_1} H(t, f^*(t))d\mu$.

Definition 2. *The Cournot-Nash correspondence of $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$ is given by the multivalued function $\Lambda : \mathbb{G} \twoheadrightarrow \widehat{M}\times\widehat{\mathcal{F}}^2$ that associates to any $\mathcal{G}\in\mathbb{G}$ the set of fixed points of $\Phi_{\mathcal{G}}$.*

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is not compact. However, if we consider that each player receives as a message the support that party a has in the population, $m = \mu(\{s \in T_1 : x_s = a\})$, then the set of equilibrium messages is equal to $[0,1]$, which is a compact set.

[5]Since action profiles are coded using the function H, there may exist several Cournot-Nash equilibria that induce a same message. Even that, this indetermination does not have *real effects* on players utility levels.

4. Essential Stability of Equilibria in $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$

We analyze how the set of Cournot-Nash equilibria of a generalized game changes when the parameters that define the game are modified. Our analysis is based in the concept of *essential stability*, that was introduced in the literature by Fort (1950), for single valued mappings, and by Jia-He (1962), for the case of correspondences.

Definition 3. *An equilibrium $(f^*, a^*) \in CN(\mathcal{G})$ is essential if for any open set $O \subset \widehat{M} \times \widehat{\mathcal{F}}^2$ such that $(m(f^*), a^*) \in O$, there exists $\epsilon > 0$ such that $\Lambda(\mathcal{G}') \cap O \neq \emptyset$, for any $\mathcal{G}' \in \mathbb{G}$ that satisfies $\rho(\mathcal{G}, \mathcal{G}') < \epsilon$. A generalized game $\mathcal{G} \in \mathbb{G}$ is essential if its Cournot-Nash equilibria are all essential.*

Hence, $\mathcal{G}$ is an essential generalized game if, and only if, messages and atomic players actions associated to a Cournot-Nash equilibrium of $\mathcal{G}$ can be approximated by equilibrium messages and actions of generalized games close to $\mathcal{G}$. Unfortunately, as the following example illustrate, not all games are essential.

Example. Suppose that, $T_1 = [0,1]$, $T_2 = \{\alpha\}$, $\widehat{K} = \{0,1\}$, $\widehat{K}_\alpha = \mathbb{R}$. Consider the generalized game $\mathcal{G}$ such that, for each $t \in T_1$, $(K_t, \Gamma_t) \equiv (\widehat{K}_t, \widehat{K}_t)$, $(K_\alpha, \Gamma_\alpha) \equiv ([0,1], [0,1])$, and $H(\cdot, x) \equiv x$. In addition, $u_\alpha(m, x) = -\|m - x\|^2$ and, for any $t \in T_1$, $(u_t(0, m, a_\alpha), u_t(1, m, a_\alpha)) = (0.5, 0.5)$.

Then, there is a continuum of Cournot-Nash equilibria and $\Lambda(\mathcal{G}) = \{(\lambda, \lambda) \in \mathbb{R}^2 : \lambda \in [0,1]\}$. On the other hand, given $\epsilon > 0$, let $\mathcal{G}_\epsilon$ be the generalized game obtaining from $\mathcal{G}$ by only change the objective functions of non-atomic players to $(u_t^\epsilon(0, m, a_\alpha), u_t^\epsilon(1, m, a_\alpha)) = (0.5(1+\epsilon), 0.5)$, for any $t \in T_1$. It follows that $\mathcal{G}_\epsilon$ has only one Cournot-Nash equilibrium and $\Lambda(\mathcal{G}_\epsilon) = \{(0,0)\}$. Since $\rho(\mathcal{G}, \mathcal{G}_\epsilon) < \epsilon$, we conclude that $\mathcal{G}$ is not essential. □

Despite the example above, essentiality of equilibrium is a generic property on $\mathbb{G}$.

Theorem 1. *The collection of essential generalized games is a dense residual subset of $\mathbb{G}$. Moreover, if $\Phi_{\mathcal{G}}$ has only one fixed point, then $\mathcal{G}$ is essential.*

Thus, the set of essential generalized games is dense and contains the intersection of a sequence of dense and open subsets of $\mathbb{G}$. In particular, given $\mathcal{G} \in \mathbb{G}$, for any $\epsilon > 0$ there exists an essential generalized game $\mathcal{G}' \in \mathbb{G}$ such that $\rho(\mathcal{G}, \mathcal{G}') < \epsilon$.

Furthermore, even unessential generalized games may have subsets of Cournot-Nash equilibria that are stable. To formalize this property, we introduce concepts of stability for subsets of equilibrium points. Indeed, we adapt the concepts of essential set and essential component that were introduced, in the context of stability of fixed point of multivalued mappings, by Jia-He (1963) and Yu and Yang (2004). These concepts were also addressed by Zhou, Yu, and Xiang (2007) to study stability of mixed strategy equilibria in non-convex games with finitely many players.

Definition 4. *Given $\mathcal{G} \in \mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t \in T_2}, H))$, fix a subset $e(\mathcal{G}) \subseteq \Lambda(\mathcal{G})$. The set $e(\mathcal{G})$ is essential if it is non-empty, compact, and for any open set $O \subset \widehat{M} \times \widehat{\mathcal{F}}^2$ we have that,*

$$[e(\mathcal{G}) \subset O] \Longrightarrow [\exists \epsilon > 0 : \rho(\mathcal{G}, \mathcal{G}') < \epsilon \Longrightarrow \Lambda(\mathcal{G}') \cap O \neq \emptyset] .$$

An essential set $e(\mathcal{G})$ is minimal if it is a minimal element ordered by set inclusion. Furthermore, $e(\mathcal{G})$ is a component if there is $(m^*, a^*) \in \Lambda(\mathcal{G})$ such that, $e(\mathcal{G})$ is the union of all connected subsets of $\Lambda(\mathcal{G})$ containing (m^*, a^*).

Since Λ is non-empty and upper hemicontinuous (see the proof of Theorem 1), it follows from the topological characterization of upper hemicontinuity that, for any generalized game $\mathcal{G} \in \mathbb{G}$ the set $\Lambda(\mathcal{G})$ is essential. Also, given $A \subset B \subseteq \Lambda(\mathcal{G})$, if A is essential and B is compact, then B is essential too. Thus, we focus the attention on the existence of minimal essentials sets.

Notice that, if for some generalized game $\mathcal{G} \in \mathbb{G}$ there is an essential Cournot-Nash equilibrium $(f^*, a^*) \in \mathrm{CN}(\mathcal{G})$, then by definition $\{(m(f^*), a^*)\}$ is a minimal essential subset of $\Lambda(\mathcal{G})$. Thus, Theorem 1 guarantees that, there exists a dense residual collection of generalized games with at least one minimal essential subset that is also connected. On the other hand, as for any $\mathcal{G} \in \mathbb{G}$ the set $\Lambda(\mathcal{G})$ is compact, any component of $\Lambda(\mathcal{G})$ is non-empty, connected and compact. Hence, when $(f^*, a^*) \in \mathrm{CN}(\mathcal{G})$ is essential, the component associated to $\{(m(f^*), a^*)\}$ is an essential subset of $\Lambda(\mathcal{G})$ (because it is compact and contains the essential set $\{(m(f^*), a^*)\}$). It follows from Theorem 1 that, for a generic set of generalized games there exists at least one essential component of the set of equilibrium messages and atomic players actions.

The following result ensures that these two properties hold for any generalized game, even for those that do not have essential equilibria.

Theorem 2. *For each generalized game $\mathcal{G} \in \mathbb{G}$ there is a minimal essential set of $\Lambda(\mathcal{G})$. In addition, if $\Lambda(\mathcal{G})$ has a connected essential set, then it has an essential component.*

Suppose that $\{\widehat{K}, (\widehat{K}_t)_{t \in T_2}\}$ are convex subsets of Banach spaces with metrics induced by norms. Then, every minimal essential set of $\Lambda(\mathcal{G})$ is connected.

5. Essential Stability for Parametrizations of $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t \in T_2}, H))$

We know that $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t \in T_2}, H))$ has a dense residual subset of essential generalized games. Moreover, any generalized game has a minimal essential set and generalized games with strategies contained in Banach spaces always have an essential component. To obtain these results, has been assumed that any characteristic of the generalized game, namely $(K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2}$, can be perturbed. However, it is interesting to discuss stability of Cournot-Nash equilibria when only some characteristics of the game are allowed to suffer perturbations.

Definition 5. *A parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of the space of generalized games is characterized by a complete metric space $(\mathbb{X}, \tau)$ of parameters and a continuous function $\kappa : \mathbb{X} \to \mathbb{G}$ that associates parameters with generalized games.*

Definition 6. *Given a parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of the space $\mathbb{G}$, fix $\mathcal{X} \in \mathbb{X}$.*

(i) A Cournot-Nash equilibrium $(f^*, a^*) \in CN(\kappa(\mathcal{X}))$ is $\mathcal{T}$-essential if for any open set $O \subset \widehat{M} \times \widehat{\mathcal{F}}^2$ such that $(m(f^*), a^*) \in O$, there exists $\epsilon > 0$ such that $\Lambda(\kappa(\mathcal{X}')) \cap O \neq \emptyset$, for any $\mathcal{X}' \in \mathbb{X}$ that satisfies $\tau(\mathcal{X}, \mathcal{X}') < \epsilon$. The generalized game $\kappa(\mathcal{X}) \in \mathbb{G}$ is essential with respect to the parametrization $\mathcal{T}$ if all of its Cournot-Nash equilibrium are $\mathcal{T}$-essential.

(ii) A subset $E \subseteq \Lambda(\kappa(\mathcal{X}))$ is $\mathcal{T}$-essential if it is non-empty, compact, and for each open set $O \subset \widehat{M} \times \widehat{\mathcal{F}}^2$ there exists $\epsilon > 0$ such that, for any $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \epsilon$, $\Lambda(\kappa(\mathcal{X}')) \cap O \neq \emptyset$. A $\mathcal{T}$-essential subset of $\Lambda(\kappa(\mathcal{X}))$ is minimal if it is a minimal element ordered by set inclusion.

Notice that, if a generalized game $\mathcal{G} \in \mathbb{G}$ is essential, then $\mathcal{G}$ is essential about any parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ such that, for some $\mathcal{X} \in \mathbb{X}$, $\mathcal{G} = \kappa(\mathcal{X})$. Furthermore, consider the particular case where $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ satisfies $\mathbb{X} \subseteq \mathbb{G}$, $\tau \equiv \rho$ and κ is the inclusion of $\mathbb{X}$ on $\mathbb{G}$. Then, for any $\mathcal{X} \in \mathbb{X}$, $\kappa(\mathcal{X})$ is $\mathcal{T}$-essential if and only if $\mathcal{X}$ is essential in the sense of Definition 3.

The following result characterizes the stability properties of Cournot-Nash equilibria when admissible perturbations are determined by a parametrization of $\mathbb{G}$.

Theorem 3. *Given a parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t \in T_2}, H))$, there is a dense residual subset $\mathbb{X}' \subseteq \mathbb{X}$ such that, $\kappa(\mathcal{X}')$ is $\mathcal{T}$-essential for any $\mathcal{X}' \in \mathbb{X}'$.*
Furthermore, for any $\mathcal{X} \in \mathbb{X}$ we have that:
(i) If $\Lambda(\kappa(\mathcal{X}))$ is a singleton, then $\kappa(\mathcal{X})$ is $\mathcal{T}$-essential.
(ii) There is a minimal $\mathcal{T}$-essential subset of $\Lambda(\kappa(\mathcal{X}))$.
(iii) Any $\mathcal{T}$-essential and connected set $m(\mathcal{X}) \subseteq \Lambda(\kappa(\mathcal{X}))$ is contained in a $\mathcal{T}$-essential component.
(iv) Suppose that $\mathbb{X}$ is a convex subset of a Banach space $\mathcal{B}$ and τ is the metric induced by its norm. Then, every minimal $\mathcal{T}$-essential subset of $\Lambda(\kappa(\mathcal{X}))$ is connected.

Proof. By assumptions $\kappa : \mathbb{X} \to \mathbb{G}$ is continuous and $(\mathbb{X}, \tau)$ is a complete metric space. In addition, Theorem 1 ensures that Λ is a closed correspondence that has non-empty and compact values. Thus, the set-valued mapping $\Lambda \circ \kappa : \mathbb{X} \twoheadrightarrow \widehat{M}$ has closed graph with non-empty and compact values. Therefore, the first two properties follow from identical arguments to those made in the proof of Theorem 1. Furthermore, properties (ii)-(iv) can be obtained by analogous arguments of those made in the proof of Theorem 2, changing $(\mathbb{G}, \rho, \Lambda)$ by $(\mathbb{X}, \tau, \Lambda \circ \kappa)$. Q.E.D.

From Theorem 3 we obtain stability results of Cournot-Nash equilibria when some but not all the characteristics that define a generalized game are allowed to change. For instance, when only the objective functions or the sets of admissible strategies can be perturbed.

Furthermore, we can allow for personalized perturbations. More formally, fix a game $\mathcal{G} = \mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2}) \in \mathbb{G}$. Given $i \in \{1, 2\}$, let $T_i^a, T_i^b, T_i^c \subseteq T_i$ be, respectively, the subsets of players in T_i for which we allow perturbations on objective functions, on strategy sets, and on correspondences of admissible strategies. Let $\mathbb{G}_{\mathcal{G}}((T_i^a, T_i^b, T_i^c)_{i \in \{1,2\}}) \subseteq \mathbb{G}$ be the set of generalized games $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}((\widetilde{K}_t, \widetilde{\Gamma}_t, \widetilde{u}_t)_{t\in T_1\cup T_2})$ such that: (1) for any $t \in (T_1 \setminus T_1^a) \cup (T_2 \setminus T_2^a)$, $\widetilde{u}_t = u_t$; (2) for any $t \in (T_1 \setminus T_1^b) \cup (T_2 \setminus T_2^b)$, $\widetilde{K}_t = K_t$; and (3) for any $t \in (T_1 \setminus T_1^c) \cup (T_2 \setminus T_2^b)$, $\widetilde{\Gamma}_t = \Gamma_t$. Then, it follows that $(\mathbb{G}_{\mathcal{G}}((T_i^u, T_i^s, T_i^a)_{i\in\{1,2\}}), \rho)$ is a complete metric space. Also, since the inclusion $\iota : \mathbb{G}_{\mathcal{G}} \hookrightarrow \mathbb{G}$ is continuous, $((\mathbb{G}_{\mathcal{G}}, \rho), \iota)$ is a parametrization of $\mathbb{G}$. Therefore, departing from the characteristics $((K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2})$, Theorem 3 characterizes several properties of essential stability when only some perturbations are allowed

As a particular case of our analysis, we can obtain stability results for non-atomic games. Indeed, a non-atomic game is a generalized game where (i) there is only non-atomic players; and (ii) admissible strategies are independent of the actions chosen by the other players. Therefore, we can identify a non-atomic game with a generalized game where there is only one atomic player, whose actions have no effect on the decisions of the other players. This identification induce a parametrization of the space of generalized games and, therefore, the properties of essential stability can be obtained as a consequence of Theorem 3.[6]

Analogously, if we fix a generalized game where actions chosen by non-atomic players have no effects on other agents decisions, then equilibrium actions associated to atomic players are Cournot-Nash equilibria of a convex generalized game with finitely many players. That is, stability properties of equilibria in convex generalized games with a finite number of players can be obtained as a particular case of our approach. Thus, we generalize the results of essential stability of Yu (1999), Yu (2009) and Yu, Yang and Xiang (2005), allowing for a great variety of admissible perturbations.

We close this section with results about stability of essential sets and components.

Given $\epsilon > 0$ and $A \subseteq \widehat{M} \times \widehat{\mathcal{F}}^2$, the ϵ-neighborhood of A is defined by

$$B[\epsilon, A] = \left\{(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2 : \exists\,(m',a') \in A,\ \widehat{\sigma}((m,a),(m',a')) \le \epsilon\right\},$$

where $\widehat{\sigma}$ is the metric associated to the product topology of $\mathbb{R}^m \times \prod_{t\in T_2} \widehat{K}_t$. Also, for any parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ let $\Lambda_m(\mathcal{T}, \mathcal{X})$ (respectively, $\Lambda_c(\mathcal{T}, \mathcal{X})$) be the collection of minimal $\mathcal{T}$-essential subsets (respectively, $\mathcal{T}$-essential components) of $\Lambda(\kappa(\mathcal{X}))$, where $\mathcal{X} \in \mathbb{X}$.

Theorem 4. *Fix a parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of $\mathbb{G}$ and let $\mathcal{X} \in \mathbb{X}$.*
(i) If $A \in \Lambda_m(\mathcal{T}, \mathcal{X})$, then for each $\epsilon > 0$ there is $\delta > 0$ such that, given $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta$, there exists $A' \in \Lambda_m(\mathcal{T}, \mathcal{X}')$ for which $A' \subseteq B[\epsilon, A]$.
(ii) Suppose that $\mathbb{X}$ is a convex subset of a Banach space $\mathcal{B}$ and τ is the metric induced by its norm. Given $A \in \Lambda_c(\mathcal{T}, \mathcal{X})$ assume that $\exists \pi > 0 : B[\pi, A] \cap B[\pi, \Lambda(\kappa(\mathcal{X})) \setminus A] = \emptyset$. Then, for each $\epsilon > 0$ there is $\delta > 0$ such that, given $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta$, there exists $A' \in \Lambda_c(\mathcal{T}, \mathcal{X}')$ for which $A' \subseteq B[\epsilon, A]$.

[6]In particular, the existence of equilibria in non-atomic games is a consequence of the existence of equilibria in large generalized games. It is not surprising, since the results of equilibrium existence of Balder (1999, 2002) are a generalization of the results of Schmeidler (1973) for large games.

Notice that, if $\Lambda(\kappa(\mathcal{X}))$ has a finite number of $\mathcal{T}$-essential components (i.e., $\Lambda_c(\mathcal{T},\mathcal{X})$ is a finite set), then there always exists $\pi > 0$ such that, $B[\pi, A] \cap B[\pi, \Lambda(\kappa(\mathcal{X})) \setminus A] = \emptyset$ for any $A \in \Lambda_c(\mathcal{T},\mathcal{X})$. Thus, in this particular case, Theorem 4(ii) ensures the stability of essential components.

6. Essential Stability as a Rationale for Electoral Participation

In a recent paper Barlo and Carmona (2011) introduce the refinement concept of strategic equilibria in large games. Intuitively, a Nash equilibrium of a large game is strategic if it is the limit of equilibria in abstract perturbed games, where players believe that have a positive impact on the social choice. As an application of their results, they give a rationale to explain why electors vote for their favorite candidate. Introducing a large game with proportional voting, they show that there is a continuum of Cournot-Nash equilibria, but only one strategic equilibrium: that in which electors vote by their favorite party (see Barlo and Carmona (2011, Example 2.1)).

Inspired by this result, we will analyze a large generalized electoral game where electors have different degrees of political interest. The Cournot-Nash equilibrium where only political engaged players vote and support their favorite party appears as the unique $\mathcal{T}$-essential equilibrium of our electoral game, for some parametrization $\mathcal{T}$.

Given a set of parties $P = \{1, ..., \overline{p}\}$ and a parameter $\mu \geq 0$, consider an electoral game $\mathcal{E}_\mu(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H), (K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2})$, where for any non-atomic player $t \in T_1 := [0,1]$ the action space is given by

$$K_t = \widehat{K} := \left\{ (x_1, \ldots, x_{\overline{p}}) \in \mathbb{Z}_+^{\overline{p}} : \sum_{p=1}^{\overline{p}} x_p \leq 1 \right\},$$

and the actions of other players do not affect her admissible allocations, i.e., $\Gamma_t \equiv \widehat{K}$. Thus, any non-atomic player $t \in T_1$ can vote for some party $p \in P$ by choosing $x \in K_t$ such that $x_p = 1$, or she can not vote by choosing $x = 0$. Each $t \in T_1$ gives an importance $v_t(p) \geq 0$ to party $p \in P$ and has a favorite party $p_t^* \in P$, i.e. $v_t(p_t^*) > v_t(p)$ for all $p \in P \setminus \{p_t^*\}$. His objective function is given by the weighted average of the utilities that she gets from individual parties, and a component that reflect the private level of satisfaction associated to her action, that is, for any $x = (x_1, \ldots, x_{\overline{p}}) \in K_t$,

$$u_t^\mu(x, a) = \sum_{p=1}^{\overline{p}} v_t(p) a_p + \mu \sum_{p=1}^{\overline{p}} (v_t(p) - \eta_t) x_p,$$

where a_p is the probability that party p has to win the election, and the coefficient $\eta_t \geq 0$ measures the electoral engagement of player t. Indeed, when $\mu > 0$, as greater η_t less interested in the election would be player t. We assume that for any $t \in T_1$ either $\eta_t > v_t(p_t^*)$ or $\eta_t < v_t(p_t^*)$. The set of politically engaged players is defined as $T_1^* = \{t \in T_1 : \eta_t < v_t(p_t^*)\}$, and we assume that it is a subset of T_1 with positive measure.

There is one atomic player $T_2 = \{e\}$, whose objective is to determine the probabilities $(a_1, \ldots, a_{\overline{p}})$. Thus, $\widehat{K}_e = \mathbb{R}^{\overline{p}}$, $K_e = \Gamma_e = \{(z_1, \ldots, z_{\overline{p}}) \in \mathbb{R}^{\overline{p}}_+ : \sum_{p=1}^{\overline{p}} z_p = 1\}$, and

$$u_e(m,a) = -\sum_{p=1}^{\overline{p}} \left(a_p \sum_{p'=1}^{\overline{p}} m_{p'} - m_p \right)^2,$$

where $m = (m_1, \ldots, m_{\overline{p}})$ is the message obtained from non-atomic players votes, assuming that $H(t,x) = x$. Thus, when a positive portion of players vote, the probability that a party p has to win is given by the proportion of issued votes that it party receives.

In any generalized game $\mathcal{E}_\mu$, with $\mu \geq 0$, the decision taken by a non-atomic player does not affect the societal choice. However, the vote of a non-atomic player $t \in T_1$ affects her own utility level when she gives a private value to her actions, i.e., when $\mu > 0$.

Consider the case where non-atomic players do not give importance to their actions, i.e., $\mu = 0$. Then, given a measurable action profile $x : T_1 \to \widehat{K}$ and a strategy $a \in \widehat{K}_e$, the vector

$$\begin{cases} \left(x, \left(\dfrac{\int_{T_1} x_p(t)dt}{\sum_{s=1}^{\overline{p}} \int_{T_1} x_s(t)dt} \right)_{p \in P} \right), & \text{if } \int_{T_1} x(t)dt \neq 0; \\ (x, a), & \text{if } \int_{T_1} x(t)dt = 0; \end{cases}$$

constitutes a Cournot-Nash equilibrium for $\mathcal{E}_0$. Thus, when electors do not give any private value to electoral participation, there is a continuum of equilibria.

On the other hand, for any $\mu > 0$, the generalized game $\mathcal{E}_\mu$ has only one Cournot-Nash equilibrium. Indeed, any player $t \in T_1^*$ votes for his favorite party, while any player in $T_1 \setminus T_1^*$ does not vote. As T_1^* has positive measure, the equilibrium vector of probabilities is well defined. Hence, it follows from Theorem 1 that $\mathcal{E}_\mu$ is an essential generalized game for any $\mu > 0$.

Since the space $([0,1], |\cdot|)$ is complete and $\kappa : [0,1] \to \mathbb{G}$ given by $\kappa(\mu) = \mathcal{E}_\mu$ is continuous, $\mathcal{T} = (([0,1], |\cdot|), \kappa)$ is a parametrization of $\mathbb{G}$, in the sense of Definition 5. Therefore, we conclude that $\mathcal{E}_0$—the electoral game where players do not give any value to their private actions—has a unique $\mathcal{T}$-essential Cournot-Nash equilibrium, the one where only politically engaged players vote supporting their favorite party.

7. Concluding Remarks

In this paper we analyzed essential stability properties of Cournot-Nash equilibria in large generalized games. Departing from the ideas of Rath (1992) and Riascos and Torres-Martínez (2012), that reduce the proof of pure strategy equilibrium existence in non-atomic (generalized) games to find fixed points of correspondences, we use the stability theory of fixed points developed by Fort (1950) and Jia-He (1962) to address the essential stability of equilibria in large games.

We guaranteed that essential stability is a generic property in the space of generalized games. Also, even unessential generalized games have essential components of the set of equilibria, which ensures that we always have local stability in a connected subset of Cournot-Nash equilibria. These connected subsets of essential Cournot-Nash equilibria are locally stable too.

Our results are compatible with general types of perturbations on the characteristics of generalized games. Indeed, stability properties still hold when (i) admissible perturbations can be captured by a continuous parametrization of the set of generalized games; and (ii) the set of parameters constitutes a complete metric space. This generality about the type of admissible perturbations allow us to obtain, as byproducts of our analysis, extensions of the results of essential stability for non-atomic games and convex games with finitely many players.

Appendix

Completeness of $(\mathbb{G}, \rho)$.

Given a metric space (S, d), consider the sets $A(S) = \{K \subseteq S : K \text{ is non-empty and compact}\}$, and $A_c(S) = \{C \in A(S) : C \text{ is convex}\}$. Denote by d_H the Hausdorff metric induced by the metric of S. If S is compact, then $(A(S), d_H)$ is a complete metric space. Also, when S is compact and convex, $(A_c(S), d_H)$ is complete.[7]

Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ a Cauchy sequence on $(\mathbb{G}, \rho)$, where $\mathcal{G}_n = \mathcal{G}_n((K_{n,t}, \Gamma_{n,t}, u_{n,t})_{t\in T_1\cup T_2})$. By the definition of $\mathbb{G}$ and ρ it follows that, for any non-atomic player $t \in T_1$, $\{K_{n,t}\}_{n\in\mathbb{N}}$ is a Cauchy sequence on $(A(\widehat{K}), d_H)$. Also, for any atomic player $s \in T_2$, $\{K_{n,s}\}_{n\in\mathbb{N}}$ is a Cauchy sequence on $(A_c(\widehat{K}_s), d_{H,s})$. Hence, there are sets $\{\overline{K}_t\}_{t\in T_1\cup T_2}$ such that: (i) $(\overline{K}_t, \overline{K}_s) \in A(\widehat{K}) \times A_c(\widehat{K}_s)$, $\forall (t,s) \in T_1 \times T_2$; and (ii) for any $(t,s) \in T_1 \times T_2$, we have that

$$\lim_{n\to+\infty} d_H(K_{n,t}, \overline{K}_t) = \lim_{n\to+\infty} d_{H,s}(K_{n,s}, \overline{K}_s) = 0.$$

The definition of the metric ρ ensures that, for any $t \in T_1$ and $(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$, the sequence $\{\Gamma_{n,t}(m,a)\}_{n\in\mathbb{N}} \subseteq A(\widehat{K})$ is Cauchy and, therefore, there exists a set $K_t(m,a) \in A(\widehat{K})$ such that $d_H(\Gamma_{n,t}(m,a), K_t(m,a))$ converges to zero as n goes to infinity. Let $\overline{\Gamma}_t : \widehat{M} \times \widehat{\mathcal{F}}^2 \twoheadrightarrow \widehat{K}$ be the set-valued mapping defined by $\overline{\Gamma}_t(m,a) = K_t(m,a)$. By analogous arguments, we can ensure that for any $s \in T_2$ there is a correspondence $\overline{\Gamma}_s : \widehat{M} \times \widehat{\mathcal{F}}^2_{-s} \twoheadrightarrow \widehat{K}_s$ such that, for each $(m, a_{-s}) \in \widehat{M} \times \widehat{\mathcal{F}}^2_{-s}$ both $\overline{\Gamma}_s(m, a_{-s}) \in A_c(\widehat{K}_s)$ and $d_{H,s}(\Gamma_{n,s}(m, a_{-s}), \overline{\Gamma}_s(m, a_{-s}))$ converges to zero as n increases.

Since $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is Cauchy on $(\mathbb{G}, \rho)$, for any $t \in T_1$ there is a function $\overline{u}_t \in \mathcal{U}(\widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2)$ such that $\{u_{n,t}\}_{n\in\mathbb{N}} \subseteq \mathcal{U}(\widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2)$ converges to it. Analogously, for any $s \in T_2$, the Cauchy sequence $\{u_{n,s}\}_{n\in\mathbb{N}} \subseteq \mathcal{U}_s(\widehat{M} \times \widehat{\mathcal{F}}^2)$ converges to some function $\overline{u}_s \in \mathcal{U}_s(\widehat{M} \times \widehat{\mathcal{F}}^2)$.

Let $\overline{\mathcal{G}} = \overline{\mathcal{G}}((\overline{K}_t, \overline{\Gamma}_t, \overline{u}_t)_{t\in T_1\cup T_2})$. It follows from arguments above that $\lim_{n\to+\infty} \rho(\mathcal{G}_n, \overline{\mathcal{G}}) = 0$. Thus, to prove that $(\mathbb{G}, \rho)$ is complete, it is sufficient to guarantee that:

(i) the function $\overline{U} : T_1 \to \mathcal{U}(\widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2)$ defined by $\overline{U}(t) = \overline{u}_t$ is measurable;

(ii) for any $(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$, the correspondence $t \in T_1 \twoheadrightarrow \overline{\Gamma}_t(m,a)$ is measurable.

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[7]Since S is a compact metric spaces, it is complete. It follows from Aliprantis and Border (2006, Theorem 3.85-(2) and Theorem 3.88-(1), pages 116 and 199) that $A(S)$ is a complete metric space under the Hausdorff metric induced by the metric of S. When the space is restricted to $A_c(S)$, $(A_c(S), d_H)$ remains a complete metric space, since the Hausdorff limit of a sequence of convex sets is still a convex set.

The definition of ρ ensures that measurable functions $U_n : T_1 \to \mathcal{U}(\widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2)$ defined by $U_n(t) = u_{n,t}$ converge to $\overline{U}$. Since T_1 is a measurable space and $\mathcal{U}(\widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2)$ is a metric space, $\overline{U}$ is measurable (see Aliprantis and Border (2006, Lemma 4.29, page 142)). Hence, item (i) holds.

Fix $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$. Given $n \in \mathbb{N}$, the correspondence that associates to any $t \in T_1$ the set $\Gamma_{n,t}(m, a)$ is measurable. Thus, it follows form Aliprantis and Border (2006, Theorem 18.10, page 598) that the function $\Theta_{n,(m,a)} : T_1 \to A(\widehat{K})$ defined by $\Theta_{n,(m,a)}(t) = \Gamma_{n,t}(m, a)$ is Borel measurable. Also, the sequence $\{\Theta_{n,(m,a)}\}_{n\in\mathbb{N}}$ converges to $\overline{\Theta}_{(m,a)} : T_1 \to A(\widehat{K})$, where $\overline{\Theta}_{(m,a)}(t) = \overline{\Gamma}_t(m, a)$. By Aliprantis and Border (2006, Lemma 4.29), $\overline{\Theta}_{(m,a)}$ is a Borel measurable function. Thus, $t \in T_1 \twoheadrightarrow \overline{\Gamma}_t(m, a)$ is measurable (cf. Aliprantis and Border (2006, Theorem 18.10)). Q.E.D

Proof of Theorem 1.

The proof of the theorem is a direct consequence of the following steps.

Step 1. The correspondence $\Lambda : \mathbb{G} \twoheadrightarrow \widehat{M} \times \widehat{\mathcal{F}}^2$ is upper hemicontinuous with compact values.

Since $\widehat{M} \times \widehat{\mathcal{F}}^2$ is compact and non-empty, we only need to prove that Graph(Λ) is closed, where Graph$(\Lambda) := \Big\{(\mathcal{G}, (m, a)) \in \mathbb{G} \times \widehat{M} \times \widehat{\mathcal{F}}^2 : (m, a) \in \Lambda(\mathcal{G})\Big\}$.

Let $\{(\mathcal{G}_n, (m_n, a_n))\}_{n\in\mathbb{N}} \subset$ Graph(Λ) such that $(\mathcal{G}_n, (m_n, a_n)) \to (\overline{\mathcal{G}}, (\overline{m}, \overline{a})) \in \mathbb{G} \times \widehat{M} \times \widehat{\mathcal{F}}^2$, where $\mathcal{G}_n = \mathcal{G}_n((K^n_t, \Gamma^n_t, u^n_t)_{t\in T_1\cup T_2})$ and $\overline{\mathcal{G}} = \overline{\mathcal{G}}((\overline{K}_t, \overline{\Gamma}_t, \overline{u}_t)_{t\in T_1\cup T_2})$. To prove that Graph$(\Lambda)$ is closed is sufficient to ensure that $(\overline{m}, \overline{a}) \in \Phi_{\overline{\mathcal{G}}}(\overline{m}, \overline{a})$.

Since $(m_n, a_n) \in \Phi_{\mathcal{G}_n}(m_n, a_n)$, for any $t \in T_1$ there exists $f_n(t) \in \Gamma^n_t(m_n, a_n)$ such that,

$$m_n = \int_{T_1} H(t, f_n(t))d\mu, \qquad\qquad u^n_t(f_n(t), m_n, a_n) = \max_{x\in\Gamma^n_t(m_n,a_n)} u^n_t(x, m_n, a_n),$$

and the function $g_n(\cdot) = H(\cdot, f_n(\cdot))$ is measurable.

Claim A. For any $t \in T_1$ there exists $\overline{f}(t) \in \widehat{K}$ such that $\overline{m} = \int_{T_1} H(t, \overline{f}(t))d\mu$.

Proof. Since H is continuous, T_1 is compact and, for each $t \in T_1$, $f_n(t) \in \widehat{K}$, it follows that the sequence $\{g_n\}_{n\in\mathbb{N}}$ is a uniformly integrable (see Hildenbrand (1974, page 52)). In addition, $\{\int_{T_1} g_n(t)d\mu\}_{n\in\mathbb{N}} \subset \mathbb{R}^m$ converges to $\overline{m}$ as n goes to infinity and, therefore, the Fatou's Lemma in m-dimension (see Hildenbrand (1974, page 69)) guarantees that there is $g : T_1 \to \mathbb{R}^m$ integrable such that, [8]

(1) $\lim\limits_{n\to\infty} \int_{T_1} g_n(t)d\mu = \int_{T_1} g(t)d\mu$

(2) There exists $\tilde{T}_1 \subseteq T_1$ such that, for any $t \in \tilde{T}_1$, $g(t) \in L_S(g_n(t))$, where $L_S(g_n(t))$ is the set of cluster points of $\{g_n(t)\}_{n\in\mathbb{N}}$ and $T_1 \setminus \tilde{T}_1$ has zero measure.

Fix $t \in \tilde{T}_1$. Then there is a subsequence $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that $g_{n_k}(t) \to g(t)$. Since $\{f_{n_k}(t)\}_{k\in\mathbb{N}} \subseteq \widehat{K}$, taking a subsequence again if it is necessary, we can ensure that there exists $f(t) \in \widehat{K}$ such that both $f_{n_k}(t) \to f(t)$ and $g(t) = \lim\limits_{k\to\infty} H(t, f_{n_k}(t)) = H(t, f(t))$.

[8] Although functions $\{g_n\}_{n\in\mathbb{N}}$ can take negative values, they are uniformly bounded from below (since H is continuous and $\{\widehat{K}, T_1\}$ are compact sets). Thus, as T_1 has finite Lebesgue measure, we can apply the Fatou's Lemma.

Let $\overline{f}: T_1 \to \widehat{K}$ such that

$$\overline{f}(t) \in \begin{cases} \{f(t)\} & \text{if } t \in \tilde{T}_1, \\ \underset{x\in\overline{\Gamma}_t(\overline{m},\overline{a})}{\operatorname{argmax}} \ \overline{u}_t(x,\overline{m},\overline{a}) & \text{if } t \notin \tilde{T}_1. \end{cases}$$

Thus, $\overline{m} = \lim_{n\to\infty} \int_{T_1} H(t, f_n(t))d\mu = \int_{T_1} H(t, \overline{f}(t))d\mu$. □

Claim B. For any $t \in T_1$, $\overline{f}(t) \in \overline{\Gamma}_t(\overline{m}, \overline{a})$.

Proof. The results follows by definition for any $t \in T_1 \setminus \tilde{T}_1$. Thus, fix $t \in \tilde{T}_1$ and let $\{f_{n_k}(t)\}_{k\in\mathbb{N}}$ the sequence that was obtained in the previous claim and that converges to $\overline{f}(t)$. We known that, for any $k \in \mathbb{N}$, $f_{n_k}(t) \in \Gamma_t^{n_k}(m_{n_k}, a_{n_k})$ and, therefore,

$$\begin{aligned} d(\overline{f}(t), \overline{\Gamma}_t(\overline{m},\overline{a})) &\le \widehat{d}(\overline{f}(t), f_{n_k}(t)) + d(f_{n_k}(t), \Gamma_t^{n_k}(m_{n_k}, a_{n_k})) + d_H(\Gamma_t^{n_k}(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(m_{n_k}, a_{n_k})) \\ &\quad + d_H(\overline{\Gamma}_t(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(\overline{m},\overline{a})) \\ &\le \widehat{d}(\overline{f}(t), f_{n_k}(t)) + \rho(\mathcal{G}_{n_k}, \overline{\mathcal{G}}) + d_H(\overline{\Gamma}_t(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(\overline{m},\overline{a})), \end{aligned}$$

where $\widehat{d}$ denotes the metric of the compact metric space $\widehat{K}$. Since $\overline{\Gamma}_t$ is continuous, by taking the limit as k goes to infinity, we obtain the result. □

Claim C. For any $t \in T_1$, $\overline{f}(t) \in \underset{x\in\overline{\Gamma}_t(\overline{m},\overline{a})}{\operatorname{argmax}} \ \overline{u}_t(x, \overline{m}, \overline{a})$.

Proof. As in the previous claim, the case $t \in T_1 \setminus \tilde{T}_1$ follows from definition. With the same notation used in the previous claim, we have that

$$d_H(\Gamma_t^{n_k}(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(\overline{m},\overline{a})) \quad \le \quad \rho(\mathcal{G}_{n_k}, \overline{\mathcal{G}}) + d_H(\overline{\Gamma}_t(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(\overline{m},\overline{a})), \quad \forall t \in \tilde{T}_1.$$

Then $\Gamma_t^{n_k}(m_{n_k}, a_{n_k}) \longrightarrow_k \ \overline{\Gamma}_t(\overline{m},\overline{a})$. Since $u_t^{n_k}$ converges uniformly to $\overline{u}_t$, it follows from Yu (1999, Lemma 2.5) and Aubin (1982, Theorem 3, page 70) that,

$$u_t^{n_k}(f_{n_k}(t), m_{n_k}, a_{n_k}) = \max_{x\in\Gamma_t^{n_k}(m_{n_k},a_{n_k})} u_t^{n_k}(x, m_{n_k}, a_{n_k}) \longrightarrow_k \max_{x\in\overline{\Gamma}_t(\overline{m},\overline{a})} \overline{u}_t(x,\overline{m},\overline{a})$$

On the other hand,

$$|u_t^{n_k}(f_{n_k}(t), m_{n_k}, a_{n_k}) - \overline{u}_t(\overline{f}(t), \overline{m}, \overline{a})| \quad \le \quad \rho(\mathcal{G}_{n_k}, \overline{\mathcal{G}}) + |\overline{u}_t(f_{n_k}(t), m_{n_k}, a_{n_k}) - \overline{u}_t(\overline{f}(t), \overline{m}, \overline{a})|.$$

Taking the limit as k goes to infinity, we obtain that $u_t^{n_k}(f_{n_k}(t), m_{n_k}, a_{n_k}) \to \overline{u}_t(\overline{f}(t), \overline{m}, \overline{a})$. Hence, it follows from Claim B that $\overline{f}(t) \in \underset{x\in\overline{\Gamma}_t(\overline{m},\overline{a})}{\operatorname{argmax}} \ \overline{u}_t(x, \overline{m}, \overline{a})$. □

Claim D. For any $t \in T_2$, $\overline{a}_t \in \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})$.

Proof. For any $(t, n) \in T_2 \times \mathbb{N}$, $a_{n,t} \in \Gamma_t^n(m_n, a_{n,-t})$ and, therefore,

$$\begin{aligned} d(\overline{a}_t, \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})) &\le \widehat{d}_t(\overline{a}_t, a_{n,t}) + d(a_{n,t}, \Gamma_t^n(m_n, a_{n,-t})) + d_{H,t}(\Gamma_t^n(m_n, a_{n,-t}), \overline{\Gamma}_t(m_n, a_{n,-t})) \\ &\quad + d_{H,t}(\overline{\Gamma}_t(m_n, a_{n,-t}), \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})) \\ &\le \widehat{d}_t(\overline{a}_t, a_{n,t}) + \rho(\mathcal{G}_n, \overline{\mathcal{G}}) + d_{H,t}(\overline{\Gamma}_t(m_n, a_{n,-t}), \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})), \end{aligned}$$

where $\widehat{d}_t$ denotes the metric of $\widehat{K}_t$. Taking the limit as n goes to infinity, we obtain the result. □

Claim E. For any $t \in T_2$, $\overline{a}_t \in \underset{x\in\overline{\Gamma}_t(\overline{m},\overline{a}_{-t})}{\operatorname{argmax}} \ \overline{u}_t(\overline{m}, x, \overline{a}_{-t})$.

Proof. Following the same arguments of Claim C, we have that

$$d_{H,t}(\Gamma_t^n(m_n,a_{n,-t}),\overline{\Gamma}_t(\overline{m},\overline{a}_{-t})) \quad \leq \quad \rho(\mathcal{G}_n,\mathcal{G})+d_{H,t}(\overline{\Gamma}_t(m_n,a_{n,-t}),\overline{\Gamma}_t(\overline{m},\overline{a}_{-t})),$$

which implies that $\Gamma_t^n(m_n,a_{n,-t})$ converges to $\overline{\Gamma}_t(\overline{m},\overline{a}_{-t})$ as n goes to infinity. Hence, Yu (1999, Lemma 2.5) ensures that,

$$u_t^n(m_n,a_n)=\max_{x\in\Gamma_t^n(m_n,a_{n,-t})} u_t^n(m_n,x,a_{n,-t}) \longrightarrow \max_{x\in\overline{\Gamma}_t(\overline{m},\overline{a}_{-t})} \overline{u}_t(\overline{m},x,\overline{a}_{-t}).$$

Since $\lim_{n\to+\infty} u_t^n(m_n,a_n)=\overline{u}_t(\overline{m},\overline{a})$,[9] it follows that $\overline{a}_t\in \operatorname*{argmax}_{x\in\overline{\Gamma}_t(\overline{m},\overline{a}_{-t})} \overline{u}_t(\overline{m},x,\overline{a}_{-t})$. □

It follows from Claims A, C and E that $(\overline{m},\overline{a})$ is a fixed point of $\Phi_{\overline{\mathcal{G}}}$. Thus, we ensure that Λ is an upper hemicontinuous correspondence with compact values.

Step 2. There is a dense residual set $Q\subseteq\mathbb{G}$ where Λ is lower hemicontinuous.

As $(\mathbb{G},\rho)$ is a complete metric space, $\mathbb{G}$ is a Baire space. Since the correspondence Λ is compact-valued and upper hemicontinuous with $\Lambda(\mathcal{G})\neq\emptyset$ for all $\mathcal{G}\in\mathbb{G}$, it follows from Lemmas 5 and 6 in Carbonell-Nicolau (2010) (see also Fort (1949) and Jia-He (1962)) that there exists a dense residual subset Q of $\mathbb{G}$ in which Λ is lower hemicontinuous.

Step 3. If $\mathcal{G}$ is a point of lower hemicontinuity of Λ, then $\mathcal{G}$ is essential.

Fix an equilibrium $(f^*,a^*)\in \mathrm{CN}(\mathcal{G})$. Then, for any open set $O\subseteq\widehat{M}\times\widehat{\mathcal{F}}^2$ such that $(m(f^*),a^*)\in O$ we have $\Lambda(\mathcal{G})\cap O\neq\emptyset$ and, therefore, the lower inverse $\Lambda^-(O):=\{\mathcal{G}'\in\mathbb{G}:\ \Lambda(\mathcal{G}')\cap O\neq\emptyset\}$ contains a neighborhood of $\mathcal{G}$. That is, there is $\epsilon>0$ such that, for any $\mathcal{G}'\in\mathbb{G}$ such that $\rho(\mathcal{G}',\mathcal{G})<\epsilon$, we have that $\Lambda(\mathcal{G}')\cap O\neq\emptyset$. That is, all Cournot-Nash equilibrium of $\mathcal{G}$ are essential.

It follows from Steps 2 and 3 that any generalized game in the dense residual set Q is essential. Finally, suppose that for a game $\mathcal{G}\in\mathbb{G}$ the correspondence $\Phi_{\mathcal{G}}$ has only one fixed point. Then, Λ is upper hemicontinuous and single valued at $\mathcal{G}$ and, therefore, it is continuous at this point. Using Step 3, we conclude that $\mathcal{G}$ is an essential generalized game. Q.E.D.

Proof of Theorem 2.

(i) Fix $\mathcal{G}\in\mathbb{G}$. Let $\mathcal{S}$ be the family of essential sets of $\Lambda(\mathcal{G})$ ordered by set inclusion. Since $\Lambda(\mathcal{G})\in\mathcal{S}$, $\mathcal{S}\neq\emptyset$. As any element of $\mathcal{S}$ is compact, any totally ordered subset of $\mathcal{S}$ has a lower bounded element. By Zorn's Lemma, $\mathcal{S}$ has a minimal element, and by definition of $\mathcal{S}$, its minimal element is an essential set of $\Lambda(\mathcal{G})$.

(ii) Suppose that there is a connected essential set of $\Lambda(\mathcal{G})$, $c(\mathcal{G})$. Fix $(\widehat{m},\widehat{a})\in c(\mathcal{G})$ and consider the set $\Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$, defined as the union of all connected subsets of $\Lambda(\mathcal{G})$ that contains $(\widehat{m},\widehat{a})$. By definition, $\Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$ is a component of $\Lambda(\mathcal{G})$. Since $\Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$ is compact and $c(\mathcal{G})\subset\Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$ is essential, the component $\Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$ is also an essential subset of $\Lambda(\mathcal{G})$.

[9] It is a direct consequence of the fact that, for any $n\in\mathbb{N}$, we have

$$|u_t^n(m_n,a_n)-\overline{u}_t(\overline{m},\overline{a})| \quad \leq \quad \rho(\mathcal{G}_n,\mathcal{G})+|\overline{u}_t(m_n,a_n)-\overline{u}_t(\overline{m},\overline{a})|.$$

(iii) Suppose that $\{\widehat{K}, (\widehat{K}_t)_{t\in T_2}\}$ are convex subsets of Banach spaces with metrics induced by the norm of the associated spaces. Fix a minimal essential set of $\Lambda(\mathcal{G})$, denoted by $m(\mathcal{G})$. We want to prove that $m(\mathcal{G})$ is connected.

By contradiction, if $m(\mathcal{G})$ is not connected, then there are closed and non-empty subsets of $\Lambda(\mathcal{G})$, A_1 and A_2 such that $m(\mathcal{G}) = A_1 \cup A_2$. Also, there are open sets V_1, V_2 such that $A_1 \subset V_1$, $A_2 \subset V_2$ and $V_1 \cap V_2 = \emptyset$. Since $m(\mathcal{G})$ is minimal, neither A_1 nor A_2 are essentials.

Fix $i \in \{1, 2\}$. Since A_i is not essential, there exists an open set O_i such that $A_i \subset O_i$ and for all $\epsilon > 0$ there exists $\mathcal{G}_i \in \mathbb{G}$ such that $\rho(\mathcal{G}, \mathcal{G}_i) < \epsilon$ and $\Lambda(\mathcal{G}_i) \cap O_i = \emptyset$. Since A_i is compact, there exists an open set U_i such that $A_i \subset U_i \subset \overline{U}_i \subset V_i \cap O_i$.

Therefore, $m(\mathcal{G}) \subset U_1 \cup U_2$ and $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. As $m(\mathcal{G})$ is essential, there exists $\nu > 0$ such that for every $\mathcal{G}' \in \mathbb{G}$ with $\rho(\mathcal{G}, \mathcal{G}') < \nu$, we have $\Lambda(\mathcal{G}') \cap (U_1 \cup U_2) \neq \emptyset$. On the other hand, given $i \in \{1, 2\}$, as $U_i \subset O_i$, there exists $\mathcal{G}'_i \in \mathbb{G}$ such that $\rho(\mathcal{G}, \mathcal{G}'_i) < \frac{\nu}{3}$ and $\Lambda(\mathcal{G}'_i) \cap U_i = \emptyset$.

Let $G : \widehat{M} \times \widehat{\mathcal{F}}^2 \twoheadrightarrow \mathbb{G}$ be the correspondence

$$G(m, a) = \lambda(m, a)\mathcal{G}'_1 + (1 - \lambda(m, a))\mathcal{G}'_2, \quad \forall (m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2,$$

where $\lambda : \widehat{M} \times \widehat{\mathcal{F}}^2 \to [0, 1]$ is the continuous function given by,

$$\lambda(m, a) = \frac{d((m, a), \overline{U}_2)}{d((m, a), \overline{U}_1) + d((m, a), \overline{U}_2)}.$$

Notice that, $(m, a) \in \overline{U}_i$ if and only if $G(m, a) = \mathcal{G}'_i$.

In addition, for any $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$, we have that,

$$\begin{aligned} \rho(G(m, a), \mathcal{G}'_1) &= \rho\left(\lambda(m, a)\,\mathcal{G}'_1 + (1 - \lambda(m, a))\,\mathcal{G}'_2, \lambda(m, a)\,\mathcal{G}'_1 + (1 - \lambda(m, a))\,\mathcal{G}'_1\right) \\ &\leq \rho(\,\mathcal{G}'_2, \mathcal{G}'_1) \leq \rho(\,\mathcal{G}'_2, \mathcal{G}) + \rho(\,\mathcal{G}, \mathcal{G}'_1) < \frac{2\nu}{3}, \end{aligned}$$

which implies that,

$$\rho(\mathcal{G}, G(m, a)) \leq \rho(\,\mathcal{G}, \mathcal{G}'_1) + \rho(\,\mathcal{G}'_1, G(m, a)) < \nu,$$

and, therefore, for each $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$, $\Lambda(G(m, a)) \cap (U_1 \cup U_2) \neq \emptyset$.[10]

Claim. There exists $(\overline{m}, \overline{a}) \in U_1$ such that, $(\overline{m}, \overline{a}) \in \Lambda(G(\overline{m}, \overline{a}))$.

Proof. Let $\tilde{A}_1 \subset U_1$ be a compact, convex and non-empty set. Define $\Theta : \tilde{A}_1 \times \tilde{A}_1 \twoheadrightarrow \tilde{A}_1 \times \tilde{A}_1$ by $\Theta((m_1, a_1), (m_2, a_2)) = \left(\Phi_{G(m_1,a_1)}(m_2, a_2) \cap \tilde{A}_1\right) \times \{(m_1, a_1)\}$. If we ensure that the correspondence $\Theta_1 : \tilde{A}_1 \times \tilde{A}_1 \twoheadrightarrow \tilde{A}_1$ given by $\Theta_1((m_1, a_1), (m_2, a_2)) = \Phi_{G(m_1,a_1)}(m_2, a_2) \cap \tilde{A}_1$ has closed graph, then the correspondence Θ is upper hemicontinuous and has non-empty, compact and convex values. Thus, applying the Kakutani's Fixed Point Theorem we can find $(\overline{m}, \overline{a}) \in \tilde{A}_1 \subset U_1$ such that, $(\overline{m}, \overline{a}) \in \Lambda(G(\overline{m}, \overline{a}))$.

Thus, let $\{(z^n_1, z^n_2, (m^n, a^n))\}_{n\in\mathbb{N}} \subset \text{Graph}(\Theta_1)$ a sequence that converges to $(\tilde{z}_1, \tilde{z}_2, (\tilde{m}, \tilde{a})) \in \tilde{A}_1 \times \tilde{A}_1 \times \tilde{A}_1$. We want to prove that $(\tilde{m}, \tilde{a}) \in \Theta_1(\tilde{z}_1, \tilde{z}_2)$.

[10] The additional assumptions about metric spaces $\widehat{K}$ and $\{\widehat{K}_t\}_{t\in T_2}$ ensure that for any $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ both $G(m, a)$ is a well defined generalized game and $\rho(G(m, a), \mathcal{G}'_1) \leq \rho(\mathcal{G}'_2, \mathcal{G}'_1)$.

Fix $t \in T_2$ and let $\gamma_t : (\widehat{M} \times \widehat{\mathcal{F}}^2_{-t}) \times \tilde{A}_1 \twoheadrightarrow \widehat{K}_t$ the correspondence characterized by

$$\gamma_t((m, a_{-t}), z) = \underset{x \in \Psi((m,a_{-t}),z)}{\operatorname{argmax}} v_t(x, (m, a_{-t}), z),$$

where

$$\begin{aligned} \Psi((m, a_{-t}), z) &= \lambda(z)\Gamma^1_t(m, a_{-t}) + (1 - \lambda(z))\Gamma^2_t(m, a_{-t}); \\ v_t(x, (m, a_{-t}), z) &= \lambda(z)u^1_t(m, x, a_{-t}) + (1 - \lambda(z))u^2_t(m, x, a_{-t}), \end{aligned}$$

and, for each $i \in \{1, 2\}$, $\mathcal{G}'_i = \mathcal{G}'_i((K^i_t, \Gamma^i_t, u^i_t)_{t\in T_1\cup T_2})$. Since $\mathcal{G}'_1, \mathcal{G}'_2 \in \mathbb{G}$ and λ is continuous, it follows that γ_t is upper hemicountinuous with non-empty and compact values. Therefore, the correspondence $\gamma : (\widehat{M} \times \widehat{\mathcal{F}}^2) \times \tilde{A}_1 \twoheadrightarrow \Pi_{t\in T_2} \widehat{K}_t$ given by $\gamma((m, a), z_2) = \prod_{t\in T_2} \gamma_t((m, a_{-t}), z)$ is upper hemicontinuous with compact and non-empty values. In particular, γ has closed graph. Therefore, as for any $n \in \mathbb{N}$, $(z^n_1, z^n_2, a^n) \in \text{Graph}(\gamma)$, it follows that $\tilde{a} \in \gamma(\tilde{z}_1, \tilde{z}_2)$.

On the other hand, for each $n \in \mathbb{N}$ there exists $f_n : T_1 \to \widehat{K}$ such that, $m_n = \int_{T_1} H(t, f_n(t))d\mu$ and, for any $t \in T_1$, $f_n(t) \in \rho_t(z^n_1, z^n_2) := \underset{x\in\Psi(z^n_1,z^n_2)}{\operatorname{argmax}} v_t(x, z^n_1, z^n_2)$, where we use analogous notations to those described above. Thus, as in the case of γ_t, the correspondences $(\rho_t; t \in T_1)$ have closed graph.

Since $m^n \to \tilde{m}$, analogous arguments to those made in Claim A of Theorem 1 ensure that, applying the multidimensional Fatou's Lemma (see Hildenbrand (1974, page 69)), there exists a zero-measure set $\dot{T}_1 \subset T_1$ and a function $\overline{f} : T_1 \to \widehat{K}$ such that,

(i) For any $t \in \dot{T}_1$, $\overline{f}(t) \in \rho_t(\tilde{z}_1, \tilde{z}_2)$;

(ii) For any $t \in T_1 \setminus \dot{T}_1$, there is a subsequence of $\{f_n(t)\}_{n\in\mathbb{N}}$ that converges to $\overline{f}(t)$;

(iii) $\tilde{m} = \int_{T_1} H(t, \overline{f}(t))d\mu$.

As for any $t \in T_1 \setminus \dot{T}_1$, the correspondence ρ_t is closed, it follows from item (ii) above that $\overline{f}(t) \in \rho_t(\tilde{z}_1, \tilde{z}_2)$. By items (i) and (iii), jointly with the fact that $\tilde{a} \in \gamma(\tilde{z}_1, \tilde{z}_2)$, we have that $(\tilde{m}, \tilde{a}) \in \Theta_1(\tilde{z}_1, \tilde{z}_2)$. This concludes the proof of the claim. □

Since $(\overline{m}, \overline{a}) \in U_1$, $G(\overline{m}, \overline{a}) = \mathcal{G}'_1$ and, therefore, by the definition of $\mathcal{G}'_1$ we have that $\Lambda(G(\overline{m}, \overline{a}))\cap U_1 = \Lambda(\mathcal{G}'_1) \cap U_1 = \emptyset$. A contradiction, since both $(\overline{m}, \overline{a}) \in U_1$ and $(\overline{m}, \overline{a}) \in \Lambda(G(\overline{m}, \overline{a}))$. Thus, the set $m(\mathcal{G})$ is connected. Q.E.D.

Proof of Theorem 4.

(i) Suppose, by contradiction, that there is $A \in \Lambda_m(\mathcal{T}, \mathcal{X})$ and $\epsilon_0 > 0$ such that, for any $\delta > 0$ there is $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta$ and $A' \cap B[\epsilon_0, A]^c \neq \emptyset$, $\forall A' \in \Lambda_m(\mathcal{T}, \mathcal{X}')$.

Since A is $\mathcal{T}$-essential, there is $\delta_0 > 0$ such that, for any $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta_0$ we have that $\Lambda(\kappa(\mathcal{X}')) \cap B(\epsilon_0, A) \neq \emptyset$, where $B(\epsilon_0, A) = \left\{(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2 : \inf_{(m',a')\in A} \widehat{\sigma}((m, a), (m', a')) < \epsilon\right\}$. Fix $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta_0$. It follows that $\Lambda(\kappa(\mathcal{X}')) \cap B[\epsilon_0, A]$ is a non-empty and closed set contained in $B[\epsilon_0, A]$ and, therefore, it is not and essential subset of $\Lambda(\kappa(\mathcal{X}'))$—a direct consequence of the property stated in the previous paragraph.

Hence, there exists $\epsilon_1 > 0$ such that, for any $n \in \mathbb{N}$ there is $\mathcal{X}_n \in \mathbb{X}$ with $\tau(\mathcal{X}', \mathcal{X}_n) < \frac{\delta_1}{n}$ and $B(\epsilon_1, \Lambda(\kappa(\mathcal{X}')) \cap B[\epsilon_0, A]) \cap \Lambda(\kappa(\mathcal{X}_n)) = \emptyset$, where $\delta_1 > 0$ satisfies $\tau(\mathcal{X}'', \mathcal{X}') < \delta_1 \Longrightarrow \tau(\mathcal{X}'', \mathcal{X}) < \delta_0$. The last property ensures that $\tau(\mathcal{X}, \mathcal{X}_n) < \delta_0$ for any $n \in \mathbb{N}$, which implies that $\Lambda(\kappa(\mathcal{X}_n)) \cap B(\epsilon_0, A)$ is non-empty. Take a sequence $\{(m_n, a_n)\}_{n\in\mathbb{N}}$ such that $(m_n, a_n) \in \Lambda(\kappa(\mathcal{X}_n)) \cap B(\epsilon_0, A)$, $\forall n \in \mathbb{N}$. Without loss of generality, there is $(m_0, a_0) \in B[\epsilon_0, A]$ such that $(m_n, a_n) \to_n (m_0, a_0)$. The upper hemicontinuity of $(\Lambda \circ \kappa)$ ensures that $(m_0, a_0) \in \Lambda(\kappa(\mathcal{X}'))$. That is, $(m_0, a_0) \in \Lambda(\kappa(\mathcal{X}')) \cap B[\epsilon_0, A]$.

However, as for any $n \in \mathbb{N}$, $(m_n, a_n) \in \Lambda(\kappa(\mathcal{X}_n))$ and $B(\epsilon_1, \Lambda(\kappa(\mathcal{X}')) \cap B[\epsilon_0, A]) \cap \Lambda(\kappa(\mathcal{X}_n)) = \emptyset$, it follows that $(m_n, a_n) \notin B(\epsilon_1, \Lambda(\kappa(\mathcal{X}')) \cap B[\epsilon_0, A])$, $\forall n \in \mathbb{N}$. A contradiction, because $(m_0, a_0) \in \Lambda(\kappa(\mathcal{X}')) \cap B[\epsilon_0, A]$.

(ii) Let $A \in \Lambda_c(\mathcal{T}, \mathcal{X})$. It follows from the proof of Theorem 2 (item (i)) that there is $A_m \in \Lambda_m(\mathcal{T}, \mathcal{X})$ such that $A_m \subseteq A$. By the previous item, for each $\epsilon > 0$ there is $\delta_1 > 0$ such that, given $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta_1$, there exists $A'_m \in \Lambda_m(\mathcal{T}, \mathcal{X}')$ for which $A'_m \subseteq B[\epsilon, A_m] \subseteq B[\epsilon, A]$. By Theorem 3(iv), minimal essential sets are connected and, therefore, following analogous arguments to those made in the proof of Theorem 2 (item (ii)) we can ensure that for any $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta_1$ there is an essential component $A' \in \Lambda_c(\mathcal{T}, \mathcal{X}')$ which contains A'_m.

Since the correspondence $\Lambda \circ \kappa$ is upper hemicontinuous, there is $\delta_2 > 0$ such that for any $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta_2$ we have that $\Lambda(\kappa(\mathcal{X}')) \subset B(\epsilon, \Lambda(\kappa(\mathcal{X}))) \subset B[\epsilon, A] \bigcup B[\epsilon, \Lambda(\kappa(\mathcal{X})) \setminus A]$.

Notice that $\Lambda(\kappa(\mathcal{X})) \setminus A$ is a compact set.[11] Let $\delta = \min\{\delta_0, \delta_1\}$ and fix $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta$. If $A' \cap B[\epsilon, A]^c \neq \emptyset$, then $A' \cap B[\epsilon, \Lambda(\kappa(\mathcal{X})) \setminus A] \neq \emptyset$ and $A' \cap B[\epsilon, A] \neq \emptyset$. In addition, when $\epsilon < \pi$ it follows that $B[\epsilon, A] \bigcap B[\epsilon, \Lambda(\kappa(\mathcal{X})) \setminus A] = \emptyset$. Since A and $\Lambda(\kappa(\mathcal{X})) \setminus A$ are compact sets, it follows that $B[\epsilon, A]$ and $B[\epsilon, \Lambda(\kappa(\mathcal{X})) \setminus A]$ are closed sets. Thus, we obtain a partition of the connected set A' into two non-empty and disjoint closed sets, $A' \cap B[\epsilon, \Lambda(\kappa(\mathcal{X})) \setminus A]$ and $A' \cap B[\epsilon, A]$, which is a contradiction. Therefore, for any $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta$ we have that $A' \subset B[\epsilon, A]$. □

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[11] Indeed, since $(\Lambda(\kappa(\mathcal{X})) \setminus A) \subset \Lambda(\kappa(\mathcal{X}))$, it is sufficient to ensure that it is closed. Let $\{(m_n, a_n)\}_{n\in\mathbb{N}} \subset (\Lambda(\kappa(\mathcal{X})) \setminus A)$ be a sequence that converges to $(m_0, a_0) \in \widehat{M} \times \widehat{\mathcal{F}}_2$. For any $n \in \mathbb{N}$, $(m_n, a_n) \in \Lambda(\kappa(\mathcal{X}))$ and $(m_n, a_n) \notin A$. Thus, $(m_0, a_0) \in \Lambda(\kappa(\mathcal{X}))$. Furthermore, if $(m_0, a_0) \in A$, then for n large enough $(m_n, a_n) \in B[\pi, A]$, a contradiction with $B[\pi, \Lambda(\kappa(\mathcal{X})) \setminus A] \cap B[\pi, A] = \emptyset$. Therefore, $(m_0, a_0) \in \Lambda(\kappa(\mathcal{X})) \setminus A$.

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