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OPERATIVE MINE PLANNING, DESIGN AND GEOLOGICAL MODELING: INTEGRATION BASED ON TOPOLOGICAL REPRESENTATIONS

TESIS PARA OPTAR AL GRADO DE DOCTOR EN INGENIERÍA DE MINAS

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Resumen

Los esfuerzos científicos y de ingeniería en la teoría de planificación minera se concentran en la mejora de las capacidades de los algoritmos existentes, en términos de velocidad y del tamaño del problema: cambiar de minutos a segundos y de cientos de miles a millones de bloques de una representación del yacimiento. Sin embargo, en la práctica, la planificación minera está tan llena de trabajo manual y decisiones personales, que las soluciones de los algoritmos son alteradas notablemente, en un proceso de operativización que además puede durar semanas.

La presente tesis doctoral pretende proponer un punto de vista diferente, uniendo algunas de tales decisiones manuales, como el diseño de mina y la modelación del yacimiento.

El trabajo desarrollado se centra en la representación paramétrica de una mina con diseño, así como también un modelo geológico del yacimiento, a través de volúmenes parametrizados y herramientas morfológicas, optimizados mediante la llamada técnica de "simulated annealing" o recocido simulado. Se muestra que es posible modelar y optimizar una mina de superficie, diseñando rampas, bancos y curvas de retorno, así como también ajustar un yacimiento con volúmenes paramétricos, incluyendo el conocimiento geológico y la información de sondajes.

Las principales aplicaciones de estos resultados son: el diseño de mina se podría automatizar, realizándose en cuestión de minutos en lugar de semanas, el valor del proyecto no cambiaría debido a las decisiones arbitrarias del diseñador, la operación de la mina y el modelo geológico tendrían un lenguaje común a través de volúmenes paramétricos, las predicciones geoestadísticas dependerían del conocimiento geológico y de los datos ajustados, la incertidumbre geológica se modelaría a partir de parámetros estocásticos, por lo que la optimización bajo incertidumbre geológica podría ser implementada a partir de simulaciones. Es más, la entrada del algoritmo de planificación minera sería ya no un modelo de bloques, sino directamente la información de los sondajes y el modelo geológico.

A pesar de que se han desarrollado algunos ejemplos numéricos, los casos reales están fuera del alcance de esta tesis. El valor de este trabajo se centra en proponer ideas y un nuevo campo de investigación en la planificación de la mina, centrado en necesidades mineras más realistas, trayendo herramientas diferentes a las que hasta hoy eran el paradigma y tratando de unir áreas profesionales que funcionan por separado.

Abstract

Scientific and engineering efforts in mine planning theory are focused on improving the speed and size capacity of existing algorithms. They look for changing from minutes to seconds, and hundreds of thousands to million blocks of an ore body representation. However, mining practices are so full of manual work and personal decisions, that algorithmic solutions are changed considerably by the mine planner, lasting weeks in this process to achieve operative final results.

This thesis proposes a different point of view, joining some of such hand work decisions, like design and ore body modeling.

The developed work concentrates on parametric representations of mine design and an ore body model, through volumes and morphological tools, optimized by simulated annealing. It is shown that it is possible to model and optimize a final open pit, with road, benches and switch-backs design and to fit an ore body with parametric volumes, which include geological knowledge.

The principal applications of such results are: mine design could be obtained in minutes instead of weeks, the project value will not change because of handmade decisions, mine operation and geological units will have a common language through parametric volumes, geostatistical predictions will depend on geological knowledge and fitted data, geological uncertainty would be modeled from parameter stochasticity, so stochastic optimization could be implemented from simulations. In fact, mine planning algorithm inputs would no longer be a block model, but directly drill hole data.

Despite that some numeral examples were developed, real cases were not the scope of this thesis work. The value of this work concentrates on proposing ideas and a new field of investigation in mine planning, focused on more realistic mining needs and bringing different tools, as those that until today were the paradigm, and trying to join professional areas that work separately.

Dedicatoria

A Esmeralda.

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1. Introduction

Mining is an important industry for some countries, representing in some cases a big part of their internal product. The ore price and the extracting and processing costs could indirectly explain local economy, as strong as money exchange indexes. This impact is defined by few big mines and several medium mines, which give a basic idea of how important is to take good decisions, on time and under accountability structure. So, any new mine, expansion even a mine closure, should be carefully valued.

In order to evaluate a mining project, several steps should be performed. First, getting sparse data, from geological exploration. Second, expanding data to obtain a geostatistical representation of the ore body, which is a hard process by itself. Third, using this model to define economic zones, optimizing what and when the minerals should be extracted. Fourth, designing an operative mine, to satisfies transport systems and facilities placement. Finally, joining all together to decide actions or to start again with the analyses. The last step considers the analysis of uncertainty in three ways: prices, operational events and lack of information in geology. This is valid from green field projects to mines that are in closing process, certainly with different focuses.

Each of these steps introduces errors to the project value estimation, due to the lack of information and because the specialist takes decisions from his/her experience and preferences, which, despite they are guided by software and algorithms, are in general qualitative.

In the last decades, several efforts to develop algorithms were focused in some of these steps. Mathematical tools from other industries were introduced in mining, mainly in geostatistics and mine planning. Geostatistics develops a blocks model, which is a discretized representation of the ore body, among others possible applications, because this is profusely studied. Mine planning takes this block model and develops an optimal (or at least a good) solution for the mine, considering which material and when will be extracted, how will be processed and proposing a production plan.

It is easy to show that geostatistical predictions for ore made by hand or using naive algorithms (like nearest neighborhood) introduce unacceptable errors. A similar situation happens if the mine planner decides, by hand or “by inspection”, the mine extraction schedule, introducing big errors for the mine value. So, big efforts to develop geostatistics and mine planning algorithms got acceptations in mining companies, becoming obligatory for mine evaluation and bank funding. Geostatistics specialists and mine planners are now users of those algorithms, while universities and research centers are developers of knowledge, mainly by mathematicians, operation researchers and computer scientists.

When those algorithm specialists are involved in mine planning modeling, they tend to concentrate to improve its hard parameters, like computing velocity or size generalization. They look for passing from minutes of calculations to seconds, from hundreds of thousands of blocks to several millions, from one metallurgic process to several, from one mine to multi-mine, from deterministic scenarios to uncertainty analysis.

However, practical mine planning face operative problems: in the final pit obtained from algorithms there are neither benches nor ramps. Furthermore, geostatistical modeling creates blocks models without a geological sense, which could define volumes of matter with the same characteristics of lithology, mineralization, alterations or structures.

This thesis work proposes a methodology attempting to push forward an operative mine planning. To reach this objective, key idea is to develop parametric objects and morphological operator modeling, for final pit design and for geological models.

The strategy that the author of this thesis attempted to follow, is trying to fit mining needs, avoiding computational interest. This means to propose a new paradigm in mine planning, with new optimization problems and some tools that could be useful to solve them, more than a numerical benchmark against actual algorithms. Mathematical tools, like stochastic geometry, topology and optimization methods that are already well known or, at least, for which there is knowledge to start. These tools could be even linked to partial differential equations and dynamical systems, opening new branches of mathematical tools.

Furthermore, this work promotes to consider a more integrated work between professional analysts in mine evaluation, mixing geological knowledge with numerical geology models, or considering mine planning scheduled to feed efficiently the mill or the plant, dumps or tailings.

It is important to consider that this thesis was developed in Chile, a country with a longstanding and strong relation with mining, whose mining culture and personality require a strong applicability of any development. Almost all the ideas presented hereafter start from interviews with miners, visiting operations and practical experiences. In this sense, to implement or show this thesis work in a commercial environment, an example with real data should be developed. However, as was advised before, this is not the objective and goal of this thesis.

So, this thesis, following this spirit, aims to offer a new point of view for mine planning modeling, in a more integrated strategy, in opposition to a new improved version of already enough fast algorithms, even if it could be not as fast or accurate as classical algorithms.

Mine planning.

Mine planning is a branch of engineering that seeks to systematize mine operating decisions and economic assessment, considering variables such as market prices, costs and information on the ore deposit to mine and on the plant processes. It tries to answer questions such as: what is the economic value of the deposit, how to temporarily order the development of the mine, which resources should be allocated.

In purely theoretical terms, it is possible to propose a model that integrates these variables, trying to decide on each portion of the deposit when the material should be mined, when it should be sent to stock or dump, when it should be processed, by what means it should be transported, etc. (Chicoisne et al., 2009). However, in most cases, algorithmic difficulties of including many variables make unfeasible a computational solution (Caccetta, 2007). This is the reason why it is often divided into sub-problems, which are used serially to achieve reasonable and operational solutions.

In open pit mines, to define the operating system phases and benches, the final pit problem is the key (Caccetta, 2007). This corresponds to the geometric border of economic efficiency, which satisfies geomechanical stability restrictions. So, first the final pit is calculated and then, within it, the phases and zones to be exploited are scheduled. In general, classical algorithms define phases as a final pit of an ore price variation; so, they resolve the final pit problem several times for nested ore price scenarios. Note that this procedure is not optimal for aggregated problems, for which all pit shells should be selected at the same time.

The analogous case in underground mines, for example with a caving system, would define the footprint of the exploitation, then the number of draw points to be open, then the best height of draw and, finally, the direction and shape of the production front.

In the present thesis, the “open pit case” will be considered, although the ideas are mainly valid for the underground case.

Figure 1.1: An example of open pit. Escondida, operated by BHP Billiton. Source: author.

Ore body modeling.

The information required by scheduling algorithms corresponds to a discrete representation of the ore body: the so-called block model. This is a regular array of voxels, i.e., pixels in three dimensions. Algorithms work by deciding whether or not to mine a block, if sending it to some process, leave it in a stock pile or send it to dump (Braticevic, 1984).

To develop a block model, rock samples are extracted through drilling. These samples are studied in a laboratory to determine the mineral or metal grades, density and other relevant geological or geometallurgical variables for processing. Certainly, these samples correspond to a tiny portion of the deposit, which must be used to represent unsampled areas by interpolating the sample values. This interpolation (spatial prediction) is often performed via geostatistical techniques, such as kriging. Unavoidably, the kriged predictions are subject to uncertainty, i.e., the interpolated values are likely to deviate from the true values (Chilès and Delfiner, 2012; Wackernagel, 2003).

Mine planning modeling.

Scheduling algorithms previously estimate the economic value of each block, based on this information: grade, density, mining and metallurgical recovery, market price, cost of mining and processing (Lerchs and Grossman, 1965). This is calculated as the greatest margin generated by processing the block versus sending it to the dump. Otherwise, blocks values may also be given directly as an input for mine planning.

There are geometric restrictions for benches, according to the stability constraints that allow the ore body geomechanics (Hustrulid et al., 2006). This is modeled as a precedence constraint in the extraction of blocks: to get a block, it is necessary to remove the top of a cone whose angle is equal to the slope angle.

The optimization problem then maximizes the total value of the blocks that are removed, subject to precedence or slope constraints. Thus, the final pit is determined, that is, the border of the deposit that is worth digging.

To define how to extract the material over time, assuming a bounded operating capacity, one usually solves several final pit problems for scaled scenarios of ore prices (Caccetta, 2007). With this set of nested pits, some of them are chosen as productive milestones or phases, depending on the volume and ability to get operationalized. Finally, the speed and production planning stages are decided, based on criteria of progress versus productivity.

Operatively designed pit.

Now, there is a pit with its phases, but not the paths or roads that make it operational. The end surface of the pit, and the nested pits, may show a wrinkled form, with several noses (material which enters to the pit, projecting its form to the pit bottom), small peaks in bottom and cannons. If transport criteria would be considered, the contours should be relatively straight for the convenience of handling trucks. If rock stability would be prevented, sharp bends should be avoided, because they could concentrate geomechanical stresses. But those situations are not considered in open pit algorithms.

This is solved by a professional, often a mining engineer with experience in road design. He/she uses software drawing, to facilitate the drawing of the paths above the pits or phases, making decisions based on one's experience, geomechanical criteria, balanced distribution of production, etc. In short, the solution relies on expert judgment and not on a mathematical optimization. The process can also last a couple of weeks, which makes difficult to repeat the analysis in other pricing scenarios, operational events or mineralogy. And, in general, there may be a loss of value of between 20% and 40%, in the opinion of interviewed mine planners asked in focus groups as mentioned before.

It is therefore of interest to develop a methodology that can define the final pit and its phases, but operationally, telling the true value of the project, reducing the uncertainty that can provide the designer and making possible to test different scenarios in order to take over the mineralogical, pricing or operational uncertainties.

Metallurgical processes.

The information used by mine planning, from the knowledge of metallurgical processes, is often summarized in the recovery obtained for each block that should be sent to the plant.

For long-term analyses, this amounts to averaging recoveries for the entire deposit, or by zones or geological / geometallurgical units. But what matters is that the recovery is assumed valid for each block, according to its constituent information (Bastid, 1978).

Once the long-term plan is defined, so the ore reserves are assessed, short or medium-term planning must conform to the mineral that is really discovered (remember that geostatistics predicts ore existence in a place, but subject to error, so true ore will be known only when it is extracted). Through stocking mineral, some properties can regulate mineral coming from the front, to homogenize the input to the process (Caccetta, 1986).

Geological and mineralogical uncertainty: closing the circle.

These scheduling decisions generating a production promise, in practical term of the operation, are never met. This is not necessarily due to a miscalculation, but to the fact that the mineral to be extracted is predicted by models that use incomplete information (drilling samples), so the spatial prediction is subject to an error.

To characterize this geological uncertainty, instead of constructing a single kriged block model, geostatistical techniques allow the construction of a large set of scenarios, through stochastic simulation, as embodiments of the distribution of spatial data (Chilès and Delfiner, 2012).

The most widely used methodology to consider the uncertainty is to solve the base estimation model, i.e. the average of the scenarios or the kriged block model, then check how the economic results or the reserve definition would change if the block model were one of the aforementioned scenarios. It is then possible to determine some indicators of robustness of the original solution, from the worst to the best cases. It should be noted that, to implement this, it is necessary to go through the design process, which, as mentioned above, can last a couple of weeks.

However, this is not a method of stochastic optimization as such. That is, the solution is not obtained by solving a problem whose formulation includes uncertainty.

1.1. Three problems

In the above introduction, the entire process of mine planning and evaluation of reserves is raised. Here the three basic problems, which are of interest in this thesis proposal, are described and the main difficulties of each problem are clarified.

Some claims presented in this section are based on a market research developed by the author. The universe represented is "*mine planners and plant planners from medium to big mines in Chile and Peru*". The objective of the study was to identify needs of mathematical models for mine planning. Ten focus groups were made, each with between 4 and 6 senior mine planners.

1.1.1. Operatively designed pit

What is the difference in value of the project, from the result of a planning software to the design of roads, ramps and switch-backs?

After defining the final pit and its phases, it is necessary to operationalize the pit, by drawing roads, ramps, switch-backs (zig-zags), berms, slopes, etc. This is done manually, but aided with design software, a process that can last a few weeks for a large mine (Araujo, 1987).

Roads need to be carved on the walls of the pit, or supported by ground. And often this should be done at the cost of either additional waste extraction or mineral left in the mine for a period that is not optimal.

In this drawing process the project may decrease its value between 20% and 40%, in the opinion of mine planners interviewed by the author.

Then there are at least these problems:

- The project value is overestimated by classical discrete algorithms (until 40% in planners' opinion).
- Expensive and senior specialist time is used in a handicraft work.
- Solutions are strongly dependent on the designer preferences, knowledge and experience.
- It is not feasible to analyze many scenarios, due to the time needed to accomplish the analysis.

1.1.2. Operative geological modeling

Is the recovery linear with the concentration?

The commonly used optimization algorithms rely on the assumption that the metallurgical recovery is linear with the fine, i.e. the product of grade and density.

But this is not necessarily correct. For example, consider the effect of arsenic, from a mineral such as enargite, on a pyrometallurgical process, which can reduce the recovery from a chemical point of view or even be not processed, under environmental restrictions (Habashi, 1969).

However, if the assumption of linearity of the recovery in the fine is invalid, it is not possible to define the value of the project as the sum of the economic values of the individual blocks. This breaks the base scheme of modern optimization algorithms pits.

Another interpretation is that the value of a block depends not only on its constituent information, but on the decision process of its destination and the rest of the deposit, but mainly from their close relative; that is, a nearly Bayesian vision.

Two questions were presented to plants specialists in several mines:

1. What it is more difficult to resolve: mineralogical or grade variability? They, in all cases, answered “mineralogy”.
2. Considering such an answer, could it be a good idea to order the mine production considering mineralogical or alteration zones? They said, in all cases, “yes”.

This small research gives the idea to consider what geostatistical models are doing. Those models in general are used to interpolate information in space, from some points where samples are collected, irrespective of whether the points represent ore or other information like animals, people, temperature, clouds and so on. So, this leads to the next question:

Do geostatistical models recognize objects like an alteration zone or a fault?

It seems that, despite they could predict the object boundary very accurately, the answer is no.

1.1.3. Geological uncertainty

There are three sources of uncertainty in classic mining: (1) the prices of ore and inputs to obtain, (2) operational events and accidents, and (3) geological uncertainty, as one ignores what is really available in the deposit.

Common to all of them, it is usual to define a set of scenarios and expose them to solve the problem of deterministic optimization. That is, the robustness of the deterministic solutions is studied, rather than a solution that is obtained in stochastic terms as the optimum.

Here we have chosen to address the problem of uncertainty associated with geological resources, for two reasons:

• The modeling of the other two issues raised here, design and integration of the plant, are related with how to represent the geometries in the pit. This is the context of geological information, not the behavior of prices or operational events.

• Impact and planning under economic uncertainty (like prices and costs) and operational events (like accidents or production unexpected stoppings) are matters that are more studied than geological variables.

1.2. Seminal idea: topological representations as an efficient strategy for integration and operativeness of mine planning, design and geological modeling.

This section shows some results and concepts presented in a mining congress. Despite that the first objective of these ideas follows the classical paradigm to improve velocity and scenarios, which is not the paradigm of this thesis, it was a seed for the ideas presented in this work.

1.2.1. Reloading floating cones as an interesting strategy

In mine planning under uncertainty, a major problem is the time needed to calculate solutions. There are two contexts where it happens:

- When new information is acquired by new drilling or by the operation itself.
- When the mine planner wants to include uncertainty in the optimization algorithm.

In both cases, more replications for scenarios of geology, economy and events, should to be done. Such a work could be expensive in time, linearly on the number of replications.

So, one idea tested when this research started (Reyes and Emery, 2012), was to develop a simulated annealing algorithm. This methodology, for certain conditions, converges in distribution to the optimum, sometimes faster than deterministic methods (Lantuéjoul, 2002). Remark: This is the old paradigm, to find a faster algorithm.

To compare time process, in the "final pit" problem, a simulated annealing determined by the method of floating cones (Carlson et al., 1966), was developed. This method looks for a set of cones, pointing downwards, whose angles satisfy the face slope and whose union approximates the economical region of the ore body. It is clarified deeper in the next chapter.

Remark: a final pit under a classical discrete algorithm, which decides in dichotomy in each block whether it would be mined or not, defines the shape of the open pit with a set of blocks on its boundary. Under a floating cone structure, the final pit is characterized just by a set of vertices of cones, therefore the representation is much simpler.

For example, consider a block model of 1.85 million blocks, representing a disseminated ore body like a porphyry copper structure. We shall compare the results of two algorithms. First ***fpsolver***, a pseudo flow type, similar to Lerchs & Grossman. Second, a floating cone under a simulated annealing structure.

Method	Blocks Model (millions)	Value	Time (secs)	Pit boundary
fpsolver	1.85	4.1*10^14	21.7	92.202 blocks
cones (annealing)	1.85	2.1*10^14	33.2	490 cones

Table 1.1: comparing results for a final pit calculation. Note that fpsolver (a kind of Lerchs & Grossman algorithm) achieves the optimum and floating cones does not, but in this case the representation of the final pit is considerably cheaper. Source: author.

Floating cones under an annealing schema achieves almost the 50% of fpsolver solution. Note that simulated annealing was stopped in time, but it could theoretically achieve in distribution, the same solution as fpsolver, if it would be free to iterate.

Does this apparently lack of capability of floating cones to achieve the optimum enough reason to discard it? Note that this difference is comparable with the mine planner's opinion that there is a difference between 20% to 40% between Lerchs & Grossman kind algorithms and the operative mine with designed roads, ramps and benches.

Is there something to rescue from the floating cones results? The representation of the final pit is considerably cheaper. Fpsolver needs 92,202 blocks to represent its boundary, while the floating cones only require 490 cones, each of which is defined by one block corresponding to its vertex. So, if it would be necessary to consider geological uncertainty, through the generation of geostatistical simulations, this could be done in a more straightforward manner by focusing on cones, not on blocks. Note that the final pit obtained by the fpsolver solution could be converted from blocks to cones, but the expensive block representation remains inexorably present during the optimization problem.

Furthermore, despite that in this case the cones were defined in a very simple fashion, with just their vertices, more complicated structures (pseudo cones) could be used, by considering for example some space for transporting material in the bottom.

A similar parametric solution based on coniform objects can be used to model geological domains (e.g., lithological, alteration, mineralogical and/or structural domains) in the ore body, instead of resorting to a discrete block model representation. The pseudo cones can have different orientations (upwards or downwards) and shapes, depending on the geological processes to be modeled, leading not only to a cheaper, but also to a geologically more meaningful representation of the ore body.

So, from here, two main ideas arise, with one starting point: **parametric pseudo cones**, which can fit transportation needs and can model the ore body in a more synthetic way.

1.2.2. Parametric cones

A cone is parameterized by its height $h = z_1 - z_0$, base radius R (or it face slope), and the position of its vertex or center. A point c in the 3D space belongs to the cone if

$$c \in C = \{(x, y, z) \in \mathbb{R}^3 / z \in [z_0, z_1], (x - x_0)^2 + (y - y_0)^2 \leq (R(1 - (z_1 - z) \operatorname{cotan} \varphi))^2\}$$

where

$$\tan \varphi = \frac{h}{R}.$$

Remark: this parameterization can model cones pointing upwards and downwards.

This definition could be slightly modified to accept space for transporting vehicles in the bottom of the pit, which is defined by introducing a lower bound r:

$$\begin{aligned} c \in C = \{(x, y, z) \in \mathbb{R}^3 / z \in [z_0, z_1], (x - x_0)^2 + (y - y_0)^2 \leq (R(1 - (z_1 - z) \operatorname{cotan} \varphi))^2, \\ R(1 - (z_1 - z) \operatorname{cotan} \varphi) \geq r\} \end{aligned}$$

or even enlarged in the XY plane, transforming the cone into something that could be called a "***pseudo cone***".

This is a first step to find an operative final pit, because we are imposing that a truck should have enough space to move in the bottom of the pit.

1.3. Problem setting

As presented before, three problems would be involved in this thesis, but only the first two will be deeply developed. They seem to be unconnected, but they are actually much related. In fact, this relationship and this joint model is the core of the thesis presented here.

Problem 1.
Find an operative final pit in terms of design (ramps, benches and slopes), by using a parametric modeling.

Problem 2.
Find an operative geological representation of lithological, alteration, mineralogical and/or structural domains, by using a parametric modeling.

Problem 3.
Solve problem 1, considering geological uncertainty.

Note the following ideas:

(1) If the geological model were a single mineralization domain with the shape of a cone pointing downwards, with a 45º slope, then the mine will match this cone. So, it is suspected that there should be a relationship between geological geometries and mine geometries.

(2) To satisfy metallurgic plant (or mill) needs, production scheduling could be ordered considering the mining of geological units, with a homogeneous mineralogy, instead of looking for a constant grade. Again, this idea suggests a match between the mine parameterization and the geological parameterization.

(3) In a dump, an acid water drainage could be generated. It could be modeled as several upwards cones, merged under morphological operators, the same as geological model.

Therefore, despite that the above problems and the ideas to solve them could be seen as separate matters of study, they in fact belong to a main idea: uses geometry and topological operators.

Chapters 3 and 4 will present algorithms to solve problems 1 and 2. Problem 3 is implicit in such solutions, considering a Sampling Average Approximation algorithm (Shapiro et al., 2009).

1.4. Key idea of the thesis

As it was presented in the introduction, this thesis tries to develop a methodology inspired in mining needs, instead of algorithmic capabilities.

Remarking this, the main contribution of this work is to show a different objective for mine planning algorithms, which, in order to be useful in mining engineering, must include design. And they should also consider geological information, because the mine must feed the plant or the mill.

In order to achieve this goal, a methodology is proposed, based on a geometrical parameterization of the mine, through pseudo cones. To find the best solution, or at least good solutions, a simulated annealing algorithm is proposed, because its properties can ensure convergence in distribution (Lantuéjoul, 2012). However, others methods could be implemented, starting from operative cones, like genetic algorithms or topology optimization (Bendsoe et al., 1995).

The second main contribution is the idea to model geological data with parameterized objects and to perform parametric topological operations over them. Again, the cone presents an interesting structure, because it could represent a geological unit, generated from processes like secondary enrichment or upwards pressing. Additionally, it is the base for an operative open pit considering the geological units (for example alteration zones) with an object-based point of view instead of a classical block model point of view. This is important for mineral processing and extractive metallurgy, because they can work efficiently when they receive homogeneous minerals, instead of homogeneous grades. And a third application is stochastic optimization, using the sampling average approximation, through multiple realizations of geological units, which can be used to calculate the main value for an operative pit (in the simulated annealing algorithm). In contrast, the Lerchs and Grossman algorithm calculates an optimum for each block model among a set of simulated models (Lantuéjoul, 2002).

So, operative open pits can be modeled with parametric pseudo cones. Geological domains, such as alteration zones, in order to be processed, can also be modeled with parametric pseudo cones. The solution for a designed open pit depends on a short representation of objects, the same for the geological model, therefore simulations for stochastic optimization methods could be considerably more efficient. And finally, modeling mine acid drainage could use the same structure, a cone of a deposition of a leach process (a kind of secondary enrichment in dump).

The tools used in this thesis, like parametric pseudo cones to model both final pits and geological domains and morphological operators, present a concrete example of how to achieve this objective. However, they are not the only ones and other methodologies could be imported from other engineering fields, like image processing and pattern recognition. In the same way, optimization algorithms based on simulated annealing are not the only possibility: genetic algorithms and coupling from the past, among others, are possible alternatives.

1.5. Objectives and scope

General objective

To make conceptual and methodological proposals to calculate an operatively designed final pit and an operative ore body model, allowing an efficient treatment of geological information and geological uncertainty.

Specific objectives

- To develop a parametric representation of an operative open pit, which includes the design of ramps, benches and slopes.
- To develop a parametric representation to model lithological, alteration, mineralogical and/or structural domains with a geological and geophysical sense.
- To propose optimization and fitting algorithms.
- To join all these concepts in an integrated formulation, which could arrange geological uncertainty.

Scope

The main objective of this thesis is to propose new ideas in mine planning, in opposition to numerical results and theoretical proofs. However, numerical examples will be developed based on synthetic data, which present the main characteristics of real cases.

For the final pit problem, the information considered is a block model, as usually used in optimization algorithms, although this is not really needed and sampling information from drill holes could be used directly. Block models have information on the grades and density for each block, prices and cost data, bounds for slopes, and process recovery. The design parameters under consideration are the road slopes, road width, and switch-back space for turning a truck.

In the geological modeling problem, the information considered is a set of drill hole samples, with a realistic design and the presence of a mineral distribution. The methodology does not start from a geostatistical model but directly from the sparse samples.

Finally, the proposed solutions do not consider the production plan, time, discount rate or scheduling.

1.6. Hypotheses

The proposed research relies on the following hypotheses:

1) Data from some sampling points within the ore body (exploration drill hole samples) allow developing a geological model and a block model that covers the ore body. These data represent the presence of one mineral, density, grade, hardness or others geological or metallurgical variables. Their statistical and spatial distributions depend on geological processes and, from these, it is possible to cluster zones that represent homogeneous geological domains or "units" (rock type domains, alteration domains, mineral zones, or structural domains).

2) A mining project has an objective function to maximize, such as the economic utility or the total production. It is possible to calculate the project value from cost, price, recovery and other economic, metallurgical, mining or geological information. For the final pit problem, no time, discount rate or scheduling will be considered for the production plan.

3) Existing "optimal" solutions for mine planning and geological modeling face problems of operativeness in relation to the definition of the final pit, which does not directly consider the definition of ramps, benches and slopes, and to the block model, which is constructed on a block-by-block basis and lacks a global logic allowing the delineation of volumes of materials with the same lithological, mineralogical or alteration characteristics.

2. Literature review

In the following, the bases of mathematical, mining and metallurgical topics for which this thesis aims at developing an integrated planning model theory, are presented. First, consider the definition of a final pit:

Definition (final pit). A ***Final Pit*** is a compact subset of the ore body that maximizes the accumulated economic value, satisfying slope constraints.

For open pit mines, the most commonly used algorithm to develop a production plan in long-term planning, consists of the following steps:

1. Calculate the final pit for expected ore prices. This is called the "***final pit problem***".

2. Calculate several final pits, with the algorithm used in step 1, considering ore price scenarios. They are called "***nested pits***".

3. Some of these nested pits are called "***push-backs***" and are special pits candidate to be chosen as "***phases***". They depend on certain characteristics, such as the volume of material, ore quantity and horizontal distance to the next push-back. This is necessary to have enough space for truck driving and reasonable volume for production planning.

So, most open pit scheduling algorithms actually are aimed at calculating the final pit. Then phases are just final pits, but for different price scenarios.

2.1. Algorithms for planning open pit mines

2.1.1. Handicraft tools

For a cross-section of the mine, one defines three lines representing the bottom and the sides, which satisfy slope constraints. Then, one moves manually the lines until the enclosed value is maximized. See (Koskiniemi, 1979; Steffen et al., 1970; Stucke, 1970; Reibel, 1969).

This process is slow and imprecise (Steffen et al., 1970). Furthermore, it considers a cross-section of the ore body, so several cut planes could not fit each other.

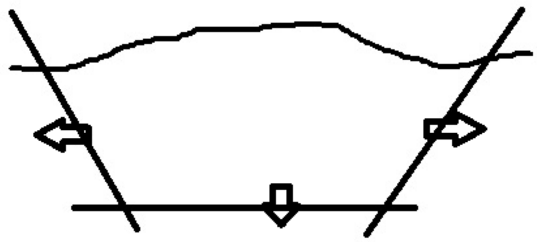

Figure 2.1: Manual procedure to look for a final pit. Source: author.

2.1.2. Floating cones

This technique dates back to the 1960s (Carlson et al., 1966). It was based on the fact that the inclined surface of an inverted cone may represent the slope of a surface mine (Kim, 1978).

The algorithm asks whether to extract a cone whose vertex is a point of the field, extending to the surface. If the net profit of the cone is greater than or equal to a desired benefit, the cone is accepted, otherwise it is rejected. The final pit is the union of the selected cones.

-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10
-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10
-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10
-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10
-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10
-10	-10	-10	-10	-10	-10	-10	-10	300	300	-10	-10	-10	-10	-10	-10	-10	-10
-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10
-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10

Figure 2.2: Ore body cross section example. Source: author.

In the above example, the left block containing +300 does not pay its upper cone, which is essentially waste, adding -350. The same happens with the right +300 block. However, when both cones are removed, the net value is positive + 300 + 300 - 400 = + 100. This happens because when the left +300 block is selected, almost all the waste for the right +300 cone is already removed, leaving few negatively-valued blocks.

It should be noted that, although this is evident in the example, in a real block model it is unlikely to identify with the naked eye such relationships. In particular, a trial-and-error method is likely to be inefficient to obtain this final pit.

Remark: this algorithm could consider geological uncertainty, following the next idea: every time a new cone is considered to be extracted, it can be accepted not only depending on the value added, but also on the uncertainty it brings. This methodology is not developed in this thesis work and no reference was found on this concept.

Gradient type algorithms.

Consider an optimization problem of the following type:

$$min_{x \in X} f(x)$$

where f is a C^1 function (i.e. continuous and with continuous first derivative) and $X \subseteq \mathbb{R}^n$ is a compact (closed and bounded) set. To solve this problem, it is common to resort to algorithms based on point sequences defined by the function gradients. Typically a Lagrangian formulation is used:

1. Start from x_0 (a guess).
2. While f evaluated in x_n decreases (or $\nabla f(x_n) \geq \varepsilon$)
    a. Consider ∇f as a direction where the sequence will be defined.
    b. Consider $h \in \mathbb{R}$ as a sequence step
    c. While f evaluated in the sequence x_n decreases
        i. Define $y_n = x_n + h$,
        ii. If $f(x_n) > f(y_n), x_{n+1} = y_n$.

The rationale is that, at a minimum of function *f*, the gradient is zero. However, if there are local minima, the sequence could be trapped. In the following example, the sequence fails to escape from the local minimum on the right.

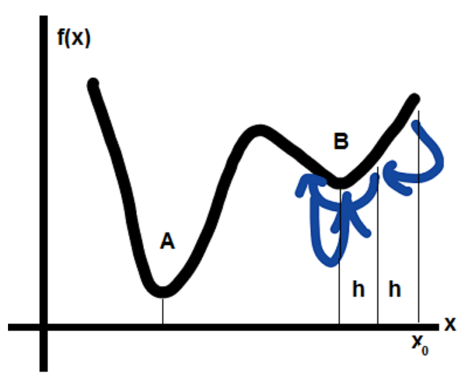

Figure 2.3: Sequence failing to escape from a local minimum. Source: author.

Suppose that the sequence, from point x_0 and its successive points x_n, decreases to a local minimum B, from right to left, step by step with h. Once it passes the minimum, the algorithm fits h by tuning a lower step, making the sequence going back to the right. This improves the solution in each step, but never achieves the global minimum A.

2.1.3. Simulated annealing

Now suppose that the algorithm, once it passes the local minimum B, does not necessarily reject the new worse point, but accepts it with a given probability. At this stage sequence gets worst, but scape from local minimum. Following this procedure, global minimum could be achieved, with a certain probability.

As a stopping criterion for iterations, it is usual to define a lower bound for a parameter that controls the probability of accepting a worse solution, which decreases with the iterations. This parameter is called "***temperature***" and its decrease is called "***cooling***". This is an analogy for a cooling process after a smelting of metals.

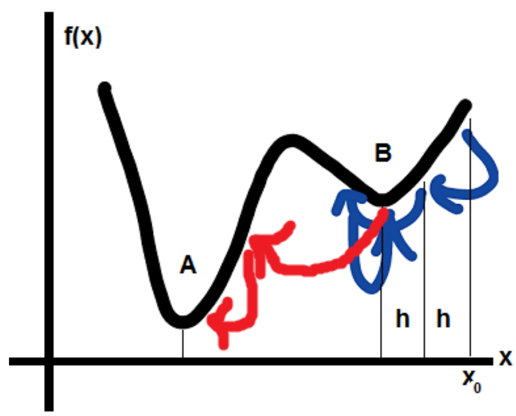

Figure 2.4: Sequence escaping from a local minimum, with a certain probability. Source: author.

Here is an example of simulated annealing algorithm.

1. Start from x_0 (a guess).
2. While f evaluated in x_n decreases (or $\nabla f(x_n) \geq \varepsilon$)
    a. Consider ∇f as a direction where the sequence will be defined.
    b. Consider $h \in \mathbb{R}$ as a sequence step
    c. While f evaluated in the sequence x_n decreases
        i. Define $y_n = x_n + h$,
        ii. If $f(x_n) > f(y_n), x_{n+1} = y_n$

iii. If $f(x_n) < f(y_n), x_{n+1} = y_n$ with probability $p\{\frac{f(x_n)-f(y_n)}{t_n}\}$, where t_n is the temperature parameter.

The proof of the algorithm convergence for given cooling schemes can be found in (Hajek, 1988).

2.1.4. Lerchs and Grossman algorithm and modern developments

Lerchs and Grossman (1965) presented an algorithm to solve the Final Pit Problem, which became the industry standard. Starting from a block model, this algorithm decides which blocks should be exploited, to be processed or discarded. As a restriction for the optimization problem, a slope constraint is imposed. The result is an open pit, with a pixelated surface. See also (Bastid, 1978; Barnes and Johnson, 1980, 1982; Braticevic, 1984; Bley et al., 2010; Coléou, 1987; Cacceta et al., 1986; Phillips, 1972; Wright, 1987).

Let B the set of n blocks that characterize the ore body. Usually, these blocks have a cubic or parallelepiped shape, do not overlap and cover the ore body. For each block, a geostatistical model assigns metal grades, rock density, hardness and other geomechanical, mineralogical and metallurgical variables of interest. A block B_i is characterized by the following parameters:

- C_m mining cost for extracting and sending it to mill, plant, stock pile or dump
- C_p processing cost per mass
- r metallurgical recovery
- p market ore price
- d_i rock density
- v_i block volume
- f_i grade (or equivalent grade) of the ore of interest.

Thus, the economic value V_i of block B_i, for $i \in \{1, \dots, n\}$, assuming that the life of mine is one period, is

$$V_i = \max\left(f_i d_i v_i r p - C_p d_i v_i, 0\right) - C_m$$

The following notation is needed to clarify the formulation of the algorithm:

- $x_i \in \{0,1\}$ represents the decision to mine or not block B_i, where 1 means "mine or remove block B_i from the ore body". Note that, if $x_i = 1$, a cone above B_i should be removed, with a slope defined by stability restrictions. So, in this case, several blocks $B_j \in B$ above B_i, must also have $x_j = 1$, for some $j \in \{1, \dots, n\}$.

- $\Gamma(B_i)$ represents the set of blocks above B_i that must to be removed to mine B_i, satisfying slope stability restrictions. Note that if $B_j \in \Gamma(B_i)$, then $\Gamma\left(B_j\right) \subseteq \Gamma(B_i)$. In term of x_i, a feasible set of blocks above B_i must to satisfy that $x_i \leq x_j,\ \forall B_j \in \Gamma(B_i)$

The Final Pit Problem is formulated as follows:

$$\max_{x_1 \dots x_n} \sum_{\substack{B_i \in B \\ x_i \leq x_j, \forall B_j \in \Gamma(B_i)}} x_i V_i$$

Note that $x_1, \dots, x_n$ are the decision variables in this problem.

Lerchs and Grossman transformed this problem into an equivalent problem of maximum closure. A closure Y is a set of blocks such that, for all $B_i \in Y$, $\Gamma(B_i) \subseteq Y$. It means, in a directed graph where each node represents a block and arcs represent the precedence relationship defined by Γ, the Lerchs and Grossman algorithm finds a group of nodes that maximize the sum of contained values under the precedence constraints.

In the past decades several improvements of this algorithm have been proposed to solve the final pit problem, based on convex analysis, operations research and mathematical programming tools (Matheron, 1975a,b; François-Bongarçon, 1978; Coléou, 1987; Wright, 1989; Underwood and Tolwinski, 1998; Hochbaum and Chen, 2000; Frimpong et al., 2002; Chandran and Hochbaum, 2009; Newman et al., 2010). These approaches allow reducing time calculations, increasing model size capabilities, incorporating varied constraints and accounting for parameter uncertainty, among others. Improvement achieved from those efforts would lead mine planners to calculate more complex scenarios, fitting closer to reality. However, in the author's point of view, the operative design of the final pit remains a critical problem.

Sequencing with Lerchs and Grossman

For sequencing, the most widespread strategy is to repeat several times the final pit problem, for different ore prices, obtaining a large number of nested pits. Then one proposes a life time for the mine, from which a production rate follows. Finally, with other arbitrary criteria, one decides to divide the mine into phases as nested grouping pits, each of which has a common characteristic. For example, the layers may have the same volume or the same value or the same ore-to-waste relationship. Finally, benches are defined so that this criterion is met, which eventually means taking into production in the same period sequenced phases.

2.2. Geostatistics and stochastic geometry

2.2.1. Kriging

Mine planning needs data in the whole ore body, not just in some places where drill hole samples are available. To extend the sampling information as an extrapolation from such data, there are some geostatistical methods that work in a similar way as linear regressions.

Kriging is a family of methods that allow fitting such regressions. They propose to predict the values of a variable Z (viewed as a spatial random field) in a point x as a weighted average of all available data $Z(x_1), \ldots, Z(x_n)$ on this variable. The weights are calculated to minimize the expected squared difference between the model prediction and the real value. The simple version of kriging is:

$$Z(x) = a_0 + \sum_{i=1}^{n} a_i * Z(x_i) + e_x$$

After error minimization, one obtains a linear system with the weights a_i as unknowns:

$$\begin{pmatrix} C(x_1 - x_1) & \cdots & C(x_1 - x_n) \\ \vdots & & \vdots \\ C(x_n - x_1) & \cdots & C(x_n - x_n) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} C(x_1 - x_0) \\ \vdots \\ C(x_n - x_0) \end{pmatrix}$$

where $C(x_i\text{-}x_j)$ is the autocorrelation function of variable Z for vector $h=x_i\text{-}x_j$, which can be modeled by a correlation or variogram analysis stage. Finally, having all these submodels and information, it is possible to predict or approximate the values of the variable, at a point support or over some volume, like a block. This modeling tool therefore allows constructing a block model and is generally enough to be useful for both continuous (e.g., metal grades) and discrete ordinal variables.

Note, however, that this model does not make difference if the data belong to a specific alteration zone in some places. That could be interpreted as the model does not “understand” the data context, missing the geological interpretation of alteration zones. There are variants of this model where nominal variables are considered, which could fit alteration zones. They predict point to point, or block to block, if it belongs to an alteration zone or not. But this is a local point (block-by-block) of view, despite that it uses global information through a covariance or variogram model.

2.2.2. Multivariable data

Grade and density data often lead to univariate geostatistical models. The modeling of multi-dimensional geological data needs to use multivariate geostatistics, becoming a more complex work. To develop such a kind of spatial modeling, in a block model or continuous supports, cokriging tools can be used (Chilès and Delfiner, 2012). To decrease the size of multivariate information set, a factorial analysis could also be implemented (Wackernagel, 2003).

2.2.3. Simulation and co-simulation

Geological variability would be underestimated by kriging prediction, which suffers from the so-called smoothing effect (Chilès and Delfiner, 2012): the block model does not reflect the actual variability and only provides a representation of the expected values of the variables of interest, conditioned to the available sampling information. One strategy to circumvent this issue is to consider geostatistical simulation techniques that aim at representing the spatial distribution of the geological variables, mimicking their spatial variability.

There are several models and algorithms to simulate scenarios (also called *realizations* or *outcomes*), which depend on the nature of the variable under study (continuous; discrete and ordinal; or nominal) and on its spatial distribution, represented through a set of finite-dimensional distributions or, more simply, spatial correlation measures such as covariance functions or variograms.

Among the more widespread models, let us mention:

- The multivariate Gaussian model, in which the variable under study is represented (up to a nonlinear transformation) by a Gaussian random field. Numerous algorithms are available for simulating such random fields, e.g., spectral approaches, turning bands, sequential, or matrix decomposition (Chilès and Delfiner, 2012, and references therein).

- The truncated Gaussian and plurigaussian models, aimed at modeling nominal variables that represent geological domains or “units”, by truncating one or more Gaussian random fields (Armstrong et al., 2011).

- Object-based models, where the random field is obtained by the superposition of objects with random shapes, sizes, orientations and values, placed at the points of a point process. Among this family of models, let us mention the Boolean, dilution and dead leaves models (Lantuéjoul, 2002).

Scenarios generation is important to study the solution behavior on worse or better situations. This is a usual idea, to analyze what would happen with the deterministic solution of optimization problem, in other possible scenarios. We will call this an "*ex post analysis*". As miners says: mine planning starts from a prediction that will never occur.

Another option is to optimize for several scenarios at the same time, through optimizing a mean objective function. This is called "*stochastic optimization*".

Another option that could be used is to simulate over parameters of 3D objects, like size, orientation, shape, merged objects, and so on. If it would be feasible to model geological units with parametric volumes, different scenarios could be performed by choosing stochastically the parameters.

In this case, it is possible to develop a co-simulation considering two major dimensions:

- Relationship between object parameters
- Relationship between objects.

This last concept could also consider the geological knowledge for the estimation of object parameters. Both concepts could be joined by using a Bayesian model.

2.2.4. Stochastic geometry: objects and fracture processes

This section is mainly based on (Stoyan et al., 1988) and (Lantuéjoul, 2002) references.

Stochastic geometry studies the behavior of objects in space, from a probabilistic point of view. It considers that the existence of such objects in space depends on distributions. In particular, if the positions and characteristics of the objects depend on their neighbors, it would be a Markov Process in space.

The simplest stochastic process in space is a Poisson process. It is characterized by an intensity parameter that defines the mean quantity of presence of objects per space unit. Despite its simplicity, this distribution is of big importance because it represents a contrast for testing as a neutral behavior.

The main theorems claim about the convergence of processes that approximate available data.

In object recognition applications, there are some geometric tools used with stochastic processes. Let A and B be two sets of points in the 3D Euclidean space, i.e., two elements of $\wp(\mathbb{R}^3)$ (parts of $\mathbb{R}^3$). We define the following operators from $\wp(\mathbb{R}^3)\times \wp(\mathbb{R}^3)$ to $\wp(\mathbb{R}^3)$:

- **Dilation**: A is dilated by B, noted by $A \oplus \check{B} = \{x - y: x\epsilon A, y\epsilon B\}$, where $\check{B} = \{-y: y\epsilon B\}$

- **Erosion**: A eroded by B, noted by $A \ominus \check{B} = (A^c \oplus \check{B})^c$, where the superscript c indicates the complementary set
- **Opening**: A opened by B, noted by $A_B = (A \ominus \check{B}) \oplus B$
- **Closing**: A closed by B, noted by $A^B = (A \oplus \check{B}) \ominus B$.

Some graphical examples are shown in Figure 2.5 (Stoyan et al., 1988). Note that $A \oplus \check{B}$ defines a kind of augmented border for A, allowing one to identify its limits, while $A \ominus \check{B}$ erodes A, allowing one to separate it from its neighborhood. These two operators could be used to identify objects, enhancing or enclosing their shapes. A_B and A^B could be used to merge two objects into a new one. Furthermore, B could be chosen so that this new object satisfies geomechanical and mining considerations, like an operative cone.

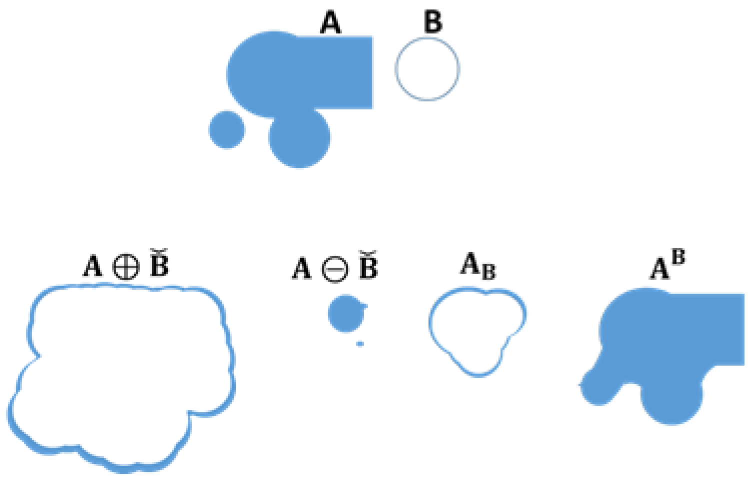

Figure 2.5: Examples of operations (Source: modified by the author from Stoyan et al., 1988). Note that under erosion and opening, components that overlap are separated, while small components and roughness vanish or are reduced. Dilations and closings produce closing up of gaps, concavities vanish or are reduced and clusters of small particles are merged.

These morphological tools could be used in three ways:

i) **Interferences** of a set of objects could be modeled. For example, when two alteration domains are present in the same place. This could be considered as co-simulation.

ii) **Object recognition**. By using erosion, which discovers enough separated objects.

iii) **Object continued fusion**. Two objects could be continuously merged by doing the following: object union, then a dilation with a small ball and finally, eroded by the same ball. This procedure will be mentioned in the automatic open pit design chapter.

Furthermore, this concept could define a Hausdorff distance between objects (Lantuéjoul, 2002). This is very interesting, because a metric space of objects could be performed and properties of density and existence of a countable base of objects studied. This could be used to ensure that the algorithms, like the ones that will be presented in this work, converge. For example, a geological unit (a subdomain of an ore body) could be approximated by a succession of set of geometrical objects.

2.3. Stochastic optimization

The concepts outlined here are presented in detail in Shapiro's book (Shapiro et al., 2009) and Birge's book (Birge et al., 1997). Other interesting references are (Artzner et al., 1999; Kleywegt et al., 2002; Linderoth et al., 2006).

They correspond to optimization methods including random variables in the objective functions or in the constraints. Consider the following general problem

$$\begin{gathered} \min_x f(x,\theta) \\ \text{s.t. } x \in X(\theta) \\ \theta \; random\; variable \end{gathered}$$

There may be uncertainty in these cases:

• **In the parameters of the objective function:** when some of the parameters associated with the variables, such as a multiplier, are random variables. For one embodiment of such random variables, one could dispose of a deterministic (conditional) version of this problem. However, the usual technique is to solve a deterministic case based on the average of the objective function. An example:

$$\begin{gathered} \min_x f(x,\theta) \\ \text{s.t. } x \in X \\ \theta \; random\; variable \end{gathered}$$

In mining engineering, there may be uncertainty in the mining cost, plant costs and mineral prices.

• In the **objective function**: it refers to the case in which the shape of the objective function is random. For example, the expectation or the variance of the project value:

$$\begin{gathered} \min_x \mathbb{E}_\theta f(x,\theta) \\ \text{s.t. } x \in X \\ \theta \; random\; variable \end{gathered}$$

In this case, the objective function is deterministic, but could be time expensive to calculate.

Most solving techniques seek to transform a problem with uncertain parameters into a deterministic equivalent problem. In some techniques, the worst case is considered, in other techniques “simulations" are averaged, besides attempting to justify some indicators as objective functions, valid for modeling risk.

In none of these techniques adaptive solutions are proposed. By contrast, the optima are good proposals for many cases.

Final pit example

Consider the final pit problem and an ore body represented by a block model. It is of interest to maximize the value of the sum of blocks to be extracted, some of which will be sent to the processing plant and others to dump. There should be an upper bound of slope angles. The classical modeling is to decide for each block, whether it is removed or left in the mine.

The value of extracting the ore to the final pit project would be calculated as the sum of the values of the blocks that should be processed, minus the costs to move those blocks necessary to clear the way to the profitable but unprocessed blocks.

The decision of extracting and eventual processing of each block is based on uncertain information. This is due to the fact that the geological and mineralogical properties assigned to the block by geostatistical models bring a stochastic error that arises from attempting to represent a universe of many points, from a few samples in space.

Also the expected price for the extracted ore is volatile and can change before starting or during the project. This variable can be known accurately in the past, but its future values are uncertain.

The third component is commonly considered to study operational uncertainty. That is, the discrete events that occur exceptionally and impact the production. For example, accidents, equipment failure, rock bursts, climatic events, strikes, etc.

Remark: the uncertainty of the demand may also be of interest, but here the overall production is assumed to be hired.

A possible optimization problem, considering uncertainty, would be to seek to maximize the expected value of the business:

$$\max_{x_i \in \{0,1\}} \mathbb{E}\left(\sum_{block\ i} \max\{Rx_i, 0\} - c_m x_i \right)$$

with R a stochastic distribution depending on the price and the geostatistical model

It is interesting to ask why using the expectation of the objective function, rather than simply outputs (simulations) generated and take the average scenario to maximize it. Or, why not optimize the worst case:

$$\max_{x_i \in \{0,1\}} \min_{R} \max\{Rx_i, 0\} - c_m x_i$$

There are two answers to that question. The first is that the choice of the form of the objective function (expected value, versus optimizing a worst case) will depend on the interest of modeling and risk aversion that has the stakeholder. The second, for the expected value, algorithms and a theory have been developed, while for the worst case it is possible to change variables and transform it into a robust optimization problem (see below).

Probability bounds

It corresponds to another classic structure optimization problem. Suppose the following example.

• Decision: *x* level of production

• Deterministic parameters: *c* unit cost, *A* production matrix, *b* demand.

• Z events in production, due to geological or geotechnical uncertainty. In this case, Z is the gap unmet demand:

$$\min_{x \in R^n} c'x \colon \mathbb{P}(Ax \leq b + Z) \leq \alpha$$

For example, one could consider that the probability of default on the production plan in an amount *Ax – b*, or higher, is at most α.

Sampling Average Approximation (SAA)

Suppose it is of interest to optimize a production plan *x*, maximizing the expected value of NPV:

$$\max_{x \in X} \mathbb{E}(\text{NPV}(x, \theta))$$

To address this problem, properties of the Central Limit Theorem and the Law of Large Numbers could be considered. The SAA method relies on these theoretical developments, as an approach to the solution of the original stochastic problem, solving problems of deterministic sequences.

For example, consider a parameter θ representing the geological uncertainty. The SAA algorithm would consist of the following: several ore body simulations (block models) are generated so as to give a set of θ values, the NPV is evaluated for each θ and *x*, and the average NPV is taken. Finally, the *x* that maximizes the NPV is chosen. This method, which is apparently very simple, could be time expensive if it requires trying a lot of cases.

Risk indicators

When the objective function is random, algorithmic developments suggest approximate approach problems from an indicator as the expected value of the objective function. In the case of a mining project, the expected NPV, expected production and expected losses.

Such an indicator, expectation, models the notion of representative value, central or point estimate of the objective function, but leaves aside the risk, variability or uncertainty.

If one would like to choose a production plan *x* that minimizes the risk of loss, it would be necessary to consider a minimization problem of variance of the NPV (or production).

The theoretical question that arises is whether such an objective function, variance, has good properties for the optimization problem, has a unique solution and can be reached with a convergent sequence.

In formulas, we can compare the approach to maximize the expected NPV versus the one that minimizes risk (variance):

$$\max_{x \in X} \mathbb{E}(\text{NPV}(x, \theta)) : Var(\text{NPV}(x, \theta) \leq t$$

$$\min_{x \in X} Var(\text{NPV}(x, \theta)) : \mathbb{E}(\text{NPV}(x, \theta) \geq r$$

The first optimization problem maximizes the expected value of NPV, subject to its variance does not exceed a certain bound *t*. The second optimization problem minimizes the risk of NPV, subject to the NPV is expected at least to a certain dimension *r*.

Note that variance has the problem of being too large when there are extreme cases that could eventually be outliers.

Other examples of risk measures

To model the notion of loss, one might consider the percentiles of a distribution, for a given level of probability.

- The α percentile level is the value z_α of a random variable such that the cumulative probability is α. That is, the probability that the random variable Z is smaller than z_α is α.

- In the context of finance, the α percentile level is called ***Value at Risk*** and denoted *V@R*.

$$V@R_\propto(Z) = inf_{t\in R}\{t : \mathbb{P}(Z \leq t) \geq 1-\alpha\}$$

- The ***Average Value at Risk*** (*AV@R*) is the average of the values of the random variables, bounded above the *V@R*.

$$AV@\mathrm{R}_\alpha(Z) = \mathbb{E}(Z/Z \geq V@\mathrm{R}_\alpha(Z))$$

or

$$AV@\mathrm{R}_\alpha(Z) = inf_{t\in R}\{t + \alpha^{-1}\mathbb{E}(Z-t)_+\}$$

It is also noted ***Conditional Value at Risk*** (*CV@AR*).

Ex post analysis

The former was the summary of stochastic optimization methods. In practice, the stochastic modeling of uncertainty is not treated in this way, but rather the sensitivity of solutions of deterministic optimization problems for the mean data is studied through different scenarios.

Specifically, a solution to the problem of final pit and phase sequencing is obtained. Then it is operationalized with mine design, Capex and Opex are added to give the value of the business. Another scenario of uncertain variables (prices, operation and resources) is then generated and the solution obtained for the original case is evaluated; i.e. the business value is determined, assuming that operating decisions are kept, even if they are not efficient in the new scenario. This process is performed for some cases, and typically concludes case scenarios in terms of their robustness.

2.4. Mine design

This is a mine engineering process that supports the calculation of the following (Araujo, 1987):

1. Mine Recovery
2. Dilution
3. Security
4. Cost
5. Economic profit

6. Reliability of production estimates
7. Road and facilities architecture.

This definition is close to mine planning. The difference is that mine design rather thinks of concepts and actions related to the operation of a mine. In the case of open pit mining, this is the design of roads, ramps and benches. For underground mining, one also adds ventilation, power transmission, communication and transport of inputs (SME, 2002).

2.4.1. About open pit design.

The concepts outlined here are presented in more details in references (Araujo, 1987; Darling, 2002; Hustrulid et al., 2006; Parlamento Chileno, 2001). Open pit mining systems are generally applied to low grade deposits. Some standard features include:

- Production rate greater than 20,000 tpd.
- Moderately selective since it has the ease of emptying the waste dumps.

The main challenges in the design of surface mines, some of them close to mine planning, are:

- Managing the ore/waste ratio and its evolution over time
- *Location of access, ramps and production*
- Design of equipment fleets
- Stability of the pit walls.

The parameters and concepts on which the mine design takes over are:

1. Model block diagram
2. Sections and plan views
3. Grade-tonnage curves to analyze the distribution of mineral
4. *<u>Definition of mineral units of interest</u>*
5. *<u>Drawing benches, ramps, access, design</u>*
6. Volume calculation
7. *Production program, preparation mining: production capacity, life of mine, ore reserves*
8. *Equipment selection*
9. *Human resources*
10. Services: ventilation, drainage
11. Cost and investment
12. Final economic assessment.

From the 1st to 12th concepts shown above, numbers 4, 5, 7, 8, 9 (in italics) require decisions that could be modeled, while the other numbers represent calculations after such decisions. And among them, the 4th and 5th concepts (underlined), which are the subject of this thesis work, are the starting point for the others.

The algorithms developed in software and papers are concentrated on the 7th concept, but the 4th and 5th are generally defined or decided by mine planner or mine designer criteria, which are biased to their experience, preferences or superior orders. For example, Minera Escondida, a copper mine in northern Chile, defines a strategy focalized on highly safe operations. In order to satisfy that strategy, ramps are wider than in other mines, requiring a low main slope and bigger transport needs, especially in gangue. So, the design strategy is a strong factor in the project value.

2.4.2. Ramps, benches and slope criteria

Ramps or roads

Ramps or roads should meet higher levels of slopes, road safety, and minimum widths. Generally, these roads are two-way and opposing pathways are separated by floes (SME, 2002). At the edge of the ramp, before the steep descent of the slope, berms are generally installed. These structures are smaller than the block size and are outside the scope of the design, but certainly interesting to consider for a finer study.

Ramps should be supported with material. This means that often one either must leave valuable mineral to base the ramp, or cut and extract additional waste from the slope or wall of the pit.

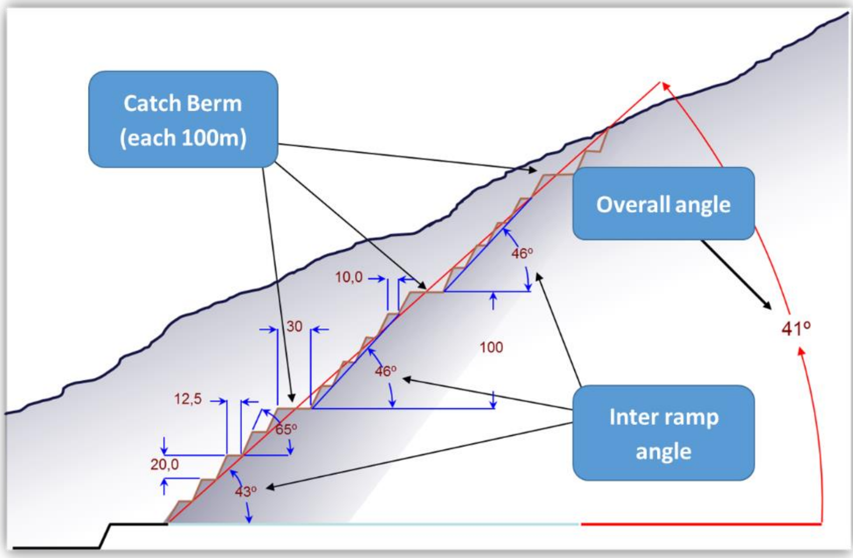

Figure 2.6: Pit design. Three catch berm stop rocks that could fall down. Several benches are defined between catch berms. A ramp is shown down. Source: modified by author from Koskiniemi B. (1979).

Benches and Catch Berm

Benches appear to be roads, but it is not possible to use them to reach the bottom, as they are parallel contours. They are necessary to ensure that the falling stones are detained by horizontal areas. Also, a slack variable defines the overall angle of the pit.

There are single and double benches. In general, they can be parameterized depending on their widths, slope and height of the bench face.

Slopes

The slope corresponds to the angle that one needs to follow, so that it is stable in operation. It depends on the geomechanical stability of rocks, which can change significantly from one area to another in the pit.

There is also a concept of global angle corresponding to the angle that is built between the basis of the pit and the top edge. This value depends not only on the stability of the slopes, but on the decision to settle roads or ramps, benches, double benches or other security structures.

2.4.3. Topological optimization

This is a set of mathematical modeling techniques for the optimized design of structures, mainly considering variables of strength, energy transfer or heat or other physical property. See (Allaire et al., 1997; Allaire et al., 1996; Bendsoe, 1995; Ben-Tal et al., 1993; Suzuki et al., 1991), and (Stoyan et al., 1988).

They are based on statement of optimal control problems, linked to a minimum of energy or other physical constant. Restrictions on functional sets where objective function lives, are required to develop and solve partial differential equations that govern the physics associated with the problem.

Linked to this thesis, two ideas could be further considered:

1. Minimize transport energy, subject to restrictions of stability. This would be a typical use of topological optimization.

2. Address a maximum closure in a continuous space, with restrictions related to the modeling of ramps. This is the reverse of point 1.

These ideas are under investigation and are presented as an alternative method based on operating floating cones coupled with simulated annealing. In fact, they are not developed or used here, but they are noted on each case that they could be used. Here some tools of topological optimization are used, like morphological operators and continuous transformations such as translation, rotation and dilation.

2.5. Metallurgy

The following concepts are extended in Habashi's book (Habashi, 1969).

Metallurgy is a crucial part of a mining project. However, in the valuation of reserves and long term mine planning, the information tends to be naive or misused, approximating recovery on each point as a main recovery of the mine, differencing just if block is mineral or gangue.

Here is presented an overview of unit operations. Some mines could have just a few of them, defining their products with different levels of quality or ending.

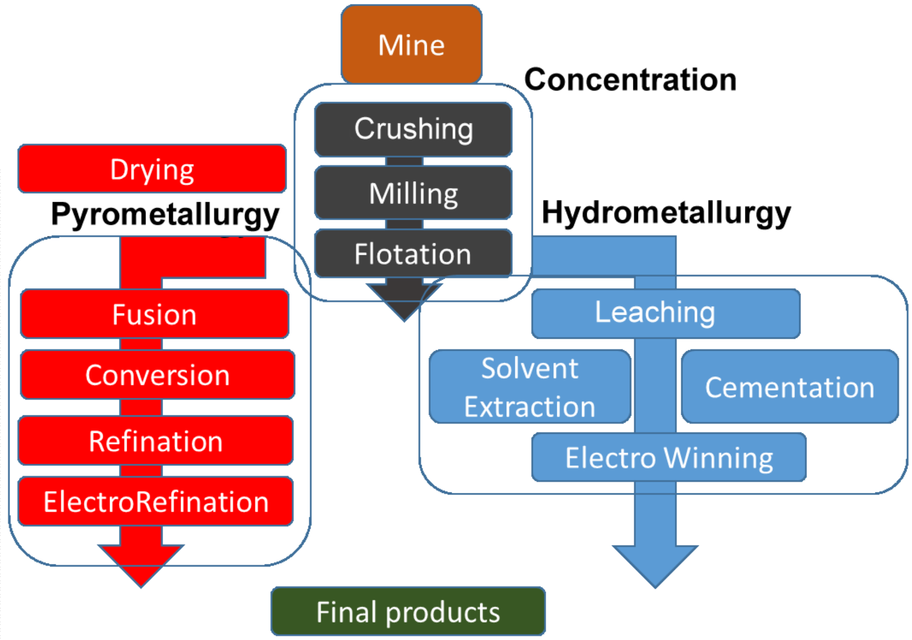

Figure 2.7: Metallurgical processes. Source: author.

2.5.1. Concentration

This process is not considered as a part of metallurgy, but its output could be the final product for several mines. It is a procedure where mineral is cleaned narrowly, obtaining a small granulometry for the next step of the process, pyrometallurgy or hydrometallurgy.

If mineral goes after this process to pyrometallurgy, it will be processed with flotation, which leads to a "***concentrated***" material. If it goes to hydrometallurgy, it will be milled without flotation.

Note this output could be sold and the process could end here. In that case, business gain depends strongly on the energy consumption, which depend on the work index (rock hardness) and the kind of mineral considered. There are some algorithms to minimize the variability of grade sent to mill. But plant workers prefer to consider the granulometry distribution, because this is important to have an efficient milling. Furthermore, mineralogy affects flotation recovery, nonlinearly on the presence of certain composites. So, this shows that efforts to minimize grade variations are not the correct goal. The big objective is mineralogy, which is a part of this thesis work.

2.5.2. Pyrometallurgy

It corresponds to the mineral processing line using high temperatures and redox process. It is characteristic of sulfide minerals, mainly because at high temperatures such compounds are more stable than oxide minerals. The latter are more stable near the ambient temperature. These processes have a high production capacity. They require a significant, but not excessive, energy consumption because they are generally exothermic. They are the oldest mining technique used by man, consisting of several separate stages of enrichment and refining, each of which has specific equipment and processes, generating products that are potentially tradable, but with different added values. The main stages are: smelting (melting), converting (conversion) and refining (refination). The following focuses on the case of copper.

Smelting

The aim of the process is to concentrate the copper content, forming a liquid phase called mate, if possible containing all fed oxidized copper and a liquid phase, called slag copper-free as possible. The concentrate smelting process of copper occurs at temperature of about 1200 ° C, either by:

• Direct sloping warming.
• Suspended in a reaction tower.
• Injection in a liquid bath.

The production of matte and slag can be represented by the following equation:

Concentrate + Flux + Energy => Matte + slag + Gas.

Figure 2.8: Pyrometallurgy. Source: Habashi, F. (1969).

Converting

It aims to eliminate unsightly and other impurities from the bush to produce metallic copper "*blister*" with 98.5 to 99.5% Cu. Conversion is performed in two chemically and physically distinct stages involving blowing air to the molten sulfur phase.

Refining

Generating a product of chemical and physical quality under standards established in electrolytic refining and marketing standards. Blister copper from converters contains about 0.05% S and 0.5% dissolved O. At these levels sulfur and oxygen combine during solidification to form blisters SO_2 (blister) on and within the metal anode casting prohibiting flat and strong surface.

In stoichiometric terms combining 0.01% by weight of S and 0.01% by weight of O produce approximately 3 cm^3 of SO_2 per cm^3 of copper.

The same as concentration happens with pyrometallurgy. The big pain is not grade variability, but mineralogy. Furthermore, process recovery is not linear on grade, because it depends on mineralogy and possible contaminants. And linearity is a basis hypothesis for Lerchs and Grossman's kind algorithms. So, it is really important to understand processes to plan a mine and it could not be taken separately.

2.5.3. Hydrometallurgy

It means for hydrometallurgical processes selective leaching (dissolution) of the valuable components of the ore and subsequent recovery of the solution by various methods. Hydrometallurgy name refers to the widespread use of aqueous solutions as dissolving agent.

The hydro-electrometallurgy includes all of leaching and precipitation through electrolysis, where the electrochemical processes are preceded by hydrometallurgical processes.

There are three main steps of hydrometallurgical processes:

1. Dissolution of the desired component present in the solid phase.

2. Concentration and / or purification of the solution obtained.

3. Precipitation of the desired metal or its compounds.

Chemical reagents used (leaching agents) must meet many properties, for example they should not be very expensive, and should be easily recoverable and be selective enough to dissolve certain compounds. The leaching and solution purification correspond to the same operations as practiced in chemical analysis only on an industrial scale.

Again, grade is not the most important variable, but mineralogy.

Here another idea could be considered, for which an analogous case will be noted for acid water drainage generation. A leaching pile could be modeled as a deposition process, where humidity evolves as a diffusion system. In a perfect pile (constant and fine granulometry), humidity frontier for an acid liquid probably would be a cone with a spherical basis, because of cylindrical symmetry, gravity and capillarity phenomena. So, this is another case where a cone parameterization could play a big role.

2.6. Environment

The concepts here are deeply developed in references (Dold 2009, 2010, 2014; Jambor et al., 1994).

2.6.1. Acid water drainage

What is acid water drainage?

They are runoffs or drainages of liquids, with a certain level of acidity, produced by materials (mineral, barren tailings) left outdoors.

It is the result of a complex series of chemical reactions involving:

• Sulfuric acid generation due to oxidation of sulfides by the combined action of oxygen and water; autocatalytic reactions accelerated by bacterial activity.

• Consumption of acid neutralization reactions with minerals consumers; these reactions result in the precipitation of calcium sulfate and metal hydroxides, oxy-hydroxides and other compounds.

Sources of acid drainage are:

• Pit walls

• Flotation tailings. To date in Chile over 1,000,000 ton / day of flotation tailings are generated.

• Waste rock dumps (<0.2% Cu). To date in Chile, more than 3,000,000 tons / day are generated.

• Extensive fractured zones ("crater") in surface Pit.

• Dumps of low grade sulfide (0.2- 0.4% Cu).

• Tailing dumps.

What problem does it have on the environment?

Acid mine drainage and its partners are one of the main causes of pollution of surface and groundwater in Europe. An estimated 4500 km of waterways in Europe are contaminated by abandoned mine drainage (Dold, 2010). Acid drainage generates two major problems for the environment:

- Acidification itself, which may make unviable life or the use of water from aquifers.
- Easier mineral penetration.

Kinetics

Little development or studies have been done, regarding the chemical interactions occurring within the tailings and waste (Jambor et al., 1994; Dold, 2010). Such interactions are an essential component model to predict the formation of acid mine drainage and to develop effective methods of prevention.

The mineralogical composition of deposited material in a damp has a strong influence on its oxidation processes. Reaction kinetics differs significantly, dependent if sulfides are oxidized by Fe (III) and the coating may hydroxide Fe (III) (Dold et al., 2009). Moreover, the presence of trace elements may generate stabilization of sulfides. Thus, if different sulfides are in contact with each other, electrochemical processes that affect their reactivity are likely to occur (Jambor, 1984; Dold et al 2009).

Most mining operations are surrounded by stockpiles, tailings and waste dumps containing powdered materials or wastes beneficiation process. Dumps generally contain low-grade material that is removed, but not ground (grain size after blasting). These materials may still contain high concentrations of sulfide oxidation and may experience being outdoors, producing acid mine drainage and runoff metals (Dold, 2010). All this together means that it is important to plan dumps and schedule its filling, which could be started from geological units, even before mining.

Multiplying acidification.

For example, if one has an A block having tailings or oxides in a dump, it can quickly release some acid solution rich in Fe (III). When this solution comes into contact with another block B having only sulfides, 16 moles of protons are produced instead of 4 in the oxidation via oxygen. This will oxidize and acidify much faster than only via oxidation of a sulfide block.

Therefore, the grouping of blocks of certain types can accelerate the generation of acid waters or overweight non-linearly with the volume or content.

Neutralizing acidification.

At the other extreme, it can also happen that two blocks will neutralize acidification each other. A real example of this case is in Cerro de Pasco, Peru (Dold, 2010). The rich Zn-Pb tailings called Quiulacocha do not generate significant amounts of acid drainage, due to the amount of carbonates that neutralize acidification through a sulfur oxidation. Therefore, the settings that are located at the dump or tailings, blocks or portions of mine can influence the costs, externalities and project appraisal.

From a mine planning point of view, once it has been decided to process a block, one may change the decision on the other, depending on whether they could neutralize or multiply acidification.

3. Problem 1: Operative open pit design

Designing an open pit mainly corresponds to defining roads or ramps from the theoretical optimized pit. As mentioned before, most algorithms give solutions that are not operative: they have no roads or ramps, they present shape with corners in plan view, which leads to geomechanical instability and difficulties for driving trucks.

The usual procedure to calculate the optimum pit is based on a discrete algorithm like Lerchs & Grossman. Its solution is taken as a guide to draw roads, ramps and benches, using other software to operatize the pit (Araujo, 1987).

This second stage, road design, strongly depends on the experience and preferences of the designer, who must also draw paths by hand in the software, in a process that could last a few weeks for large mines. He/she must decide where a ramp should pass, sometimes leaving valuable ore to sustain the road, sometimes extracting additional waste material, both situations that lead to a loss of value. So, in this design work or path drawing, the shape of the pit, therefore the ore reserves, may vary significantly.

Therefore, an algorithm to optimize the pit should consider not only the economic value, but also the feasibility of its operation.

So, if it could be possible to automatize the design, mine planners could save expensive development time, taking advantage for mining analysis, management or considering other design alternatives. Furthermore, the optimized value could be closer to reality.

Asking users

As a part of the market research developed by the author, mentioned in introduction chapter, ten focus groups with senior long-term mine planners were implemented, in different medium to large mines of Chile and Peru.

This qualitative tool allows researching about general and specific needs, discovering non-obvious insights. The goal is not to validate numerically the confidence of any claim, but to identify new ideas.

In each focus group, between 4 and 6 people participate. They answer open questions and discuss ideas or answers presented by others, always moderated by the author.

They were asked about how ***interesting could be automatizing the design***, answering these concepts:

1. More than getting an optimal and rigid design solution, it would be interesting to get several options to take a qualitative decision.
2. It is difficult to give one optimal solution considering design, because there are branches of styles, depending on the risk aversion, mine strategy and designer's preferences and experiences.

These ideas lead to conclude that more relevant than to optimize, is to give reasonable good design solutions. This opens a line of research to understand design preferences, styles and parameterize mine strategies.

A second remarkable question was "***how big could be the difference between the output from an optimizing open pit software and the output from a manually made design***". They answer

1. I think between 20% and 40%.
2. I don't know.

It is very interesting to note this, because uncertainty from economy (prices, costs) and operative events (strikes, accidents) are always considered and evaluated problems. Until now they did not ask themselves how much manual design affects.

A third question was "***how many times do you need to design the mine, after the optimized open pit***". The answer

1. Between 2 and 3 weeks (in general interviewed people were planners in large mines, with between 4 and 10 million blocks in each block model).
2. If I have to redesign a part of the mine, between 3 and 10 days. We have to do this frequently, but the entire mine just once each year.

This leads to conclude that they are designing manually a big percent of their working time, which give them short time to analyze.

Fourth, "***how much does the design depend on the designer***?" They answer

1. Under the same block model and *ceteris paribus*, each designer will give a different solution, depending on his/her preferences and experiences.
2. I am a risk adverse modeler, so I will draw several bifurcations on roads, to have more than one exit, if something happens. Furthermore, because my company prefers to maximize security, the roads will be wide. All this leads to a flatter and expensive, but safe, design.
3. I want a quick design, so I draw ramps always with the maximum slope, minimum bifurcation, thin roads. This leads to a more vertical and cheap, but unsafe design.

An idea of solution: parameterization of a pit in cylindrical coordinates, considering its main features: roads, ramps, benches and slopes.

Consider a pit like in the next figures. It shows a great symmetry and discards the rough result of a standard software.

Figure 3.1: Palabora open pit, north-eastern South Africa. Source: corporate web page.

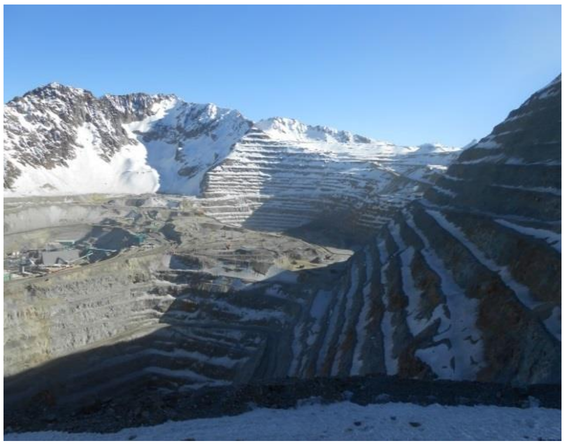

Figure 3.2: Los Bronces open pit, AngloAmerican. Chile. Source: corporate web page.

Note that the roads are mainly straight wards, having few corners. The same happens with the benches. This is because

1. Corners, both concave or convex, are points where rock tensions are concentrated so they should to be avoided.
2. Straight roads are more secure and comfortable for driving.

The overall shape of these mines, due to their regularity, could be approximated by a pseudo-cone, with a pseudo-elliptical basis. Benches are represented by imposing a module condition with a frequency equal to the number of benches, by vertical meters. Ramps are constructed from respecting the slope bounds, using a criterion to respect critical points to be linked or avoided, imposing to go inside or outside of the pseudo-cylinder. Switch-backs are defined by changing the azimuth pattern notation in cylindrical coordinates.

All these ideas will be developed more deeply in the next subsections in a proposed parameterization, which is not the only possibility. These ideas could be a seed for improved or specific structures.

3.1. Reality

From the theoretical point of view, a final pit problem tries to maximize the value extraction and to satisfy constraints of benches and slopes. To be general enough, an optimized pit could have a very irregular slope, with peaks and valleys, where roads could not necessary be implemented nor a truck be put in there.

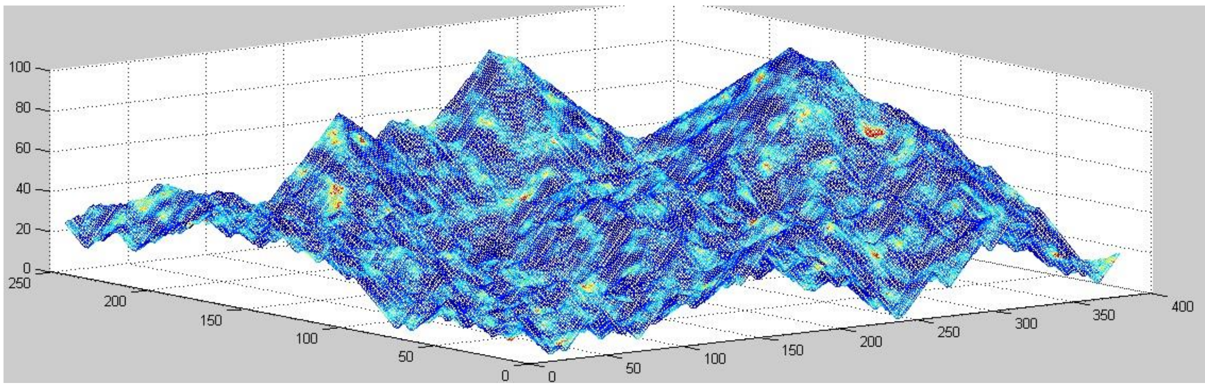

Figure 3.3: A possible final pit from classical algorithms. Source: author.

However, in practice the open pits are more regular, mainly with rectangle-oval forms. This responds to the operational needs of transportation, blasting, pre-cut and extraction with scoops.

Consider the following image of a portion of an open pit:

Figure 3.4: Mina Escondida, operated by BHP-Billiton. Chile. Source: author.

Two faces or walls are shown in those figures, with straight wards benches, no strong concavity nor convexity zones. The roads also follow a very linear pattern, with few curves.

Below is a sample image using a software for design and mine planning. This is the phase 1 of a pit, where the following parameters have been decided arbitrarily: place of entrance, direction of rotation, oval shape that encloses the mineral, no switch-back, slopes of the road (in elevation), etc.

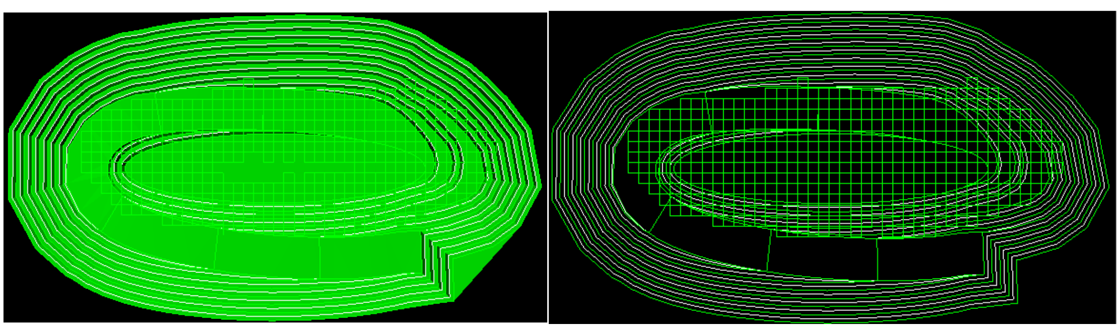

Figure 3.5: Phase 1, plan view. Developed with a design and mine planning software. Note that entrance, road direction and shape are decided arbitrary and drawn by hand. Source: author.

Note that, in order to link phases of the ramp and given the limited available space, part of the mineral cannot be extracted in this phase. For the counterpart, a portion of ore cannot be mined to sustain the road. This generates a significant gap between the promised value and the manual design value, which justifies working to automatize the design of ramps, switch-backs, slopes and benches.

3.2. Model

Let (B_i) with $i = 1, \dots, I$ be a block model. Optimization algorithms decide whether to remove the block and, if so, to what destination: dump, plant, stock, etc. (Chicoisne et al 2009). Therefore, decisions are eminently micro, at a block level.

The slopes to be respected, are macro information, but are satisfied from a micro approach: if one wants to remove a block, all nodes in the upper cone with a slope less than or equal to the acceptable slope should be removed.

This has implications vertically, but not horizontally. No geometric relationship exists between blocks of the same height level. So, there could be a solution with convex or concave features in plan view, which is hazardous for rock stability.

An idea to solve this is to impose a condition of lateral dependence, implemented from the parameterization of curves or closed arcs joining the edge of the pit, in plan view. However, such restrictions break the classic optimization algorithms like Lerchs and Grossmann (Lerchs and Grossmann, 1965).

3.3. Cones as parametric objects.

The floating cones, an ancient technique (Carlson et al., 1966; Phillips, 1972), but still in use, has the flexibility to be parameterized in cylindrical coordinates. Thus, it is possible to construct figures that resemble the classic form of the actual operative pits.

For example, a cone with vertex at (x_0, y_0, z_0) and slope theta, is parameterized as:

$$(x - x_0)^2 + (y - y_0)^2 \leq r(z)^2$$

where (x, y, z) represent the coordinates of a generic point in the three-dimensional space, and $r(z) = k_0 z + z_0$. This is a very symmetrical cone, with a ground absolutely round. Parameter k_0 could be interpreted as the pit slope. In practice, most pits rather have an elliptical shape at their bases. This can be parameterized easily by dividing the squared addends, with the sizes of the axes of the ellipse.

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}\leq r(z)^2$$

Or perhaps a rectangle with smooth edges:

$$\frac{(x-x_0)^\alpha}{a^\alpha}+\frac{(y-y_0)^\alpha}{b^\alpha}\leq r(z)^\alpha, \qquad \alpha>2$$

Let us call this object a pseudo-ellipse (more like a rectangle with rounded edges).

However, in the method of floating cones, they are chosen depending on the value they add. Here a different idea is proposed, which is not definitive, but it is a first approach: consider a single large operative, parameterized, cone.

A second idea is to choose iteratively operative cones, whose roads connect with the roads that were chosen at the previous iteration.

A third idea is the same as the second one, but not trying to connect the road on each iteration, but at the end. Despite that this procedure could be implemented over a Lerchs and Grossman output, in fact it has the problem that after an optimum this will change the solutions when drawing the road.

3.4. Parameterizing an open pit design

3.4.1. Benches

Benches are important for the slopes and rock fall elements. It is possible to parameterize them from discrete changes in the radius of the pseudo-cone, as the depth increases. In formulas, if n is the number of blocks in a vertical face defining a bench, then $floor\ (z\ /n)$ grows by jumps with z, and one can define a radius that varies with z in the following fashion:

$$r(z)=k_0z+k_1floor(z/n)+z_0$$

This definition should depend on the desired height and width of the bench. In general, benches cannot be driven by trucks (because of their size there is no entrance for trucks). If a big rock falls, the benches stop it and prevent possibilities of jammed roads. The width of benches (k_1) will depend on the security strategy and we will take it as a parameter.

3.4.2. Ramps

This is the essential element. It is a downhill or horizontal road were trucks drive to carry load. There is a bound imposed for safe driving. In general, horizontal roads are avoided, because they lose driving time.

First, let us consider a spiral line in the plane, which could be parameterized as

$$x^2 + y^2 = r(\theta)^2,$$

where θ stands for the azimuth and $r(\theta) = k_2\theta$. Parameter k_2 could be interpreted as a spiral growing.

Then, this spiral could be elevated to form a cone, by adding this condition

$$z(\theta) = k_3\theta.$$

Here parameter k_3 could be interpreted as a frequency, vertical advance per round. Mixing this definition, the road slope could be calculated as

$$\frac{dz}{ds} = \frac{k_3}{\sqrt{k_3^2 + k_2^2}},$$

where s denotes the road length. This result depends on the shape of the cone, which is regular and symmetric. Bus it could be more complex. In any case, it could be included as a restriction for an optimization problem, bounding with slope definition.

The pit slope, which is performed by an imaginary cone, with a three-dimensional spiral drawing the road, is defined by

$$\frac{dz}{dr} = \frac{k_3}{k_2}$$

Finally, an operative pit could be defined as a restriction which considers general pit slope, benches and the ramp. The ramp could be modeled as a surface between two parallel spirals, one interpreted as the bottom of a part of the pit wall and the other as the edge of the cliff, separated by the width of the road.

3.4.3. Switch-backs

It is a concept that gives flexibility in the design, allowing changes of direction of the ramp. This allows, for example, standing on one side of the pit to access the pit bottom, using just one face of the pit. It is possible to parameterize the change of the azimuth, at the point where the change of direction should be done. It could be configured based on several criteria:

- Pit height
- Traveled distance on ramps
- Distance to a certain area, for example a blocked zone of instability
- Interpolating critical points, such as a primary mill or a conveyor belt entrance.

In formulas, considering ϑ as the new azimuth variable from the original cylindrical coordinate variable θ

$$\vartheta \coloneqq 2pi - \theta$$

Under this parameterization, one entire pit can be modeled as a vector or arrangement representing the road, plus a set of rings with different slopes and containing benches. So, a ramp is a surface in three dimensions, satisfying the following conditions:

- Ramp width
- Slope bound
- Projecting to the XY plane its curvature is bounded (for turning the trucks and avoiding rock instability).

An example of a road description: a ramp starting from a point (X_1, Y_1, Z_1) on the upper surface of the pit (input). Proceeds according to general pit (pseudo-ellipse) clockwise (looking at the floor), with a slope of 5% (being the upper bound of 10%) over 100m, until a point in Cartesian coordinates $X_2Y_2Z_2$ (or $R_2\theta_2Z_2$ in cylindrical coordinates). After that, continue horizontally attached to the pit during 500m where point 2 in that location has been reached, that coordinate is a switch-back, so the advance azimuth would be in the opposite direction, i.e. against time. After 100m, another switch-back, advance other 500m and return to make a switch-back.

Note that in this example switch-back decisions were verbalized depending on traveled distance. But they could also be defined by areas, points to interpolate distances or strategic locations. Such concepts can be entered as restrictions in an optimization problem, even in probability or punishment.

In the next images, some examples of an operative mine are shown. They were calculated and drawn by the algorithm developed in this thesis work. They consist of a road, a switch back and several ramps.

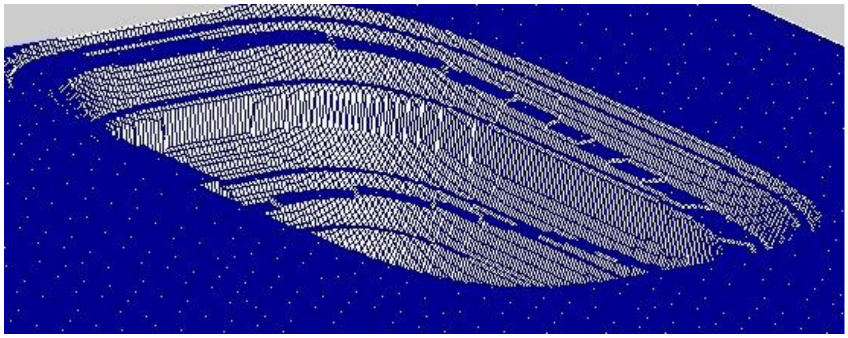

Figure 3.6: Final pit parameterized with benches and ramp. Note that in this pit presents a ramp with a switch-back. Source: author.

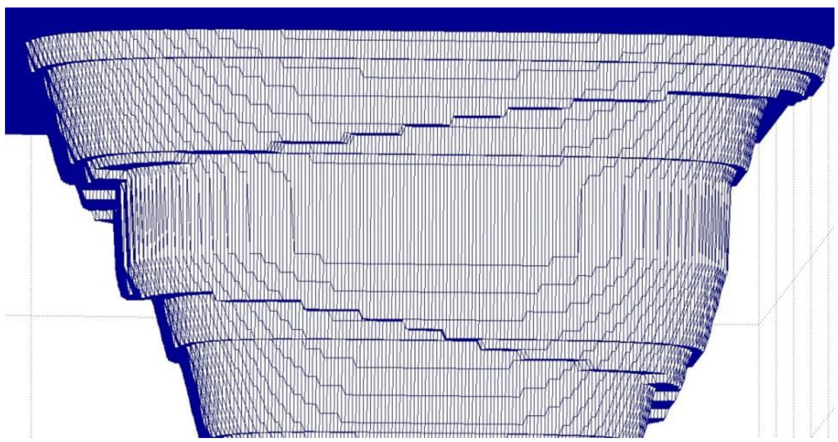

Figure 3.7: Final pit parameterized with benches and ramp. Source: author.

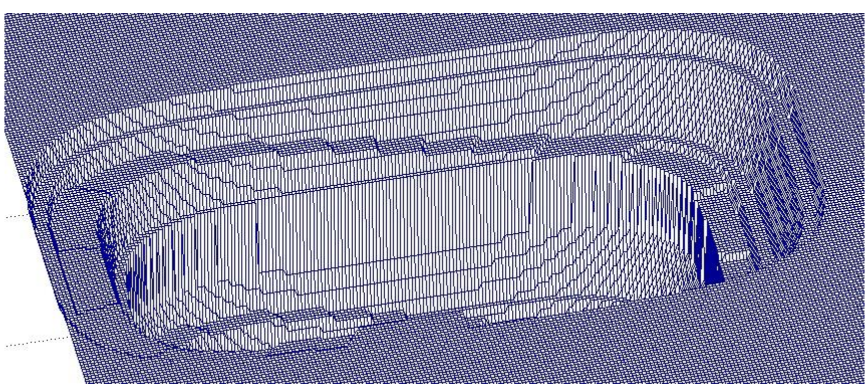

Figure 3.8: Final pit parameterized with benches and ramp. Note in this pit there are no switch-backs. Source: author.

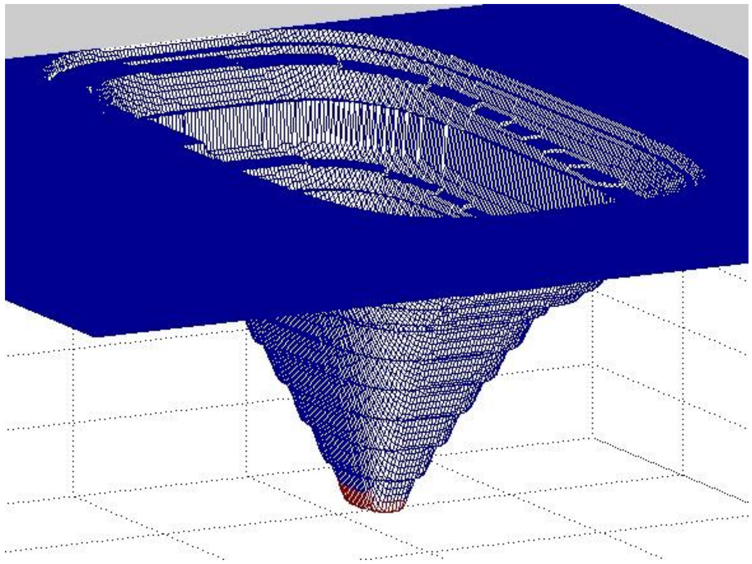

Figure 3.9: Final pit parameterized with benches and ramp. Source: author.

The two simple examples of operative open pits presented above contain a single cone, with parameterized ramps, benches and switchbacks. An open pit mine could be modeled as a union of several cones like this. However, the problem is to ensure that this union of cones has a ramp connecting the entire pit, which could not be the case if the ramps associated with each cone are not forced to meet each other, see next subsection.

In (Brazil et al., 2008), there is a formulation to define and optimize roads in underground mines. In this methodology, a path is defined from parameters of curvature and length of piecewise paths, which define the road once connected. Then, an optimization algorithm attempts to minimize the operation cost or to maximize the revenue or reliability. The path connects points satisfying operative constraints, like road slope and geomechanical stability.

This methodology is different to what is proposed in this thesis work, first because of the feasible space where solutions are searched (paths for Brazil et al. (2008) and merged cones using stochastic geometry tools in this thesis work). Second, for mine applications, (Brazil et al., 2008) define paths for an already defined reserve, while this thesis work attempts to define reserves from the resources. Although the context for (Brazil et al., 2008) is underground mining, their solution could perhaps be used in open pit, defining the reserves to be extracted as the union of cones moving over the path, satisfying slope constraint for the open pit faces. This could be a mix between (Brazil et al., 2008) and some ideas from this thesis work.

3.5. Optimizing with several pseudo cones

The previous section described how to parameterize a pseudo cone to model a mine as a single unit, which could be a poor approximation. One idea to solve this is to merge several pseudo cones, like in the floating cones algorithm. But there is a problem: the union of two or more operative pseudo cones is not necessary operative.

Definition. Let A and B two operative pseudo cones, with a non-empty intersection. Let us call "*contact of A on B*" to $C_{AB} = B \cap \partial A$, where ∂ is the boundary operator.

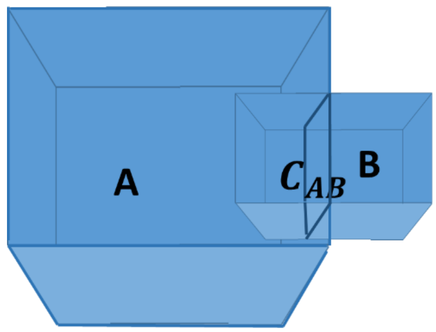

Figure 3.10: Contact of pseudo cone A on pseudo cone B. Source: author.

Let us consider two functions $s_A, s_B: S \to \mathbb{R}^3$ representing ramps, where $S \subseteq \mathbb{R}$ is the parametric space that defines the ramp evolution. For simplicity we will say that ramps start from above and go downward.

If s_A and s_B would match in the right point and with the correct slope, there would be no problem. But this is probably not the case.

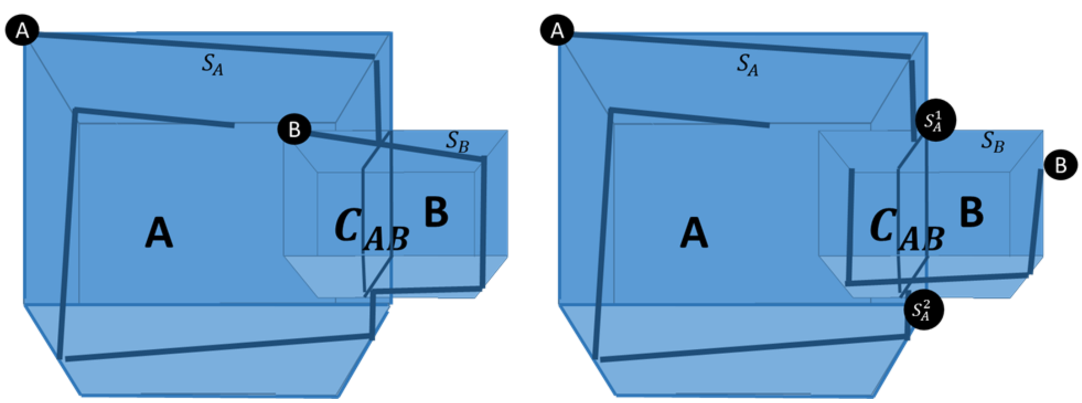

Figure 3.11: Left, ramps s_A and s_B connected a priori, an improbable event.
Right, unconnected ramps. The starting points for A and B are respective black balls. Source: author.

Let us consider A and suppose that it is bigger than B. C_{AB} stops the road defined by s_A in two points ${s_A}^1$, ${s_A}^2$, for which $\langle {s_A}^1 | e_3 \rangle \geq \langle {s_A}^2 | e_3 \rangle$, where $\langle \cdot | e_3 \rangle$ is the projection onto the Z-coordinate.

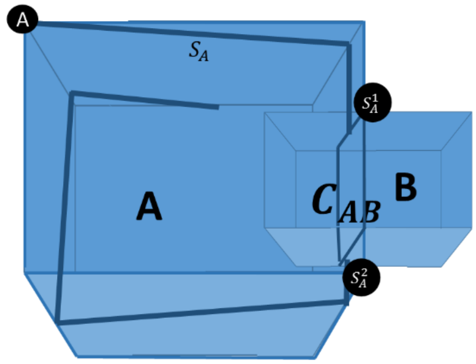

Figure 3.12 Two point where s_A is stopped by C_{AB}. Source: author.

The slope of s is bounded by the driving security slope. Now, to merge these two pseudo cones A and B, the following options could be considered, among others:

1. Eliminate s_B, and redefine s_A in ∂B as a ramp linking points ${s_A}^1$, ${s_A}^2$ with a constant slope on its gradient. Remark: this will decrease the average slope of the ramp, making a slower road for trucks.

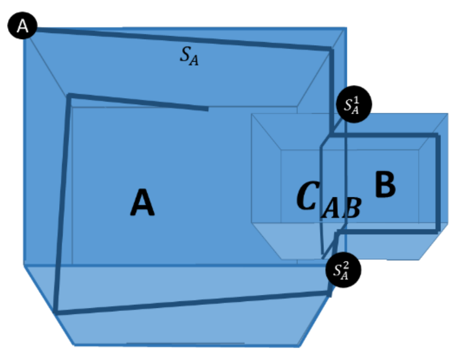

Figure 3.13: Defining a ramp in ∂B. Source: author.

2. Eliminate s_B, and redefine s_A in ∂B as a ramp starting from point ${s_A}^1$, and going to the deepest bottom point. Remark: this will maintain the ramp slope, but could need to travel more path in the pit face to reach the bottom, possibly enlarging the pit, what could be more expensive.

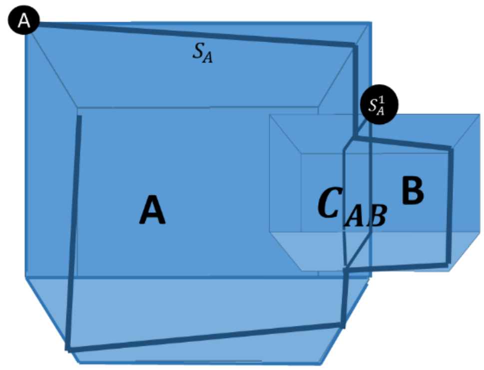

Figure 3.14: Eliminating s_A^2, respecting the road slope. Source: author.

These statements call for the following comments.

i. The merging process may fail if B is not deep enough, as it is possible that s_A goes below the bottom of B and does not reach its external face. This problem can be fixed by adding a constraint to choose B with a depth that makes the road s_A to go through the external face of B.

ii. $\partial(A \cup B)$ is not smooth on C_{AB} (In other words $\nabla\partial A|_{C_{AB}} \neq \nabla\partial B|_{C_{AB}}$), which could generate rock instability in the pit face, among an unsafe road for trucks. Controlling this gradient in the algorithm could define better pseudo cones in a birth (and death) process.

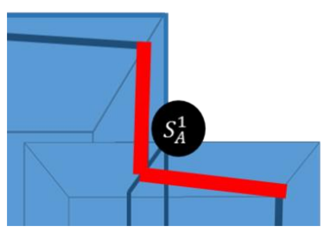

Figure 3.15: Non-smooth corner, an instability problem. Source: author.

iii. Having several possibilities to decide in the road construction could lead to different solutions, which is an advantage for the miner interest.

iv. When the ramps are already defined, $\partial(A \cup B)$ could be smoothed by a merging operator like enclosing $(A \cup B)^b$, where b is a ball whose radius depends on the stability bounds for the pit face. This minds that $\partial(A \cup B)$ is first dilated in b and then eroded in b.

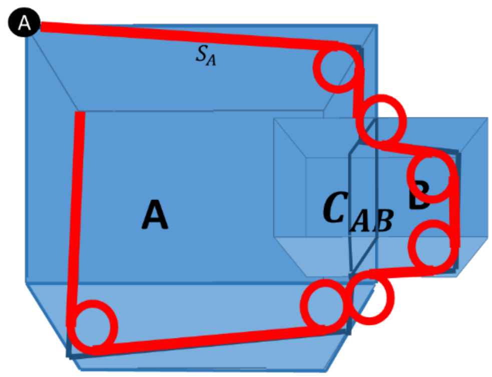

Figure 3.16: Smoothing with a ball. Source: author.

3.5.1. Floating cones with simulated annealing

In the previous section, it was explained how to join two pseudo cones. Under a simulated annealing algorithm, this process could be done iteratively, creating a new possible pseudo cone like in the floating cone procedure. Here is the main idea for an iteration of the algorithm:

- Propose a new pseudo cone, choosing it with a probability proportional to the value added in the remaining part of the ore body for this iteration.
- Accept the new cone always if its value is positive, but if it is negative accept it with a probability proportional to the inverse of the destroyed value and of the temperature parameter of the annealing.

3.5.2. Automatic design from an output of Lerchs & Grossman algorithm

Now the question is: ***could this method be used to create an operative final pit starting from the Lerchs and Grossman solution?*** There could be at least two ways to implement this:

- Define a new ore body, with null or negative value outside the Lerchs and Grossman final pit, and with the original value inside. Then use the above algorithm with parameterized pseudo cones.

- Use the parameterized pseudo cones algorithm, by changing the objective function "*maximize the total extracted value*" for "*minimize the gap between the Lerchs and Grossman's output and the parameterized design optimization*".

Then, it is possible to design from the Lerchs and Grossman solutions, avoiding manual work. However, as claimed before, this algorithm starts by not considering the design, which leads to a solution far from the desirable application, so trying to design over such solutions could be even worse.

3.5.3. Sampling Average Approximation scheme

Simulations are useful to study the behavior of deterministic solutions in different scenarios, which is different to find out the optimum under uncertainty.

Let us consider the following problem

$$min_{\,x\in X}\mathbb{E}_{\theta}(\text{value}(x,\theta))$$

Here X is the set of final pits and θ a geological model. The expected mine value is a proposed objective function, but others alternatives can be considered, like V@R.

For this stochastic optimization problem, a SAA (Sampling Average Approximation) algorithm could be a good option (Shapiro et al., 2009).

But this cannot be implemented under a Lerchs and Grossman scheme, because this algorithm requires a fixed block model for its calculations. Let us consider a block model $(B_i)_{i\in I}$ that contains geological information (position, grade, density, rock type) and defines a valued 3D matrix $(V_i)_{i\in I}$. Rock stability requires satisfying slope bounds for the open pit walls, which defines an ordered relation between blocks called *precedence*. The Lerchs and Grossman algorithm and its variants transform the valued block model and precedence relations into a flux on a tree equivalent representation, where blocks are now nodes and precedence is defined by tree links on branches. Its root, nodes and leaf nodes strongly depend on the difference between block values. So, if the block values change, precedence on trees changes, making impossible to define a unique final pit.

In contrast, the proposed simulated annealing algorithm allows calculating the main value for a set of pseudo cones (and morphological operations) and a set of simulated geological models, with a defined and unique final pit (which could be accepted or not, as a part of the algorithm). So, it is feasible to implement a SAA under this floating pseudo cones algorithm based on simulated annealing.

3.6. Conclusions from automatic open pit design

It is possible to parameterize an open pit mine, including information on its operating design, in particular benches, slopes, ramps and switch-backs. The strategy used is parameterizing under cylindrical coordinates, modeling ramps evolution, switch-backs and slope changes.

The overall plant shape of the pit, because it usually has some regularity, can be obtained by using a pseudo-cone, with a pseudo elliptic basis, or a union of such pseudo-cones. Benches are represented by imposing a condition module with a frequency equal to the number of benches that one wants to impose, by vertical meters. Ramps are bounded with their maximum slopes, but they could be lower to reach certain points. Switch-backs are obtained by changing the azimuth pattern notation in cylindrical coordinates. They could be modeled as a stochastic birth process.

Such tunable parameters could be optimized using a Simulated Annealing algorithm, deciding where to start ramps, which point they should fit, in particular where to define switch-backs. Furthermore, a birth process of pseudo cones could be implemented, as in the "*floating cones*" algorithm. Several techniques could be used to merge pseudo cones, allowing flexibility on the solutions. In particular, topologic operations could be implemented to ensure rock stability and safe driving, with a parametric operator *enclosing*.

Finally, despite their use, support and social validation, the Lerchs and Grossman algorithm and similar tools should be revised and, eventually, strongly modified to consider operation and design, because of the great difference in value against operative solution and developing time of handmade design. In contrast, here a new field of research, under a topological point of view, was proposed. This field of research has been introduced from a conceptual point of view and still requires being tested and evaluated in real-world cases, which is left for future works.

4. Problem 2: Geological modeling

4.1. Introduction: modeling alteration domains

A conceptual modeling of the layout of the alteration domains is needed at a first stage to develop a numerical representation of an ore body. Such a work is usually made by geologists, who divide the ore body into disjoint volumes representing the alteration domains (Ridley, 2013), but this approach is somehow biased to their experience and knowledge, what could be seen as a weakness. An alternative approach is to define the alteration domains through the geostatistical modeling of categorical variables, which seems more rational and quantitative (Chilès and Delfiner, 2012). Here, the ore body is represented by a block model, i.e., a discretization into elementary volumes in which the properties of interest (alteration, rock type, mineralization, grades, etc.) are interpolated from the available sampling information. In other words, the first approach considers the alteration domains as objects that are interacting with each other, whereas geostatistics fits numerical data in a local scope and does not take awareness of modeling objects, missing the geological interpretation of alteration zones (Figure 4.1).

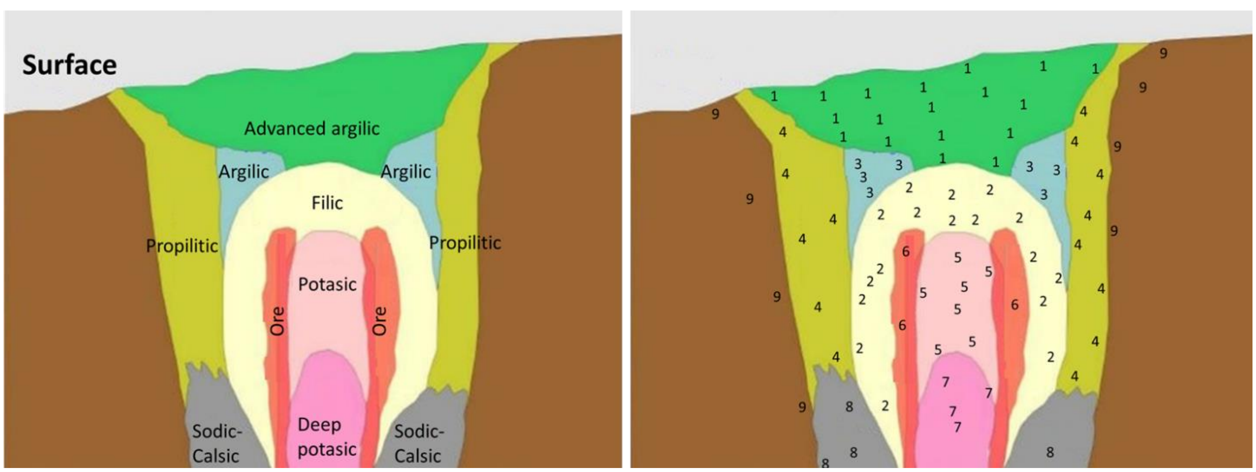

Figure 4.1: Same group of alteration domains, seen from geological and geostatistical points of view. Source: modified by author from Ridley, J. (2013).

Both approaches therefore have weaknesses and strengths. This gives the idea to develop a model of alteration domains (geological point of view) by fitting numerical parameterized objects (Figure 4.2). Such a model could be useful to understand the ore body, to simulate the distribution of mineral resources, and to find out the optimal pit and phases for mine planning, e.g., using a parameterized floating cones algorithm as the one proposed in the previous chapter.

Furthermore, joining two worlds, geostatistics and mine planning, in a common conceptual representation of the reality: parameterized.

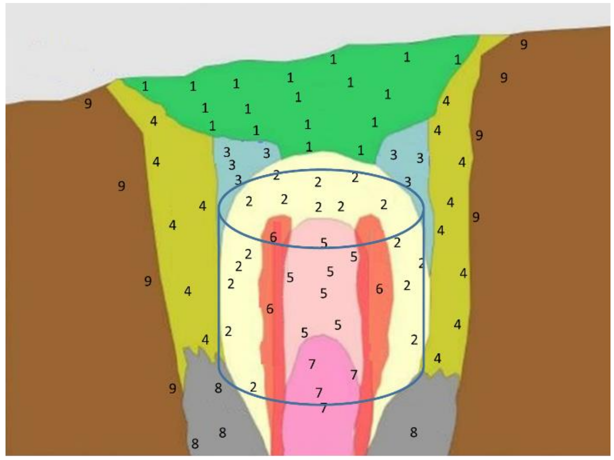

Figure 4.2: An alteration domain fitted by a parameterized cylinder. Source: modified by author from Ridley, J. (2013).

Here is another example from a real ore body, where the geological units are delineated and can be modeled as geometrical objects.

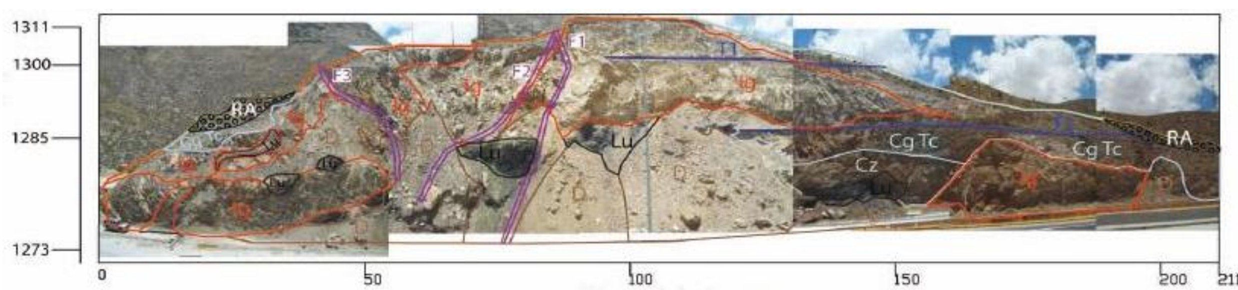

Figure 4.3: Geological units from a real ore body. Image taken from web www.scielo.org.mx, from the Hidalgo mine. Unit for horizontal distance is meters. Height is defined as meter above the sea level.

4.2. Concepts

4.2.1. Geological modeling

Using the information from drill hole loggings and assays, the geologist has to characterize an ore body, considering the following four dimensions (Ridley, 2013):

Dimension 1: ***Lithology***

Dimension 2: ***Alteration***

Dimension 3: ***Mineralogy***

Dimension 4: ***Structures***

These four dimensions show an actual photography of an old geological massive process, which includes big pressure, fracturing, secondary enrichment for oxides and sulfides. It is created from a mixture of data and conceptual geological modeling, in a more qualitative than quantitative way. These models are not considered in mine planning algorithms. Instead, block models created by geostatistical approaches such as kriging or conditional simulation, do not really care about the actual origin or context of the information. So, an interesting question is whether or not this four-dimension information could be used to improve mine planning.

Definition. Let Ω be a compact region of $\mathbb{R}^3$ containing the ore body, and $(\Omega_i)^j \subseteq \mathbb{R}^3$ a set of compact volumes covering $\Omega \subseteq \bigcup_i \Omega_i^j$, for $j = 1$ (lithology), 2 (alteration), 3 (mineralogy) and 4 (structures). Here i denotes a generic point in $\mathbb{R}^3$ and j the geological context. We will say that $(\Omega_i)^j$ is a geological model when its volumes are disjoint on i, for all j.

Remark 1. This definition allows intersection between volumes from different j.

Remark 2. In the fourth dimension "structures", two kinds of behaviors can actually be measured: first, fragmentation, which eventually leads to enrichment and can be geometrically parameterized (frequency of fragmentation, size and direction). Second, fault, which can be modeled as a mantle separating two regions, which would be some $(\Omega_i)^4$ volumes.

We will look for a geological model whose volumes can be interpreted, have small inner variance and big inter volumes variance for the geo-metallurgical properties of interest. Such a kind of partitions of a region Ω is known as clustering or segmentation, for which there are plenty of available methods (Wackernagel, 2003). The difference here is that we are in the presence of a geological process, with volumes of material affected by many geophysical forces, which leads to spatial concentration, homogeneity and heterogeneity. So, these clustering methods could be used partially and mixed with tools used in image or object recognition.

4.2.2. Geomechanical stability

Rock competence defines feasible slope angles. This parameter is classically considered as a restriction for mine design. In a block model that discretizes the ore body, it defines connections between parent blocks. There is a relationship between rock competence and hardness, which impacts the income in an opposite form: hardness implies more energy to mill, but allows more flexible slope angles, therefore less waste to move.

4.2.3. Metallurgy and processing

The decision to send material to the processing plant is often taken individually on each block, giving a value of the block depending just on its own content. This is a strong and false assumption, that the metallurgical recovery is linear in the blended components. Classical models are based on this claim, because they leave the mixing work for short term planning. However, this lead to misestimating the ore value, plant needs and acid mine drainage, among others.

There have been some efforts to optimize in the long term, considering a stable grade feed to the plant. However, plants engineers explain that the variability in the mineral (mineralogy) is the real problem, rather than the variability in the grade, because specific processing parameters are unfeasible to get just in time. Some examples are chemical additives, temperature and pressure. Furthermore, plant engineers accept as a good idea to feed the plant with mineral from the same alteration zone at a time. Those two ideas are the key of this chapter:

1. Variability in the mineral is the real problem, not variability in the grade.
2. One should feed the plant with mineral from the same alteration zone at a time.

4.3. Proposal

4.3.1. Problem description

Problem 1: Obtain a geological model with parameterized 3D objects, from drill hole sample information, using stochastic geometry tools.

Definition (clustering). Let $K \subseteq \wp(Orebody)$ (parts of Ore body), we will say that K is a "***clustering***" if and only if it satisfies the following three conditions for any $A, B \in K$:

1. The elements of A are homogeneous (for example, grade). The same with B.

2. Comparing elements of A with elements of B, they are significantly different.
3. A and B are not trivial sets (not empty sets, and such that their minimum volume, mass or dimension are defined by the modeler); they can be disjoint or partially overlap.

Remarks:

- Points 1 and 2 can be determined by a statistical test, as commented below.
- In statistics applications, a "*good clustering*" has comprehensive sets, i.e. sets that could be linked to a concept and baptized with a reasonable name.
- K is defined on the basis of sampling information, like drill hole samples in 3D.

Problem 2: Solve problem 1 by maximizing the differences between objects that represent geological units.

Remark: This problem allows solutions of sets that could have non-empty intersections, which could be a problem.

Remark: Differences could be defined in several ways. There are two concepts to consider:

1. **Point Metric**: distance between points in sets of the geological model, could be defined by: difference between concentration, presence of minerals, variability, concentration weighted by distance.
2. **Set Metric:** distance between sets, from point information, could be defined by: mean difference of grade, biggest difference of grade, distribution difference (chi-squared measure, Kolmogorov-Smirnov measure), mean differences weighted by distance, and so on.

Problem 3: Solve problem 2, bounding intersections between sets, with a mean difference metric.

Remark: Note that this last problem looks for solutions with sets of strongly different matter, each of them homogeneous, but they could have intersection. This is not a problem, from a geological point of view, because there is often the presence of more than one mineral, alteration zone or geological units at the same place, generally in contacts and secondary enrichment zones.

Definition. Problem 3 will be named as "***weak geological clustering***".

In the four-dimensional information system generated by a geological model, there are volumes representing alteration domains or units, and surfaces representing fractures. Such volumes are drawn by geologists under their criteria (and biases).

Hypothesis: Such volumes are homeomorphisms of cones, pointing upward if the geological process represented is a "going down process" (for example, run-off, deposition or oxidation), and downward if it is a process under pressure. Homeomorphisms mean that an alteration zone could be modeled by a perturbation of a cone: translation, rotation, dilation along the main axes (X, Y or Z), or dilation along a curve.

4.3.2. Parameterization of alteration, lithology and mineralogy domains

A possibility for the concepts $j = 1$ (alteration), 2 (lithology) and 3 (mineralogy) of a geological model is a cone-based representation, the orientation of which would be related to the geological processes:

- Cone pointing upwards: infiltration by gravity.
- Cone pointing downwards: material emerging due to high pressure.

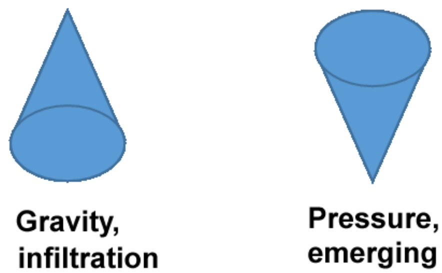

Figure 4.4 Basic objects approximating main geological processes. Source: author.

As a cone-based representation could not fit as a partition of a subset of $\mathbb{R}^3$, it would be useful to join them for covering geological objects by using geometrical tools, like opening or closing. In this case, such geometrical operations could be parameterized. Furthermore, from a rigid cones-based representation, to improve fitting, disturbances could be used, e.g.:

- **Position**: movement of a cone (or volume) by adding a vector.
- **Scale**: multiply by a constant depending on the direction along X, Y and Z.
- **Orientation**: rotating in azimuth and dip.
- **Interaction**: allowed, not allowed, or allowed under probability presence of two volumes.

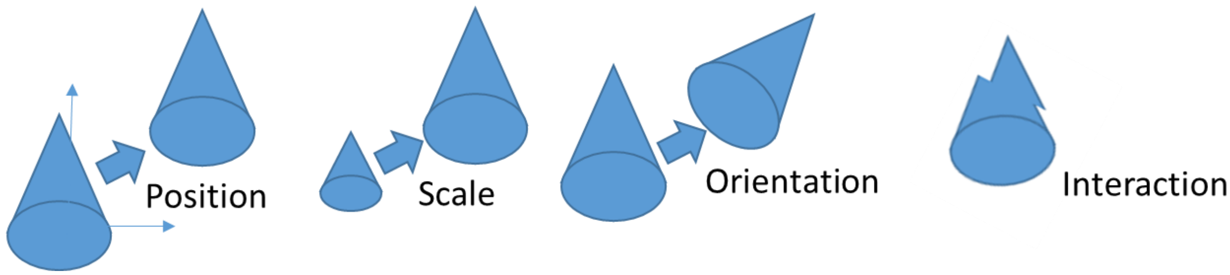

Figure 4.5: Usual perturbations in geometrical pattern recognition. Source: author.

All these tools can be parameterized. The problem is now to develop an algorithm that can fit geological objects by using a cone-based representation, fitted by opening, closing or position, scale, orientation and interaction disturbances.

An example of algorithm: stochastic disturbances.

Input: information from drill hole samples.

Objective function to be minimized: mean absolute difference between sample information and projection of the representation onto the drill hole positions.

Output: objects (cones) and applied disturbances.

Remark: fitting objects to a geological model, in agreement with drill hole data: supposing that the geological model is a realization of a stochastic process, one can reach this process iteratively by an annealing procedure. To demonstrate that the desired distribution is the limit of the distribution at each iteration (convergence in distribution), perturbation rules and specific volumes (cones) should be helpful.

- While the optimizing problem convergence condition is not reached:
  - Choose a new state for process (birth of a new object): an upwards or downwards cone
  - Merge the new object with the old ones, under an opening operator
  - Evaluate the added value
  - While the improvement perturbation of cone convergence condition is not reached
    - Change position, scale, orientation
    - Merge the new object with the old ones, under an opening operator
    - Evaluate the added value
  - While the improvement perturbation of cone convergence condition is not reached
    - Interaction
    - Merge the new object with the old ones, under an opening operator

- Evaluate the added value.

Remark: difficulties will be found when specific algorithms will be developed. There are at least two lines: engineering and mathematical problems to be solved.

- Geometric **modeling of disturbances**: this is already solved with pattern recognition tools.

- Estimate correlations **(interaction) between objects**: this is an interesting and not trivial problem. Geophysical interaction should be modeled, to define an allowable probability of combination of objects. Furthermore, information from other tools, like kriging with a variogram model, could be considered.

- Demonstrate or test **convergence speed**: depending on the optimization strategy used, convergence could be proved or estimated with examples. In each case it would be interesting to discover the solution properties, like uniqueness, bounds, concentration, among others.

- Application: generate different realizations with annealing. This would be analogous to geostatistical conditional simulation.

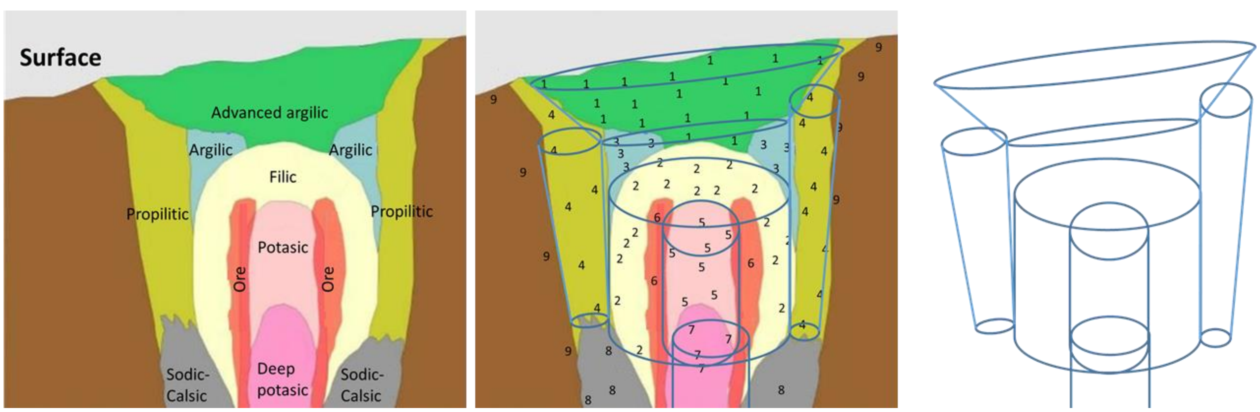

Figure 4.6: General geological model of a porphyry ore body. Left: the model developed by geologist, middle: fitted parameterized model by cone representation, right: final model used for mine planning. Source: modified by author from Ridley, J. (2013).

4.3.3. Parameterization of structures

Structures are defined by faults and fractures. Faults could be modeled by a plane and perturbations of such a plane using a topological space of 2D functions (for example, polynomials or Fourier series in 2D) and an internal product (like the L2 operator, which induces metrics and convergence). Fractures could be modeled by a stochastic process of small planes, e.g., a marked Poisson process, with density and interactions.

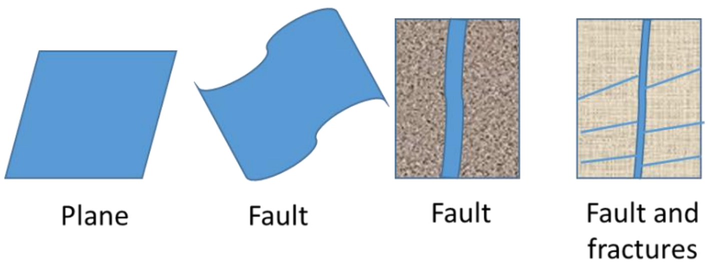

Figure 4.7: Examples of faults and fractures. Source: author.

Two kinds of relations between planes and volumes could be found: a fracture cutting a volume (for instance, an alteration domain) or surrounding it. Both possibilities could be considered to fit volumes. In the first one (parsimonious model), one will join the two volumes as one alteration domain. In the second one (saturated model), one would split as two different alteration domains.

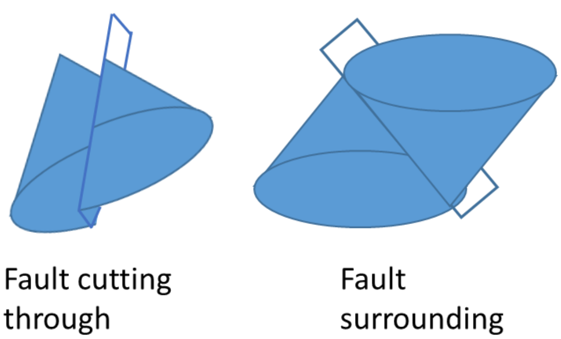

Figure 4.8: Interactions between volumes and faults. Source: author.

4.3.4. Topological fitting

Let A be a compact (volume), continuously differentiable on its surface boundary. If A has a parameterized geometric structure and if there is information of A in some places, such as samples in space that belong to A, one can estimate its parameters by using geometric and/or inferential methods. If A is not a parametric set, it can be approximated by the union of finitely many parameterized, differentiable convex volumes, eventually filtered with geometrical operations such as opening or closing.

Geometrical methods

It is possible to define metrics for a volume A, such as the mean absolute difference between its points and drill hole data. Some examples of metrics are (see figure 4.9):

- **Volume difference minimization**: Minimize the number of data points that are outside A and should actually be inside A, or minimize the number of data points that are located inside A and should not, or minimize the sum of both numbers.

- **Min-min distance between volumes**: Calculate the distance between each data point and the nearest point in A. Minimize the smallest of such distances for the data that should be in A, maximize the smallest of such distances for the data that should not be in A, or use a combination of both metrics.

- **Min-max distance between volumes**: Calculate the distance between each data point and the nearest point in A. Minimize the largest of such distances for the data that should be in A, maximize the largest of such distances for data that should not be in A, or use a combination of both metrics.

- **Min of mean distance between volumes**: Calculate the distance between each data point and the nearest point in A. Minimize the mean of such distances for the data that should be in A, maximize the mean of such distances for the data that should not be in A, or use a combination of both metrics.

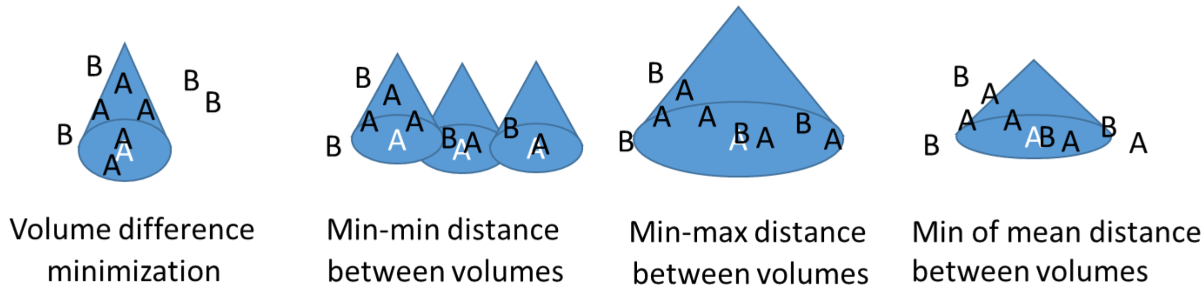

Figure 4.9 Examples of metric-based fittings. Source: author.

Inferential methods

- **Maximum likelihood**: If one assumes that the drill hole data are Bernoulli random variables, with value 1 if it should be in A and 0 otherwise, one can choose the parameters of volume A that maximize the likelihood of such variables. That is, maximize $f(data|A)$, where f is the likelihood function.

- **Maximum conditional probability:** choose the most probable volume A, given the data information. That is, $f(A|data) * \pi(data)$, where $\pi(data)$ is the prior probability for geological data. For example, π could be obtained from a conceptual geological model; or, by default, as a simple Poisson process.

All these methods can be solved, because their objective functions are bounded and continuous.

A complex volume can be fitted for binding parameterized volumes. Moreover, this union can be disjoint if parametric volumes are cut by surfaces whose boundaries correspond to the edge of the intersection between adjacent parameterized volumes. In this case, it is possible to extend the assessment methods, based on Monte Carlo algorithms, Metropolis, Simulated Annealing, Hidden Markov Process or Coupling from the Past.

What is interesting is the difference of results when compared to a classical geostatistical approach. Geostatistical fitting reflects a small-scale, local behavior, with a global influence governed by variogram tools, while in the geometric method proposed here, the setting is macro and one fitted volume is likely to interact with others. Even more, it is possible to include (or to replace) in the objective function the geophysical behavior that has led to the ore body. For example, energy minimization, conditioned on geological forces, resulting in volumes representing geological domains, alteration, mineralization, lithology, faults and structures in general.

Relationship between volumes

The geophysical behavior that led to the anomaly corresponding to the ore body results in a joint distribution of geometric volumes. Describing such behavior is complex, even in qualitative terms. However, basic laws can be modeled from diffusion equations and plastic deformations. The technical difficulty in this case is to prove the existence of solutions and then bring them closer to convergent sequences in a larger space, with a weak metric. For practical purposes, one could define exception rules and probability that allow or do not allow the existence of volumes, in the presence of others. This comes from the knowledge of the mineralogy and geological theory. Specific implementations of this are subject of further development.

4.3.5. Fitting with Simulated Annealing

General formulation

Let M be the set of volumes (cones) modeling the ore body. Let $\Omega \subseteq \wp(\mathrm{M})$ be a sigma algebra on M. We will define a stochastic time process (X_n) on Ω with a transition probability function P (sometimes called *transition kernel*), irreducible, aperiodic and non-reversible, with an ergodic limit π. This distribution should rule the behavior of the ore body as a geological process, and the ore body itself would be a realization of such a distribution.

Let a be an acceptance function and E be an energy function on M. In general, E is a function measuring the distance between the current state x_n and the conditioning data on the ore body: as the objective function decreases, one gets closer to the desired state. The acceptance function is of the form:

$$a(X_n, X_{n+1}) = \min(\exp\left(\frac{E(X_n) - E(X_{n+1})}{\boldsymbol{t}}\right), 1)$$

where $\boldsymbol{t}$ is a positive parameter called "*temperature*".

It is possible to prove that the transition function of the Markov chain has an ergodic limit π_t (punctual convergence on $\boldsymbol{t}$) that depends on $\boldsymbol{t}$, and that π_t converges to π_c (distribution π restricted to the states that fulfill the conditioning data on the ore body) as $\boldsymbol{t}$ tends to zero (Tierney, 1994; Lantuéjoul, 2002). This suggests that, to simulate a conditional distribution π_t, one should decrease the temperature to zero while being carried out the iterations. In practice, it is important to pay attention to the way in which one decreases the temperature: if it is decreased too quickly, the chain can be trapped in a local minimum of the objective function. A suggested method is to take the temperature at iteration k proportional to $1/\ln(k+1)$ or to e^{k+1} , with $e \in (0,1)$ (Hajek, 1988; Lantuéjoul, 2002).

Application to ore body fitting

The idea is to define the transition probability function as a random generation of cones, defined by its parameters. At each iteration, a new cone would be generated, randomly or with some criterion (for example, places where nothing is still covering). This is a birth process idea, which could be modified to a birth and death process.

The new cone is appended to the existing set of cones. Then, an opening or closing operation is applied over the new set of cones, to get a smoothed set. This opening or closing could be random as well. If this new structure of smoothed cones improves the fitting (i.e., if it decreases the objective function), then the new cone is accepted. Otherwise, it may be accepted anyway but with the abovementioned acceptance probability that depends on the temperature parameter.

When the algorithm stops (after finitely many iterations), there will be a set of smoothed cones constituting an approximate realization of the limit distribution, so it could be considered a fitting of the real ore body conditional to the known data on this ore body.

Relationship between volumes

Certain configurations of volumes are allowed or are accepted with some probability, because there exists a geological behavior or physical laws that govern them. Such information could be entered as a restriction. Here the geological knowledge plays an important role and could be modeled as a prior probability in a Bayesian context. For example, certain volumes modeling alteration domains should intersect; some of them should be close together, others no; some should be over and other under; some should point upwards whereas the others point downwards.

4.3.6. Algorithm

Here a solution for Problem 3 is presented.

The proposed topological method starts from sparse and raw data from drill holes. The algorithm does not know how many geological units there are, nor their positions, sizes and shapes. Let us suppose that the geologist can give an idea about how large could be the geological units, which can be taken as a priori information that could be improved from the available data.

Remark: note that the input for this algorithm is a set of drill hole data, not a block model.

Now, a birth process starts by proposing new pseudo cones. At the first step, the algorithm considers to locate a pseudo cone in a place with a probability proportional to the grade of the main element of interest, starting with a fixed size and orientation. The idea is to choose a pseudo cone which concentrates more mass of the mineral ore under consideration. Because the generation of a cone with probabilities depending on the position could be time expensive, this could be approximated by considering probabilities in three axes, separately.

Once the pseudo cone is placed, the algorithm can modify its properties and simulate its radius, height and orientation (pointing downwards or upwards) in search of an improvement. This is suboptimal, considering a problem that looks for the best pseudo cone, but allows having a heuristic solution. Note that this part of the problem could be solved with a simulated annealing.

Another improvement could be introduced here: a stochastic geometry operator, merging this new cone with the group of already formed cones, could be chosen. This could be, for example, the *closing* operator, parameterized by a set B that could represent a diffusion process between the geological units.

Then, the algorithm tests whether or not this new cone adds relevant information. To this end, three tests are considered:

1. **Significant value added**. Remark in this context “value” means “matter”, not economic value. This test verifies that the new pseudo cone has enough matter (volume) to be considered. Without this, each drill hole data could be taken as a small pseudo cone. So in this test, the geologists’ knowledge is relevant: they should say how massive should be the geological units. In clustering problems, this concept is “*the cluster should be big enough*”. This test is made using a T-Student with the null hypothesis that “the mean added value is equal to zero”.

2. **Significant difference from the remaining of the ore body**. The mean in the pseudo cone considered should be different to the matter that is surrounding it. If not, a constant ore body could be modeled by a set of equal cones, covering it, one next to the others. In clustering problems, this concept is “*the cluster should be significantly different from the universe*”. This test is made using a T-Student with the null hypothesis that “the mean added value is equal to the remaining value”.

3. **Significant difference from the already accepted cone list.** The mean in the pseudo cone considered should be different from the other cones already considered. If not, a large geological unit could be covered by several (many) nearby cones. In clustering

problems this concept is "*the cluster should be significantly different from already accepted cones*". This test is made using a T-Student with the null hypothesis that "the mean added value is equal to the already accepted clusters".

Starting from the above ideas, an algorithm has been implemented, including some other details to improve the fitting and processing time. In the next lines, it is presented using a pseudo programming language.

Step 0: load data

i. Drill hole data. Remark: this algorithm is not using a block model, but sparse conditioning data.
ii. Initial temperature (high).
iii. Significance parameter (95%, 99%, for testing the mean differences).
iv. A priori geological knowledge, such as the mean size of the geological units (say, alteration zones).

Step 1: initialize parameters:

i. Cone list (empty at the beginning).
ii. Matrix with aggregated value (null at the beginning, empty box enclosing the ore body).
iii. Matrix with remaining value from the ore body (the entire ore body at the beginning).

Step n: Iterate (while temperature is warm enough)

iv. Create a cone, placed with a probability proportional to the mean grade, size defined by point iv of the above step 0, and optimized (place, size).
v. Calculate the added value of this cone.
vi. Test the mean difference between the cone and the cone list (T-test or Analysis of Variance).
vii. Test the mean difference between the cone and the ore body (T-test or Analysis of Variance).
viii. If (*this new cone adds a significant value)* and (*it is different to ore body*) and (*it is different to the cone list*), accept it as a new cone in the cone list.
ix. Otherwise accept it as a new cone in the cone list with a probability proportional to an objective function of (the added value) and (the difference against the ore body) and (the difference against the cone list).
x. If the cone is not accepted, redefine the cone rotated 180º in the Z-axis. Repeat from step ix.

xi. Decrease the temperature according to a pre-specified cooling schedule.

4.4. Implementation and possible impact

4.4.1. Synthetic drill hole data

For simplicity, we will consider an ore body containing a single cone pointing upward. This could represent a secondary enrichment and deposition process. The parameters of this cone are (X_0, Y_0, Z_0, Z_1, R)=(200,200,200,100,50), where Z_0 is the vertical position of the vertex, Z_1 the basis and R its radius.

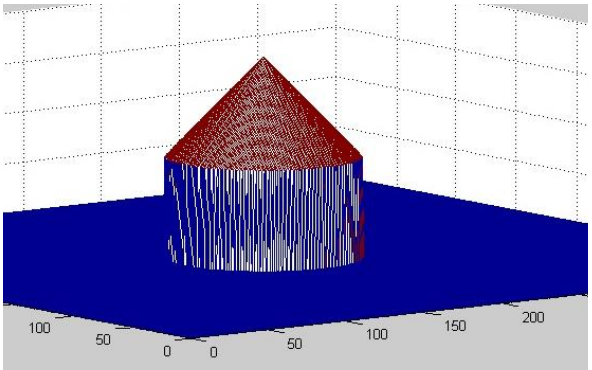

Figure 4.10: synthetic ore body containing a single alteration zone as a cone (secondary enrichment). It notation is (X_0,Y_0,Z_0,Z_1,R)=(200,200,200,100,50). Source: author.

For this ore body, a drill hole network is implemented with a spacing of 20m in the XY plane and samples are taken each 5m along Z. The position of each drill hole is altered with a noise with mean zero and standard deviation of 1m. Also, the drill hole angles are altered. This is a synthetic example, but this configuration is close to reality.

Now, a data base is created from these drill hole samples, containing on its rows the sample points and in its columns the variables: X, Y, Z (position), L (grade). Note that this is not a block model, nor a set of points obtained from a geostatistical model, but raw data from a drill hole campaign.

In the next graphs, it is possible to see the mineral resources and their distribution (grade-tonnage curve). This was inferred directly from the drill hole samples, not from a geostatistical model.

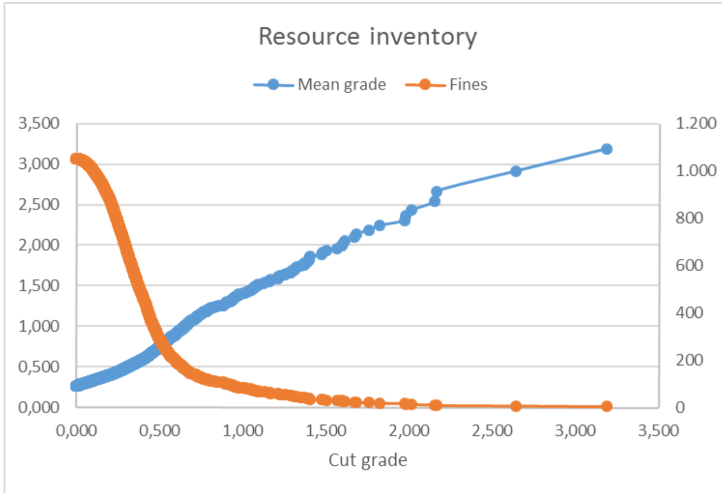

Figure 4.11: Resource inventory for the synthetic example. Source: author.

4.4.2. Problem

For this case study, we will look for ***weak geological clustering*** solutions.

The available information consists of sparse samples. The scale of the variables could be dichotomist (presence/absence), percent (mineral grade), discrete (rock types) or continuous (work index). This information comes from the field measurements or laboratory tests.

So, identifying a geological unit from a single variable, for this context, means surrounding a volume, supposing that the matter inside it is of the same kind and is significantly different from the outside matter.

This can be solved by drawing in the 3D space, and surrounding by hand the geological units, taking certain assumptions that depend on the geologists' experience.

In statistics this problem is called clustering or segmentation. There are some tools and methods for this and the simplest one in this case is the k-means method. This method looks for zones whose inner points are similar, but different from the points outside. The zones are defined by their mean values and separated by hyperplanes.

With geostatistical tools, this could be done by interpolating the sparse data, with a linear model like kriging. In order to implement this, the drill hole samples should be marked with a dichotomy variable, with a 1 inside and a 0 outside of the geological unit. Then, kriging determines for each block or point if it is inside or outside the geological unit. This tool uses a point of view focused on micro relations, it means between neighboring points, although it uses large-scale correlations.

Under the proposed methodology, the solution looks for parameterized objects, a macro point of view. The cones should be very different from the surrounding matter, inside the ore body, to be considered as a geological unit. Furthermore, its matter should be homogeneous, at least at a certain level. Finally, the matter of this cone should be recognized from a geological, mineralogical or metallurgical point of view. In such a sense, it could be claimed that this model "understands" that there is a geological unit at this place, with these characteristics (parametric values). Furthermore, its solutions can estimate values (interpolate), by kriging. Remark: despite that it could be less accurate, the added value of this approach is threefold:

1) Parametric cones discover objects, which could be seen as "understanding the ore body".
2) They depend only on a few parameters. This allows an easy and fast way to simulate scenarios (outcomes).
3) And finally, they try to represent geological knowledge. This creates a research line, connecting geology, geophysics and mathematical modeling.

4.4.3. Results of the proposed topological method

Two solutions of the proposed algorithm are shown. First, with a 95% confidence in testing, it found two cones, placed next to the true place, one pointing upward, which is correct, but the other one pointing downward, which is wrong. This second cone tries to include matter that the first did not include.

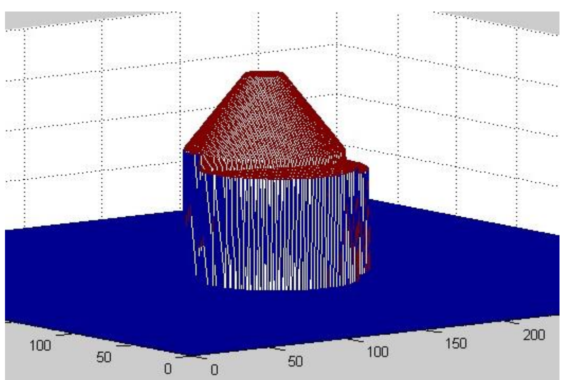

Figure 4.12: Solution 1, 95% confidence on null intersection. Source: author.

The second solution is a single cone, bigger than the cones found in the first solution. This cone is placed in the right place (200,200,100) for the base center, but is taller and wider than the original cone. This happens because the algorithm is trying to add value, trying not to lose any drill hole sample with a positive value, which is the case for all points. Remember that in every sample point there is a non-zero grade, because the noise of synthetic data. So, in a compromise solution, the algorithm prefers to be biased to generate cones that cover the geological unit. Getting solutions biased to be smaller than the geological unit that one is trying to fit, could be done by forcing a distance between objects or by applying an erosion operator (see *Chapter 3. State of the art*).

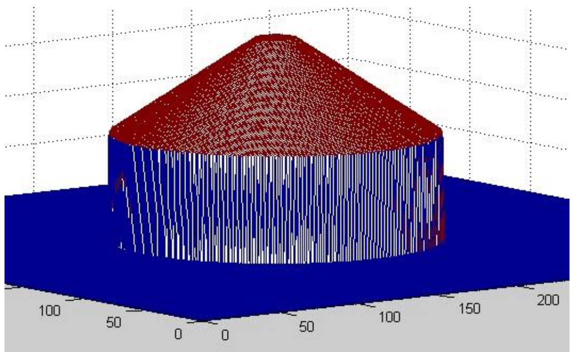

Figure 4.13: solution 2, 99% confidence on null intersection. Source: author.

To achieve this solution a CPU 2.00GHz (Inter® Core™ i7-3537U), with a RAM 8.00Gb, OS x64 bits and the algorithm programmed with Matlab, needed 10 seconds. This cone was the first one considered and 10 iterations were performed, which means that it really needed about 1 minute to calculate this solution.

Although simple, this example includes two main difficulties: sparse information and noise. Its extension to more complex objects is left for future works.

4.4.4. Possible impact

A big application of modeling an ore body with parametric volumes, is to mine it sorting by alteration or mineralization domains. This will be more efficient for the plant processing, which will access to homogeneous inputs, avoiding cost to fit the process to a variable mineralogy.

In a group of eight interviews with plant specialists, made by the author, the question "***what is more difficult to solve in plant processing: mineral variability or grade variability?"***, they answer

1. **Mineral variability** (7), because it is impossible to fit additive needs to reach the desired recovery under mineral variability.
2. In only one mine, they said that more important than mineralogy is the WI (**Work Index**), which defines the energy needed to mill the rock. This last case sustains the interest to model geological units, because WI is associated with lithology and structures.

Furthermore, if the ore body is mined by an open pit, its shape could be parameterized, as presented in the last chapter. Then, both parameterizations, of the ore body and of the open pit, could be mixed in order to model the mine and to evaluate the project value. Note that, in the parametric open pit, the roads could also be parameterized, which is very important as the real shape of the mine strongly depends on them.

Although the parametric volume modeling of an ore body could be locally less accurate than the prediction with a classical geostatistical modeling, this formulation works together with a parametric representation of the mine. Therefore, the impact of the proposed modeling is to improve the evaluation of mining projects, working jointly with geology, mining and metallurgy.

4.5. An alternative algorithm

Another solution for Problem 3 is to consider a Boolean model, which is constructed from the following two components (Lantuéjoul, 2002):

1. A point process
2. A family of independent and identically distributed objects.

At each point of the point process, one places an object. The Boolean model is the union of the objects so placed. In the present case, the objects would be pseudo cones with random sizes and orientations, as in the previous algorithm, and the resulting Boolean model would be a geological unit of interest.

As for the point process, the simplest model is the Poisson process, with a constant intensity over space, or with a regionalized intensity. Lantuéjoul (1997, 2002) proposes an iterative algorithm for simulating such a Poisson-Boolean model conditionally to a set of existing data (observations made at a set of drill hole samples), based on Monte Carlo Markov Chain simulation.

A limitation of the Poisson-Boolean model is that it requires a perfect knowledge of the Poisson intensity, which controls how many objects are generated and where they are placed. In practice, for geological reasons, this intensity cannot be considered as constant over space. For instance, mineralogical domains present a vertical zonation, so that more objects are needed close to the surface when simulating a leach zone or an oxide domain, while the reverse happens for simulating a secondary enrichment or a primary sulfide domain. Likewise, in porphyry copper deposits, alteration domains present vertical and lateral zonations, where potassic and phyllic alteration domains are surrounded by sericitic, argillic and, finally, propylitic alteration domains (Guilbert and Park, 1986). Having a spatially-varying intensity poses some problems for the inference of such an intensity, in particular, in areas where few drill holes are available, insofar as the Poisson-Boolean model does not allow including any uncertainty in the underlying Poisson intensity.

An alternative is to use a Cox process, for which the intensity is a spatial random field (therefore, it varies in space but its actual value is uncertain and changes from one realization to another), in lieu of a Poisson process (Lantuéjoul, 2002). The random intensity represents all the geological or geophysical factors that affect the distribution of the objects and measures the propensity of any region to be covered by the geological unit of interest. This yields a Cox-Boolean model for the simulation of a geological unit. However, to this author's knowledge, to date, no algorithm is available for the conditional simulation of such a model, which therefore constitutes an area for future research.

More complex models could also be designed, for instance, by lifting the independence assumption between the objects. In this case, the placement of an object will depend on the presence, size and orientation of other neighboring objects.

5. Discussion and Conclusions

5.1. Discussion

This research was focused on developing a mine planning methodology that should consider operative evaluation: both the design and the geological information. The proposal integrates both problems into a single paradigm focusing on operative mine: model using parametric objects and topological structures.

The design problem, which is mainly the definition of roads and facilities placement, has so a big impact in the project value, that it is absolutely necessary to consider it in a long term mine planning or in the ore body economic evaluation. As it was mentioned above, mine planners' perception places the difference between optimization and posterior design, until 40% in economic value. Furthermore, this design process could last weeks, for a normal mine. So, it is nonsense to develop faster algorithms to calculate in seconds what today is calculated in minutes, considering that the design will last weeks. As it is useless trying to calculate more accurately, passing to smaller blocks, if the designer will take intuitive decisions, changing notably the optimal results.

But today the focus on mathematical development is the speed and precision for the same problem, without including design. And if mine planners want to consider geological uncertainty, the situation could be even worse, because each geological simulated scenario needs the same time of calculations. This makes time expensive to compare scenarios like the worst, conservative, VAR (Value at risk), because for each one a new design should be implemented (it means, weeks of drawing). And definitely this denies options to develop a SAA (Sampling Average Approximation) algorithm, to solve a stochastic optimization problem, which is better than solving deterministic medium scenario and then verify variations on bad scenarios.

Then, the original final pit problem, defined in almost every mine plan algorithm, without design, is a wrong problem, a wrong goal. Until today, such algorithms and manual design procedure, have obtained uncertain solutions, affecting the project value and its risks.

Two more considerations to reinforce the idea to study and systematically model the design. First, mines need frequently to redesign some mine section, because there is new information or there are operational incidents. They spend a lot of time in this design work. When they were asked about "*how interesting could it be to optimize design*?" they answer that "*it is better to have several good options, than a rigid optimum*". Second, there are preferences about how to design depending on experiences, knowledge heritage, school of work, universities, and so on. From a set of drill holes or a block model, two senior designers could achieve really different solutions, both perfectly acceptable.

So, modeling the design, understanding what is a "*good option for design*", defining which mathematical tools work better (discrete or continuous, deterministic or stochastic), and determining design styles and their influence on the project mine value, are crucial problems to consider.

The solution proposed in this work has some advantages comparing to the most widely used algorithms of today, which are variations or improvements from Lerchs and Grossman. First, this proposed model considering design as a priority problem in mine planning, gives a feasible way to model and optimize mines, roads, ramps, benches, switch-backs. Second, the parametric model depends on a small amount of parameters, which leads to simulate scenarios by changing few variables.

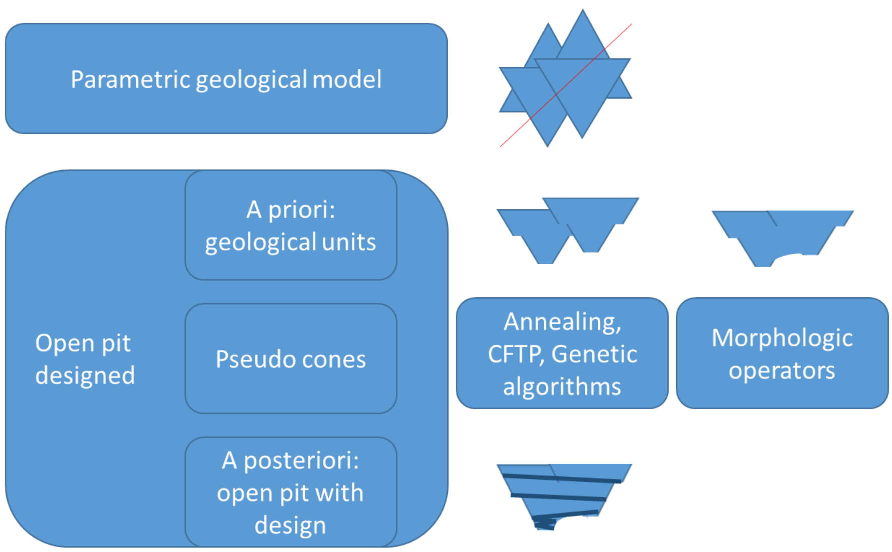

Figure 5.1: Proposed scheme in this thesis. Source: author.

5.2. Conclusions

This thesis started from a search of needs in real mine planning, through group and individual interviews, in opposition to looking for improvement in the speed and size capabilities of today's algorithms, which are in general variations of Lerchs and Grossman. Senior mine planners need to include design in their algorithms, because this concept could explain up to 40% of the project value, and because they spend weeks to design their mines.

After a first study, which tried to use an old algorithm, the floating cones, but with a different optimization tactic, simulated annealing, it was concluded that, despite this floating cone method achieves a worse solution for the final pit problem, comparing with Lerchs and Grossman variation, it showed flexibility to represent some design concepts (benches and roads) and its representation is cheap, which leads use simulations for stochastic algorithms. So, the hypothesis that this method could obtain an entirely final open pit is made.

It was demonstrated that it is possible to represent an open pit with a union of pseudo cones, with few parameters and topological filters. This open pit is operative and could be optimized with a simulated annealing procedure, based on a floating pseudo cones approach. This was programed to probe in a simple mine, obtaining an operative open pit in less than 1 minute, for a block model containing 400x400x200 blocks.

Despite their use, support and social validation, the Lerchs and Grossman algorithm and similar tools should be revised and, eventually, strongly modified to consider the operation, because of the great difference in value to the operative solution and developing time of manual design. In contrast, here a new line of research, under a topological point of view, was proposed.

For the other part, it is possible to model the ore body as a set of parametric cones, filtered with topologic operators. This paradigm allows conceptually fitting geological units such as alteration zones, a methodology that could be locally less accurate than kriging, but that better "understands" or "identifies" the geological units.

This was probed by fitting a single cone, having just drill hole data, not a block model. In a 400×400×200 points network, two solutions were presented, which needed less than 1 minute to be calculated. These two solutions show that there are possibilities to choose, which is a requirement from mine planners.

Note that the coincidence between modeling an ore body with topologic cones and filters and modeling open pits with pseudo cones, is not casual. This is a main result of this research.

Furthermore, such a cheap representation of the ore body and the open pit allows easily incorporating simulated outcomes for uncertainty analysis. For example, this could be used under a Sampling Average Approximation scheme, which needs scenario generation for the ore body and the economic parameters.

The final conclusion is that a mine project could be modeled with topological tools (like parameters, filters, fitting, stochastic process), which joint mine planning, geology, design and uncertainty.

Or in other words ***topology could be the key to join the mine and the ore body, to optimally feed the process***. Three worlds in general far separated.

5.3. Future Research

From this research more questions arise than were answered. Because it is a paradigm far away from current research efforts, the principal value added is to put the problem, not to solve every one of them.

Some questions and research lines that could be considered are the following:

1. Determine design styles, from experiences and preferences of mine designers. Determine what they understand as "*good enough solutions*".

2. Define a topology of objects, which could be dense with a mine space (which should be defined as well).

   Remarks: an example for this is a polynomial basis $(x^k)_{k\geq 0}$, which defines a topology under several functional metrics. Taylor theorem states that any continuous function could be approximated by a succession of linear combination of this basis. In this case we say that "*polynomials are dense in continuous function space*". As an analogous case for geological modeling, a cone basis, a metric and a topology would be defined, to look for a property which could ensure convergence to an ore body.

   Metrics and possible bases of vectors should be defined. Naturally there could be several options, which gives more flexibility and could better fit some design styles than others. Density means that there always exists a succession of object that converges in topology to any point of the mine space; this gives a theoretical construction to prove the existence of solutions for optimization problems.

3. Idem, to better construct a geological model.

4. Simulate the ore body geology from parametric modeling, including spatial correlations (variogram information).

5. Study properties of SAA solutions considering simulation from parametric cones.

6. Model the ore body with Bayesian tools.

7. Include geophysical knowledge to define cone interactions.

8. Study waste dump to determine how to model with parametric objects acid drainage and percolation.

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