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# Pushing amplitude equations far from threshold: application to the supercritical Hopf bifurcation in the cylinder wake

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**Abstract**

The purpose of this review article is to push amplitude equations as far as possible from threshold. We focus on the Stuart–Landau amplitude equation describing the supercritical Hopf bifurcation of the flow in the wake of a cylinder for critical Reynolds number $Re_c \approx 46$. After having reviewed Stuart's weakly nonlinear multiple-scale expansion method, we first demonstrate the crucial importance of the choice of the critical parameter. For the wake behind a cylinder considered in this paper, choosing $\epsilon^2 = Re_c^{-1} - Re^{-1}$ instead of $\epsilon'^2 = \frac{Re - Re_c}{Re_c^2}$ considerably improves the prediction of the Landau equation. Although Sipp and Lebedev (2007 *J. Fluid Mech* **593** 333–58) correctly identified the adequate bifurcation parameter $\epsilon$, they have plotted their results adding an additional linearization, which amounts to using $\epsilon'$ as approximation to $\epsilon$. We then illustrate the risks of calculating 'running' Landau constants by projection formulas at arbitrary values of the control parameter. For the cylinder wake case, this scheme breaks down and diverges close to $Re \approx 100$. We propose an interpretation based on the progressive loss of the non-resonant compatibility condition, which is the cornerstone of Stuart's multiple-scale expansion method. We then briefly review a self-consistent model recently introduced in the literature and demonstrate a link between its properties and the above-mentioned failure.

Keywords: bifurcation, amplitude equation, hydrodynamic instability

(Some figures may appear in colour only in the online journal)

Interest has recently regrown in the *a priori* approximation of limit cycles: methods include POD (Noack *et al* 2003), Central manifold reduction (Carini *et al* 2015), amplitude equations (Meliga *et al* 2009, 2012, Gupta and Chokshi 2015), semi-linear approaches (Farrell and Ioannou 2003). One of the most prominent and well studied flow serving as archetype of supercritical Hopf bifurcation is the flow around a cylinder. Above a threshold of $Re > Re_c$ with $Re_c \approx 46$, the Bénard-von Karman vortex street sets in and a limit cycle is reached. While linear stability conducted around the base flow $\mathbf{U}_b$, i.e., the laminar solution of the steady Navier–Stokes equations, can predict the critical Reynolds number, it fails in predicting three important aspects of the bifurcation: (i) the correct limit cycle frequency (except in the very vicinity of the threshold), (ii) the limit cycle amplitude (which remains unspecified) and (iii) the mean flow distortion, which is known to be the essential link causing nonlinear amplitude saturation and nonlinear frequency correction (Maurel *et al* 1995, Zielinska *et al* 1997).

More precisely, the frequency prediction resulting from the stability analysis of the base flow fails to a large amount, even for Reynolds numbers as low as $Re = 100$. However, for this range of Reynolds number, Pier (2002) and Barkley (2006) have shown, respectively through a weakly non parallel and a global stability analysis, that a correct frequency prediction can be obtained by considering the linear stability of the mean flow $\mathbf{U}_m$. In addition, the growth rate of the least stable mode of the mean flow stability analysis was found to vanish. This observed neutral stability of the mean flow is reminiscent of the marginal stability criterion of Malkus (1956) developed in the context of turbulent flows. Physically, this means that perturbations to an unstable flow induce mean flow modifications that increase while perturbations grow, until the point of saturation when the mean flow becomes marginally stable (Maurel *et al* 1995, Zielinska *et al* 1997, Thiria and Wesfreid 2007). The remarkable fact that linear stability of the mean flow of the cylinder's wake predicts the correct shedding frequency with an almost zero growth rate was theoretically rationalized by Sipp and Lebedev (2007) through a weakly nonlinear analysis, following Stuart's (1960) multiple-scale expansion method.

However, despite their appealing properties, such mean flow stability approaches bear an intrinsic limitation: they require the knowledge of the mean flow from DNS or experimental measurements. More importantly, they are not predictive as far as the amplitude of the disturbance is concerned. In this article, we focus on the Landau (1944) equation as an accurate model to the nonlinear dynamics. The latter was first obtained experimentally, using several experimental procedures to obtain the two Reynolds-dependent complex coefficients $\lambda$ and $\nu$ of the Landau equation (Mathis *et al* 1984, Provansal *et al* 1987, Dusek *et al* 1994)

$$\mathrm{d}A/\mathrm{d}t = \lambda A - \nu A|A|^2. \tag{1}$$

It was found that $\nu_r(Re_c) > 0$, where the subscript $r$ (resp. $i$) designates the real part $\Re$ (resp. the imaginary part $\Im$), in which case the Landau equation is indeed an excellent model for a supercritical Hopf bifurcation. It predicts amplitude saturation and frequency correction. It was found however that both $\lambda$ and $\nu$ were dependent on the spatial location at which the coefficients were tuned, in contrast to the apparent spatial coherence and the resulting expected universality of this supercritical Hopf bifurcation.

In the rigorous multiple scale expansion, as introduced by Stuart (1960) and applied to the cylinder wake case by Sipp and Lebedev (2007), these coefficients are spatially independent: $\lambda(Re)$ is approximated as an affine function of the bifurcation parameter (which is itself a function of the control parameter $Re$) while $\nu(Re)$ is found as the sum of two complex scalars. The resulting four complex constants are computed at $Re_c$ through formulas which involve scalar products of a certain number of fields that all have to be calculated only at threshold. We will review this approach in section 1.

Note that Stuart–Landau equations have also been derived for other hydrodynamic instabilities, like Rayleigh–Bnard convection (Segel 1962, Newell and Whitehead 1969) or Taylor–Couette vortices (Davey 1962), to cite some canonical examples. We should also mention that the terminology 'amplitude equation' can equally well refer to spatio-temporal nonlinear scalar equations like the cubic Ginzburg–Landau equation (see Schumm *et al* 1994 for an application to the cylinder wake). Such equations describing the slow modulations in space and time can be rigorously derived and have helped understanding pattern formation and front propagation in nonlinear unstable media (Fauve 1998). They have also helped defining the concept of nonlinear global mode and its spatial distribution (Zielinska and Wesfreid 1995, Chomaz 2005). The present article is however restricted to pure temporal amplitude equations.

We will then highlight in section 2 that the best suited bifurcation parameter is not $Re - Re_c$ but $Re_c^{-1} - Re^{-1}$, as correctly identified by Sipp and Lebedev (2007) and suggested by the linearity of the Navier–Stokes equations in $Re^{-1}$. Using the latter extends the domain of validity of the amplitude equation.

In section 3, we will then show that, while it is tempting to extrapolate these formulas at any Reynolds number (see Gupta and Chokshi 2015 for a recent example), such an approach, that we will call running amplitude equation, can sometimes radically fail. This is the case for the wake behind a cylinder where the approximation diverges for $Re \approx 100$. The origin of this failure is the vanishing of the real part $\tilde{\nu}_r(Re)$ of this running estimation of the Landau coefficient, resulting in a divergence of the frequency and amplitude.

We will then introduce in section 4 another, computationally far more expensive, approach to go further beyond threshold by keeping the spirit of the amplitude equation and using the semi-linear self-consistent (SC) monochromatic approximation of the limit cycle recently introduced by Mantic-Lugo *et al* (2014, 2015). The latter is seen to be valid at least until $Re = 120$.

A detailed analysis of this set of semi-linear equations in section 5 will finally help shedding further light onto the failure of the ad hoc procedure described in section 3. While it is true that the sign of the real part of the Landau coefficient $\nu(Re_c)$ determines the sub-critical/supercritical nature of the instability, its running value $\tilde{\nu}_r(Re)$ does not bear such an immediate sense, but can still be interpreted in the context of the semi-linear model.

## 1. Flow description and Landau equation

In 1921, Noether (1921) suggested that in wall-bounded shear flows the unstable mean-flow profile should differ from the undisturbed profile because of the presence of a steady wave and considered the role of the Reynolds stresses in this change. He concluded by calling for a self-consistent approach to characterize the instability: the growth rate determines the unstable character of the system and the Reynolds tensor stabilizes the profile, so that the growth rate value diminishes until it vanishes. Following Farrell's mechanical analogy of a system with regulatory feedback presented at BIFD2015, the unstable mode is the throttle-valve while the mean flow correction is Watt's governor.

The mean flow distortion induced by the Reynolds stresses is also the main idea behind Stuart's (1958) initial simplified model wherein the mean flow is only affected by the

Reynolds stress divergence of the most unstable eigenmode of the unperturbed base flow. It is also central in Stuart's (1960) multiple-scale weakly-nonlinear rigorous expansion method to obtain an amplitude equation, the so-called Stuart–Landau equation. The mean flow distortion is also a crucial component of recent semi-linear models as the stochastic structural stability theory of Farrell and Ioannou (2003) or the SC model of Mantic-Lugo *et al* (2014), that will be analyzed in more detail in section 4.

Stuart's multiple-scale method was followed by Sipp and Lebedev (2007) for the cylinder wake flow, as briefly summarized in the sequel of this section, with minor adaptations of the notations. Let us consider a fluid of density $\rho$, dynamic viscosity $\eta$, flowing at velocity $U_\infty$ around a cylinder of diameter $D$ in a two-dimensional domain $\Omega$. The Reynolds number is defined $Re = \frac{\rho U_\infty D}{\eta}$. The dimensionless incompressible unsteady 2D Navier–Stokes equations write

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \frac{1}{Re}\nabla^2 \mathbf{u} = 0, \quad \nabla \cdot \mathbf{u} = 0. \tag{2}$$

We then introduce the base flow $\mathbf{U}_\mathrm{b}$ which is solution of the steady Navier–Stokes equations

$$(\mathbf{U}_\mathrm{b} \cdot \nabla)\mathbf{U}_\mathrm{b} + \nabla P_\mathrm{b} - \frac{1}{Re}\nabla^2 \mathbf{U}_\mathrm{b} = 0, \qquad \nabla \cdot \mathbf{U}_\mathrm{b} = 0, \tag{3}$$

in short notation

$$\mathbf{N}(\mathbf{U}_\mathrm{b}) = 0. \tag{4}$$

The linear stability problem simply writes for an eigenvalue, eigenvector pair $(\sigma + \mathrm{i}\omega; \mathbf{u}_1)$ (the 1 refers to the most unstable or least stable eigenvalue)

$$(\sigma + \mathrm{i}\omega)\mathbf{u}_1 + \mathbf{L}(\mathbf{U}_\mathrm{b})\mathbf{u}_1 = \mathbf{0}, \tag{5}$$

where

$$\mathbf{L}(\mathbf{U}_\mathrm{b})\mathbf{u}_1 = (\mathbf{U}_\mathrm{b} \cdot \nabla)\mathbf{u}_1 + (\mathbf{u}_1 \cdot \nabla)\mathbf{U}_\mathrm{b} + \nabla p_1 - \frac{1}{Re}\nabla^2 \mathbf{u}_1 = 0 \tag{6}$$

and $\nabla \cdot \mathbf{u}_1 = 0$ is assumed implicitly. At threshold $Re_\mathrm{c}$, the eigenvalue is neutral and $\sigma = 0$ and $\omega = \omega_c$.

Let us now introduce the small parameter $\epsilon^2 = (Re_\mathrm{c}^{-1} - Re^{-1})$, a slow time-scale $T = \epsilon^2 t$ and the frequency at threshold $\omega_\mathrm{c}$. The following decomposition is used

$$\mathbf{u} = \mathbf{U}_\mathrm{b} + \epsilon^2(\mathbf{u}_2^0 + |A|^2\mathbf{u}_2^{|A|^2}) + \epsilon(A(T)\mathbf{u}_1 \exp^{\mathrm{i}\omega_\mathrm{c} t} + \mathrm{c.c.}) + \epsilon^2(A(T)^2\mathbf{u}_2^{A^2} \exp^{\mathrm{i}2\omega_\mathrm{c} t} + \mathrm{c.c.}), \tag{7}$$

where c.c. designates the complex conjugate and where the first three terms have been grouped because of their time-invariance. Their sum in fact approximates $\boldsymbol{U}_\mathrm{m} \approx \mathbf{U}_\mathrm{b} + \epsilon^2(\mathbf{u}_2^0 + |A|^2\mathbf{u}_2^{|A|^2})$.

When introducing this expression in the Navier–Stokes equations and solving in sequential order the linear problems appearing at each power of epsilon, a compatibility condition is found to be required at order $\mathcal{O}(\epsilon^3)$ to avoid the forcing of the linear operator $\mathbf{L}(\mathbf{U}_\mathrm{b})$ governing $\mathbf{u}_3$ by a resonant term oscillating at the eigenfrequency $\omega_c$. This compatibility condition is a necessary condition to maintain the consistency of the asymptotic scheme. Fredholm's alternative then stipulates that secular terms will be avoided only if the forcing term is orthogonal to the adjoint $\mathbf{u}_1^+$ of the eigenmode of $\mathbf{L}(\mathbf{U}_\mathrm{b})$ corresponding to the resonating frequency. Enforcing this orthogonality condition through scalar products yields

$$\frac{\mathrm{d}A}{\mathrm{d}T} = \Lambda A - (\nu_0 + \nu_2)A|A|^2, \tag{8}$$

where

$$\Lambda = \frac{(\mathbf{u}_1^\dagger \mid \mathbf{F}_3^A)}{(\mathbf{u}_1^\dagger \mid \mathbf{u}_1)}, \quad \nu_0 = -\frac{(\mathbf{u}_1^\dagger \mid \mathbf{F}_3^{A|A|^2})}{(\mathbf{u}_1^\dagger \mid \mathbf{u}_1)}, \quad \nu_2 = -\frac{(\mathbf{u}_1^\dagger \mid \mathbf{F}_3^{A^*A^2})}{(\mathbf{u}_1^\dagger \mid \mathbf{u}_1)}, \tag{9}$$

$$\mathbf{F}_3^A = -(\mathbf{u}_1 \cdot \boldsymbol{\nabla})\mathbf{u}_2^0 - (\mathbf{u}_2^0 \cdot \boldsymbol{\nabla})\mathbf{u}_1 - \boldsymbol{\nabla}^2\mathbf{u}_1, \tag{10}$$

$$\mathbf{F}_3^{A|A|^2} = -(\mathbf{u}_1 \cdot \boldsymbol{\nabla})\mathbf{u}_2^{|A|^2} - (\mathbf{u}_2^{|A|^2} \cdot \boldsymbol{\nabla})\mathbf{u}_1, \tag{11}$$

$$\mathbf{F}_3^{A^*A^2} = -(\mathbf{u}_1{}^* \cdot \boldsymbol{\nabla})\mathbf{u}_2^{A^2} - (\mathbf{u}_2^{A^2} \cdot \boldsymbol{\nabla})\mathbf{u}_1{}^*, \tag{12}$$

where the fields $\mathbf{u}_2^0$, $\mathbf{u}_2^{|A|^2}$ and $\mathbf{u}_2^{A^2}$ are respectively solution of

$$\mathbf{L}(\mathbf{U}_\mathrm{b})\mathbf{u}_2^0 = -\boldsymbol{\nabla}^2\mathbf{U}_\mathrm{b}, \tag{13}$$

$$\mathbf{L}(\mathbf{U}_\mathrm{b})\mathbf{u}_2^{|A|^2} = -(\mathbf{u}_1{}^* \cdot \boldsymbol{\nabla})\mathbf{u}_1 - (\mathbf{u}_1 \cdot \boldsymbol{\nabla})\mathbf{u}_1{}^* = -\psi(\mathbf{u}_1), \tag{14}$$

$$2\mathrm{i}\omega\mathbf{u}_2^{A^2} + \mathbf{L}(\mathbf{U}_\mathrm{b})\mathbf{u}_2^{A^2} = -(\mathbf{u}_1 \cdot \boldsymbol{\nabla})\mathbf{u}_1. \tag{15}$$

Note that if $\mathbf{u}_1$ is multiplied by $\alpha e^{\mathrm{i}\theta}$ then $\Lambda$ does not vary while $\nu_0$ and $\nu_2$ are multiplied by $\alpha^2$. Therefore, the saturation amplitude

$$|A_\mathrm{LC}| = \sqrt{\frac{\Lambda_r}{\nu_{0,r} + \nu_{2,r}}} \tag{16}$$

does also depend on the normalization. The $L_2$ norm of fluctuating fields extracted from DNS

$$E = \frac{1}{2}\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\int_\mathcal{V} |\mathbf{u} - \mathbf{U}_\mathrm{m}|^2 \,\mathrm{d}V\mathrm{d}t \tag{17}$$

should therefore be directly compared to

$$E_\mathrm{LC} = \epsilon^2|A_\mathrm{LC}|^2 \tag{18}$$

for the chosen normalization, as done in figure 2. However the actual limit-cycle velocity at a given point $\epsilon A_\mathrm{LC}\mathbf{u}_1(x, y)\exp^{\mathrm{i}\omega_c t}$ does not depend on the normalization.

Defining now $a = A(T)\exp^{\mathrm{i}\omega_c t}$, the amplitude equation writes

$$\frac{\mathrm{d}a}{\mathrm{d}t} = (\mathrm{i}\omega_\mathrm{c} + \epsilon^2\Lambda)a - \epsilon^2(\nu_0 + \nu_2)a|a|^2, \tag{19}$$

which directly highlights the nonlinear frequency correction

$$\omega = \omega_\mathrm{c} + \epsilon^2\left(\Lambda_i - \Lambda_r\frac{\nu_{0,i} + \nu_{2,i}}{\nu_{0,r} + \nu_{2,r}}\right). \tag{20}$$

## 2. Choice of the bifurcation parameter in the Landau equation

Sipp and Lebedev (2007) have correctly identified the appropriate bifurcation parameter as $\epsilon = (Re_\mathrm{c}^{-1} - Re^{-1})^{1/2}$ but have apparently plotted their results fortuitously using

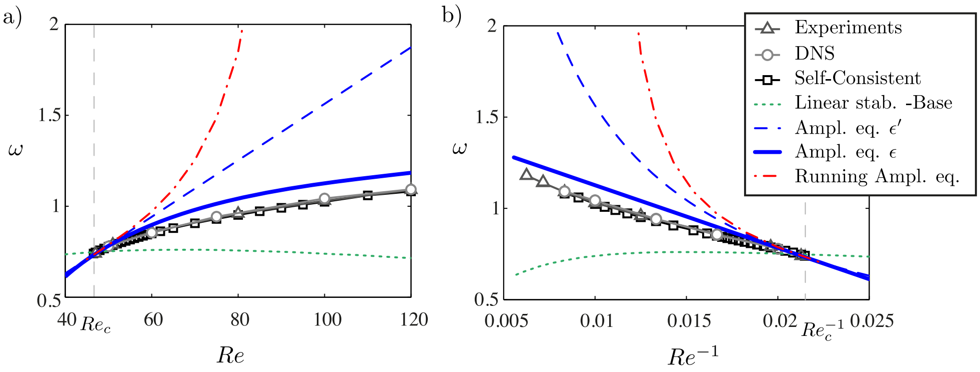


**Figure 1.** Limit cycle frequencies, as determined by linear analysis, amplitude equation, amplitude equation with linearized small parameter, self consistent model, experiments, DNS and running amplitude equation. In (a) they are reported as a function of *Re* and in (b) as a function of $1/Re$.

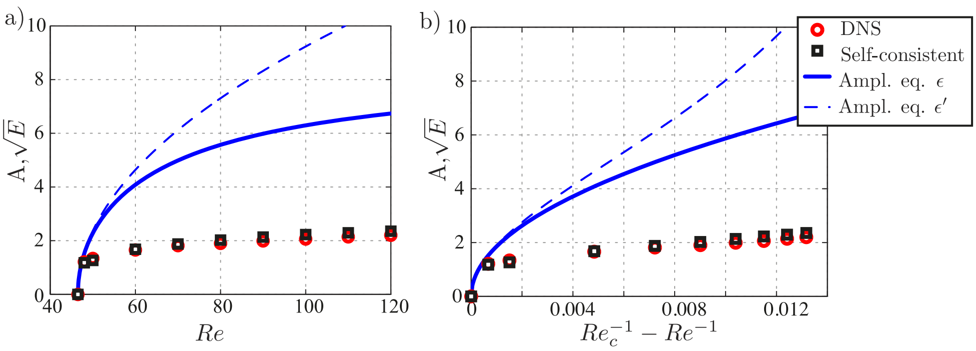


**Figure 2.** Limit cycle amplitude $\sqrt{E}$ (see equation (17)–(18)) as determined by amplitude equation, amplitude equation with linearized bifurcation parameter, self consistent model and DNS. In (a) they are reported as a function of *Re* and in (b) as a function of $Re_c^{-1}{-}Re^{-1}$ to highlight the square-root dependence.

$\epsilon' = \left(\frac{Re - Re_c}{Re_c^2}\right)^{1/2}$. This explains why their amplitude equation frequency prediction is linear in the $Re{-}\omega$ plane (figure 1(a)) while it should be linear in the $Re^{-1} - \omega$ plane (figure 1(b)). Of course, very close to threshold $\epsilon'^2 \sim \epsilon^2$, but further away from threshold, using $\epsilon'$ introduces an artificial additional nonlinearity which quickly ruins the otherwise acceptable prediction. At $Re = 100$ for instance, $\epsilon'^2 \simeq 2\epsilon^2$.

In figure 1(a), we have first replotted Sipp and Lebedev (2007)'s curves in the $Re - \omega$ plane together with the $\epsilon$-based Stuart–Landau prediction (continuous line), which appears as nonlinear once mapped from the $Re^{-1} - \omega$ plane to the $Re - \omega$. The dashed line is the $\epsilon'$-based Stuart–Landau prediction, while the dotted line is the frequency prediction from the linear analysis of the base flow $\omega(Re)$, the circles correspond to DNS and the triangles to experimental measurements. The squares correspond to the SC model that is introduced in section 4 while the dotted-dashed curve is the frequency prediction resulting from a running Landau equation, as detailed next in section 3.

In figure 1(b), the same approximations are plotted in the $Re^{-1} - \omega$ plane where the Landau approximation is bound to be linear. Both DNS and experimental data follow indeed a reasonably linear trend. These two figures demonstrate that Sipp and Lebedev have actually underestimated the quality of their approximation by correctly obtaining the amplitude equation (19) while adding unfortunately a superfluous linearization step when plotting the nonlinear frequency correction.

Figure 2 compares the effective amplitude $\sqrt{E}$ (equation (17), as measured from DNS to the prediction by the amplitude equation $\sqrt{E_{\mathrm{LC}}}$ (equation (18), using both the correct bifurcation parameter $\epsilon$ and its linearized approximation $\epsilon'$ in the $Re - A$ plane (figure 2(a)) and in the $(Re_c^{-1} - Re^{-1}) - A$ plane (figure 2(b)). The comparison is seen to be reasonably fair only close to threshold while it quickly degrades further away from threshold. This seems to indicate that, in order to approximate the mean flow, which was seen to have a correct frequency shift, the mean flow equation has to be forced by the Reynolds stresses associated with the eigenmode prevailing at threshold multiplied by an artificially large amplitude. Somewhat surprisingly, the Landau prediction of a larger fluctuating amplitude than necessary creates stronger Reynolds stresses which yields an approximate mean flow that correctly predicts the nonlinear frequency correction.

More importantly, and beyond this quantitative disagreement, the following qualitative difference is important to notice. While the amplitude grows like the square root of the bifurcation parameter in the $(Re_c^{-1} - Re^{-1}) - A$ plane and does therefore not saturate away from threshold, it does saturate to a finite value in the $Re - A$ plane. Our DNS data shows that this saturation sets in rapidly after threshold, in qualitative agreement with the Landau prediction, once formulated with the correct bifurcation parameter.

In summary, when defining the bifurcation parameter experimentally (or numerically), one should look for the largest possible domain of validity of a linear approximation of the frequency. For the theoreticians, the same holds true and the parameter should enter linearly the governing equation, as $Re^{-1}$ enters linearly in the Navier–Stokes equations. Of course all choices of bifurcation parameter are equivalent in the very near vicinity of the bifurcation point where the linear approximation is accurate.

## 3. A tempting but risky generalization

Landau's equation can also be interpreted as a truncated Galerkin projection, in which case it is tempting to determine its constants using the same formulas as equation (9) but using the running values of the growth-rate, eigenmodes and adjoint eigenmodes (i.e. values evaluated for any given $Re$ rather than $Re_c$), a procedure we shall refer to as a running amplitude equation. For example, $\mathrm{i}\omega_c + \epsilon^2\Lambda$ is simply replaced by $\sigma + \mathrm{i}\omega$ in equation (19). Similarly $\nu_0$ is replaced by

$$\tilde{\nu}_0(Re) = -\frac{(\mathbf{u}_1^\dagger \mid \mathbf{F}_3^{A|A|^2})}{(\mathbf{u}_1^\dagger \mid \mathbf{u}_1)}, \tag{21}$$

where $\mathbf{u}_1^\dagger$, $\mathbf{u}_1$, as well as $\mathbf{F}_3^{A|A|^2}$ are all determined at the chosen value of the Reynolds number. We believe that this truncation procedure is less robust than the compatibility condition obtained through the multiple-scale expansion approach of Stuart.

Indeed, one should keep in mind that the compatibility condition results from a non-resonance condition of a forced linear equation when the linear eigenvalue $\lambda = \mathrm{i}\omega_c$ matches the forcing frequency $\mathrm{i}\omega_c$. This condition is necessary to avoid the growth of secular terms. Away from threshold, when the linear growth-rate $\sigma$ is not incorporated in the asymptotic

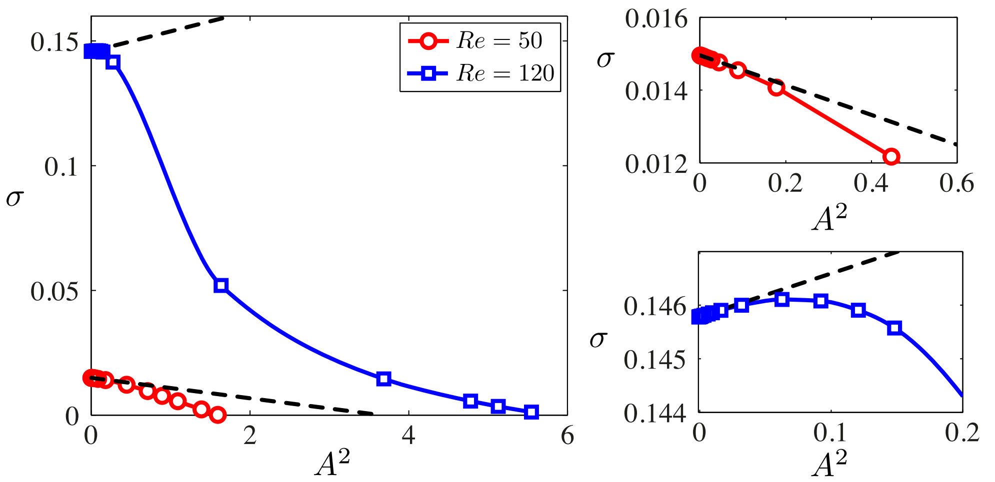


**Figure 3.** $\sigma(A^2)$ curves for $Re = 50$ and $Re = 120$ as obtained from the self-consistent model (SC) defined in equations (22)–(24) and their zooms.

expansion framework, the eigenvalue $\lambda = i\omega + \sigma$ of the linear operator $\mathbf{L}(\mathbf{U}_b)$ governing $\mathbf{u}_3$ and the frequency $i\omega + 3\sigma$ of the forcing term $\mathbf{F}_3^{A|A|^2}$ do not strictly resonate. This forcing does therefore not induce a divergent contribution at large time but merely yields an important amplification of the response. The projection on the adjoint becomes then only a truncation procedure and looses its necessary nature and therefore possibly its robustness.

Figure 1 shows that this projection procedure can sometimes appear as a very risky approach. The real part of the $\nu$ coefficient is seen to vanish close to $Re \approx 100$, resulting in a totally unphysical behavior of both frequency (figure 1) and amplitude (not shown).

## 4. A semi-linear self-consistent model

Despite the consistency of the multiple-scale expansion method, its perturbative nature implies that the spatial structure of the growing unstable mode is in large part fixed by the unperturbed base flow. There are flows, including the Bénard-von Karman vortex street (Dusek *et al* 1994), in which the spatial structure of the saturated mode differs considerably from that of the linear mode, limiting the validity of the Stuart–Landau amplitude equation.

This has led Mantic-Lugo *et al* (2014) to propose a SC model which consists of a semi-linear coupling where the perturbation equation is linearized while the quadratic Reynolds stresses are kept in the mean flow equation. This model is not perturbative, it is merely a single-harmonic (i.e. monochromatic) approximation of the limit cycle. Although the model is closed by enforcing the marginality condition (Malkus 1956) $\sigma = 0$, it is convenient to start from the following family of SC systems, characterized by a single free parameter $A$

$$\mathbf{N}(\mathbf{U}_m) = -A^2 \psi(\mathbf{u}), \tag{22}$$

$$(\sigma + i\omega)\mathbf{u} + \mathbf{L}(\mathbf{U}_m)\mathbf{u} = \mathbf{0}, \tag{23}$$

$$||\mathbf{u}|| = 1, \tag{24}$$

where $\psi(\mathbf{u}) = \mathbf{u}^* \cdot \nabla\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u}^*$.

This free parameter $A$ is the amplitude of the linear disturbance field on which the solution quadruplet $(\mathbf{U}_{\mathrm{m}}, \mathbf{u}, \sigma, \omega)$ solely depends, for a fixed Reynolds number. The iterative solution procedure of this coupled system of equations is detailed in (Mantic-Lugo *et al* 2014, 2015). Starting from the base flow, $A = 0$, the amplitude is gradually increased, the growth rate $\sigma$ decreases to zero and the flow approaches the mean flow. Two examples of curves $\sigma(A^2)$ are depicted in figure 3: while this curve decays monotonically at $Re = 50$, it does display a small overshoot at small values of $A$ for $Re = 120$. More importantly, in both cases, the curve $\sigma(A^2)$ indeed reaches 0 for a specific value $A^*$. This is precisely the closure condition of the SC model, following Malkus (1956)' marginal stability condition, found to be valid for the cylinder flow (Barkley 2006, Pier 2002).

The quality of the approximation resulting from this model has been assessed in detail for the cylinder flow (Mantic-Lugo *et al* 2014, 2015), not only as far as the limit cycle frequency and amplitude are concerned, but also for the disturbance spatial structure and the resulting mean flow approximation. The SC frequency and amplitude predictions are compared to the full DNS results in figures 1 and 2 respectively, showing good agreement.

This model therefore appears as a possible approach to extend amplitude equations further away beyond threshold, despite its computational cost. There are however additional difficulties that potentially arise. First, as the control parameter increases, the iterative scheme becomes more and more prohibitive. Second, and more importantly, there are flows where the second harmonic generation contributes significantly to the limit cycle saturation, an effect which is captured by $\nu_2$ in the Landau equation. Such flows are characterized by a limit cycle which is distorted by higher harmonics: the fluctuations are time-periodic but non-monochromatic (or in other words an-harmonic), the mean flow being then strongly linearly unstable. Such cases, that do not satisfy the mean flow marginal stability condition, were encountered in cavity flows (Sipp and Lebedev 2007), in turbulent wakes at high Reynolds number (Meliga *et al* 2012) or more recently for standing waves in double solutal convection (Turton *et al* 2015). Note that in this latter flow configuration, traveling waves do in contrast verify the marginal stability condition, therein called RZIF (real zero imaginary frequency) condition.

## 5. Significance of the running Landau coefficient

We are interested in taking a closer look at the slope of the curve $\sigma(A^2)$ shown in figure 3, in particular in $A = 0$. We consider the SC model (22)–(24) at intermediate stages before convergence to marginal stability, i.e. when the growth rate is still positive. We consider now a small change in amplitude $\delta A^2$, and wish to express the resulting first-order variation in growth rate as

$$\delta\sigma = (\boldsymbol{\nabla}_{A^2}\sigma \mid \delta A^2). \tag{25}$$

An expression for $\boldsymbol{\nabla}_{A^2}\sigma$, the sensitivity of the growth rate to the squared amplitude, can be derived by introducing a Lagrangian functional for the growth rate subjected to the constraints expressed in the SC model (22)–(24):

$$\begin{aligned}\mathcal{L} = \sigma &- (\mathbf{U}^\dagger \mid \mathbf{N}(\mathbf{U}_{\mathrm{m}}) + A^2\boldsymbol{\psi}(\mathbf{u}))\\ &- (\mathbf{u}^\dagger \mid (\sigma + \mathrm{i}\omega)\mathbf{u} + \mathbf{L}(\mathbf{U}_{\mathrm{m}})\mathbf{u}) - (\mathbf{u}^{\dagger *} \mid (\sigma - \mathrm{i}\omega)\mathbf{u}^* + \mathbf{L}(\mathbf{U}_{\mathrm{m}})\mathbf{u}^*)\\ &- \beta(1 - (\mathbf{u} \mid \mathbf{u})).\end{aligned} \tag{26}$$

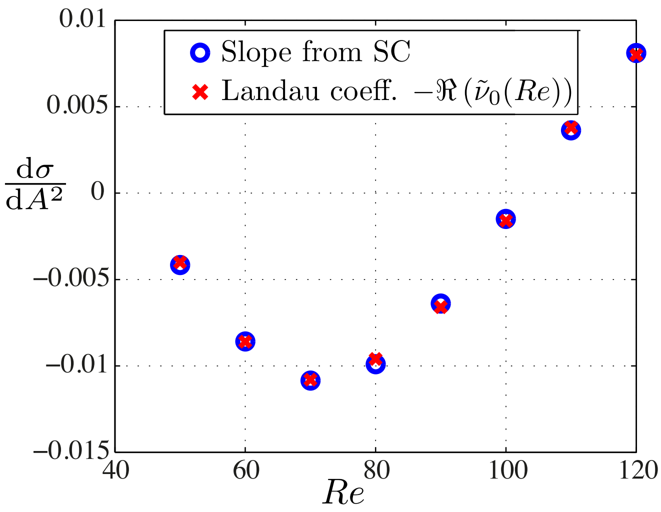


**Figure 4.** $d\sigma/dA^2|_{A=0}$ as predicted by the running Landau coefficient and as retrieved by finite differences from figure 3.

Imposing the first-order stationarity condition on the Lagrangian yields

$$\nabla_{A^2}\sigma = \frac{\partial \mathcal{L}}{\partial A^2} = -(\mathbf{U}^\dagger \mid \psi(\mathbf{u})), \tag{27}$$

where $\mathbf{U}^\dagger$, $\mathbf{u}^\dagger$ are a solution of the coupled system

$$(\mathbf{u}^\dagger \mid \mathbf{u}) = 1/2, \tag{28}$$

$$\mathbf{L}^\dagger(\mathbf{U}_\mathrm{m})\mathbf{U}^\dagger = -2\Re(\mathbf{u}^{\dagger *} \cdot \nabla\mathbf{u}^T - \mathbf{u} \cdot \nabla\mathbf{u}^{\dagger *}), \tag{29}$$

$$(\sigma - \mathrm{i}\omega)\mathbf{u}^\dagger + \mathbf{L}^\dagger(\mathbf{U}_\mathrm{m})\mathbf{u}^\dagger = -A^2(\mathbf{U}^\dagger \cdot \nabla\mathbf{u}^T - \mathbf{u} \cdot \nabla\mathbf{U}^\dagger) + \beta\mathbf{u}. \tag{30}$$

Additionally, a compatibility condition for (30) is obtained

$$0 = (-A^2(\mathbf{U}^\dagger \cdot \nabla\mathbf{u}^T - \mathbf{u} \cdot \nabla\mathbf{U}^\dagger) + \beta\mathbf{u} \mid \mathbf{u}), \tag{31}$$

$$\Leftrightarrow \quad \beta = A^2(\mathbf{U}^\dagger \cdot \nabla\mathbf{u}^T - \mathbf{u} \cdot \nabla\mathbf{U}^\dagger \mid \mathbf{u}) \tag{32}$$

$$= A^2(\mathbf{U}^\dagger \mid \psi(\mathbf{u})) = -A^2\nabla_{A^2}\sigma. \tag{33}$$

In the particular case where $A = 0$, the SC model reduces to the steady base flow equation for $\mathbf{U}_\mathrm{m} = \mathbf{U}_\mathrm{b}$ and eigenvalue problem for $\mathbf{u} = \mathbf{u}_1$:

$$\mathbf{N}(\mathbf{U}_\mathrm{b}) = \mathbf{0} \tag{34}$$

$$(\sigma + \mathrm{i}\omega)\mathbf{u}_1 + \mathbf{L}(\mathbf{U}_\mathrm{b})\mathbf{u}_1 = \mathbf{0} \tag{35}$$

and $\mathbf{U}^\dagger$ is a solution of

$$(\mathbf{u}_1^\dagger \mid \mathbf{u}_1) = 1/2, \tag{36}$$

$$\mathbf{L}^\dagger(\mathbf{U}_\mathrm{b})\mathbf{U}^\dagger = -2\Re(\mathbf{u}_1^{\dagger *} \cdot \nabla\mathbf{u}_1^T - \mathbf{u}_1 \cdot \nabla\mathbf{u}_1^{\dagger *}), \tag{37}$$

$$(\sigma - \mathrm{i}\omega)\mathbf{u}_1^\dagger + \mathbf{L}^\dagger(\mathbf{U}_\mathrm{b})\mathbf{u}_1^\dagger = \mathbf{0}, \tag{38}$$

i.e. $\mathbf{u}_1^\dagger$ is simply the adjoint mode associated with the eigenmode $\mathbf{u}_1$. From (27) and (37), the slope of $\sigma(A^2)$ in $A = 0$ is

$$\left.\frac{\partial\sigma}{\partial A^2}\right|_{A=0} = -(\mathbf{U}^\dagger \mid \psi(\mathbf{u}_1)) \tag{39}$$

$$= (\mathbf{L}^\dagger(\mathbf{U}_\mathrm{b})^{-1}(2\Re(\mathbf{u}_1^{\dagger *}\cdot\nabla\mathbf{u}_1^T - \mathbf{u}_1\cdot\nabla\mathbf{u}_1^{\dagger *})) \mid \psi(\mathbf{u}_1)) \tag{40}$$

$$= (2\Re(\mathbf{u}_1^{\dagger *}\cdot\nabla\mathbf{u}_1^T - \mathbf{u}_1\cdot\nabla\mathbf{u}_1^{\dagger *}) \mid \mathbf{L}(\mathbf{U}_b)^{-1}\psi(\mathbf{u}_1)) \tag{41}$$

$$= -\Re(\tilde{\nu}_0). \tag{42}$$

The last equality has been found after rearrangement of terms and reminding the definition of the running Landau coefficient $\tilde{\nu}_0$ defined back in equation (21). The validity of this expression is ascertained in figure 4 by comparing the expression (42) to the slope $\mathrm{d}\sigma/\mathrm{d}A^2|_{A=0}$, retrieved from SC iterations as those described in figure 3. For a given Reynolds number, the real part of running value of the mean-flow correction component of the Landau coefficient $\Re(\tilde{\nu}_0(Re))$ can be interpreted as the opposite of the initial slope $\mathrm{d}\sigma/\mathrm{d}A^2|_{A=0}$ of the SC family of solutions $\sigma(A)$.

As this slope changes sign for $Re \approx 100$, $\Re(\tilde{\nu}_0)$ vanishes and, since $\tilde{\nu}_2 \ll \tilde{\nu}_0$ for the cylinder flow, it induces the divergence of the running Landau equation prediction.

## 6. Conclusion

In this paper, we have discussed how Stuart–Landau equations, which are remarkable approximations of super-critical Hopf bifurcations close to threshold, can be pushed further away from threshold. We have first highlighted the choice of the relevant bifurcation parameter. From a theoretical point of view, it should enter the governing equations linearly, while from an experimental viewpoint, its dependence upon the limit-cycle frequency should appear as linear as possible. For the wake behind a cylinder considered in this paper, choosing $Re_\mathrm{c}^{-1} - Re^{-1}$ instead of $Re - Re_\mathrm{c}$ considerably improves the prediction of the Landau equation.

We have then illustrated the risks of calculating running values of the Landau constants by projection formulas at arbitrary values of the control parameter. For the cylinder wake, this scheme breaks down and diverges close to $Re \approx 100$. We have proposed two interpretations. The first one is based on the progressive loss of the non-resonant compatibility condition, which is the cornerstone of Stuart's multiple-scale expansion method. The other is related to the variation of the growth-rate of the SC model that couples the mean flow to its dominant eigenmode via the Reynolds stresses, assuming an asymptotically small amplitude.

More generally, this semi-linear and SC model, closed by imposing the marginal stability criterion, keeps the spirit of the Landau equation while improving its predictive power, to the price of a significant computational cost. Additionally, it fails to describe an-harmonic (i.e. non monochromatic) limit cycles. Whether it can be generalized to such situations remains at present an open issue.

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