UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

INCLUSIONES DIFERENCIALES CON CONOS NORMALES DE CONJUNTOS NO REGULARES EN ESPACIOS DE HILBERT

TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA
EN COTUTELA CON LA UNIVERSIDAD DE BORGOÑA
FRANCO-CONDADO

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Este trabajo ha sido parcialmente financiado por CONICYT-Beca Doctorado Nacional 2013

SANTIAGO DE CHILE
2017

RESUMEN

INCLUSIONES DIFERENCIALES CON CONOS NORMALES DE CONJUNTOS NO REGULARES EN ESPACIOS DE HILBERT

Esta tesis está dedicada al estudio de inclusiones diferenciales con conos normales de conjuntos no regulares en espacios de Hilbert. En particular, nos interesa el proceso de arrastre y sus variantes. El proceso de arrastre es una inclusión diferencial restringida con conos normales que aparece naturalmente en varias aplicaciones tales como elastoplasticidad, histéresis, circuitos eléctricos, movimiento de multitudes, etc.

Este trabajo está dividido conceptualmente en tres partes: Estudio de los conjuntos "α-far", existencia de soluciones para las inclusiones diferenciales con conos normales y caracterizaciones de los pares de Lyapunov para el proceso de arrastre en espacios de Hilbert separable.

En la primera parte (Capítulo 2), investigamos la clase de conjuntos positivamente "α-far". Esta clase de conjuntos no regulares es muy general e incluye los conjuntos convexos, uniformemente prox-regulares y uniformemente sub-lisos, entre otros. Esta clase de conjuntos es la mejor adaptada al estudio de inclusiones diferenciales con conos normales.

En la segunda parte (Capítulo 3 hasta la primera parte del Capítulo 8), se entregan varios resultados de existencia para el proceso de arrastre y sus variantes. Para ello, consideramos tres enfoques: el algoritmo de rectificación (Catching-up algorithm), el método de tipo Galerkin y la regularización de Moreau-Yosida.

El primer método es el más clásico en el estudio de inclusiones diferenciales gobernadas por conos normales. Aquí es utilizado en el caso donde el conjunto considerado es fijo.

El segundo método (de tipo Galerkin) consiste en aproximar el problema original proyectando el estado sobre un espacio de Hilbert de dimensión finita, pero no la velocidad. Los problemas aproximados siempre tienen una solución y, bajo ciertas condiciones de compacidad, se demuestra que ellos convergen fuertemente (salvo subsucesión) a una solución de la inclusión diferencial original. Más aún, se muestra que este método está bien adaptado para tratar inclusiones diferenciales con conos normales, proporcionando resultados generales de existencia para el proceso de arrastre generalizado. En consecuencia, se obtiene la existencia de soluciones para el proceso de arrastre de primer y segundo orden. Adicionalmente, este método es utilizado para mostrar la existencia de soluciones del proceso de arrastre con condiciones iniciales no locales.

El tercer método es la técnica de regularización de Moreau-Yosida que consiste en aproximar una inclusión diferencial por una penalizada, en función de un parámetro positivo, para luego pasar al límite cuando el parámetro tiende a cero. Este método es utilizado para tratar el proceso de arrastre dependiente del estado gobernado por conjuntos uniformemente sub-lisos.

Finalmente, en la tercera parte (segunda parte del Capítulo 8 y Capítulo 9), se proporcionan algunas caracterizaciones de los pares de Lyapunov débiles y la invariancia débil para el proceso de arrastre perturbado con conjuntos uniformemente sub-lisos.

Palabras Clave: Inclusión diferencial, procesos de arrastre, cono normal, función distancia, método de tipo Galerkin, regularización de Moreau-Yosida, conjuntos positivamente "α-far", subdiferencial de Clarke, Funciones de Lyapunov.

ABSTRACT
DIFFERENTIAL INCLUSIONS INVOLVING NORMAL CONES OF NONREGULAR SETS IN HILBERT SPACES

This thesis is dedicated to the study of differential inclusions involving normal cones of nonregular sets in Hilbert spaces. In particular, we are interested in the sweeping process and its variants. The sweeping process is a constrained differential inclusion involving normal cones which appears naturally in several applications such as elastoplasticity, electrical circuits, hysteresis, crowd motion, etc.

This work is divided conceptually in three parts: Study of positively α-far sets, existence results for differential inclusions involving normal cones and characterizations of Lyapunov pairs for the sweeping process.

In the first part (Chapter 2), we investigate the class of positively α-far sets. This class of nonregular sets is very general and includes convex, uniformly prox-regular and uniformly subsmooth sets, among others. It turns out that this class is the best suited to the study of differential inclusions involving normal cones.

In the second part (Chapter 3 to the first part of Chapter 8), we provide several existence results for the sweeping process and its variants. In order to do that, we consider three approaches: The Catching-up algorithm, the Galerkin-like method and the Moreau-Yosida regularization.

The first method is the most classic in the study of differential inclusions involving normal cones. We used it in the case where the set considered is fixed.

The second method (Galerkin-like) consists in approximating the original problem by projecting the state into a finite-dimensional Hilbert space, but not the velocity. Approximate problems always have a solution and, under some compactness conditions, we prove that they converge strongly pointwisely (up to a subsequence) to a solution of the original differential inclusion. Moreover, it is shown that this method is well adapted to deal with differential inclusions involving normal cones, by providing general existence results for the generalized sweeping process. As a result, existence of solutions for the first order and second order sweeping process is obtained. Furthermore, this method is used to show existence of solutions of the perturbed sweeping process with nonlocal initial conditions.

The third method is the Moreau-Yosida regularization technique which consists in approximating a given differential inclusion by a penalized one, depending on a positive parameter and then to pass to the limit when the parameter goes to zero. This method is used to deal with state-dependent sweeping processes governed by uniformly subsmooth sets.

Finally, in the third part (Second part of Chapter 8 and Chapter 9), we give some characterizations of weak Lyapunov pairs and weak invariance for the perturbed sweeping process with uniformly subsmooth sets.

Keywords: Differential inclusions, sweeping process, normal cone, Distance function, Galerkin-like method, Moreau-Yosida regularization, positively α-far sets, Clarke subdifferential, Lyapunov functions.

RÉSUMÉ
INCLUSIONS DIFFÉRENTIELLES SUR LES ESPACES DE HILBERT AVEC DES CÔNES NORMAUX À DES ENSEMBLES NON RÉGULIERS

Cette thèse est consacrée à l'étude des inclusions différentielles sur les espaces de Hilbert séparables avec des cônes normaux à des ensembles non réguliers. En particulier, nous nous sommes intéressés à l'étude des processus de rafle et de ses variantes. Le processus de rafle est une inclusion différentielle contrainte avec des cônes normaux qui apparaissent naturellement dans plusieurs applications telles que l'elastoplasticité, les circuits électriques, l'hystérésis, le mouvement de foule, etc.

Ce travail est divisé conceptuellement en trois parties : Étude des ensembles positivement "α-far", existence de solutions pour les inclusions différentielles avec des cônes normaux et caractérisations des paires de Lyapunov pour le processus de rafle dans des espaces de Hilbert séparables.

Dans la première partie (Chapitre 2), nous étudions la classe d'ensembles positivement "α-far". Cette classe d'ensembles non réguliers est très générale et comprend des ensembles convexes, uniformément prox-réguliers et uniformément sous-lisses, entre autres. Il se trouve que cette classe est la mieux adaptée à l'étude des inclusions différentielles avec des cônes normaux.

Dans la deuxième partie (Chapitre 3 à la première partie du Chapitre 8), nous fournissons plusieurs résultats d'existence pour le processus de rafle et ses variantes. Pour cela, nous considérons trois approches : L'algorithme de rattrapage (Catching-up algorithm), la méthode de type Galerkin et la régularisation de Moreau-Yosida.

La première méthode est la plus classique dans l'étude des inclusions différentiels gouvernées par des cônes normaux. On l'a utilisé dans le cas où l'ensemble considéré est fixe.

La deuxième méthode (de type Galerkin) consiste à approcher le problème original en projetant l'état sur un espace de Hilbert de dimension finie, mais pas la vitesse. Les problèmes approchés ont toujours une solution et, sous certaines conditions de compacité, on montre qu'ils convergent fortement (via une sous-suite) vers une solution de l'inclusion différentielle initiale. En outre, on montre que cette méthode est bien adaptée pour traiter les inclusions différentielles avec des cônes normaux, en fournissant des résultats généraux d'existence pour le processus de rafle généralisé. En conséquence, l'existence de solutions pour le processus de rafle de premier ordre et de deuxième ordre est obtenue. En plus, cette méthode est utilisée pour montrer l'existence de solutions du processus de rafle perturbé avec des conditions initiales non locales.

La troisième méthode est la technique de régularisation de Moreau-Yosida qui consiste à approcher une inclusion différentielle donnée par une pénalisée, en fonction d'un paramètre positif, puis passer à la limite lorsque le paramètre tend vers zéro.

Cette méthode est utilisée pour traiter les processus de rafle dépendants de l'état régis par des ensembles uniformement sous-lisses.

Finalement, dans la troisième partie (Deuxième partie du Chapitre 8 et Chapitre 9), on fournit des caractérisations des paires de Lyapunov faibles et l'invariance faible pour le processus de rafle perturbé avec des ensembles uniformement sous-lisses.

Mots clés: Inclusion différentielle, processus de rafle, cône normal, fonction distance, méthode de type Galerkin, régularisation de Moreau-Yosida, ensembles positivement "α-far", Sous-différentiel de Clarke, fonctions de Lyapunov.

A mi madre, por su apoyo y amor incondicional.

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[5] E. Vilches. Existence and Lyapunov pairs for the perturbed sweeping process governed by a fixed set. *Submitted*, 2017.

[6] A. Jourani and E. Vilches. Constrained differential inclusions with nonlocal initial conditions. *Submitted*, 2017.

[7] A. Hantoute and E. Vilches. Lyapunov pairs for perturbed sweeping processes. *Submitted*, 2017.

Agradecimientos

Je tiens tout d'abord à exprimer ma profonde gratitude à Rafael Correa, Abderrahim Hantoute et Abderrahim Jourani, mes directeurs de thèse. Sans eux ce travail n'aurait jamais pu être réalisé.

Je suis très honoré que Messieurs les Professeurs Samir Adly et Fabián Flores-Bazán aient accepté d'être rapporteurs de ma thèse. Je les prie de trouver ici l'expression de ma plus grande reconnaissance.

Je suis très sensible à l'honneur que m'ont fait les Professeurs Alexandre Cabot et Lionel Thibault, qui a beaucoup fait pour le Chili, de participer au jury de ma thèse. Je les en remercie chaleureusement.

Je désire remercier les membres de l'Institut de Mathématiques de Bourgougne pour leur gentillesse, et plus particulièrement Caroline Singvorada-Gerin, Anissa Bellaassali, Francis Léger, Véronique De Biasio, Jean-Baptiste Caillau, Hervé le Ferrand, Charlie Petitjean, Rémi Bignalet-Cazalet, Jeremy Rouot, Olga Assainova, Michael Mignard, Adriana Da-Luz et Mario Shannon.

Agradezco profundamente a mis directores de tesis: Rafael Correa, Abderrahim Hantoute y Abderahim Jourani, por confiar en mí, por ayudarme en el desarrollo de este proyecto de largo aliento, por su paciencia, su disponibilidad y su sabiduría. Su amabilidad me ha permitido trabajar en las mejores condiciones durante estos años.

Agradezco a todos los funcionarios del Departamento de Ingeniería Matemática de la Facultad de Ciencias Físicas y Matemáticas de la Universidad de Chile, especialmente, a Eterin Jaña, Natacha Astromujoff, Luis Mella, Oscar Mori y Silvia Mariano.

Agradezco a todos los amigos con quienes compartí en Francia: Patricia Díaz, Sebastián Riffo, Sandra Astete, Jean-Marc Montanier, Anne Vesin, Johan Van der Molen, Daniela Rodriguez, Ignacio Vergara, Roberto Morales, Camila Romero, Anton Svensson.

Agradezco a Rubén Poblete y Claudia Martínez, por su hermosa forma de ver la vida con alegría y por recibirme en Londres.

Agradezco a Cristopher Hermosilla, Sebastián Donoso, Nicolás Hernández, Orlando Rivera y Matías Godoy por su amistad y compañía desde nuestros estudios de pregrado.

Agradezco a David Salas y Pedro Pérez Aros con quienes compartimos proyectos, ideales y sueños.

Agradezco a mi gran amigo Pedro Montealegre que desde el principio de esta tesis me animó a viajar a Francia, aprender francés, extender mis horizontes y ser mejor persona.

Agradezco a Estefania, Susana, Tomás y Arturo, por ser mi familia querida.

Agradezco a mi madre, por su amor y apoyo incondicional, por mostrarme el valor del esfuerzo, por tener el valor de salir adelante sin importar los obstáculos. Todo mi amor y admiración hacia ella.

Finalmente, no tengo palabras suficientes para agradecer a Valentina, mi amor, que me ha acompañado, animado, apoyado y escuchado durante todos estos años.

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Notations

Operations and Symbols

$:=$	Equal by definition
$\equiv$	Identically equal
$\langle \cdot, \cdot \rangle$	Inner product on a Hilbert space
$\Vert \cdot \Vert$, $\Vert \cdot \Vert_H$	Norm, norm of space H
$x_n \rightharpoonup x$	x_n converges to x weakly (in weak topology)
co	Convex hull of a set
cl co	Closed convex hull of a set
$\alpha(A)$	Kuratowski measure of noncompactness
$\beta(A)$	Hausdorff measure of noncompactness
$\gamma(A)$	Measure of noncompactness of a set
$\mathrm{Haus}(A, B)$	Hausdorff distance between sets

Spaces

$L^1([T_0, T];)$	H-valued Lebesgue integrable functions over $[T_0, T]$
$L^1_w([T_0, T];)$	$L^1([T_0, T];)$ endowed with the weak topology
$W^{1,1}([T_0, T]; H)$	H-valued absolutely continuous functions
$W^{2,1}([T_0, T]; H)$	H-valued functions with derivative in $W^{1,1}([T_0, T]; H)$
$\mathrm{Lip}([T_0, T]; H)$	H-valued Lipschitz functions
$\mathrm{BV}([T_0, T]; H)$	H-valued functions with bounded variation
$\mathrm{CBV}([T_0, T]; H)$	H-valued continuous functions with bounded variation

Sets

H, U, V, X, Y	Hilbert spaces
H_w	H equipped with the weak topology
$\mathbb{B}_H$	closed unit ball of space H
$\mathbb{B}$	closed unit ball of the space in question
$T(S;x)$	Clarke tangent cone to S at x
$T^w(S;x)$ or $T^w_S(x)$	Weak tangent cone to S at x
$T^B(S;x)$	Bouligand tangent cone to S at x
$N(S;x)$	Clarke normal cone to S at x
$N^P(S;x)$ or$N^P_S(x)$	Proximal normal cone to S at x
$\partial f(x)$	Clarke subdifferential of f at x
$\partial_P f(x)$	Proximal subdifferential of f at x
$\partial_L f(x)$	Limiting proximal subdifferential of f at x
$\operatorname{epi} f$	Epigraph of an extended real valued function f
$\operatorname{Proj}_S(x)$	Set of projections of S at x
$\operatorname{proj}^\gamma_S(x)$	Set of γ-approximate projections of S at x
$\mathcal{B}(A)$	Borel sets of a real set A
$U_\rho(S)$	Open ρ-tube around a set S

Functions

$f^\circ(x;\cdot)$	Generalized directional derivative of f at x
$\sigma(\cdot;S)$	Support function of a set S
$d_S(\cdot)$ or $d(\cdot,S)$	Distance function
I_S	Indicator function of a set S
P_n	Projection from H into $\{e_1,\ldots,e_n\}$
φ_C	Asplund function associated with C
$\operatorname{Var}(u,J)$	Variation of a function u over J
ℓ_u	Normalized arc-length of u
$\lvert\mu\rvert$	Variation measure of μ
Du	Differential measure associated with u

Mappings

$f\colon X\to Y$	Single-valued map from X to Y
$F\colon X\rightrightarrows X$	Set-valued map X to Y

General Introduction

The aim of this thesis is to give some contributions to theory of differential inclusions involving normal cones from the point of view of nonsmooth and variational analysis. In particular, we are interested in the perturbed sweeping process:

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0). \end{cases}$$

Here $C\colon [T_0, T] \rightrightarrows H$ is a set-valued map with nonempty and closed values, $N(S; \cdot)$ denotes the Clarke normal cone to S and $F\colon [T_0, T] \times H \rightrightarrows H$ is a given set-valued map with nonempty, closed and convex values. Roughly speaking, a point is swept by a moving closed set. The sweeping process, introduced by Moreau [113,114] to model an elastoplastic mechanical system, is a constrained differential inclusion involving normal cones which appears naturally in several applications such as elastoplasticity, electrical circuits, hysteresis, crow motion, etc.

This work, which is based on [79,87–90,143,144], is divided conceptually in three parts: Study of positively α-far sets (Chapter 2), existence results for differential inclusions involving normal cones (Chapter 3 to the first part of Chapter 8) and characterizations of Lyapunov pairs for the perturbed sweeping process (Second part of Chapter 8 and Chapter 9).

Chapter 2: Positively α-far sets

In this chapter, which is based on [87], we study the class of positively α-far sets in Hilbert spaces X. Given $\alpha \in (0, 1]$ and $\rho \in (0, +\infty)$, we say that a closed set $S \subseteq H$, with $S \neq X$, is positively α-far for some $\rho > 0$ if

$$\alpha \leq \inf_{x \in U_\rho(S)} d(0, \partial d_S(x)),$$

where $U_\rho(S) := \{x \in X \colon 0 < d_S(x) < \rho\}$ denotes the open ρ-tube around S and $\partial d_S(\cdot)$ is the Clarke subdifferential of the distance function. From this definition it follows that every r-uniformly prox-regular set (see Definition 1.21) is positively 1-far for $\rho = \frac{1}{r}$. Moreover, the class of positively α-far sets is strictly bigger than the class of uniformly prox-regular sets (see Example 2.2) .

This class, introduced in [75], as a localization of the class of subdifferentially behaved sets in [68], is very general and encompasses several other class of sets.

This chapter begins by giving some necessary and sufficient conditions to assure the positively α-farness. Our first result is the following (See Proposition 2.2)

Proposition *Let S be a closed subset of X and $\rho > 0$.*

(i) Assume that the following property holds: For every $x \in U_\rho(S)$ there exists $\gamma(x) > 0$ such that for all $\gamma \in]0, \gamma(x)[$

$$u_1^*, u_2^* \in \frac{x - \operatorname{proj}_S^\gamma(x)}{d_S(x)} \quad \Rightarrow \quad \langle u_1^*, u_2^* \rangle \geq \alpha^2 + \theta(\gamma, x),$$

where $\lim_{\gamma \downarrow 0} \theta(\gamma, x) = 0$ for all $x \in U_\rho(S)$. Then, S is positively α-far for $\rho > 0$.

(ii) Assume that S is positively α-far for some $\rho > 0$, then the following property holds:

$$\forall x \in U_\rho(S) \quad u_1^*, u_2^* \in \partial d_S(x) \quad \Rightarrow \quad \langle u_1^*, u_2^* \rangle \geq 2\alpha^2 - 1.$$

The first part of this proposition will be very useful to prove that uniformly subsmooth sets are positively α-farness.

Next, we give a characterization of positively α-far sets in terms of the existence of a pseudo gradient (see Proposition 2.5). Then, we give some necessary and sufficient geometrical conditions (see Proposition 2.6).

Then, we present one fundamental result of the chapter (see Proposition 2.8):

Proposition *Let $S \subseteq X$ be a closed and uniformly subsmooth set. Then, for all $\varepsilon \in]0, 1[$ there exists $\rho \in]0, +\infty[$ such S is positively $\sqrt{1-\varepsilon}$-far for $\rho > 0$, i.e.,*

$$\sqrt{1-\varepsilon} \leq \inf_{y \in U_\rho(S)} d\left(0, \partial d_S(y)\right).$$

This important result will be used in Chapters 3, 4 and 5 to show existence for the perturbed state-dependent sweeping process. Moreover, in Chapter 9, we used this proposition to give a characterization of Lyapunov pairs for the sweeping process.

Afterwards, in Proposition 2.9, we prove that α-paraconvex sets are positively $1-\alpha$-far (for any ρ). Then, we discuss about the preservation of uniform subsmoothness under union, intersection and inverse images. We end this chapter by giving some sufficient conditions to assure the equi-uniformly subsmoothness of a family of moving sets (see Proposition 2.13 and Corollary 2.14).

Chapter 3: Galerkin-Like method and applications

Let H, U and V be separable Hilbert spaces, T_0, T be two non-negative real numbers with $T_0 < T$. In this chapter, which is based on [89], we present a new method to solve differential inclusions. In this method we approach the original problem by projecting the state into a n-dimensional Hilbert space but not the velocity. We prove that the approached problem always has a solution (see Proposition 3.3) and that, under some compactness conditions, the approached problems converges strongly pointwisely (up to a subsequence) to a solution of the original differential inclusion (see Theorem 3.4).

More explicitly, consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{a.e } t \in [T_0, T], \\ x(T_0) = x_0. \end{cases} \tag{1}$$

For each $n \in \mathbb{N}$ we approach (1) by the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t, P_n(x(t))) & \text{a.e } t \in [T_0, T], \\ x(T_0) = P_n(x_0), \end{cases} \tag{2}$$

where, given an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ de H, P_n is the projector from H into the linear span of $\{e_1, \ldots, e_n\}$. We will call this method *Galerkin-like method*. We will show how this method is well adapted to deal with constrained differential inclusions by providing existence of solutions to the following differential inclusion:

$$\begin{cases} -\dot{u}(t) = Bv(t) & \text{a.e. } t \in [T_0, T], \\ -\dot{v}(t) \in N\left(C(t, u(t), v(t)); v(t)\right) + F(t, u(t), v(t)) + Au(t) & \text{a.e. } t \in [T_0, T], \\ u(T_0) = u_0, v(T_0) = v_0 \in C(T_0, u_0, v_0), \end{cases} \tag{3}$$

where $A\colon U \to V$ and $B\colon V \to U$ are two bounded linear operators, $N(S; \cdot)$ denotes the Clarke normal cone to a closed set $S \subseteq V$ and $F\colon [T_0, T] \times U \times V \rightrightarrows V$ is a set-valued mapping with nonempty closed and convex values satisfying some appropriate conditions. We call the differential inclusion (3) Generalized Sweeping Process because it includes the perturbed state-dependent sweeping process, the Moreau's sweeping process and the perturbed second-order sweeping process.

The chapter is organized as follows. In Sections 3.1 and 3.2, respectively, we collect the hypotheses and give some lemmas used along the chapter. The Galerkin-like method is studied in Section 3.3, where we prove the existence of solutions to the approached problems (2) (see Proposition 3.3) and its convergence strongly pointwisely (up to a subsequence) to a solution of (1) (see Theorem 3.4). In Section 3.4 we established the existence of solutions of the Generalized Sweeping Process via the Galerkin-like method. Then, in Sections 3.5, 3.6 and 3.7 we obtain, respectively, existence for the state-dependent, Moreau's and second order sweeping process. Finally, we end this chapter with an example which shows the importance of the ball compactness of the sets moving sets.

We emphasize that our main contributions, in this chapter, are two: First, we introduce a new method to solve differential inclusions in Hilbert spaces. Second, we introduce and show existence of the Generalized Sweeping Process, which includes the state-dependent, Moreau's and second order sweeping process.

Chapter 4: A Variant of the Perturbed State-Dependent Sweeping Process

Let X and Y be two finite-dimensional Hilbert spaces and $C\colon [T_0,T]\times X \rightrightarrows Y$ be an absolutely continuous set-valued map with nonempty and closed values. In this chapter, we are interested in the following variation of the perturbed state-dependent sweeping process:

$$\begin{cases} \dot{x}(t) \in F(t,x(t)) - g(x(t))N\left(C(t,x(t));h(x(t))\right) \quad \text{a.e. } t\in[T_0,T],\\ x(T_0) = x_0 \in h^{-1}\left(C(T_0,x_0)\right), \end{cases} \tag{4}$$

Here $N(S;\cdot)$ denotes the Clarke normal cone to a set $S\subseteq Y$ and $g\colon X\to \mathcal{L}\left(Y,X\right)$ and $h\colon X\to Y$ are two functions.

The motivation to study (4) is that several differential inclusions can be written in this way. For example, perturbed state-dependent sweeping process ($X=Y$, $h\equiv \mathrm{Id}$ and $g\equiv Dh$), complementarity dynamical systems (CDS) (see Section 4.3 below), some control systems describing hysteresis (see Section 4.4 below), projected dynamical systems [38], some differential variational inequalities (see [121, section 2.5]), etc.

This chapter is devoted to show existence of solutions of (4) and to give some applications to complementarity dynamical systems and control systems describing hysteresis. Indeed, after some preliminaries, in Section 4.2, in Theorem 4.1, we show existence of solutions to (4) for equi-uniformly subsmooth moving sets.

In section 4.3, we describe complementarity dynamical systems (CDS), which consists of ordinary differential equations coupled with complementarity conditions, which can be specified by functions $F\colon [T_0,T]\times X\to Y$, $g\colon X\to\mathcal{L}\left(Y;X\right)$ and $H\colon X\to Y$. The defining equations for the CDS, corresponding to F, g and H, are

$$\begin{cases} \dot{x}(t) = F(t,x(t)) + g(x(t))u(t),\\ y(t) = H(t,x(t),u(t)),\\ K\ni y(t)\perp u(t)\in K^*, \end{cases} \tag{5}$$

where $K\subseteq Y$ is a closed convex cone and $K^* := \{d\in Y\colon \langle v,d\rangle \geq 0\ \forall v\in K\}$ denotes the dual cone of K. The third line in (5) is a complementarity relation between $y(t)$ and $u(t)$ which are forced to remain always orthogonal one to each other. This fact can be expressed in an equivalent way as

$$K\ni y(t)\perp u(t)\in K^* \quad \Leftrightarrow \quad -u(t)\in N\left(K,y(t)\right). \tag{6}$$

Therefore, by using (6), we can write (5) as (4).

CDS have been the object of strong interest because of their applications in various fields like mechanics, electrical circuits, transportation science, control systems, etc (see [39] and the references therein). The CDS formalism includes the so-called *linear complementarity systems* (LCS), widely used to deal with some electrical problems (see [1]), and the so called *gradient complementarity system* (GCS), which corresponds to the particular case $g \equiv [Dh]^*$.

CDS has been studied in [39], where the authors consider an "input-output property" to perform a change of state variable allowing them to write (5) as a GCS which is transformed into a perturbed sweeping process. These transformations are made by using the identity

$$I_{C(t)}\left(h(x)\right) = I_{h^{-1}(C(t))}(x) \quad \forall x \in X,$$

where $I_S\colon X \to \mathbb{R} \cup \{+\infty\}$ is the indicator function of a set S. Then, the authors show that the sets $h^{-1}\left(C(t)\right)$ are r-uniformly prox-regular and use a nonsmooth chain rule, for which they must assume some regularity and constraint qualification conditions on h. Thus, existence is obtained from known results in the literature of sweeping process [65,135]. We will follow this path but only in the particular case of the GCS, where we give conditions on the sets $C(t)$ to assure that $h^{-1}\left(C(t)\right)$ be equi-uniformly subsmooth and then we apply Theorem 3.10 (see Theorem 4.4). In the general case we will transform the CDS into (4) and we show existence directly from Theorem 4.1 (see Theorem 4.3). This avoids the assumption of a special structure on the functions g and h, as the input-output property used in [39]. We emphasize that our main contributions in Section 4.3 are Theorems 4.3 and 4.4, where we prove the existence for CDS and GCS, respectively.

In Section 4.4 we study existence of solutions of the following control system:

$$\left\{\begin{array}{ll} \dot{w}(t) + \partial I_{K(v(t))}(w(t)) \ni h_1(w(t),v(t))u^1(t) & \text{a.e } t \in [T_0, T], \\ \dot{v}(t) + c(w(t),v(t))\dot{w}(t) = h_2(w(t),v(t))u^2(t) & \text{a.e. } t \in [T_0, T], \\ w(T_0) = w_0, v(T_0) = v_0, & \end{array}\right. \tag{7}$$

with the constraint

$$u(t) = \left(u^1(t), u^2(t)\right) \in U(t, v(t), w(t)) \quad \text{a.e. } t \in [T_0, T], \tag{8}$$

where $K(v) = [f_*(v), f^*(v)]$. Here c, h_1, h_2, f_*, f^* are given functions satisfying some mild hypotheses (see Assumption 4), $(w_0, v_0) \in \mathbb{R}^2$ is a given initial conditions with $w_0 \in K(v_0)$, and U is a set-valued map with closed, convex and bounded values in $\mathbb{R}^2$ satisfying standard hypotheses (see Assumption 5).

The system (7)-(8) describes many controlled input-output relations $u \mapsto w$ which are physically relevant, for example, solid-liquid phase transition with supercooling effect and martensite-austenite phase transition in shape memory alloys (see [109] and references therein), among others. The problem (7)-(8) has been studied by several authors (see, for example, [93,108,109,139]). In all these papers, the authors use the Moreau-Yosida regularization technique to obtain a family of approximated problems which converges to a solution of (7)-(8). We follow a different path by using Theorem 4.1, which is based on the reduction technique for the sweeping process. Furthermore, we consider the following particular case of (7)-(8):

$$\begin{cases} \dot{w}(t) + \partial I_{K(v(t))}(w(t)) \ni h_1(w(t), v(t)) & \text{a.e } t \in [T_0, T], \\ \dot{v}(t) + c(w(t), v(t))\dot{w}(t) = h_2(w(t), v(t)) & \text{a.e. } t \in [T_0, T], \\ v(T_0) = v_0, w(T_0) = w_0, \end{cases} \tag{9}$$

By considering the system (9) as a sweeping process, we obtain the following numerical algorithm of catching-up type, to solve this system:

$$\begin{cases} w_{i+1}^n = \operatorname{proj}_{K(v_i^n)}\left(w_i^n + \mu_n h_1^{n,i}\right), \\ v_{i+1}^n = v_i^n + \mu_n h_2^{n,i} - c^{n,i}\left(w_{i+1}^n - w_i^n\right). \end{cases}$$

This algorithm is different from the used in [109], where the authors discretize the Moreau-Yosida regularization of the normal cone and they obtain a numerical algorithm depending on two parameters. We illustrate our existence result by performing some numerical simulations with this new algorithm.

The main contribution of Section 4.4 is Theorem 4.7, which assure the existence of solutions to the problem (4.4)-(4.5) under mild assumptions. We emphasize that this result is new and improves the results given in [108, 109] by weakening the regularity on the functions f_*, f^*, c, h_1 and h_2. Moreover, Theorem 4.7 can by seen as a complement of [93, 139], where the authors assume that the set-valued map $(v, w) \to \operatorname{co} U(t, v, w)$ is Lipschitz continuous.

Chapter 5: Moreau-Yosida regularization of state-dependent sweeping process

In this chapter, which is based on [90], we study the state-dependent sweeping process, which correspond to the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N\left(C(t, x(t)); x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0, x_0). \end{cases} \tag{10}$$

The purpose of this chapter is to give an existence result of (10) for equi-uniformly subsmooth sets, under some compactness conditions. To do that, we use the Moreau-Yosida regularization technique, which consists in approaching the given differential inclusion by a penalized one, depending on a parameter, whose existence is easier to establish (for example, by using the classical Cauchy-Lipschitz theorem), and then to study the limit when the parameter goes to zero. More specifically, let $\lambda > 0$ and consider the differential inclusion:

$$\begin{cases} -\dot{x}_\lambda(t) \in \dfrac{1}{2\lambda}\partial d^2_{C(t,x_\lambda(t))}(x_\lambda(t)) & \text{a.e. } t \in [T_0, T], \\ x_\lambda(T_0) = x_0 \in C(T_0, x_0). \end{cases}$$

The first main contribution of this chapter is Theorem 5.9, which asserts the convergence (up to a subsequence) of $(x_\lambda)_\lambda$ to a Lipschitz solution of (10). Then, by using a reparametrization technique, in Theorem 5.10, we obtain the existence of solutions for (10) in the continuous bounded variation case (see Definition 5.5 for the meaning of solution in this case).

Furthermore, as a consequence of Theorems 5.9 and 5.10, we obtain the existence of solutions for the sweeping process with positively α-far sets in the Continuous Bounded Variation case (see Section 5.5). We end this chapter with an application to Hysteresis (see Section 5.6).

We emphasize that the Moreau-Yosida regularization technique has been used to deal only with convex or uniformly prox-regular sets (see [96, 105, 110, 111, 113, 130, 136] for more details), although it has never been used, even in the convex case, to study the state-dependent sweeping process.

Chapter 6: Moreau-Yosida regularization of perturbed state-dependent sweeping processes

In this chapter, which is based on [144], we study the state-dependent sweeping process, which correspond to the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N\left(C(t,x(t)); x(t)\right) + f(t,x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0, x_0), \end{cases} \tag{11}$$

where for any subset $S \subseteq H$ the set $N\left(S; u\right)$ denotes the Clarke normal cone to S at $u \in S$ and $f\colon [T_0, T] \times H \to H$ is a mapping which is measurable with respect to the first variable and either Lipschitzian or monotone with respect to the second variable.

The purpose of this chapter is to give an existence result of (11) for equi-uniformly subsmooth sets, under some compactness conditions. To do that, we use the Moreau-Yosida regularization technique, which consists in approaching the given differential inclusion by a penalized one, depending on a parameter, whose existence is easier to establish (for example, by using the classical Cauchy-Lipschitz theorem), and then to study the limit when the parameter goes to zero. More specifically, let $\lambda > 0$ and consider the differential inclusion:

$$\begin{cases} \dot{x}_\lambda(t) \in -\dfrac{1}{2\lambda}\partial d^2_{C(t,x_\lambda(t))}(x_\lambda(t)) + f(t, x_\lambda(t)) & \text{a.e. } t \in [T_0, T], \\ x_\lambda(T_0) = x_0 \in C(T_0, x_0). \end{cases}$$

The first main contribution of this chapter is Theorem 6.7, which asserts the convergence (up to a subsequence) of $(x_\lambda)_\lambda$ to a Lipschitz solution of (11). Furthermore, by using the same technique, we obtain the existence of solutions for the sweeping process with positively α-far sets (see Theorem 6.9).

Let us consider a lower semicontinuous convex function $\Phi\colon [T_0,T]\times H\to\mathbb{R}$. We say that a function $\Phi\colon [T_0,T]\times H\to\mathbb{R}$ is boundedly Lipschitz-continuous if for all $r>0$, there exists $L_r>0$ such that for $t\in[T_0,T]$, for all $(x,y)\in\bar{B}(0,r)\times\bar{B}(0,r)$

$$|\Phi(t,x)-\Phi(t,y)|\le L_r\|x-y\|.$$

Let us consider the following differential inclusion:

$$\begin{cases} -\dot{x}(t)\in N\left(C(t,x(t));x(t)\right)+\partial\Phi(t,x(t)) & \text{a.e. } t\in[T_0,T],\\ x(T_0)=x_0\in C(T_0,x_0), \end{cases} \tag{12}$$

The second main contribution of this chapter is the following result (see Theorem 6.10).

Theorem *Assume, in addition to $(\mathcal{H}_4)$, $(\mathcal{H}_6)$ and $(\mathcal{H}_8)$, that Φ is a positive, boundedly Lipschitz-continuous and convex function with $\Phi(t,0)\le m$, for some $m\ge 0$. Then, there exists at least one solution $x\in \operatorname{Lip}([T_0,T];H)$ of (12).*

Chapter 7: Perturbed Sweeping Processes with Nonlocal Initial Conditions

In this chapter, which is based on [88], we study differential inclusions with nonlocal initial conditions. We show existence for the perturbed sweeping process with nonlocal initial conditions. Moreover, through the concept of bounding functions and some tangential conditions, we prove existence for abstract differential inclusions with nonlocal initial conditions.

While existence for the sweeping process with Cauchy initial conditions is well known (see Section 3.6), the sweeping process with nonlocal initial condition has received relatively little attention. In the context of periodic sweeping processes, we can mention the works of Castaing and Monteiro-Marques [46, 48] for convex sets in Hilbert spaces and Gavioli [71] for wedges sets in finite dimensions.

The first part of this chapter is devoted to establishing some sufficient conditions for the existence of perturbed sweeping processes with nonregular sets and nonlocal initial conditions, that is, we consider the following differential inclusion:

$$\begin{cases} \dot{x}(t)\in -N\left(C(t);x(t)\right)+F(t,x(t)) & \text{a.e. } t\in[T_0,T],\\ x(T_0)=Mx, \end{cases} \tag{13}$$

where H is a separable Hilbert space, $C\colon [T_0, T] \rightrightarrows H$ is a set-valued map with nonempty and closed values, $N(S;\cdot)$ denotes the Clarke normal cone to S and $F\colon [T_0,T]\times H \rightrightarrows H$ is a given set-valued map with nonempty, closed and convex values. Here $M\colon C([T_0,T];H) \to H$ is an operator (possibly nonlinear) satisfying

$$\|Mx - My\| \le m\|x-y\|_\infty \quad \text{for all } x,y \in C([T_0,T];H), \tag{14}$$

with $m \in [0,1]$. The class of operators M satisfying the condition (14) is sufficiently large and includes the following well-known nonlocal initial conditions:

(i) $Mx = x_0$ (general Cauchy initial condition $x(T_0) = x_0$);

(ii) $Mx = \pm x(T)$ (periodic and anti-periodic initial conditions);

(iii) $Mx = \frac{1}{T-T_0}\int_{T_0}^{T} x(s)ds$ (mean value initial condition);

(iv) $Mx = \sum_{i=1}^{k_0} \alpha_i x(t_i)$ with $\alpha_i \in \mathbb{R}$ and $\sum_{i=1}^{k_0} |\alpha_i| \le 1$, where $T_0 < t_1 < \cdots < t_{k_0} \le T$ (multi-point initial condition).

We combine the Galerkin-Like method (see Chapter 3) with the reduction technique for the sweeping process (see, e.g., [75, 135]). The reduction technique associates to every sweeping process an unconstrained differential inclusion, whose solutions are also solutions of the sweeping process. In order to apply this method, the moving sets must to be positively α-far (see Chapter 2).

We distinguish between the contractive case: there exists $m \in [0,1)$ such that

$$\|Mx - My\| \le m\|x-y\|_\infty \quad \text{for all } x,y \in C([T_0,T];H), \tag{15}$$

and the nonexpansive case:

$$\|Mx - My\| \le \|x-y\|_\infty \quad \text{for all } x,y \in C([T_0,T];H). \tag{16}$$

Moreover, due to the constrained nature of the sweeping process, our results are associated with the existence of a convex set D so that $M\mathcal{C} \subseteq D \subseteq C(T_0)$, where

$$\mathcal{C} := \{x \in C([T_0,T]) : x(t) \in C(t) \text{ for all } t \in [T_0,T]\} \tag{17}$$

The following result, which is the main contribution of Section 7.2, asserts the existence of solutions for (13) (See Theorems 7.2 and 7.3).

Theorem *Let $F\colon [T_0,T]\times H \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$ and $C\colon [T_0,T] \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$. Assume that M is sequentially weakly upper semicontinuous and that there exists a convex set D such that $M\mathcal{C} \subseteq D \subseteq C(T_0)$, where $\mathcal{C}$ is given by (17) and*

$$\left(1 + \frac{1}{\alpha_0^2}\right)\int_{T_0}^{T} \left(|\dot{\zeta}(s)| + \beta(s)\right) ds < \rho. \tag{18}$$

Assume that one of the following two conditions is satisfied:

i) (15) *holds.*

ii) D *is bounded and* (16) *holds.*

Then, there exists at least one solution of (13)*. Moreover,*

$$\|\dot{x}(t)\| \leq \frac{1}{\alpha_0^2}|\dot{\zeta}(t)| + \left(1 + \frac{1}{\alpha_0^2}\right)\beta(t) \quad \textit{a.e. } t \in [T_0, T].$$

When M is a linear and continuous operator, M is sequentially weakly upper semicontinuous (see hypothesis $(\mathcal{H}_3^M)$) and condition (18) in the last theorem can be removed (see Theorem 7.4).

Theorem *Let* $F\colon [T_0, T] \times H \rightrightarrows H$ *be a set-valued map satisfying* $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ *and* $(\mathcal{H}_3^F)$ *and* $C\colon [T_0, T] \rightrightarrows H$ *be a set-valued map satisfying* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ *and* $(\mathcal{H}_3)$. *Assume that* M *is linear and continuous and there exists a convex set* D *such that* $M\mathcal{C} \subseteq D \subseteq C(T_0)$, *where* $\mathcal{C}$ *is given by* (17)*. Assume that one of the following two conditions is satisfied:*

i) (15) *holds.*

ii) D *is bounded and* (16) *holds.*

Then, there exists at least one solution of (13)*.*

The second part of the chapter is concerned with existence of abstract differential inclusions with nonlocal initial conditions. To deal with it, we use the concept of bounding functions and some tangential conditions. We say that V is a (weak/strong) bounding function for a differential inclusion (see Definition below), when the existence of this function implies the existence of an a priori bound for the solutions of the differential inclusion. Typically, the bounding function has to satisfy some conditions involving the derivatives of V (in some sense) and the right-hand side of the differential inclusion. The idea of bounding functions was introduced by Mawhin [103] to deal with second order boundary value problems. Since then, it was systematically used for the study of various boundary problems (see [21, 120] and the references therein). In [103], Mawhin imposes a specific condition relative to the second order derivatives of V, which implies the boundedness of the solution for the second order boundary value problem. For the case of first order differential inclusions, the concept of bounding function involves conditions on the first order derivatives of V and the right-hand side of the differential inclusion, in some ring or localized in the boundary of some bounded set. Thus, the concept of bounding function is vague and highly depending on the method to deal with the differential inclusion. Our definition of weak bounding function (see Definition below) is taken from [21]. The use of bounding functions, generally, is related to the Leray-Schauder continuation principle and the topological degree theory (see [21] for more details). We emphasize that our approach make no appeal to these tools from nonlinear analysis but merely basic elements of set-valued and variational analysis.

Let $F\colon [T_0, T] \times H \rightrightarrows H$ be a set-valued map satisfying hypotheses $(\mathcal{H}_1^F)$ and $(\mathcal{H}_2^F)$. In Section 7.3 we study existence of solutions for the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = Mx, \end{cases} \tag{19}$$

where $M\colon C([T_0, T]; H) \to H$ is a (possibly nonlinear) operator and F satisfies the additional hypothesis (see Section 7.1): $(H, \|\cdot\|_H)$ is compactly embedded in a separable Banach space $(E, \|\cdot\|_E)$ (for example, $H = H^1(\Omega)$ and $E = L^2(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is an open domain with Lipschitz boundary) and

$(\mathcal{H}_5^F)$ For a.e. $t \in [T_0, T]$, $F(t, \cdot)$ is closed from E into E_w, that is, $\operatorname{graph} F(t, \cdot)$ is closed in $E \times E_w$.

We emphasize that several control problem for first-order partial integro-differential equations (e.g., with $H = H^1(\Omega)$ and $E = L^2(\Omega)$) can be formulated as (19) (see, e.g., [20, 21]). To show existence for (19) we use the notion of bounding functions.

Definition *Let $V\colon H \to \mathbb{R}$ be a locally Lipschitz function such that $V(x) = 0$ for $\|x\|_H = R_0$ and $V(x) < 0$ for $r_0 < \|x\|_H < R_0$.*

a) We say that V is a weak bounding function *if V is C^1 in the ring $\{x \in H : r_0 < \|x\|_H < R_0\}$ and there exists a sequence $(n_m)_m \subseteq \mathbb{N}$ converging to $+\infty$ such that for a.e. $t \in [T_0, T]$*

$$\inf_{d \in F(t, P_{n_m}(x))} \langle \nabla V(P_{n_m}(x)), P_{n_m}(d) \rangle \le 0 \quad \textit{for all } r_0 < \|P_{n_m}(x)\|_H < R_0.$$

b) We say that V is a strong bounding function *if there exists a sequence $(n_m)_m \subseteq \mathbb{N}$ converging to $+\infty$ such that for a.e. $t \in [T_0, T]$ and all $r_0 < \|P_{n_m}(x)\|_H < R_0$*

$$\sup_{d \in F(t, P_{n_m}(x))} \min\{DV(P_{n_m}(x); P_{n_m}(d)), D(-V)(P_{n_m}(x); -P_{n_m}(d))\} \le 0.$$

By using the notion of bounding function, we can prove an existence result for (19). The statement (ii) of the following theorem extends the results of [20] by allowing to M be a nonlinear map. Moreover, statement (iii) of the following theorem extends [101, Theorem 7] to infinite dimensions and extends the main result of [21] by allowing to M to be a nonlinear map and F to be multivalued and upper semicontinuous from E into E_w.

Theorem (Theorem 7.8) *Assume that H is compactly embedded in E. Assume that $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_5^F)$ hold, M is sequentially weakly upper semicontinuous and that one of the following conditions is verified:*

(i) $(\mathcal{H}_3^F)$ and (15) *hold.*

(ii) $(\mathcal{H}_4^F)$ and (16) *hold, $M(C([T_0, T]; R_0\mathbb{B}_H)) \subseteq R_0\mathbb{B}_H$ and there exists a weak bounding function V for F.*

(iii) $(\mathcal{H}_4^F)$ *and* (16) *hold,* $M\left(C\left([T_0,T];R_0\mathbb{B}_H\right)\right) \subseteq R_0\mathbb{B}_H$ *and there exists a strong bounding function* V *for* F.

Then, there exists at least one solution of (19).

In Section 7.4, we use some tangential conditions to get the existence of abstract nonlocal differential inclusion in finite dimensions. These tangential conditions, related with the weak invariance of differential inclusions, typically, involves the intersection between the Bouligand tangent cone and the right-hand side of the differential inclusion. Since we apply a fixed point theorem to the solution map of the differential inclusion, a strong property is needed, namely, the intersection between the Clarke tangent cone and the right-hand side of the differential inclusion is nonempty (see Remark 7.8). The following result is the main contribution of Section 7.4.

Theorem (Theorem 7.10) *Let* H *be a finite-dimensional Hilbert space. Assume that* $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ *and* $(\mathcal{H}_4^F)$ *hold. Let* M *be a Lipschitz map such that there exists a closed, contractible, positively* α*-far and bounded set* D *such that* $M\left(C\left([T_0,T];D\right)\right) \subseteq D$ *and*

$$F(t,x)\cap T(D;x)\neq\emptyset \quad \text{for all } (t,x)\in[T_0,T]\times D$$

Then, there exists at least one solution of (19). *Moreover,* $x(t)\in D$ *for all* $t\in[T_0,T]$.

Finally, in Sections 7.5 and 7.6, we give, respectively, some applications to nonlocal differential complementarity systems and vector hysteresis.

Chapter 8: Existence and Lyapunov pairs for the perturbed sweeping process governed by a fixed set

In this chapter, which is based on [143], we study the perturbed sweeping process governed by a fixed set, that is, the following differential inclusion:

$$\begin{cases} \dot{x}(t)\in -N\left(S;x(t)\right)+F(t,x(t)) & \text{a.e. } t\in[T_0,T],\\ x(T_0)=x_0\in S, \end{cases} \tag{20}$$

where $S\subseteq H$ is a merely closed and ball-compact set, $N\left(S;x\right)$ denotes the Clarke normal cone to S at x and $F\colon[T_0,T]\times H\rightrightarrows H$ is a given set-valued map with nonempty closed and convex values.

The first part of this chapter (Section 8.1) is devoted to give an existence result of (10) for a merely ball compact set S. To do that, we use the Catching-up algorithm, which is the following algorithm: Put $x_0^n := x_0\in S$ and for $k=0,\ldots,n-1$ we consider

$$x_{k+1}^n\in \operatorname{Proj}_S(x_k^n+\mu_n f(t_k^n,x_k^n)),$$

where $\mu_n := (T-T_0)/n$ and $f(t,x) := \operatorname{proj}_{F(t,x)}(0)$. The contribution of Section 8.1 is Theorem 8.1, which asserts the convergence of the Catching-up (up to a subsequence) to a solution of (20) for a merely closed and ball-compact set S. This new result is in line with [19, 57, 135] and extends the result given in [131] for bounded sleek sets.

The second part of this chapter (Section 8.2) is concerned with an explicit characterization of Lyapunov pairs for (20). Let $V\colon H \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function and $W\colon H \to \mathbb{R}$ be continuous. We say that (V, W) forms a weak Lyapunov pair for the perturbed sweeping process (20) if for every $x_0 \in S$ the exists x solution of (20) such that

$$V(x(t)) + \int_{T_0}^{t} W(x(s))ds \leq V(x_0) \quad \text{for all } t \in [T_0, T].$$

The main contribution of Section 8.2 is Theorem 8.4, which, under appropriate conditions on F, affirms the equivalence of the following assertions:

(i) For a.e. $t \in [T_0, T]$, $x \in \operatorname{dom} V$ and $\zeta \in \partial^P V(x)$

$$\inf\{\langle v, \zeta\rangle : v \in -\alpha(t)h(x)\partial d_S(x) + F(t,x)\} \leq -W(x),$$

(ii) (V, W) forms a weak Lyapunov pair for the sweeping process (8.1).

As a direct consequence of this characterization, we obtain, by taking $V = I_S$ and $W \equiv 0$, the existence of solutions for (8.1) (see Theorem 8.5) and, by taking $V = I_K$ and $W \equiv 0$, the following characterization for weak invariance (see Theorem 8.7): Let $K \subseteq S$ be a closed set. Then the following conditions are equivalent:

(i) For a.e. $t \in [T_0, T]$, for all $x \in K$ and $\zeta \in N^P(K; x)$

$$\inf\{\langle v, \zeta\rangle : v \in -\alpha(t)h(x)\partial d_S(x) + F(t,x)\} \leq 0,$$

(ii) For all $x_0 \in K$ there exists a solution of the sweeping process (8.1) with $x(T_0) = x_0$ and $x(t) \in K$ for all $t \in [T_0, T]$.

We end this chapter with applications to Hysteresis and Crowd Motion (see Section 8.3).

Chapter 9: Lyapunov pairs for Perturbed Sweeping Processes

In this chapter, we investigate Lyapunov pairs for the perturbed sweeping processes:

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in S, \end{cases} \tag{21}$$

where $C\colon [T_0, T] \rightrightarrows H$ is a set-valued map with nonempty and closed values, $N(S; x)$ denotes the Clarke normal cone to S at x and $F\colon [T_0, T] \times H \rightrightarrows H$ is a given set-valued map with nonempty closed and convex values.

This chapter is concerned with an explicit characterization of Lyapunov pairs for (21). Let $V\colon [T_0, T] \times H \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function with $\operatorname{dom} V(t, \cdot) \subseteq C(t)$ for all $t \in [T_0, T]$ and $W\colon [T_0, T] \times H \to \mathbb{R}$ be a proper lower semicontinuous function. We say that (V, W) forms a weak Lyapunov pair for the perturbed sweeping process (21) if for every $x_0 \in C(T_0)$ the exists x solution of (20) such that

$$V(t, x(t)) + \int_{T_0}^{t} W(s, x(s))ds \leq V(T_0, x_0) \quad \text{for all } t \in [T_0, T].$$

The main contribution of this Chapter is Theorem 9.2, which, under appropriate conditions on F, affirms the equivalence of the following assertions:

(i) For all $(t, x) \in \operatorname{dom} V$ and $(\theta, \zeta) \in \partial^P V(t, x)$

$$\theta + \inf\{\langle v, \zeta\rangle : v \in -\left(\kappa + \alpha(t)h(x)\right)\partial d_{C(t)}(x) + F(t, x)\} \leq -W(t, x).$$

(ii) (V, W) forms a weak Lyapunov pair for the sweeping process (8.1).

As a direct consequence of this characterization, we obtain, by taking $V = I_{C(t)}$ and $W \equiv 0$, the existence of solutions for (21) and, by taking $V = I_{K(t)}$ and $W \equiv 0$, the following characterization for weak invariance (see Theorem 9.6): Assume that $\operatorname{graph} K \subseteq \operatorname{graph} C$. Then the following conditions are equivalent:

(i) For all $(\theta, \zeta) \in N^P(\operatorname{graph} K; (t, x))$

$$\theta + \inf\{\langle v, \zeta\rangle : v \in -\left(\kappa + \alpha(t)h(x)\right)\partial d_{C(t)}(x) + F(t, x)\} \leq 0.$$

(ii) For all $x_0 \in K$ there exists a solution of the sweeping process (9.1) with $x(T_0) = x_0$ and $x(t) \in K(t)$ for all $t \in [T_0, T]$.

We illustrate our results with an application to gradient complementarity dynamical systems.

Chapter 1

Preliminaries

In this chapter we describe the notation, the definitions and basic results that are going to be used throughout the thesis.

From now on H, U, V, X and Y stand for separable Hilbert spaces whose norm is denoted by $\|\cdot\|$. The closed ball centered at x with radius r is defined by $\bar{B}(x,\rho) := \{y \in H \colon \|x-y\| \leq \rho\}$ and the closed unit ball is denoted by $\mathbb{B}$. The notation H_w stands for H equipped with the weak topology and $x_n \rightharpoonup x$ denotes the weak of a sequence $(x_n)_n$ to x (similar notation for U_w, V_w, X_w and Y_w).

Given $S \subseteq H$, we say that S is ball compact if, for any, $r > 0$, the set $S \cap r\mathbb{R}$ is compact.

Recall that a vector $h \in H$ belongs to the Clarke tangent cone $T(S;x)$ (see [52]) when for every sequence $(x_n)_n$ in S converging to x and every sequence of positive numbers $(t_n)_n$ converging to 0, there exists some sequence $(h_n)_n$ in H converging to h such that $x_n + t_n h_n \in S$ for all $n \in \mathbb{N}$. This cone is closed and convex, and its negative polar $N(S;x)$ is the Clarke normal cone to S at $x \in S$, that is,

$$N\left(S;x\right) = \left\{v \in H \colon \langle v, h\rangle \leq 0 \quad \forall h \in T(S;x)\right\}.$$

As usual, $N(S;x) = \emptyset$ if $x \notin S$. Through that normal cone, the Clarke subdifferential of a function $f \colon H \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\partial f(x) := \left\{v \in H \colon (v,-1) \in N\left(\operatorname{epi} f, (x, f(x))\right)\right\},$$

where $\operatorname{epi} f := \{(y,r) \in H \times \mathbb{R} \colon f(y) \leq r\}$ is the epigraph of f. When the function f is finite and locally Lipschitzian around x, the Clarke subdifferential is characterized (see [55]) in the following simple and amenable way

$$\partial f(x) = \left\{v \in H \colon \langle v, h\rangle \leq f^{\circ}(x;h) \text{ for all } h \in H\right\},$$

where

$$f^{\circ}(x;h) := \limsup_{(t,y)\to(0^+,x)} t^{-1}\left[f(y+th) - f(y)\right],$$

is the *generalized directional derivative* of the locally Lipschitzian function f at x in the direction $h \in H$. The function $f^{\circ}(x;\cdot)$ is in fact the support function of $\partial f(x)$. That characterization easily yields that the Clarke subdifferential of any locally Lipschitzian function has the important property of upper semicontinuity from H into H_w (see definition below).

The support function of $S \subseteq H$, is defined, for any $v \in H$, by

$$\sigma(v, S) := \sup_{s \in S} \langle v, s \rangle .$$

The weak tangent cone to a set S at $x \in S$ is defined by

$$T^w(S;x) := \{v \in H\colon \text{ there exists } t_n \searrow 0, v_n \rightharpoonup v \text{ such that } x + t_n v_n \in S\}.$$

Given $x \in S$, we say that $v \in H$ belong to the Bouligand tangent cone $T^B(S;x)$ (see [52]), when there exist $v_n \to v$ and $t_n \to 0^+$ such that $x + t_n v_n \in S$ for all $n \in \mathbb{N}$. By the very definition of $T^B(S;x)$, it is clear that

$$T^C(S;x) \subseteq T^B(S;x) \quad \text{for all } x \in S.$$

Moreover, equality holds when S is convex.

Given a lower semicontinuous function $f\colon H \to \mathbb{R}$, we define the *Dini directional derivative* of f at x in the direction v, denoted $Df(x;v)$, as

$$Df(x;v) := \liminf_{w \to v, t \downarrow 0} \frac{f(x+tw) - f(x)}{t}.$$

Moreover, if f is locally Lipschitz, then $Df(x;v) = \liminf_{t \downarrow 0} t^{-1}\left[f(x+tv) - f(x)\right]$.

For $x \in H$ and $S \subseteq H$ the distance function is defined by $d_S(x) := \inf_{y \in S} \|x - y\|$. We denote $\mathrm{Proj}_S(x)$ the (possibly empty) set of projections over S, i.e.,

$$\mathrm{Proj}_S(x) := \{y \in S\colon d_S(x) = \|x - y\|\}.$$

The equality (see [55])

$$N(S;x) = \overline{\mathbb{R}_+ \partial d_S(x)} \quad \text{for } x \in S, \tag{1.1}$$

gives an expression of the Clarke normal cone in terms of the distance function. As usual, it will be convenient to write $\partial d(x, S)$ in place of $\partial d(\cdot, S)(x)$.

Remark 1.1 In this thesis, we will calculate the Clarke subdifferential of the distance function to a moving set. By doing so, the subdifferential will be always calculated with respect to the variable involved in the distance function by assuming that the set is fixed. More explicitly, $\partial d_{C(t,y)}(x)$ means the subdifferential of the function $d_{C(t,y)}(\cdot)$ (here $C(t,y)$ is fixed) calculated at the point x, i.e., $\partial\left(d_{C(t,y)}(\cdot)\right)(x)$. In the same way, $\partial d_{C(t,x)}(x)$ means the subdifferential of the function $d_{C(t,x)}(\cdot)$ (here $C(t,x)$ is a fixed set) calculated at the point x, i.e., $\partial\left(d_{C(t,x)}(\cdot)\right)(x)$.

The following formula gives a representation of the subdifferential of the distance to a closed set $S \subseteq H$ (see [84]).

$$\partial d_S(x) = \bigcap_{\gamma>0} \overline{\mathrm{co}} \left(\frac{x - \mathrm{proj}_S^\gamma(x)}{d_S(x)} \right), \tag{1.2}$$

where $\mathrm{proj}_S^\gamma(x) = \{z \in S \colon \|x - z\| < d_S(x) + \gamma\}$.

Let $f \colon H \to \mathbb{R} \cup \{+\infty\}$ be an lower semicontinuous function and $x \in \mathrm{dom}\, f$. An element ζ belongs to the proximal subdifferential (see [55, Chapter 1]) $\partial_P f(x)$ of f at x if there exist two positive numbers σ and η such that

$$f(y) \geq f(x) + \langle \zeta, y - x\rangle - \sigma\|y - x\|^2 \quad \forall y \in B(x; \eta).$$

Moreover, the following equivalence holds (see [55, Chapter 1]):

$$\zeta \in \partial^P f(x) \quad \Leftrightarrow \quad (\zeta, -1) \in N^P\left(\mathrm{epi}\, f; (x, f(x))\right). \tag{1.3}$$

The following result (see [138, Proposition 2.6]) will be used in Chapters 8 and 9.

Proposition 1.1 *Let $f \colon H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, $x \in \mathrm{dom}\, f$ and let $x^* \in H$ with $(x^*, 0) \in N^P\left(\mathrm{epi}\, f; (x, f(x))\right)$. Then, for any $\varepsilon > 0$ there exist $x_\varepsilon \in \mathrm{dom}\, f$ and $(x_\varepsilon^*, -r_\varepsilon) \in N^P\left(\mathrm{epi}\, f; (x_\varepsilon, f(x_\varepsilon))\right)$ with $r_\varepsilon > 0$ along with*

$$\|x_\varepsilon - x\| + |f(x_\varepsilon) - f(x)| < \varepsilon \text{ and } \|(x_\varepsilon^*, -r_\varepsilon) - (x^*, 0)\| < \varepsilon.$$

The limiting proximal subdifferential (see [55, Chapter 1]) is defined by

$$\partial_L f(x) := \left\{w\text{-}\lim \zeta_i \colon \zeta_i \in \partial_P f(x_i), x_i \to x, f(x_i) \to f(x)\right\}.$$

When f is locally Lipschitz, the following formula holds: $\partial f(x) = \mathrm{cl}\,\mathrm{co}\,\partial_L f(x)$.

The following lemma will be used in the proof of Lemma 5.1.

Lemma 1.2 *Let $S \subseteq H$ be a closed set. Then, for $x \notin S$ and $s \in \mathrm{Proj}_S(x)$ we have $\frac{x-s}{\|x-s\|} \in \partial_L d_S(s)$.*

Proof. According to the definition of the proximal normal cone and [55, Proposition 1.1.3], we have $x - s \in N^P(S; s)$. Also, by exact penalization (see [55, Proposition 1.6.3]), for all $\varepsilon > 0$, $\frac{x-s}{\|x-s\|+\varepsilon} \in \partial_P d_S(s)$. Then, by taking $\varepsilon \downarrow 0$, we get the result. □

1.1 Spaces of functions

We denote by $L^1([T_0, T]; H)$ the space of H-valued Lebesgue integrable functions defined over $[T_0, T]$. We write $L_w^1([T_0, T]; H)$ to mean the space $L^1([T_0, T]; H)$ endowed with the weak topology. A set $K \subseteq L^1([T_0, T]; H)$ is uniformly integrable if

$$\lim_{\lambda\to+\infty}\left[\sup_{f\in K}\int_{\{\|f\|\geq\lambda\}}\|f(s)\|ds\right]=0.$$

Moreover, if there exists $\psi \in L^1(T_0, T)$ such that for all $f \in K$

$$\|f(t)\| \leq \psi(t) \quad \text{a.e. } t \in [T_0, T],$$

then the set K is uniformly integrable. We recall the Dunford-Pettis theorem (see [70, Theorem 2.3.24]), which characterizes relatively weakly compact subsets of $L^1(\Omega)$.

Theorem 1.3 (Dunford-Pettis theorem) *Let H be a Hilbert space. A bounded set $K \subseteq L^1([T_0, T]; H)$ is relatively weakly compact in $L^1([T_0, T]; H)$ if and only if it is uniformly integrable.*

We recall the following characterization of weak convergence in $C([T_0, T]; H)$ (see [24, Theorem 4.2]).

Lemma 1.4 *$(x_n)_n \subseteq C([T_0, T]; H)$ weakly converges in $C([T_0, T]; H)$ to x if and only if $(x_n)_n$ is bounded in $C([T_0, T]; H)$ and $x_n(t) \rightharpoonup x(t)$ for all $t \in [T_0, T]$.*

We say that $u \in W^{1,1}([T_0, T]; H)$ if there exists $f \in L^1([T_0, T]; H)$ and $u_0 \in H$ such that

$$u(t) = u_0 + \int_{T_0}^{t} f(s)ds \text{ for all } t \in [T_0, T].$$

Moreover, we say that $u \in \mathrm{W}^{2,1}([T_0, T]; H)$ if $\dot{u} \in W^{1,1}([T_0, T]; H)$. Furthermore, for $u\colon [T_0, T] \to H$ we define

$$\mathrm{Lip}(u) := \sup_{t\neq s}\|u(t) - u(s)\|/|t - s|.$$

and

$$\mathrm{Lip}([T_0, T]; H) := \{u\colon [T_0, T] \to H\colon \ \mathrm{Lip}(u) < +\infty\},$$

the space of H-valued Lipschitz functions.

We recall the classical Arzela-Ascoli theorem (see [70, Theorem 2.3.2]), which characterizes the relatively compact subsets of $C([T_0, T]; H)$.

Theorem 1.5 (Arzela-Ascoli theorem) *A set $K \subseteq C([T_0, T]; H)$ is relatively compact if and only if*

a) for every $t \in [T_0, T]$, the set $K(t) := \{u(t)\colon u \in K\}$ is relatively compact in X; and

b) K is uniformly equicontinuous, i.e., for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that, if $t, s \in [T_0, T]$ and $|t - s| < \delta$, then

$$\|u(t) - u(s)\| < \varepsilon \quad \textit{for all } u \in K.$$

The following Lemma, proved in [90], is a sufficient condition for compactness of absolutely continuous functions.

Lemma 1.6 *Let $(x_n)_n$ be a sequence of absolutely continuous functions from $[T_0, T]$ into H with $x_n(T_0) = x_0^n$. Assume that for all $n \in \mathbb{N}$*

$$\|\dot{x}_n(t)\| \leq \psi(t) \quad a.e\ t \in [T_0, T], \tag{1.4}$$

where $\psi \in L^1(T_0, T)$ and that $x_0^n \to x_0$ as $n \to +\infty$. Then, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and an absolutely continuous function x such that

(i) $x_{n_k}(t) \rightharpoonup x(t)$ in H as $k \to +\infty$ for all $t \in [T_0, T]$,

(ii) $x_{n_k} \rightharpoonup x$ in $L^1([T_0, T]; H)$ as $k \to +\infty$,

(iii) $\dot{x}_{n_k} \rightharpoonup \dot{x}$ in $L^1([T_0, T]; H)$ as $k \to +\infty$,

(iv) $\|\dot{x}(t)\| \leq \psi(t)$ a.e. $t \in [T_0, T]$.

Proof. On one hand, let us consider $K := \{\dot{x}_n \colon n \in \mathbb{N}\} \subseteq L^1([T_0, T]; H)$. According to (1.4), the set K is bounded and uniformly integrable (see [70, Theorem A.2.5]). Thus, as a result of the Dunford-Pettis theorem (see Theorem 1.3), K is compact in $L^1_w([T_0, T]; H)$. Therefore, there exists a subsequence of $(\dot{x}_{n_k})_k$ of $(\dot{x}_n)_n$ converging to some v in $L^1_w([T_0, T]; H)$. Define $S := \{x_{n_k} \colon k \in \mathbb{N}\} \subseteq L^1([T_0, T]; H)$. Thus, due to (1.4), for every $x_{n_k} \in S$ we have

$$\|x_{n_k}(t)\| \leq \|x_0^{n_k}\| + \int_{T_0}^{t} \psi(s) ds \quad t \in [T_0, T], \tag{1.5}$$

which implies, by virtue of the Dunford-Pettis theorem, that S is compact subset of $L^1_w([T_0, T]; H)$. Consequently, there exists a subsequence $(x_{n_k})_k$ (without relabeling) of $(x_{n_k})_k$ converging to some x in $L^1_w([T_0, T]; H)$.
On the other hand, due to (1.4) and (1.5), the sequence $(x_{n_k})_k$ is uniformly bounded in $W^{1,1}([T_0, T]; H)$ and in $L^\infty([T_0, T]; H)$. Therefore, due to [111, Theorem 0.2.2.1], there exists a subsequence $(x_{n_k})_k$ (without relabeling) of $(x_{n_k})_k$ and a function $\tilde{x}$ such that $\|\tilde{x}(t)\| \leq \psi(t)$ a.e. $t \in [T_0, T]$ and

$$x_{n_k}(t) \to \tilde{x}(t) \text{ weakly as } k \to +\infty \text{ for all } t \in [T_0, T]. \tag{1.6}$$

Moreover, by virtue of [70, Proposition 2.3.31], $x \equiv \tilde{x}$, which proves (iv). Now, we prove that $v = \dot{x}$. Indeed, let $w \in H$ and $t \in [T_0, T]$ be fixed. Then,

$$\langle x_{n_k}(t) - x_0^{n_k}, w\rangle = \int_{T_0}^{t} \langle \dot{x}_{n_k}(s), w\rangle\, ds = \int_{T_0}^{T} \left\langle \dot{x}_{n_k}(s), w \cdot \mathbb{1}_{[T_0,t]}(s)\right\rangle ds, \tag{1.7}$$

where

$$\mathbb{1}_{[T_0,t]}(s) := \begin{cases} 1, & \text{if } s \in [T_0, t], \\ 0, & \text{if } s \in]t, T], \end{cases}$$

belongs to $L^\infty([T_0,T];H)$. Hence, using (1.6), the weak of $\dot{x}_{n_k}$ to v in $L^1([T_0,T];H)$ and passing to the limit in (1.7), we obtain

$$\langle x(t)-x_0, w\rangle = \int_{T_0}^t \langle v(s), w\rangle\, ds \quad \text{for all } w\in H,$$

which implies that $x(t)-x_0=\int_{T_0}^t v(s)ds$ for all $t\in[T_0,T]$. Hence $v=\dot{x}$. Therefore, (i), (ii), (iii) and (iv) hold. □

Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H. For every $n\in\mathbb{N}$ we consider the linear operator P_n from H into span $\{e_1,\ldots,e_n\}$ defined by

$$P_n\left(\sum_{k=1}^{\infty}\langle x, e_k\rangle e_k\right)=\sum_{k=1}^{n}\langle x,e_k\rangle e_k.$$

The following lemma summarize the main properties of the linear operator P_n.

Lemma 1.7

(i) $\|P_n(x)\|\le\|x\|$ *for all* $x\in H$*;*

(ii) $\langle P_n(x), x-P_n(x)\rangle = 0$ *for all* $x\in H$*;*

(iii) $P_n(x)\to x$ *as* $n\to+\infty$ *for all* $x\in H$*;*

(iv) if $(x_n)_n$ *is a bounded sequence with* $x_n\rightharpoonup x$ *as* $n\to+\infty$ *then* $P_n(x_n)\rightharpoonup x$ *as* $n\to+\infty$*;*

(v) if $B\subseteq H$ *is relatively compact then* $\sup_{x\in B}\|x-P_n(x)\|\to 0$ *as* $n\to+\infty$.

Proof. It is enough to prove (iv): Let $j\in\mathbb{N}$. Then, for $n\ge j$:

$$\langle x_n-P_n(x_n), e_j\rangle=\sum_{k=n+1}^{+\infty}\langle x_n,e_k\rangle\langle e_k,e_j\rangle=0.$$

Thus, by linearity,

$$\lim_{n\to+\infty}\langle x_n-P_n(x_n), v\rangle = 0 \quad \forall v\in \operatorname{span}\left(\{e_j\}_{j\in\mathbb{N}}\right).$$

Let $v\in H$. Then, there is $v_m\to v$ with $v_m\in\operatorname{span}(\{e_j\}_{j\in\mathbb{N}})$. Hence,

$$\begin{aligned}|\langle x_n-P_n(x_n),v\rangle| &\le |\langle x_n-P_n(x_n), v-v_m\rangle|+|\langle x_n-P_n(x_n),v_m\rangle|\\ &\le \|x_n-P_n(x_n)\|\cdot\|v_m-v\|+|\langle x_n-P_n(x_n),v_m\rangle|\\ &\le 2\sup_{n\in\mathbb{N}}\|x_n\|\cdot\|v_m-v\|+|\langle x_n-P_n(x_n),v_m\rangle|.\end{aligned}$$

Therefore, taking the limit $n\to+\infty$ and then the limit $m\to+\infty$ we get the result. □

1.2 Measures of noncompactness

Let A be a bounded subset of H. We define the Kuratowski measure of noncompactness of A, $\alpha(A)$, as

$$\alpha(A) = \inf\{d > 0 \colon A \text{ admits a finite cover by sets of diameter } \leq d\},$$

and the Hausdorff measure of noncompactness of A, $\beta(A)$, as

$$\beta(A) = \inf\{r > 0 \colon A \text{ can be covered by finitely many balls of radius } r\}.$$

The following result gives the main properties of the Kuratowski and Hausdorff measure of noncompactness (see [62, Proposition 9.1 from Section 9.2]).

Proposition 1.8 *Let H be a Hilbert space and B, B_1, B_2 be bounded subsets of H. Let γ be the Kuratowski or the Hausdorff measure of noncompactness. Then,*

(i) $\gamma(B) = 0$ *if and only if* $\operatorname{cl}(B)$ *is compact.*

(ii) $\gamma(\lambda B) = |\lambda|\gamma(B)$ *for every* $\lambda \in \mathbb{R}$.

(iii) $\gamma(B_1 + B_2) \leq \gamma(B_1) + \gamma(B_2)$.

(iv) $B_1 \subseteq B_2$ *implies* $\gamma(B_1) \leq \gamma(B_2)$.

(v) $\gamma(\operatorname{conv} B) = \gamma(B)$.

(vi) $\gamma(\operatorname{cl}(B)) = \gamma(B)$.

The following lemma (see [62, Proposition 9.3]) is a useful rule for the interchange of γ and integration.

Lemma 1.9 *Let (v_n) be a sequence of measurable functions $v_n \colon [T_0, T] \to H$ such that $\sup_n \|v_n(t)\| \leq \psi(t)$ a.e. $t \in [T_0, T]$, where $\psi \in L^1(T_0, T)$. Then*

$$\gamma\left(\left\{\int_t^{t+h} v_n(s)ds \colon n \in \mathbb{N}\right\}\right) \leq \int_t^{t+h} \gamma\left(\{v_n(s) \colon n \in \mathbb{N}\}\right) ds,$$

for $T_0 \leq t < t + h \leq T$.

1.3 Set-valued maps

The following definitions related to set-valued mappings will be needed in the sequel.

Definition 1.10 *Let $\mathcal{L}$ be the Lebesgue σ-field on $[T_0, T]$ and $\Phi \colon [T_0, T] \rightrightarrows H$. We say that Φ is measurable if the set*

$$\Phi^{-1}(C) := \{t \in [T_0, T] \colon \Phi(t) \cap C \neq \emptyset\} \in \mathcal{L},$$

for any closed subset $C \subseteq H$.

Moreover, if $\Phi\colon [T_0, T] \rightrightarrows H$ has nonempty, closed, convex and bounded values, $\Phi\colon [T_0, T] \rightrightarrows H$ is measurable if and only if its support function $t \mapsto \sigma(v, \Phi(t))$ is $\mathcal{L}$-measurable for all $v \in H$. Furthermore, if $\mathcal{B}$ denotes the Lebesgue Borel σ-field on H and $\Psi\colon [T_0, T] \times H \rightrightarrows H$ is a set-valued map with nonempty, closed, convex and bounded values, Ψ is said $\mathcal{L} \otimes \mathcal{B}$-measurable if its support function $(t, x) \mapsto \sigma(v, \Psi(t, x))$ is $\mathcal{L} \otimes \mathcal{B}$-measurable for all $v \in H$.

Definition 1.11 *A set-valued map $\Psi\colon H \rightrightarrows H$ is said to be upper semicontinuous from H into H_w if the set*

$$\Psi^{-1}(C) := \{x \in H\colon \Psi(x) \cap C \neq \emptyset\}$$

is norm-closed for any C weakly closed set of H.

Furthermore, if $\Psi\colon H \rightrightarrows H$ has nonempty, closed, convex and bounded values, Ψ is upper semicontinuous from H into H_w if and only if its support function $x \mapsto \sigma\left(v, \Psi(x)\right)$ is upper semicontinuous for all $v \in H$.

We recall the Kakutani-Fan-Glicksberg fixed point (see [9, Corollary 17.55]), which will be used in the sequel.

Theorem 1.12 (Kakutani-Fan-Glicksberg) *Let K be a nonempty compact convex subset of a locally convex Hausdorff space, and let $\mathcal{F}\colon K \rightrightarrows K$ be a set-valued map with closed graph and nonempty convex values. Then the set of fixed point of $\mathcal{F}$ is compact and nonempty.*

A metric space X is called contractible if there exist a point $x_0 \in X$ and a continuous map (homotopy) $h\colon X \times [0, 1] \to X$ such that $h(x, 0) = x$ and $h(x, 1) = x_0$ for all $x \in X$. It is clear that convex sets are contractible. Moreover, a compact metric space A is called an R_δ-set if there exists a decreasing sequence $\{A_n\}_n$ of compact contractible sets such that

$$A = \bigcap_{n \geq 1} A_n.$$

Furthermore, we say that $\Phi\colon X \rightrightarrows Y$ is an R_δ-map if it is upper semicontinuous and takes R_δ-values. The following result is a generalization of the Bohnenblust-Karlin fixed point theorem (see [120, Proposition 1.23]).

Proposition 1.13 *Let X be a nonempty, compact and contractible topological space, $\Phi\colon X \rightrightarrows Y$ an R_δ-map and $f\colon Y \to X$ a continuous function. If $\mathcal{P}\colon X \rightrightarrows X$ is the composition map $x \rightrightarrows f(\Phi(x))$, then $\mathcal{P}$ admits a fixed point.*

The following result (see [27, Theorem 4]) will be used in Chapter 5.

Lemma 1.14 *Let $f\colon [T_0, T] \times H \to H$ be a function satisfying*

1. *For every* $x \in H$ $f(\cdot, x)$ *is measurable.*

2. *For every* $t \in [T_0, T]$ $f(t, \cdot)$ *is continuous.*

3. *For all* $x, y \in H$ *and all* $t \in [T_0, T]$

$$\langle f(t,x) - f(t,y), x - y \rangle \leq \omega(t)\|x - y\|^2,$$

where $\omega \in L^1(T_0, T)$.

4. *There exists* $d \geq 0$ *such that, for all* $t \in [T_0, T]$ *and all* $x, y \in H$

$$\|f(t,x)\| \leq d(1 + \|x\|).$$

Let $F\colon [T_0, T] \times H \rightrightarrows H$ *be a set-valued map with nonempty closed and convex values satisfying:*

(i) *For every* $x \in H$ $F(\cdot, x)$ *is measurable,*

(ii) *for every* $t \in [T_0, T]$ $F(t, \cdot)$ *is upper semicontinuous from* H *into* H_w,

(iii) *for a.e.* $t \in [T_0, T]$ *and every* $A \subseteq H$ *bounded*

$$\gamma(F(t, A)) \leq k(t)\gamma(A),$$

for some $k \in L^1(T_0, T)$, *where* $\gamma = \alpha$ *or* $\gamma = \beta$ *is either the Kuratowski or the Hausdorff measure of non-compactness.*

Then, the differential inclusion

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) + f(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0, \end{cases}$$

has at least one solution $x \in W^{1,1}([T_0, T]; H)$.

1.4 The Hausdorff distance

Let $A, B \subseteq H$ be two closed sets. We define the excess of A over B as

$$\operatorname{exc}(A, B) = \sup_{x \in A} d_B(x).$$

The excess may well be $+\infty$ (for example, this will occur if B is bounded and A is unbounded). It is not difficult to prove that (see [18, Section 1.5])

$$\operatorname{exc}(A, B) = \sup_{x \in H} (d_B(x) - d_A(x)) = \inf\{\varepsilon > 0\colon A \subseteq S_\varepsilon(B)\},$$

where $S_\varepsilon(B) := \{x \in H : d_B(x) < \varepsilon\}$. Furthermore, we define the Hausdorff distance between A and B as the uniform distance between $d_A(\cdot)$ and $d_B(\cdot)$:

$$\mathrm{Haus}(A, B) = \sup_{x \in H} |d_A(x) - d_B(x)| . \tag{1.8}$$

Of course, the Hausdorff distance is not a metric because it might be $+\infty$.

The Hausdorff distance can be expressed in terms of the excess of A over B (see [18, Section 3.2]):

$$\begin{aligned}\mathrm{Haus}(A, B) &= \max\{\mathrm{exc}(A, B), \mathrm{exc}(B, A)\}\\ &= \inf\{\varepsilon > 0 \colon A \subseteq S_\varepsilon(B) \text{ and } B \subseteq S_\varepsilon(A)\}.\end{aligned}$$

In this thesis we will consider some continuity properties of moving sets with respect to the Hausdorff distance. Thus, given a set valued map $C \colon [T_0, T] \rightrightarrows X$ we say that C is absolutely continuous (Lipschitz) if there exists $\zeta \in W^{1,1}([T_0, T]; \mathbb{R})$ ($\zeta \in \mathrm{Lip}([T_0, T]; \mathbb{R})$) such that

$$\mathrm{Haus}(C(t), C(s)) \leq |\zeta(t) - \zeta(s)| \quad \text{for all } t \in [T_0, T].$$

According to (1.8), this is equivalent to

$$\sup_{x \in H} |d_{C(t)}(x) - d_{C(s)}(x)| \leq |\zeta(t) - \zeta(s)| \quad \text{for all } t \in [T_0, T].$$

1.5 Some useful lemmas

In this section we give some useful lemmas that will be used in the following chapters. They are related to properties of the distance function and set-valued maps.

Lemma 1.15 *Let $C \colon [T_0, T] \times H \rightrightarrows H$ be a set-valued map with nonempty and closed values. Assume that there exists $k \in L^1(T_0, T)$ such that for every $t \in [T_0, T]$, every $r > 0$ and every bounded set $A \subseteq H$,*

$$\gamma(C(t, A) \cap r\mathbb{B}) \leq k(t)\gamma(A),$$

where $\gamma = \alpha$ or $\gamma = \beta$ is either the Kuratowski or the Hausdorff measure of non-compactness and $k(t) < 1$ for all $t \in [T_0, T]$. Let $t \in [T_0, T]$, $y \in H$ and $x \notin C(t, y)$. Then,

$$\partial d_{C(t,y)}(x) = \frac{x - \mathrm{cl}\,\mathrm{co}\,\mathrm{Proj}_{C(t,y)}(x)}{d_{C(t,y)}(x)}.$$

Proof. Let $t \in [T_0, T]$ and $y \in H$ be given. It is not difficult to see that $C(t, y)$ is ball compact. To simplify the rest of the proof let us denote $C := C(t, y)$. According to [84],

$$\partial d_C(x) = \frac{x - \partial \varphi_C(x)}{d_C(x)},$$

where $\varphi_C(x) := \sup_{c\in C}\left\{\langle x,c\rangle - \frac{1}{2}\|c\|^2\right\}$ is the Asplund function associated with C. Moreover, due to [122, Proposition 4.51] and the ball compactness of C, $\partial\varphi_C(x) = \operatorname{cl}\operatorname{co}\operatorname{Proj}_C(x)$, which shows the result. □

Since $-d(\cdot,S)$ has a directional derivative that coincides with the Clarke's directional derivative of $-d(\cdot,S)$ whenever $x\notin S$ (see [25]), we obtain the following lemma.

Lemma 1.16 *Let $S\subseteq X$ be a closed set, $x\notin S$ and $v\in X$. Then*

$$\lim_{h\downarrow 0}\frac{d(x+hv,S)-d(x,S)}{h} = \min_{y^*\in\partial d(x,S)}\langle y^*,v\rangle .$$

To get a priori bounds on the solutions of differential inclusions we need the following consequence of Grönwall's inequality (see [10, Proposition 2.4.1]).

Lemma 1.17 *Let $I=[T_0,T]$, α and u be continuous functions and β be a non-negative integrable function, all defined on I. Assume that α is non-increasing and that*

$$u(t)\leq \alpha(t)+\int_{T_0}^{t}\beta(s)u(s)ds \quad t\in[T_0,T].$$

Then

$$u(t)\leq\alpha(t)\exp\left(\int_{T_0}^{t}\beta(s)ds\right)\quad t\in[T_0,T].$$

The next result gives some properties of the distance function composed with differentiable functions along a curve.

Lemma 1.18 *Assume that X,Y are two separable Hilbert spaces. Let $x\colon[T_0,T]\to X$ be an absolutely continuous function and let $C\colon[T_0,T]\rightrightarrows Y$ be a set-valued map with nonempty closed values satisfying: there exists $\zeta\colon[T_0,T]\to\mathbb{R}_+$ absolutely continuous such that*

$$\sup_{x\in H}\left|d_{C(t)}(x)-d_{C(s)}(x)\right|\leq|\zeta(t)-\zeta(s)|, \tag{1.9}$$

for all $s,t\in[T_0,T]$. Let $h\colon X\to Y$ be a differentiable function with bounded derivative. Then

1. *The function $t\to d_{C(t)}(h(x(t)))$ is absolutely continuous over $[T_0,T]$.*

2. *For all $t\in[T_0,T]$, where $\dot\zeta(t)$ exists,*

$$\begin{aligned}\limsup_{s\downarrow 0}&\frac{d_{C(t+s)}(h\left(x(t+s)\right))-d_{C(t)}(h\left(x(t)\right))}{s}\\ &\leq|\dot\zeta(t)|+\limsup_{s\downarrow 0}\frac{d_{C(t)}(h\left(x(t+s)\right))-d_{C(t)}(h\left(x(t)\right))}{s}.\end{aligned}\tag{1.10}$$

3. *For all $t \in [T_0, T]$, where $\dot{x}(t)$ exists,*

$$\begin{aligned}\limsup_{s\downarrow 0} \frac{1}{s}\left(d_{C(t)}(h\left(x(t+s)\right)) - d_{C(t)}(h\left(x(t)\right))\right)\\ \leq \max_{y^*\in\partial d_{C(t)}(h(x(t)))} \left\langle y^*, Dh(x(t))\,\dot{x}(t)\right\rangle .\end{aligned} \tag{1.11}$$

4. *For all $t \in \{t \in [T_0,T] \colon h\left(x(t)\right) \notin C(t)\}$ where $\dot{x}(t)$ exists, we have*

$$\lim_{s\downarrow 0} \frac{d_{C(t)}(h\left(x(t+s)\right)) - d_{C(t)}(h\left(x(t)\right))}{s} = \min_{y^*\in\partial d_{C(t)}(h(x(t)))} \left\langle y^*, Dh(x(t))\,\dot{x}(t)\right\rangle .$$

5. *For every $x \in X$ the set-valued map $t \rightrightarrows \partial d_{C(t)}(h(x))$ is measurable.*

Proof. Let $\psi \colon [T_0,T] \to \mathbb{R}$ be the function defined by $\psi(t) := d_{C(t)}(h\left(x(t)\right))$.

1. It follows directly from (1.9) and the boundedness of the derivative of h.

2. Let $t \in [T_0, T]$ be such that $\dot{\zeta}(t)$ exists. Then

$$\begin{aligned}\frac{\psi(t+s)-\psi(t)}{s} &= \frac{d_{C(t+s)}(h(x(t+s))) - d_{C(t)}(h(x(t+s)))}{s}\\ &+ \frac{d_{C(t)}(h(x(t+s))) - d_{C(t)}(h(x(t)))}{s}\\ &\leq \frac{|\zeta(t+s)-\zeta(t)|}{s} + \frac{d_{C(t)}(h(x(t+s))) - d_{C(t)}(h(x(t)))}{s},\end{aligned}$$

and taking superior limit, we get (1.10).

3. Let $t \in [T_0, T]$ be such that $\dot{x}(t)$ exists. Let $s_n \to 0^+$ be such

$$\begin{aligned}\limsup_{s\downarrow 0^+} &\frac{d_{C(t)}(h(x(t+s))) - d_{C(t)}(h(x(t)))}{s}\\ &= \lim_{n\to+\infty} \frac{d_{C(t)}(h(x(t+s_n))) - d_{C(t)}(h(x(t)))}{s_n}.\end{aligned}$$

By virtue of Lebourg's mean value theorem [55, Theorem 2.2.4], there exist $z_n \in]h(x(t)), h(x(t+s_n))[$ and $\xi_n \in \partial d_{C(t)}(z_n)$ such that

$$\frac{d_{C(t)}(h(x(t+s_n))) - d_{C(t)}(h(x(t)))}{s_n} = \left\langle \xi_n, \frac{h(x(t+s_n)) - h(x(t))}{s_n}\right\rangle,$$

since $\|\xi_n\| \leq 1$, there is a subsequence (without relabeling) of $(\xi_n)_n$ such that $\xi_n \rightharpoonup \xi \in \partial d_{C(t)}(h(x(t)))$, and taking limit in the last equality we get (1.11).

4. Let $t \in \{t \in [T_0, T] \colon x(t) \notin C(t)\}$ where $\dot{x}(t)$ exists. Then, for $s > 0$ small enough,

$$\begin{aligned}&\frac{d_{C(t)}(h(x(t+s))) - d_{C(t)}(h(x(t)))}{s}\\&= \frac{d_{C(t)}(h(x(t)) + sDh(x(t))\dot{x}(t) + s\varepsilon(s,t)) - d_{C(t)}(h(x(t)))}{s}\\&= \frac{d_{C(t)}(h(x(t)) + sDh(x(t))\dot{x}(t)) - d_{C(t)}(h(x(t)))}{s} + \eta(s,t),\end{aligned}$$

for some mappings $\varepsilon(\cdot, t)$ and $\eta(\cdot, t)$ with $\lim_{s\downarrow 0} \varepsilon(s,t) = 0$ and $\lim_{s\downarrow 0} \eta(s,t) = 0$. Then, using Lemma 1.16, we get

$$\begin{aligned}&\lim_{s\downarrow 0} \frac{d_{C(t)}(h(x(t+s))) - d_{C(t)}(h(x(t)))}{s}\\&= \lim_{s\downarrow 0} \frac{d_{C(t)}(h(x(t)) + sDh(x(t))\dot{x}(t)) - d_{C(t)}(h(x(t)))}{h}\\&= \min_{y^* \in \partial d_{C(t)}(h(x(t)))} \langle y^*, Dh(x(t))\dot{x}(t)\rangle .\end{aligned}$$

5. Let $\Gamma(t) := \partial d_{C(t)}(h(x))$. Then, Γ takes weakly compact and convex values. Fixing any $d \in Y$, by virtue of [86, Proposition 2.2.39], it is enough to verify that the support function $t \mapsto \sigma(d, \Gamma(t))$ is measurable, where

$$\sigma(d, \Gamma(t)) := \sup_{v\in\Gamma(t)} \langle v, d\rangle.$$

Recalling that $\sigma(d, \Gamma(t)) = d^\circ_{C(t)}(h(x); d)$ and fixing a countable dense subset Δ of $\mathbb{B}$, we have

$$\begin{aligned}d^\circ_{C(t)}(h(x); d) &= \inf_{n\in\mathbb{N}} \sup_{s\in]0,\frac{1}{n}[, y\in\frac{1}{n}\mathbb{B}} \frac{1}{s}\left[d_{C(t)}(y + h(x) + sd) - d_{C(t)}(y + h(x))\right]\\&= \inf_{n\in\mathbb{N}} \sup_{s\in]0,\frac{1}{n}[\cap\mathbb{Q}, y\in\frac{1}{n}\Delta} \frac{1}{s}\left[d_{C(t)}(y + h(x) + sd) - d_{C(t)}(y + h(x))\right].\end{aligned}$$

This and the continuity of the function $(t, x) \mapsto d_{C(t)}(h(x))$, due to the continuity of h and (1.9), guarantees the desired measurability of $t \mapsto \sigma(d, \Gamma(t))$ which finishes the proof.

□

1.6 Some elements of measure theory

Given a measure ν over $[T_0, T]$, we denote by $L^1_\nu([T_0, T]; H)$ the space of H-valued ν-integrable functions defined over $[T_0, T]$. When ν is the Lebesgue measure we simply write $L^1([T_0,T];H)$ and, in this case, we write $L^1_w([T_0,T];H)$ to mean the space $L^1([T_0,T];H)$ endowed with the weak topology.

Given a function $u\colon [T_0,T]\to H$ and a subinterval $J\subseteq[T_0,T]$, the variation of u on J is defined by

$$\operatorname{Var}(u,J):=\sup\left\{\sum_{j=1}^{m}\|u(t_j)-u(t_{j-1})\|\colon m\in\mathbb{N}, t_j\in J,\ t_0<\cdots<t_m\right\}.$$

If $\operatorname{Var}(u,[T_0,T])<+\infty$ we say that u has *bounded variation* on $[T_0,T]$. The space of functions with bounded variation is denoted by $\operatorname{BV}([T_0,T];H)$. The set of H-valued continuous functions defined on $[T_0,T]$ is denoted by $C([T_0,T];H)$. For convenience we set

$$\operatorname{CBV}([T_0,T];H):=\operatorname{BV}([T_0,T];H)\cap C([T_0,T];H).$$

We recall the concept of (normalized) arc-length ℓ_u (see [67, Section 2.5.16]). For $u\in\operatorname{CBV}([T_0,T];H)$, let $\ell_u\colon[T_0,T]\to[T_0,T]$ be defined by

$$\ell_u(t)=\begin{cases}T_0+\frac{(T-T_0)}{\operatorname{Var}(u,[T_0,T])}\operatorname{Var}(u,[T_0,t]), & \text{if } \operatorname{Var}(u,[T_0,T])\neq 0,\\ T_0, & \text{if } \operatorname{Var}(u,[T_0,T])=0,\end{cases}$$

for $t\in[T_0,T]$.

The following result is the key element of the reparametrization technique used in Chapter 5 (see for instance [127, Proposition 2.1]).

Proposition 1.19 *For every $u\in\operatorname{CBV}([T_0,T];H)$ there exists a unique function $U\in\operatorname{Lip}([T_0,T];H)$ such that $u=U\circ\ell_u$. Moreover, $\operatorname{Lip}(U)\leq\frac{\operatorname{Var}(u,[T_0,T])}{(T-T_0)}$.*

Given a vector measure $\mu\colon\mathcal{B}([T_0,T])\to H$, where $\mathcal{B}([T_0,T])$ are the Borel sets of $[T_0,T]$, its variation measure $|\mu|\colon\mathcal{B}([T_0,T])\to\mathbb{R}$ is defined for any Borel set $A\subseteq[T_0,T]$ as $|\mu|(A):=\sup\sum_{n\in\mathbb{N}}\|\mu(B_n)\|$, where the supremum is taken over all sequences $(B_n)_{n\in\mathbb{N}}$ of mutually disjoint Borel subsets of $[T_0,T]$ such that $A=\bigcup_{n\in\mathbb{N}}B_n$. We say that μ has bounded variation if $|\mu|([T_0,T])$ is finite (see [125, 137]). Also, given $u\in\operatorname{BV}([T_0,T];H)$ it is known that its distributional derivative $Du\colon\mathcal{B}([T_0,T])\to H$ is a measure with bounded variation, i.e., $|Du|([T_0,T])<\infty$ and $-\int_{\mathbb{R}}\varphi'(t)\bar{u}(t)dt=\int_{\mathbb{R}}\varphi dD\bar{u}$ for all $\varphi\in C^1_c(\mathbb{R};\mathbb{R})$, where $\bar{u}\colon\mathbb{R}\to H$ is defined by

$$\bar{u}(t):=\begin{cases}u(T_0), & t<T_0.\\ u(t), & t\in[T_0,T],\\ u(T), & t>T.\end{cases}$$

We recall that Du is the differential measure associated with u.

The next proposition is a chain rule for BV functions (see [127, Proposition 2.2] and [125, Lemma 6.4 and Theorem 6.1] for more details).

Proposition 1.20 *Let $I, J \subseteq \mathbb{R}$ be intervals and let $h\colon I \to J$ be nondecreasing and continuous. Then,*

(i) $Dh\,(h^{-1}(B)) = \mathcal{L}^1(B)$ *for every* $B \in \mathcal{B}\,(h(I))$, *where* $\mathcal{L}^1$ *is the Lebesgue measure and* $\mathcal{B}(h(I))$ *are the Borel sets of* $h(I)$.

(ii) If $g \in \operatorname{Lip}(J; H)$ *then* $g \circ h \in \operatorname{BV}(I; H)$ *and* $D\,(g \circ h) = (g' \circ h)\,Dh$, *where* g' *is any representative of the distributional derivative of* g.

1.7 Some classes of sets

In this section we give the definition of some classes of sets which generalize the class of convex sets.

1.7.1 Uniformly prox-regular sets

Definition 1.21 ([123]) *Let $S \subseteq H$ be a closed set and $\rho > 0$. We say that S is ρ-uniformly prox-regular if for all $x \in S$ and all $v \in N\,(S; x)$ with $\|v\| < 1$, x is the unique nearest point of S to $x + \rho^{-1}v$, i.e.,*

$$x = \operatorname{proj}_S(x + \rho^{-1}v).$$

Here $\operatorname{proj}_S(u)$ *denotes the unique nearest point of S to u.*

The notion of uniformly prox-regular sets is related to the differentiability of the distance function. The following theorem gives a complete characterization of this notion in Hilbert spaces (see [123, Theorem 4.1]).

Theorem 1.22 *Let S be a closed subset of H and $\rho > 0$. The following properties are equivalent:*

a) S is $1/\rho'$ uniformly prox-regular for every $0 < \rho' < \rho$.

b) d_S is continuously differentiable on $U_\rho(S) := \{u \in H\colon 0 < d_S(u) < \rho\}$.

c) d_S is Fréchet differentiable on $U_\rho(S)$.

d) d_S is Gâteaux differentiable on $U_\rho(S)$ and proj_S *is nonempty-valued on $U_\rho(S)$.*

e) d_S^2 is differentiable with locally Lipschitz continuous derivative on $U_\rho(S)$.

f) Every nonzero proximal normal to S at any point x of S can be realized by an ρ-ball, i.e., for every $x \in S$ and every nonzero $v \in N^P(S; x)$

$$\left\langle \frac{v}{\|v\|}, y - x \right\rangle \leq \frac{1}{2\rho}\|y - x\|^2 \quad \forall y \in S.$$

g) Whenever $x_i \in S$ *and* $v_i \in N_S^P(x_i) \cap \rho\mathbb{B}$*, one has*

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq -\|x_1 - x_2\|^2.$$

h) proj_S *is single-valued and strongly-weakly continuous on* $U_\rho(S)$*.*

i) $d_{T^B(S;x)}(x'-x) \leq \dfrac{1}{2\rho}\|x'-x\|^2$ *whenever* x', x *are in* S*, where* $T^B(S;x)$ *denotes the Bouligand tangent cone to* S *at* x*.*

Moreover, if S *is weakly closed, then one can add the following to the list of equivalent properties:*

j) proj_S *is single-valued on* $U_\rho(S)$*.*

1.7.2 Uniformly subsmooth sets

In this subsection we describe the class of uniformly subsmooth sets. The notion of subsmoothness of a set was introduced in [11] as a extension of convexity, related to the submonotonicity of the truncated Clarke normal cone.

Definition 1.23 *Let* S *be a closed subset of* H*. We say that* S *is* uniformly subsmooth*, if for every* $\varepsilon > 0$ *there exists* $\delta > 0$*, such that*

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon\|x_1 - x_2\|, \tag{1.12}$$

holds for all $x_1, x_2 \in S$ *satisfying* $\|x_1 - x_2\| < \delta$ *and all* $x_i^* \in N(S; x_i) \cap \mathbb{B}$ *for* $i = 1, 2$*. Also, if* E *is a given nonempty set, we say that the family* $(S(t))_{t\in E}$ *is* equi-uniformly subsmooth*, if for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that (1.12) holds for each* $t \in E$ *and all* $x_1, x_2 \in S(t)$ *satisfying* $\|x_1 - x_2\| < \delta$ *and all* $x_i^* \in N(S(t); x_i) \cap \mathbb{B}$ *for* $i = 1, 2$*.*

Remark 1.2 In the above definition we can replace $N(S; x) \cap \mathbb{B}$ by $N^P(S; x) \cap \mathbb{B}$, $N^F(S; x) \cap \mathbb{B}$ or $N^L(S; x) \cap \mathbb{B}$ (see [11, Proposition 3.9] and [138, Proposition 3.10]).

The class of uniformly subsmooth sets includes the class of uniformly prox-regular sets. Moreover, every subsmooth set is Fréchet normally regular and, consequently, tangentially regular (see [11, Proposition 3.4]).

Chapter 2

Positively α-far sets

Let X be a Hilbert space. In this chapter, which is based on [87], we study the class of positively α-far sets. This family of nonregular sets is the class of closed sets for which the origin is kept uniformly positively far from the Clarke subdifferential of the distance function, say $\partial d_S(\cdot)$, in some tube $\{x \in X : 0 < d_S(x) < \rho\}$ for some $\rho \in (0, +\infty)$. This class was introduced in [75], as a localization of the class of subdifferentially behaved sets in [68], to study some differential inclusions of sweeping process type in finite dimensions. The aim of this chapter is to study some properties of these sets. In particular, we will show that this class includes several others as uniformly prox-regular sets, uniformly subsmooth sets and paraconvex sets, among others. Moreover, we discuss about the preservation of positively α-farness under union, intersection and inverse images. We end this chapter, by providing sufficient conditions to assure the equi-uniform subsmoothness of the inverse image, under a smooth function, of a family of moving sets.

Let $S \subseteq X$ be a closed set. For $\rho \in (0, +\infty)$ we define the open ρ-tube around the set S by (see Figure 2.1)

$$U_\rho(S) := \{x \in X : 0 < d_S(x) < \rho\}.$$

Definition 2.1 *Let $\alpha \in (0, 1]$ and $\rho \in (0, +\infty)$. Let S be a nonempty closed subset of X with $S \neq X$. We say that the Clarke subdifferential of the distance function $d_S(\cdot)$ keeps the origin α-far-off on the open ρ-tube around S provided*

$$\alpha \leq \inf_{x \in U_\rho(S)} d(0, \partial d_S(x)), \tag{2.1}$$

where $\partial d_S(\cdot)$ denotes the Clarke subdifferential of the distance function. Moreover, if E is a given nonempty set, we say that the family $(S(t))_{t\in E}$ is positively α-far if every set $S(t)$ satisfies (2.1) *with the same $\alpha > 0$ and $\rho > 0$, for some ρ.*

The class of these sets will be called the class of *positively α-far sets*. From this definition, it follows directly that every r-uniform prox-regular set (see Definition

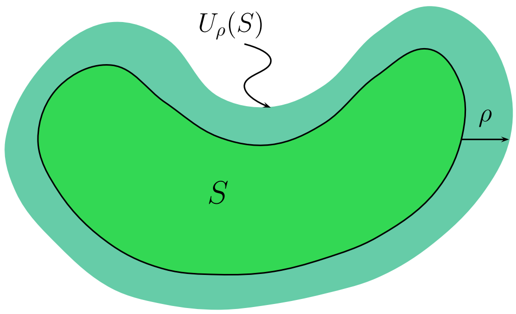


Figure 2.1: ρ-tube around S.

1.21) is positively α-far for $\rho = 1/\rho$. Moreover, the class of positively α-far sets includes strictly the class of uniform prox-regular sets (See Example 2.2).

Since, the notion of positively α-farness involves the Clarke subdifferential of the distance, which can be characterized via the formula (see formula (1.2))

$$\partial d_S(x) = \bigcap_{\gamma>0} \overline{\text{co}} \left(\frac{x - \text{proj}_S^\gamma(x)}{d_S(x)} \right), \tag{2.2}$$

where $\text{proj}_S^\gamma(x) = \{z \in S \colon \|x - z\| < d_S(x) + \gamma\}$, we get the following result.

Proposition 2.2 *Let S be a closed subset of X and $\rho > 0$.*

(i) Assume that the following property holds: For every $x \in U_\rho(S)$ there exists $\gamma(x) > 0$ such that for all $\gamma \in]0, \gamma(x)[$

$$u_1^*, u_2^* \in \frac{x - \text{proj}_S^\gamma(x)}{d_S(x)} \quad \Rightarrow \quad \langle u_1^*, u_2^* \rangle \geq \alpha^2 + \theta(\gamma, x), \tag{2.3}$$

where $\lim_{\gamma \downarrow 0} \theta(\gamma, x) = 0$ for all $x \in U_\rho(S)$. Then the origin is kept positively α-far from the Clarke subdifferential of the distance function $d_S(\cdot)$ on $U_\rho(S)$.

(ii) Assume that the origin is kept positively α-far from the Clarke subdifferential of the distance function $d_S(\cdot)$ on $U_\rho(S)$. Then the following property holds:

$$\forall x \in U_\rho(S) \quad u_1^*, u_2^* \in \partial d_S(x) \quad \Rightarrow \quad \langle u_1^*, u_2^* \rangle \geq 2\alpha^2 - 1. \tag{2.4}$$

Proof. (i) Fix $x \in U_\rho(S)$. If $u^* \in \text{co}\left(\frac{x - \text{proj}_S^\gamma(x)}{d_S(x)}\right)$, then $u^* = \sum_{i=1}^n \lambda_i u_i^*$ for some $u_i^* \in \frac{x-\text{proj}_S^\gamma(x)}{d_S(x)}$, $\lambda_i \geq 0$ for $i = 1, \ldots, n$ with $\sum_{i=1}^n \lambda_i = 1$ and some $n \in \mathbb{N}$.

Thus,

$$\begin{aligned}
\|u^*\|^2 &= \left\langle \sum_{i=1}^{n} \lambda_i u_i^*, \sum_{j=1}^{n} \lambda_j u_j^* \right\rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i\lambda_j \left\langle u_i^*, u_j^* \right\rangle \\
&\geq \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i\lambda_j \left(\alpha^2 + \theta(\gamma, x)\right) \\
&\geq \alpha^2 + \theta(\gamma, x).
\end{aligned}$$

Hence, (2.3) holds for $u_1^*, u_2^* \in \operatorname{co}\left(\frac{x-\operatorname{proj}_S^\gamma(x)}{d_S(x)}\right)$, and, since the weak and strong closure of convex sets coincide, (2.3) holds for $u_1^*, u_2^* \in \overline{\operatorname{co}}\left(\frac{x-\operatorname{proj}_S^\gamma(x)}{d_S(x)}\right)$. Therefore, due to the formula (2.2), if $x^* \in \partial d_S(x)$

$$\|x^*\|^2 \geq \alpha^2 + \theta(\gamma, x),$$

for all $\gamma < \gamma(x)$. Finally, taking $\gamma \downarrow 0$, we obtain that $\|x^*\| \geq \alpha$.

(ii) Take $x \in U_\rho(S)$ and $u_1^*, u_2^* \in \partial d_S(x)$. Since $u_1^*, u_2^* \in \partial d_S(x)$ we obtain $\frac{1}{2}(u_1^* + u_2^*) \in \partial d_S(x)$. Thus,

$$\begin{aligned}
\alpha^2 &\leq \left\| \frac{u_1^* + u_2^*}{2} \right\|^2 \\
&= \frac{\|u_1^*\|^2}{4} + \frac{1}{2}\langle u_1^*, u_2^*\rangle + \frac{\|u_1^*\|^2}{4} \\
&\leq \frac{1}{4} + \frac{1}{2}\langle u_1^*, u_2^*\rangle + \frac{1}{4},
\end{aligned}$$

which entails (2.4).

□

The next example, taken from [75], shows that the inequality in (2.4) is attained.

Example 2.1 Consider the set $S = \{(x, y) \in \mathbb{R}^2 \colon y \geq -|x|\}$.

For $(r, s) \notin S$ we have $d_S((r, s)) = \frac{\sqrt{2}}{2}\,||r| + s|$ and

$$\partial d_S((r,s)) = \begin{cases} \frac{\sqrt{2}}{2}(-1,-1) & \text{if } r > 0, \\ \left\{\frac{\sqrt{2}}{2}(1-2\lambda, -1) \in \mathbb{R}^2 \colon \lambda \in [0,1]\right\} & \text{if } r = 0, \\ \frac{\sqrt{2}}{2}(1,-1) & \text{if } r < 0, \end{cases}$$

therefore the origin is kept positively $\frac{\sqrt{2}}{2}$-far from the Clarke subdifferential of the distance function on $X \setminus S$. Then, if $(0, s) \notin S$, $u_1^* = \frac{\sqrt{2}}{2}(-1,-1)$, $u_2^* = \frac{\sqrt{2}}{2}(1,-1)$ belong to $\partial d_S((0,s))$ and satisfy $\langle u_1^*, u_2^*\rangle = 0$, attaining the equality in (2.4).

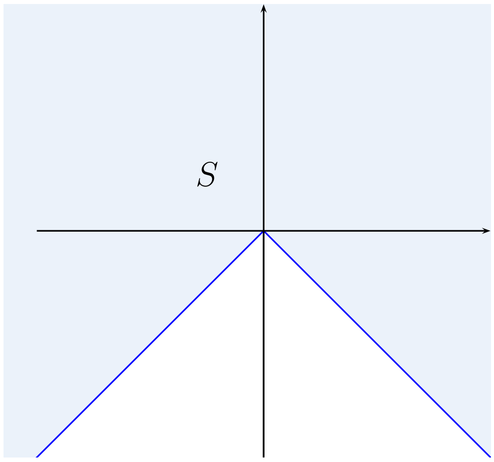


Figure 2.2: $S = \{(x, y) \in \mathbb{R}^2 \colon y \geq -|x|\}$.

The following result is a direct consequence of Proposition 2.2 and Lemma 1.15.

Corollary 2.3 *Assume that S is ball compact.*

i) If for all $u_1, u_2 \in \operatorname{Proj}_S(x)$

$$\langle x - u_1, x - u_2 \rangle \geq \alpha^2 d_S^2(x) \quad \forall x \in U_\rho(S),$$

then S is positively α-far for $\rho > 0$.

ii) If S is positively α-far for $\rho > 0$, then

$$\langle x - u_1, x - u_2 \rangle \geq (2\alpha^2 - 1) d_S^2(x) \quad \forall x \in U_\rho(S),$$

for all $u_1, u_2 \in \operatorname{Proj}_S(x)$.

According to Theorem 1.22, a weak closed set S is $1/\rho$-uniformly prox-regular if and only if the Clarke subdifferential of the distance $\partial d_S(\cdot)$ is a singleton for every $x \in U_\rho(S)$. Hence, using Proposition 2.2, we can give the following characterization of uniformly prox-regularity.

Corollary 2.4 *Let S be a closed subset of X with $S \neq X$ and $\rho > 0$. Then S is $1/\rho$-uniformly prox-regular if and only if the origin is kept positive 1-far from the Clarke subdifferential of the distance function $d_S(\cdot)$ on $U_\rho(S)$.*

Proof. The necessity is direct from the definition of positively α-far sets. For the sufficiency we take $x \in U_\rho(S)$ and $u_1^*, u_2^* \in \partial d_S(x)$. Then, due to Proposition 2.2, we have $\langle u_1^*, u_2^* \rangle = 1$. Therefore

$$\|u_1^* - u_1^*\|^2 = \|u_1^*\|^2 - 2\langle u_1^*, u_2^* \rangle + \|u_1^*\|^2 \leq 1 - 2 + 1 = 0,$$

Thus, $\partial d_S(x)$ is a singleton for every $x \in U_\rho(S)$, which proves (see [52]) that the distance function d_S is strictly differentiable in $U_\rho(S)$. Therefore, due to [123, Theorem 4.1], S is $1/\rho$-uniformly prox-regular. □

The next proposition gives a characterization of the concept of positively α-farness in terms of the existence of a function called pseudo-gradient. We use the well-known Lau theorem [99] which asserts the density of the set of nearest points to a given closed set.

Proposition 2.5 *Let S be a closed subset of X and let $\rho > 0$ and $\alpha \in (0,1]$. Then the following assertions are equivalent:*

i) The origin is kept positively α-far from the Clarke subdifferential of the distance function $d_S(\cdot)$ on $U_\rho(S)$.

ii) For all $\varepsilon \in]0,\alpha[$, there exists a locally Lipschitz function $V : X \setminus S \to X$ such that

$$\forall x \in U_\rho(S),\ \|V(x)\| \leq 1+\varepsilon,\ \inf_{x^* \in \partial d_S(x)} \langle x^*, V(x)\rangle \geq (\alpha - \varepsilon).$$

iii) For all $\varepsilon \in]0,\alpha[$, there exists a locally Lipschitz function $V : X \setminus S \to X$ such that $\forall x \in U_\rho(S)$

$$\|V(x)\| \leq 1+\varepsilon, \quad \langle u - \mathrm{proj}_S(u), V(u)\rangle \geq (\alpha-\varepsilon) d_S(u) \quad \forall u \in U_\rho(S) \cap D.$$

Here D is the dense set of those points which have a nearest point to S.

Proof. *i)* $\Rightarrow$ *ii)*: This implication is an adaptation of the proof of Theorem 4.1 in [61]. The equivalence *ii)* $\Leftrightarrow$ *iii)* follows from the characterization of Clarke's subdifferential of the distance function (see [55, Theorem 1.6.1])

$$\partial d_S(x) = \overline{\mathrm{co}} \left\{ w\text{-}\lim_{i \to +\infty} \frac{x_i - \mathrm{proj}_S(x_i)}{\|x_i - \mathrm{proj}_S(x_i)\|} \ :\ D \ni x_i \to x \right\},$$

Finally the implication *ii)* $\Rightarrow$ *i)* is direct. □

Remark 2.1 The equivalence *i)* $\Leftrightarrow$ *ii)* holds true in any Banach space.

The following proposition gives another characterization of positively α-far sets.

Proposition 2.6 *Let $S \subseteq X$ be a closed set, $\alpha \in]0,1[$ and $\rho > 0$.*

i) If the origin is kept positively α-far from the Clarke subdifferential of the distance function $d_S(\cdot)$ on $U_\rho(S)$, then

$$\forall x \in U_\rho(S), \quad \forall x^* \in \partial d_S(x); \quad d_S(x - x^* d_S(x)) \leq d_S(x)\sqrt{1-\alpha^2}. \qquad (2.5)$$

ii) Conversely, if (2.5) *is satisfied, then the origin is kept positively $1-\sqrt{1-\alpha^2}$-far from the Clarke subdifferential of the distance function $d_S(\cdot)$ on $U_\rho(S)$.*

The proof of the Proposition 2.6 is based on the following geometrical lemma (see Figure 2.3)

Lemma 2.7 *Let $r > 0$ and $x \in X$ be such that $S \cap B(x,r) \neq \emptyset$. Then*

$$\forall u \in \mathrm{co}\,\big(S \cap B(x,r)\big), \quad \exists v \in S \cap B(x,r); \quad \|u-v\|^2 + \|x-u\|^2 \leq r^2. \tag{2.6}$$

Consequently,

$$\forall u \in \overline{\mathrm{co}}\big(S \cap B(x,r)\big), \quad d_S(u) \leq \sqrt{r^2 - \|x-u\|^2}. \tag{2.7}$$

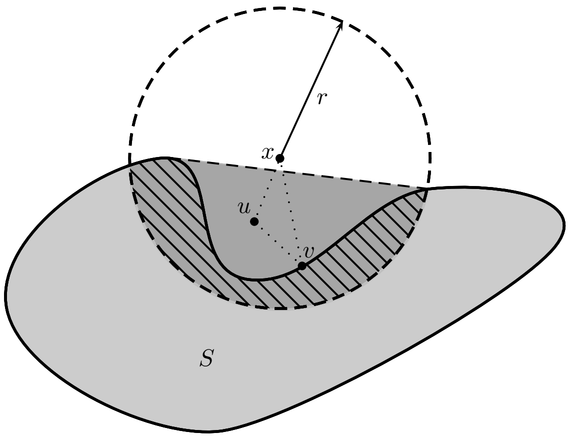


Figure 2.3: Geometrical interpretation of Lemma 2.7.

Proof of Lemma 2.7. The second assertion follows directly from the first one. Let us prove (2.6). By contradiction, assume that there exists $u \in \mathrm{co}\,(S \cap B(x,r))$ such that for all $v \in S \cap B(x,r)$

$$\|u-v\|^2 + \|x-u\|^2 > r^2.$$

Then, for all $v \in S \cap B(x,r)$

$$\begin{aligned} r^2 &\geq \|x-v\|^2 \\ &= \|x-u+u-v\|^2 \\ &= \|x-u\|^2 + 2\langle x-u, u-v\rangle + \|u-v\|^2 \\ &> 2\langle x-u, u-v\rangle + r^2, \end{aligned}$$

So, using the last inequality and passing to the convex hull, we obtain

$$\langle x-u, u-v\rangle < 0 \qquad \text{for all } v \in \mathrm{co}\,(S \cap B(x,r)),$$

which is a contradiction because $u \in \mathrm{co}\,(S \cap B(x,r))$. □

Now we proceed to prove Proposition 2.6.

Proof. The assertion *ii*) is obvious. Let us prove *i*). Let $x \in U_\rho(S)$ and $x^* \in \partial d_S(x)$ or equivalently $x^* = \dfrac{x-u}{d_S(x)}$, for some $u \in \bigcap_{\gamma>0} \overline{\mathrm{co}} P_S^\gamma(x)$ (see (2.2)). Then, for all $\gamma > 0$, $u = x - x^* d_S(x) \in \overline{\mathrm{co}} P_S^\gamma(x)$. So, by (2.7),

$$d_S(x - x^* d_S(x)) \leq \sqrt{(d_S(x)+\gamma)^2 - \|x^*\|^2 d_S^2(x)}$$

and since the origin is kept positively α-far from the Clarke subdifferential of the distance function $d_S(\cdot)$ on $U_\rho(S)$, we get

$$d_S(x - x^* d_S(x)) \leq \sqrt{(d_S(x)+\gamma)^2 - \alpha^2 d_S^2(x)}.$$

As $\gamma > 0$ is arbitrary, we obtain (2.5). □

2.1 Relation with other classes

We start this section by showing that the class of positively α-far sets included strictly the class of uniformly subsmooth sets (see Section 1.7.2).

Proposition 2.8 *Let $S \subseteq X$ be a closed and uniformly subsmooth set. Then, for all $\varepsilon \in]0,1[$ there exists $\rho \in]0,+\infty[$ such that the origin is kept positively $\sqrt{1-\varepsilon}$-far from the Clarke subdifferential of the distance function $d_S(\cdot)$ on the open ρ-tube $U_\rho(S)$, i.e.,*

$$\sqrt{1-\varepsilon} \leq \inf_{y \in U_\rho(S)} d\left(0, \partial d_S(y)\right).$$

Proof. Let $\varepsilon \in]0,1[$ and define $\rho = \min\{\delta, 2\}/4$, where δ is given by the uniform subsmoothness of S for ε. Fix $x \in U_\rho(S)$. Due to Proposition 2.2, it is sufficient to show that for all $\gamma < \gamma(x) := \min\left\{\frac{(1-\varepsilon)^2}{100}\frac{d_S(x)^4}{(2+d_S(x))^4}, \frac{\delta}{4}\right\}$ the following property holds:

$$u_1^*, u_2^* \in \frac{x - \operatorname{proj}_S^\gamma(x)}{d_S(x)} \quad \Rightarrow \quad \langle u_1^*, u_2^* \rangle \geq 1 - \varepsilon - 10\sqrt{\gamma}\left(\frac{2}{d_S(x)} + 1\right)^2. \qquad (2.8)$$

For $i \in \{1,2\}$ we take $u_i^* = \frac{x - z_i}{d_S(x)}$ with $z_i \in \operatorname{proj}_S^\gamma(x)$, i.e., $z_i \in S$ and $\|x - z_i\| \leq d_S(x) + \gamma$. Next, we apply Ekeland's variational principle [26, Theorem 2.1.3] to the function $u \mapsto I_S(u) + \|u - x\|$, to obtain the existence of $u_i \in S$ such that

(i) $\|z_i - u_i\| \leq \sqrt{\gamma}$,

(ii) $\|u_i - x\| + \sqrt{\gamma}\|z_i - u_i\| \leq \|z_i - x\|$,

(iii) the function $u \mapsto I_S(u) + \|u - x\| + \sqrt{\gamma}\|u - u_i\|$ attains a minimum at $u_i \in S$.

Next, using exact penalization [55, Proposition 1.6.3], (iii) is equivalent to the fact that the function $u \mapsto \left(1 + 2\sqrt{\gamma}\right) d_S(u) + \|u - x\| + \sqrt{\gamma}\|u - u_i\|$ attains a minimum at u_i. Note that $u_i \neq x$ because $u_i \in S$ and $x \notin S$. Hence,

$$0 \in (1 + 2\sqrt{\gamma})\,\partial d_S(u_i) + \frac{u_i - x}{\|u_i - x\|} + \sqrt{\gamma}\mathbb{B}.$$

Therefore, there exists $b_i \in \mathbb{B}$ such that $v_i^* = \frac{x - u_i}{\|x - u_i\|} + \sqrt{\gamma} b_i \in \left(1 + 2\sqrt{\gamma}\right)\partial d_S(u_i)$ for $i = 1, 2$. Then, for $i = 1, 2$

$$u_i^* - v_i^* = \frac{x - u_i}{d_S(x)} - \frac{x - u_i}{\|x - u_i\|} + \frac{u_i - z_i}{d_S(x)} - \sqrt{\gamma} b_i.$$

Thus,

$$\|u_i^* - v_i^*\| \leq \frac{\gamma}{d_S(x)} + \frac{\sqrt{\gamma}}{d_S(x)} + \sqrt{\gamma} \leq \sqrt{\gamma}\left(\frac{2}{d_S(x)} + 1\right)$$

Moreover, as a result of (ii), $\|u_1 - u_2\| \leq \|u_1 - x\| + \|u_2 - x\| \leq 2d_S(x) + 2\gamma < \delta$ and consequently, by uniform subsmoothness of S,

$$\langle v_1^* - v_2^*, u_1 - u_2\rangle \geq -\varepsilon\left(1 + 2\sqrt{\gamma}\right)\|u_1 - u_2\|. \tag{2.9}$$

Next, on the one hand

$$\begin{aligned}\langle v_1^* - v_2^*, u_1 - u_2\rangle &\leq \left\langle \frac{x - u_1}{\|x - u_1\|} - \frac{x - u_2}{\|x - u_2\|}, u_1 - u_2 \right\rangle + 2\sqrt{\gamma}\|u_1 - u_2\| \\ &= \left(\|x - u_1\| + \|x - u_2\|\right)\left[-1 + \langle v_1^* - \sqrt{\gamma}b_1, v_2^* - \sqrt{\gamma}b_2\rangle\right] \\ &\quad + 2\sqrt{\gamma}\|u_1 - u_2\| \\ &\leq \left(-1 + 5\sqrt{\gamma} + \langle v_1^*, v_2^*\rangle\right)\left(\|x - u_1\| + \|x - u_2\|\right).\end{aligned}$$

On the other hand,

$$-\varepsilon\left(1 + 2\sqrt{\gamma}\right)\|u_1 - u_2\| \geq -\varepsilon\left(1 + 2\sqrt{\gamma}\right)\left(\|x - u_1\| + \|x - u_2\|\right).$$

Therefore, combining these two last inequalities with (2.9) and dividing by

$$\left(\|x - u_1\| + \|x - u_2\|\right),$$

we obtain

$$-1 + 5\sqrt{\gamma} + \langle v_1^*, v_2^*\rangle \geq -\varepsilon\left(1 + 2\sqrt{\gamma}\right),$$

which entails

$$\langle v_1^*, v_2^*\rangle \geq 1 - \varepsilon(1 + 2\sqrt{\gamma}) - 5\sqrt{\gamma}. \tag{2.10}$$

On the other hand, we notice that

$$\begin{aligned}\langle v_1^*, v_2^*\rangle &= \langle v_1^* - u_1^*, v_2^* - u_2^*\rangle + \langle v_1^* - u_1^*, u_2^*\rangle + \langle u_1^*, v_2^* - u_2^*\rangle + \langle u_1^*, u_2^*\rangle \\ &\leq \gamma\left(\frac{2}{d_S(x)} + 1\right)^2 + 2\sqrt{\gamma}\left(\frac{2}{d_S(x)} + 1\right)\left(\frac{\gamma}{d_S(x)} + 1\right) + \langle u_1^*, u_2^*\rangle \\ &\leq 3\sqrt{\gamma}\left(\frac{2}{d_S(x)} + 1\right)^2 + \langle u_1^*, u_2^*\rangle.\end{aligned}$$

Finally, using this last inequality and (2.10), we get

$$\begin{aligned}\langle u_1^*, u_2^*\rangle &\geq \langle v_1^*, v_2^*\rangle - 3\sqrt{\gamma}\left(\frac{2}{d_S(x)} + 1\right)^2 \\ &\geq 1 - \varepsilon(1 + 2\sqrt{\gamma}) - 5\sqrt{\gamma} - 3\sqrt{\gamma}\left(\frac{2}{d_S(x)} + 1\right)^2 \\ &\geq 1 - \varepsilon - 10\sqrt{\gamma}\left(\frac{2}{d_S(x)} + 1\right)^2,\end{aligned}$$

which proves (2.8). □

The following example shows that the class of positively α-far sets is strictly bigger than the class of uniformly prox-regular sets and the class of subsmooth sets (see Example 2.1).

Example 2.2 Consider the set $S = \{(x, y) \in \mathbb{R}^2 \colon y \geq -|x|\}$. Then the origin is kept positively $\frac{\sqrt{2}}{2}$-far from the Clarke subdifferential of the distance function $d_S(\cdot)$ on $X \setminus S$ but S is not uniformly prox-regular neither uniformly subsmooth.

The class of positively α-far sets also contains the class of paraconvex sets, introduced by Michael [106]. Following Michael, a set S is α-paraconvex for some $\alpha \in [0, 1[$ if, whenever $x \in X$ and $r > 0$ are such that $d_S(x) < r$, then

$$d_S(u) \leq \alpha r \quad \forall\, u \in \operatorname{co}[B(x, r) \cap S].$$

This implies immediately that

$$\forall \rho > 0, \forall x \in U_\rho(S), \forall \gamma > 0, \quad \alpha(d_S(x) + \gamma) \geq d_S(u) \quad \forall u \in \operatorname{co} \operatorname{proj}_S^\gamma(x). \quad (2.11)$$

This remark allows us to get the following result.

Proposition 2.9 *Let $S \subseteq X$ be a closed set which is α-paraconvex for some $\alpha \in [0, 1[$. Then for each $\rho > 0$, the Clarke subdifferential of the distance function $d_S(\cdot)$ keeps the origin $(1 - \alpha)$-far-off on the open ρ-tube around S.*

Proof. Let $\rho > 0$ and $x \in U_\rho(S)$. First note that $d(0, \partial d_S(x)) = \frac{\|x-u\|}{d_S(x)}$ for some $u \in \bigcap_{\gamma>0} \overline{\operatorname{co}} P_S^\gamma(x)$. As $u \in \overline{\operatorname{co}} P_S^\gamma(x)$ for all $\gamma > 0$, relation (2.11) ensures that

$$\|x - u\| \geq d_S(x) - d_S(u) \geq d_S(x) - \alpha(d_S(x) + \gamma).$$

Thus

$$d(0, \partial d_S(x)) \geq (1 - \alpha) - \frac{\alpha\gamma}{d_S(x)} \quad \forall \gamma > 0,$$

which completes the proof. □

2.2 Preservation of positively α-far sets

In this section we discuss about the preservation of positively α-far sets under union, intersection and inverse images.

2.2.1 Union of positively α-far sets

We start this section by showing that the union of two convex sets is not positively α-far.

Example 2.3 Let us consider the sets (see Figure 2.4)

$$C = \{(x, y) \in \mathbb{R}^2 : \exp(x) \le y\},$$
$$D = \{(x, y) \in \mathbb{R}^2 : y \le -\exp(x)\}.$$

Then $S := C \cup D$ is not positively α-far. To see this, define the function $f(x)$ as

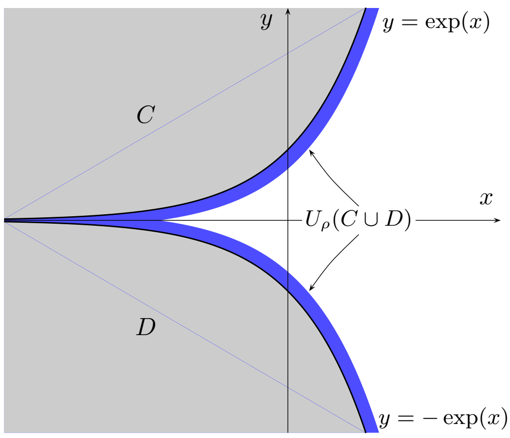


Figure 2.4: The union of two convex sets is not positively α-far.

the unique solution of the equation $y + \exp(2y) = x$. Thus, if $x \to -\infty$ $f(x) \to -\infty$ and $d_S((x, 0)) = \exp(f(x))\sqrt{1 + \exp(2f(x))}$. Moreover,

$$x^* = \frac{1}{\sqrt{1 + \exp(2f(x))}} (\exp(f(x)), 0) \in \partial d_S((x, 0)),$$

and

$$\|x^*\| = \frac{\exp(f(x))}{\sqrt{1 + \exp(2f(x))}} \to 0 \text{ if } x \to -\infty,$$

which shows that S is not positively α-far.

The distance between two sets C and D is defined by $d(C, D) := \inf\{\|x - y\| : x \in D, y \in D\}$. When the distance between the sets is not zero we have the following result.

Proposition 2.10 *Let C, D be two closed sets of H with $d(C, D) > 0$. If C is positively α-far for $\alpha \in]0, 1]$ and D is positively β-far for $\beta \in]0, 1]$ then $C \cup D$ is positively* $\min\{\alpha, \beta\}$*-far.*

2.2.2 Intersection of positively α-far sets

It will be very interesting to know, probably under some constraint qualification, when the intersection of two positively α-far sets remains positively β-far, for some $\alpha, \beta \in]0, 1]$. Although this problem is quite complicated, it is not difficult to prove, under some constraint qualification, that the intersection of two uniformly subsmooth sets remains uniformly subsmooth, and therefore positively α-far. The following result was stated in [118, Theorem 0.2.5] without a proof.

Proposition 2.11 *Let C_1 and C_2 be two uniformly subsmooth sets and assume that there is $\beta > 0$ such that for all $x \in C_1 \cap C_2$*

$$N(C_1 \cap C_2; x) \cap \mathbb{B} \subseteq N(C_1; x) \cap \beta\mathbb{B} + N(C_2; x) \cap \beta\mathbb{B}, \tag{2.12}$$

called in [59] *as "normal cone intersection property". Then $C_1 \cap C_2$ is uniformly subsmooth and therefore positively α-far.*

Proof. Let $\varepsilon > 0$ and define $\delta = \min\{\delta_1, \delta_2\}$, where $\delta_1 > 0$ and $\delta_2 > 0$ are given, respectively, by the uniform subsmoothness of C_1 and C_2 for $\frac{\varepsilon}{2\beta}$. Take $x_1, x_2 \in C_1 \cap C_2$ with $\|x_1 - x_2\| < \delta$ and $x_i^* \in N(C_1 \cap C_2; x_i)$ for $i = 1, 2$. Thus, due to (2.12), there are $u_i^* \in N(C_i; x_1) \cap \beta\mathbb{B}$ and $v_i^* \in N(C_i; x_2) \cap \beta\mathbb{B}$ for $i = 1, 2$, such that $x_1^* = u_1^* + u_2^*$ and $x_2^* = v_1^* + v_2^*$. Therefore, by uniformly subsmoothness of C_1 and C_2,

$$\begin{aligned} \langle u_1^* - v_1^*, x_1 - x_2 \rangle &\geq -\frac{\varepsilon}{2}\|x_1 - x_2\|, \\ \langle u_2^* - v_2^*, x_1 - x_2 \rangle &\geq -\frac{\varepsilon}{2}\|x_1 - x_2\|, \end{aligned}$$

which implies $\langle x_1^* - x_1^*, x_1 - x_2 \rangle \geq -\varepsilon\|x_1 - x_2\|$, i.e, $C_1 \cap C_2$ is uniformly subsmooth. □

2.2.3 Inverse images of positively α-far sets

Let $h\colon X \to Y$ be a differentiable mapping between two Hilbert spaces X and Y. In this subsection we give some conditions to assure the uniform subsmoothness of the family $(h^{-1}(D(t)))_{t\in[T_0,T]}$, where $(D(t))_{t\in[T_0,T]}$ is a given family.

Let $C \subseteq X$, $D \subseteq Y$ be two closed sets and $\bar{x} \in h^{-1}(D) \cap C$. We say that the inverse image set $h^{-1}(D) \cap C$ has the normal cone inverse image property at $\bar{x} \in h^{-1}(D) \cap C$ with respect to the Clarke normal cone (see [59]) if there exists some $k > 0$ and some neighborhood U of $\bar{x}$ such that for all $x \in U \cap (h^{-1}(D) \cap C)$

$$N\left(h^{-1}(D) \cap C; x\right) \cap \mathbb{B}_X \subseteq Dh(x)^* \left(N(D; h(x)) \cap k\mathbb{B}_Y\right) + N(C; x). \tag{2.13}$$

Also, we say that the inverse image set $h^{-1}(D) \cap C$ has the *uniform normal cone inverse image* property (UNCII) property if there exists some $k > 0$ such that (2.13) holds for all $x \in h^{-1}(D) \cap C$.

The next proposition, which will be useful in the Section 4.3, gives some sufficient conditions to assure the UNCII property and the Lipschitz continuity of the set-valued map $w \rightrightarrows h^{-1}(D - w) \cap C$,

Proposition 2.12 *Let $C \subseteq X$ and $D \subseteq Y$ be two closed convex sets and $h\colon X \to Y$ be a differentiable function. Consider the following statements:*

(i) There exists $k > 0$ such that for all $x \in C$

$$\mathbb{B}_Y \subseteq Dh(x)\left(T(C;x) \cap k\mathbb{B}_X\right) - D.$$

(ii) There exists $k > 0$ such that for all $w \in Y$ and all $x \in C$

$$d\left(x, h^{-1}(D-w) \cap C\right) \leq kd\left(h(x)+w, D\right).$$

(iii) There exists $k > 0$ such that for all $w \in Y$ and all $x \in h^{-1}(D-w) \cap C$.

$$\mathbb{B}_Y \subseteq T(D; h(x)+w) - kDh(x)\left(T(C;x) \cap \mathbb{B}_X\right).$$

(iv) There exists $k > 0$ such that for all $w \in Y$ and all $x \in h^{-1}(D-w) \cap C$

$$N\left(h^{-1}(D-w) \cap C; x\right) \cap \mathbb{B}_X \subseteq Dh(x)^* \left[N(D; h(x)+w) \cap k\mathbb{B}_Y\right] + N(C;x).$$

Then the following hold:

1. *(ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv)*
2. *If D is a cone then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv)*
3. *Under (ii) the set-valued map $w \rightrightarrows h^{-1}(D-w) \cap C$ is k'-Lipschitz continuous for every $k' > k$.*

Proof. 1. *(ii)* $\Rightarrow$ *(iii)*: Take $b \in \mathbb{B}_Y$ and $x \in h^{-1}(D-w) \cap C$. Then, by *(ii)*, for all $t > 0$

$$d\left(x, h^{-1}(D-w-tb) \cap C\right) \leq kd\left(h(x)+w+tb, D\right).$$

Thus, for every $\varepsilon > 0$, there exists $v_\varepsilon(t) \in h^{-1}(D-w-tb) \cap C$ such that

$$\begin{aligned}\|x - v_\varepsilon(t)\| &\leq d\left(x, h^{-1}(D-w-tb) \cap C\right) + t\varepsilon \\ &\leq kd\left(h(x)+w+tb, D\right) + t\varepsilon \\ &\leq kt\|b\| + t\varepsilon \\ &\leq t\left(k+\varepsilon\right).\end{aligned}$$

Hence, since $v_\varepsilon(t) \in h^{-1}(D-w-tb) \cap C$,

$$\begin{aligned}tb &\in D - w - h(v_\varepsilon(t)) \\ &\subseteq D - w - h(x) - Dh(x)\left(v_\varepsilon(t) - x\right) + \theta\left(\|v_\varepsilon(t) - x\|, \varepsilon\right) \\ &\subseteq D - w - h(x) - t\left(k+\varepsilon\right)Dh(x)\left(T(C;x) \cap \mathbb{B}_X\right) + \eta\left(t, \varepsilon\right),\end{aligned}$$

for some mappings $\theta(\cdot,\varepsilon)$ and $\eta(\cdot,\varepsilon)$ with

$$\lim_{s\downarrow 0} s^{-1}\theta(s,\varepsilon)=0 \text{ and } \lim_{s\downarrow 0} s^{-1}\eta(s,\varepsilon)=0 \text{ in } Y.$$

From where we obtain

$$\mathbb{B}_Y \subseteq \frac{D-w-h(x)}{t} - (k+\varepsilon)\, Dh(x)\left(T(C;x)\cap \mathbb{B}_X\right) + \frac{\eta\,(t,\varepsilon)}{t},$$

then, taking $t\downarrow 0$, we get

$$\mathbb{B}_Y \subseteq \overline{T(D;h(x)+w)-(k+\varepsilon)\, Dh(x)\left(T(C;x)\cap \mathbb{B}_X\right)}.$$

Next, consider the set-valued map $M\colon z \rightrightarrows T(D;h(x)+w)-(k+\varepsilon)\,Dh(x)z$ and define the closed convex set

$$E=\{(z,y)\in T(C;x)\cap \mathbb{B}_X\times Y : y\in M(z)\}.$$

Hence, $P_X(E)=T(C;x)\cap\mathbb{B}_X$ is bounded and

$$P_Y(E)=T(D;h(x))-(k+\varepsilon)\,Dh(x)\left(T(D;x)\cap\mathbb{B}_X\right).$$

Next, using Robinson's Lemma [129, Lemma 1], we obtain that $P_Y(E)$ is closed which implies (iii).

$(iii)\Rightarrow(ii)$: By contradiction, suppose that for every $n\in\mathbb{N}$ there exist $x_n\in C$ and $w_n\in Y$ such that

$$d\left(x_n, h^{-1}\left(D-w_n\right)\cap C\right) > nd\left(h(x_n)+w_n, D\right). \tag{2.14}$$

For every $n\in\mathbb{N}$, consider the function $f_n(x)=d(h(x)+w_n,D)+I_C(x)$. Then, applying Ekeland's variational principle [26, Theorem 2.1.2] with $\lambda_n=n\varepsilon_n$ and $\varepsilon_n=d(h(x_n)+w_n,D)$, there exists $u_n\in C$ such that

(a) $\|u_n-x_n\|\leq\lambda_n$,

(b) $f_n(u_n)+\frac{1}{n}\leq f_n(x_n)$,

(c) the function $u\mapsto f_n(u)+\frac{1}{n}\|u-u_n\|$ attains a minimum at $u=u_n$.

We claim that $h(u_n)+w_n\notin D$. Indeed, if $h(u_n)+w_n\in D$, due to (a) and (2.14), we have

$$\begin{aligned}\|u_n-x_n\| &\leq nd(h(x_n)+w_n,D)\\ &< d\left(x_n,h^{-1}\left(D-w_n\right)\cap C\right)\\ &\leq \|u_n-x_n\|,\end{aligned}$$

which is a contradiction, hence $h(u_n)+w_n\notin D$. Next, on the one hand, as a result of (c), we obtain the existence of $b_n\in\mathbb{B}_X$ and $y_n^*\in X$ with $\|y_n^*\|=1$

and $y_n^* \in \partial d\left(h(u_n) + w_n, D\right)$ such that $0 \in Dh(u_n)^*\left(y_n^*\right) + \frac{1}{n}b_n + N(C; u_n)$. This last inclusion implies that

$$\left\langle Dh(u_n)^*\left(y_n^*\right) + \frac{1}{n}b_n, c\right\rangle \geq 0 \quad \forall c \in T(C; u_n), \tag{2.15}$$

and since $y_n^* \in \partial d(h(u_n) + w_n, D)$ and $h(u_n) + w_n \notin D$, $y_n^* \in N(D; v_n)$ with v_n the projection of $h(u_n) + w_n$ into D. On the other hand, due to (iii), for every $b \in \mathbb{B}_Y$ there exist $c_n \in T(C; u_n) \cap k\mathbb{B}_X$ and $d_n \in T(D; v_n)$ such that $b = -Dh(u_n)c_n + d_n$. Thus, by virtue of (2.15),

$$\begin{aligned}
\langle y_n^*, b\rangle &= \langle y_n^*, -Dh(u_n)c_n + d_n\rangle \\
&= \langle -Dh(u_n)^*\left(y_n^*\right), c_n\rangle + \langle y_n^*, d_n\rangle \\
&\leq \left\langle \frac{1}{n}b_n, c_n\right\rangle + \langle y_n^*, d_n\rangle \\
&\leq \frac{k}{n},
\end{aligned}$$

from where we get that $\|y_n^*\| \leq \frac{k}{n}$, which is a contradiction. Therefore (ii) holds.

$(iii) \Rightarrow (iv)$: For every $x \in X$ we have the following identity

$$I_{h^{-1}(D-w)\cap C}(x) = I_D(h(x) + w) + I_C(x).$$

Then, due to (iii), we can apply the chain rule for nonsmooth functions [112, Theorem 3.4.1] to obtain

$$N\left(h^{-1}\left(D - w\right) \cap C; x\right) = Dh(x)^* N(D; h(x) + w) + N(C; x),$$

for all $x \in h^{-1}\left(D - w\right) \cap C$. Next, take $x^* \in N\left(h^{-1}\left(D - w\right) \cap C; x\right) \cap \mathbb{B}_X$ so $x^* = Dh(x)^* y^* + z^*$ for some $y^* \in N(D; h(x) + w)$ and $z^* \in N(C; x)$. Then, by (iii), for every $v \in \mathbb{B}_Y$ there exist $w^* \in T(D; h(x) + w)$ and $u \in T(C; x) \cap \mathbb{B}_X$ such that $v = w^* - kDh(x)u$. Hence,

$$\begin{aligned}
\langle y^*, v\rangle &= \langle y^*, w^* - kDh(x)u\rangle \\
&= \langle y^*, w^*\rangle - k\left\langle Dh(x)^* y^*, u\right\rangle \\
&= \langle y^*, w^*\rangle - k\left\langle x^* - z^*, u\right\rangle \\
&\leq k,
\end{aligned}$$

which implies that $\|y^*\| \leq k$. Therefore (iv) holds.

2. Assume that D is a cone. We proceed to prove that $(i) \Rightarrow (ii)$: By contradiction, suppose that for every $n \in \mathbb{N}$ there exist $x_n \in C$ and $w_n \in Y$ such that

$$d\left(x_n, h^{-1}\left(D - w_n\right) \cap C\right) > nd\left(h(x_n) + w_n, D\right). \tag{2.16}$$

For every $n \in \mathbb{N}$ consider the function $f_n(x) = d(h(x) + w_n, D) + I_C(x)$. Then, applying Ekeland's variational principle [26, Theorem 2.1.2] for every $n \in \mathbb{N}$, with $\varepsilon_n = d(h(x_n) + w_n, D)$ and $\lambda_n = n\varepsilon_n$, there exists $u_n \in C$ such that (*a*) $\|u_n - x_n\| \leq \lambda_n$,

(*b*) $f_n(u_n) + \frac{1}{n} \leq f_n(x_n)$,

(*c*) the function $u \mapsto f_n(u) + \frac{1}{n}\|u - u_n\|$ attains a minimum at $u = u_n$.

We claim that $h(u_n) + w_n \notin D$. Indeed, if $h(u_n) + w_n \in D$, due to (*a*) and (2.16), we have

$$\|u_n - x_n\| \leq nd(h(x_n) + w_n, D) < d\left(x_n, h^{-1}(D - w_n) \cap C\right) \leq \|u_n - x_n\|,$$

which is a contradiction, hence $h(u_n) + w_n \notin D$. Next, on one hand, as a result of (*c*), we obtain the existence of $b_n \in \mathbb{B}_X$ and $y_n^* \in X$ with $\|y_n^*\| = 1$ and $y_n^* \in \partial d(h(u_n) + w_n, D)$ such that $0 \in Dh(u_n)^*(y_n^*) + \frac{1}{n}b_n + N(C; u_n)$. This last inclusion implies that

$$\left\langle Dh(u_n)^*(y_n^*) + \frac{1}{n}b_n, c\right\rangle \geq 0 \quad \forall c \in T(C; u_n), \tag{2.17}$$

and since D is a cone $\langle y_n^*, d\rangle \leq 0$ for every $d \in D$. On the other hand, due to (*ii*), for every $b \in \mathbb{B}_Y$ there exist $c_n \in T(C; u_n) \cap k\mathbb{B}_X$ and $d_n \in D$ such that $b = -Dh(u_n)c_n + d_n$. Thus, by virtue of (2.17),

$$\begin{aligned} \langle y_n^*, b\rangle &= \langle y_n^*, -Dh(u_n)c_n + d_n\rangle \\ &= \langle -Dh(u_n)^*(y_n^*), c_n\rangle + \langle y_n^*, d_n\rangle \\ &\leq \left\langle \frac{1}{n}b_n, c_n\right\rangle + \langle y_n^*, d_n\rangle \\ &\leq \frac{k}{n}, \end{aligned}$$

from which we get that $\|y_n^*\| \leq \frac{k}{n}$, and this is a contradiction because $\|y_n^*\| = 1$. Therefore (*i*) holds.

3. Consider $w_1, w_2 \in Y$ and $x \in h^{-1}(D - w_1) \cap C$. Then, due to (*ii*),

$$d\left(x, h^{-1}(D - w_2) \cap C\right) \leq kd(h(x) + w_2, D) \leq k\|w_1 - w_2\|, \tag{2.18}$$

Thus, for $\varepsilon < k' - k$ there exists $y_\varepsilon \in h^{-1}(D - w_2) \cap C$ such that

$$\|x - y_\varepsilon\| \leq d\left(x, h^{-1}(D - w_2) \cap C\right) + \varepsilon\|w_1 - w_2\|.$$

Therefore, as a result of (2.18), $\|x - y_\varepsilon\| \leq (k + \varepsilon)\|w_1 - w_2\|$, which implies that

$$h^{-1}(D - w_1) \cap C \subseteq h^{-1}(D - w_2) \cap C + k'\|w_1 - w_2\|\mathbb{B}_X.$$

□

Now we give some conditions to assure the equi-uniform subsmoothness of inverse images of a given family of sets under a differentiable mapping $h\colon X \to Y$. The first statement of the next proposition was stated in [118] without a proof.

Proposition 2.13 *Let $h\colon X \to Y$ be a continuously differentiable mapping with Dh uniformly continuous. Assume that one of the following conditions is verified:*

i) Assume that the function h is Lipschitz, the family $(D(t))_{t\in[T_0,T]} \subseteq Y$ is equi-uniformly subsmooth and the UNCII property holds for all $D(t)$, $t \in [T_0, T]$ with the same $k > 0$.

ii) Assume that the set $D(t) \subseteq Y$ is convex for every $t \in [T_0, T]$ and UNCII property holds for all $D(t)$, $t \in [T_0, T]$ with the same $k > 0$.

iii) Assume that the function h is Lipschitz, the set $D \subseteq Y$ is uniformly subsmooth, $d\colon [T_0, T] \to Y$ is a function and UNCII property holds for all $D(t) := D - d(t)$, $t \in [T_0, T]$ with the same $k > 0$.

iv) Assume that the set $D \subseteq Y$ is convex, $d\colon [T_0, T] \to Y$ is a function and UNCII property holds for all $D(t) := D - d(t)$, $t \in [T_0, T]$ with the same $k > 0$.

Then the family $(h^{-1}(D(t)))_{t\in[T_0,T]}$ is equi-uniformly subsmooth and, hence, positively α-far for some $\alpha \in]0, 1]$.

Proof. i) Since h is Lipschitz, there exist $L > 0$ and $\delta_1 > 0$ such that

$$\|x - y\| < \delta_1 \text{ then } \|h(x) - h(y)\| \leq L\|x - y\|.$$

For $\varepsilon > 0$ take $\delta = \min\{\frac{\delta_3}{L}, \delta_2, \delta_1\}$, where δ_3 is given by the equi-uniform subsmoothness of $(D(t))_{t\in[T_0,T]}$ for $\frac{\varepsilon}{2kL}$, δ_2 is given by the uniform continuity of Dh for $\frac{\varepsilon}{4k}$. Next, fix $t \in [T_0, T]$ and take $u_1, u_2 \in h^{-1}(D(t))$ with $\|u_1 - u_2\| < \delta$ and $x_i^* \in N(h^{-1}(D(t)); u_i) \cap \mathbb{B}_X$ for $i = 1, 2$. We have to prove that

$$\langle x_1^* - x_2^*, u_1 - u_2\rangle \geq -\varepsilon\|u_1 - u_2\|. \tag{2.19}$$

Indeed, by the UNCII property, there exist $y_i^* \in N(D(t); h(u_i)) \cap \mathbb{B}_Y$ for $i = 1, 2$ such that $x_i^* = kDh(u_i)^* y_i^*$ for $i = 1, 2$. Then, since

$$\|h(u_1) - h(u_2)\| \leq L\|u_1 - u_2\| < \delta_3,$$

we can use the equi-uniform subsmoothness of $D(t)$ to get

$$\langle y_1^* - y_2^*, h(u_1) - h(u_2)\rangle \geq -\frac{\varepsilon}{2kL}\|h(u_1) - h(u_2)\|. \tag{2.20}$$

Next, since $\|u_2 + s(u_1 - u_2) - u_1\| < \delta_2$ for all $s \in [0, 1]$ and due to the uniform continuity of Dh we have

$$\|Dh(u_2 + s(u_1 - u_2)) - Dh(u_1)\| \leq \frac{\varepsilon}{4k}.$$

Therefore, using this last inequality

$$
\begin{aligned}
\langle ky_1^*, h(u_1) - h(u_2)\rangle &= \left\langle ky_1^*, \int_0^1 Dh\left(u_2 + s(u_1 - u_2)\right)(u_1 - u_2)ds \right\rangle \\
&= \langle ky_1^*, Dh(u_1)(u_1 - u_2)\rangle \\
&+ \left\langle ky_1^*, \int_0^1 \left[Dh\left(u_2 + s(u_1 - u_2)\right) - Dh(u_1)\right](u_1 - u_2)ds \right\rangle \\
&\leq \langle ky_1^*, Dh\left(u_1\right)\left(u_1 - u_2\right)\rangle + k \cdot \frac{\varepsilon}{4k}\|u_1 - u_2\| \\
&= \langle kDh\left(u_1\right)^* y_1^*, u_1 - u_2\rangle + \frac{\varepsilon}{4}\|u_1 - u_2\| \\
&= \langle x_1^*, u_1 - u_2\rangle + \frac{\varepsilon}{4}\|u_1 - u_2\|
\end{aligned}
$$

Therefore,

$$
\langle ky_1^*, h(u_1) - h(u_2)\rangle \leq \langle x_1^*, u_1 - u_2\rangle + \frac{\varepsilon}{4}\|u_1 - u_2\|,
$$

and similarly,

$$
\langle ky_2^*, h(u_2) - h(u_1)\rangle \leq \langle x_2^*, u_2 - u_1\rangle + \frac{\varepsilon}{4}\|u_1 - u_2\|,
$$

Next, by adding up these two inequalities we obtain

$$
\langle ky_1^* - ky_2^*, h(u_1) - h(u_2)\rangle \leq \langle x_1^* - x_2^*, u_1 - u_2\rangle + \frac{\varepsilon}{2}\|u_1 - u_2\|,
$$

which combined with (2.20) and the Lipschitzianity of h entails (2.19), i.e., the equi-uniform subsmoothness of the family $(h^{-1}(D(t)))_{t\in[T_0,T]}$.

ii) For $\varepsilon > 0$ take δ given by the uniform continuity of Dh for $\frac{\varepsilon}{2k}$. Next, fix $t \in [T_0, T]$ and take $u_1, u_2 \in h^{-1}\left(D(t)\right)$ with $\|u_1 - u_2\| < \delta$ and

$$
x_i^* \in N\left(h^{-1}\left(D(t)\right); u_i\right) \cap \mathbb{B}_X \quad \text{for } i = 1, 2.
$$

We have to prove that

$$
\langle x_1^* - x_2^*, u_1 - u_2\rangle \geq -\varepsilon\|u_1 - u_2\|. \tag{2.21}
$$

Indeed, by the UNCII property, there exist $y_i^* \in N(D(t); h(u_i)) \cap \mathbb{B}_Y$ for $i = 1, 2$ such that $x_i^* = kDh(u_i)^* y_i^*$ for $i = 1, 2$. Then, due to the convexity of $D(t)$, we obtain

$$
\langle y_1^* - y_2^*, h(u_1) - h(u_2)\rangle \geq 0. \tag{2.22}
$$

Next, since $\|u_2 + s(u_1 - u_2) - u_1\| < \delta_2$ for all $s \in [0, 1]$ and due to the uniform continuity of Dh, we have

$$
\|Dh\left(u_2 + s(u_1 - u_2)\right) - Dh(u_1)\| \leq \frac{\varepsilon}{2k}.
$$

Therefore, using this last inequality

$$\begin{aligned}
\langle ky_1^*, h(u_1) - h(u_2)\rangle &= \left\langle ky_1^*, \int_0^1 Dh\left(u_2 + s(u_1 - u_2)\right)(u_1 - u_2)ds \right\rangle \\
&= \langle ky_1^*, Dh(u_1)(u_1 - u_2)\rangle + \\
&\left\langle ky_1^*, \int_0^1 \left[Dh\left(u_2 + s(u_1 - u_2)\right) - Dh(u_1)\right](u_1 - u_2)ds \right\rangle \\
&\leq \langle ky_1^*, Dh\left(u_1\right)(u_1 - u_2)\rangle + k \cdot \frac{\varepsilon}{2k}\|u_1 - u_2\| \\
&= \langle kDh\left(u_1\right)^* y_1^*, u_1 - u_2\rangle + \frac{\varepsilon}{2}\|u_1 - u_2\| \\
&= \langle x_1^*, u_1 - u_2\rangle + \frac{\varepsilon}{2}\|u_1 - u_2\|
\end{aligned}$$

Therefore,

$$\langle ky_1^*, h(u_1) - h(u_2)\rangle \leq \langle x_1^*, u_1 - u_2\rangle + \frac{\varepsilon}{2}\|u_1 - u_2\|,$$

and similarly,

$$\langle ky_2^*, h(u_2) - h(u_1)\rangle \leq \langle x_2^*, u_2 - u_1\rangle + \frac{\varepsilon}{2}\|u_1 - u_2\|,$$

Next, by adding up these two inequalities we obtain

$$\langle ky_1^* - ky_2^*, h(u_1) - h(u_2)\rangle \leq \langle x_1^* - x_2^*, u_1 - u_2\rangle + \varepsilon\|u_1 - u_2\|,$$

which combined with (2.22) entails (2.21).

iii) It follows from the proof of i) and the formula

$$N(D - d(t); x) = N(D; x + d(t)) \quad \text{for all } t \in [T_0, T]. \tag{2.23}$$

iv) It follows from the proof of ii) and the formula (2.23).

□

The next corollary will be useful in Section 4.3.

Corollary 2.14 *Let $h\colon X \to Y$ be a continuously differentiable mapping with Dh uniformly continuous and let $d\colon [T_0, T] \to Y$ be a function. Assume that there exists $k > 0$ such that*

$$\mathbb{B}_Y \subseteq Dh(x)k\mathbb{B}_X - K \quad \text{for all } x \in X,$$

where $K \subseteq Y$ is a closed convex cone. Then the family $\left(h^{-1}\left(K - d(t)\right)\right)_{t\in[T_0,T]}$ is equi-uniformly subsmooth, hence, positively α-far for some $\alpha \in]0, 1]$ and the set-valued map $t \rightrightarrows h^{-1}\left(K - d(t)\right)$ is absolutely continuous over $[T_0, T]$.

Proof. For the first assertion, by Proposition 2.13, it is sufficient to prove that the set $h^{-1}\left(K - d(t)\right)$ satisfies the UNCII property for all $t \in [T_0, T]$ with the same k which follows directly from the first statement of Proposition 2.12. The second assertion follows from the third statement of Proposition 2.12 combined with the absolute continuity of d. □

Chapter 3

Galerkin-Like method and applications

Let H, U and V be separable Hilbert spaces, T_0, T be two non-negative real numbers with $T_0 < T$. In this chapter, which is based on [89], we present a new method to solve differential inclusions. In this method we approach the original problem by projecting the state into a n-dimensional Hilbert space but not the velocity. We prove that the approached problem always has a solution (see Proposition 3.3) and that, under some compactness conditions, the approached problems converges strongly pointwisely (up to a subsequence) to a solution of the original differential inclusion (see Theorem 3.4).

More explicitly, consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{a.e } t \in [T_0, T], \\ x(T_0) = x_0. \end{cases} \tag{3.1}$$

For each $n \in \mathbb{N}$ we approach (3.1) by the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t, P_n(x(t))) & \text{a.e } t \in [T_0, T], \\ x(T_0) = P_n(x_0), \end{cases} \tag{3.2}$$

where, given an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ de H, P_n is the projector from H into the linear span of $\{e_1, \ldots, e_n\}$. We will call this method *Galerkin-like method*. We will show how this method is well adapted to deal with constrained differential inclusions by providing existence of solutions to the following differential inclusion:

$$\begin{cases} -\dot{u}(t) = Bv(t) & \text{a.e. } t \in [T_0, T], \\ -\dot{v}(t) \in N\left(C(t, u(t), v(t)); v(t)\right) + F(t, u(t), v(t)) + Au(t) & \text{a.e. } t \in [T_0, T], \\ u(T_0) = u_0, v(T_0) = v_0 \in C(T_0, u_0, v_0), \end{cases} \tag{3.3}$$

where $A\colon U \to V$ and $B\colon V \to U$ are two bounded linear operators, $N(S; \cdot)$ denotes the Clarke normal cone to a closed set $S \subseteq V$ and $F\colon [T_0, T] \times U \times V \rightrightarrows V$ is a set-valued mapping with nonempty closed and convex values satisfying some appropriate conditions. We call the differential inclusion (3.3) Generalized Sweeping Process because it includes the perturbed state-dependent sweeping process, the Moreau's sweeping process and the perturbed second-order sweeping process.

The chapter is organized as follows. In Sections 3.1 and 3.2, respectively, we collect the hypotheses and give some lemmas used along the chapter. The Galerkin-like method is studied in Section 3.3, where we prove the existence of solutions to the approached problems (3.2) (see Proposition 3.3) and its convergence strongly pointwisely (up to a subsequence) to a solution of (3.1) (see Theorem 3.4). In Section 3.4 we established the existence of solutions of the Generalized Sweeping Process via the Galerkin-like method. Then, in Sections 3.5, 3.6 and 3.7 we obtain, respectively, existence for the state-dependent, Moreau's and second order sweeping process. Finally, we end this chapter with an example which shows the importance of the ball compactness of the sets moving sets.

3.1 Technical assumptions

For the sake of readability, in this section we collect the hypotheses used along the chapter.

Hypotheses on the set-valued map $C\colon [T_0,T]\times U\times V \rightrightarrows V$ C is a set-valued map with nonempty and closed values. Also, we will consider the following conditions:

($\mathcal{H}_1$) There exist $\zeta \in W^{1,1}([T_0,T];\mathbb{R})$, $L_1 \geq 0$ and $L_2 \in [0,1[$ such that for all $s,t\in[0,T]$ and all $x,y\in U$ and $u,v,w\in V$

$$|d(w,C(t,x,u))-d(w,C(s,y,v))| \leq |\zeta(t)-\zeta(s)| + L_1\|x-y\| + L_2\|u-v\|.$$

($\mathcal{H}_2$) There exist two constants $\alpha_0 \in]0,1]$ and $\rho \in]0,+\infty]$ such that for every $(u,v)\in U\times V$

$$0<\alpha_0 \leq \inf_{x\in U_\rho(C(t,u,v))} d\left(0,\partial d(\cdot,C(t,u,v))(x)\right) \quad \text{a.e. } t\in[T_0,T],$$

where $U_\rho\left(C(t,u,v)\right) = \{x\in V\colon 0<d(x,C(t,u,v))<\rho\}$.

($\mathcal{H}_3$) The family $\{C(t,u,v)\colon (t,u,v)\in[T_0,T]\times U\times V\}$ is equi-uniformly subsmooth.

($\mathcal{H}_4$) For every $t\in[T_0,T]$, every $r>0$ and every pair of bounded sets $A\subseteq U$ and $B\subseteq V$, the set $C(t,A,B)\cap r\mathbb{B}$ is relatively compact.

Hypotheses on the set-valued map $C\colon [T_0,T]\times H \rightrightarrows H$ C is a set-valued map with nonempty and closed values. Also, we will consider the following conditions:

($\mathcal{H}_5$) There exist $\zeta \in W^{1,1}([T_0, T]; \mathbb{R})$ and $L_2 \in [0, 1[$ such that for all $s, t \in [0, T]$ and all $x, y, z \in H$
$$|d(z, C(t, x)) - d(z, C(s, y))| \leq |\zeta(t) - \zeta(s)| + L_2\|x - y\|.$$

($\mathcal{H}_6$) The family $\{C(t, v) \colon (t, v) \in [T_0, T] \times H\}$ is equi-uniformly subsmooth.

($\mathcal{H}_7$) For every $t \in [T_0, T]$, every $r > 0$ and every bounded set $A \subseteq H$ the set $C(t, A) \cap r\mathbb{B}$ is relatively compact.

Hypotheses on the set-valued map $C \colon [T_0, T] \rightrightarrows H$ C is a set-valued map with nonempty and closed values. Also, we will consider the following conditions:

($\mathcal{H}_8$) There exists $\zeta \in W^{1,1}([T_0, T]; \mathbb{R})$ such that for all $s, t \in [0, T]$ and all $x \in H$
$$|d(x, C(t)) - d(x, C(s))| \leq |\zeta(t) - \zeta(s)|.$$

($\mathcal{H}_9$) There exist two constants $\alpha_0 \in]0, 1]$ and $\rho \in]0, +\infty]$ such that
$$0 < \alpha_0 \leq \inf_{x \in U_\rho(C(t))} d(0, \partial d(x, C(t))) \quad \text{a.e. } t \in [T_0, T],$$
where $U_\rho(C(t)) = \{x \in H \colon 0 < d(x, C(t)) < \rho\}$ for all $t \in [T_0, T]$.

($\mathcal{H}_{10}$) For all $t \in [T_0, T]$ the set $C(t)$ is ball compact, that is, for every $r > 0$ the set $C(t) \cap r\mathbb{B}$ is compact in H.

Remark 3.1 Under ($\mathcal{H}_4$), the set $\text{Proj}_{C(t,u,v)}(v) \neq \emptyset$ for all $(t, u, v) \in [T_0, T] \times U \times V$. Indeed, let $(z_n)_n \subseteq C(t, u, v)$ such that $\|v - z_n\| \to d_{C(t,u,v)}(v)$ as $n \to +\infty$. Then, $(z_n)_n \subseteq r\mathbb{B} \cap C(t, \{u\}, \{v\})$ for some $r > 0$, which implies, by virtue of ($\mathcal{H}_4$), that $(z_n)_n$ is relatively compact. Thus, a subsequence of (z_n) converges to an element of $\text{Proj}_{C(t,u,v)}(v)$.

Remark 3.2 Let $L_2 \in [0, 1[$. Under ($\mathcal{H}_3$) for every $\alpha_0 \in]\sqrt{L_2}, 1]$ there exists $\rho > 0$ such that ($\mathcal{H}_2$) holds. This is a consequence of Proposition 2.8.

Hypotheses on the set-valued map $F \colon [T_0, T] \times U \times V \rightrightarrows V$ F is a set-valued map with nonempty, closed and convex values. Also, we will consider the following conditions:

($\mathcal{H}_1^F$) For each $(u, v) \in U \times V$, $F(\cdot, u, v)$ is measurable.

($\mathcal{H}_2^F$) For a.e. $t \in [T_0, T]$, $F(t, \cdot, \cdot)$ is upper semicontinuous from $U \times V$ into V_w.

($\mathcal{H}_3^F$) There exist $c, d \in L^1(T_0, T)$ such that
$$d(0, F(t, u, v)) := \inf\{\|w\| \colon w \in F(t, u, v)\} \leq c(t)\|(u, v)\| + d(t),$$
for all $(u, v) \in U \times V$ and a.e. $t \in [T_0, T]$.

Hypotheses on the set-valued map $F\colon [T_0,T]\times H \rightrightarrows H$ F is a set-valued map with nonempty, closed and convex values. Moreover, we will consider the following conditions:

$(\mathcal{H}_4^F)$ For each $v\in H$, $F(\cdot,v)$ is measurable.

$(\mathcal{H}_5^F)$ For a.e. $t\in[T_0,T]$, $F(t,\cdot)$ is upper semicontinuous from H into H_w.

$(\mathcal{H}_6^F)$ There exist $c,d\in L^1(T_0,T)$ such that

$$d\left(0,F(t,v)\right) := \inf\{\|w\|\colon w\in F(t,v)\} \le c(t)\|v\| + d(t),$$

for all $v\in H$ and a.e. $t\in[T_0,T]$.

3.2 Preliminary results

The following lemma will be used in the proof of Proposition 3.3.

Lemma 3.1 *Assume that* $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$ *and* $(\mathcal{H}_6^F)$ *hold and let* $r\colon [T_0,T]\to\mathbb{R}_+$ *be a continuous function. Then, the set-valued map* $G\colon [T_0,T]\times H\rightrightarrows H$ *defined by*

$$G(t,x) := F(t,p_{r(t)}(x)) \cap \left(c(t)\|p_{r(t)}(x)\| + d(t)\right)\mathbb{B} \quad (t,x)\in[T_0,T]\times H,$$

where $p_{r(t)}(x) = \begin{cases} x & \text{if } \|x\|\le r(t),\\ r(t)\frac{x}{\|x\|} & \text{if } \|x\|> r(t),\end{cases}$ *, satisfies:*

(i) $G(t,x)$ *is nonempty, closed and convex for all* $(t,x)\in[T_0,T]\times H$*,*

(ii) for each $x\in H$*,* $G(\cdot,x)$ *is measurable,*

(iii) for a.e. $t\in[T_0,T]$*,* $G(t,\cdot)$ *is upper semicontinuous from* H *into* H_w*,*

(iv) for all $x\in H$ *and a.e.* $t\in[T_0,T]$

$$\|G(t,x)\| := \sup\{\|w\|\colon w\in G(t,x)\} \le c(t)r(t) + d(t).$$

Proof. (i) is direct. (iii) follows from $(\mathcal{H}_5^F)$ and [9, Theorems 17.23 and 17.25]. Also, due to $(\mathcal{H}_6^F)$, we have

$$\begin{aligned}\|G(t,x)\| &= \sup\{\|w\|\colon w\in G(t,x)\}\\ &\le c(t)\|p_{r(t)}(x)\| + d(t)\\ &\le c(t)r(t)+d(t)\end{aligned}$$

which proves (iv). Thus, by virtue of (i) and (iv), G takes weakly compact and convex values. Therefore, (ii) follows from $(\mathcal{H}_4^F)$ and [85, Proposition 2.2.37]. □

The following result may be proved in much the same way as Lemma 1.18.

Lemma 3.2 *Let $x, z\colon [T_0, T] \to V$ and $y\colon [T_0, T] \to U$ be three absolutely continuous functions and let $C\colon [T_0, T] \times U \times V \rightrightarrows V$ be a set-valued map with nonempty closed values satisfying $(\mathcal{H}_1)$. Then*

(i) The function $t \to d(x(t); C(t, y(t), z(t)))$ is absolutely continuous over $[T_0, T]$.

(ii) For all $t \in [T_0, T]$, where $\dot{\zeta}(t)$, $\dot{y}(t)$ and $\dot{z}(t)$ exist,

$$\begin{aligned}
&\limsup_{s \downarrow 0} \frac{1}{s} \left[d_{C(t+s, y(t+s), z(t+s))}(x(t+s)) - d_{C(t, y(t), z(t))}(x(t)) \right] \\
&\leq |\dot{\zeta}(t)| + L_1 \|\dot{y}(t)\| + L_2 \|\dot{z}(t)\| \\
&\qquad\qquad + \limsup_{s \downarrow 0} \frac{1}{s} \left[d_{C(t, y(t), z(t))}(x(t+s)) - d_{C(t, y(t), z(t))}(x(t)) \right].
\end{aligned}$$

(iii) For all $t \in [T_0, T]$, where $\dot{x}(t)$ exists,

$$\begin{aligned}
&\limsup_{s \downarrow 0} \frac{1}{s} \left[d_{C(t, y(t), z(t))}(x(t+s)) - d_{C(t, y(t), z(t))}(x(t)) \right] \\
&\qquad\qquad \leq \max_{y^* \in \partial d_{C(t, y(t), z(t))}(x(t))} \langle y^*, \dot{x}(t) \rangle .
\end{aligned}$$

(iv) For all $t \in \{s \in [T_0, T] \colon x(s) \notin C(s, y(s), z(s))\}$, where $\dot{x}(t)$ exists,

$$\begin{aligned}
&\lim_{s \downarrow 0} \frac{1}{s} \left[d_{C(t, y(t), z(t))}(x(t+s)) - d_{C(t, y(t), z(t))}(x(t)) \right] \\
&\qquad\qquad = \min_{y^* \in \partial d(x(t), C(t, y(t), z(t)))} \langle y^*, \dot{x}(t) \rangle .
\end{aligned}$$

(v) For every $x \in V$ the set-valued map $t \rightrightarrows \partial d(x, C(t, y(t), z(t)))$ is measurable.

3.3 Galerkin-like method

In this section we study existence of solutions to the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0, \end{cases} \tag{3.4}$$

where $F\colon [T_0, T] \times H \rightrightarrows H$ is a set-valued map with nonempty closed and convex values. For every $n \in \mathbb{N}$ let us consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t, P_n(x(t))) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = P_n(x_0), \end{cases} \tag{3.5}$$

where $P_n\colon H \to \operatorname{span}\{e_1, \ldots, e_n\}$ is the linear operator defined in Lemma 1.7. The next proposition asserts the existence of solutions for the approximate problem (3.5).

Proposition 3.3 *Assume that* $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$ *and* $(\mathcal{H}_6^F)$ *hold. Then, for each* $n \in \mathbb{N}$ *there exists at least one solution* $x_n \in W^{1,1}([T_0,T];H)$ *of* (3.5). *Moreover, for all* $t \in [T_0,T]$

$$\|x_n(t)\| \leq r(t) := \left(\|x_0\| + \int_{T_0}^t d(s)ds\right) \exp\left(\int_{T_0}^t c(s)ds\right), \tag{3.6}$$

and

$$\|\dot{x}_n(t)\| \leq \psi(t) := c(t)r(t) + d(t) \quad \text{a.e. } t \in [T_0,T]. \tag{3.7}$$

Proof. Let us consider $G(t,x) := F(t, p_{r(t)}(x)) \cap \big(c(t)\|p_{r(t)}(x)\| + d(t)\big)$, where

$$p_{r(t)} \colon H \to H$$

is given by

$$p_{r(t)}(x) = \begin{cases} x & \text{if } \|x\| \leq r(t), \\ r(t)\frac{x}{\|x\|} & \text{if } \|x\| > r(t), \end{cases}.$$

Then, due to Lemma 3.1, G satisfies $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$ and

$$\|G(t,x)\| := \sup\{\|w\| \colon w \in G(t,x)\} \leq c(t)r(t) + d(t), \tag{3.8}$$

for all $x \in H$ and a.e. $t \in [T_0,T]$.

Consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in G(t, P_n(x(t))) & \text{a.e. } t \in [T_0,T], \\ x(T_0) = P_n(x_0). \end{cases} \tag{3.9}$$

Let $K \subseteq L^1([T_0,T];H)$ be defined by

$$K := \left\{ f \in L^1([T_0,T];H) : \|f(t)\| \leq \psi(t) \text{ a.e. } t \in [T_0,T] \right\},$$

where ψ is defined by (3.7). This set is nonempty, closed and convex. In addition, since $\psi \in L^1(T_0,T)$, K is bounded and uniformly integrable, hence, it is compact in $L^1_w([T_0,T];H)$ (see Theorem 1.3). Since $L^1([T_0,T];H)$ is separable, we also note that K, endowed with the relative $L^1_w([T_0,T];H)$ topology is a metric space (see [64, Theorem V.6.3]). Define the map $\mathcal{F}_n \colon K \rightrightarrows L^1([T_0,T];H)$ by

$$\mathcal{F}_n(f) := \left\{ v \in L^1([T_0,T];H) : v(t) \in G(t, P_n(x_0 + \int_{T_0}^t f(s)ds)) \text{ a.e. } t \in [T_0,T] \right\},$$

for $f \in K$. By $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$, (3.8) and [7, Lemma 6], we conclude that $\mathcal{F}_n(f)$ has nonempty, closed and convex values. Moreover, $\mathcal{F}_n(K) \subseteq K$. Indeed, let $f \in K$ and $v \in \mathcal{F}_n(f)$. Then,

$$\begin{aligned} \|v(t)\| &\leq \sup\{\|w\| \colon w \in G(t, P_n(x_0 + \int_{T_0}^t f(s)ds))\} \\ &\leq c(t)r(t) + d(t) \\ &= \psi(t). \end{aligned}$$

We denote K_w the set K seen as a compact convex subset of $L^1_w([T_0,T];H)$.

Claim 1: $\mathcal{F}_n$ is upper semicontinuous from K_w into K_w.

Proof of Claim 1: By virtue of [85, Proposition 1.2.23] it is sufficient to prove that its graph graph$(\mathcal{F}_n)$ is sequentially closed in $K_w \times K_w$.

Let $(f_m, v_m) \in \text{graph}(\mathcal{F}_n)$ with $f_m \to f$ and $v_m \to v$ in $L^1_w([T_0,T];H)$ as $m \to +\infty$. We have to show that $(f,v) \in \text{graph}(\mathcal{F}_n)$. To do that, let us define

$$u_m(t) := P_n(x_0) + \int_{T_0}^t f_m(s)ds \quad \text{for every } t \in [T_0,T].$$

Thus,

$$v_m(t) \in G(t, P_n(u_m(t))) \text{ for a.e. } t \in [T_0,T]. \tag{3.10}$$

Also, since $f_m \in K$, we have that

$$\|\dot{u}_m(t)\| \leq \psi(t) \quad \text{a.e. } t \in [T_0,T].$$

Hence, due to Lemma 1.6, there exists a subsequence of $(u_m)_m$ (without relabeling) and an absolutely continuous function $u\colon [T_0,T] \to H$ such that

$$\begin{aligned} u_m(t) &\to u(t) \text{ weakly } \text{ for all } t \in [T_0,T], \\ \dot{u}_m &\to \dot{u} \text{ in } L^1_w([T_0,T];H), \end{aligned}$$

which implies that $\dot{u} = f$. Moreover, since $(u_m(t))_m$ is bounded for every $t \in [T_0,T]$, $P_n(u_m(t)) \to P_n(u(t))$ for every $t \in [T_0,T]$. Consequently, by virtue of [70, Proposition 2.3.1], (3.10) and the upper semicontinuity of G from H into H_w, for a.e. $t \in [T_0,T]$

$$\begin{aligned} v(t) &\in \overline{\text{conv}}\, w\text{-}\limsup_{m\to+\infty}\{v_m(t)\} \\ &\subseteq \overline{\text{conv}}\, G(t, P_n(u(t))) \\ &= G(t, P_n(u(t))), \end{aligned}$$

which shows that $(f,v) \in \text{graph}(\mathcal{F}_n)$, as claimed. □

Now, we can invoke the Kakutani-Fan-Glicksberg fixed point theorem (see [9, Corollary 17.55]) to the set-valued map $\mathcal{F}_n\colon K_w \rightrightarrows K_w$ to deduce the existence of $\widehat{f}_n \in K$ such that $\widehat{f}_n \in \mathcal{F}_n(\widehat{f}_n)$. Then, the function $x_n \in W^{1,1}([T_0,T];H)$ defined for every $t \in [T_0,T]$ as:

$$x_n(t) = P_n(x_0) + \int_{T_0}^t \widehat{f}_n(s)ds,$$

is a solution of (3.9). Moreover, $x_n \in W^{1,1}([T_0,T];H)$ is a solution of (3.5). Indeed, for a.e. $t \in [T_0,T]$,

$$\begin{aligned} \|\dot{x}_n(t)\| &\leq c(t)\|p_{r(t)}(P_n(x_n(t)))\| + d(t) \\ &\leq c(t)\|x_n(t)\| + d(t), \end{aligned}$$

which, by Gronwall's inequality (see Lemma 1.17) and the fact that $\|P_n(x_0)\| \leq \|x_0\|$, implies (3.6). Finally, $\|P_n(x_n(t))\| \leq r(t)$ and $p_{r(t)}(P_n(x_n(t))) = P_n(x_n(t))$ for all $t \in [T_0,T]$, which finishes the proof. □

The following theorem asserts the existence of solution of (3.4) under a compactness condition on the sequence $(P_n(x_n(t)))_n$ for every $t \in [T_0, T]$.

Theorem 3.4 *Let assumptions* $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$ *and* $(\mathcal{H}_6^F)$ *hold. Assume that the sequence* $(P_n(x_n(t)))_n$ *is relatively compact for all* $t \in [T_0, T]$. *Then, there exists a subsequence* $(x_{n_k})_k$ *of* $(x_n)_n$ *converging strongly pointwisely to a solution* $x \in W1,1([T_0, T]; H)$ *of* (3.4). *Moreover,*

$$\|x(t)\| \leq r(t) := \left(\|x_0\| + \int_{T_0}^t d(s)ds\right) \exp\left(\int_{T_0}^t c(s)ds\right) \quad \text{for all } t \in [T_0, T],$$

and

$$\|\dot{x}(t)\| \leq \psi(t) := c(t)r(t) + d(t) \quad \text{a.e. } t \in [T_0, T].$$

Proof. We will show the existence of the subsequence via Lemma 1.6.

Claim 1: There exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and an absolutely continuous function x such that (i), (ii), (iii) and (iv) from Lemma 1.6 hold with ψ defined as in the statement of the theorem.

Proof of Claim 1: According to Proposition 3.3, $\|\dot{x}_n(t)\| \leq \psi(t) = c(t)r(t) + d(t)$ for a.e. $t \in [T_0, T]$, which shows that (1.4) holds with the function ψ defined as above. Also, $P_n(x_0) \to x_0$ as $n \to +\infty$. Therefore, the claim follows from Lemma 1.6. □

By simplicity we denote $x_k := x_{n_k}$ for $k \in \mathbb{N}$.

Claim 2: $P_k(x_k(t)) \rightharpoonup x(t)$ as $k \to +\infty$ for all $t \in [T_0, T]$.

Proof of Claim 2: Since $x_k(t) \rightharpoonup x(t)$ as $k \to +\infty$ for all $t \in [T_0, T]$, the result follows from (iv) of Lemma 1.7. □

Claim 3: $P_k(x_k(t)) \to x(t)$ as $k \to +\infty$ for all $t \in [T_0, T]$.

Proof of Claim 3: The result follows from Claim 2 and the relative compactness of the sequence $(P_n(x_n(t)))_n$ for a.e. $t \in [T_0, T]$. □

Summarizing, we have

(i) For each $x \in H$, $F(\cdot, x)$ is measurable.

(ii) For a.e. $t \in [T_0, T]$, $F(t, \cdot)$ is upper semicontinuous from H into H_w.

(iii) $\dot{x}_k \rightharpoonup \dot{x}$ in $L^1([T_0, T]; H)$.

(iv) $P_k(x_k(t)) \to x(t)$ as $k \to +\infty$ for a.e. $t \in [T_0, T]$.

(v) For all $k \in \mathbb{N}$, $\dot{x}_k(t) \in F(t, P_k(x_k(t)))$ for a.e. $t \in [T_0, T]$.

These conditions and the convergence theorem (see [7, Proposition 5] for more details) implies that $x \in W^{1,1}([T_0, T]; H)$ is a solution of (3.4), which finishes the proof. □

3.4 A Generalized Perturbed Sweeping Process

In this section we study the generalized perturbed sweeping process:

$$\begin{cases} -\dot{u}(t) = Bv(t) & \text{a.e. } t \in [T_0, T], \\ -\dot{v}(t) \in N\left(C(t, u(t), v(t)); v(t)\right) & \\ \qquad + F(t, u(t), v(t)) + Au(t) & \text{a.e. } t \in [T_0, T], \\ u(T_0) = u_0, v(T_0) = v_0 \in C(T_0, u_0, v_0), \end{cases} \tag{3.11}$$

where $A\colon U \to V$ and $B\colon V \to U$ are two bounded linear operators and $F\colon [T_0, T] \times U \times V \rightrightarrows V$ is a set-valued mapping with nonempty closed and convex values.

The following lemma can be proved in the same way as Lemma 5.1.

Lemma 3.5 *Assume that $(\mathcal{H}_1)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ hold. Then, for all $t \in [T_0, T]$ the set-valued map $(u, v) \rightrightarrows \partial d(\cdot, C(t, u, v))(v)$ is upper semicontinuous from $U \times V$ into V_w.*

The next theorem, which is the main result of this section, gives an existence result for (3.11).

Theorem 3.6 *Assume, in addition to $(\mathcal{H}_1)$, $(\mathcal{H}_3)$, $(\mathcal{H}_4)$, that $A\colon U \to V$ and $B\colon V \to U$ are two bounded linear operators. Let $F\colon [T_0, T] \times U \times V \rightrightarrows V$ be a set-valued mapping with nonempty closed and convex values satisfying $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$. Then, for all $\alpha_0 \in]\sqrt{L_2}, 1]$ there exists at least one solution $(u, v) \in \mathrm{W}^{2,1}([T_0, T]; U) \times W^{1,1}([T_0, T]; V)$ of (3.11) satisfying*

$$\|(u(t), v(t))\| \le \mu(t) := \left(\|(u_0, v_0)\| + \int_{T_0}^t \widetilde{d}(s) ds\right) \exp\left(\int_{T_0}^t \widetilde{c}(s) ds\right),$$

for all $t \in [T_0, T]$, where

$$\widetilde{c}(t) := \frac{\alpha_0^2 + 1}{\alpha_0^2 - L_2}(c(t) + \|A\|) + \left(1 + \frac{L_1}{\alpha_0^2 - L_2}\right)\|B\|,$$

$$\widetilde{d}(t) := \frac{\alpha_0^2 + 1}{\alpha_0^2 - L_2} d(t) + \frac{1}{\alpha_0^2 - L_2}|\dot{\zeta}(t)|,$$

for all $t \in [T_0, T]$.

Proof. The proof will be divided into two steps.

Step 1: We first prove the theorem under the additional assumption:

$$\frac{\alpha_0^2 + 1}{\alpha_0^2 - L_2} \int_{T_0}^T \left(|\dot{\zeta}(s)| + L_1\|B\|\mu(s) + (1 + L_2)(c(s)\mu(s) + d(s) + \|A\|\mu(s))\right) ds < \rho, \tag{3.12}$$

where $\rho > 0$ is defined by Remark 3.2.

Let $m\colon [T_0, T] \times U \times V \to \mathbb{R}$ be defined by

$$m(t,u,v) := \frac{1}{\alpha_0^2 - L_2}\left(|\dot{\zeta}(t)| + L_1\|B\|\|v\| + (1+L_2)(c(t)\|(u,v)\| + d(t) + \|A\|\|u\|)\right), \tag{3.13}$$

for all $(t,u,v) \in [T_0,T] \times U \times V$.

Define the set-valued map $G\colon [T_0,T] \times U \times V \rightrightarrows U \times V$ as

$$G(t,u,v) = (-Bv, -m(t,u,v)\partial d_{C(t,u,v)}(v) - F(t,u,v) - Au),$$

for all $(t,u,v) \in [T_0,T] \times U \times V$. We will show, by using Theorem 3.4, that the following differential inclusion has at least one solution:

$$\begin{cases} (\dot{u}(t), \dot{v}(t)) \in G(t, u(t), v(t)) \quad \text{a.e. } t \in [T_0,T], \\ (u(T_0), v(T_0)) = (u_0, v_0). \end{cases} \tag{3.14}$$

Claim 1:

(i) For each $(u,v) \in U \times V$, $G(\cdot,u,v)$ is measurable.

(ii) For a.e. $t \in [T_0,T]$, $G(t,\cdot,\cdot)$ is upper semicontinuous from $U \times V$ into $U_w \times V_w$.

(iii) For all $(u,v) \in U \times V$ and a.e. $t \in [T_0,T]$

$$d(0, G(t,u,v)) \leq \widetilde{c}(t)\|(u,v)\| + \widetilde{d}(t),$$

where $\widetilde{c}$ and $\widetilde{d}$ are defined as in the statement of the theorem.

Proof of Claim 1: (i) follows from Lemma 3.2 and $(\mathcal{H}_1^F)$. Also, (ii) follows from Lemma 3.5 and $(\mathcal{H}_2^F)$. To prove (iii) let $(u,v) \in U \times V$ and $t \in [T_0,T]$. Then, by virtue of $(\mathcal{H}_3^F)$,

$$\begin{aligned} d(0, G(t,u,v)) &= \inf\{\|w\|\colon w \in G(t,u,v)\} \\ &\leq \|B\|\|v\| + m(t,u,v) + \inf\{\|w\|\colon w \in F(t,u,v)\} + \|A\|\|u\| \\ &\leq \|B\|\|v\| + m(t,u,v) + c(t)\|(u,v)\| + d(t) + \|A\|\|u\| \\ &\leq \widetilde{c}(t)\|(u,v)\| + \widetilde{d}(t), \end{aligned}$$

which finishes the proof of Claim 1. □

For each $n \in \mathbb{N}$, let us consider the following differential inclusion:

$$\begin{cases} (\dot{u}(t), \dot{v}(t)) \in G(t, P_n(u(t)), Q_n(v(t))) \quad \text{a.e. } t \in [T_0,T], \\ (u(T_0), v(T_0)) = (P_n(u_0), Q_n(v_0)), \end{cases} \tag{3.15}$$

where $(P_n)_n$ and $(Q_n)_n$ are, respectively, orthonormal basis of U and V. By virtue of Proposition 3.3, the differential inclusion (3.15) has at least one solution $(u_n, v_n) \in W^{1,1}([T_0,T];U) \times W^{1,1}([T_0,T];V)$. Moreover,

$$\|(u_n(t), v_n(t))\| \le \mu(t) \quad \text{for all } t \in [T_0,T], \tag{3.16}$$

and

$$\|(\dot{u}_n(t), \dot{v}_n(t))\| \le \widetilde{c}(t)\mu(t) + \widetilde{d}(t) \quad \text{a.e. } t \in [T_0,T], \tag{3.17}$$

where μ, $\widetilde{c}$ and $\widetilde{d}$ are defined as in the statement of the theorem.
To simplify the notation, we write

$$\begin{aligned} m_n(t) &:= m(t, P_n(u_n(t)), Q_n(v_n(t))) \\ \Gamma_n(t) &:= \partial d_{C(t,P_n(u_n(t)),Q_n(v_n(t)))}(Q_n(v_n(t))), \end{aligned}$$

and we note that (see (3.16) and (3.17))

$$\begin{aligned} m_n(t) \le \delta(t) :=& \frac{1}{\alpha_0^2 - L_2}\Big(|\dot{\zeta}(t)| + L_1\|B\|\mu(t)\Big) \\ &+ \frac{1}{\alpha_0^2 - L_2}\left((1+L_2)(c(t)\mu(t) + d(t) + \|A\|\mu(t))\right), \end{aligned} \tag{3.18}$$

for a.e. $t \in [T_0,T]$. Moreover, there exist $f_n(t) \in F(t, P_n(u_n(t)), Q_n(v_n(t)))$ and $d_n(t) \in \Gamma_n(t)$ such that

$$\begin{cases} -\dot{u}_n(t) = B(Q_n(v_n(t))) & \text{a.e. } t \in [T_0,T], \\ -\dot{v}_n(t) = m_n(t)d_n(t) + f_n(t) + A(Q_n(v_n(t))) & \text{a.e. } t \in [T_0,T]. \end{cases}$$

Define $\varphi_n(t) = d_{C(t,P_n(u_n(t)),Q_n(v_n(t)))}(Q_n(v_n(t)))$ for $t \in [T_0,T]$.

Claim 2: For all $t \in [T_0,T]$

$$\varphi_n^3(t) \le \varphi_n^3(T_0) + 3\int_{T_0}^t \delta(s) \sup_{x\in D(s)} \|x - Q_n(x)\|^2 ds,$$

where by $(\mathcal{H}_4)$ the set $D(t) := \overline{\text{co}}\,(C(t, \mu(t)\mathbb{B}, \mu(t)\mathbb{B}) \cap (\rho + \mu(t))\mathbb{B})$ is relatively compact for every $t \in [T_0,T]$.

Proof of Claim 2: The idea of the proof is to use $(\mathcal{H}_2)$ (see Remark 3.2). To do that, we proceed to show first that $\varphi_n(t) < \rho$ for all $t \in [T_0,T]$. Indeed, let $t \in [T_0,T]$ where $\dot{u}_n(t)$ and $\dot{v}_n(t)$ exist. Then, due to Lemma 3.2, (3.18) and (3.13),

$$\begin{aligned} \dot{\varphi}_n(t) &\le |\dot{\zeta}(t)| + L_1\|P_n(\dot{u}_n(t))\| + L_2\|Q_n(\dot{v}_n(t))\| + \max_{y^*\in\Gamma_n(t)} \langle y^*, Q_n(\dot{v}_n(t))\rangle \\ &\le |\dot{\zeta}(t)| + L_1\|B\|\|Q_n(v_n(t))\| + (1+L_2)\|Q_n(\dot{v}_n(t))\| \\ &\le |\dot{\zeta}(t)| + L_1\|B\|\|Q_n(v_n(t))\| + (1+L_2)\left[m_n(t) + c(t)\|(P_n(u_n(t)), Q_n(v_n(t)))\|\right. \\ &\left. + d(t) + \|A\|\|P_n(u_n(t))\|\right] \\ &= (\alpha_0^2+1)m_n(t) \\ &\le (\alpha_0^2+1)\delta(t). \end{aligned}$$

Therefore, according to (3.12), $\varphi_n(t) < \rho$ for all $t \in [T_0, T]$.

Now, let $t \in \Omega_n := \{t \in [T_0, T] \colon Q_n(v_n(t)) \notin C(t, P_n(u_n(t)), Q_n(v_n(t)))\}$ where $\dot{u}_n(t)$ and $\dot{v}_n(t)$ exist. Then, due to Lemma 3.2,

$$\begin{aligned}\dot{\varphi}_n(t) &\leq |\dot{\zeta}(t)| + L_1\|P_n(\dot{u}_n(t))\| + L_2\|Q_n(\dot{v}_n(t))\| + \min_{y^*\in\Gamma_n(t)} \langle y^*, Q_n(\dot{v}_n(t))\rangle \\ &= |\dot{\zeta}(t)| + L_1\|B\|\|Q_n(v_n(t))\| + L_2(m_n(t) + c(t)\|(P_n(u_n(t)), Q_n(v_n(t)))\| \\ &+ d(t) + \|A\|\|P_n(u_n(t))\|) + \min_{y^*\in\Gamma_n(t)} \langle y^*, Q_n(\dot{v}_n(t))\rangle\end{aligned}$$

Also, since $d_n(t) \in \Gamma_n(t)$,

$$\begin{aligned}\min_{y^*\in\Gamma_n(t)} \langle y^*, Q_n(\dot{v}_n(t))\rangle &\leq \langle d_n(t), Q_n(\dot{v}_n(t))\rangle \\ &= \langle d_n(t), Q_n\left(-m_n(t)d_n(t) - f_n(t) - A(P_n(u_n(t)))\right)\rangle \\ &\leq \|f_n(t)\| + \|A\|\|P_n(u_n(t))\| - m_n(t)\left\langle d_n(t), Q_n(d_n(t))\right\rangle \\ &\leq c(t)\|(P_n(u_n(t)), Q_n(v_n(t)))\| + d(t) + \|A\|\|P_n(u_n(t))\| \\ &- m_n(t)\left\langle d_n(t), Q_n(d_n(t))\right\rangle.\end{aligned}$$

Hence, by using the last two estimations and (3.13), we obtain

$$\dot{\varphi}_n(t) \leq m_n(t)\left(\alpha_0^2 - \langle d_n(t), Q_n(d_n(t))\rangle\right).$$

Moreover, due to $(\mathcal{H}_2)$,

$$\begin{aligned}\langle d_n(t), -Q_n(d_n(t))\rangle &= \langle d_n(t), d_n(t) - Q_n(d_n(t))\rangle + \langle d_n(t), -d_n(t)\rangle \\ &\leq \langle d_n(t), d_n(t) - Q_n(d_n(t))\rangle - \alpha_0^2 \\ &= \|d_n(t) - Q_n(d_n(t))\|^2 - \alpha_0^2.\end{aligned}$$

Then,

$$\begin{aligned}\dot{\varphi}_n(t) &\leq m_n(t)\left(\alpha_0^2 - \langle d_n(t), -Q_n(d_n(t))\rangle\right) \\ &\leq m_n(t)\|d_n(t) - Q_n(d_n(t))\|^2 \\ &\leq \delta(t)\|d_n(t) - Q_n(d_n(t))\|^2.\end{aligned}$$

Furthermore, for $t \in \Omega_n$, since $d_n(t) \in \Gamma_n(t)$, Lemma 1.15 ensures the existence of $g_n(t) \in \overline{\text{co}}\,\text{Proj}_{C(t,P_n(u_n(t)),Q_n(v_n(t)))}(Q_n(v_n(t)))$ such that

$$d_n(t) = \frac{1}{\varphi_n(t)}\left(Q_n(v_n(t)) - g_n(t)\right). \tag{3.19}$$

Then,

$$\begin{aligned}\|g_n(t)\| &\leq \varphi_n(t) + \|Q_n(v_n(t))\| \\ &\leq \rho + \mu(t),\end{aligned}$$

which entails that $g_n(t) \in D(t)$ for all $t \in \Omega_n$. Thus, for every $t \in \Omega_n$ (see (3.19))

$$\begin{aligned}\varphi_n(t)^2\|d_n(t) - Q_n(d_n(t))\|^2 &= \|g_n(t) - Q_n(g_n(t))\|^2 \\ &\leq \sup_{x\in D(t)} \|x - Q_n(x)\|^2.\end{aligned}$$

Let $t \in [T_0, T]$. Then,

$$\begin{aligned}\varphi_n^3(t) &= \varphi_n^3(T_0) + 3\int_{T_0}^{t} \varphi_n^2(s)\dot{\varphi}_n(s)ds \\ &\leq \varphi_n^3(T_0) + 3\int_{T_0}^{t} \delta(s) \sup_{x\in D(s)} \|x - Q_n(x)\|^2 ds,\end{aligned}$$

as claimed. □

Claim 3: $\lim_{n\to+\infty} \varphi_n(t) = 0$ for all $t \in [T_0, T]$.

Proof of Claim 3: Fix $t \in [T_0, T]$. Then, since $D(t)$ is relatively compact and (v) from Lemma 1.7,

$$\lim_{n\to+\infty} \sup_{x\in D(t)} \|x - Q_n(x)\| = 0.$$

Hence, by Fatou's lemma and Claim 2,

$$\begin{aligned}\limsup_{n\to+\infty} \varphi_n^3(t) &\leq 3 \limsup_{n\to+\infty} \int_{T_0}^{t} \delta(s) \sup_{x\in D(s)} \|x - Q_n(x)\|^2 ds \\ &\leq 3\int_{T_0}^{t} \delta(s) \limsup_{n\to+\infty} \sup_{x\in D(s)} \|x - Q_n(x)\|^2 ds \\ &= 0,\end{aligned}$$

as required. □

Claim 4: $(P_n(u_n(t)))_n$ and $(Q_n(v_n(t)))_n$ are relatively compact for all $t \in [T_0, T]$.

Proof of Claim 4: Let $\gamma = \alpha$ or $\gamma = \beta$ be either the Kuratowski or the Hausdorff measure of noncompactness. On the one hand, let

$$s_n(t) \in \operatorname{Proj}_{C(t,P_n(u_n(t)),Q_n(v_n(t)))} \left(Q_n(v_n(t))\right).$$

Then, $s_n(t) \in (\rho + \mu(t))\mathbb{B}$ and, due to Claim 3,

$$\begin{aligned}\gamma\left(\{Q_n(v_n(t)) \colon n \in \mathbb{N}\}\right) &= \gamma\left(\{s_n(t) \colon n \in \mathbb{N}\}\right) \\ &\leq \gamma\left(C\left(t, \mu(t)\mathbb{B}, \mu(t)\mathbb{B}\right) \cap (\rho + \mu(t))\mathbb{B}\right) \\ &= 0,\end{aligned}$$

which shows that $(Q_n(v_n(t)))_n$ is relatively compact. On the other hand, by using Lemma 1.9 and the relative compactness of $(Q_n(v_n(t)))_n$ for all $t \in [T_0, T]$, we obtain

$$\begin{aligned}\gamma\left(\{u_n(t) \colon n \in \mathbb{N}\}\right) &= \gamma(\{P_n(u_0) + \int_{T_0}^{t} \dot{u}_n(s)ds \colon n \in \mathbb{N}\}) \\ &\leq \gamma(\{P_n(u_0) \colon n \in \mathbb{N}\}) + \gamma\left(\left\{\int_{T_0}^{t} \dot{u}_n(s)ds \colon n \in \mathbb{N}\right\}\right) \\ &= \gamma\left(\left\{-\int_{T_0}^{t} B(Q_n(v_n(s)))ds \colon n \in \mathbb{N}\right\}\right) \\ &= 0,\end{aligned}$$

which shows that $(u_n(t))_n$ is relatively compact for all $t \in [T_0, T]$. Therefore, the sequence $(P_n(u_n(t)))_n$ is relatively compact for all $t \in [T_0, T]$, as claimed. □

Hence, we have verified all the hypotheses of Theorem 3.4. Therefore, there exists at least one solution $(u, v) \in W^{1,1}([T_0, T]; U) \times W^{1,1}([T_0, T]; V)$ of (3.14). Now it remains to show that (u, v) is a solution of (3.11).

Claim 5: For all $t \in [T_0, T]$, $v(t) \in C(t, u(t), v(t))$.

Proof of Claim 5: Fix $t \in [T_0, T]$. Then, as in the proof of Theorem 3.4, $P_k(u_k(t)) \to u(t)$ and $Q_k(v_k(t)) \to v(t)$, where $(u_k, v_k)_k$ is a subsequence of $(u_n, v_n)_n$. Thus, due to Claim 3,

$$\begin{aligned} &d_{C(t,u(t),v(t))}(v(t)) \\ &= \limsup_{k\to+\infty} \left(d_{C(t,u(t),v(t))}(v(t)) - \varphi_k(t) + \varphi_k(t)\right) \\ &\leq \limsup_{k\to+\infty} ((1 + L_2)\|v(t) - Q_k(v_k(t))\| + L_1\|u(t) - P_k(u_k(t))\| + \varphi_k(t)) \\ &= 0, \end{aligned}$$

as claimed. □

Finally, by virtue of (1.1) and Claim 5, (u, v) is also a solution of (3.11).

<u>*Step 2.*</u> In the general case, without any restriction on the length of T, let us consider $\{T_0, T_1, \ldots, T_N\}$ be a partition of $[T_0, T]$ such that for every $k \in \{1, \ldots, N\}$

$$\frac{\alpha_0^2 + 1}{\alpha_0^2 - L_2} \int_{T_{k-1}}^{T_k} \left(|\dot{\zeta}(s)| + L_1\|B\|\mu(t) + (1 + L_2)(c(t)\mu(t) + d(t) + \|A\|\mu(t))\right) ds < \rho.$$

For $k = 1$, due to Step 1, let (u^1, v^1) be a solution of (3.11) over $[T_0, T_1]$. Then, $v^1(t) \in C(t, u^1(t), v^1(t))$ for all $t \in [T_0, T_1]$ and $u^1(T_0) = u_0$ and $v^1(T_0) = v_0 \in C(T_0, u_0, v_0)$.
Inductively, for $k = 2, \ldots, N$, since $v^{k-1}(T_{k-1}) \in C(T_{k-1}, u^{k-1}(T_{k-1}), v^{k-1}(T_{k-1}))$, let (u^k, v^k) be a solution of (3.11) over $[T_{k-1}, T_k]$. Then, $v^k(t) \in C(t, u^k(t), v^k(t))$ for all $t \in [T_{k-1}, T_k]$ and $v^k(T_{k-1}) = v^{k-1}(T_{k-1})$.
Finally, we define $u(t) = u^k(t)$ and $v(t) = v^k(t)$ over $[T_{k-1}, T_k]$, for $k = 1, \ldots, N$. Then (u, v) is a solution of (3.11), which finishes the proof of the theorem. □

According to the proof of Theorem 3.6, we observe that $(\mathcal{H}_3)$ was used only to obtain the upper semicontinuity of $\partial d_{C(t,\cdot,:)}(:)$ from $U \times V$ into V_w for all $t \in [T_0, T]$. Since, when $C(t, u, v) \equiv C(t)$ for all $(u, v) \in U \times V$ and $t \in [T_0, T]$ the subdifferential $\partial d_{C(t)}(\cdot)$ is always upper semicontinuous from V into V_w for all $t \in [T_0, T]$, we have the following existence result for (3.11) with positively α_0-far sets.

Theorem 3.7 *Assume, in addition to* $(\mathcal{H}_8)$, $(\mathcal{H}_9)$, $(\mathcal{H}_{10})$, $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ *and* $(\mathcal{H}_3^F)$, *that* $A\colon U \to V$ *and* $B\colon V \to U$ *are two bounded linear operators. Then, there exists* *at least one solution* $(u,v) \in W^{1,1}([T_0,T];U) \times W^{1,1}([T_0,T];V)$ *of*

$$\begin{cases} -\dot{u}(t) = Bv(t) & a.e.\ t \in [T_0,T], \\ -\dot{v}(t) \in N\left(C(t);v(t)\right) + F(t,u(t),v(t)) + Au(t) & a.e.\ t \in [T_0,T], \\ u(T_0) = u_0, v(T_0) = v_0 \in C(T_0), \end{cases}$$

satisfying

$$\|(u(t),v(t))\| \leq \mu(t) := \left(\|(u_0,v_0)\| + \int_{T_0}^t \widetilde{d}(s)ds\right) \exp\left(\int_{T_0}^t \widetilde{c}(s)ds\right),$$

for all $t \in [T_0,T]$, *where*

$$\begin{aligned} \widetilde{c}(t) &:= \frac{\alpha_0^2+1}{\alpha_0^2}\left(c(t) + \|A\|\right) + \|B\|, \\ \widetilde{d}(t) &:= \frac{\alpha_0^2+1}{\alpha_0^2} d(t) + \frac{1}{\alpha_0^2}|\dot{\zeta}(t)|, \end{aligned}$$

for all $t \in [T_0,T]$.

3.5 Perturbed state-dependent sweeping process

The perturbed state-dependent sweeping process is the following differential inclusion:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t;x(t));x(t)\right) + F(t,x(t)) & \text{a.e. } t \in [T_0,T], \\ x(T_0) = x_0 \in C(T_0,x_0), \end{cases} \tag{3.20}$$

where for any subset S in H the set $N(S;\cdot)$ denotes the Clarke normal cone to S and $F\colon [T_0,T] \times H \rightrightarrows H$ is a set-valued mapping, called perturbation term, with nonempty closed and convex values. This differential inclusion includes the state-dependent sweeping process:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t,x(t));x(t)\right) & \text{a.e. } t \in [T_0,T], \\ x(T_0) = x_0 \in C(T_0,x_0), \end{cases} \tag{3.21}$$

and the perturbed Moreau's sweeping process:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t);x(t)\right) + F(t,x(t)) & \text{a.e. } t \in [T_0,T], \\ x(T_0) = x_0 \in C(T_0). \end{cases} \tag{3.22}$$

The study of this kind of differential inclusions was initiated by Moreau [113–116], for (3.22), to deal with problems arising in mechanics (see [98] for a general introduction to the subject). Since then, several authors have been interested in the existence and uniqueness of solutions in the convex and nonconvex case (see [2, 19, 34, 34, 44, 57, 65, 66, 75, 87, 90, 135, 136]).

Concerning (3.21), as far as we know, it has been introduced and studied for the first time, for convex sets $C(t,x)$ in $\mathbb{R}^3$, by Chraibi Kaadoud [51] to model certain mechanical problems and later generalized to (3.20) in the convex and nonconvex setting.

In the convex setting, Kunze and Monteiro-Marques [97] proved the existence of solutions to (3.21) when the set-valued satisfies the following Lipschitz condition: There exist $L_1 \geq 0$ and $L_2 \in [0,1[$ such that

$$|d(x, C(t,u)) - d(x, C(s,v))| \leq L_1|t-s| + L_2\|u-v\|, \tag{3.23}$$

for $t,s \in [T_0,T]$ and $x,u,v \in H$. Also, they showed that when $L_2 \geq 1$ no solution of (3.21) can be expected. The authors used Darbo's fixed point theorem to show the convergence of the following semi-implicit discretization scheme:

$$x_{i+1}^n = \operatorname{proj}\left(x_i^n; C(t_{i+1}^n, x_{i+1}^n)\right). \tag{3.24}$$

The discretization scheme (3.24) comes from an implicit discretization of (3.21) and can be seen as a generalization of the well known Moreau's Catching-up algorithm [113, 115, 116]. Next, Haddad and Haddad [74] showed, using an explicit discretization scheme, the existence of solutions to (3.20) in the particular case $C(t,x) := C(x)$ and $F(t,x) = Ax + f(t)$, where A is a linear bounded operator and f is a continuous and bounded function. This result was used to show the existence of solutions to a superconductivity model. Later, Haddad [73] showed the existence of solutions of (3.20) with upper semicontinuous perturbation by using the explicit discretization scheme:

$$x_{i+1}^n = \operatorname{proj}(x_i^n - \frac{T-T_0}{n} f_i^n; C(t_{i+1}^n, x_i^n)) \text{ and } f_i^n \in F(t_i^n, x_i^n). \tag{3.25}$$

Finally, Bounkhel and Castaing [32], by using (3.25), showed the existence of solutions to (3.21) in uniformly smooth and uniformly convex Banach spaces.

In the nonconvex case, Chemetov and Monteiro-Marques [49] proved the existence of solutions to (3.20) for uniformly prox-regular sets $C(t,x)$ with absolutely continuous variation in space and Lipschitz variation in time with a single-valued perturbation. They construct the operator $w = P(v)$ where w is the unique solution of (3.22) with $C(t) := C(t, v(t))$ and they show the existence of a fixed point of P via Schauder's fixed point theorem. Then, the same authors [50] proved the existence of solutions to (3.21) by using a fixed point argument in ordered spaces. Next, Castaing, Ibrahim and Yarou [45] used an extended version of Schauder's theorem and the discretization scheme (3.24) to show the existence of solutions to (3.21) in the uniformly prox-regular case. Later, Azzam-Laouir, Izza and Thibault [14] and Haddad, Kecis and Thibault [76] showed the existence of solutions to (3.20) in the finite-dimensional and uniformly prox-regular setting with a perturbation term defined as the sum of an u.s.c and a mixed semicontinuous set-valued mapping with closed and convex values satisfying a linear growth condition. They reduce the constrained differential inclusion (3.21) to the following unconstrained one

$$\begin{cases} -\dot{x}(t) \in \dfrac{|\dot{\zeta}(t)|}{1-L_2}\partial d\left(x(t);C(t,x(t))\right) & \text{a.e. on } [T_0,T], \\ x(T_0) = x_0 \in C(T_0,x_0), \end{cases}$$

where $L_2 \in [0,1[$ is the constant in (3.23) and ζ is the variation in time of C. Next, Noel [118] and Noel and Thibault [119] showed, respectively, the existence of solutions of (3.20) with equi-uniformly subsmooth and uniformly prox-regular sets for scalarly upper semicontinuous set-valued perturbations with closed and convex values. By using an extended Schauder theorem, they showed the convergence of the following semi-implicit discretization scheme:

$$x_{k+1}^n \in \operatorname{Proj}(x_k^n + \frac{T-T_0}{p^n} g(t_k^n, x_k^n); C(t_{k+1}^n, x_{k+1}^n)) \text{ and } g(t,x) = \operatorname{Proj}_{F(t,x)}(0),$$

Finally, Jourani and Vilches [90] showed the existence of solutions to (3.21) and (3.22) (with $F \equiv 0$), respectively, for subsmooth and positively α-far sets by using the Moreau-Yosida regularization techniques.

The perturbed state-dependent sweeping process (3.20) includes, as a special case, the Bensoussan-Lions-Mosco problem (see [132]): Find $v \in [T_0,T] \to H$ with $v(t) \in \Gamma(v(t))$ such that

$$a(v(t), u - v(t)) + \langle \dot{v}(t), u - v(t)\rangle \geq \langle l(t), u - v(t)\rangle, \tag{3.26}$$

for a.e. $t \in [T_0,T]$ and for all $u \in \Gamma(v(t))$, $v(T_0) = v_0 \in \Gamma(v_0)$. In the above parabolic quasi-variational inequality $a(\cdot,\cdot)$ is a real bilinear, symmetric, bounded and elliptic form on $H \times H$, $l \in L^1([T_0,T];H)$ and $\Gamma(\cdot) \subseteq H$ is a convex set of constraints. The interest in the study of (3.26) arises in connection with quasi-static problems, sandpile growth and superconductivity models, among others (see [132,133] for more details).

Now we give an existence result for the perturbed state-dependent sweeping process.

$$\begin{cases} -\dot{v}(t) \in N\left(C(t,v(t)); v(t)\right) + F(t,v(t)) & \text{a.e. } t \in [T_0,T], \\ v(T_0) = v_0 \in C(T_0,v_0), \end{cases} \tag{3.27}$$

The following result, consequence of Theorem 3.6, gives a very general existence result for the perturbed state-dependent sweeping process. The following theorem is related to [90, Theorem 6.1] and improves the results given in [118, 119].

Theorem 3.8 *Assume that $(\mathcal{H}_5)$, $(\mathcal{H}_6)$ and $(\mathcal{H}_7)$ hold. Let $F\colon [T_0,T] \times H \rightrightarrows H$ be a set-valued mapping with nonempty closed and convex values satisfying $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$ and $(\mathcal{H}_6^F)$. Then, for all $\alpha_0 \in]\sqrt{L_2}, 1]$ there exists at least one solution $v \in W^{1,1}([T_0,T];H)$ of (3.27) satisfying:*

$$\|v(t)\| \leq \left(\|v_0\| + \int_{T_0}^{t} \widetilde{d}(s)ds\right) \exp\left(\int_{T_0}^{t} \widetilde{c}(s)ds\right) \quad \textit{for all } t \in [T_0,T],$$

where for all $t \in [T_0, T]$

$$\widetilde{c}(t) := \frac{\alpha_0^2 + 1}{\alpha_0^2 - L_2} c(t),$$
$$\widetilde{d}(t) := \frac{\alpha_0^2 + 1}{\alpha_0^2 - L_2} d(t) + \frac{1}{\alpha_0^2 - L_2} |\dot{\zeta}(t)|.$$

Remark 3.3 It is important to mention that:

(i) The hypothesis $L_2 \in [0, 1[$ in Theorem 3.8 cannot be improved. In fact, there are counterexamples to the existence of solutions to (3.27) when $L_2 \geq 1$ (see [97]).

(ii) It is an open problem to know if the compactness assumption $(\mathcal{H}_7)$ in Theorem 3.8 can be removed.

(iii) It is well known that under the conditions of Theorem 3.8, uniqueness of solution to (3.27) (even for convex sets) does not necessarily hold (see [16, 97] for more details). However, Krejčí and Schnabel [40] have proved existence of solutions to (3.21) when the dependence of the Minkowski function and its gradient are Lipschitz functions.

(iv) Existence results for the state-dependent sweeping process with uniformly subsmooth sets have been proved in [118] under very strong conditions. In fact, in [118] it is assumed that for any bounded set A, the set $C([T_0, T], A)$ is relatively ball compact, C has a Lipschitz variation in both variables and the perturbation term F is upper semicontinuous from $[T_0, T] \times H$ into H_w, with bounded perturbation term F.

As a consequence of Theorem 3.8, we obtain get of solutions for the Bensoussan-Lions-Mosco problem (3.26). The following result improves [74, Proposition 17.5] where the authors assume that $\Gamma(\cdot) \subseteq K$ for some convex compact set K and $l \in W^{1,2}([T_0, T]; H) \cap L^\infty([T_0, T]; H)$.

Proposition 3.9 *Let* $a(\cdot, \cdot)$ *be a bilinear, symmetric, bounded and elliptic form and* $l \in L^1([T_0, T]; H)$*. Assume that* $\Gamma \colon H \rightrightarrows H$ *is Lipschitz continuous with constant* $0 < L < 1$*, takes closed convex values and for any bounded set* A*, the set* $\Gamma(A)$ *is relatively ball compact. Then, for every* $v_0 \in \Gamma(v_0)$*, there exists at least one solution of* (3.26).

3.6 The perturbed sweeping process

The sweeping process is a first-order differential inclusion involving the normal cone to a moving set depending on time. Roughly speaking, a point is swept by a moving closed set. This differential inclusion was introduced and deeply studied by Moreau in a series of papers (see [113–116]) to model an elasto-plastic mechanical system. Since then, many other applications of the sweeping process have been given, namely in electrical circuits [1], crowd motion [102], hysteresis in elasto-plastic models [94], etc.

The seminal work of Moreau was the starting point of many other developments, as the state-dependent sweeping process, the second-order sweeping process [29], the generalized sweeping process [87], etc.

The perturbed sweeping process

$$\begin{cases} -\dot{v}(t) \in N\left(C(t); v(t)\right) + F(t, v(t)) & \text{a.e. } t \in [T_0, T], \\ v(T_0) = v_0 \in C(T_0), \end{cases} \tag{3.28}$$

is well known in the literature and has been studied by several authors. Its importance comes from the study of problems in mechanics of elastoplastic materials [117], non-smooth mechanics [37], dynamics of systems with inelastic shocks [111], modeling and simulation of switched electrical circuits [1], crowd motion modeling [102], etc. When $F \equiv 0$ and the sets $C(t)$ are convex, the system above is referred to as a *sweeping process* because it can be visualized as a point $x(t)$ moving inside $C(t)$ and being pushed by the boundary of this convex set when contact is established. It was introduced in the seventies by Moreau in a series of chapters [113–115] to study contact problems in mechanical systems. Since then, many improvements have been made by weakening the convexity assumption and by considering the perturbed version of (3.28). In [42], Castaing dealt with sweeping process associated with sets $C(t) = S + v(t)$, where S is a fixed closed nonconvex set, and v is a mapping with finite variation. Then, Valadier [141] extended the study of the absolutely continuous case to a more general situation of complements of convex sets in finite dimension. Afterward, Benabdellah [19] and Colombo and Goncharov [57] proved independently and almost at the same time the existence of solutions for general nonconvex sets $C(t)$ in $\mathbb{R}^n$ moving in a Lipschitz continuous way. Extensions of existence and uniqueness of solution of (3.28) when $C(t)$ is a uniformly prox-regular set of a Hilbert space X, moving in a Lipschitz continuous way and in an absolutely continuous way, have been obtained by Colombo and Goncharov [57] and Bounkhel and Thibault [34] (see also the chapter [136] by Thibault for a regularization of the process in such a context). In the case where the uniformly prox-regular sets $C(t)$ move with bounded variation we refer the reader to the chapter [65] by Edmond and Thibault.

The study of the differential inclusions which are perturbations of sweeping processes began, as far as we know, with [82]. Then, Henry, to deal with planning procedures in mathematical economy, introduced the differential inclusion

$$\begin{cases} -\dot{x}(t) \in P_{T_C(x(t))}(F(x(t))) & \text{a .e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C, \end{cases}$$

where $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an upper semicontinuous set-valued map with nonempty compact convex values, C is a fixed closed convex set, $T_C(\cdot)$ is the tangent cone to

C and $P_{T_C(x(t))}$ denotes the projection mapping into the closed convex set $T_C(x(t))$. This differential inclusion has been also considered by Cornet [60] with a Clarke tangentially regular set C of $\mathbb{R}^n$, reducing the problem as in [82] to the existence of a solution of the perturbed differential inclusion

$$\begin{cases} -\dot{x}(t) \in N(C; x(t)) + F(x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C. \end{cases}$$

Since then the case of moving sets $C(t)$ has been developed. We refer to Castaing, Duc Ha and Valadier [44] and Castaing and Monteiro-Marques [47] for the study of perturbed sweeping processes in the form of (3.28) in the cases where all the sets $C(t)$ are either convex or complements of open convex sets. For the case of general nonconvex closed sets of $\mathbb{R}^n$ we refer to Thibault [135] (see also the chapter of Haddad, Jourani and Thibault [75] for reduction of sweeping process to unconstrained differential inclusions). Several other chapters deal with perturbed sweeping processes, in the Hilbert setting, under uniform prox-regularity assumptions, such as the works of Bounkhel and Thibault [34], Edmond and Thibault [65], Mazade and Thibault [104, 105] and Sene and Thibault [130] (see [29] for a general account on sweeping process and the prox-regularity).

Now we present an existence theorem for set-valued map taking positively α_0-far values. The following result, consequence of Theorem 3.6, was established in [87] by using a completely different approach.

Theorem 3.10 *Assume that* $(\mathcal{H}_8)$, $(\mathcal{H}_9)$ *and* $(\mathcal{H}_{10})$ *hold. Let* $F\colon [T_0, T] \times H \rightrightarrows H$ *be a set-valued mapping with nonempty closed and convex values satisfying* $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$ *and* $(\mathcal{H}_6^F)$. *Then, there exists at least one solution* $v \in W^{1,1}([T_0, T]; H)$ *of* (3.28) *satisfying*

$$\|v(t)\| \leq \left(\|v_0\| + \int_{T_0}^{t} \widetilde{d}(s)ds \right) \exp\left(\int_{T_0}^{t} \widetilde{c}(s)ds \right) \quad \textit{for all } t \in [T_0, T],$$

where for all $t \in [T_0, T]$

$$\begin{aligned} \widetilde{c}(t) &:= \frac{\alpha_0^2 + 1}{\alpha_0^2} c(t), \\ \widetilde{d}(t) &:= \frac{\alpha_0^2 + 1}{\alpha_0^2} d(t) + \frac{1}{\alpha_0^2} |\dot{\zeta}(t)|. \end{aligned}$$

Related to uniqueness for (3.28) with positively α_0-far sets, we have the following counterexample.

Example 3.1 Let us consider the set-valued map $C\colon [0,1] \rightrightarrows \mathbb{R}^2$ defined by $C(t) = S - (t, 0)$ for $t \in [0,1]$, where $S = \{(x, y) \in \mathbb{R}^2 \colon |y| \geq x\} \cap \mathbb{B}$ (see Figure 3.1). Then, $C(t)$ is $\sqrt{2}/2$-far. Also, $v_1(t) = (-t/2, t/2)$ and $v_2(t) = (-t/2, -t/2)$ defined for $t \in [0,1]$ are solutions of (3.28) with $F \equiv 0$. Thus, in general, there is no uniqueness of solutions to (3.28) with positively α_0-far sets. This is not the case when the sets C are convex or r-uniformly prox-regular (see for instance [29]).

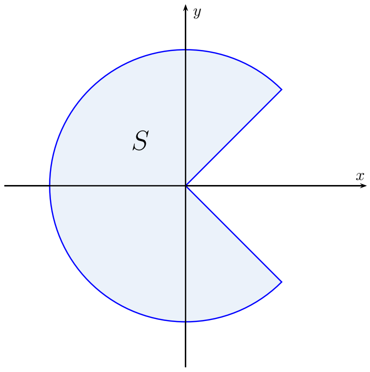


Figure 3.1: Set S in Example 3.1.

3.7 Perturbed second-order sweeping process

The perturbed second-order sweeping process is the following differential inclusion:

$$\begin{cases} -\ddot{u}(t) \in N\left(C(t,u(t),\dot{u}(t));\dot{u}(t)\right) + F(t,u(t),\dot{u}(t)) & \text{a.e. } t \in [T_0,T], \\ u(T_0) = u_0, \dot{u}(T_0) = v_0 \in C(T_0,u_0,v_0). \end{cases} \tag{3.29}$$

The study of this kind of differential inclusions was initiated by Castaing [43], where the moving set depends on the state with convex and compacts values. Since then, several works deal with second-order sweeping process with convex/prox-regular sets in Hilbert/Banach spaces (see [5, 8, 12, 13, 28, 30, 31, 33, 44]).

The second-order sweeping process (3.29) includes the dynamic analogue of the Signorini problem: Find $u\colon [T_0,T] \to H$, $u(T_0) = u_0$, $\dot{u}(T_0) = v_0 \in C(T_0)$ such that $\dot{u}(t) \in C(t)$ for a.e. $t \in [T_0,T]$ and

$$\langle l(t) - \ddot{u}(t), y - \dot{u}(t)\rangle \leq a\left(u(t), y - \dot{u}(t)\right) + J(t,y) - J(t,\dot{u}(t)), \tag{3.30}$$

for all $y \in C(t)$ and a.e. $t \in [T_0,T]$. Here $a(\cdot,\cdot) := \langle A(\cdot),\cdot\rangle$ is a real bilinear, symmetric, bounded and elliptic form on $H \times H$, $l \in L^1([T_0,T];H)$ and $J\colon [T_0,T] \times H \to \mathbb{R}$ is a convex and locally Lipschitz continuous function. We observe that (3.30) can be written in the following form:

$$-\ddot{u}(t) \in \partial J\left(t,\dot{u}(t)\right) + N\left(C(t);\dot{u}(t)\right) + Au(t) - l(t) \quad \text{a.e. } t \in [T_0,T].$$

This differential inclusion can be studied in a more general context, namely, the convexity of J and $C(\cdot)$ can be removed.

In this section we give a very general existence result to (3.29) (see Theorem 3.11) where the moving sets are assumed to be nonempty, closed and subsmooth or positively α_0-far with absolutely continuous variation in time and Lipschitz variation in the state. The perturbation term is supposed to be upper semicontinuous from $H \times H$ into H_w with nonempty, closed and convex values satisfying a weak linear growth condition which enables us to deal with unbounded perturbation terms. We emphasize that the novelty of our work resides as much in the method as in the great generality in that the second-order sweeping process is treated. In fact, this is the first time in that the moving set depends jointly on the state and on the velocity.

The following result, consequence of Theorem 3.6, extends several works in the literature [8, 12, 13, 28, 30, 31, 44], where the authors assume that the set-valued map takes convex or uniformly prox-regular values.

Theorem 3.11 *Assume that* $(\mathcal{H}_1)$, $(\mathcal{H}_3)$ *and* $(\mathcal{H}_4)$ *hold. Let* $F\colon [T_0,T]\times H\times H \rightrightarrows H$ *be a set-valued mapping with nonempty closed and convex values satisfying* $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ *and* $(\mathcal{H}_3^F)$. *Then, for all* $\alpha_0 \in]\sqrt{L_2}, 1]$ *there exists at least one solution* $u \in \mathrm{W}^{2,1}([T_0,T];H)$ *of* (3.29) *satisfying*

$$\|(u(t),\dot{u}(t))\| \leq \left(\|(u_0,v_0)\| + \int_{T_0}^t \widetilde{d}(s)ds\right)\exp\left(\int_{T_0}^t \widetilde{c}(s)ds\right) \quad \text{for all } t \in [T_0,T],$$

where for all $t \in [T_0,T]$

$$\widetilde{c}(t) := \frac{\alpha_0^2+1}{\alpha_0^2-L_2}c(t) \quad \text{and} \quad \widetilde{d}(t) := \frac{\alpha_0^2+1}{\alpha_0^2-L_2}d(t) + \frac{1}{\alpha_0^2-L_2}|\dot{\zeta}(t)|.$$

By using Theorem 3.6, we can get existence of solutions for a variant of the second-order sweeping process with perturbation considered by Bounkhel and Haddad [33]. The next proposition greatly extends [33, Theorem 3.1], where the authors assume that $C(\cdot)$ is uniformly prox-regular, $C(\cdot) \subseteq K$ for some convex compact set K and $F\colon [T_0,T]\times H \rightrightarrows H$ is an upper semicontinuous set-valued mapping from $[T_0,T]\times H$ into H_w with nonempty closed convex values satisfying the stronger linear growth condition: There exists $L > 0$ such that $F(t,x) \subseteq L(1+\|x\|)$ for all $(t,x) \in [T_0,T]\times H$.

Proposition 3.12 *Let* $C\colon [T_0,T] \rightrightarrows H$ *be a set-valued map satisfying* $(\mathcal{H}_9)$, $(\mathcal{H}_8)$ *and* $(\mathcal{H}_{10})$, $A\colon H \to H$ *be a linear bounded operator and let* $F\colon [T_0,T]\times H \rightrightarrows H$ *be a set-valued map with nonempty closed and convex values satisfying* $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$ *and* $(\mathcal{H}_6^F)$. *Then, there exists at least one solution* $u \in \mathrm{W}^{2,1}([T_0,T];H)$ *of the problem*

$$\begin{cases} -\ddot{u}(t) \in N(C(t);\dot{u}(t)) + F(t,\dot{u}(t)) + Au(t) & \text{a.e. } t \in [T_0,T], \\ u(T_0) = u_0, \dot{u}(T_0) = v_0 \in C(T_0). \end{cases}$$

As a consequence of Proposition 3.12 we obtain the existence of solutions for the dynamic analogue of the Signorini problem (3.30) which extends [33, Corollary 1], where the authors assume that $C(\cdot) \subseteq K$ for some convex compact set K, l is uniformly bounded and J is time-independent and uniformly Lipschitz continuous.

Corollary 3.13 *Assume that* $C\colon [T_0,T] \rightrightarrows H$ *is a set-valued map with closed and convex values satisfying hypotheses* $(\mathcal{H}_8)$ *and* $(\mathcal{H}_{10})$. *Assume that* $l \in L^1([T_0,T];H)$ *and* $J\colon [T_0,T]\times H \to \mathbb{R}$ *is a locally Lipschitz function such that* ∂J *satisfies* $(\mathcal{H}_4^F)$, $(\mathcal{H}_5^F)$ *and* $(\mathcal{H}_6^F)$. *Then, for every* $u_0 \in H$ *and any* $v_0 \in C(T_0)$, *there exists at least one solution of* (3.30).

3.8 The necessity of the compactness assumptions

The existence of a solution of unperturbed sweeping process has been established in the literature in the case where the sets $C(\cdot)$ are uniformly prox-regular. But the situation becomes more complicate in presence of a perturbation. As shown by the following counter-example, the compactness assumptions $(\mathcal{H}_4)$, $(\mathcal{H}_{10})$ and $(\mathcal{H}_4)$ in the previous theorems cannot be removed. It shows that a sweeping process governed by a single-valued continuous perturbation mapping and a normal cone to a closed bounded convex and autonomous set may have no solution. This example is based on the reference [77] where the authors have shown that in every separable Banach space X there is a continuous function $f\colon X \to X$ such that the autonomous differential equation

$$\dot{x}(t) = f(x(t)) \tag{3.31}$$

has no solutions in any interval of the real line (see [77, Theorem 8]). Since f is continuous at 0, we may assume that f is bounded on $r\mathbb{B}$, for some $r > 0$. By considering this function we define

$$F(x) = \begin{cases} f(x) & \text{if } \|x\| \leq r, \\ f\left(r\frac{x}{\|x\|}\right) & \text{if } \|x\| > r, \end{cases}$$

which is continuous and uniformly bounded on X. Now, consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N\left((M+1)\mathbb{B}; x(t)\right) + F(x(t)) & \text{a.e. } t \in [0,T], \\ x(0) = 0, \end{cases} \tag{3.32}$$

where $M := \sup\limits_{x \in X} \|F(x)\|$. Assume that (3.32) has a solution $x\colon [0,T] \to X$ for some $T > 0$. Then,

$$\langle -\dot{x}(t) + F(x(t)), y - x(t) \rangle \leq 0 \quad \forall y \in (M+1)\mathbb{B}.$$

Since $x(t) \in (M+1)\mathbb{B}$ for all $t \in [0,T]$, we have for every $t \in [0,T]$ where $\dot{x}(t)$ exists

$$\langle -\dot{x}(t) + F(x(t)), \dot{x}(t) \rangle = 0.$$

Thus $\|\dot{x}(t)\| \leq M$ for a.e. $t \in [0,T]$ and hence $\|x(t)\| \leq MT$ for all $t \in [0,T]$. Hence, if $T < \frac{r}{M}$, $x(t) \in \operatorname{int} \mathbb{B}$ for all $t \in [0,T]$. Therefore,

$$\dot{x}(t) = F(x(t)) \quad \text{a.e. } t \in [0,T],$$

which, since F and x are continuous and $\|x\|_\infty < r$, implies that x is a solution of (3.31). Therefore, the system (3.32) has no solutions.

Remark 3.4 The function $F\colon X \to X$ is continuous and uniformly bounded on X. Thus, F satisfies all the assumptions of Theorem 3.10 showing that the compactness on the sets $C(\cdot)$ is needed.

Chapter 4

A Variant of the Perturbed State-Dependent Sweeping Process

Let X and Y be two finite-dimensional Hilbert spaces and $C\colon [T_0,T]\times X \rightrightarrows Y$ be an absolutely continuous set-valued map with nonempty and closed values. In this chapter, we are interested in the following variation of the perturbed state-dependent sweeping process:

$$\begin{cases} \dot{x}(t) \in F(t,x(t)) - g(x(t))N\left(C(t,x(t));h(x(t))\right) & \text{a.e. } t\in[T_0,T],\\ x(T_0)=x_0 \in h^{-1}\left(C(T_0,x_0)\right), \end{cases} \tag{4.1}$$

Here $N(S;\cdot)$ denotes the Clarke normal cone to the a $S\subseteq Y$ and $g\colon X\to \mathcal{L}\left(Y,X\right)$ and $h\colon X\to Y$ are two functions.

The motivation to study (4.1) is that several differential inclusions can be written in this way. For example, perturbed state-dependent sweeping process ($X=Y$, $h\equiv \mathrm{Id}$ and $g\equiv Dh$), complementarity dynamical systems (CDS) (see Section 4.3 below), some control systems describing hysteresis (see Section 4.4 below), projected dynamical systems [38], some differential variational inequalities (see [121, section 2.5]), etc.

The aim of this chapter is to show existence of solutions of (4) and to give some applications to complementarity dynamical systems and control systems describing hysteresis.

A complementarity dynamical system (CDS) consists of ordinary differential equations coupled with complementarity conditions which can be specified by functions $F\colon [T_0,T]\times X\to Y$, $g\colon X\to \mathcal{L}\left(Y;X\right)$ and $H\colon X\to Y$. The defining equations for the CDS corresponding to F, g and H are

$$\begin{cases} \dot{x}(t) = F(t, x(t)) + g(x(t))u(t), \\ y(t) = H(t, x(t), u(t)), \\ K \ni y(t) \perp u(t) \in K^*, \end{cases} \tag{4.2}$$

where $K \subseteq Y$ is a closed convex cone and $K^* := \{d \in Y : \langle v, d\rangle \geq 0\, \forall v \in K\}$ denotes the dual cone of K. The third line in (4.2) is a complementarity relation between $y(t)$ and $u(t)$ which are forced to remain always orthogonal one to each other. This fact can be expressed in an equivalent way as

$$K \ni y(t) \perp u(t) \in K^* \quad \Leftrightarrow \quad -u(t) \in N\left(K, y(t)\right). \tag{4.3}$$

Therefore, by using (4.3), we can write (4.2) as (4.1).

CDS have been the object of strong interest because of their applications in various fields like mechanics, electrical circuits, transportation science, control systems, etc (see [39] and the references therein). The CDS formalism includes the so-called *linear complementarity systems* (LCS), widely used to deal with some electrical problems (see [1]), and the so called *gradient complementarity system* (GCS), which corresponds to the particular case $g \equiv [Dh]^*$.

CDS has been studied in [39], where the authors consider an "input-output property" to perform a change of state variable allowing them to write (4.2) as a GCS which is transformed into a perturbed sweeping process. These transformations are made by using the identity

$$I_{C(t)}\left(h(x)\right) = I_{h^{-1}(C(t))}(x) \quad \forall x \in X,$$

where $I_S : X \to \mathbb{R} \cup \{+\infty\}$ is the indicator function of a set S. Then, the authors show that the sets $h^{-1}\left(C(t)\right)$ are r-uniformly prox-regular and use a nonsmooth chain rule, for which they must assume some regularity and constraint qualification conditions on h. Thus, existence is obtained from known results in the literature of sweeping process [65, 135]. We will follow this path but only in the particular case of the GCS, where we give conditions on the sets $C(t)$ to assure that $h^{-1}\left(C(t)\right)$ be equi-uniformly subsmooth and then we apply Theorem 3.10. In the general case we will transform the CDS into (4.1) and we show existence directly from Theorem 4.1. This avoids the assumption of a special structure on the functions g and h, as the input-output property used in [39].

In Section 4.4 we study existence of solutions of the following control system:

$$\begin{cases} \dot{w}(t) + \partial I_{K(v(t))}(w(t)) \ni h_1(w(t), v(t))u^1(t) & \text{a.e } t \in [T_0, T], \\ \dot{v}(t) + c(w(t), v(t))\dot{w}(t) = h_2(w(t), v(t))u^2(t) & \text{a.e. } t \in [T_0, T], \\ v(T_0) = v_0, w(T_0) = w_0, \end{cases} \tag{4.4}$$

with the constraint

$$u(t) = \left(u^1(t), u^2(t)\right) \in U(t, v(t), w(t)) \quad \text{a.e. } t \in [T_0, T], \tag{4.5}$$

where $K(v) = [f_*(v), f^*(v)]$. Here c, h_1, h_2, f_*, f^* are given functions satisfying Assumption 4 below, $(w_0, v_0) \in \mathbb{R}^2$ is a given initial conditions with $w_0 \in K(v_0)$, and U is a set-valued map with closed, convex and bounded values in $\mathbb{R}^2$ satisfying Assumption 5 below.

The system (4.4)-(4.5) describes many controlled input-output relations $u \mapsto w$ which are physically relevant, for example, solid-liquid phase transition with supercooling effect and martensite-austenite phase transition in shape memory alloys (see [109] and references therein), among others. The problem (4.4)-(4.5) has been studied by several authors (see, for example, [93, 108, 109, 139]). In all these papers, the authors use the Moreau-Yosida regularization technique to obtain a family of approximated problems which converges to a solution of (4.4)-(4.5). We follow a different path by using Theorem 4.1, which is based on the reduction technique for the sweeping process. Furthermore, we consider the following particular case of (7)-(8):

$$\begin{cases} \dot{w}(t) + \partial I_{K(v(t))}(w(t)) \ni h_1(w(t), v(t)) & \text{a.e } t \in [T_0, T], \\ \dot{v}(t) + c(w(t), v(t))\dot{w}(t) = h_2(w(t), v(t)) & \text{a.e. } t \in [T_0, T], \\ v(T_0) = v_0, w(T_0) = w_0, \end{cases} \tag{4.6}$$

By considering the problem (4.6) as a sweeping process, we obtain the following numerical algorithm of catching-up type, to solve this system:

$$\begin{cases} w_{i+1}^n = \operatorname{proj}_{K(v_i^n)}\left(w_i^n + \mu_n h_1^{n,i}\right), \\ v_{i+1}^n = v_i^n + \mu_n h_2^{n,i} - c^{n,i}\left(w_{i+1}^n - w_i^n\right). \end{cases}$$

This algorithm is different from the used in [109], where the authors discretize the Moreau-Yosida regularization of the normal cone and they obtain a numerical algorithm depending on two parameters. We illustrate our existence result by performing some numerical simulations with this new algorithm.

We emphasize that result is new and improves the results given in [108, 109] by weakening the regularity on the functions f_*, f^*, c, h_1 and h_2. Moreover, Theorem 4.7 can by seen as a complement of [93, 139], where the authors assume that the set-valued map $(v, w) \to \operatorname{co} U(t, v, w)$ is Lipschitz continuous.

4.1 Technical assumptions

For the sake of readability, in this section we collect the hypothesis used along this chapter.

Hypotheses on the set-valued map $C\colon [T_0, T] \times X \rightrightarrows Y$**:** C is a set-valued map with nonempty and closed values. Moreover, we will consider the following conditions:

$(\mathcal{H}_1)$ There exists $\zeta \in W^{1,1}([T_0,T];\mathbb{R})$ and $L \geq 0$ such that

$$\sup_{y\in Y} |d(y, C(t,u)) - d(y, C(s,v))| \leq L\|u-v\| + |\zeta(t)-\zeta(s)|,$$

for all $s,t \in [0,T]$ and all $u,v \in X$.

$(\mathcal{H}_2)$ The family $\{C(t,x)\colon (t,x) \in [T_0,T]\times X\}$ is equi-uniformly subsmooth.

Furthermore, when $C(t,x) \equiv C(t)$ for all $t \in [T_0,T]$, we will consider:

$(\mathcal{H}_3)$ There exist two constants $\alpha_0 \in]0,1]$ and $\rho \in]0,+\infty)$ such that

$$0 < \alpha_0 \leq \inf_{x\in U_\rho(C(t))} d\left(0, \partial d(x, C(t))\right) \quad \text{a.e. } t \in [T_0,T].$$

where $U_\rho(C(t)) := \{y \in Y \colon 0 < d(y, C(t)) < \rho\}$ for all $t \in [T_0,T]$.

Hypotheses on the functions $g\colon X \to \mathcal{L}(Y,X)$ and $h\colon X \to Y$:

$(\mathcal{H}_4)$ (a) The function g is continuous with

$$\|g(x)\| \leq M \quad \text{for all } x \in X,$$

for some $M > 0$.

(b) $h\colon X \to Y$ is differentiable with $\sup_{x\in X}\|Dh(x)\| \leq M$.

(c) There exists $\lambda > 0$ such that for all $x \in X$

$$\langle Dh(x) \circ g(x)y, y\rangle \geq \lambda\|y\|^2 \quad \text{for all } y \in Y.$$

Hypotheses on the set-valued map $F\colon [T_0,T]\times X \rightrightarrows X$: F is a set-valued map with nonempty, closed and convex values. Furthermore, we will consider the following conditions:

$(\mathcal{H}_1^F)$ For each $v \in X$, $F(\cdot, v)$ is measurable.

$(\mathcal{H}_2^F)$ For a.e. $t \in [T_0,T]$, $F(t,\cdot)$ is upper semicontinuous from X into X.

$(\mathcal{H}_3^F)$ There exist $c,d \in L^1(T_0,T)$ such that

$$d\left(0, F(t,x)\right) := \inf\{\|w\| \colon w \in F(t,x)\} \leq c(t)\|x\| + d(t),$$

for all $x \in X$ and a.e. $t \in [T_0,T]$.

4.2 Existence result

In this section we prove existence of solutions of (4.1).

Theorem 4.1 *Assume, in addition to* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ *and* $(\mathcal{H}_4)$, *that* $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ *and* $(\mathcal{H}_3^F)$ *hold. If* $L \cdot M < \lambda$, *then there exists at least one solution of* (4.1). *Moreover,*

$$\|\dot{x}(t)\| \leq \frac{\lambda\alpha_0^2 + M}{\lambda\alpha_0^2 - LM}\left(c(t)\sigma(t) + d(t)\right) + \frac{M}{\lambda\alpha_0^2 - LM}|\dot{\zeta}(t)|,$$

where α_0 *is such that* $\frac{LM}{\lambda} < \alpha_0^2 < 1$ *and*

$$\begin{aligned}\sigma(t) &:= \eta(t)\exp\left(\frac{\lambda\alpha_0^2 + M}{\lambda\alpha_0^2 - LM}\int_{T_0}^t c(s)ds\right),\\ \eta(t) &:= \|x_0\| + \frac{\lambda\alpha_0^2 + M}{\lambda\alpha_0^2 - LM}\int_{T_0}^t d(s)ds + \frac{M}{\lambda\alpha_0^2 - LM}\int_{T_0}^t |\dot{\zeta}(s)|ds.\end{aligned}$$

Proof. Let α_0 be such that $\frac{LM}{\lambda} < \alpha_0^2 < 1$. Then, according to Proposition 2.8, there exists $\rho > 0$ such that

$$\alpha_0^2 \leq \inf_{y \in U_\rho(C(t,x))} d\left(0, \partial d_{C(t,x)}(y)\right) \quad \text{for all } t \in [T_0, T], x \in X.$$

To prove existence, we reduce (4.1) to the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) - \mu(t, x(t))g(x(t))\partial d_{C(t,x(t))}(h(x(t))) & \text{a.e. } t \in [T_0, T],\\ x(T_0) = x_0 \in h^{-1}\left(C(T_0, x_0)\right). \end{cases} \tag{4.7}$$

where $\mu\colon [T_0, T] \times X \to \mathbb{R}^+$ is defined by

$$\mu(t, x) := \frac{1 + L}{\lambda\alpha_0^2 - LM}\left(c(t)\|x\| + d(t)\right) + \frac{|\dot{\zeta}(t)|}{\lambda\alpha_0^2 - LM}.$$

Since we are working in finite-dimensional spaces, the existence of (4.7) is a direct consequence of Proposition 3.3. To finish the proof it is enough to prove that $h(x(t)) \in C(t, x(t))$ for all $t \in [T_0, T]$. The proof will be divided into two steps.
<u>Step 1</u>: We first prove the theorem under the additional assumption:

$$\int_{T_0}^T \kappa(s)ds < \rho, \tag{4.8}$$

where

$$\kappa(t) := \frac{\lambda\alpha_0^2 + M^2}{\lambda\alpha_0^2 - LM}|\dot{\zeta}(t)| + (L + M)\frac{\lambda\alpha_0^2 + M}{\lambda\alpha_0^2 - LM}\left(c(t)\sigma(t) + d(t)\right).$$

Claim 1:

$$\|x(t)\| \leq \sigma(t) \quad \text{for all } t \in [T_0, T].$$

Proof of Claim 1:

$$\begin{aligned}\|\dot{x}(t)\| &\leq c(t)\|x(t)\| + d(t) + M\mu(t, x)\\ &= \frac{\lambda\alpha_0^2 + M}{\lambda\alpha_0^2 - LM}\left(c(t)\|x(t)\| + d(t)\right) + \frac{M}{\lambda\alpha_0^2 - LM}|\dot{\zeta}(t)|.\end{aligned}$$

Therefore, by using Lemma 1.17, we obtain the claim. ☐

Define $\varphi(t) := d_{C(t,x(t))}(h(x(t)))$ for $t \in [T_0, T]$. To simplify the notation, we write

$$\Gamma(t) := \partial d_{C(t,x(t))}(h(x(t))).$$

Claim 2: $\varphi(t) < \rho$ for all $t \in [T_0, T]$.

Proof of Claim 2: Let $t \in [T_0, T]$ where $\dot{x}(t)$ and $\dot{\zeta}(t)$ exist. Then, due to Lemma 1.18 and (4.2),

$$\begin{aligned}
\dot{\varphi}(t) &\leq L\|\dot{x}(t)\| + |\dot{\zeta}(t)| + \max_{v\in\Gamma(t)} \langle v, Dh(x(t))\dot{x}(t)\rangle \\
&\leq L\|\dot{x}(t)\| + |\dot{\zeta}(t)| + M\|\dot{x}(t)\| \\
&= |\dot{\zeta}(t)| + (L+M)\|\dot{x}(t)\| \\
&\leq |\dot{\zeta}(t)| + (L+M)\left(\frac{\lambda\alpha_0^2 + M}{\lambda\alpha_0^2 - LM}(c(t)\|x(t)\| + d(t)) + \frac{M}{\lambda\alpha_0^2 - LM}|\dot{\zeta}(t)|\right) \\
&\leq \frac{\lambda\alpha_0^2 + M^2}{\lambda\alpha_0^2 - LM}|\dot{\zeta}(t)| + (L+M)\frac{\lambda\alpha_0^2 + M}{\lambda\alpha_0^2 - LM}(c(t)\sigma(t) + d(t)) \\
&= \kappa(t).
\end{aligned}$$

Therefore, according to (4.8), $\varphi(t) < \rho$ for all $t \in [T_0, T]$. ☐

Define $\Omega := \{s \in [T_0, T] \colon h(x(s)) \notin C(s, x(s))\}$. Then, Ω is open. We proceed to prove that $\Omega = \emptyset$. Indeed, otherwise, let $t \in \Omega$ where $\dot{x}(t)$ and $\dot{\zeta}(t)$ exist. Hence, due to Lemma 1.18,

$$\begin{aligned}
\dot{\varphi}(t) &\leq L\|\dot{x}(t)\| + |\dot{\zeta}(t)| + \min_{x^*\in\Gamma(t)} \langle x^*, Dh(x(t))\dot{x}(t)\rangle \\
&\leq L\|\dot{x}(t)\| + |\dot{\zeta}(t)| + (c(t)\|x(t)\| + d(t)) - \lambda\alpha_0^2\mu(t, x(t)) \\
&\leq L\left(c(t)\|x(t)\| + d(t) + M\mu(t, x(t))\right) + |\dot{\zeta}(t)| \\
&\quad + (c(t)\|x(t)| + d(t)) - \lambda\alpha_0^2\mu(t, x(t)) \\
&= 0.
\end{aligned}$$

Take $(\tau_-, \tau_+) \subseteq \Omega$ such that $\varphi(\tau_-) = 0$. Then, for every $t \in (\tau_-, \tau_+)$,

$$\varphi(t) = \varphi(\tau_-) + \int_{\tau_-}^{\tau_+} \leq 0,$$

which gives a contradiction. Therefore, $\Omega = \emptyset$, i.e., $h(x(t)) \in C(t, x(t))$ for all $t \in [T_0, T]$.

<u>Step 2</u>: In the general case, without any restriction on the length of $T - T_0$, let us consider $\{T_0, T_1, \ldots, T_N\}$ be a partition of $[T_0, T]$ such that for every $k \in \{1, \ldots, N\}$

$$\int_{T_{k-1}}^{T_k} \kappa(s)ds < \rho.$$

For $k = 1$, due to Step 1, let x^1 be a solution of (4.1) over $[T_0, T_1]$. Then, $h(x^1(t)) \in C(t, x^1(t))$ for all $t \in [T_0, T_1]$ and $x^1(T_0) = x_0 \in h^{-1}(C(T_0, x_0))$.
Inductively, for $k = 2, \ldots, N$, since $h(x^{k-1}) \in C\left(T_{k-1}(T_{k-1}), x^{k-1}(T_{k-1})\right)$ let x^k be a solution of (4.1) over $[T_{k-1}, T_k]$ with $x^k(T_{k-1}) = x^{k-1}(T_{k-1})$. Then, $h(x^k(t)) \in C(t, x^k(t))$ for all $t \in [T_{k-1}, T_k]$ and $x^k(T_{k-1}) = x^{k-1}(T_{k-1})$.
Finally, we define $x(t) = x^k(t)$ over $[T_{k-1}, T_K]$, for $k = 1, \ldots, N$. Thus, x is a solution of (4.1), which finishes the proof of the theorem. □

Remark 4.1 If $X = Y$, $h \equiv \mathrm{Id}$ and $g \equiv Dh$, then (4.1) becomes the state-dependent sweeping process (See Section 3.5) and the condition $LM < \lambda$, in Theorem 4.1, becomes $L < 1$. It was shown in [97] (See also Remark 3.3) that the state-dependent sweeping process could have no solutions for $L \geq 1$. Therefore, the condition $LM < \lambda$ cannot be improved.

If $C(t, x) \equiv C(t)$, then we can relax hypothesis $(\mathcal{H}_2)$ by $(\mathcal{H}_3)$. The following result was proved by Jourani and Vilches [87] by a different method.

Theorem 4.2 *Assume, in addition to* $(\mathcal{H}_1)$, $(\mathcal{H}_3)$ *and* $(\mathcal{H}_4)$, *that* $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ *and* $(\mathcal{H}_3^F)$ *hold. Then, there exists at least one solution of* (4.1). *Moreover,*

$$\|\dot{x}(t)\| \leq \frac{\lambda\alpha_0^2 + M}{\lambda\alpha_0^2}\left(c(t)\sigma(t) + d(t)\right) + \frac{M}{\lambda\alpha_0^2}|\dot{\zeta}(t)|,$$

where α_0 *is given by* $(\mathcal{H}_3)$ *and*

$$\sigma(t) := \eta(t)\exp\left(\frac{\lambda\alpha_0^2 + M}{\lambda\alpha_0^2}\int_{T_0}^{t} c(s)ds\right),$$
$$\eta(t) := \|x_0\| + \frac{\lambda\alpha_0^2 + M}{\lambda\alpha_0^2}\int_{T_0}^{t} d(s)ds + \frac{M}{\lambda\alpha_0^2}\int_{T_0}^{t} |\dot{\zeta}(s)|ds.$$

Remark 4.2

(i) If the moving sets are convex, then we can take $\alpha_0 = 1$ in Theorem 4.1 and 4.2.

(ii) When $X = Y$, $h \equiv \mathrm{Id}$ and $g \equiv Dh$, Theorems 4.1 and 4.2 give, respectively, the existence of solutions of the perturbed state-dependent sweeping process and perturbed sweeping process (see Sections 3.6 and 3.5).

4.3 Complementarity Dynamical Systems

In this section we show existence of solutions of (4.2) in two different ways. First, by using Theorem 4.1 for CDS and second, by using Theorem 3.10 for GCS. This section is based in [87]. We will consider the following assumptions:

Assumption 1. *The system* (4.2) *satisfies:* $H(t, x(t), u(t)) = h(x(t)) + \zeta(t)$ *where* $h\colon X \to Y$ *is a differentiable function and* $\zeta\colon [T_0, T] \to Y$ *is an absolutely continuous function.*

Under Assumption 1 we can write (4.2) as (4.1). Indeed, due to (4.3) and the identity

$$N\left(K; h(x(t)) + \zeta(t)\right) = N\left(K - \zeta(t); h(x(t))\right),$$

(4.2) is formally equivalent to

$$\dot{x}(t) \in F(t, x(t)) - g(x(t))N\left(C(t); h(x(t))\right),$$

where $C(t) = K - \zeta(t)$, which has the structure of (4.1). In order to apply Theorem 4.1, we make the following assumption:

Assumption 2. *The functions* h *and* g *satisfy* $(\mathcal{H}_4)$

(i) h *is a differentiable function with* $\sup_{x\in X} \|Dh(x)\| \leq M$*, for some* $M > 0$.

(ii) g *is a continuous function satisfying*

$$\|g(x)\| \leq M \quad \text{for all } x \in X.$$

(iii) There exists $\lambda > 0$ *such that for all* $x \in X$

$$\langle Dh(x)g(x)h, h\rangle \geq \lambda \|h\|^2 \quad \forall h \in Y.$$

Theorem 4.3 *In addition to Assumptions* 1 *and* 2*, assume that* F *satisfies* $(\mathcal{H}_1^F)$*,* $(\mathcal{H}_2^F)$ *and* $(\mathcal{H}_3^F)$*. Then, for every* x_0 *with* $h(x_0) + \zeta(T_0) \in K$ *there exists at least one absolutely continuous solution* x *of* (4.2) *satisfying* $x(T_0) = x_0$ *and*

$$\|\dot{x}(t)\| \leq \frac{\lambda + M}{\lambda}\left(c(t)\sigma(t) + d(t)\right) + \frac{M}{\lambda}|\dot{\zeta}(t)|,$$

where

$$\sigma(t) := \eta(t)\exp\left(\frac{\lambda+M}{\lambda}\int_{T_0}^t c(s)ds\right),$$
$$\eta(t) := \left(\|x_0\| + \frac{\lambda+M}{\lambda}\int_{T_0}^t d(s)ds + \frac{M}{\lambda}\int_{T_0}^t |\dot{\zeta}(s)|ds\right).$$

Proof. Let us consider the set $C(t) = K - \zeta(t)$ for every $t \in [T_0, T]$. It is clear that $C(t)$ is convex, so $(\mathcal{H}_2)$ holds. Moreover, since ζ is absolutely continuous, C satisfies $(\mathcal{H}_1)$. Due to Assumption 2, $(\mathcal{H}_4)$ holds. Therefore, the result follows from Theorem 4.2. □

Now we consider the so called *Gradient Complementarity Dynamical systems* (GCS), which correspond to the particular case where $g \equiv [Dh]^*$. In order to obtain the existence of solutions we will make the following assumption:

Assumption 3. *The mapping h is differentiable with uniformly continuous derivative Dh, $g \equiv [Dh]^*$ and there exists $k > 0$ such that*

$$\mathbb{B}_Y \subseteq Dh(x)k\mathbb{B}_X - K \quad \textit{for all } x \in h^{-1}(K).$$

Under this assumption, we transform (4.2) into a perturbed sweeping process, from which we obtain existence of solutions. Indeed, consider the following identity:

$$I_{h^{-1}(K-\zeta(t))}(x) = I_K(h(x) + \zeta(t)) \quad \text{for all } x \in X, t \in [T_0, T].$$

Then, Assumption 3 allows us to apply the chain rule for nonsmooth functions (see [112, Theorem 1.17]) to obtain:

$$N\left(h^{-1}(K - \zeta(t)), x\right) = [Dh(x)]^* N(K - \zeta(t); h(x)).$$

Therefore, (4.2) can be rewritten as

$$\dot{x}(t) \in F(t, x(t)) - N\left(h^{-1}(K - d(t)); x(t)\right).$$

Moreover, due to Corollary 2.14, the family $(h^{-1}(K - d(t)))_{t\in[T_0,T]}$ is equi-uniformly subsmooth and the set-valued map $C(t) = h^{-1}(K - \zeta(t))$ satisfies

$$|d(x, C(t)) - d(x, C(s))| \leq k'|\zeta(t) - \zeta(s)| \quad \text{for all } x \in X \text{ and } s, t \in [T_0, T],$$

with $k' > k$. Therefore, from Theorem 3.10, we obtain the following result which extends [38, Theorem 1] and [39, Theorem 4.3].

Theorem 4.4 *In addition to Assumptions 1 and 3, assume that F satisfies $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$. Then, for every x_0 with $h(x_0) + \zeta(T_0) \in K$, there exists at least one absolutely continuous solution x of (4.2) satisfying $x(T_0) = x_0$ and*

$$\|\dot{x}(t)\| \leq \frac{\lambda + M}{\lambda}(c(t)\sigma(t) + d(t)) + \frac{Mk'}{\lambda}|\dot{\zeta}(t)|,$$

where $k' > k$ is given by Assumption 3 and

$$\begin{aligned}
\sigma(t) &:= \eta(t) \exp\left(\frac{\lambda + M}{\lambda} \int_{T_0}^{t} c(s)ds\right), \\
\eta(t) &:= \left(\|x_0\| + \frac{\lambda + M}{\lambda} \int_{T_0}^{t} d(s)ds + \frac{Mk'}{\lambda} \int_{T_0}^{t} |\dot{\zeta}(s)|ds\right).
\end{aligned}$$

Remark 4.3 Theorem 4.4 extends [39, Theorem 4.3] by weakening the regularity of the function h from C^2 to merely differentiable functions with uniformly continuous differential.

4.4 A control system describing hysteresis effects

In this section we show existence of solutions to the problem (4.4)-(4.5). We will consider the following assumptions:

Assumption 4.

(i) *The functions $f_*\colon \mathbb{R} \to \mathbb{R}$, $f^*\colon \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous with Lipschitz constants $L_1, L_2 > 0$, respectively and $f_* \le f^*$.*

(ii) *There exist constants $m, a > 0$ such that*

$$\max\{|h_1(w,v)|, |h_2(w,v)|\} \le m + a\|(w,v)\|, \quad (w,v) \in \mathbb{R}^2,$$

(iii) *There exists $c_0 \in (0,2)$ with*

$$\max\{L_1, L_2\} < \frac{2 - c_0}{2 + c^2 + \sqrt{c^4 + 4c^2}}$$

such that

$$|c(w,v)| \le c_0 \quad (w,v) \in \mathbb{R}^2.$$

(iv) *The functions c, h_1, h_2 are continuous on $\mathbb{R}^2$.*

Assumption 5. *The set-valued map $U\colon [T_0, T] \times \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ has nonempty, closed and convex values. Moreover,*

(i) *For each $(v,w) \in \mathbb{R}^2$, $U(\cdot, v, w)$ is measurable.*

(ii) *For a.e. $t \in [T_0, T]$, $U(t, \cdot, \cdot)$ is upper semicontinuous from $\mathbb{R}^2$ into $\mathbb{R}^2$.*

(iii) *There exists $d > 0$ such that*

$$d\,(0, U(t,v,w)) := \inf\{\|w\| \colon w \in U(t,v,w)\} \le d,$$

for all $(v,w) \in \mathbb{R}^2$ and for a.e. $t \in [T_0, T]$.

Lemma 4.5 *If Assumptions 4 and 5 hold, then, the set-valued map $F\colon [T_0,T] \times \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ defined by*

$$F(t,v,w) := \left\{ \begin{pmatrix} h_2(w,v)u^2 - c(w,v)h_1(w,v)u^1 \\ h_1(w,v)u^1 \end{pmatrix} : (u^1, u^2) \in U(t,v,w) \cap d\,\mathbb{B}_{\mathbb{R}^2} \right\}$$

satisfies $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$.

Proof. It is easy to see that F has nonempty, closed and convex values. $(\mathcal{H}_1^F)$ follows directly from Assumptions 4 and 5. To prove $(\mathcal{H}_2^F)$, it is sufficient to show that graph $F(t,\cdot,:)$ is closed. Let $w_n \to w$, $v_n \to v$ and

$$\zeta_n^1 = h_2(w_n,v_n)u_n^2 - c(w_n,v_n)h_1(w_n,v_n)u_n^1 \to \zeta^1,$$
$$\zeta_n^2 = h_1(w_n,v_n)u_n^1 \to \zeta^2,$$

with $(u_n^1,u_n^2) \in U(t,v_n,w_n)$ for all $n \in \mathbb{N}$. We have to prove that $(\zeta^1,\zeta^2) \in F(t,v,w)$. Indeed, since $U(t,v,w)\cap d\,\mathbb{B}_{\mathbb{R}^2}$ is compact and $U(t,\cdot,:)$ is upper semicontinuous from $\mathbb{R}^2$ into $\mathbb{R}^2$, without loss of generality, we can assume that $(u_n^1,u_n^2) \to (u^1,u^2) \in U(t,v,w)$. Thus, by the continuity of h_1, h_2 and c, it follows that

$$\zeta^1 = h_2(w,v)u^2 - c(w,v)h_1(w,v)u^1 \text{ and } \zeta^2 = h_1(w,v)u^1,$$

i.e., $(\zeta^1,\zeta^2) \in F(t,v,w)$, which establishes $(\mathcal{H}_2^F)$. Finally, to prove $(\mathcal{H}_3^F)$ we take $(u^1,u^2) \in U(t,v,w)\cap d\,\mathbb{B}_{\mathbb{R}^2}$. Then, by using Assumption 5,

$$\begin{aligned} d(0,F(t,v,w)) &\le \left\|(h_2(w,v)u^2 - c(w(t),v(t))h_1(w(t),v(t))u^1, h_2(w(t),v(t))u^1)\right\| \\ &\le (1+2c_0)d\left(m + c\|(v,w)\|\right), \end{aligned}$$

which proves $(\mathcal{H}_3^F)$. □

Lemma 4.6 *Let $f_*, f^*\colon \mathbb{R}\to\mathbb{R}$ be two functions. Then set valued map $K\colon \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $K(v) = [f_*(v), f^*(v)]$ satisfies: For all $v,w \in \mathbb{R}$*

$$\sup_{x\in\mathbb{R}} |d(x,K(v)) - d(x,K(w))| \le \max\{|f_*(v)-f_*(w)|, |f^*(v)-f^*(w)|\}.$$

Now we present the main result of this section.

Theorem 4.7 *If Assumptions 4 and 5 hold, then, for every $(v_0,w_0) \in \mathbb{R}^2$ with $w_0 \in K(v_0)$, there exists at least one absolutely continuous solution (v,w) of (4.4)-(4.5) satisfying $(v(T_0),w(T_0)) = (v_0,w_0)$.*

Proof. We observe that (4.4) is equivalent to the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t,x(t)) - g(x(t))N\left(C(x(t));x(t)\right) & \text{a.e. } t \in [T_0,T], \\ x(T_0) = x_0 \in C(x_0), \end{cases} \tag{4.9}$$

where $x(t) := (v(t),w(t))$, F was defined in Lemma 4.5, $C(v,w) = \mathbb{R}\times K(v)$ and g is defined by

$$g(v,w) = \begin{pmatrix} 1 & -c(w,v) \\ 0 & 1 \end{pmatrix}.$$

To prove the theorem, it is enough to verify the hypotheses from Theorem 4.1. Indeed, according to Lemma 4.6, C satisfies $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ with $L = \max\{L_1,L_2\}$ and $\zeta(t) \equiv 0$. Furthermore,

$$\|g(v,w)\|^2 = \frac{1}{2}\left(c^2(w,v) + \sqrt{c^4(w,v)+4c^2(w,v)} + 2\right) \le \frac{1}{2}\left(c_0^2 + \sqrt{c_0^4 + 4c_0^2} + 2\right).$$

and for all $h \in \mathbb{R}^2$

$$\begin{aligned}\langle g(v,w)h,h\rangle &\geq \left(1-\frac{|c(w,v)|}{2}\right)\|h\|^2\\ &\geq \left(1-\frac{c_0}{2}\right)\|h\|^2,\end{aligned}$$

Hence, $(\mathcal{H}_4)$ holds with $\lambda := 1-\frac{c_0}{2}$ and $M := \frac{1}{2}\left(c_0^2+\sqrt{c_0^4+4c_0^2}+2\right)$. Thus, $LM < \lambda$. Therefore, by virtue of Theorem 4.1, there exists at least one solution of (4.9), i.e., there exists at least one solution of (4.4)-(4.5). □

Numerical Algorithm

Now we provide a new numerical algorithm to solve the following particular case of (4.1):

$$\begin{cases} \dot{w}(t)+\partial I_{K(v(t))}(w(t)) \ni h_1(w(t),v(t)) & \text{a.e } t\in[T_0,T],\\ \dot{v}(t)+c(w(t),v(t))\dot{w}(t)=h_2(w(t),v(t)) & \text{a.e. } t\in[T_0,T],\\ v(T_0)=v_0, w(T_0)=w_0, \end{cases} \tag{4.10}$$

Let $n\in\mathbb{N}$, define $\mu_n := \frac{T-T_0}{n}$ and consider the partition $\pi_n=\{t_0^n,\ldots,t_n^n\}$ of $[T_0,T]$ defined by $t_i^n = T_0+i\cdot\mu_n$ for $i=0,\ldots,n$. To solve (4.10), we consider the following approximations:

$$\begin{aligned} \dot{w}(t_i^n) &\approx \frac{w_{i+1}^n-w_i^n}{\mu_n} & \dot{v}(t_i^n) &\approx \frac{v_{i+1}^n-v_i^n}{\mu_n}\\ h_1^{n,i} &:= h_1(w(t_i^n),v(t_i^n)) \approx h_1(w_i^n,v_i^n) & h_2^{n,i} &:= h_2(w(t_i^n),v(t_i^n)) \approx h_2(w_i^n,v_i^n)\\ c^{n,i} &:= c(w(t_i^n),v(t_i^n)) \approx c(w_i^n,v_i^n). \end{aligned}$$

For the normal cone, we consider the following semi-implicit approximation:

$$\partial I_{K(v(t_i^n))}(w(t_i^n)) \approx \partial I_{K(v_i^n)}(w_{i+1}^n).$$

Therefore, according to the formula:

$$z\in \partial I_{K(v_i^n)}\left(w_{i+1}^n\right)+w_{i+1}^n \quad \Leftrightarrow \quad w_{i+1}^n = \operatorname{proj}_{K(v_i^n)}(z),$$

we obtain the following catching-up like algorithm to solve (4.10):

$$\begin{cases} w_{i+1}^n = \operatorname{proj}_{K(v_i^n)}\left(w_i^n+\mu_n h_1^{n,i}\right),\\ v_{i+1}^n = v_i^n+\mu_n h_2^{n,i} - c^{n,i}\left(w_{i+1}^n-w_i^n\right). \end{cases} \tag{4.11}$$

Numerical Simulations

We present numerical simulations to illustrate the algorithm (4.11). To perform the numerical simulations, we consider the functions (see Figure 4.1) f_* and f^*, borrowed from [109].

$$f_*(v) = \begin{cases} -1 & \text{if } v < 0.4 \\ 5v^2 - 4v - 0.2 & \text{if } 0.4 \leq v < 0.6 \\ 2v - 2 & \text{if } 0.6 \leq v < 1.4 \\ -5v^2 + 16v - 11.8 & \text{if } 1.4 \leq v < 1.6 \\ 1 & \text{if } 1.6 \leq v, \end{cases}$$

$$f^*(v) = \begin{cases} -1 & \text{if } v < -1.6 \\ 5v^2 + 16v + 11.8 & \text{if } -1.6 \leq v < -1.4 \\ 2v + 2 & \text{if } -1.4 \leq v < -0.6 \\ -5v^2 - 4v + 0.2 & \text{if } -0.6 \leq v < -0.4 \\ 1 & \text{if } -0.4 \leq v, \end{cases}$$

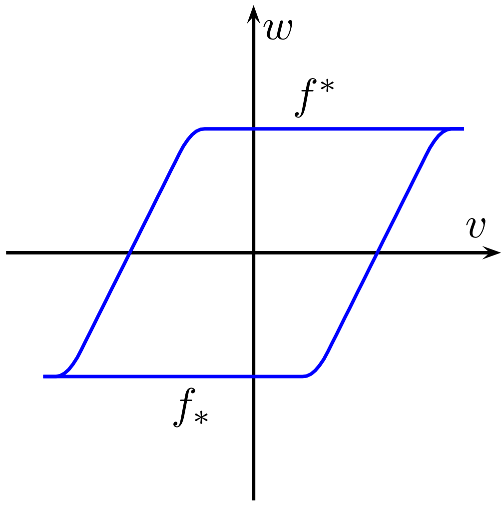
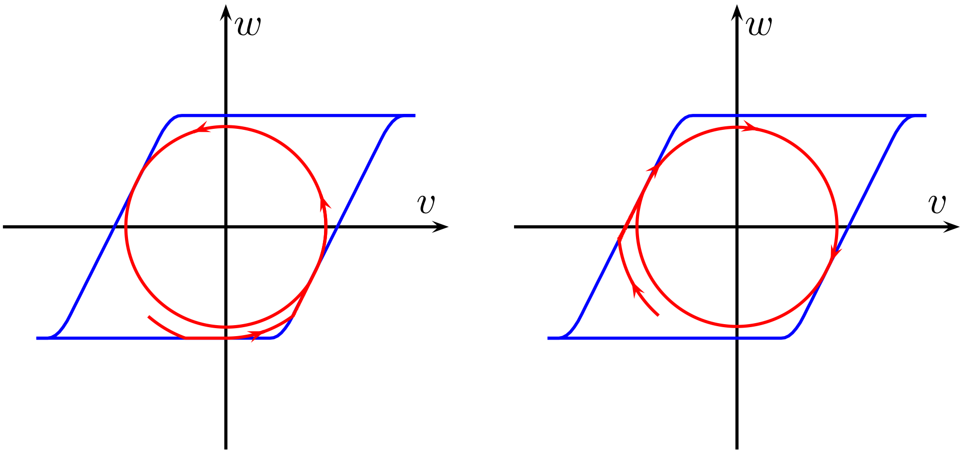


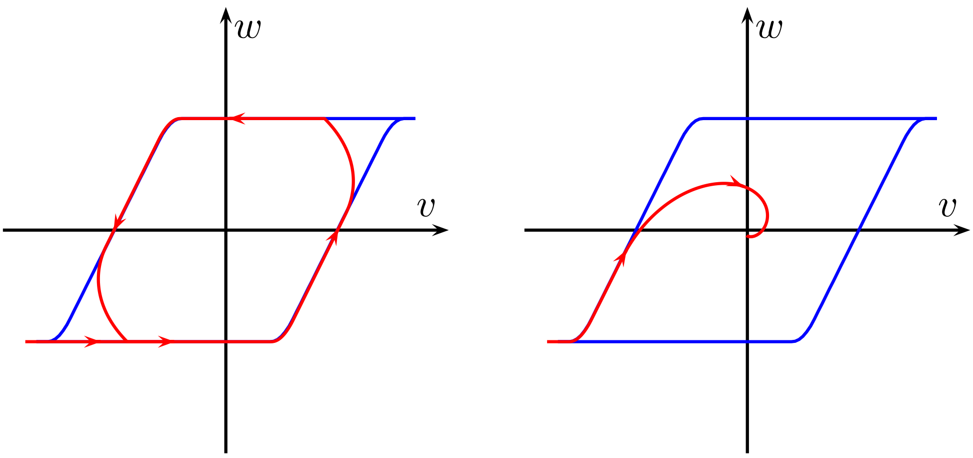
Figure 4.1: Functions f_* and f^*.

We obtain several simulations for different continuous functions c, h_1 and h_2. It is observed that the functions v and w are always piecewise smooth. Moreover, the hysteresis curve $v \mapsto w$ is continuous and piecewise smooth. Furthermore, since the hysteresis curve sticks to the boundary, we can not expect better smoothness than the regularity of f_* and f^*. The behavior of the hysteresis curve is complex, presenting periodic orbits (see Figures 4.2 and 4.3), stable and instable points (see Figures 4.2 to 4.5), bifurcation (see Figure 4.4), etc.



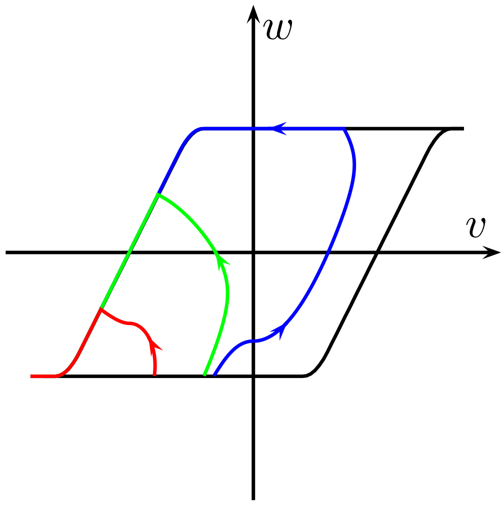
(a) $h_1(w,v) = v$, $h_2(w,v) = v - w$. (b) $h_1(w,v) = -v$, $h_2(w,v) = -v + w$.

Figure 4.2: Numerical simulations with $c(w,v) = 1$ and $(v_0, w_0) = (-0.7, -0.8)$.

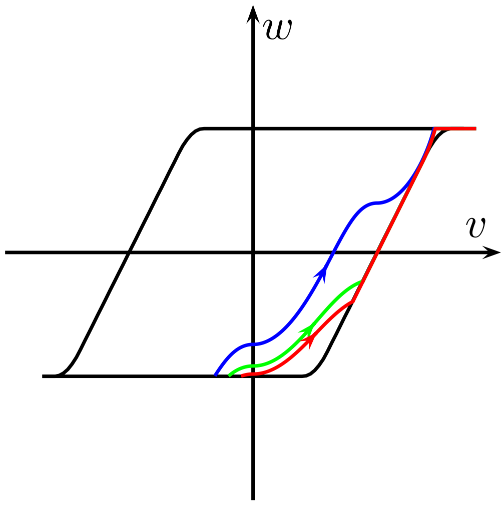


(a) $h_1(w,v) = v + \frac{1}{2}w$, $h_2(w,v) = v - 2w$. (b) $h_1(w,v) = -v - \frac{1}{2}w$, $h_2(w,v) = -v + w$.

Figure 4.3: Numerical simulations with $c(w,v) = 0.2$ and $(v_0, w_0) = (-1.8, -1.0)$.

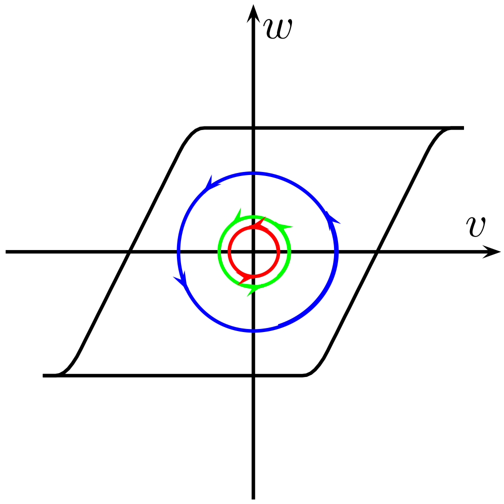


(a) $(v_0, w_0) = (-0.32, -1)$ (blue)
$(v_0, w_0) = (-0.4, -1)$ (green)
$(v_0, w_0) = (-0.8, -1)$ (red).

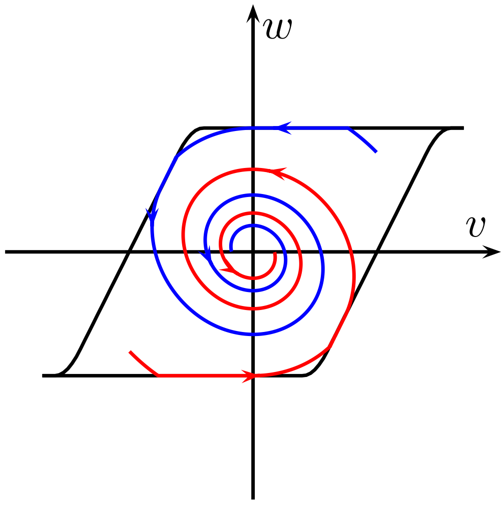


(b) $(v_0, w_0) = (-0.3, -1)$ (blue)
$(v_0, w_0) = (-0.2, -1)$ (green)
$(v_0, w_0) = (-0.1, -1)$ (red).

Figure 4.4: Numerical simulations for $c(w, v) = 0.2$, $h_1(w, v) = \sin((v - [v])\pi)$ and $h_2(w, v) = v - w$.



(a) $c(w, v) = 1 + \frac{1}{4}\sin(w)$
$(v_0, w_0) = (0.2, -0.6)$ (blue)
$(v_0, w_0) = (0.2, -0.2)$ (green)
$(v_0, w_0) = (0.2, 0.1)$ (red).



(b) $c(w, v) = 1 + \frac{1}{4}\cos(w)$
$(v_0, w_0) = (1, 0.8)$ (blue)
$(v_0, w_0) = (-1, -0.8)$ (red).

Figure 4.5: Numerical simulations for $h_1(w, v) = v$ and $h_2(w, v) = v - w$.

Chapter 5

Moreau-Yosida regularization of state-dependent sweeping process

In this chapter, which is based on [90], we are interested in the study of the state-dependent sweeping process:

$$\begin{cases} \dot{x}(t) \in -N\left(C(t,x(t)); x(t)\right) \quad \text{a.e. } t \in [T_0,T], \\ x(T_0) = x_0 \in C(T_0,x_0). \end{cases}$$

This differential inclusion has been motivated by quasi-variational inequalities arising e.g. in the evolution of sandpiles, quasistatic evolution problems with friction, micromechanical damage models for iron materials, among others (see [98] and the references therein).

The study of the state-dependent sweeping process was initiated by Chraibi Kaadoud [51], for convex sets in three dimension to deal with a mechanical problem with unilateral contact and friction, and generalized to a (possibly multivalued) perturbed form in the convex and nonconvex setting.

In the convex setting and by using a semi-implicit discretization scheme, Kunze and Monteiro-Marques [97] obtained the existence of solutions when the sets have a Lipschitz variation. Using an explicit discretization scheme, Haddad and Haddad [74] proved the existence of solutions of a perturbed state-dependent sweeping process with time-independent sets. Later, Bounkhel and Castaing [32] considered state-dependent sweeping process in uniformly smooth and uniformly convex Banach spaces.

In the nonconvex case and by using Schauder's fixed point theorem, Chemetov and Monteiro-Marques [49] established existence of solutions of perturbed state-dependent sweeping processes with uniformly prox-regular sets. Using a fixed point argument in ordered spaces, the same authors [50] proved the existence of solutions of the perturbed state-dependent sweeping process. Next, Castaing, Ibrahim and Yarou [45] showed the existence of solutions of the state-dependent sweeping process in the uniformly prox-regular case by using an extended version of Schauder's theorem and a discretization scheme. Later, Azzam-Laouir, Izza and Thibault [14] and Haddad, Kecis and Thibault [76] showed the existence of solution of the multivalued perturbed state-dependent sweeping process in the finite-dimensional setting with uniformly prox-regular sets. Finally, Noel [118] and Noel and Thibault [119] showed the existence of multivalued perturbed versions of the state-dependent sweeping process with equi-uniformly subsmooth and uniformly prox-regular sets.

The purpose of this chapter is twofold: first, to show the existence of solutions of the state-dependent sweeping process, by using the Moreau-Yosida regularization Technique, and second, to show the existence of solutions of the state-dependent sweeping process in the bounded variation continuous case by using a reparametrization technique.

The Moreau-Yosida regularization is a quite old approach to deal with differential inclusions. It consists in approaching the given differential inclusion by a penalized one, depending on a parameter, whose existence is easier to establish (for example, by using the classical Cauchy-Lipschitz theorem), and then to study the limit when the parameter goes to zero.

In the case of sweeping processes, the Moreau-Yosida regularization has been used to deal only with convex or uniformly prox-regular sets (see [96, 105, 110, 111, 113, 130, 136] for more details), although it has never been used, even in the convex case, to study the state-dependent sweeping process.

To deal with the state-dependent sweeping process in the bounded variation continuous case, we use the reparametrization technique from [113, 126–128] to reduce the bounded variation continuous case to the Lipschitz one. The application of the reparametrization technique is possible due to the rate-independence property of the sweeping process.

This chapter is organized as follows. In Section 5.1 we collect the hypotheses used throughout the chapter. In Section 5.3, we introduce the notion of solution for the state-dependent sweeping process with bounded variation. Next, in Section 5.4 and 5.5, we present the main results of this chapter (Theorems 5.9 and 5.10), namely the convergence (up to a subsequence) of the Moreau-Yosida regularization for the state-dependent sweeping process and the existence of solutions for bounded variation continuous state-dependent sweeping process.

5.1 Technical Assumptions

In this section, we list the hypotheses used throughout this chapter.

Hypotheses on the set-valued map $C\colon [T_0,T] \rightrightarrows H$: C is a set-valued map with nonempty and closed values. Also, the following hypotheses will be considered in Section 5.5.

($\mathcal{H}_1$) There exists $v \in \mathrm{CBV}\left([T_0,T];\mathbb{R}\right)$ such that for $s,t \in [T_0,T]$ and $x \in H$

$$|d(x,C(t)) - d(x,C(s))| \leq |v(t)-v(s)|.$$

($\mathcal{H}_2$) There exists $\kappa \geq 0$ such that for all $s,t \in [T_0,T]$ and all $x \in H$

$$|d(x,C(t)) - d(x,C(s))| \leq \kappa|t-s|.$$

($\mathcal{H}_3$) There exist two constants $\alpha \in]0,1]$ and $\rho \in]0,+\infty]$ such that

$$0 < \alpha \leq \inf_{x \in U_\rho(C(t))} d\left(0, \partial d(x,C(t))\right) \quad \text{a.e. } t \in [T_0,T],$$

where $U_\rho\left(C(t)\right) = \{x \in H \colon 0 < d(x,C(t)) < \rho\}$ for all $t \in [T_0,T]$.

($\mathcal{H}_4$) For a.e. $t \in [T_0,T]$ the set $C(t)$ is ball compact, that is, for every $r > 0$ the set $C(t) \cap r\mathbb{B}$ is compact in H.

($\mathcal{H}_5$) For a.e. $t \in [T_0,T]$ the set $C(t)$ is r-uniformly prox-regular for some $r > 0$.

Hypotheses on the set-valued map $C\colon [T_0,T]\times H \rightrightarrows H$: C is a set-valued map with nonempty and closed values. Also, we will consider the following conditions:

($\mathcal{H}_6$) There exists $v \in \mathrm{CBV}\left([T_0,T];\mathbb{R}\right)$ and $L \in [0,1[$ such that for $s,t \in [T_0,T]$ and $x,y,z \in H$

$$|d(z,C(t,x)) - d(z,C(s,y))| \leq |v(t)-v(s)| + L\|x-y\|.$$

($\mathcal{H}_7$) There exist $\kappa \geq 0$ and $L \in [0,1[$ such that for $s,t \in [T_0,T]$ and $x,y,z \in H$

$$|d(z,C(t,x)) - d(z,C(s,y))| \leq \kappa|t-s| + L\|x-y\|.$$

($\mathcal{H}_8$) There exist constants $\alpha \in]0,1]$ and $\rho \in]0,+\infty]$ such that for every $y \in H$

$$0 < \alpha \leq \inf_{x \in U_\rho(C(t,y))} d\left(0, \partial d(\cdot,C(t,y))(x)\right) \quad \text{a.e. } t \in [T_0,T],$$

where $U_\rho\left(C(t,y)\right) = \{x \in H \colon 0 < d(x,C(t,y)) < \rho\}$.

($\mathcal{H}_9$) The family $\{C(t,v)\colon (t,v) \in [T_0,T] \times H\}$ is equi-uniformly subsmooth.

($\mathcal{H}_{10}$) There exists $k \in L^1(T_0,T)$ such that for every $t \in [T_0,T]$, every $r > 0$ and every bounded set $A \subseteq H$,

$$\gamma\left(C(t,A) \cap r\mathbb{B}\right) \leq k(t)\gamma(A),$$

where $\gamma = \alpha$ or $\gamma = \beta$ is either the Kuratowski or the Hausdorff measure of noncompactness (see Proposition 1.8) and $k(t) < 1$ for all $t \in [T_0,T]$.

Remark 5.1 The following comments are important.

(i) Let $L \in [0,1[$. Under $(\mathcal{H}_9)$ for every $\alpha \in]\sqrt{L}, 1]$ there exists $\rho > 0$ such that $(\mathcal{H}_8)$ holds. This follows from Proposition 2.8.

(ii) It is not difficult to prove that $(\mathcal{H}_5)$ implies $(\mathcal{H}_3)$ with $\alpha = 1$ and $\rho = r$.

(iii) If $C(t,x) := C(t)$ for every $(t,x) \in [T_0,T] \times H$. Then $(\mathcal{H}_{10})$ implies $(\mathcal{H}_4)$. Indeed, fix $t \in [T_0,T]$ and $r > 0$. Then, for a fixed $x \in H$, we have

$$\gamma\left(C(t,\{x\}) \cap r\mathbb{B}\right) = \gamma\left(C(t) \cap r\mathbb{B}\right) \leq k(t)\gamma\left(\{x\}\right) = 0,$$

which implies, since $C(t)$ is closed, that $C(t) \cap r\mathbb{B}$ is compact.

(iv) The condition $L \in [0,1[$ in $(\mathcal{H}_6)$ and $(\mathcal{H}_7)$ cannot be dispensed with as it is shown in [97].

5.2 Preliminary lemmas

In this section we give some preliminary results needed through this chapter.

Lemma 5.1 *If $(\mathcal{H}_6)$, $(\mathcal{H}_9)$ and $(\mathcal{H}_{10})$ hold then, for all $t \in [T_0,T]$, the set-valued map $x \rightrightarrows \partial d(\cdot, C(t,x))(x)$ is upper semicontinuous from H into H_w.*

Proof. Fix $t \in [T_0,T]$ and $x \in H$.

I) Assume that $x \in C(t,x)$: Due to [9, Theorem 17.35], it is enough to prove that $x \rightrightarrows \partial_L d(\cdot, C(t,x))(x)$ is sequentially upper semicontinuous from H into H_w at x. Let $x_n \to x$ and $x_n^* \rightharpoonup x^*$ with $x_n^* \in \partial_L d_{C(t,x_n)}(x_n)$. We have to prove that $x^* \in \partial_L d_{C(t,x)}(x)$. Indeed, for every $n \in \mathbb{N}$ where $x_n \notin C(t,x_n)$ (see Lemma 1.2) we have

$$x_n^* = \frac{x_n - y_n}{d_{C(t,x_n)}(x_n)} \in \partial_L d_{C(t,x_n)}(y_n), \tag{5.1}$$

for some $y_n \in \operatorname{Proj}_{C(t,x_n)}(x_n)$. Then, for each $n \in \mathbb{N}$, we define

$$\hat{x}_n = \begin{cases} x_n, & \text{if } x_n \in C(t,x_n), \\ y_n, & \text{if } x_n \notin C(t,x_n), \end{cases}$$

where $y_n \in H$ is given by (5.1). Thus, $\hat{x}_n \to x$, $x_n^* \rightharpoonup x^*$, $\hat{x}_n \in C(t,x_n)$ and $x_n^* \in \partial_L d_{C(t,x_n)}\left(\hat{x}_n\right)$. Therefore, using $(\mathcal{H}_9)$ and [118, Lemma 2.2.2], we obtain that $x^* \in \partial d_{C(t,x)}(x)$.

II) Assume that $x \notin C(t,x)$: Due to Lemma 1.15 and [9, Theorem 17.35], it is enough to prove that $x \rightrightarrows \operatorname{Proj}_{C(t,x)}(x)$ is sequentially upper semicontinuous from H into H_w at x. Indeed, let $x_n \to x$ and $x_n^* \rightharpoonup x^*$ with $x_n^* \in \operatorname{Proj}_{C(t,x_n)}(x_n)$. We have to prove that $x^* \in \mathrm{Proj}_{C(t,x)}(x)$. Indeed, due to $(\mathcal{H}_{10})$, $(x_n^*)_n$ is relatively compact and, thus, $x_n^* \to x^*$ up to a subsequence. Moreover,

$$\begin{aligned}\|x - x^*\| &\leq \|x - x_n\| + d_{C(t,x_n)}(x_n) + \|x_n^* - x^*\| \\ &\leq (1+L)\|x - x_n\| + d_{C(t,x)}(x_n),\end{aligned}$$

which shows that $\|x - x^*\| \leq d_{C(t,x)}(x)$. Also,

$$d_{C(t,x)}(x^*) = d_{C(t,x)}(x^*) - d_{C(t,x_n)}(x_n^*) \leq L\|x - x_n\| + \|x^* - x_n^*\|,$$

which shows that $x^* \in C(t,x)$. □

The next lemma gives some properties and estimations of the distance function to a moving set depending on the state.

Lemma 5.2 *Let $x, y \in W^{1,1}([T_0,T]; H)$ and let $C\colon [T_0,T] \times H \rightrightarrows H$ be a set-valued map with nonempty closed values satisfying $(\mathcal{H}_7)$. Then,*

(i) The function $t \to d(x(t), C(t,y(t)))$ is absolutely continuous over $[T_0,T]$.

(ii) For all $t \in [T_0,T]$, where $\dot{y}(t)$ exists,

$$\begin{aligned}&\limsup_{s\downarrow 0} \frac{d_{C(t+s,y(t+s))}(x(t+s)) - d_{C(t,y(t))}(x(t))}{s} \\ &\leq \kappa + L\|\dot{y}(t)\| + \limsup_{s\downarrow 0} \frac{d_{C(t,y(t))}(x(t+s)) - d_{C(t,y(t))}(x(t))}{s}.\end{aligned}$$

(iii) For all $t \in [T_0,T]$, where $\dot{x}(t)$ exists,

$$\limsup_{s\downarrow 0} \frac{d_{C(t,y(t))}(x(t+s)) - d_{C(t,y(t))}(x(t))}{s} \leq \max_{y^* \in \partial d(x(t),C(t,y(t)))} \langle y^*, \dot{x}(t)\rangle .$$

(iv) For all $t \in \{s \in [T_0,T]\colon x(s) \notin C(s,y(s))\}$, where $\dot{x}(t)$ exists,

$$\lim_{s\downarrow 0} \frac{d_{C(t,y(t))}(x(t+s)) - d_{C(t,y(t))}(x(t))}{s} = \min_{y^* \in \partial d(x(t),C(t,y(t)))} \langle y^*, \dot{x}(t)\rangle .$$

(v) For every $x \in H$ the set-valued map $t \rightrightarrows \partial d(\cdot, C(t,y(t)))(x)$ is measurable.

Proof. Let $\psi\colon [T_0,T] \to \mathbb{R}$ be the function defined by $\psi(t) := d(x(t), C(t,y(t)))$.

(i) It follows directly from $(\mathcal{H}_7)$.

(ii) Let $t \in]T_0, T[$ where $\dot{y}(t)$ exists. Then, for $s > 0$ small enough,

$$\begin{aligned}\frac{\psi(t+s)-\psi(t)}{s} &= \frac{d(x(t+s),C(t+s,y(t+s)))-d(x(t+s),C(t,y(t)))}{s} \\ &+ \frac{d(x(t+s),C(t,y(t)))-d(x(t),C(t,y(t)))}{s} \\ &\leq \kappa + L\frac{\|y(t+s)-y(t)\|}{s} \\ &+ \frac{d(x(t+s),C(t,y(t)))-d(x(t),C(t,y(t)))}{s},\end{aligned}$$

and taking the superior limit, we get the desired inequality.

(iii) Let $t \in [T_0, T]$ be such that $\dot{x}(t)$ exists. Let $s_n \downarrow 0$ be such that

$$\begin{aligned}&\limsup_{s\downarrow 0}\frac{d(x(t+s),C(t,y(t)))-d(x(t),C(t,y(t)))}{s} \\ &\qquad = \lim_{n\to+\infty}\frac{d(x(t+s_n),C(t,y(t)))-d(x(t),C(t,y(t)))}{s_n}.\end{aligned}$$

By virtue of Lebourg's mean value theorem [55, Theorem 2.2.4], there exist $z_n \in]x(t), x(t+s_n)[$ and $\xi_n \in \partial d(z_n, C(t, y(t)))$ such that

$$\frac{1}{s_n}\left(d(x(t+s_n),C(t,y(t)))-d(x(t),C(t,y(t)))\right) = \frac{1}{s_n}\left\langle \xi_n, x(t+s_n)-x(t)\right\rangle.$$

Since $\|\xi_n\| \leq 1$, there is a subsequence (without relabeling) of $(\xi_n)_n$ such that $\xi_n \rightharpoonup \xi \in \partial d(x(t), C(t, y(t)))$. Thus, taking the limit in the last equality we obtain the result.

(iv) Let $t \in \{s \in [T_0, T] \colon x(s) \notin C(s, y(s))\}$ where $\dot{x}(t)$ exists. Then, for $s > 0$ small enough,

$$\begin{aligned}&\frac{1}{s}\left(d(x(t+s),C(t,y(t)))-d(x(t),C(t,y(t)))\right) \\ &= \frac{1}{s}\left(d(x(t)+s\dot{x}(t)+s\varepsilon(s,t),C(t,y(t)))-d(x(t),C(t,y(t)))\right) \\ &= \frac{1}{s}\left(d(x(t)+s\dot{x}(t),C(t,y(t)))-d(x(t),C(t,y(t)))\right)+\eta(s,t),\end{aligned}$$

for some mappings $\varepsilon(\cdot, t)$ and $\eta(\cdot, t)$ with $\lim_{s\downarrow 0} \varepsilon(s, t) = \lim_{s\downarrow 0} \eta(s, t) = 0$. Then, by using Lemma 1.16, we get

$$\begin{aligned}&\lim_{s\downarrow 0}\frac{d(x(t+s),C(t,y(t)))-d(x(t),C(t,y(t)))}{s} \\ &= \lim_{s\downarrow 0}\frac{d(x(t)+s\dot{x}(t),C(t,y(t)))-d(x(t),C(t,y(t)))}{s} \\ &= \min_{y^*\in\partial d(x(t),C(t,y(t)))}\left\langle y^*, \dot{x}(t)\right\rangle.\end{aligned}$$

(v) See Lemma 1.18.

□

The following result shows that the set-valued map $(t,x) \rightrightarrows \frac{1}{2}\partial d^2_{C(t,x)}(x)$ satisfies the conditions of Theorem 1.14.

Proposition 5.3 *Assume that $(\mathcal{H}_6)$, $(\mathcal{H}_9)$ and $(\mathcal{H}_{10})$ hold. Then, the set-valued map $G\colon [T_0,T]\times H \rightrightarrows H$ defined by $G(t,x) := \frac{1}{2}\partial d^2_{C(t,x)}(x)$ satisfies:*

(i) for all $x \in H$ and all $t \in [T_0,T]$, $G(t,x) = x - \operatorname{cl}\operatorname{co}\operatorname{Proj}_{C(t,x)}(x)$.

(ii) for every $x \in H$ the set-valued map $G(\cdot,x)$ is measurable.

(iii) for every $t \in [T_0,T]$, $G(t,\cdot)$ is upper semicontinuous from H into H_w.

(iv) for every $t \in [T_0,T]$ and $A \subseteq H$ bounded,

$$\gamma\left(G(t,A)\right) \leq (1+k(t))\gamma\left(A\right),$$

where $\gamma = \alpha$ or $\gamma = \beta$ is the Kuratowski or the Hausdorff measure of noncompactness of A and $k \in L^1(T_0,T)$ is given by $(\mathcal{H}_{10})$.

(v) Let $x_0 \in C(T_0,x_0)$. Then, for all $t \in [T_0,T]$ and $x \in H$,

$$\|G(t,x)\| := \sup\left\{\|w\| \colon w \in G(t,x)\right\} \leq (1+L)\|x-x_0\| + |v(t)-v(T_0)|.$$

Proof. (i), (ii) and (iii) follow, respectively, from Lemma 1.15, (v) of Lemma 5.2 and Lemma 5.1. To prove (iv), let $A \subseteq H$ be a bounded set included in the ball $r\mathbb{B}$, for some $r > 0$. Define the set-valued map $F(t,x) := \operatorname{Proj}_{C(t,x)}(x)$. Then, for every $t \in [T_0,T]$, $\|F(t,A)\| := \sup\{\|w\| \colon w \in F(t,A)\} \leq \tilde{r}(t)$, where $\tilde{r}(t) := (2+L)r + (1+L)\|x_0\| + |v(t)-v(T_0)|$. Indeed, let $z \in F(t,A)$, then there exists $x \in A$ such that $z \in \operatorname{Proj}_{C(t,x)}(x)$. Thus,

$$\begin{aligned}\|z\| &\leq d_{C(t,x)}(x) - d_{C(T_0,x_0)}(x_0) + \|x\| \\ &\leq (1+L)\|x-x_0\| + |v(t)-v(T_0)| + \|x\| \\ &\leq (2+L)r + (1+L)\|x_0\| + |v(t)-v(T_0)| = \tilde{r}(t).\end{aligned}$$

Therefore,

$$\begin{aligned}\gamma\left(G(t,A)\right) &\leq \gamma(A) + \gamma\left(\operatorname{cl}\operatorname{co}F(t,A)\right) \\ &= \gamma(A) + \gamma\left(F(t,A) \cap \tilde{r}(t)\mathbb{B}\right) \\ &\leq \gamma(A) + \gamma\left(C(t,A) \cap \tilde{r}(t)\mathbb{B}\right) \\ &\leq (1+k(t))\gamma(A),\end{aligned}$$

where the last equality is due to $(\mathcal{H}_{10})$.

To prove (v), define $\tilde{G}(t,x) := x - \operatorname{Proj}_{C(t,x)}(x)$. Then, due to $(\mathcal{H}_6)$,

$$\|\tilde{G}(t,x)\| = d(x, C(t,x)) - d(x_0, C(T_0,x_0)) \leq (1+L)\|x-x_0\| + |v(t)-v(T_0)|.$$

By passing to the closed convex hull in the last inequality, we get the result. □

When the sets $C(t,x)$ are independent of x, the subsmoothness in Proposition 5.3 is no longer required. The following result follows in the same way as Proposition 5.3.

Proposition 5.4 *Assume that* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_4)$ *hold. Then, the set-valued map* $G\colon [T_0,T]\times H \rightrightarrows H$ *defined by* $G(t,x) := \frac{1}{2}\partial d^2_{C(t)}(x)$ *satisfies:*

(i) for all $x\in H$ *and all* $t\in[T_0,T]$, $G(t,x) = x - \operatorname{cl}\operatorname{co}\operatorname{Proj}_{C(t)}(x)$.

(ii) for every $x\in H$ *the set-valued map* $G(\cdot,x)$ *is measurable.*

(iii) for every $t\in[T_0,T]$, $G(t,\cdot)$ *is upper semicontinuous from* H *into* H_w.

(iv) for all $t\in[T_0,T]$ *and all* $A\subseteq H$ *bounded,*

$$\gamma\left(G(t,A)\right) \leq \gamma\left(A\right),$$

where $\gamma=\alpha$ *or* $\gamma=\beta$ *is either the Kuratowski or the Hausdorff measure of noncompactness of* A.

(v) Let $x_0\in C(T_0)$. *Then, for all* $t\in[T_0,T]$ *and* $x\in H$,

$$\|G(t,x)\| := \sup\left\{\|w\|\colon w\in G(t,x)\right\} \leq \|x-x_0\| + |v(t)-v(T_0)|.$$

5.3 The concept of Solution

In this section, we define the notion of solution for the state-dependent sweeping process in the sense of differential measures. Through this section, we put $I = [T_0,T]$. Let $x\colon I\to H$ be a function of bounded variation and denote by dx the differential vector measure associated with x (see [63]). If x is right continuous, this measure satisfies $x(t) = x(s) + \int_{]s,t]} dx$ for all $s,t\in I$ with $z\leq t$. Conversely, if there exists some mapping $\hat{x}\in L^1_\nu(I;H)$ such that $x(t) = x(T_0) + \int_{]T_0,t]}\hat{x}d\nu$ for all $t\in I$, then x is of bounded variation and right continuous. For the associated differential vector measure dx it is known that its variation measure $|dx|$ satisfies $|dx|(]s,t]) = \int_{]s,t]}\|\hat{x}(\tau)\|d\nu(\tau)$ for all $s,t\in I$ with $s\leq t$, dx is absolutely continuous with respect to ν and admits $\hat{x}$ as a density relative to ν, that is, $dx = \hat{x}(\cdot)d\nu$.

Now we define the notion of solution of the state-dependent sweeping process in the sense of differential measures. The following definition is based on [137, Definition 2.1].

Definition 5.5 *Let $C\colon I\times H \rightrightarrows H$ be a set-valued map with nonempty closed values. We say that $x\colon I\to H$ is a solution of the state-dependent sweeping process*

$$\begin{cases} -dx \in N\left(C(t,x(t));x(t)\right), \\ x(T_0)=x_0\in C(T_0,x_0), \end{cases} \tag{$\mathcal{BVSP}$}$$

in the sense of differential measure, provided there is $L\in[0,1[$ and a positive Radon measure μ on I satisfying, for all $s\le t$ in I and $x,y\in H$,

$$\sup_{z\in H}\left|d_{C(s,x)}(z)-d_{C(t,y)}(z)\right|\le \mu(]s,t])+L\|x-y\|,$$

and such that the following conditions hold:

(i) The mapping $x(\cdot)$ is of bounded variation on I, right continuous, and satisfies $x(T_0)=x_0$ and $x(t)\in C(t,x(t))$ for all $t\in I$.

(ii) There exists a positive Radon measure ν absolutely continuously equivalent to μ and with respect to which the differential measure dx of $x(\cdot)$ is absolutely continuous with $\frac{dx}{d\nu}(\cdot)$ as an $L^1_\nu(I;H)$-density and

$$-\frac{dx}{d\nu}(t)\in N(C(t,x(t));x(t)) \quad \nu\text{-a.e. } t\in[T_0,T].$$

5.4 Existence through Moreau-Yosida regularization

In this section, we prove the existence of solutions in the sense of differential measures for ($\mathcal{BVSP}$). To do that, we prove the existence of Lipschitz solutions of the classical state-dependent sweeping process

$$\begin{cases} -\dot{x}(t)\in N\left(C(t,x(t));x(t)\right) \quad \text{a.e. } t\in[T_0,T], \\ x(T_0)=x_0\in C(T_0,x_0). \end{cases} \tag{$\mathcal{SP}$}$$

Then, by means of a reparametrization technique we obtain the existence of ($\mathcal{BVSP}$).

Let $\lambda>0$ and consider the following differential inclusion

$$\begin{cases} -\dot{x}_\lambda(t)\in \dfrac{1}{2\lambda}\partial d^2_{C(t,x_\lambda(t))}\left(x_\lambda(t)\right) \quad \text{a.e. } t\in[T_0,T], \\ x_\lambda(T_0)=x_0\in C(T_0,x_0). \end{cases} \tag{$\mathcal{P}_\lambda$}$$

The following proposition follows from Theorem 1.14 and Proposition 5.3.

Proposition 5.6 *Assume that ($\mathcal{H}_7$), ($\mathcal{H}_9$) and ($\mathcal{H}_{10}$) hold. Then, for every $\lambda>0$ there exists at least one absolutely continuous solution x_λ of ($\mathcal{P}_\lambda$).*

Let us define $\varphi_\lambda(t) := d_{C(t,x_\lambda(t))}(x_\lambda(t))$ for $t \in [T_0, T]$.

Remark 5.2 Recall that under $(\mathcal{H}_9)$, according to Proposition 2.8, for every $\alpha \in]\sqrt{L}, 1]$ there exists $\rho > 0$ such that $(\mathcal{H}_8)$ holds.

Proposition 5.7 *Under the hypotheses of Proposition* 5.6*, if* $\lambda < \frac{(\alpha^2 - L)\rho}{\kappa}$*, then*

$$\dot{\varphi}_\lambda(t) \leq \kappa + \frac{L - \alpha^2}{\lambda}\varphi_\lambda(t) \quad a.e.\ t \in [T_0, T], \tag{5.2}$$

where $\alpha \in]\sqrt{L}, 1]$ *and* $\rho > 0$ *are given by Remark* 5.2*. Moreover,*

$$\varphi_\lambda(t) \leq \frac{\kappa\lambda}{\alpha^2 - L} \quad \text{for all } t \in [T_0, T]. \tag{5.3}$$

Proof. According to Proposition 5.6, the function x_λ is absolutely continuous. Thus, due to $(\mathcal{H}_7)$, for every $t, s \in [T_0, T]$

$$|\varphi_\lambda(t) - \varphi_\lambda(s)| \leq (1 + L)\|x_\lambda(t) - x_\lambda(s)\| + \kappa|t - s|,$$

which implies the absolute continuity of φ_λ. On one hand, let $t \in [T_0, T]$ where $\varphi_\lambda(t) \in]0, \rho[$ and $\dot{x}_\lambda(t)$ exists. Then, by using Lemma 5.2, we have

$$\begin{aligned}
\dot{\varphi}_\lambda(t) &\leq \kappa + L\|\dot{x}_\lambda(t)\| + \min_{w \in \partial d_{C(t,x_\lambda(t))}(x_\lambda(t))} \langle w, \dot{x}_\lambda(t)\rangle \\
&\leq \kappa + \frac{L}{\lambda}\varphi_\lambda(t) - \frac{\alpha^2}{\lambda}\varphi_\lambda(t) \\
&= \kappa - \frac{\alpha^2 - L}{\lambda}\varphi_\lambda(t),
\end{aligned}$$

where we have used $(\mathcal{H}_9)$ and Proposition 2.8.

On the other hand, let $t \in \varphi_\lambda^{-1}(\{0\})$ where $\dot{x}_\lambda(t)$ exists. Then, according to $(\mathcal{P}_\lambda)$, $\|\dot{x}_\lambda(t)\| = 0$. Indeed,

$$\|\dot{x}_\lambda(t)\| \leq \frac{1}{2\lambda}\sup\{\|z\| \colon z \in \partial d^2_{C(t,x_\lambda(t))}(x_\lambda(t))\} \leq \frac{\varphi_\lambda(t)}{\lambda} = 0,$$

where we have used the identity $\partial d_S^2(x) = 2d_S(x)\partial d_S(x)$. Then, due to $(\mathcal{H}_7)$,

$$\begin{aligned}
\dot{\varphi}_\lambda(t) &= \lim_{h\downarrow 0}\frac{1}{h}\left(d_{C(t+h,x_\lambda(t+h))}(x_\lambda(t+h)) - d_{C(t,x_\lambda(t))}(x_\lambda(t+h))\right) \\
&\quad + d_{C(t,x_\lambda(t))}(x_\lambda(t+h)) \\
&\leq \kappa + L\|\dot{x}_\lambda(t)\| + \lim_{h\downarrow 0}\frac{1}{h}d_{C(t,x_\lambda(t))}(x_\lambda(t+h)) \\
&\leq \kappa + (1 + L)\|\dot{x}_\lambda(t)\| \\
&\leq \kappa + \frac{1 + L}{\lambda}\varphi_\lambda(t) \\
&= \kappa - \frac{\alpha^2 - L}{\lambda}\varphi_\lambda(t).
\end{aligned}$$

Also, we have that $\varphi_\lambda(t) < \rho$ for all $t \in [T_0, T]$. Otherwise, since $\varphi_\lambda^{-1}(]-\infty,\rho[)$ is open and $T_0 \in \varphi_\lambda^{-1}(]-\infty,\rho[)$, there would exist $t^* \in]T_0, T]$ such that $[T_0, t^*[\subseteq \varphi_\lambda^{-1}(]-\infty,\rho[)$ and $\varphi_\lambda(t^*) = \rho$. Then,

$$\dot{\varphi}_\lambda(t) \leq \kappa - \frac{\alpha^2 - L}{\lambda}\varphi_\lambda(t) \quad \text{a.e. } t \in [T_0, t^*[,$$

which, by virtue of Grönwall's inequality (see Lemma 1.17), entrain that, for every $t \in [T_0, t^*[$

$$\varphi_\lambda(t) \leq \frac{\kappa\lambda}{\alpha^2 - L}\left(1 - \exp\left(-\frac{\alpha^2 - L}{\lambda}t\right)\right) \leq \frac{\kappa\lambda}{\alpha^2 - L} < \rho,$$

that implies that $\varphi_\lambda(t^*) < \rho$, which is not possible.

Thus, we have proved that φ_λ satisfies (5.2) and (5.3). □

As a corollary to the last proposition, we obtain that x_λ is $\frac{\kappa}{\alpha^2-L}$-Lipschitz.

Corollary 5.8 *For every $\lambda > 0$ the function x_λ is $\frac{\kappa}{\alpha^2-L}$-Lipschitz.*

Proof. Since x_λ satisfies $(\mathcal{P}_\lambda)$, we have

$$\|\dot{x}_\lambda(t)\| \leq \frac{1}{2\lambda}\sup\{\|z\| \colon z \in \partial d^2_{C(t,x_\lambda(t))}(x_\lambda(t))\} \leq \frac{\varphi_\lambda(t)}{\lambda},$$

where we have used the identity $\partial d_S^2(x) = 2d_S(x)\partial d_S(x)$. Consequently, by using (5.3), for a.e. $t \in [T_0, T]$, $\|\dot{x}_\lambda(t)\| \leq \frac{\varphi_\lambda(t)}{\lambda} \leq \frac{\kappa}{\alpha^2-L}$, which proves that x_λ is $\frac{\kappa}{\alpha^2-L}$-Lipschitz.

□

Let $(\lambda_n)_n$ be a sequence converging to 0. The next result shows the existence of a subsequence $(\lambda_{n_k})_k$ of $(\lambda_n)_n$ such that $\left(x_{\lambda_{n_k}}\right)_k$ converges (in the sense of Lemma 1.6) to a solution of $(\mathcal{SP})$. A similar result was proved by Noel in [118, Theorem 5.2.1] (with a stronger compactness condition on the sets $C(t,x)$) by using a very different approach.

Theorem 5.9 *Assume that $(\mathcal{H}_7)$, $(\mathcal{H}_9)$ and $(\mathcal{H}_{10})$ hold. Then, there exists at least one solution $x \in \operatorname{Lip}([T_0, T]; H)$ of $(\mathcal{SP})$. Moreover, $\|\dot{x}(t)\| \leq \frac{\kappa}{\alpha^2-L}$ for a.e. $t \in [T_0, T]$.*

Proof. According to Proposition 5.7, the sequence $(x_{\lambda_n})_n$ satisfies the hypotheses of Lemma 1.6 with $\psi(t) := \frac{k}{\alpha^2-L}$. Therefore, there exists a subsequence $(x_{\lambda_{n_k}})_k$ of $(x_{\lambda_n})_n$ and a function $x \colon [T_0, T] \to H$ satisfying the hypotheses (i)-(iv) of Lemma 1.6. For simplicity, we write x_k instead of $x_{\lambda_{n_k}}$ for all $k \in \mathbb{N}$.

Claim 1: $(x_k(t))_k$ is relatively compact in H for all $t \in [T_0, T]$.

Proof of Claim 1: Let $t \in [T_0, T]$. Let us consider $y_k(t) \in \operatorname{Proj}_{C(t,x_k(t))}(x_k(t))$. Then, $\|x_k(t) - y_k(t)\| = d_{C(t,x_k(t))}(x_k(t))$. Thus,

$$\begin{aligned}\|y_k(t)\| &\leq d_{C(t,x_k(t))}(x_k(t)) + \|x_k(t)\| \\ &\leq \frac{\kappa\lambda_{n_k}}{\alpha^2 - L} + \|x_k(t) - x_0\| + \|x_0\| \\ &\leq \tilde{r}(t) := \frac{\kappa}{\alpha^2 - L}(\lambda_{n_k} + (t - T_0)) + \|x_0\|.\end{aligned}$$

Also, since $(x_k(t) - y_k(t))$ converges to 0,

$$\gamma(\{x_k(t) \colon k \in \mathbb{N}\}) = \gamma(\{y_k(t) \colon k \in \mathbb{N}\}).$$

Therefore, if $A := \{x_k(t) \colon k \in \mathbb{N}\}$,

$$\gamma(A) = \gamma(\{y_k(t) \colon k \in \mathbb{N}\}) \leq \gamma(C(t, A) \cap \tilde{r}(t)\mathbb{B}) \leq k(t)\gamma(A),$$

where we have used $(\mathcal{H}_{10})$. Finally, since $k(t) < 1$, we obtain that $\gamma(A) = 0$, which shows the result. □

Claim 2: $x(t) \in C(t, x(t))$ for all $t \in [T_0, T]$.

Proof of Claim 2: As a result of Claim 1 and the weak convergence $x_k(t) \rightharpoonup x(t)$ for all $t \in [T_0, T]$ (due to (i) of Lemma 1.6), we obtain the strong convergence of $(x_k(t))_k$ to $x(t)$ for all $t \in [T_0, T]$. Therefore, due to $(\mathcal{H}_7)$,

$$\begin{aligned}d_{C(t,x(t))}(x(t)) &\leq \liminf_{k\to\infty} \left(d_{C(t,x_k(t))}(x_k(t)) + (1 + L)\|x_k(t) - x(t)\|\right) \\ &\leq \liminf_{k\to\infty} \left(\frac{\kappa\lambda_{n_k}}{\alpha^2 - L} + (1 + L)\|x_k(t) - x(t)\|\right) = 0,\end{aligned}$$

which shows the claim. □

Now, we prove that x is a solution of $(\mathcal{SP})$. Define

$$\tilde{F}(t, x) := \operatorname{cl}\operatorname{co}\left(\frac{\kappa}{\alpha^2 - L}\partial d_{C(t,x)}(x) \cup \{0\}\right),$$

for $(t, x) \in [T_0, T] \times H$. Then, for a.e. $t \in [T_0, T]$

$$-\dot{x}_k(t) \in \frac{1}{2\lambda}\partial d^2_{C(t,x_k(t))}(x_k(t)) \subseteq \tilde{F}(t, x_k(t)),$$

where we have used Proposition 5.7.

Claim 3: $\tilde{F}$ has closed and convex values and satisfies:

(i) for each $x \in H$, $\tilde{F}(\cdot, x)$ is measurable;

(ii) for all $t \in [T_0, T]$, $\tilde{F}(t, \cdot)$ is upper semicontinuous from H into H_w;

(iii) if $x \in C(t, x)$ then $\tilde{F}(t, x) = \frac{\kappa}{\alpha^2 - L}\partial d_{C(t,x)}(x)$.

Proof of Claim 3: Define $G(t,x) := \frac{\kappa}{\alpha^2-L}\partial d_{C(t,x)}(x) \cup \{0\}$. We note that $G(\cdot,x)$ is measurable as the union of two measurable set-valued maps (see [9, Lemma 18.4]). Let us define $\Gamma(t) := \tilde{F}(t,x)$. Then, Γ takes weakly compact convex values. Fixing any $d \in H$, by virtue of [85, Proposition 2.2.39], is enough to verify that the support function $t \mapsto \sigma(d,\Gamma(t)) := \sup\{\langle v,d\rangle : v \in \Gamma(t)\}$ is measurable. Thus,

$$\sigma(d,\Gamma(t)) := \sup\{\langle v,d\rangle : v \in \Gamma(t)\} = \sup\{\langle v,d\rangle : v \in G(t,x)\},$$

is measurable because $G(\cdot,x)$ is measurable. Thus (i) holds. Assertion (ii) follows directly from [9, Theorem 17.27 and 17.3]. Finally, if $x \in C(t,x)$ then $0 \in \partial d_{C(t,x)}(x)$. Hence, using the fact that the subdifferential of a locally Lipschitz function is closed and convex,

$$\tilde{F}(t,x) = \operatorname{cl}\operatorname{co}\left(\frac{\kappa}{\alpha^2-L}\partial d_{C(t,x)}(x)\right) = \frac{\kappa}{\alpha^2-L}\partial d_{C(t,x)}(x),$$

which shows (iii). □

In summary, we have

(i) for each $x \in H$, $\tilde{F}(\cdot,x)$ is measurable.

(ii) for all $t \in [T_0,T]$, $\tilde{F}(t,\cdot)$ is upper semicontinuous from H into H_w.

(iii) $\dot{x}_k \rightharpoonup \dot{x}$ in $L^1([T_0,T];H)$ as $k \to +\infty$.

(iv) $x_k(t) \to x(t)$ as $k \to +\infty$ for all $t \in [T_0,T]$.

(v) $-\dot{x}_k(t) \in \tilde{F}(t,x_k(t))$ for a.e. $t \in [T_0,T]$.

These conditions and the convergence theorem (see [10, p.60] for more details) imply that x satisfies

$$\begin{cases} -\dot{x}(t) \in \tilde{F}(t,x(t)) & \text{a.e. } t \in [T_0,T],\\ x(T_0) = x_0 \in C(T_0,x_0), \end{cases}$$

which, according to Claim 3, implies that x is a solution of

$$\begin{cases} -\dot{x}(t) \in \dfrac{\kappa}{\alpha^2-L}\partial d_{C(t,x(t))}(x(t)) & \text{a.e. } t \in [T_0,T],\\ x(T_0) = x_0 \in C(T_0,x_0). \end{cases}$$

Therefore, by virtue of (1.1) and Claim 2, x is a solution of $(\mathcal{SP})$. Finally, since $\|\dot{x}(t)\| \leq \frac{\kappa}{\alpha^2-L}$ for a.e. $t \in [T_0,T]$, x is $\frac{\kappa}{\alpha^2-L}$-Lipschitz continuous. □

Now, from Theorem 5.9 and by means of a reparametrization technique, we will deduce the existence of solutions for $(\mathcal{BVSP})$. The following theorem extends all the known existence results for $(\mathcal{BVSP})$.

Theorem 5.10 *Assume that* $(\mathcal{H}_6)$, $(\mathcal{H}_9)$ *and* $(\mathcal{H}_{10})$ *hold. Then, there exists at least one solution* $x \in \operatorname{CBV}([T_0,T];H)$ *of* $(\mathcal{BVSP})$*. Moreover, this solution satisfies* $\operatorname{Var}(x,[T_0,T]) \leq \frac{\operatorname{Var}(v,[T_0,T])}{\alpha^2-L}$.

Proof. Without loss of generality, we can assume that the function v from $(\mathcal{H}_6)$ is strictly increasing. Indeed, if $T_0 \leq t_1 \leq t_2 \leq T$

$$\begin{aligned}|v(t_1) - v(t_2)| &\leq \operatorname{Var}(v, [t_1, t_2]) \\ &= \operatorname{Var}(v, [T_0, t_2]) - \operatorname{Var}(v, [T_0, t_1]) \\ &\leq v_\varepsilon(t_2) - v_\varepsilon(t_1),\end{aligned}$$

where $v_\varepsilon(t) := \operatorname{Var}(v, [T_0, t]) + \varepsilon(t - T_0)$, for $\varepsilon > 0$, is a strictly increasing function. Accordingly, by Proposition 1.19, there exists $V \in \operatorname{Lip}([T_0, T]; H)$ such that $v = V \circ \ell_v$ and $\operatorname{Lip}(V) \leq \frac{\operatorname{Var}(v,[T_0,T])}{(T-T_0)}$. Moreover, as v is continuous and strictly increasing, the arc-length ℓ_v is continuous, strictly increasing and $\ell_v([T_0, T]) = [T_0, T]$. Therefore, $\ell_v^{-1} \colon [T_0, T] \to [T_0, T]$ is continuous, strictly increasing and with bounded variation.

Let us consider $\tilde{C} \colon [T_0, T] \times H \rightrightarrows H$ defined by $\tilde{C}(t, x) = C(\ell_v^{-1}(t), x)$. Then, $\tilde{C}$ satisfies $(\mathcal{H}_7)$ with $\kappa = \operatorname{Lip}(V)$. Indeed, for $t \in [T_0, T]$ and $x, y, z \in H$,

$$\begin{aligned}\left|d(z, \tilde{C}(t, x)) - d(z, \tilde{C}(s, y))\right| &\leq |v \circ \ell_v^{-1}(t) - v \circ \ell_v^{-1}(s)| + L\|x - y\| \\ &= |V(t) - V(s)| + L\|x - y\| \\ &\leq \operatorname{Lip}(V)|t - s| + L\|x - y\|.\end{aligned}$$

Thus, due to Theorem 5.9, there exists at least one solution $u \in \operatorname{Lip}([T_0, T]; H)$ of the differential inclusion:

$$\begin{cases} -\dot{u}(t) \in N(\tilde{C}(t, u(t)), u(t)) & \text{a.e. } t \in [T_0, T], \\ u(T_0) = u_0 \in \tilde{C}(T_0, x_0), \end{cases} \tag{5.4}$$

with $\operatorname{Lip}(u) \leq \frac{\operatorname{Lip}(V)}{\alpha^2 - L}$. Let us consider the mapping $x \colon [T_0, T] \to H$ defined by $x(t) = u \circ \ell_v(t)$. Then, x is continuous with bounded variation. Indeed,

$$\begin{aligned}\operatorname{Var}(x, [T_0, T]) &\leq \operatorname{Lip}(u) \operatorname{Var}(\ell_v, [T_0, T]) \\ &\leq \frac{\operatorname{Lip}(V)}{\alpha^2 - L} \operatorname{Var}(\ell_v, [T_0, T]) \\ &\leq \frac{\operatorname{Var}(\ell_v, [T_0, T])}{T - T_0} \frac{\operatorname{Var}(v, [T_0, T])}{\alpha^2 - L} \\ &\leq \frac{\operatorname{Var}(v, [T_0, T])}{\alpha^2 - L}.\end{aligned}$$

Also, due to Proposition 1.20, $Dx = D(u \circ \ell_v) = (\dot{u} \circ \ell_v) D\ell_v$. Let us define $w := \dot{u} \circ \ell_v$ and $Z := \{t \in [T_0, T] \colon -\dot{u}(t) \notin N(\tilde{C}(t, u(t)); u(t))\}$.

Then, $\mathcal{L}^1(Z) = 0$ because of (5.4). Moreover,

$$\begin{aligned}&D\ell_v(\{t \in [T_0, T] \colon -w(t) \notin N(C(t, x(t)); x(t))\}) \\ &= D\ell_v\left(\{t \in [T_0, T] \colon -\dot{u}(\ell_v(t)) \notin N(\tilde{C}(\ell_v(t), u(\ell_v(t))); u(\ell_v(t)))\}\right) \\ &= D\ell_v(\{t \in [T_0, T] \colon \ell_v(t) \in Z\}) \\ &= D\ell_v\left(\ell_v^{-1}(Z)\right) \\ &= \mathcal{L}^1(Z) = 0,\end{aligned}$$

where we have used (i) from Proposition 1.20. Therefore, $x \in \mathrm{CBV}([T_0,T];H)$ is a solution of $(\mathcal{BVSP})$ in the sense of differential measures. $\square$

5.5 The Case of the sweeping process

This section is devoted to the measure differential inclusion:

$$\begin{cases} -dx \in N\left(C(t); x(t)\right), \\ x(T_0) = x_0 \in C(T_0), \end{cases} \tag{5.5}$$

and the classical sweeping process:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) & \text{a.e. } t \in [T_0,T], \\ x(T_0) = x_0 \in C(T_0). \end{cases} \tag{5.6}$$

These two differential inclusions can be seen, respectively, as a particular case of $(\mathcal{BVSP})$ and $(\mathcal{SP})$ when the sets $C(t,x)$ do not depend on the state. We show that Theorem 5.9 and Theorem 5.10 are valid under the weaker hypothesis $(\mathcal{H}_3)$ instead of $(\mathcal{H}_9)$. A similar result of the following was proved by Jourani and Vilches in [87] by using a very different approach.

Theorem 5.11 *Assume that $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ hold. Then, there exists at least one solution $x \in \mathrm{Lip}([T_0,T];H)$ of (5.6). Moreover, $\mathrm{Lip}(x) \leq \frac{\kappa}{\alpha^2}$.*

Proof. According to the proof of Theorem 5.9, we observe that $(\mathcal{H}_9)$ was used to obtain $(\mathcal{H}_8)$ and the upper semicontinuity of $\partial d_{C(t,\cdot)}(\cdot)$ from H into H_w for all $t \in [T_0,T]$. Since in the present case these two properties hold under $(\mathcal{H}_3)$ (see Proposition 5.4), it is sufficient to adapt the proof of Theorem 5.9 to get the result. $\square$

The following result follows in the same way as in the proof of Theorem 5.10.

Theorem 5.12 *Assume that $(\mathcal{H}_1)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ hold. Then, there exists at least one solution $x \in \mathrm{CBV}([T_0,T];H)$ of (5.5). Moreover, this solution satisfies $\mathrm{Var}(x,[T_0,T]) \leq \frac{\mathrm{Var}(v,[T_0,T])}{\alpha^2}$.*

Remark 5.3 When the sets $C(t)$ are convex or r-uniformly prox-regular it has been proved that the Moreau-Yosida regularization generates a family $(x_\lambda)_\lambda$ which converges uniformly in $C([T_0,T];H)$, as $\lambda \downarrow 0$, to the unique solution of (5.6) (see [96, 105, 130, 136] for more details). In particular, the following theorem holds.

Theorem 5.13 *Assume that $(\mathcal{H}_1)$ and $(\mathcal{H}_5)$ hold. Then, there exists a unique solution $x \in \mathrm{CBV}([T_0,T];H)$ of (5.5). Moreover, this solution satisfies*

$$\mathrm{Var}(x,[T_0,T]) \leq \mathrm{Var}(v,[T_0,T]).$$

5.5.1 The Finite-Dimensional Case

When H is a finite-dimensional Hilbert space, Benabdellah [19] and Colombo and Goncharov [57] proved, at almost the same time, the existence of solutions for the sweeping process (5.6) under merely $(\mathcal{H}_2)$ (see [135] for similar results).

Theorem 5.14 ([19, 57]) *Assume that* $(\mathcal{H}_2)$ *holds. Then, there exists at least one solution* $x \in \mathrm{Lip}([T_0, T]; H)$ *of* (5.6)*. Moreover,* $\mathrm{Lip}(x) \leq \kappa$*.*

From Theorem 5.14 and the reparametrization technique used in the proof of Proposition 5.10, we can prove the following result, which extends Theorem 5.12 to completely nonregular sets with continuous bounded variation.

Theorem 5.15 *Assume that* $(\mathcal{H}_1)$ *holds. Then, there exists at least one solution* $x \in \mathrm{CBV}([T_0, T]; H)$ *of* (5.5)*. Moreover,* $\mathrm{Var}(x, [T_0, T]) \leq \mathrm{Var}(v, [T_0, T])$*.*

Remark 5.4 When the sets $C(t)$ are r-uniformly prox-regular, Theorem 5.15 is well known, even in infinite-dimensional spaces (see [65, 127, 128] for more details).

5.6 An Application to Hysteresis

In this section, we study the so-called Play operator which arises in hysteresis and we extend the results given in [72, 124] to the class of positively α-far sets in finite-dimensional Hilbert spaces. Hysteresis occurs in phenomena such as plasticity, ferromagnetism, ferroelectricity, porous media filtration and behavior of thermostats (see [92] for more details). Several properties in hysteresis can be described in terms of some hysteresis operators. One of these hysteresis operators is the so-called Play operator [94, 124]. This operator can be defined as the solution of a differential inclusion associated with a fixed set $Z \subseteq H$. The case where the set Z is convex has been thoroughly studied (see for instance [95, 107, 124, 125]), whereas the nonconvex case has been only considered in [72] for uniformly prox-regular sets. The use of nonconvex sets is important in applications because, as Gudovich and Quincampoix stated in [72, Remark 3.7], when "the elastic properties change with plastic deformation, then a nonconvex yield surface cannot be excluded from consideration" and "its nonconvexity can be explained physically allowing irregularities, elastic-plastic interaction, and the granular character of the material" (see [72] and the references given there for a deeper discussion on the nonconvexity of the set under consideration and an example of a multidimensional Play operator). In the aforementioned chapter [72] the authors construct the Play operator, with Z uniformly prox-regular set, for only Lipschitz inputs while by using Theorem 5.12 we can easily define the Play operator, with Z positively α-far, for BV continuous inputs.

Let $Z \subseteq H$ be a positively α-far set. Let $y \in \mathrm{CBV}([T_0, T]; H)$ and consider the following differential inclusion:

$$\begin{cases} du \in dy - N(Z; u(t)), \\ u(T_0) = y(T_0) - x_0, \end{cases} \tag{5.7}$$

where $x_0 \in y(T_0) - Z$. Then $x := y - u$ is a solution of $(\mathcal{BVSP})$ with $C(t) = y(t) - Z$ for all $t \in [T_0, T]$ if and only if $u = y - x$ is a solution of (5.7). Moreover, the sets $C(t) = y(t) - Z$ are positively α-far for all $t \in [T_0, T]$ and

$$|d(x, C(t)) - d(x, C(s))| = |d(y(t) - x, Z) - d(y(s) - x, Z)| \leq |y(t) - y(s)|,$$

for every $x \in H$ and $t, s \in [T_0, T]$. Then, $(\mathcal{H}_1)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ hold. Therefore, Theorem 5.12 shows that there is at least one solution $x \in \mathrm{CBV}([T_0, T]; H)$ of (5.5). This allows us to define the hysteresis operator

$$P\colon \mathrm{CBV}([T_0, T]; H) \rightrightarrows \mathrm{CBV}([T_0, T]; H),$$

which to every function y associates the set of solutions of (5.7). Therefore, the Play operator is well defined for inputs in $\mathrm{CBV}([T_0, T]; H)$ generalizing the results given in [72, 124] to the class of positively α-far.

Remark 5.5

(i) If Z is uniformly prox-regular, due to the uniqueness of solution of (5.7), the Play operator is single valued.

(ii) Let us consider $y \in \mathrm{CBV}([T_0, T]; H)$ and $C(t) := y(t) - Z$ for all $t \in [T_0, T]$, where $Z \subseteq H$ is a convex set. Let $x \in \mathrm{CBV}([T_0, T]; H)$ be a solution of (5.5). Then $u := y - x$ satisfies $u(t) - y(t) \in Z$ for all $t \in [T_0, T]$ and

$$\int_{T_0}^{t} \langle u(s) - y(s) - z(s), dy \rangle \geq 0 \quad \forall z \in C([T_0, T]; Z) \;\; \forall t \in [T_0, T],$$

which corresponds to the classical formulation of the evolution variational inequality associated with the Play operator (see [124]).

Chapter 6

Regularization of perturbed state-dependent sweeping processes with nonregular sets

Let $T_0 < T$ be two nonnegative numbers and let $C\colon [T_0, T] \times H \rightrightarrows H$ be a set-valued mapping with nonempty closed values of a separable Hilbert space H. In this chapter, which is based on [144], we will be concerned, for any $x_0 \in C(T_0, x_0)$, with the differential inclusion

$$\begin{cases} \dot{x}(t) \in -N\left(C(t, x(t)); x(t)\right) + f(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0, x_0), \end{cases} \tag{$\mathcal{PSP}$}$$

where for any subset $S \subseteq H$ the set $N(S; u)$ denotes the Clarke normal cone to S at $u \in S$ and $f\colon [T_0, T] \times H \to H$ is a mapping which is measurable with respect to the first variable and either Lipschitzian or monotone with respect to the second variable. The differential inclusion ($\mathcal{PSP}$) is know as perturbed state-dependent sweeping process and includes the classical sweeping process

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) + f(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0), \end{cases} \tag{$\mathcal{SP}$}$$

introduced and thoroughly studied by Moreau [113–116] to deal with contact problems in mechanical systems (see [98] for an introduction to the subject).

The study of ($\mathcal{PSP}$) is motivated through quasi-variational inequalities arising, e.g. in the evolution of sandpiles, quasistatic evolution problems with friction, micromechanical damage models for iron materials, among others (see [98] and the references given there). It has been considered by several authors [14, 32, 45, 49, 50, 74, 76, 90, 97, 118, 119] in the convex/prox-regular setting. The used approach to deal with ($\mathcal{PSP}$) are implicit/explicit discretization scheme [32, 45, 74, 97, 118, 119], fixed point arguments [14, 49, 50, 76] and recently Moreau-Yosida regularization for ($\mathcal{PSP}$) without perturbation [90].

In the case $(\mathcal{SP})$ the use of Moreau-Yosida regularization is more extended [96, 105, 130, 136] but always restricted to the convex/prox-regular setting because in this case the function $x \mapsto \frac{1}{2}d^2_{C(t)}(x)$ is Lipschitz continuously differentiable on a neighborhood of $C(t)$ (see [123, Theorem 4.1]) which allows to apply the classical Cauchy-Lipschitz theorem.

Thus, the novelty of this work is to show that the Moreau-Yosida regularization is still valid under rather general conditions, namely subsmooth sets (for $(\mathcal{PSP})$ or positively α-far sets for $(\mathcal{SP})$).

The Moreau-Yosida regularization technique is a quite old approach to deal with differential equations and inclusions. It consists in approaching the given differential equation/inclusion by a penalize one, depending on a parameter $\lambda > 0$, whose existence is easier to establish (for example, by using the classical Cauchy-Lipschitz theorem) and then to pass to the limit $\lambda \downarrow 0$. To deal with $(\mathcal{PSP})$ we will consider, for $\lambda > 0$, the following differential inclusion:

$$\begin{cases} -\dot{x}_\lambda(t) \in \dfrac{1}{2\lambda}\partial d^2_{C(t,x_\lambda(t))}(x_\lambda(t)) + f(t,x_\lambda(t)) & \text{a.e. } t \in [T_0,T], \\ x_\lambda(T_0) = x_0 \in C(T_0,x_0). \end{cases} \tag{6.1}$$

We emphasize that when the sets $C(t,x)$ are state-independent and r-uniformly prox-regular, the function $x \mapsto \frac{1}{2}d^2_{C(t)}(x)$ is Lipschitz continuously differentiable on a neighborhood of $C(t)$. Thus, the existence of solution for (6.1) is a direct application of the Cauchy-Lipschitz theorem. When the sets $C(t,x)$ are either nonregular or state-dependent, due to the loss of smoothness of the function $x \mapsto \frac{1}{2}d^2_{C(t,x)}(x)$, the Cauchy-Lipschitz is no longer applicable. We overcome this problem by using an existence result for differential inclusions due to Bothe [27, Theorem 4]. Then, we show the convergence strongly pointwisely (up to a subsequence), as $\lambda \downarrow 0$, of solutions of (6.1) to a solution of $(\mathcal{PSP})$. The preceding result is established for two classes of nonregular sets, namely, the class of uniformly subsmooth sets (for $(\mathcal{PSP})$) and the class of positively α-far sets (for $(\mathcal{SP})$). These two classes includes strictly the class of uniformly prox-regular sets ([87] for more details).

The chapter is organized as follows. After some preliminaries, in Section 6.1 we collect the hypotheses used along the chapter. In section 6.2 we display some lemmas needed in the proof of the main result. In Section 6.3 we present and prove the main result of this work, namely Theorem 5.9. Finally, in Section 6.4 we mention some consequences of Theorem 5.9 to sweeping process and subdifferentially perturbed state-dependent sweeping process.

6.1 Technical assumptions

For the sake of readability, in this section we collect the hypotheses used along the chapter.

Hypotheses on the set-valued map $C\colon [T_0, T] \rightrightarrows H$: C is a set-valued map with nonempty and closed values. Moreover, we will consider the following conditions:

($\mathcal{H}_1$) There exists $\kappa \geq 0$ such that for all $s, t \in [T_0, T]$ and all $x \in H$

$$|d(x, C(t)) - d(x, C(s))| \leq \kappa |t - s|.$$

($\mathcal{H}_2$) There exist two constants $\alpha \in]0, 1]$ and $\rho \in]0, +\infty[$ such that

$$0 < \alpha \leq \inf_{x \in U_\rho(C(t))} d\left(0, \partial d(x, C(t))\right) \quad \text{a.e. } t \in [T_0, T],$$

where $U_\rho\left(C(t)\right) = \{x \in H\colon 0 < d(x, C(t)) < \rho\}$ for all $t \in [T_0, T]$.

($\mathcal{H}_3$) For a.e. $t \in [T_0, T]$ the set $C(t)$ is ball-compact, that is, for every $r > 0$ the set $C(t) \cap r\mathbb{B}$ is compact in H.

Hypotheses on the set-valued map $C\colon [T_0, T] \times H \rightrightarrows H$: C is a set-valued map with nonempty and closed values. Moreover, we will consider the following conditions:

($\mathcal{H}_4$) There exists $\kappa \geq 0$ and $L \in [0, 1[$ such that for all $s, t \in [T_0, T]$ and all $x, y, z \in H$

$$|d(z, C(t, x)) - d(z, C(s, y))| \leq \kappa |t - s| + L\|x - y\|.$$

($\mathcal{H}_5$) There exist two constants $\alpha \in]0, 1]$ and $\rho \in]0, +\infty[$ such that for every $y \in H$

$$0 < \alpha \leq \inf_{x \in U_\rho(C(t,y))} d\left(0, \partial d(\cdot, C(t, y))(x)\right) \quad \text{a.e. } t \in [T_0, T],$$

where $U_\rho\left(C(t, y)\right) := \{x \in H\colon 0 < d(x, C(t, y)) < \rho\}$.

($\mathcal{H}_6$) The family of sets $\{C(t, v)\colon (t, v) \in [T_0, T] \times H\}$ is equi-uniformly subsmooth.

($\mathcal{H}_7$) There exists $k \in L^1(T_0, T)$ such that for every $t \in [T_0, T]$, every $r > 0$ and every bounded set $A \subseteq H$

$$\gamma\left(C(t, A) \cap r\mathbb{B}\right) \leq k(t)\gamma(A),$$

where $\gamma = \alpha$ or $\gamma = \beta$ is either the Kuratowski or the Hausdorff measure of non-compactness (see Proposition 1.8) and $k(t) < 1$ for all $t \in [T_0, T]$.

($\mathcal{H}_8$) The assumption ($\mathcal{H}_7$) holds and there exists $R \geq 0$ such that for all $(t, x) \in [T_0, T] \times H$, $\text{Proj}_{C(t,x)}(x) \subseteq R\,\mathbb{B}$.

Hypotheses on the mapping $f\colon [T_0,T]\times H \to H$: We will consider the following conditions on the mapping $f\colon [T_0,T]\times H\to H$:

$(\mathcal{F}_1)$ For every $x\in H$ $f(\cdot,x)$ is measurable.

$(\mathcal{F}_2)$ For every $t\in[T_0,T]$ $f(t,\cdot)$ is continuous.

$(\mathcal{F}_3)$ For all $x,y\in H$ and all $t\in[T_0,T]$

$$\langle f(t,x)-f(t,y),x-y\rangle \le \omega(t)\|x-y\|^2,$$

where $\omega\in L^1(T_0,T)$.

$(\mathcal{F}_4)$ There exists $\beta>0$ such that, for all $t\in[T_0,T]$ and all $x,y\in H$

$$\|f(t,x)-f(t,y)\|\le\beta\|x-y\|,$$

and $\|f(\cdot,0)\|\in L^1(T_0,T)$.

$(\mathcal{F}_5)$ There exists $d\ge 0$ such that, for all $t\in[T_0,T]$ and all $x\in H$

$$\|f(t,x)\|\le d\left(1+\|x\|\right).$$

$(\mathcal{F}_6)$ There exists $d\ge0$ such that, For all $t\in[T_0,T]$ and all $x\in H$

$$\|f(t,x)\|\le d.$$

Remark 6.1

(i) Let $L\in[0,1[$. Under $(\mathcal{H}_6)$ for every $\alpha\in]\sqrt{L},1]$ there exists $\rho>0$ such that $(\mathcal{H}_5)$ holds. This follows from Proposition 2.8.

(ii) Assume that $(\mathcal{H}_7)$ holds and fix $t\in[T_0,T]$ and $r>0$. Then, for a fixed $y\in H$, we have

$$\begin{aligned}\gamma\left(C(t,\{y\})\cap r\mathbb{B}\right) &\le k(t)\gamma\left(\{y\}\right)\\ &=0,\end{aligned}$$

which, since $C(t,y)$ is closed, implies that $C(t,y)\cap r\mathbb{B}$ is compact. Therefore, $\mathrm{Proj}_{C(t,y)}(x)$ is nonempty for all $x\in H$.

(iii) If $C(t,x):=C(t)$ for every $(t,x)\in[T_0,T]\times H$, then $(\mathcal{H}_7)$ implies $(\mathcal{H}_3)$. Indeed, fix $t\in[T_0,T]$ and $r>0$. Then, for a fixed $x\in H$, we have

$$\begin{aligned}\gamma\left(C(t,\{x\})\cap r\mathbb{B}\right) &= \gamma\left(C(t)\cap r\mathbb{B}\right)\\ &\le k(t)\gamma\left(\{x\}\right)\\ &=0,\end{aligned}$$

which, since $C(t)$ is closed, implies that $C(t)\cap r\mathbb{B}$ is compact.

(iv) The condition $L\in[0,1[$ in $(\mathcal{H}_4)$ cannot be dispensed with as it is shown in [97].

6.2 Preparatory lemmas

In this section we give some preliminary lemmas that will be used in the following sections. They are related to differential inclusions, set-valued maps and properties of the distance function.

We will need the following consequence of Grönwall's inequality (see [10, Proposition 2.4.1]).

Lemma 6.1 *Let α, β two positive numbers and $\varphi\colon [T_0, T] \to \mathbb{R}$ be an absolutely continuous function. Assume that*

$$\dot{\varphi}(t) + \beta\varphi(t) \leq \alpha \quad \text{for a.e. } t \in [T_0, T].$$

Then, for all $t \in [T_0, T]$

$$\varphi(t) \leq \varphi(T_0) \exp\left(-\beta(t - T_0)\right) + \frac{\alpha}{\beta}\left(1 - \exp\left(-\beta(t - T_0)\right)\right).$$

The following result shows that the set-valued map $(t, x) \mapsto \frac{1}{2}\partial d^2_{C(t,x)}(x)$ satisfies the conditions of Lemma 1.14.

Proposition 6.2 *Assume that $(\mathcal{H}_4)$, $(\mathcal{H}_6)$ and $(\mathcal{H}_7)$ hold. Then the set-valued map $G\colon [T_0, T] \times H \rightrightarrows H$ defined by $G(t, x) := \frac{1}{2}\partial d^2_{C(t,x)}(x)$ satisfies:*

(i) For all $x \in H$ and all $t \in [T_0, T]$

$$G(t, x) = x - \overline{\text{co}}\, \text{Proj}_{C(t,x)}(x).$$

(ii) For every $x \in H$ the set-valued map $G(\cdot, x)$ is measurable.

(iii) For every $t \in [T_0, T]$ the set-valued map $G(t, \cdot)$ is upper semicontinuous from H into H_w.

(iv) For every $t \in [T_0, T]$ and $A \subseteq H$ bounded

$$\gamma\left(G(t, A)\right) \leq (1 + k(t))\gamma\left(A\right),$$

where $\gamma = \alpha$ or $\gamma = \beta$ is the Kuratowski or the Hausdorff measure of non-compactness of A and $k \in L^1(T_0, T)$ is given by $(\mathcal{H}_7)$.

(v) For all $t \in [T_0, T]$ and $x \in H$

$$\|G(t, x)\| := \sup\left\{\|w\|\colon w \in G(t, x)\right\} \leq (1 + L)\|x - x_0\| + \kappa\left|t - T_0\right|,$$

where $x_0 \in C(T_0, x_0)$.

Proof. (i) It follows from Lemma 1.15.

(ii) It follows from (v) of Lemma 5.2 because

$$\frac{1}{2}\partial d^2_{C(t,x)}(x) = d_{C(t,x)}(x)\partial d_{C(t,x)}(x).$$

(iii) It follows from Lemma 5.1.

(iv) Let $A \subseteq H$ be a bounded set included in the ball $r\mathbb{B}$ for some $r > 0$. Let us consider $x_0 \in H$ fixed and define $F(t,x) := \text{Proj}_{C(t,x)}(x)$. Then, for every $t \in [T_0, T]$

$$\begin{aligned}\|F(t,A)\| &:= \sup\{\|w\| \colon w \in F(t,A)\} \\ &\leq \tilde{r}(t) := (2+L)r + (1+L)\|x_0\| + \kappa|t - T_0|.\end{aligned}$$

Indeed, let $z \in F(t,A)$, then there exists $x \in A$ with $\|x\| \leq r$ such that $z \in \text{Proj}_{C(t,x)}(x)$. Thus,

$$\begin{aligned}\|z\| &\leq d_{C(t,x)}(x) + \|x\| \\ &= d_{C(t,x)}(x) - d_{C(T_0,x_0)}(x_0) + d_{C(T_0,x_0)}(x_0) + \|x\| \\ &\leq (1+L)\|x - x_0\| + \kappa|t - T_0| + d_{C(T_0,x_0)}(x_0) + \|x\| \\ &\leq (2+L)r + (1+L)\|x_0\| + \kappa|t - T_0| + d_{C(T_0,x_0)}(x_0) \\ &= \tilde{r}(t).\end{aligned}$$

Therefore,

$$\begin{aligned}\gamma\left(G(t,A)\right) &= \gamma\left(A - \overline{\text{co}}F(t,A)\right) \\ &\leq \gamma(A) + \gamma\left(\overline{\text{co}}F(t,A)\right) \\ &= \gamma(A) + \gamma\left(F(t,A)\right) \\ &= \gamma(A) + \gamma\left(F(t,A) \cap \tilde{r}(t)\mathbb{B}\right) \\ &\leq \gamma(A) + \gamma\left(C(t,A) \cap \tilde{r}(t)\mathbb{B}\right) \\ &\leq (1 + k(t))\gamma(A),\end{aligned}$$

where the last equality is due to $(\mathcal{H}_7)$.

(v) Define $\tilde{G}(t,x) := x - \text{Proj}_{C(t,x)}(x)$. Then, due to $(\mathcal{H}_4)$,

$$\begin{aligned}\|\tilde{G}(t,x)\| &= \sup\left\{\|w\| \colon w \in \tilde{G}(t,x)\right\} \\ &= d(x, C(t,x)) \\ &= d(x, C(t,x)) - d(x_0, C(T_0,x_0)) \\ &\leq (1+L)\|x - x_0\| + \kappa\left|t - T_0\right|.\end{aligned}$$

Therefore, by using the last inequality, we can pass to the closed convex hull and get the result.

□

When the sets $C(t,x)$ are independent of x, the subsmoothness in Proposition 6.2 is no longer required. The following result may be proved in much the same way as Proposition 6.2.

Proposition 6.3 *Assume that* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_3)$ *hold. Then* $G\colon [T_0,T]\times H \rightrightarrows H$ *defined by* $G(t,x) := \frac{1}{2}\partial d^2_{C(t)}(x)$ *satisfies:*

(i) For all $x\in H$ *and all* $t\in[T_0,T]$

$$G(t,x) = x - \overline{\text{co}}\,\text{Proj}_{C(t)}(x).$$

(ii) For every $x\in H$ *the set-valued map* $G(\cdot,x)$ *is measurable.*

(iii) For every $t\in[T_0,T]$ *the set-valued map* $G(t,\cdot)$ *is upper semicontinuous from* H *into* H_w.

(iv) For all $t\in[T_0,T]$ *and* $A\subseteq H$ *bounded*

$$\gamma\left(G(t,A)\right) \leq \gamma\left(A\right),$$

where $\gamma=\alpha$ *or* $\gamma=\beta$ *is either the Kuratowski or the Hausdorff measure of non-compactness of* A.

(v) For all $t\in[T_0,T]$ *and* $x\in H$

$$\|G(t,x)\| := \sup\{\|w\|\colon w\in G(t,x)\} \leq \|x-x_0\| + \kappa\,|t-T_0|\,,$$

where $x_0\in C(T_0)$.

6.3 An existence result for the perturbed state-dependent sweeping process

In this section we prove the existence of solutions for the perturbed state-dependent sweeping process $(\mathcal{PSP})$ via Moreau-Yosida regularization.

Let $\lambda>0$ and consider the following differential inclusion

$$\begin{cases} \dot{x}_\lambda(t) \in -\dfrac{1}{2\lambda}\partial d^2_{C(t,x_\lambda(t))}\left(x_\lambda(t)\right) + f(t,x_\lambda(t)) & \text{a.e. } t\in[T_0,T],\\ x(T_0) = x_0 \in C(T_0,x_0). \end{cases} \tag{$\mathcal{P}_\lambda$}$$

Proposition 6.4 *Assume, in addition to* $(\mathcal{H}_4)$, $(\mathcal{H}_6)$ *and* $(\mathcal{H}_7)$, *that* $(\mathcal{F}_1)$ *and* $(\mathcal{F}_4)$ *hold. Then, for every* $\lambda>0$ *there exists at least one solution* $x_\lambda\in W^{1,1}([T_0,T];H)$ *of* $(\mathcal{P}_\lambda)$.

Proof. It follows from Lemma 1.14 and Proposition 6.2. □

Let us define $\varphi_\lambda(t) := d_{C(t,x_\lambda(t))}(x_\lambda(t))$ for $t \in [T_0, T]$.

Remark 6.2 Recall that under $(\mathcal{H}_6)$, according to Proposition 2.8, for every $\alpha \in]\sqrt{L}, 1]$ there exists $\rho > 0$ such that $(\mathcal{H}_5)$ holds.

The following assumption will be useful.
Assumption 6. *There exists $M \geq 0$ such that, for all $t \in [T_0, T]$ and all $x \in H$*

$$\sup_{y \in D(t,x)} \|f(t, y)\| \leq M,$$

where $D(t, x) = \mathrm{Proj}_{C(t,x)}(x)$ (see Remark 6.1).

Proposition 6.5 *Assume, in addition to $(\mathcal{H}_4)$, $(\mathcal{H}_6)$ and $(\mathcal{H}_7)$, that $(\mathcal{F}_1)$, $(\mathcal{F}_4)$ and Assumption 6 hold. Then, if $\lambda < \lambda^*$, for a.e. $t \in [T_0, T]$*

$$\dot{\varphi}_\lambda(t) \leq \kappa + (1 + L)M - \frac{\alpha^2 - L - (1 + L)\beta\lambda}{\lambda}\varphi_\lambda(t), \tag{6.2}$$

where $\alpha \in]\sqrt{L}, 1]$ and $\rho > 0$ are given by Remark 6.2 and

$$\lambda^* = \frac{1}{2}\min\left\{\frac{\rho(\alpha^2 - L)}{\kappa + M(1 + L) + \rho\beta(1 + L)}, \frac{\alpha^2 - L}{(1 + L)\beta}\right\}.$$

Consequently,

$$\varphi_\lambda(t) \leq \frac{(\kappa + (1 + L)M)\lambda}{\alpha^2 - L - (1 + L)\beta\lambda} \text{ for all } t \in [T_0, T]. \tag{6.3}$$

Proof. Due to $(\mathcal{H}_4)$ the function φ_λ is absolutely continuous. Indeed, for every $t, s \in [T_0, T]$

$$|\varphi_\lambda(t) - \varphi_\lambda(s)| \leq (1 + L)\|x_\lambda(t) - x_\lambda(s)\| + \kappa|t - s|,$$

which implies the absolute continuity of φ_λ.
On the one hand, let $t \in \varphi_\lambda^{-1}(]0, \rho[)$ where $\dot{x}_\lambda(t)$ exists. Since $(\mathcal{H}_7)$ holds (see Remark 6.1), we can take $z_\lambda(t) \in \mathrm{Proj}_{C(t,x_\lambda(t))}(x_\lambda(t))$. Then, by using Lemma 5.2, we have

$$\begin{aligned}
\dot{\varphi}_\lambda(t) &\leq \kappa + L\|\dot{x}_\lambda(t)\| + \min_{w \in \partial d_{C(t,x_\lambda(t))}(x_\lambda(t))} \langle w, \dot{x}_\lambda(t)\rangle \\
&\leq \kappa + \frac{L}{\lambda}\varphi_\lambda(t) + L\|f(t, x_\lambda(t))\| - \frac{\alpha^2}{\lambda}\varphi_\lambda(t) + \|f(t, x_\lambda(t))\| \\
&= \kappa + \frac{L - \alpha^2}{\lambda}\varphi_\lambda(t) + (1 + L)\|f(t, x_\lambda(t))\| \\
&\leq \kappa + \frac{L - \alpha^2}{\lambda}\varphi_\lambda(t) + (1 + L)\|f(t, x_\lambda(t)) - f(t, z_\lambda(t))\| \\
&\quad + (1 + L)\|f(t, z_\lambda(t))\| \\
&\leq \kappa + \frac{L - \alpha^2}{\lambda}\varphi_\lambda(t) + (1 + L)\beta\varphi_\lambda(t) + (1 + L)M \\
&= \kappa + (1 + L)M - \frac{\alpha^2 - L - (1 + L)\beta\lambda}{\lambda}\varphi_\lambda(t),
\end{aligned}$$

where we have used hypothesis $(\mathcal{H}_5)$ (see Remark 6.2), Assumption 6 and Proposition 2.8.

On the other hand, let $t \in \varphi_\lambda^{-1}(\{0\})$ where $\dot{x}_\lambda(t)$ exists. Then, due to $(\mathcal{H}_4)$,

$$\begin{aligned}
\dot{\varphi}_\lambda(t) &= \lim_{h\downarrow 0} \frac{1}{h}\left(d_{C(t+h,x_\lambda(t+h))}(x_\lambda(t+h)) - d_{C(t,x_\lambda(t))}(x_\lambda(t))\right)\\
&\leq \kappa + L\|\dot{x}_\lambda(t)\| + \lim_{h\downarrow 0}\frac{1}{h} d_{C(t,x_\lambda(t))}(x_\lambda(t+h))\\
&\leq \kappa + (1+L)\|\dot{x}_\lambda(t)\|\\
&\leq \kappa + \frac{1+L}{\lambda}\varphi_\lambda(t) + (1+L)\|f(t,x_\lambda(t))\|\\
&\leq \kappa + \frac{1+L}{\lambda}\varphi_\lambda(t) + (1+L)M\\
&= \kappa + (1+L)M - \frac{\alpha^2 - L - (1+L)\beta\lambda}{\lambda}\varphi_\lambda(t),
\end{aligned}$$

where we have used that $t \in \varphi_\lambda^{-1}(\{0\})$.

Claim : $\varphi_\lambda^{-1}(]-\infty,\rho[) = [T_0, T]$

Proof of Claim : Otherwise, since the set $\varphi_\lambda^{-1}(]-\infty,\rho[)$ is open relative to $[T_0,T]$ and $T_0 \in \varphi_\lambda^{-1}(]-\infty,\rho[)$, there would exist $t^* \in]T_0,T]$ such that

$$[T_0, t^*[\subseteq \varphi_\lambda^{-1}(]-\infty,\rho[)$$

and $\varphi_\lambda(t^*) = \rho$. Then,

$$\dot{\varphi}_\lambda(t) \leq \kappa + (1+L)M - \frac{\alpha^2 - L - (1+L)\beta\lambda}{\lambda}\varphi_\lambda(t) \quad \text{a.e. } t \in [T_0, t^*[,$$

which, by virtue of Lemma 6.1, entails that for every $t \in [T_0, t^*[$

$$\begin{aligned}
\varphi_\lambda(t) &\leq \frac{(\kappa + (1+L)M)\lambda}{\alpha^2 - L - (1+L)\beta\lambda}\left(1 - \exp\left(-\frac{\alpha^2 - L - (1+L)\beta\lambda}{\lambda}t\right)\right)\\
&\leq \frac{(\kappa + (1+L)M)\lambda}{\alpha^2 - L - (1+L)\beta\lambda}\\
&\leq 2\frac{(\kappa + (1+L)M)\lambda^*}{\alpha^2 - L}\\
&< \rho,
\end{aligned}$$

which implies that $\varphi_\lambda(t^*) < \rho$, which is impossible. □

Thus, we have proved that φ_λ satisfies (6.2) and (6.3), which shows the proposition. □

As a corollary of the last proposition we obtain that x_λ is uniformly Lipschitz.

Corollary 6.6 *Under the assumptions of Proposition 6.5, for every $\lambda > 0$ the function x_λ is $\frac{\kappa+(1+L)M}{\alpha^2-L-(1+L)\beta\lambda}(1+\lambda\beta)+M$-Lipschitz.*

Let $(\lambda_n)_n$ be a sequence converging to 0 with $\lambda_n < \lambda^*$ for all $n \in \mathbb{N}$. In view of the Proposition 6.4 and Lemma 1.6, the next result show the existence of a subsequence $(\lambda_{n_k})_k$ of $(\lambda_n)_n$ such that $\left(x_{\lambda_{n_k}}\right)_k$ converges (in the sense of Lemma 1.6) to a solution of the state-dependent sweeping process $(\mathcal{PSP})$. A similar result of the following one was proved by Noel in [118, Theorem 5.2.1] by using a very different approach and a more restrictive compactness condition on the sets C.

Theorem 6.7 *Assume, in addition to $(\mathcal{H}_4)$, $(\mathcal{H}_6)$ and $(\mathcal{H}_7)$, that $(\mathcal{F}_1)$, $(\mathcal{F}_4)$ and Assumption 6 hold. Then, there exists at least one solution $x \in \operatorname{Lip}([T_0,T];H)$ of the state-dependent sweeping process $(\mathcal{PSP})$. Moreover,*

$$\|\dot{x}(t)\| \leq \frac{\kappa+(1+L)M}{\alpha^2-L}+M \quad a.e.\ t \in [T_0,T]. \tag{6.4}$$

Proof. According to Proposition 6.5,

$$\begin{aligned}
\|\dot{x}_\lambda(t)\| &\leq \frac{1}{\lambda}\varphi_\lambda(t)+\|f(t,x_\lambda(t))\| \\
&\leq \frac{1}{\lambda}\varphi_\lambda(t)+\beta\varphi_\lambda(t)+M \\
&\leq \frac{(\kappa+(1+L)M)}{\alpha^2-L-(1+L)\beta\lambda}(1+\beta\lambda)+M \\
&\leq \tilde{\kappa} := 2\frac{(\kappa+(1+L)M)}{\alpha^2-L}(1+\beta\lambda^*)+M
\end{aligned}$$

Thus, the sequence $(x_{\lambda_n})_n$ satisfies the hypotheses of Lemma 1.6 with $\psi(t)=\tilde{\kappa}$. Therefore, there exists a subsequence $(x_{\lambda_{n_k}})_k$ of $(x_{\lambda_n})_n$ and a function $x\colon [T_0,T] \to H$ satisfying (i)-(iv) from Lemma 1.6. For simplicity, we write x_k instead of $x_{\lambda_{n_k}}$ for all $k \in \mathbb{N}$.

Claim 1: $(x_k(t))_k$ is relatively compact in H for all $t \in [T_0,T]$.

Proof of Claim 1: Let $t \in [T_0,T]$ and take $y_k(t) \in \operatorname{Proj}_{C(t,x_k(t))}(x_k(t))$. Then, $\|x_k(t)-y_k(t)\| = d_{C(t,x_k(t))}(x_k(t))$. Thus,

$$\begin{aligned}
\|y_k(t)\| &\leq d_{C(t,x_k(t))}(x_k(t))+\|x_k(t)\| \\
&\leq \varphi_{\lambda_{n_k}}(t)+\int_{T_0}^{t}\|\dot{x}_k(s)\|ds+\|x_0\| \\
&\leq \tilde{r} := 2\frac{\kappa+(1+L)M}{\alpha^2-L}\left(\lambda^*+(1+\beta\lambda^*)(t-T_0)\right)+M(t-T_0)+\|x_0\|,
\end{aligned}$$

Also, since $(x_k(t)-y_k(t))$ converges to 0,

$$\gamma\left(\{x_k(t)\colon k \in \mathbb{N}\}\right) = \gamma\left(\{y_k(t)\colon k \in \mathbb{N}\}\right). \tag{6.5}$$

Indeed, on the one hand,

$$\begin{aligned}\gamma\left(\{x_k(t)\colon k\in\mathbb{N}\}\right) &\le \gamma\left(\{x_k(t)-y_k(t)\colon k\in\mathbb{N}\}\right)+\gamma\left(\{y_k(t)\colon k\in\mathbb{N}\}\right)\\ &= \gamma\left(\{y_k(t)\colon k\in\mathbb{N}\}\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\gamma\left(\{y_k(t)\colon k\in\mathbb{N}\}\right) &\le \gamma\left(\{y_k(t)-x_k(t)\colon k\in\mathbb{N}\}\right)+\gamma\left(\{x_k(t)\colon k\in\mathbb{N}\}\right)\\ &= \gamma\left(\{x_k(t)\colon k\in\mathbb{N}\}\right),\end{aligned}$$

which shows (6.5). Therefore,

$$\begin{aligned}\gamma\left(\{x_k(t)\colon k\in\mathbb{N}\}\right) &= \gamma(\{y_k(t)\colon k\in\mathbb{N}\})\\ &\le \gamma\left(C\left(t,\{x_k(t)\colon k\in\mathbb{N}\}\right)\cap\tilde{r}\mathbb{B}\right)\\ &\le k(t)\gamma\left(\{x_k(t)\colon k\in\mathbb{N}\}\right),\end{aligned}$$

where we have used $(\mathcal{H}_7)$. Finally, since $k(t)<1$, we obtain that

$$\gamma\left(\{x_k(t)\colon k\in\mathbb{N}\}\right)=0,$$

which shows the result. □

Claim 2: $x(t)\in C(t,x(t))$ for all $t\in[T_0,T]$.

Proof of Claim 2: As a result of the weak convergence $x_k(t)\rightharpoonup x(t)$ for all $t\in[T_0,T]$ (due to (i) of Lemma 1.6) and Claim 1, we obtain that

$$x_k(t)\to x(t)\quad \text{for all } t\in[T_0,T].$$

Therefore, due to $(\mathcal{H}_4)$ and Proposition 6.5, we have

$$\begin{aligned}d_{C(t,x(t))}(x(t)) &\le \liminf_{k\to\infty}\left(d_{C(t,x_k(t))}\left(x_k(t)\right)+(1+L)\|x_k(t)-x(t)\|\right)\\ &\le \liminf_{k\to\infty}\left(2\frac{\kappa+(1+L)M}{\alpha^2-L}\lambda_{n_k}+(1+L)\|x_k(t)-x(t)\|\right)\\ &=0,\end{aligned}$$

as claimed. □

Now we prove that x is a solution of $(\mathcal{PSP})$. Define

$$\tilde{F}(t,x):=-\overline{\text{co}}\left(\mu\partial d_{C(t,x)}(x)\cup\{0\}\right)+f(t,x),$$

for $(t,x)\in[T_0,T]\times H$, where $\mu:=2\frac{\kappa+(1+L)M}{\alpha^2-L}$. Then, for a.e. $t\in[T_0,T]$

$$\begin{aligned}\dot{x}_k(t) &\in -\frac{1}{2\lambda_{n_k}}\partial d^2_{C(t,x_k(t))}\left(x_k(t)\right)+f(t,x_k(t))\\ &= -\frac{d_{C(t,x_k(t))}\left(x_k(t)\right)}{\lambda_{n_k}}\partial d_{C(t,x_k(t))}\left(x_k(t)\right)+f(t,x_k(t))\\ &\subseteq \tilde{F}(t,x_k(t)),\end{aligned}$$

where we have used Proposition 6.5.

Claim 3: $\tilde{F}$ has closed convex values and satisfies:

(i) For each $x \in H$, $\tilde{F}(\cdot, x)$ is measurable;

(ii) for all $t \in [T_0, T]$, $\tilde{F}(t, \cdot)$ is upper semicontinuous from H into H_w;

(iii) if $x \in C(t,x)$ then $\tilde{F}(t,x) = -\mu\partial d_{C(t,x)}(x) + f(t,x)$.

Proof of Claim 3: Define $G(t,x) := -\mu\partial d_{C(t,x)}(x) \cup \{0\}$ for $(t,x) \in [T_0, T] \times H$. We note that $G(\cdot, x)$ is measurable as the union of two measurable set-valued maps (see [9, Lemma 18.4]). Let us define $\Gamma(t) := \tilde{F}(t,x)$. Then, Γ takes weakly compact convex values. Fix any $d \in H$, by virtue of [85, Proposition 2.2.39], it is enough to verify that the support function $t \mapsto \sigma(d, \Gamma(t)) := \sup_{v \in \Gamma(t)} \langle v, d \rangle$ is measurable. Thus,

$$\sigma(d, \Gamma(t)) := \sup\{\langle v, d\rangle : v \in \Gamma(t)\} = \sup\{\langle v, d\rangle : v \in G(t,x) + f(t,x)\},$$

is measurable because $G(\cdot, x)$ and $f(\cdot, x)$ are measurable. Hence (i) holds. Assertion (ii) follows from [9, Theorem 17.27 and 17.3]. Finally, if $x \in C(t,x)$ then $0 \in \partial d_{C(t,x)}(x)$. Hence, by using that the subdifferential of a locally Lipschitz function is closed and convex,

$$\begin{aligned}\tilde{F}(t,x) &= \overline{\text{co}}\left(-\mu\partial d_{C(t,x)}(x) \cup \{0\}\right) + f(t,x)\\ &= \overline{\text{co}}\left(-\mu\partial d_{C(t,x)}(x)\right) + f(t,x)\\ &= -\mu\partial d_{C(t,x)}(x) + f(t,x),\end{aligned}$$

which shows (iii). □

Summarizing, we have

(i) For each $x \in H$, $\tilde{F}(\cdot, x)$ is measurable;

(ii) for all $t \in [T_0, T]$, $\tilde{F}(t, \cdot)$ is upper semicontinuous from H into H_w;

(iii) $\dot{x}_k \rightharpoonup \dot{x}$ in $L^1([T_0, T]; H)$ as $k \to +\infty$;

(iv) $x_k(t) \to x(t)$ as $k \to +\infty$ for all $t \in [T_0, T]$;

(v) $\dot{x}_k(t) \in \tilde{F}(t, x_k(t))$ for a.e. $t \in [T_0, T]$.

These conditions and the Convergence Theorem (see [7, Proposition 5] for more details) implies that

$$\begin{cases} \dot{x}(t) \in \tilde{F}(t, x(t)) & \text{a.e. } t \in [T_0, T],\\ x(T_0) = x_0 \in C(T_0, x_0), \end{cases}$$

which, according to Claim 3, implies that x is a solution of

$$\begin{cases} \dot{x}(t) \in -\mu\partial d_{C(t,x(t))}(x(t)) + f(t, x(t)) & \text{a.e. } t \in [T_0, T],\\ x(T_0) = x_0 \in C(T_0, x_0). \end{cases}$$

Therefore, by virtue of (1.1) and Claim 2, x is a solution of $(\mathcal{PSP})$. Finally, due to Corollary 6.6, we have

$$\|x_k(t) - x_k(s)\| \leq \left(\frac{\kappa + (1+L)M}{\alpha^2 - L - (1+L)\beta\lambda_{n_k}}(1 + \beta\lambda_{n_k}) + M \right) |t-s|,$$

for all $t, s \in [T_0, T]$, which, by taking $k \to +\infty$, gives (6.4) and the theorem is proved. □

Remark 6.3 As far as we know, Theorem 6.7 is the most general result for Lipschitz perturbed state-dependent sweeping process.

Theorem 6.8 may be proved in much the same way as Theorem 6.7.

Theorem 6.8 *Assume, in addition to* $(\mathcal{H}_4)$ *and* $(\mathcal{H}_6)$*, that the function* $f\colon [T_0, T] \times H \to H$ *satisfies* $(\mathcal{F}_1)$, $(\mathcal{F}_2)$, $(\mathcal{F}_3)$ *and one of the following conditions is satisfied:*

i) $(\mathcal{H}_7)$ *and* $(\mathcal{F}_6)$ *hold.*

ii) $(\mathcal{H}_8)$ *and* $(\mathcal{F}_5)$ *hold.*

Then, there exists at least one solution $x \in \operatorname{Lip}([T_0, T]; H)$ *of the state-dependent sweeping process* $(\mathcal{PSP})$*. Moreover,*

$$\|\dot{x}(t)\| \leq \frac{\kappa + (1+L)\tilde{d}}{\alpha^2 - L} + \tilde{d} \quad \text{a.e. } t \in [T_0, T],$$

where $\tilde{d} := d$ *in the first case and* $\tilde{d} := d(1+R)$ *in the second case.*

6.4 Sweeping process and Subdifferentially Perturbed Sweeping Process

In this section we mention two important consequences of Theorem 6.7.

6.4.1 The case of the sweeping process

When the sets $C(t, x)$ do not depend on the state, that is, $C(t, x) = C(t)$ for all $(t, x) \in [T_0, T] \times H$, the perturbed differential inclusion $(\mathcal{SP})$ can be obtained as a particular case of $(\mathcal{PSP})$. In this particular case, Theorem 6.7 is valid under weaker hypotheses, namely, $(\mathcal{H}_2)$ instead of $(\mathcal{H}_6)$. A similar result was proved by Jourani and Vilches in [87] by using a very different approach.

Theorem 6.9 *Assume, in addition to* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ *and* $(\mathcal{H}_3)$*, that* $(\mathcal{F}_1)$, $(\mathcal{F}_4)$ *and Assumption* 1 *hold. Then, there exists at least one Lipschitz solution* $x \in \operatorname{Lip}([T_0, T]; H)$ *of sweeping process* $(\mathcal{SP})$*. Moreover, this solution satisfies* $\operatorname{Lip}(x) \leq \frac{\kappa + M}{\alpha^2} + M$.

Proof. According to the proof of Theorem 6.7, we observe that $(\mathcal{H}_6)$ was used to obtain $(\mathcal{H}_5)$ and the upper semicontinuity of $\partial d_{C(t,\cdot)}(\cdot)$ from H into H_w for all $t \in [T_0, T]$. Since in the present case these two properties holds under $(\mathcal{H}_2)$ (see Proposition 6.3), it is sufficient to adapt the proof of Theorem 6.7 to get the result. □

6.4.2 Subdifferentially state-dependent perturbed sweeping process

Let us consider a lower semicontinuous convex function $\Phi\colon [T_0, T] \times H \to \mathbb{R}$. We say that a function $\Phi\colon [T_0, T] \times H \to \mathbb{R}$ is boundedly Lipschitz-continuous if for all $r > 0$, there exists $L_r > 0$ such that for $t \in [T_0, T]$, for all $(x, y) \in \bar{B}(0, r) \times \bar{B}(0, r)$

$$|\Phi(t, x) - \Phi(t, y)| \leq L_r \|x - y\|. \tag{6.6}$$

Let us consider the following differential inclusion:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t, x(t)); x(t)\right) + \partial\Phi(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0, x_0), \end{cases} \tag{6.7}$$

This differential inclusion has been considered in [36, 140] and includes several physical models, e.g., includes parabolic variational inequalities and electrical circuits with ideal diodes. We will show how Theorem 6.7 leads to an existence result (6.7).

Theorem 6.10 *Assume, in addition to $(\mathcal{H}_4)$, $(\mathcal{H}_6)$ and $(\mathcal{H}_8)$, that Φ is a positive, boundedly Lipschitz-continuous and convex function with $\Phi(t, 0) \leq m$, for some $m \geq 0$. Then, there exists at least one solution $x \in \operatorname{Lip}([T_0, T]; H)$ of (6.7).*

Proof. For $\lambda \in]0, 1[$, let us consider the Moreau-Yosida envelope of the function $x \mapsto \Phi(t, x)$ (see [17, Chapter 12]), defined as:

$$\Phi_\lambda(t, x) := \inf_{y \in H} \left\{ \Phi(t, y) + \frac{1}{2\lambda}\|x - y\|^2 \right\} \quad x \in [T_0, T] \times H.$$

The Moreau-Yosida envelope enjoys the following properties:

(i) For every $x \in H$ and $t \in [T_0, T]$ there exists a unique point $J_\lambda^t(x)$ such that

$$\Phi_\lambda(t, x) = \Phi(t, J_\lambda^t(x)) + \frac{1}{2\lambda}\|x - J_\lambda^t(x)\|^2.$$

Also, $J_\lambda^t\colon H \to H$ is 1-Lipschitz continuous for all $t \in [T_0, T]$.

(ii) For all $t \in [T_0, T]$ the function $x \mapsto \Phi_\lambda(t, x)$ is Fréchet differentiable. Moreover, its gradient, given by

$$\nabla\Phi_\lambda(t, x) = \frac{1}{\lambda}(x - J_\lambda^t(x)),$$

is $\frac{1}{\lambda}$-Lipschitz continuous.

Thus, we can consider the following differential inclusion:

$$\begin{cases} \dot{x}_\lambda(t) \in -N\left(C(t, x_\lambda(t)); x_\lambda(t)\right) - \nabla\Phi_\lambda(t, x_\lambda(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0, x_0), \end{cases} \tag{$\mathcal{Q}_\lambda$}$$

We will verify that the hypotheses of Theorem 6.7 hold. Obviously, $(\mathcal{F}_1)$ and $(\mathcal{F}_4)$ hold. So, it remains to prove Assumption 6.

Claim : For all $t \in [T_0, T]$ and all $x \in H$

$$\sup_{y \in \operatorname{Proj}_{C(t,x)}(x)} \|\nabla\Phi_\lambda(t, y)\| \leq 2L_{\tilde{R}}, \tag{6.8}$$

where $\tilde{R} := R + \sqrt{2(L_R R + m)}$ and L_R and $L_{\tilde{R}}$ are given by (6.6).

Proof of Claim : Let $y \in \operatorname{Proj}_{C(t,x)}(x) \subseteq R\mathbb{B}$. Then,

$$\Phi_\lambda(t, y) = \Phi(t, J_\lambda^t(y)) + \frac{1}{2\lambda}\|y - J_\lambda^t(y)\|^2 \leq \Phi(t, y). \tag{6.9}$$

Hence,

$$\begin{aligned} \|y - J_\lambda^t(y)\|^2 &\leq 2\lambda\Phi(t, y) \\ &\leq 2\lambda(L_R\|y\| + \Phi(t, 0)) \\ &\leq 2\lambda(L_R R + m), \end{aligned}$$

which implies that $\|J_\lambda^t(y)\| \leq R + \sqrt{2(L_R R + m)} = \tilde{R}$. Therefore, due to (6.9),

$$\begin{aligned} \|y - J_\lambda^t(y)\|^2 &\leq 2\lambda\left(\Phi(t, y) - \Phi(t, J_\lambda^t(y))\right) \\ &\leq 2\lambda L_{\tilde{R}}\|y - J_\lambda^t(y)\|. \end{aligned}$$

Thus, $\frac{1}{\lambda}\|y - J_\lambda^t(y)\| \leq 2L_{\tilde{R}}$, as claimed. □

Therefore, according to Theorem 6.7, $(\mathcal{Q}_\lambda)$ has at least one solution $x_\lambda \colon [T_0, T] \to H$ with $x_\lambda(t) \in C(t, x_\lambda(t))$ for all $t \in [T_0, T]$ and

$$\|\dot{x}_\lambda(t)\| \leq \frac{\kappa + 2(1 + L)L_{\tilde{R}}}{\alpha^2 - L} + 2L_{\tilde{R}} \quad \text{a.e. } t \in [T_0, T]. \tag{6.10}$$

Let $(\lambda_n)_n$ be a sequence converging to 0 with $\lambda_n \in]0, 1[$ for all $n \in \mathbb{N}$. In view of (6.10) and Lemma 1.6, there exists a subsequence $(x_{\lambda_{n_k}})_k$ of $(x_{\lambda_n})_n$ and a function $x \colon [T_0, T] \to H$ satisfying (i)-(iv) from Lemma 1.6. For simplicity, we write x_k instead of $x_{\lambda_{n_k}}$ for all $k \in \mathbb{N}$. Also, in the same way as in the proof of Theorem 5.1 (see Claim 1), we can deduce that $x_k(t) \to x(t)$ for all $t \in [T_0, T]$. Moreover, due to (6.10), $(x_{\lambda_{n_k}})$ is bounded. Thus, by a similar argument to the one given in the proof of (6.8), we have that $J_{\lambda_{n_k}}^t(x_k(t)) \to x(t)$ for all $t \in [T_0, T]$. Hence, for a.e. $t \in [T_0, T]$

$$-\dot{x}_k(t) \in \frac{\kappa + 2(1 + L)\tilde{R}}{\alpha^2 - L}\partial d_{C(t, x_k(t))}(x_\lambda(t)) + \partial\Phi(t, J_{\lambda_{n_k}}^t(x_k(t))).$$

Therefore, as in the proof of Theorem 6.7, by using the Convergence Theorem (see [7, Proposition 5]), we obtain that x is a solution of (6.7). □

Chapter 7

Perturbed Sweeping Processes with Nonlocal Initial Conditions

In this chapter, which is based on [88], we study differential inclusions with nonlocal initial conditions. We show existence for the perturbed sweeping process with nonlocal initial conditions. Moreover, through the concept of bounding functions and some tangential conditions, we prove existence for abstract differential inclusions with nonlocal initial conditions.

While the study of the sweeping process with Cauchy initial conditions is well known (see Chapter 3.6), the sweeping process with nonlocal initial condition has received relatively little attention. In the context of periodic sweeping processes, we can mention the works of Castaing and Monteiro-Marques [46, 48] for convex sets in Hilbert spaces and Gavioli [71] for wedges sets in finite dimensions.

The first part of this chapter is devoted to establishing some sufficient conditions for the existence of perturbed sweeping processes with nonregular sets and nonlocal initial conditions, that is, we consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = Mx, \end{cases} \tag{7.1}$$

where H is a separable Hilbert space, $C\colon [T_0, T] \rightrightarrows H$ is a set-valued map with nonempty and closed values, $N\left(S; \cdot\right)$ denotes the Clarke normal cone to S and $F\colon [T_0, T] \times H \rightrightarrows H$ is a given set-valued map with nonempty, closed and convex values. Here $M\colon C\left([T_0, T]; H\right) \to H$ is an operator (possibly nonlinear) satisfying

$$\|Mx - My\| \leq m\|x - y\|_\infty \quad \text{for all } x, y \in C\left([T_0, T]; H\right), \tag{7.2}$$

with $m \in [0, 1]$. The class of operators M satisfying the condition (7.2) is sufficiently large and includes the following well-known nonlocal initial conditions:

(i) $Mx = x_0$ (general Cauchy initial condition $x(T_0) = x_0$);

(ii) $Mx = \pm x(T)$ (periodic and anti-periodic initial conditions);

(iii) $Mx = \frac{1}{T-T_0}\int_{T_0}^{T} x(s)ds$ (mean value initial condition);

(iv) $Mx = \sum_{i=1}^{k_0} \alpha_i x(t_i)$ with $\alpha_i \in \mathbb{R}$ and $\sum_{i=1}^{k_0} |\alpha_i| \leq 1$, where $T_0 < t_1 < \cdots < t_{k_0} \leq T$ (multi-point initial condition).

Our study is achieved through the Galerkin-Like method (see Chapter 3). This method to solve differential inclusions, consists in approaching the original problem by projecting the state into a n-dimensional Hilbert space but not the velocity. The approached problems always have a solution and, under some compactness conditions, they converge strongly pointwisely (up to a subsequence) to a solution of the original differential inclusion. We combine the Galerkin-Like method with the reduction technique for the sweeping process (see, e.g., [75, 135]). The reduction technique associates to every sweeping process an unconstrained differential inclusion, whose solutions are also solutions of the sweeping process. In order to apply this method, the moving sets must to be positively α-far (see Chapter 2).

In Section 7.2, we present the main results of the first part of the chapter, namely, the existence for perturbed sweeping process with nonlocal initial conditions. As a consequence, we obtain the existence of periodic solutions for the perturbed sweeping process, which extends the results from [46, 48, 71]. We believe that these results can be used for further developments in the theory of periodic perturbations and stability for the sweeping process (see [91]).

The second part of the chapter is concerned with existence of abstract differential inclusions with nonlocal initial conditions. To deal with it, we use the concept of bounding functions and some tangential conditions. We say that V is a (weak/strong) bounding function for a differential inclusion (see Definition 7.7), when the existence of this function implies the existence of an a priori bound for the solutions of the differential inclusion. Typically, the bounding function has to satisfy some conditions involving the derivatives of V (in some sense) and the right-hand side of the differential inclusion. The idea of bounding functions was introduced by Mawhin [103] to deal with second order boundary value problems. Since then, it was systematically used for the study of various boundary problems (see [21, 120] and the references therein). In [103], Mawhin imposes a specific condition relative to the second order derivatives of V, which implies the boundedness of the solution for the second order boundary value problem. For the case of first order differential inclusions, the concept of bounding function involves conditions on the first order derivatives of V and the right-hand side of the differential inclusion, in some ring or localized in the boundary of some bounded set. Thus, the concept of bounding function is vague and highly depending on the method to deal with the differential inclusion. Our definition of weak bounding function (see Definition 7.7) is taken from [21]. The use of bounding functions, generally, is related to the Leray-Schauder continuation principle and the topological degree theory (see [21] for more details). We emphasize that our approach make no appeal to these tools from nonlinear analysis but merely basic elements of set-valued and variational analysis.

In Section 7.3 we use bounding functions to study a first order differential inclusion with nonlocal initial conditions when H is compactly embedded in a separable Banach space E.

In Section 7.4, we use some tangential conditions to get the existence of abstract nonlocal differential inclusion in finite dimensions. These tangential conditions, related with the weak invariance of differential inclusions, typically, involves the intersection between the Bouligand tangent cone and the right-hand side of the differential inclusion. Since we apply a fixed point theorem to the solution map of the differential inclusion, a strong property is needed, namely, the intersection between the Clarke tangent cone and the right-hand side of the differential inclusion is nonempty (see Remark 7.8).

Finally, in Sections 7.5 and 7.6, we give, respectively, some applications to nonlocal differential complementarity systems and vector hysteresis.

7.1 Technical Assumptions

For the sake of readability, in this section we collect the hypotheses used along the chapter. Through this chapter H stands for a separable Hilbert space whose norm is denoted by $\|\cdot\|$. Moreover, we say that $(H, \|\cdot\|)_H$ is compactly embedded in a separable Banach space $(E, \|\cdot\|_E)$, we write $H \hookrightarrow E$, if there exists $C > 0$ such that $\|x\|_E \leq C\|x\|_H$ for all $x \in H$ and every bounded set in H is relatively compact in E.

Hypotheses on the set-valued map $C\colon [T_0, T] \rightrightarrows H$: C is a set-valued map with nonempty and closed values. Moreover, we will consider the following conditions:

($\mathcal{H}_1$) There exists $\zeta \in W^{1,1}(T_0, T)$ such that for $s, t \in [T_0, T]$ and all $x \in H$

$$|d(x, C(t)) - d(x, C(s))| \leq |\zeta(t) - \zeta(s)|.$$

($\mathcal{H}_2$) There exist two constants $\alpha_0 \in (0, 1]$ and $\rho \in (0, +\infty)$ such that

$$0 < \alpha_0 \leq \inf_{x \in U_\rho(C(t))} d\left(0, \partial d(x, C(t))\right) \quad \text{a.e. } t \in [T_0, T],$$

where $U_\rho(C(t)) := \{y \in H\colon 0 < d(y, C(t)) < \rho\}$ for all $t \in [T_0, T]$.

($\mathcal{H}_3$) For all $t \in [T_0, T]$, the set $C(t)$ is ball compact, that is, for every $r > 0$ the set $C(t) \cap r\mathbb{B}$ is compact in H.

Remark 7.1 The condition ($\mathcal{H}_2$) holds true for a big family of sets, e.g., convex sets, r-uniformly prox-regular sets, equi-uniformly subsmooth sets, etc (see [87]).

Hypotheses on the set-valued map $F\colon [T_0,T]\times H \rightrightarrows H$*:* F is a set-valued map with nonempty, closed and convex values. Moreover, we will consider the following conditions:

($\mathcal{H}_1^F$) For all $x\in H$, $F(\cdot,x)$ is measurable.

($\mathcal{H}_2^F$) For a.e. $t\in[T_0,T]$, $F(t,\cdot)$ is upper semicontinuous from H into H_w.

($\mathcal{H}_3^F$) There exists $\beta\in L^1(T_0,T)$ such that

$$d\,(0,F(t,x)) := \inf\{\|w\|\colon w\in F(t,x)\}\le \beta(t),$$

for all $x\in H$ and a.e. $t\in[T_0,T]$.

($\mathcal{H}_4^F$) For all $r>0$ there exists $v_r\in L^1(T_0,T)$ such that for a.e. $t\in[T_0,T]$ and all $x\in H$ with $\|x\|\le r$

$$d\,(0,F(t,x)) := \inf\{\|w\|\colon w\in F(t,x)\}\le v_r(t).$$

Moreover, in the case where $(H,\|\cdot\|_H)$ is compactly embedded in a separable Banach space $(E,\|\cdot\|_E)$, we will consider the following hypothesis on F:

($\mathcal{H}_5^F$) For a.e. $t\in[T_0,T]$, $F(t,\cdot)$ is closed from E into E_w, that is, $\operatorname{graph} F(t,\cdot)$ is closed in $E\times E_w$.

Hypotheses on the map $M\colon C\,([T_0,T];H)\to H$*:*

($\mathcal{H}_1^M$) There exists $m\in[0,1)$ such that

$$\|Mx-My\|\le m\|x-y\|_\infty \quad \text{for all } x,y\in C\,([T_0,T];H)\,,$$

($\mathcal{H}_2^M$) For all $x,y\in C\,([T_0,T];H)$

$$\|Mx-My\|\le \|x-y\|_\infty.$$

($\mathcal{H}_3^M$) $M\colon C\,([T_0,T];H)\to H$ is sequentially weakly upper semicontinuous, that is, if $x_n \rightharpoonup x$ in $C\,([T_0,T];H)$ (see Lemma 1.4), then $Mx_k\rightharpoonup Mx$ in H, for some subsequence $(x_k)_k$ of $(x_n)_n$.

Remark 7.2

a) If $M\colon C\,([T_0,T];H)\to H$ is linear and continuous, then ($\mathcal{H}_3^M$) holds.

b) The conditions ($\mathcal{H}_2^M$) and ($\mathcal{H}_3^M$) hold for the following operators:

  i) $Mx = x_0$;

  ii) $Mx=\pm x(T)$;

iii) $Mx = \frac{1}{T-T_0}\int_{T_0}^{T} x(s)ds$;

iv) $Mx = \sum_{i=1}^{k_0} \alpha_i x(t_i)$ with $\alpha_i \in \mathbb{R}$ and $\sum_{i=1}^{k_0} |\alpha_i| \leq 1$, where $T_0 < t_1 < \cdots < t_{k_0} \leq T$.

7.2 Perturbed Sweeping Process with Nonlocal Initial Conditions

In this section, we prove existence results for (7.1) in infinite dimensional Hilbert spaces. We distinguish between the contractive case $(\mathcal{H}_1^M)$ (see Theorem 7.2) and the nonexpansive case $(\mathcal{H}_2^M)$ (see Theorem 7.3). Our results are associated with the existence of a convex set D so that $M\mathcal{C} \subseteq D \subseteq C(T_0)$, where

$$\mathcal{C} := \{x \in C([T_0, T]) : x(t) \in C(t) \text{ for all } t \in [T_0, T]\} \tag{7.3}$$

This condition seems very natural because the constrained nature of the sweeping process. Moreover, unlike the contractive case, we have to impose a boundedness condition on the set D to assure the existence of solutions of (7.1).

Before presenting the main results of this section, we want to emphasize the role of condition $(\mathcal{H}_3)$. Indeed, the compactness hypothesis $(\mathcal{H}_3)$ seems to be a strong assumption, but it is not. We refer to [89] for an example of a perturbed sweeping process with Cauchy initial condition, governed by a ball, without solutions.

The following lemma will be used in the construction of the fixed point operator used in the proof of Theorem 7.2.

Lemma 7.1 *Assume that $(\mathcal{H}_1^M)$ holds. If $f \in L^1([T_0, T]; H)$, then there exists a unique solution of the following differential equation:*

$$\begin{cases} \dot{x}(t) = f(t) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = Mx. \end{cases}$$

Moreover, $\|x(t)\| \leq \dfrac{1}{1-m}\left(\|M0\| + \int_{T_0}^{T} \|f(s)\| ds\right)$ *for all* $t \in [T_0, T]$.

Proof. Fix $x_0 \in H$. For each $n \in \mathbb{N}$ we define

$$x_{n+1}(t) = Mx_n + \int_{T_0}^{t} f(s)ds \quad \text{for all } t \in [T_0, T]. \tag{7.4}$$

Then, for all $n \geq 1$

$$\|x_{n+1}(t) - x_n(t)\| = \|Mx_n - Mx_{n-1}\| \leq m\|x_n - x_{n-1}\|_\infty.$$

Therefore, $\|x_{n+1} - x_n\|_\infty \leq m\|x_n - x_{n-1}\|_\infty$ for all $n \geq 1$, which proves, since $m \in [0, 1)$, that $(x_n)_n$ is a Cauchy sequence in $C([T_0, T]; H)$. Therefore, by passing to the limit in (7.4), we obtain the result. □

The following result asserts the existence of solutions for (7.1), when the operator M is a contraction.

Theorem 7.2 *Let $F\colon [T_0,T]\times H \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$ and $C\colon [T_0,T]\rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$. Assume, in addition to $(\mathcal{H}_1^M)$, $(\mathcal{H}_3^M)$, that there exists a convex set D such that $M\mathcal{C}\subseteq D\subseteq C(T_0)$, where $\mathcal{C}$ is given by (7.3) and*

$$\left(1+\frac{1}{\alpha_0^2}\right)\int_{T_0}^{T}\left(|\dot{\zeta}(s)|+\beta(s)\right)ds<\rho. \tag{7.5}$$

Then, there exists at least one solution of (7.1). Moreover,

$$\|\dot{x}(t)\|\leq\frac{1}{\alpha_0^2}|\dot{\zeta}(t)|+\left(1+\frac{1}{\alpha_0^2}\right)\beta(t)\quad \text{a.e. } t\in[T_0,T].$$

Proof. Let us define the set-valued map $G\colon [T_0,T]\times H\rightrightarrows H$ by:

$$G(t,x):=-\frac{1}{\alpha_0^2}\left(|\dot{\zeta}(t)|+\beta(t)\right)\partial d_{C(t)}(x)+F(t,x)\cap\beta(t)\mathbb{B}.$$

It is clear that G satisfy $(\mathcal{H}_1^F)$ and $(\mathcal{H}_2^F)$.
The idea of the proof is to use the reduction technique for the sweeping process together with the Galerkin-like method. The reduction technique consists in showing the existence of solutions of the following unconstrained differential inclusion:

$$\begin{cases}\dot{x}(t)\in G(t,x(t)) & \text{a.e. } t\in[T0,T],\\ x(T_0)=Mx,\end{cases} \tag{7.6}$$

Thus, by formula (1.1), every solution of (7.6) together with the condition $x(t)\in C(t)$ for all $t\in[T_0,T]$, is a solution of (7.1). Since it is not possible to prove directly the existence of (7.6), we use the Galerkin like-method, that is, we approach (7.6) by projecting the state into a n-dimensional Hilbert space.
The proof will be divided into several steps.

Step 1: For each $n\in\mathbb{N}$ there exists at least one solution x_n of

$$\begin{cases}\dot{x}(t)\in G(t,P_n(x(t))) & \text{a.e. } t\in[T_0,T],\\ x(T_0)=\operatorname{proj}_D\left(P_n\left(Mx\right)\right),\end{cases} \tag{7.7}$$

where $P_n\colon H\to \operatorname{span}\{e_1,\ldots,e_n\}$ is the linear operator defined in Lemma 1.7.
Proof of Step 1: Let $K\subseteq L^1([T_0,T];H)$ be the set defined by

$$K:=\left\{f\in L^1([T_0,T];H):\|f(t)\|\leq\psi(t) \text{ a.e. } t\in[T_0,T]\right\},$$

where

$$\psi(t):=\frac{1}{\alpha_0^2}|\dot{\zeta}(t)|+\left(1+\frac{1}{\alpha_0^2}\right)\beta(t)\quad\text{for all } t\in[T_0,T]. \tag{7.8}$$

It is clear that K is nonempty, closed and convex. In addition, since $\psi \in L^1(T_0,T)$, K is bounded and uniformly integrable, hence, it is compact in $L^1_w([T_0,T];H)$ (see Theorem 1.3). We observe that K, endowed with the relative $L^1_w([T_0,T];H)$ topology is a metric space (see [64, Theorem V.6.3]). Define the map $\mathcal{F}_n\colon K \rightrightarrows L^1([T_0,T];H)$ as

$$\mathcal{F}_n(f) := \{v \in L^1([T_0,T];H) : v(t) \in G(t, P_n(x_f(t))) \text{ a.e. } t \in [T_0,T]\},$$

where for each $f \in K$, x_f is the unique solution (see Lemma 7.1) of

$$\begin{cases} \dot{x}(t) = f(t) & \text{a.e. } t \in [T_0,T], \\ x(T_0) = \operatorname{proj}_D(P_n(Mx)). \end{cases}$$

By $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and [7, Lemma 6], we conclude that $\mathcal{F}_n(f)$ has nonempty, closed and convex values. Moreover, $\mathcal{F}_n(K) \subseteq K$. Indeed, if $f \in K$ and $v \in \mathcal{F}_n(f)$ then,

$$\|v(t)\| \le \sup\{\|w\|\colon w \in G(t, P_n(x_f(t)))\} \le \psi(t) \quad \text{for a.e. } t \in [T_0,T].$$

We denote K_w the set K seen as a compact and convex subset of $L^1_w([T_0,T];H)$.
Claim 1.: $\mathcal{F}_n$ is upper semicontinuous from K_w into K_w.
Proof of Claim 1.: By virtue of [85, Proposition 1.2.23], it is sufficient to prove that the graph graph$(\mathcal{F}_n)$ of $\mathcal{F}_n$ is sequentially closed in $K_w \times K_w$.

Let $((f_j,v_j))_j \subseteq \operatorname{graph}(\mathcal{F}_n)$ with $f_j \to f$ and $v_j \to v$ in $L^1_w([T_0,T];H)$ as $j \to +\infty$. We have to show that $(f,v) \in \operatorname{graph}(\mathcal{F}_n)$. We first note that,

$$v_j(t) \in G(t, P_n(x_{f_j}(t))) \quad \text{for a.e. } t \in [T_0,T]. \tag{7.9}$$

Moreover, since $f_j \in K$ and Lemma 7.1, we have that

$$\|\dot{x}_{f_j}(t)\| = \|f_j(t)\| \le \psi(t) \quad \text{a.e. } t \in [T_0,T]. \tag{7.10}$$

and

$$\|x_{f_j}(t)\| \le \frac{1}{1-m}\left(\|\operatorname{proj}_D(P_n(M0))\| + \int_{T_0}^{T} \psi(s)ds\right) \quad \forall t \in [T_0,T]. \tag{7.11}$$

On the one hand, let us consider $P := \{\dot{x}_{f_j}\colon j \in \mathbb{N}\} \subseteq L^1([T_0,T];H)$. According to (7.10), the set P is bounded and uniformly integrable. Thus, as a result of the Dunford-Pettis theorem (see Theorem 1.3), P is relatively compact in $L^1_w([T_0,T];H)$. Therefore, there exists a subsequence of $(\dot{x}_{f_j})$ (without relabeling) converging to some $v \in L^1_w([T_0,T];H)$. Now, let $S := \{x_{f_j}\colon j \in \mathbb{N}\} \subseteq L^1([T_0,T];H)$. Then, due to (7.11) and the Dunford-Pettis theorem (see Theorem 1.3), S is relatively compact in $L^1_w([T_0,T];H)$. Consequently, there exists a subsequence of $(x_{f_j})_j$ (without relabeling) converging to some $x \in L^1_w([T_0,T];H)$.

On the other hand, due to (7.10) and (7.11), the sequence $(x_{f_j})_j$ is uniformly bounded in $W^{1,1}([T_0,T];H)$ and in $L^\infty([T_0,T];H)$. Therefore, by result of [111, Theorem 0.2.2.1], there exists a subsequence of $(x_{f_j})_j$ (without relabeling) and a function $\tilde{x}$ such that

$$x_{f_j}(t) \to \tilde{x}(t) \text{ weakly as } j \to +\infty \text{ for all } t \in [T_0,T]. \tag{7.12}$$

Moreover, by virtue of [70, Proposition 2.3.31], $x \equiv \tilde{x}$. Now, we prove that $v = \dot{x}$. Indeed, let $w \in H$ and $t \in [T_0, T]$ be fixed. Then,

$$\left\langle x_{f_j}(t) - x_{f_j}(T_0), w \right\rangle = \int_{T_0}^{t} \left\langle \dot{x}_{f_j}(s), w \right\rangle = \int_{T_0}^{T} \left\langle \dot{x}_{f_j}(s), w \cdot \mathbb{1}_{[T_0,t]}(s) \right\rangle ds, \tag{7.13}$$

where

$$\mathbb{1}_{[T_0,t]}(s) := \begin{cases} 1, & \text{if } s \in [T_0, t], \\ 0, & \text{if } s \in]t, T], \end{cases}$$

belongs to $L^{\infty}([T_0, T]; H)$. Moreover, due to $(\mathcal{H}_3^M)$ and Lemma 1.4, $Mx_{f_j} \rightharpoonup Mx$ in H (without relabeling), which implies, by the definition of P_n, that $P_n\left(Mx_{f_j}\right) \to P_n(Mx)$. Thus, $x_{f_j}(T_0) \to \operatorname{proj}_D(P_n(Mx))$. Therefore, using (7.12), the weak convergence of $\dot{x}_{f_j}$ to v in $L^1([T_0, T]; H)$ and passing to the limit in (7.13), we obtain

$$\langle x(t) - \operatorname{proj}_D(P_n(Mx)), w \rangle = \int_{T_0}^{t} \langle v(s), w \rangle \, ds \quad \text{for all } w \in H.$$

Thus

$$x(t) = \operatorname{proj}_D(P_n(Mx)) + \int_{T_0}^{t} v(s) ds \quad \text{for all } t \in [T_0, T].$$

Therefore, we have proved the existence of a subsequence of $(x_{f_j})_j$ (without relabeling) and an absolutely continuous function $x\colon [T_0, T] \to H$ such that

$$\begin{aligned} &x_{f_j}(t) \to x(t) \text{ weakly for all } t \in [T_0, T], \\ &x_{f_j} \to x \text{ in } L^1_w([T_0, T]; H), \\ &\dot{x}_{f_j} \to \dot{x} \text{ in } L^1_w([T_0, T]; H), \\ &x(t) = \operatorname{proj}_D(P_n(Mx)) + \int_{T_0}^{t} f(s) ds \qquad \text{for all } t \in [T_0, T]. \end{aligned}$$

Moreover, by the definition of P_n, $P_n(x_{f_j}(t)) \to P_n(x(t))$ for every $t \in [T_0, T]$. Consequently, by virtue of (7.9), the upper semicontinuity of $G(t, \cdot)$ from H into H_w and [70, Proposition 2.3.1], we obtain, for a.e. $t \in [T_0, T]$

$$v(t) \in \overline{\operatorname{conv}}\, w\text{-}\limsup_{m \to +\infty}\{v_m(t)\} \subseteq \overline{\operatorname{conv}}\, G(t, P_n(x(t))) = G(t, P_n(x(t))),$$

which shows that $(f, v) \in \operatorname{graph}(\mathcal{F}_n)$, as claimed. □

Now, we invoke the Kakutani-Fan-Glicksberg fixed point theorem (see Theorem 1.12) to the set-valued map $\mathcal{F}_n\colon K_w \rightrightarrows K_w$, to deduce the existence of $\hat{f}_n \in K$ such that $\hat{f}_n \in \mathcal{F}_n\left(\hat{f}_n\right)$. Thus, the function $x_n := x_{\hat{f}_n} \in W^{1,1}([T_0, T]; H)$ is a solution of (7.7), which proves Step 1. □

Step 2.: There exists $x \in W^{1,1}([T_0, T]; H)$ solution of

$$\begin{cases} \dot{x}(t) \in G(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(t) \in C(t) & \text{for all } t \in [T_0, T], \\ x(T_0) = \operatorname{proj}_D(Mx), \end{cases} \tag{7.14}$$

Proof of Step 2.: For each $n \in \mathbb{N}$, let x_n be a solution of (7.7) and for all $t \in [T_0, T]$ define

$$\varphi_n(t) := d_{C(t)}(P_n(x_n(t))) \quad \text{and} \quad \Gamma_n(t) := \partial d_{C(t)}(P_n(x_n(t))).$$

Then, according to Step 1, there exist $f_n(t) \in F(t, P_n(x_n(t))) \cap \beta(t)\mathbb{B}$ and $d_n(t) \in \Gamma_n(t)$ such that

$$\begin{cases} \dot{x}_n(t) = -\dfrac{1}{\alpha_0^2}\left(|\dot{\zeta}(t)| + \beta(t)\right) d_n(t) + f_n(t) & \text{a.e. } t \in [T_0, T], \\ x_n(T_0) = \operatorname{proj}_D(P_n(Mx_n)). \end{cases} \tag{7.15}$$

Moreover, according to (7.11),

$$\|x_n(t)\| \leq \frac{1}{1-m}\left(\|\operatorname{proj}_D(P_n(M0))\| + \int_{T_0}^{T} \varphi(s)ds\right) \quad \text{for all } t \in [T_0, T], \tag{7.16}$$

where ψ is defined by (7.8). Therefore, $(x_n)_n$ and $(P_n(x_n))$ are uniformly bounded in $C([T_0, T]; H)$.

Claim 2. $\lim_{n\to+\infty} \varphi_n(T_0) = 0$.

Proof of Claim 2.: Indeed, since $(x_n(T_0))_n$ is bounded (see (7.16)), there exists a positive number $\tilde{R}$ such that $(x_n(T_0))_n \subseteq D \cap \tilde{R}\mathbb{B} \subseteq C(T_0) \cap \tilde{R}\mathbb{B}$. Hence, due to the ball compactness of $C(T_0)$ and Lemma 1.7,

$$\begin{aligned} \limsup_{n\to+\infty} \varphi_n(T_0) &= \limsup_{n\to+\infty} \left[d_{C(T_0)}(P_n x_n(T_0)) - d_{C(T_0)}(x_n(T_0))\right] \\ &\leq \limsup_{n\to+\infty} \|x_n(T_0) - P_n(x_n(T_0))\| \\ &\leq \limsup_{n\to+\infty} \sup_{x \in D\cap\tilde{R}\mathbb{B}} \|x - P_n(x)\| \\ &= 0, \end{aligned}$$

which proves the claim.

From now on, without loss of generality, due to Claim 2 and condition (7.5), we may assume that for all $n \in \mathbb{N}$

$$\varphi_n(T_0) + \left(1 + \frac{1}{\alpha_0^2}\right)\int_{T_0}^{T} \left(|\dot{\zeta}(s)| + \beta(s)\right) ds < \rho. \tag{7.17}$$

Claim 3.: For all $t \in [T_0, T]$

$$\varphi_n^3(t) \leq \varphi_n(T_0)^3 + \frac{3}{\alpha_0^2}\int_{T_0}^{t} \left(|\dot{\zeta}(s)| + \beta(s)\right) \sup_{x\in A(s)} \|x - P_n(x)\|^2 ds, \tag{7.18}$$

where

$$R := \rho + \frac{1}{1-m}\left(\sup_{n\in\mathbb{N}} \|\operatorname{proj}_D(P_n(M0))\| + \int_{T_0}^{T} \psi(s)ds\right)$$

and, due to $(\mathcal{H}_3)$, $A(t) := \overline{\text{co}}\,(C(t) \cap R\mathbb{B})$ is relatively compact for every $t \in [T_0, T]$.
Proof of Claim 3.: The idea of the proof is to use $(\mathcal{H}_2)$. To do that, we proceed to show first that $\varphi_n(t) < \rho$ for all $t \in [T_0, T]$. Indeed, let $t \in [T_0, T]$ where $\dot{x}_n(t)$ exists. Then, due to (ii) and (iii) of Lemma 1.18 and (7.15),

$$\begin{aligned}\dot{\varphi}_n(t) &\le |\dot{\zeta}(t)| + \max_{y^* \in \Gamma_n(t)} \langle y^*, P_n(\dot{x}_n(t))\rangle \\ &\le |\dot{\zeta}(t)| + \|\dot{x}_n(t)\| \\ &\le \left(1 + \frac{1}{\alpha_0^2}\right)\left(|\dot{\zeta}(t)| + \beta(t)\right).\end{aligned}$$

Therefore, according to (7.17), for all $t \in [T_0, T]$

$$\varphi_n(t) \le \varphi_n(T_0) + \left(1 + \frac{1}{\alpha_0^2}\right)\int_{T_0}^{t}\left(|\dot{\zeta}(s)| + \beta(s)\right)ds < \rho,$$

as claimed. □

Now, let $t \in \Omega_n := \{t \in [T_0, T] \colon P_n(x_n(t)) \notin C(t)\}$ where $\dot{x}_n(t)$ exists. Then, due to Lemma 1.18,

$$\begin{aligned}\dot{\varphi}_n(t) &\le |\dot{\zeta}(t)| + \min_{y^* \in \Gamma_n(t)} \langle y^*, P_n(\dot{x}_n(t))\rangle \\ &\le |\dot{\zeta}(t)| - \frac{1}{\alpha_0^2}\left(|\dot{\zeta}(t)| + \beta(t)\right)\langle d_n(t), P_n(d_n(t))\rangle \\ &+ \langle d_n(t), P_n(f_n(t))\rangle \\ &\le \left(|\dot{\zeta}(t)| + \beta(t)\right)\left(1 - \frac{1}{\alpha_0^2}\langle d_n(t), P_n(d_n(t))\rangle\right).\end{aligned}$$

Moreover, due to $(\mathcal{H}_2)$,

$$\begin{aligned}-\langle d_n(t), P_n(d_n(t))\rangle &= \langle d_n(t), d_n(t) - P_n(d_n(t))\rangle + \langle d_n(t), -d_n(t)\rangle \\ &\le \langle d_n(t), d_n(t) - P_n(d_n(t))\rangle - \alpha_0^2 \\ &= \|d_n(t) - P_n(d_n(t))\|^2 - \alpha_0^2.\end{aligned}$$

Hence, for a.e. $t \in \Omega_n$,

$$\dot{\varphi}_n(t) \le \frac{1}{\alpha_0^2}\left(|\dot{\zeta}(t)| + \beta(t)\right)\|d_n(t) - P_n(d_n(t))\|^2.$$

Furthermore, for $t \in \Omega_n$, since $d_n(t) \in \Gamma_n(t)$, Lemma 1.15 ensures the existence of

$$g_n(t) \in \overline{\text{co}}\ \text{Proj}_{C(t)}(P_n(x_n(t)))$$

such that

$$d_n(t) = \frac{1}{\varphi_n(t)}\left(P_n(x_n(t)) - g_n(t)\right).$$

Thus, by virtue of (7.16) and the inequality $\varphi_n(t) < \rho$ for all $t \in [T_0, T]$,

$$\begin{aligned}\|g_n(t)\| &\le \varphi_n(t) + \|P_n(x_n(t))\| \\ &\le \rho + \frac{1}{1-m}\left(\|\operatorname{proj}_D(P_n(M0))\| + \int_{T_0}^{t}\psi(s)ds\right) \\ &\le R,\end{aligned}$$

which entails that $g_n(t) \in A(t)$ for all $t \in \Omega_n$. Thus, for every $t \in \Omega_n$

$$\varphi_n(t)^2 \|d_n(t) - P_n(d_n(t))\|^2 = \|g_n(t) - P_n(g_n(t))\|^2 \leq \sup_{x \in A(t)} \|x - P_n(x)\|^2.$$

Therefore, for $t \notin \Omega_n$, we obtain that for $t \in [T_0, T]$

$$\begin{aligned}\varphi_n^3(t) &= \varphi_n^3(T_0) + 3\int_{T_0}^t \varphi_n^2(s)\dot{\varphi}_n(s)ds \\ &\leq \varphi_n^3(T_0) + \frac{3}{\alpha_0^2}\int_{T_0}^t \left(|\dot{\zeta}(s)| + \beta(s)\right) \sup_{x \in A(s)} \|x - P_n(x)\|^2 ds,\end{aligned}$$

as claimed. □

Claim 4.: $\lim_{n \to +\infty} \varphi_n(t) = 0$ for all $t \in [T_0, T]$.

Proof of Claim 4.: Fix $t \in [T_0, T]$. Since $A(t)$ is relatively compact, Lemma 1.7 (v) asserts that

$$\lim_{n \to +\infty} \sup_{x \in A(t)} \|x - P_n(x)\| = 0.$$

Hence, by Fatou's lemma and (7.18),

$$\begin{aligned}\limsup_{n \to +\infty} \varphi_n^3(t) &\leq \frac{3}{\alpha_0^2} \limsup_{n \to +\infty} \int_{T_0}^t \left(|\dot{\zeta}(s)| + \beta(s)\right) \sup_{x \in A(s)} \|x - P_n(x)\|^2 ds \\ &\leq \frac{3}{\alpha_0^2} \int_{T_0}^t \left(|\dot{\zeta}(s)| + \beta(s)\right) \limsup_{n \to +\infty} \sup_{x \in A(s)} \|x - P_n(x)\|^2 ds \\ &= 0,\end{aligned}$$

as required. □

Claim 5.: $(P_n(x_n(t)))_n$ is relatively compact for all $t \in [T_0, T]$.

Proof of Claim 5.: Fix $t \in [T_0, T]$ and let $s_n(t) \in \operatorname{Proj}_{C(t)}(P_n(x_n(t)))$ (this projection is well defined because $(P_n(x_n))_n$ is uniformly bounded in $C([T_0, T]; H)$). Then, as a result of (7.16),

$$\begin{aligned}\|s_n(t)\| &\leq \varphi(t) + \|P_n(x_n(t))\| \\ &\leq \rho + \frac{1}{1-m}\left(\|\operatorname{proj}_D(P_n(M0))\| + \int_{T_0}^T \psi(s)ds\right) \\ &\leq R,\end{aligned}$$

where we have used (7.16) and the definition of R. Hence, $s_n(t) \in C(t) \cap R\mathbb{B}$. Thus, due to the ball compactness of $C(t)$, there exists a subsequence of $(s_n(t))_n$ (without relabeling) such that $s_n(t) \to s(t)$ as $n \to +\infty$. Therefore, by virtue of Claim 4,

$$\begin{aligned}\limsup_{n \to +\infty} \|P_n(x_n(t)) - s(t)\| &\leq \limsup_{n \to +\infty} [\|P_n(x_n(t)) - s_n(t)\| + \|s_n(t) - s(t)\|] \\ &\leq \limsup_{n \to +\infty} [\varphi_n(t) + \|s_n(t) - s(t)\|] \\ &= 0,\end{aligned}$$

which proves the claim. □

Claim 6.: There exists a subsequence $(x_k)_k$ of $(x_n)_n$ and an absolutely continuous function x such that

(i) $x_k(t) \rightharpoonup x(t)$ in H as $k \to +\infty$ for all $t \in [T_0, T]$,

(ii) $x_k \rightharpoonup x$ in $L^1([T_0, T]; H)$ as $k \to +\infty$,

(iii) $\dot{x}_k \rightharpoonup \dot{x}$ in $L^1([T_0, T]; H)$ as $k \to +\infty$,

(iv) $\|\dot{x}(t)\| \leq \psi(t)$ a.e. $t \in [T_0, T]$, where ψ is the function defined in (7.8).

Proof of Claim 6.: It follows from similar arguments given in Claim 1. □

Claim 7.: $P_k(x_k(t)) \to x(t)$ as $k \to +\infty$ for all $t \in [T_0, T]$.

Proof of Claim 7.: Fix $t \in [T_0, T]$. Since $x_k(t) \rightharpoonup x(t)$ as $k \to +\infty$, from (iv) of Lemma 1.7, it follows that $P_k(x_k(t)) \rightharpoonup x(t)$. Hence, due to the relative compactness of $(P_k(x_k(t)))_k$ (see Claim 5), the claim is proved. □

Claim 8.: For all $t \in [T_0, T]$, $x(t) \in C(t)$.

Proof of Claim 8.: Fix $t \in [T_0, T]$. Then, due to Claim 4 and Claim 7,

$$\begin{aligned} d_{C(t)}(x(t)) &= \limsup_{k\to+\infty} \left(d_{C(t)}(x(t)) - d_{C(t)}(P_k(x_k(t))) + d_{C(t)}(P_k(x_k(t)))\right) \\ &\leq \limsup_{k\to+\infty} \left(\|x(t) - P_k(x_k(t))\| + \varphi_k(t)\right) \\ &= 0, \end{aligned}$$

which proves the claim. □

Summarizing, we have

(i) For each $x \in H$, $G(\cdot, x)$ is measurable,

(ii) for a.e. $t \in [T_0, T]$, $G(t, \cdot)$ is upper semicontinuous from H into H_w,

(iii) $\dot{x}_k \rightharpoonup \dot{x}$ in $L^1([T_0, T]; H)$,

(iv) $P_k(x_k(t)) \to x(t)$ as $k \to +\infty$ for a.e. $t \in [T_0, T]$,

(v) for all $k \in \mathbb{N}$, $\dot{x}_k(t) \in G(t, P_k(x_k(t)))$ for a.e. $t \in [T_0, T]$.

These conditions and the convergence theorem (see [7, Proposition 5] for more details) imply that x is a solution of (7.14), which finishes the proof of Step 2. □

Step 3: The theorem holds.

Proof of Step 3: Since $x(t) \in C(t)$ for all $t \in [T_0, T]$, $x \in \mathcal{C}$ (see (7.3)). Thus, $Mx \in D$ and, hence, $x(T_0) = \text{proj}_D(Mx) = Mx$, which proves the theorem. □

Remark 7.3

1. The hypothesis $(\mathcal{H}_3^F)$ in Theorem 7.2 can be replaced by the following more general condition: There exist $\alpha, \beta \in L^1(T_0, T)$ with $\int_{T_0}^T \alpha(s)ds < 1 - m$ such

that

$$d(0, F(t, x(t))) := \inf\{\|w\| \colon w \in F(t, x)\} \leq \alpha(t)\|x\| + \beta(t),$$

for all $x \in H$ and a.e. $t \in [T_0, T]$. Indeed, by virtue of Gronwall's inequality, it is possible to prove that every solution of (7.6) satisfies

$$\|x\|_\infty \leq R := \frac{1}{1 - m - \int_{T_0}^{T} \alpha(s)ds} \left(\|M0\| + \int_{T_0}^{T} \psi(s)ds\right),$$

where ψ is given by (7.8). Define

$$p_R(x) = \begin{cases} x & \text{if } \|x\| \leq R, \\ R\frac{x}{\|x\|} & \text{if } \|x\| > R. \end{cases}$$

Then, by using the set-valued map $\tilde{G}(t, x) := G(t, p_R(x))$ instead of G in (7.6), we have that $\tilde{G}$ satisfies $(\mathcal{H}_3^F)$ and the same proof applies.

2. When H is a finite dimensional Hilbert space, the condition $(\mathcal{H}_3^M)$ in Theorem 7.2 can be removed. Indeed, if suffices to use Arzela-Ascoli's theorem in Step 1 and Step 6 instead of [111, Theorem 0.2.2.1].

The following result deals with the nonexpansive case. We emphasize that contrary to the contractive case (Theorem 7.2), it is not posible to assure the boundedness of solutions of (7.1) without any additional condition. Therefore, to overcome this difficulty, we assume the boundedness of the convex set D.

Theorem 7.3 *Let $F \colon [T_0, T] \times H \rightrightarrows H$ be satisfying $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$ and $C \colon [T_0, T] \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$. Assume, in addition to $(\mathcal{H}_2^M)$, $(\mathcal{H}_3^M)$, that there exists a convex and bounded set D such that $M\mathcal{C} \subseteq D \subseteq C(T_0)$, where $\mathcal{C}$ is given by* (7.3) *and*

$$\left(1 + \frac{1}{\alpha_0^2}\right) \int_{T_0}^{T} \left(|\dot{\zeta}(s)| + \beta(s)\right) ds < \rho. \tag{7.19}$$

Then, there exists at least one solution of (7.1). *Moreover,*

$$\|\dot{x}(t)\| \leq \frac{1}{\alpha_0^2}|\dot{\zeta}(t)| + \left(1 + \frac{1}{\alpha_0^2}\right)\beta(t) \quad \text{a.e. } t \in [T_0, T].$$

Proof. For each $k \in \mathbb{N}$, let x_k be a solution (whose existence is guaranteed by Step 2. of the proof of Theorem 7.2) of the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -\frac{1}{\alpha_0^2}\left(|\dot{\zeta}(t)| + \beta(t)\right)\partial d_{C(t)}(x(t)) + F(t, x(t)) \cap \beta(t)\mathbb{B} & \text{a.e. } t \in [T_0, T], \\ x(t) \in C(t) & \text{for all } t \in [T_0, T], \\ x(T_0) = \text{proj}_D\left(\frac{k}{k+1}Mx\right). \end{cases}$$

Then, $(x_k(T_0))_k \subseteq D \subseteq C(T_0)$. Thus, since D is bounded, there exists $R > 0$ such that $(x_k(T_0))_k \subseteq D \subseteq R\mathbb{B}$. Hence, for all $k \in \mathbb{N}$ and all $t \in [T_0, T]$

$$\|x_k(t)\| \leq \|x_k(T_0)\| + \int_{T_0}^{t} \|\dot{x}_k(s)\| ds \leq R + \int_{T_0}^{t} \left(\frac{|\dot{\zeta}(s)|}{\alpha_0^2} + \left(1 + \frac{1}{\alpha_0^2}\right) \beta(s) \right) ds.$$

This inequality shows that $(x_k)_k$ is bounded in $C([T_0, T]; H)$ and, due to $(\mathcal{H}_3)$, this gives that the sequence $(x_{k(t)})_k$ is relatively compact for all $t \in [T_0, T]$. Therefore, by using Arzela-Ascoli and Dunford-Pettis theorems, we obtain the existence of a subsequence of $(x_k)_k$ (without relabeling) and an absolutely continuous function $x \colon [T_0, T] \to H$ such that

(i) $(x_k)_k$ converges uniformly to x on $[T_0, T]$,

(ii) $\dot{x}_k \rightharpoonup \dot{x}$ in $L^1([T_0, T]; H)$.

These conditions and the convergence theorem (see [7, Proposition 5] for more details) imply that x satisfies

$$\begin{cases} \dot{x}(t) \in -\dfrac{1}{\alpha_0^2} \left(|\dot{\zeta}(t)| + \beta(t) \right) \partial d_{C(t)}(x(t)) + F(t, x(t)) \cap \beta(t)\mathbb{B} & \text{a.e. } t \in [T_0, T], \\ x(t) \in C(t) & \text{for all } t \in [T_0, T], \\ x(T_0) = \operatorname{proj}_D(Mx). \end{cases}$$

Moreover, since $x(t) \in C(t)$ for all $t \in [T_0, T]$, $x \in \mathcal{C}$ (see (7.3)). Thus, $Mx \in D$. Therefore, $x(T_0) = \operatorname{proj}_D(Mx) = Mx$, which proves the theorem. □

Remark 7.4 When H is a finite dimensional Hilbert space, the condition $(\mathcal{H}_3^M)$ in Theorem 7.3 can be removed (see Remark 7.3).

When M is positively homogeneous, conditions (7.5) and (7.19) in Theorems 7.2 and 7.3, respectively, can be removed.

Theorem 7.4 *Let $F \colon [T_0, T] \times H \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$ and $C \colon [T_0, T] \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$. Assume that M is positively homogeneous, satisfies $(\mathcal{H}_3^M)$ and there exists a convex set D such that $M\mathcal{C} \subseteq D \subseteq C(T_0)$, where $\mathcal{C}$ is given by (7.3). Assume that one of the following two conditions is satisfied:*

i) $(\mathcal{H}_1^M)$ holds.

ii) D is bounded and $(\mathcal{H}_2^M)$ holds.

Then, there exists at least one solution of (7.1).

Proof. Let us consider the set-valued map $C_\lambda(t) := \frac{1}{\lambda} C(t)$ and $\tilde{F}(t, x) := \frac{1}{\lambda} F(t, \lambda x)$, where $\lambda > 0$ is such that

$$\left(1 + \frac{1}{\alpha_0^2}\right) \int_{T_0}^{T} \left(|\dot{\zeta}(s)| + \beta(s) \right) ds < \lambda\rho.$$

Then, for all $s, t \in [T_0, T]$ and $x \in H$

$$\left|d_{C_\lambda(t)}(x) - d_{C_\lambda(s)}(x)\right| = \frac{1}{\lambda}\left|d_{C(t)}(\lambda x) - d_{C(s)}(\lambda x)\right| \leq \frac{1}{\lambda}|\zeta(t) - \zeta(s)|.$$

Therefore, according to Theorem 7.2, in the first case, and Theorem 7.3, in the second case, there exists a solution x_λ of

$$\begin{cases} \dot{x}_\lambda(t) \in -N\left(C_\lambda(t); x_\lambda(t)\right) + \tilde{F}_\lambda(t, x(t)) \cap \dfrac{\beta(t)}{\lambda}\mathbb{B} & \text{a.e. } t \in [T_0, T], \\ x_\lambda(T_0) = Mx_\lambda. \end{cases}$$

Define $x(t) := \lambda x_\lambda(t)$. Then, since M is positively homogeneous, it is not difficult to verify that x is a solution of (7.1). $\square$

Remark 7.5 The argument given in the proof of Theorem 7.4 shows that there are infinitely many solutions of the nonlocal problem (7.1).

As a consequence of Theorem 7.4, we obtain the existence of periodic solutions of the perturbed sweeping process. The following corollary extends the results given in [71] and [46,48], where the authors showed the existence, respectively, for wedged and convex sets compact sets.

Corollary 7.5 *Let $F\colon [T_0, T] \times H \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$ and $C\colon [T_0, T] \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$. Assume that there exists a convex and bounded set D such that $C(T) \subseteq D \subseteq C(T_0)$. Then, there exists at least one solution of*

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) + F(t, x(t)) & a.e.\ t \in [T_0, T], \\ x(T_0) = x(T). \end{cases}$$

Proof. Let $Mx = x(T)$. Then, M satisfies $(\mathcal{H}_2^M)$, $(\mathcal{H}_3^M)$ and $M\mathcal{C} \subseteq C(T) \subseteq D$. Therefore, the result follows from Theorem 7.4. $\square$

The following result, which is a direct consequence of Theorem 7.4, deals with several common nonlocal initial conditions for the sweeping process governed by a fixed set C.

Corollary 7.6 *Let $F\colon [T_0, T] \times H \rightrightarrows H$ satisfying $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$ and $C \subseteq H$ be a fixed compact and convex set. Assume that $M\colon C\left([T_0, T]; H\right) \to H$ is one of the following operators:*

(i) $Mx = x(T)$ (periodic initial condition);

(ii) $Mx = \frac{1}{T-T_0}\int_{T_0}^{T} x(s)ds$ (mean value initial condition);

(iii) $Mx = \sum_{i=1}^{k_0} \alpha_i x(t_i)$ with $\alpha_i \in \mathbb{R}^+$ and $\sum_{i=1}^{k_0} \alpha_i = 1$, where $T_0 < t_1 < \cdots < t_{k_0} \leq T$ (multi-point initial condition).

Then, there exists at least one solution of

$$\begin{cases} \dot{x}(t) \in -N\left(C; x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = Mx. \end{cases}$$

Moreover, $\|\dot{x}(t)\| \le 2\beta(t)$ *for a.e.* $t \in [T_0, T]$.

7.3 The Case H is Compactly Embedded in a Banach Space E

In this section we assume that $(H, \|\cdot\|_H)$ is compactly embedded in a separable Banach space $(E, \|\cdot\|_E)$ (for example, $H = H^1(\Omega)$ and $E = L^2(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is an open domain with Lipschitz boundary).

Let $F\colon [T_0, T] \times H \rightrightarrows H$ be a set-valued map satisfying hypotheses $(\mathcal{H}_1^F)$ and $(\mathcal{H}_2^F)$. In this section we study existence of solutions for the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = Mx, \end{cases} \tag{7.20}$$

where $M\colon C([T_0, T]; H) \to H$ is a (possibly nonlinear) operator and F satisfies the additional hypothesis $(\mathcal{H}_5^F)$ (see Section 7.1). We emphasize that several control problem for first-order partial integro-differential equations (e.g., with $H = H^1(\Omega)$ and $E = L^2(\Omega)$) can be formulated as (7.20) (see, e.g., [20, 21]).

Now we introduce the concept of bounding function. We distinguish between weak and strong bounding function according to whether the infimum or the supremum over F is considered. We point out that our definition of weak bounding function coincides with the given in [21] under the name of merely "bounding function".

Definition 7.7 *Let* $V\colon H \to \mathbb{R}$ *be a locally Lipschitz function such that* $V(x) = 0$ *for* $\|x\|_H = R_0$ *and* $V(x) < 0$ *for* $r_0 < \|x\|_H < R_0$.

a) We say that V *is a* weak bounding function *if* V *is* C^1 *in the ring* $\{x \in H\colon r_0 < \|x\|_H < R_0\}$ *and there exists a sequence* $(n_m)_m \subseteq \mathbb{N}$ *converging to* $+\infty$ *such that for a.e.* $t \in [T_0, T]$

$$\inf_{d \in F(t, P_{n_m}(x))} \langle \nabla V(P_{n_m}(x)), P_{n_m}(d) \rangle \le 0 \quad \text{for all } r_0 < \|P_{n_m}(x)\|_H < R_0. \tag{7.21}$$

b) We say that V *is a* strong bounding function *if there exists a sequence* $(n_m)_m \subseteq \mathbb{N}$ *converging to* $+\infty$ *such that for a.e.* $t \in [T_0, T]$ *and all* $r_0 < \|P_{n_m}(x)\|_H < R_0$

$$\sup_{d \in F(t, P_{n_m}(x))} \min\{DV(P_{n_m}(x); P_{n_m}(d)), D(-V)(P_{n_m}(x); -P_{n_m}(d))\} \le 0. \tag{7.22}$$

Remark 7.6

(i) If V is differentiable at x, then

$$\min\{DV(x;d), D(-V)(x;-d)\} = \langle \nabla V(x), d\rangle .$$

Thus, if V is differentiable in the ring $\{x \in H\colon r_0 < \|x\|_H < R_0\}$, then every strong bounding function is indeed a weak bounding function.

(ii) The bounding function is unaffected by changing F outside the ball $R_0\mathbb{B}$.

(iii) When $V(x) := \frac{1}{2}(\|x\|_H^2 - R_0^2)$, the notion of weak bounding function is equivalent to the well known "Hartman's type condition" (see Example 7.1): For a.e. $t \in [T_0, T]$

$$\inf_{d\in F(t,x)} \langle \nabla V(x), d\rangle \le 0 \quad \text{for all } r_0 < \|x\|_H < R_0.$$

By using the notion of bounding function, we can prove an existence result for (7.20). The statement (ii) of the following theorem extends the results of [20] by allowing to M be a nonlinear map. Moreover, statement (iii) of the following theorem extends [101, Theorem 7] to infinite dimensions and extends the main result of [21], by allowing to M to be a nonlinear map and F to be multivalued upper semicontinuous from E into E_w.

Theorem 7.8 *Assume that H is compactly embedded in E. Let $F\colon [T_0,T]\times H \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_5^F)$. Assume that one of the following conditions is verified:*

(i) $(\mathcal{H}_3^F)$, $(\mathcal{H}_1^M)$ and $(\mathcal{H}_3^M)$ hold.

(ii) $(\mathcal{H}_4^F)$, $(\mathcal{H}_2^M)$ and $(\mathcal{H}_3^M)$ hold, $M(C([T_0,T];R_0\mathbb{B}_H)) \subseteq R_0\mathbb{B}_H$ and there exists a weak bounding function V for F.

(iii) $(\mathcal{H}_4^F)$, $(\mathcal{H}_2^M)$ and $(\mathcal{H}_3^M)$ hold, $M(C([T_0,T];R_0\mathbb{B}_H)) \subseteq R_0\mathbb{B}_H$ and there exists a strong bounding function V for F.

Then, there exists at least one solution of (7.20).

Proof. (i) According to Step 1 from the proof of Theorem 7.2, for each $n \in \mathbb{N}$, there exists x_n solution of

$$\begin{cases} \dot{x}_n(t) \in F(t, P_n(x_n(t))) \cap \beta(t)\mathbb{B}_H & \text{a.e. } t \in [T_0,T], \\ x_n(T_0) = P_n(Mx_n). \end{cases}$$

Define

$$L := \frac{1}{1-m}\left(\|M0\|_H + \int_{T_0}^{T} \beta(s)ds\right).$$

Then, $\|\dot{x}_n(t)\|_H \leq \beta(t)$ for a.e. $t \in [T_0, T]$ and $\|x_n(t)\|_H \leq L$ for all $t \in [T_0, T]$. Indeed, for all $t \in [T_0, T]$

$$\begin{aligned} \|x_n(t)\|_H &\leq \|x_n(T_0)\|_H + \int_{T_0}^t \beta(s)ds \\ &\leq \|Mx_n\|_H + \int_{T_0}^t \beta(s)ds \\ &\leq m \sup_{t\in[T_0,T]} \|x_n(t)\|_H + \|M0\|_H + \int_{T_0}^t \beta(s)ds. \end{aligned}$$

Therefore, as in the proof of Claim 6 from Theorem 7.2, there exists a subsequence of $(x_n)_n$ (without relabeling) and a absolutely continuous function $x\colon [T_0, T] \to H$ such that

$$\begin{aligned} &x_n(t) \to x(t) \text{ weakly in } H \text{ for all } t \in [T_0, T], \\ &x_n \to x \text{ in } L^1_w\left([T_0, T]; H\right), \\ &\dot{x}_n \to \dot{x} \text{ in } L^1_w\left([T_0, T]; H\right). \end{aligned}$$

Moreover, due to the compactness of the embedding $H \hookrightarrow E$, $x_n(t) \to x(t)$ in E for every $t \in [T_0, T]$. These conditions, $(\mathcal{H}_5^F)$ and the convergence theorem (see [7, Proposition 5] for more details) imply that x satisfies $\dot{x}(t) \in F(t, x(t))$ for a.e. $t \in [T_0, T]$. Finally, due to $(\mathcal{H}_3^M)$, $P_n M x_n \to Mx$ weakly in H (up to a subsequence), which finishes the proof.

(ii) Define

$$\begin{aligned} \tilde{F}(t,x) &:= \{d \in F(t,x)\colon \alpha(x)\langle \nabla V(x), d\rangle_H \leq 0\}, \\ G(t,x) &:= \tilde{F}\left(t, \operatorname{proj}_{R_0\mathbb{B}_H}(x)\right) \cap v_{R_0}(t)\mathbb{B}_H, \end{aligned}$$

where

$$\alpha(x) = \begin{cases} 1 & \text{if } r_0 < \|x\|_H < R_0, \\ 0 & \text{otherwise.} \end{cases}$$

By similar arguments as in [20], the set-valued map G satisfies $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$.

Fix $r \in (r_0, R_0)$. For each $n \in \mathbb{N}$ let x_n be a solution (whose existence is guaranteed by Step 1 from the proof of Theorem 7.2) of

$$\begin{cases} \dot{x}_n(t) \in G(t, P_n(x_n(t))) & \text{a.e. } t \in [T_0, T], \\ x_n(T_0) = \operatorname{proj}_{r\mathbb{B}_H}\left(\dfrac{r}{R_0} P_n(M(P_n x_n))\right). \end{cases}$$

Therefore, for all $t \in [T_0, T]$,

$$\|x_n(t)\|_H \leq r + \int_{T_0}^t v_{R_0}(s)ds.$$

After taking a subsequence (without relabeling), we can assume that (7.21) holds.
Now we proceed to prove that $P_n(x_n(t)) \in R_0\mathbb{B}_H$. Indeed, otherwise, since

$$\|P_n(x_n(T_0))\|_H \leq r,$$

we can find $t_0 \in (T_0, T]$ and $\varepsilon > 0$ such that $\|P_n(x_n(t_0))\|_H = R_0$ and $r_0 < \|P_n(x_n(t))\|_H < R_0$ for $t \in (t_0 - \varepsilon, t_0)$. We observe that for all $t \in (t_0 - \epsilon, t_0)$

$$\begin{aligned} G(t, P_n(x(t))) &= \tilde{F}(t, \mathrm{proj}_{R_0\mathbb{B}}(P_n(x_n(t)))) \cap v_{R_0}(t)\mathbb{B}_H \\ &= \tilde{F}(t, P_n(x_n(t))) \cap v_{R_0}(t)\mathbb{B}_H \\ &\subseteq F(t, P_n(x_n(t))) \cap v_{R_0}(t)\mathbb{B}_H. \end{aligned} \tag{7.23}$$

Define $g_n(t) := V\left(P_n(x_n(t))\right)$ in $(t_0 - \delta, t_0)$, where $\delta \in (0, \varepsilon)$ is such that g_n is absolutely continuous in $(t_0 - \delta, t_0)$. Then, $\dot{g}_n(t)$ exists for a.e. $t \in (t_0 - \delta, t_0)$. On the one hand,

$$\int_{t_0-\delta}^{t_0} \dot{g}_n(s)ds = V(P_n(x_n(t_0))) - V(P_n(x_n(t_0 - \delta))) = -V(P_n(x_n(t_0 - \delta))) > 0. \tag{7.24}$$

On the other hand, for a.e. $t \in (t_0 - \delta, t_0)$,

$$\dot{g}_n(t) = \langle \nabla V\left(P_n(x_n(t))\right), P_n(\dot{x}_n(t)) \rangle_H \quad \leq 0,$$

where we have used the definition of G, (7.23) and the definition of weak bounding function. Thus, $\int_{t_0-\delta}^{t_0} \dot{g}_n(s)ds \leq 0$, which gives a contradiction with (7.24). Hence, $P_n(x_n(t)) \in R_0\mathbb{B}_H$ for all $t \in [T_0, T]$.

So, by the assumptions of (ii), $M(P_n x_n) \in R_0\mathbb{B}_H$, which implies that

$$\frac{r}{R_0} P_n M(P_n x_n) \in r\mathbb{B}_H.$$

Thus,

$$x_n(T_0) = \frac{r}{R_0} P_n M(P_n x_n).$$

Therefore, for each $n \in \mathbb{N}$, there exists x_n solution of

$$\begin{cases} \dot{x}_n(t) \in F(t, P_n(x_n(t))) \cap v_{R_0}(t)\mathbb{B}_H & \text{a.e. } t \in [T_0, T], \\ x_n(T_0) = \dfrac{r}{R_0} P_n(M(P_n x_n)). \end{cases}$$

Then, by passing to the limit (up to a subsequence), as in (i), we obtain the existence of a solution $x\colon [T_0, T] \to R_0\mathbb{B}_H$ of

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) \cap v_{R_0}(t)\mathbb{B}_H & \text{a.e. } t \in [T_0, T], \\ x(T_0) = \dfrac{r}{R_0} Mx. \end{cases}$$

Let $(r_k)_k$ be a sequence converging to R_0 with $r_k \in (r_0, R_0)$. Then, for each $k \in \mathbb{N}$, there exists x_k solution of

$$\begin{cases} \dot{x}_k(t) \in F(t, x_k(t)) \cap v_{R_0}(t)\mathbb{B}_H & \text{a.e. } t \in [T_0, T], \\ x_k(T_0) = \dfrac{r_k}{R_0} M x_k, \end{cases}$$

with $x_k(t) \in R_0\mathbb{B}_H$ for all $t \in [T_0, T]$. Therefore, by passing to the limit (up to a subsequence), as in (i), we obtain the existence of a solution x of

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) \cap v_{R_0}(t)\mathbb{B}_H & \text{a.e. } t \in [T_0, T], \\ x(T_0) = Mx, \end{cases}$$

which finishes the proof.

(iii) Fix $r \in (r_0, R_0)$. For each $n \in \mathbb{N}$ let x_n be a solution (whose existence is guaranteed by Step 1 from the proof of Theorem 7.2) of

$$\begin{cases} \dot{x}_n(t) \in F(t, \text{proj}_{R_0\mathbb{B}_H}(P_n(x_n(t))) \cap v_{R_0}(t)\mathbb{B}_H & \text{a.e. } t \in [T_0, T], \\ x_n(T_0) = \text{proj}_{r\mathbb{B}_H}\left(\dfrac{r}{R_0} P_n(M(P_n x_n))\right). \end{cases}$$

Therefore, for all $t \in [T_0, T]$,

$$\|x_n(t)\|_H \leq r + \int_{T_0}^t v_{R_0}(s)ds.$$

After taking a subsequence (without relabeling), we can assume that (7.22) holds.

We proceed to prove that $P_n(x_n(t)) \in R_0\mathbb{B}_H$. Indeed, otherwise, since

$$\|P_n(x_n(T_0))\|_H \leq r,$$

we can find $t_0 \in (T_0, T]$ and $\varepsilon > 0$ such that $\|P_n(x(t_0))\|_H = R_0$ and $r_0 < \|P_n(x_n(t))\|_H < R_0$ for $t \in (t_0 - \varepsilon, t_0)$. We observe that for all $t \in (t_0 - \varepsilon, t_0)$

$$F(t, \text{proj}_{R_0\mathbb{B}}(P_n(x_n(t)))) \cap v_{R_0}(t)\mathbb{B}_H = F(t, P_n(x_n(t))) \cap v_{R_0}(t)\mathbb{B}_H \subseteq F(t, P_n(x_n(t))). \tag{7.25}$$

Define $g_n(t) := V(P_n(x_n(t)))$ in $(t_0 - \delta, t_0)$, where $\delta \in (0, \varepsilon)$ is such that g_n is absolutely continuous in $(t_0 - \delta, t_0)$. Then, $\dot{g}_n(t)$ exists for a.e. $t \in (t_0 - \delta, t_0)$. On the one hand,

$$\int_{t_0-\delta}^{t_0} \dot{g}_n(s)ds = V(P_n(x_n(t_0))) - V(P_n(x_n(t_0 - \delta))) = -V(P_n(x_n(t_0 - \delta))) > 0, \tag{7.26}$$

because $\|P_n(x(t_0))\|_H = R_0$. On the other hand, for a.e. $t \in (t_0 - \delta, t_0)$,

$$\begin{aligned} \lim_{h \to 0} \frac{g_n(t+h) - g_n(t)}{h} &= \lim_{h \to 0} \frac{V(P_n(x_n(t+h))) - V(P_n(x_n(t)))}{h} \\ &= \lim_{h \to 0} \frac{V(P_n(x_n(t)) + hP_n\dot{x}_n(t)) - V(P_n(x_n(t)))}{h} \\ &= DV\left(P_n(x_n(t)); P_n\dot{x}_n(t)\right) \\ &= D(-V)\left(P_n(x_n(t)); -P_n\dot{x}_n(t)\right) \\ &\leq 0, \end{aligned}$$

where we have used (7.25) and the definition of the strong bounding function for F. Thus, $\int_{t_0-\delta}^{t_0} \dot{g}_n(s)ds \leq 0$, which gives a contradiction with (7.26). The rest of the proof, follows as in (ii).

□

Remark 7.7 If $M\colon C([T_0,T];H) \to H$ satisfies $M(0)=0$, which is true if, e.g., M is linear, then with the notation of Theorem 7.8,

$$M\left(C\left([T_0,T];R_0\mathbb{B}\right)\right) \subseteq R_0\mathbb{B}.$$

Indeed, if $x \in C([T_0,T];R_0\mathbb{B})$, then $\|Mx\| \leq \|x\| \leq R_0$.

7.4 Tangential conditions

In this section, we give an abstract result for the abstract problem (7.20) in finite dimensions.

Let us consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t,x(t)) & \text{a.e. } t \in [T_0,T], \\ x(t) \in D & \text{for all } t \in [T_0,T], \\ x(T_0) = x_0 \in D, \end{cases} \tag{7.27}$$

and the set-valued map $\mathcal{S}\colon D \rightrightarrows D$ defined for each $x_0 \in D$ as, $\mathcal{S}(x_0)$, the set of solutions of (7.27). A classical approach to find solutions of (7.20) is to apply some fixed point theorem to the set-valued map $M \circ \mathcal{S}$. Of course, some conditions are needed to obtain the nonemptiness of the values of $\mathcal{S}$. These conditions, generally, are of tangential type for some appropriate tangent cone. In fact, it is well known that existence solutions of (7.27) can be obtained when the set-valued map F has a nonempty intersection with the Bouligand tangent cone (see [15, Theorem 2]). However, in order to get some topological properties of the values of $\mathcal{S}$, we will ask for a strong property, namely, the intersection of the set-valued map F with the Clarke tangent cone is nonempty, i.e.,

$$F(t,x) \cap T(D;x) \neq \emptyset \quad \text{for all } (t,x) \in [T_0,T] \times D. \tag{7.28}$$

The following proposition is a direct consequence of [15, Theorem 16].

Proposition 7.9 *Let H be a finite-dimensional Hilbert space. Let $D \subseteq H$ be a closed and bounded set. Assume that D is positively α-far and* (7.28) *holds. Then, for any $x_0 \in D$, the set $\mathcal{S}(x_0)$ of solutions of* (7.27) *is nonempty, compact and an R_δ-set. Moreover, the set-valued map $\mathcal{S}\colon D \rightrightarrows C([T_0,T];D)$ is upper semicontinuous.*

Remark 7.8

i) If F is single-valued with $F(t,\cdot)$ continuous for all $t \in [T_0,T]$, then (7.28) is equivalent to

$$F(t,x) \cap T^B(D;x) \neq \emptyset \quad \text{for all } (t,x) \in [T_0,T] \times D.$$

Indeed, let $(x_n)_n \subseteq D$ converging to $x \in D$. Then, for all $t \in [T_0,T]$, $F(t,x_n) \in T^B(D;x_n)$ and, $F(t,x) = \lim_{n\to+\infty} F(t,x) \in \liminf_{y\to x, y\in D} T^B(D;y) = T(D;x)$.

ii) See [15, Example 4] for an example of a positively α-far set D and a set-valued map F whose intersection with the Bouligand tangent cone is nonempty but the solution map $\mathcal{S}$ does not have R_δ-values.

Now we can state an existence result for (7.20).

Theorem 7.10 *Let H be a finite-dimensional Hilbert space. Assume that $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_4^F)$ hold. Let M be a Lipschitz map such that there exists a closed, contractible, positively α-far and bounded set D, satisfying* (7.28)*, such that $M\left(C\left([T_0,T];D\right)\right) \subseteq D$. Then, there exists at least one solution of* (7.20)*. Moreover, $x(t) \in D$ for all $t \in [T_0,T]$.*

Proof. It is enough to apply Proposition 1.13 with $X := D$, $\Phi := \mathcal{S}$ and $f := M$. $\square$

The following result gives a characterization of the tangential condition (7.28) for convex sets.

Proposition 7.11 ([83]) *Let $S \neq H$ be a closed convex set and $t \in [T_0,T]$. Then, the following conditions are equivalent:*

i) $F(t,x) \cap T(S;x) \neq \emptyset$ *for all* $x \in S$.

ii) $\inf_{v\in F(t,x)} \langle v,\zeta\rangle \leq 0$ *for all* $\zeta \in N\left(S;x\right)$ *and* $x \in S$

iii) $\inf_{v\in F(t,x)} \langle v,\zeta\rangle \leq 0$ *for all* $\zeta \in \partial d_S(x)$ *and* $x \in S$.

iv) $\inf_{v\in F(t,x)} \langle v,\zeta\rangle \leq 0$ *for all* $\zeta \in \partial \Delta_S(x)$ *and* $x \in \operatorname{bd} S$, *where* $\Delta_S(x) = d_S(x) - d_{S^c}(x)$.

Example 7.1 Let us consider $S := R_0\mathbb{B}$. Then, the condition $F(t,x) \cap T\left(S;x\right) \neq \emptyset$ is equivalent to

$$\inf_{v\in F(t,x)} \langle v,x\rangle \leq 0 \quad \text{for all } x \text{ with } \|x\| = R_0. \tag{7.29}$$

Inequality (7.29) is known in the literature as Hartman's condition and was first used by Hartman in the context of second order systems (see [80]). Since then, it has been used to deal with periodic problems (see, e.g., [6, 20]).

Example 7.2 Let $V\colon H \to \mathbb{R}$ be a convex function such that $S := \{x \in H\colon V(x) \leq 0\}$ is bounded with nonempty interior. Then, the condition (7.28) is equivalent to

$$\inf_{v \in F(t,x)} \langle v, \zeta \rangle \leq 0 \quad \text{for all } \zeta \in \partial V(x) \text{ and } x \in S \text{ with } V(x) = 0.$$

Example 7.3 Let $V\colon H \to \mathbb{R}$ be a C^1 function. Define $S := \{x \in H\colon V(x) \leq 0\}$ and assume that S is bounded, $\operatorname{bd} S = \{x \in H\colon V(x) = 0\}$ and that $\nabla V(x) \neq 0$ for all $x \in \partial S$. Then, S is positively α-far and condition (7.28) is equivalent to

$$\inf_{v \in F(t,x)} \langle v, \nabla V(x) \rangle \leq 0 \quad \text{for all } x \in \operatorname{bd} S. \tag{7.30}$$

If (7.30) holds for all $x \in H$, it is said that V is a weak Lyapunov function for F (see [55]). Therefore, the existence of a weak Lyapunov function for F, with bounded level sets, implies the existence of solutions for (7.20).

7.5 An application to nonlocal differential complementarity systems

Let K be a closed convex cone in $\mathbb{R}^m$ and K^* be its dual cone. In this section, we consider differential complementarity systems (CDSs), which are differential equations coupled with complementarity conditions (see [121] for more details). More specifically,

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) & \text{a.e. } t \in [T_0, T], \\ K \ni u(t) \perp (G(t, x(t)) + F(u(t))) \in K^* & \text{a.e. } t \in [T_0, T], \\ x(T_0) = Mx, \end{cases} \tag{7.31}$$

where $f\colon [T_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $G\colon [T_0, T] \times \mathbb{R}^n \to \mathbb{R}^m$ and $F\colon \mathbb{R}^m \to \mathbb{R}^m$ are continuous mappings; $M\colon C([T_0, T]; \mathbb{R}^n) \to \mathbb{R}^n$ is a (possible nonlinear) operator.

In order to give sufficient conditions for the existence of solutions of (7.31), we consider the following hypotheses:

($\mathcal{A}_1$) for each $(t, z) \in [T_0, T] \times \mathbb{R}^n$ the set

$$f(t, z, \Omega) := \{f(t, z, y)\colon y \in \Omega\}$$

is convex for every convex subset $\Omega \subseteq \mathbb{R}^m$.

($\mathcal{A}_2$) for every bounded subset $Z \subseteq \mathbb{R}^n \times \mathbb{R}^m$ there exists $\alpha_Z > 0$ such that $\|f(t, z, w)\| \leq \alpha_Z$ for $(t, z, w) \in [T_0, T] \times Z$.

($\mathcal{A}_3$) for every bounded subset $\Omega \subseteq \mathbb{R}^n$there exists $\gamma_\Omega > 0$ such that

$$\|G(t, z)\| \leq \gamma_\Omega \quad \text{for } (t, z) \in [T_0, T] \times \Omega.$$

($\mathcal{A}_4$) F is monotone and there exists $a_* > 0$ such that

$$\langle x, F(x)\rangle \geq a_* |x\|^2 \quad \text{for all } x \in K.$$

Remark 7.9 A common example is $f(t,x,y) \equiv \tilde{f}(t,x) + B(t,x)u$ (see [121] for more details).

Let $U\colon [T_0,T] \times \mathbb{R}^n \rightrightarrows K$ be the set-valued map defined as

$$U(t,z) := \mathrm{SOL}\,(K, G(t,z) + F) = \{w \in K\colon \langle w, G(t,z) + F(w)\rangle = 0\}.$$

According to [101, Lemma 9], under ($\mathcal{A}_4$), for every $z \in \mathbb{R}^n$, the set $U(t,z)$ is nonempty, convex and closed. Consider $\Phi\colon [T_0,T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ the set-valued map

$$\Phi(t,x) := \{f(t,x,w)\colon w \in U(t,x)\}.$$

Then, according to [101, Lemma 10], under ($\mathcal{A}_1$)-($\mathcal{A}_4$), Φ satisfies ($\mathcal{H}_1^F$), ($\mathcal{H}_2^F$) and ($\mathcal{H}_4^F$). Thus, the existence of solutions for (7.31) can be obtained from the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in \Phi(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = Mx. \end{cases}$$

Therefore, by virtue of Theorems 7.3 and 7.8, we obtain the following result, which improves [101, Theorem 12] by allowing to M be a nonlinear map. Moreover, the statements (i) and (ii), in the following theorem, are new.

Theorem 7.12 *Assume, in addition to* ($\mathcal{A}_1$)-($\mathcal{A}_4$), *that one of the following conditions is verified:*

(i) ($\mathcal{H}_1^M$) *and* ($\mathcal{H}_3^F$) *hold.*

(ii) ($\mathcal{H}_2^M$) *holds and there exists a weak bounding function* V *for* Φ.

(iii) ($\mathcal{H}_2^M$) *holds and there exists a strong bounding function* V *for* Φ.

Then, there exists at least one solution of (7.31).

7.6 An application to Hysteresis

In this section, we illustrate our results by giving an application to the existence of periodic solutions for the Play operator (see [94] for more details).

We denote by $W^{1,1}_{\text{per}}([T_0,T]; H)$ the space of periodic absolutely continuous functions. Let $g\colon [T_0,T] \times H \times H \times H \to H$ be a continuous function such that

$$\|g(t,x,y,w)\| \leq \beta(t) \quad \text{a.e. } (t,x,y,w) \in [T_0,T] \times H \times H \times H. \tag{7.32}$$

Given $y \in W^{1,1}_{\text{per}}([T_0,T];H)$ we consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N\left(K(y(t)); x(t)\right) + g(t, x(t), y(t), \dot{y}(t)) & \text{a.e. } t \in [T_0,T], \\ x(T_0) = x(T), \end{cases} \tag{7.33}$$

where $K\colon H \rightrightarrows H$ is a κ-Lipschitz set-valued map with nonempty, compact and convex values satisfying $K(y) \subseteq R\,\mathbb{B}$ for some $R > 0$.

Proposition 7.13 *Under the above conditions, for all $y \in W^{1,1}_{per}([T_0,T];H)$ there exists at least one solution $x \in W^{1,1}_{per}([T_0,T];H)$ of (7.33). Moreover,*

$$\|\dot{x}(t)\| \leq \kappa\|\dot{y}(t)\| + 2\beta(t) \quad \textit{a.e. } t \in [T_0,T].$$

Proof. Define $C(t) := K(y(t))$ and $F(t,x) = g(t,x,y(t),\dot{y}(t))$. To show existence of solutions, we verify the hypotheses of Corollary 7.5.

- $(\mathcal{H}_1)$ holds: Let $x \in H$ and $t, s \in [T_0,T]$. Then

$$\begin{aligned} |d(x,C(t)) - d(x,C(s))| &= |d(x,C(y(t))) - d(x,C(y(s)))| \\ &\leq \kappa\|y(t) - y(s)\| \\ &\leq \kappa\left|\int_{T_0}^{t}\|\dot{y}(\tau)\|d\tau - \int_{T_0}^{s}\|\dot{y}(\tau)\|d\tau\right|, \end{aligned}$$

  which shows that $(\mathcal{H}_1)$ holds with $\zeta(t) := \kappa\int_{T_0}^{t}\|\dot{y}(\tau)\|d\tau$.

- $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$ hold: It follows from the continuity of g, the fact that $y \in W^{1,1}_{\text{per}}([T_0,T];H)$ and (7.32).

- $C(T) \subseteq D \subseteq C(T_0)$ for some set D convex and bounded: Indeed, if $D := C(T_0) \cap R\,\mathbb{B}$ then,

$$C(T) = K(y(T)) = K(y(T_0)) = C(T_0) \cap R\mathbb{B},$$

  where we have used that $y \in W^{1,1}_{\text{per}}([T_0,T];H)$.

Thus, the existence for (7.33) follows from Corollary 7.5. □

Remark 7.10 Proposition 7.13 allows us to define the set-valued Play operator

$$P\colon W^{1,1}_{\text{per}}([T_0,T];H) \rightrightarrows W^{1,1}_{\text{per}}([T_0,T];H),$$

which to every function y associates the set of solutions of (7.33). Thus, the Play operator is well defined for inputs in $W^{1,1}_{\text{per}}([T_0,T];H)$.

Chapter 8

Existence and Lyapunov pairs for the perturbed sweeping process governed by a fixed set

In this chapter, which is based on [143], we study existence and Lyapunov pairs for the perturbed sweeping process governed by a fixed set, that is, the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N(S; x(t)) + F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in S, \end{cases} \tag{8.1}$$

where $S \subseteq H$ is a merely closed and ball compact set, $N(S; x)$ denotes the Clarke normal cone to S at x and $F\colon [T_0, T] \times H \rightrightarrows H$ is a given set-valued map with nonempty closed and convex values.

The study of differential inclusions involving normal cones goes back a long time. In the convex case, they are included in the so-called evolution equations governed by maximal monotone operators, which is a well known subject (see [35, 85] and the references therein). They also appears in the theory of projected dynamical systems which, as far as we know, began with the works of Henry [81, 82]. In these papers, to study some planning procedures in economy, Henry introduced the following differential inclusion:

$$\begin{cases} -\dot{x}(t) \in \operatorname{proj}_{T_S(x(t))}(F(x(t))) & \text{a .e. } t \in [T_0, T], \\ x(T_0) = x_0 \in S, \end{cases} \tag{8.2}$$

where $S \subseteq \mathbb{R}^n$ is a closed convex set, $T_S(\cdot)$ denotes tangent cone to C and $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an upper semicontinuous set-valued map. Henry showed existence and equivalence results for (8.1) and (8.2). Next, Cornet [60] relaxed the convexity assumption on S to tangential regularity. Since then, projected dynamical systems has been studied by several authors (see for instance [38,56,69,131]) and the equivalence with differential variational inequalities and sweeping processes is well known.

Two years before the work of Henry, Moreau, in his seminal papers [113, 114], introduced the so-called sweeping process, which correspond to the differential inclusion (8.1) with a convex moving set $S(t)$ without perturbation. In these papers, to study some mechanical problems arising in elastoplasticity, Moreau introduced the so-called catching-up algorithm to deal with the existence of solutions. Since then, this algorithm has been used by several authors to show existence of solutions for the perturbed sweeping process. We can mention, e.g., [22, 34] for uniformly prox-regular sets, [118, 119] for uniformly subsmooth sets, among others. In the first part of the chapter, we show existence through this algorithm for merely closed and ball compact sets.

The work of Moreau was the starting point of several developments related to perturbed sweeping process with regular and nonregular moving sets. We refer to [19, 29, 34, 44, 47, 57, 65, 75, 87, 104, 105, 115, 130, 135] for more details. In this respect, it is worth pointing out that existence results for the sweeping process with merely closed moving sets already exist in the literature. We can mention the work of Benabdellah [19] and Colombo and Goncharov [57] for the unperturbed sweeping process and the work of Thibault [135] for the perturbed sweeping process in finite dimensions. Thus, our aim is to show existence of the sweeping process through the catching-up, which could be useful to deal with practical problems.

The second part of the chapter is devoted to Lyapunov pairs for the sweeping process (8.1). Lyapunov pairs are the central idea behind the Lyapunov method. This indirect approach is relevant because it does not require an explicit expression for the solutions of the dynamical system. This is especially useful when dealing with complex real-world applications. Moreover, the Lyapunov method allows to address several stability properties of differential inclusions as asymptotic stability, existence of equilibria, stabilization among others (see, for example, [53–55]).

Characterizations of smooth and nonsmooth Lyapunov pairs has been considered for different dynamical systems by several authors (see [10, 53–55] and the references given there). In the present case, Adly, Hantoute and Théra [3, 4] give explicit criteria for Lyapunov pairs for maximal monotone evolution equations, which includes the sweeping process driven by a fixed convex set. Then, Hantoute and Mazade [78] give explicit criteria for Lyapunov functions for the sweeping process driven by a fixed uniformly prox-regular set. Unfortunately, it is well known that some dynamical systems do not admit smooth Lyapunov pairs (see [54]). Thus, it is very important to deal with the nonsmooth Lyapunov pairs. Here is where the subdifferential theory has been very helpful. In this setting, the work of Clarke et al [55] has became a benchmark because they characterize Lyapunov pairs for differential inclusions by using the proximal subdifferential. The proximal subdifferential is the smallest reasonable subdifferential that allows a characterization of lower semicontinuous Lyapunov pairs. We follow this path and give an explicit criteria, involving the proximal subdifferential, of weak Lyapunov pairs for the sweeping process. It is worth pointing out that our result, in contrast with [3, 4, 78], does not involve the singular (horizon) subdifferential which gives a simpler criterion.

The chapter is organized as follows. After some preliminaries, in Section 8.1 we give an existence result through the catching-up algorithm. Then, in Section 8.2 we give a criteria for weak Lyapunov pairs for the sweeping process. As a result, we also give a criteria for weak invariance for the sweeping process. Finally, in Section 8.3, we give some applications to hysteresis and crowd motion.

8.1 The Catching-up Algorithm

In this section, we show existence for the sweeping process (8.1). More specifically, given a closed ball compact set S, we show that the catching-up algorithm converges uniformly (up to a subsequence) to a solution of (8.1).

Throughout this section, $F\colon [T_0, T] \times H \rightrightarrows H$ will be a set-valued map with nonempty, closed and convex values. Moreover, we will consider the following conditions:

($\mathcal{H}_1^F$) F is upper semicontinuous from $[T_0, T] \times H$ into H_w.

($\mathcal{H}_2^F$) There exists $h\colon H \to \mathbb{R}^+$ Lipschitz continuous such that

$$d(0, F(t, x)) := \inf\{\|w\| \colon w \in F(t, x)\} \leq h(x),$$

for all $x \in H$ and a.e. $t \in [T_0, T]$.

The following theorem, which is the main result of this section, asserts the existence of solutions for the sweeping process (8.1) for a merely closed set and ball compact set S. This result is in line with [19, 57, 135] and extends the result given in [131] for bounded sleek sets.

Theorem 8.1 *Assume that S is a closed and ball compact subset of H and that $F\colon [T_0, T] \times H \rightrightarrows H$ satisfies ($\mathcal{H}_1^F$) and ($\mathcal{H}_2^F$). Then, for any $x_0 \in S$, there exists at least one Lipschitz solution x of the sweeping process* (8.1)*. Moreover,*

$$\|\dot{x}(t)\| \leq 2h(x(t)) \quad a.e.\ t \in [T_0, T].$$

Proof. Let $n \in \mathbb{N}$ and define $\mu_n := (T - T_0)/n$. Consider the partition of $[T_0, T]$ defined by $t_k^n := T_0 + k \cdot \mu_n$ for $k = 0, \ldots, n$. For each $(t, x) \in [T_0, T] \times H$ denote by $f(t, x)$ the element of minimal norm of the closed convex set $F(t, x)$, that is,

$$f(t, x) := \text{proj}_{F(t,x)}(0).$$

It is clear from ($\mathcal{H}_2^F$) that $\|f(t, x)\| \leq h(x)$ for all $(t, x) \in [T_0, T] \times H$.
We will construct a sequence of Lipschitz functions $(x_n)_n$ which converges (up to a subsequence) to a solution of the sweeping process (8.1).
Define the functions δ_n and θ_n as

$$\delta_n(t) = \begin{cases} t_k^n & \text{if } t \in [t_k^n, t_{k+1}^n[\\ t_{n-1}^n & \text{if } t = T, \end{cases}$$

and

$$\theta_n(t) = \begin{cases} t_{k+1}^n & \text{if } t \in [t_k^n, t_{k+1}^n[\\ T & \text{if } t = T. \end{cases}$$

It is clear that $\theta_n(t) \to t$ and $\delta_n(t) \to t$ uniformly as $n \to \infty$.

Put $x_0^n := x_0 \in S$ and for $k = 0, \ldots, n-1$ we define

$$x_{k+1}^n \in \text{Proj}_S \left(x_k^n + \mu_n \cdot f(t_k^n, x_k^n) \right),$$

where the right-hand side is non empty because S is ball compact. Moreover, due to $(\mathcal{H}_2^F)$, we observe that for $k = 0, \ldots, n-1$

$$\|x_{k+1}^n - x_k^n\| \leq d_S \left(x_k^n + \mu_n f(t_k^n, x_k^n) \right) + \mu_n \|f(t_k^n, x_k^n)\| \leq 2\mu_n h(x_k^n).$$

Thus, if L_h is the Lipschitz constant of h,

$$\|x_{k+1}^n\| \leq (1 + 2\mu_n L_h)\|x_k^n\| + 2\mu_n h(0) \quad \text{for all } k = 0, \ldots, n-1,$$

which, due to [55, p. 183], entails

$$\begin{aligned} \|x_{k+1}^n\| &\leq (\|x_0\| + 2(k+1)\mu_n h(0)) \exp\left(2L_f(k+1)\mu_n\right) \\ &\leq M := (\|x_0\| + 2(T - T_0)h(0)) \exp\left(2L_f(T - T_0)\right). \end{aligned}$$

For any $t \in [t_k^n, t_{k+1}^n]$ with $k = 0, \ldots, n-1$, we put

$$x_n(t) := \frac{t_{k+1}^n - t}{\mu_n} x_k^n + \frac{t - t_k^n}{\mu_n} x_{k+1}^n.$$

Then, for a.e. $t \in [t_k^n, t_k^{n+1}]$, $\|x_n(t)\| \leq M$ and

$$\|\dot{x}_n(t)\| = \|x_{k+1}^n - x_k^n\|/\mu_n \leq 2h(x_n(\delta_n(t))).$$

Since $x - y \in \|x - y\|\partial d_S(y)$ for any $y \in \text{Proj}_S(x)$ (see Lemma 1.2), we obtain

$$\dot{x}_n(t) \in -2h(x_k^n)\partial d_S \left(x_n(t_{k+1}^n) \right) + f(t_k^n, x_n(t_k^n)) \quad \text{a.e. } t \in [t_k^n, t_{k+1}^n]. \tag{8.3}$$

Furthermore, the definitions of δ_n and θ_n together with (8.3) gives, for a.e. $t \in [T_0, T]$

$$\dot{x}_n(t) \in -2h(x_n(\delta_n(t)))\partial d_S \left(x_n(\theta_n(t)) \right) + f\left(\delta_n(t), x_n(\delta_n(t)) \right). \tag{8.4}$$

Moreover, due to the definition of x_n, for all $t \in [T_0, T]$

$$\begin{aligned} d_S(x_n(t)) &\leq 2\mu_n h(x_n(\delta_n(t))) \\ &\leq 2\mu_n L_h \|x_n(\delta_n(t))\| + 2\mu_n h(0) \\ &\leq 2\mu_n \left(L_h M + h(0) \right). \end{aligned} \tag{8.5}$$

Fix $t \in [T_0, T]$ and define $K(t) := \{x_n(t) \colon n \in \mathbb{N}\}$. We claim that $K(t)$ is relatively compact. Indeed, let $(x_m(t))_m \subseteq K(t)$ and take $y_m(t) \in \text{Proj}_S(x_m(t))$ (the projection exists due to the ball compactness of S). Moreover, according to (8.5),

$$\|y_n(t)\| \leq d_S(x_n(t)) + \|x_n(t)\| \leq R := 2\mu_n \left(L_h M + h(0) \right) + M.$$

This entails that $y_n(t) \in S \cap R\mathbb{B}$. Thus, by the ball compactness of S, there exists a subsequence $(y_{m_k}(t))_{m_k}$ of $(y_m(t))_m$ converging to some y as $k \to +\infty$. Then,

$$\|x_{m_k}(t) - y\| \leq d_S(x_{m_k}(t)) + \|y_{m_k}(t) - y\|,$$

which implies that $K(t)$ is relatively compact.

Therefore, by virtue of (8.5), Arzela-Ascoli and Dunford-Pettis theorems (see Theorems 1.5 and 1.3), we obtain the existence of a Lipschitz function x and a subsequence $(x_k)_k$ of $(x_n)_n$ such that

(i) x_k converges uniformly to x on $[T_0, T]$.

(ii) $\dot{x}_k \rightharpoonup \dot{x}$ in $L^1([T_0, T]; H)$.

(iii) $x_k(\theta_k(t)) \to x(t)$ for all $t \in [T_0, T]$.

(iv) $x_k(\delta_k(t)) \to x(t)$ for all $t \in [T_0, T]$.

These conditions, the convergence theorem (see [7, Proposition 5] for more details) and (8.4) imply that x satisfies

$$\begin{cases} \dot{x}(t) \in -2h(x(t))\partial d_S(x(t)) + F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0. \end{cases}$$

Furthermore, according to (8.5), we obtain that $x(t) \in S$ for all $t \in [T_0, T]$. Finally, x is a solution of (8.1) because $\partial d_S(x) \subseteq N(S; x)$ for all $x \in S$. □

Remark 8.1 If the set S is r-uniformly prox-regular and F is single-valued, then there exists a unique solution of (8.1) which satisfies

$$\|\dot{x}(t)\|^2 = \langle \dot{x}(t), F(t, x(t)) \rangle \quad \text{a.e. } t \in [T_0, T].$$

Thus, in particular,

$$\|\dot{x}(t) - F(t, x(t))\| \leq \|F(t, x(t))\| \quad \text{a.e. } t \in [T_0, T].$$

These facts are well known and the compactness of S is not needed here.

8.2 Lyapunov Pairs and Invariance

In this section we give an explicit criterion for weak Lyapunov pairs and weak Lyapunov functions for the sweeping process (8.1). Throughout this section we assume that $F\colon [T_0, T] \times H \rightrightarrows H$ with nonempty, closed and convex values. Moreover, we will consider the following conditions:

$(\mathcal{H}_3^F)$ $F(\cdot, \cdot)$ is scalarly $\mathcal{L} \otimes \mathcal{B}$ measurable on $[T_0, T] \times H$.

($\mathcal{H}_4^F$) For a.e. $t \in [T_0, T]$, $F(t, \cdot)$ is upper semicontinuous from H into H_w.

($\mathcal{H}_5^F$) There exist $\alpha \in L^1(T_0, T)$ and $h \colon H \to \mathbb{R}^+$ Lipschitz such that

$$\|F(t, x)\| := \sup\{\|w\| \colon w \in F(t, x)\} \leq \alpha(t)h(x),$$

for all $x \in H$ and a.e. $t \in [T_0, T]$.

Definition 8.2 *Let $V \colon H \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function and $W \colon H \to \mathbb{R}$ be continuous. We say that (V, W) forms a* weak Lyapunov par *for the sweeping process (8.1) if for every $x_0 \in S$ there exists x solution of (8.1) such that*

$$V(x(t)) + \int_{T_0}^{t} W(x(s))ds \leq V(x_0) \quad \textit{for all } t \in [T_0, T].$$

We will consider the following Hypotheses on V and W:

($\mathcal{H}_1^V$) $V \colon H \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous function with $\operatorname{dom} V \subseteq S$.

($\mathcal{H}_2^V$) $W \colon H \to \mathbb{R}$ is an lower semicontinuous function with

$$0 \leq W(x) \leq \beta\,(1 + \|x\|) \quad \text{for all } x \in H,$$

for some $\beta \geq 0$.

Proposition 8.3 *Assume, in addition to ($\mathcal{H}_3^F$), ($\mathcal{H}_4^F$) and ($\mathcal{H}_5^F$), that S is ball compact and ($\mathcal{H}_1^V$) and ($\mathcal{H}_2^V$) hold. Then (V, W) forms a weak Lyapunov pair for* (8.1) *if and only if for all $n \in \mathbb{N}$, (V, W_n) forms a weak Lyapunov pair for* (8.1), *where*

$$W_n(x) := \inf\{W(y) + n\|x - y\| \colon y \in H\}.$$

Proof. If (V, W) forms a weak Lyapunov pair for (8.1), then for $x_0 \in S$ there exists a solution x of (8.1) such that

$$V(x(t)) + \int_{T_0}^{t} W_n(x(s))ds \leq V(x(t)) + \int_{T_0}^{t} W(x(s))ds \leq V(x_0),$$

that is, (V, W_n) forms a weak Lyapunov pair for (8.1). Reciprocally, if for all $n \in \mathbb{N}$, (V, W_n) forms a weak Lyapunov pair for (8.1), for all $x_0 \in S$, there exist x_n solution of (8.1) such that

$$V(x_n(t)) + \int_{T_0}^{t} W_n(x_n(s))ds \leq V(x_0), \tag{8.6}$$

Since S is compact, the set of solutions of the sweeping process is compact in $\operatorname{Lip}([T_0, T]; H)$. Therefore, there exists a subsequence $(x_k)_k$ of $(x_n)_n$ converging uniformly to a solution x of the sweeping process. Then, by passing to the limit in (8.6), we obtain that (V, W) is forms a Lyapunov pair for (8.1). □

The following result, which is the main result of this section, gives a fully characterization of the weak Lyapunov pairs for the sweeping process (8.1).

Theorem 8.4 *Assume, in addition to* $(\mathcal{H}_3^F)$, $(\mathcal{H}_4^F)$ *and* $(\mathcal{H}_5^F)$, *that* S *is ball compact and* $(\mathcal{H}_1^V)$ *and* $(\mathcal{H}_2^V)$ *hold. Then the following conditions are equivalent:*

(i) For a.e. $t \in [T_0, T]$, $x \in \operatorname{dom} V$ *and* $\zeta \in \partial^P V(x)$

$$\inf\{\langle v, \zeta\rangle : v \in -\alpha(t)h(x)\partial d_S(x) + F(t,x)\} \leq -W(x).$$

(ii) (V, W) *forms a weak Lyapunov pair for the sweeping process* (8.1).

Proof. According to Proposition 8.3, without loss of generality, we can assume that W is continuous. Let $G\colon [T_0, T] \times H \times \mathbb{R} \to H \times \mathbb{R}$ defined by

$$G(t, x, y) = \begin{pmatrix} -\alpha(t)h(x)\partial d_S(x) + F(t,x) \\ -W(x). \end{pmatrix}$$

Then G has closed and convex values. Moreover, for a.e. $t \in [T_0, T]$ $G(t, \cdot, :)$ is upper semicontinuous from $H \times \mathbb{R}$ into $H_w \times \mathbb{R}$ and for a.e. $t \in [T_0, T]$ and all $(x, y) \in [T_0, T] \times H \times \mathbb{R}$

$$\begin{aligned} \|G(t,x,y)\| &:= \sup\{\|v\| \colon v \in G(t,x,y)\} \\ &\leq \alpha(t)h(x) + \|F(t,x)\| + |W(x)| \\ &\leq 2\alpha(t)h(x) + \beta\,(1 + \|x\|) \\ &\leq (2\alpha(t)L_h + \beta)\|x\| + (2\alpha(t)L_h h(0) + \beta), \end{aligned}$$

where L_h is the Lipschitz constant of h. Moreover, since $\operatorname{epi} V \subseteq S \times \mathbb{R}$ and S is ball compact, $\operatorname{epi} V$ is also ball compact. Therefore, due to [41, Theorem 3.3], the following conditions are equivalent:

(a) For a.e. $t \in [T_0, T]$ and $(x, r) \in \operatorname{epi} V$

$$G(t, x, r) \cap T^w_{\operatorname{epi} V}(x, r) \neq \emptyset.$$

(b) For a.e. $t \in [T_0, T]$ and $(x, r) \in \operatorname{epi} V$

$$G(t, x, r) \cap \overline{\operatorname{co}}\, T^w_{\operatorname{epi} V}(x, r) \neq \emptyset.$$

(c) For a.e. $t \in [T_0, T]$ and $(\zeta, \theta) \in N^P(\operatorname{epi} V; (x, r))$

$$\inf\{\langle v, \zeta\rangle + s\theta \colon (v, s) \in G(t, x, r)\} \leq 0.$$

(d) $(\operatorname{epi} V, G)$ is weakly invariant, that is, for any $(x_0, r_0) \in \operatorname{epi} V$ there exists a solution (x, r) of the differential inclusion $(\dot{x}(t), \dot{r}(t)) \in G(t, x(t), r(t))$ on $[T_0, T]$ with $(x(T_0), r(T_0)) = (x_0, r_0)$ such that $(x(t), r(t)) \in \operatorname{epi} V$ for all $t \in [T_0, T]$.

To finish the proof, it suffices to show that (c) is equivalent to (i).

(c) $\Rightarrow$ (i): Let $t \in [T_0, T]$ and $\zeta \in \partial^P V(x)$. Then, by virtue of (1.3),

$$(\zeta, -1) \in N^P(\operatorname{epi} V; (x, V(x))).$$

Therefore, by using (c),

$$\inf\{\langle v, \zeta\rangle - s\colon (v,s) \in G(t,x,V(x))\} \leq 0,$$

which implies (i).

(i) $\Rightarrow$ (c): Let $t \in [T_0, T]$ and $(\zeta, \theta) \in N^P(\operatorname{epi} V; (x, r))$. Then, $\theta \leq 0$ and

$$(\zeta, \theta) \in N^P(\operatorname{epi} V; (x, V(x))).$$

First case: $\theta < 0$:
It is not difficult to prove that $r = V(x)$. Then, due to (1.3) and (i), we obtain

$$\begin{aligned}
&\inf\{\langle v, \zeta\rangle + s\theta\colon (v,s) \in G(t,x,V(x))\}\\
&= \inf\{\langle v, \frac{\zeta}{|\theta|}\rangle\colon v \in -\alpha(t)h(x)\partial d_S(x) + F(t,x)\}|\theta| - \theta W(x)\\
&\leq -W(x)|\theta| - \theta W(x)\\
&= 0,
\end{aligned}$$

which proves (c).
Second case $\theta = 0$:
According to Proposition 1.1, for all $n \in \mathbb{N}$ there exist

$$(\zeta_n, \theta_n) \in N^P(\operatorname{epi} V; (x_n, V(x_n))),$$

with $x_n \to x$, $V(x_n) \to V(x)$, $\zeta_n \to \zeta$, $\theta_n \to 0$ and $\theta_n < 0$. Thus, by the argument given in the first case, for all $n \in \mathbb{N}$

$$\inf\{\langle v, \zeta_n\rangle + s\theta_n\colon (v,s) \in G(t,x_n,V(x_n))\} \leq 0. \tag{8.7}$$

Moreover, since $G(t, x_n, V(x_n))$ is closed, convex and bounded, the infimum in (8.7) is attained at some points (v_n, s_n) with $s_n = -W(x_n)$ and $v_n \in -\alpha(t)h(x_n)\partial d_S(x_n) + F(t, x_n)$. This implies that

$$v_n \in 2\alpha(t)h(x_n)\mathbb{B} \subseteq 2\alpha(t)\left(L_h\|x_n\| + h(0)\right)\mathbb{B},$$

where L_h is the Lipschitz constant of h. Hence, since $(x_n)_n$ is bounded, $(v_n)_n$ is bounded and we can assume that $v_n \rightharpoonup \bar{v}$. The upper semicontinuity from H into H_w of F and $\partial d_S(\cdot)$, shows that $\bar{v} \in -\alpha(t)h(x)\partial d_S(x) + F(t,x)$. Therefore, by using (8.7), we get

$$\begin{aligned}
\inf\{\langle v, \zeta\rangle : (v,s) \in{}& G(t,x,r)\}\\
&= \inf\{\langle v, \zeta\rangle : (v,s) \in G(t,x,V(x))\}\\
&\leq \langle \bar{v}, \zeta\rangle\\
&= \lim_{n\to\infty} \left(\langle v_n, \zeta_n\rangle + s_n\theta_n\right)\\
&= \lim_{n\to\infty} \inf\{\langle v, \zeta_n\rangle + s\theta_n\colon (v,s) \in G(t,x_n,V(x_n))\}\\
&\leq 0,
\end{aligned}$$

which proves (c). □

As an immediate consequence of Theorem 8.4, by taking V as the indicator function of S and W equals to 0, we obtain the existence for the sweeping process (8.1). The following result improves Theorem 8.1.

Theorem 8.5 *Assume that S is a closed and ball compact subset of H and that $F\colon [T_0,T]\times H \rightrightarrows H$ satisfies $(\mathcal{H}_2^F)$, $(\mathcal{H}_3^F)$ and $(\mathcal{H}_4^F)$. Then, for any $x_0 \in S$, there exists at least one Lipschitz solution x of the sweeping process* (8.1). *Moreover,*

$$\|\dot{x}(t)\| \leq 2h(x(t)) \quad a.e.\ t \in [T_0,T].$$

Proof. Define $\tilde{F}(t,x) := F(t,x)\cap h(x)\mathbb{B}$. It is clear that $\tilde{F}$ satisfies $(\mathcal{H}_3^F)$, $(\mathcal{H}_4^F)$ and $(\mathcal{H}_5^F)$. Let $V\colon H \to \mathbb{R}\cup\{+\infty\}$ be defined by $V(x) = I_S$ and $W \equiv 0$. Then, for a.e. $t \in [T_0,T]$ and $\zeta \in N^P(S;x)\setminus\{0\}$

$$\inf\{\langle v,\zeta\rangle : v \in -h(x)\partial d_S(x) + \tilde{F}(t,x)\} \leq -\frac{h(x)}{\|\zeta\|}\langle \zeta,\zeta\rangle + h(x)\|\zeta\| \leq 0.$$

Therefore, all the conditions of Theorem 8.4 hold. Thus, for all $x_0 \in S$, there exists at least one solution x of the sweeping process (8.1). $\square$

Example 8.1 Let $V\colon H \to \mathbb{R}$ be a $C^{1,1}$ function (i,e, V is differentiable and ∇V is Lipschitz continuous) and S a merely closed set. Consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -\nabla V(x(t)) - N(S;x(t)) & \text{a.e. } t \in [T_0,T], \\ x(T_0) = x_0 \in S. \end{cases} \tag{8.8}$$

Consider the function $\tilde{V}(x) := V(x) + I_S(x)$ and fix $\zeta \in \partial^P\tilde{V}(x)$. Then, by the classical sum rule, $\zeta \in \nabla V(x) + N^F(S;x)$ and

$$\inf\{\langle v,\zeta\rangle\colon v \in -\|\nabla V(x)\|\partial d_S(x) - \nabla V(x)\} \leq 0,$$

which, by virtue of Theorem 8.4, shows that V is a Lyapunov function for (8.8). This result had been already obtained in [100, Proposition 3.1] for r-uniformly prox-regular sets.

8.2.1 Weak invariance

In this subsection, as a consequence of Theorem 8.4, we give a characterization of weak invariance for the sweeping process.

Definition 8.6 (weak invariance) *We say that K is weakly invariant with respect to the sweeping process* (8.1) *if for all $x_0 \in K$ there exists a solution of the sweeping process* (8.1) *with $x(T_0) = x_0$ and $x(t) \in K$ for all $t \in [T_0,T]$.*

The following result is an improvement of [58, Theorem 4.3] for a fixed set.

Theorem 8.7 *Assume, in addition to* $(\mathcal{H}_3^F)$, $(\mathcal{H}_4^F)$ *and* $(\mathcal{H}_5^F)$, *that* S *is ball compact and* $(\mathcal{H}_1^V)$ *and* $(\mathcal{H}_2^V)$ *hold. Let* $K \subseteq S$ *be a closed set. Then the following conditions are equivalent:*

(i) For a.e. $t \in [T_0, T]$, *for all* $x \in K$ *and* $\zeta \in N^P(K; x)$

$$\inf\{\langle v, \zeta \rangle : v \in -\alpha(t)h(x)\partial d_S(x) + F(t,x)\} \leq 0,$$

(ii) For all $x_0 \in K$ *there exists a solution of the sweeping process* (8.1) *with* $x(T_0) = x_0$ *and* $x(t) \in K$ *for all* $t \in [T_0, T]$.

8.3 Applications

In this section we give some applications of our existence results (Theorems 8.1 and 8.5) to hysteresis and to the modeling of crowd motion in emergency evacuation.

8.3.1 Hysteresis

In this subsection, we study the so-called Play operator, which arises in hysteresis (see, for instance [72,124]). Several properties in hysteresis can be described in terms of some hysteresis operators. One of these hysteresis operators is the so-called Play operator [124], which to a given Lipschitz function y associates the set of solutions of the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N(S; x(t)) + \dot{y}(t) & \text{a.e. } t \in [T_0, T]; \\ x(T_0) \in x_0 \in y(T_0) - S. \end{cases} \tag{8.9}$$

The case where S is convex, uniformly prox-regular and α-far has been studied, respectively, in [72,90,124]. For a general closed and ball compact set S and given y Lipschitz with $x_0 \in y(T_0) - S$, due to Theorem 8.5 and since $(\mathcal{H}_3^F)$, $(\mathcal{H}_4^F)$ and $(\mathcal{H}_5^F)$ trivially hold, there exists at least one solution of (8.9) with $\|\dot{x}(t)\| \leq 2\|\dot{y}(t)\|$ for a.e. $t \in [T_0, T]$. Therefore, the Play operator is well defined in $\operatorname{Lip}([T_0, T]; H)$.

8.3.2 Crowd motion

In this subsection, we consider a model of crowd motion in emergency evacuation. We refer to [22,102,142] for a detailed description. Our discussion is based on [22].

The model handles contacts in order to deal with local interactions between people and to describe the whole dynamics of the pedestrian traffic. This model for crowd motion (where people are identified to rigid disks) rests on two principles. On the

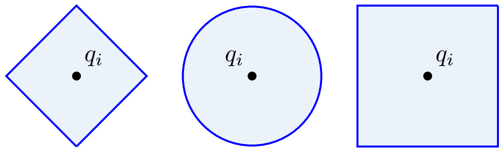


Figure 8.1: Disks, respectively, for $d = \|\cdot\|_1$. $d = \|\cdot\|_2$ and $d = \|\cdot\|_\infty$.

one hand, each individual has a spontaneous velocity that he would like to have in the absence of other people. On the other hand, the actual velocity must take into account congestion. Those two principles lead to defining the actual velocity as the Euclidean projection of the spontaneous velocity over the set of admissible velocities (regarding the non overlapping constrains between sets).

More precisely, we consider N persons identified to rigid disks (for some distance d in $\mathbb{R}^2$). For convenience, the disks are supposed to have the same radius r. The center of the ith disk is denoted by $q_i \in \mathbb{R}^2$. Since overlapping is forbidden, the vector of positions $q = (q_1, \ldots, q_n) \in \mathbb{R}^{2N}$ has to belong to the "set of feasible configurations", defined by

$$Q := \{q \in \mathbb{R}^{2N} \colon D_{ij}(q) \geq 0 \quad \forall i \neq j\},$$

where $D_{ij}(q) = d(q_i, q_j) - 2r$ is the distance between the disk i and j and d is some distance in $\mathbb{R}^2$ (see Figure 8.1).

It is worth emphasizing that Q is not uniformly prox-regular if, for instance, $d(x, y) = \|(x, y)\|_1$ or $d(x, y) = \|(x, y)\|_\infty$.

If the global spontaneous velocity of the crowd is denoted by

$$V(t, q) = (V_1(t, q_1), \ldots, V_N(t, q_N)) \in \mathbb{R}^{2N},$$

the previous crowd motion model can be described by the following differential inclusion:

$$\frac{dq}{dt} \in -N(Q; q) + V(t, q),$$

which fits in our context. Therefore, Theorems 8.1 and 8.5 give the existence for the crowd motion model. Moreover, one solution for this model can be obtained through the catching-up algorithm, described in the proof of Theorem 8.1.

Chapter 9

Lyapunov pairs for Perturbed Sweeping Processes

The aim of this chapter, which is based on HV-Lyapunov, is to study Lyapunov pairs for the perturbed sweeping process, that is, we consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0), \end{cases} \tag{9.1}$$

where $C\colon [T_0, T] \rightrightarrows H$ is a set-valued map with nonempty and closed values, $N\left(S; x\right)$ denotes the Clarke normal cone to S at x and $F\colon [T_0, T] \times H \rightrightarrows H$ is a given set-valued map with nonempty closed and convex values.

The existence theory for the sweeping process began, as far as we know, with the seminal work of Moreau [113]. Since then, it has been studied for convex and nonconvex moving sets. We refer to Section 3.6 for more details.

Lyapunov pairs are the central idea behind the Lyapunov method. This indirect approach is relevant because it does not require explicit calculations of the solutions of the dynamical system. This is especially useful when dealing with complex real-world applications. Moreover, the Lyapunov method allows to address several stability properties of differential inclusions as finite or asymptotic stability, existence of equilibria, stabilization, etc. (see, for example, [53–55]).

Characterizations of smooth and nonsmooth Lyapunov pairs has been considered for different dynamical systems by several authors (see [10, 53–55] and the references given there). In the present case, Adly, Hantoute and Théra [3,4] give explicit criteria for Lyapunov pairs for maximal monotone evolution equations, which includes the sweeping process driven by a fixed convex set. Then, Hantoute and Mazade [78] give explicit criteria for Lyapunov functions for the sweeping process driven by a fixed uniformly prox-regular set.

Unfortunately, it is well known that some dynamical systems do not admit smooth

Lyapunov pairs (see [54]). Thus, it very important to deal with nonsmooth Lyapunov pairs. Here is where the subdifferential theory has been very helpful. In this setting, the work of Clarke et al [55] has became a benchmark because they characterize Lyapunov pairs for differential inclusions by using the proximal subdifferential. In fact, the proximal subdifferential is the smallest reasonable subdifferential that allows a characterization of lower semicontinuous Lyapunov pairs. We follow this path and give an explicit criteria, involving the proximal subdifferential, of weak Lyapunov pairs for the sweeping process. It is worth pointing out that our result, in contrast with [3, 4, 78], does not involve the singular (horizon) subdifferential.

The chapter is organized as follows. In Section 9.1 we give a criteria for weak Lyapunov pairs for the sweeping process. As a result, we give an existence result and a criterion for weak invariance for the sweeping process. We illustrate our result with an application to gradient complementarity dynamical systems.

Let $V\colon [T_0, T] \times H \to \mathbb{R} \cup \{+\infty\}$ and $W\colon [T_0, T] \times H \to \mathbb{R}$ be two proper and lower semicontinuous functions. We say that (V, W) forms a *weak Lyapunov pair* for the sweeping process (9.1) if for every $x_0 \in C(T_0)$ there exists x solution of (9.1) such that

$$V(t, x(t)) + \int_{T_0}^{t} W(s, x(s))ds \leq V(T_0, x_0) \quad \text{for all } t \in [T_0, T].$$

Moreover, we say that V is a *weak Lyapunov function* for (9.1) if $(V, 0)$ is a Lyapunov pair for (9.1).

We will consider the following Hypotheses on C, V and W

($\mathcal{H}_1$) There exists $\kappa \geq 0$ such that for $s, t \in [T_0, T]$ and all $x \in H$

$$|d(x, C(t)) - d(x, C(s))| \leq \kappa|t - s|.$$

($\mathcal{H}_2$) The family $\{C(t)\colon t \in [T_0, T]\}$ is equi-uniformly subsmooth.

($\mathcal{H}_3$) For all $t \in [T_0, T]$, the set $C(t)$ is ball compact, that is, for every $r > 0$ the set $C(t) \cap r\mathbb{B}$ is compact in H.

($\mathcal{H}_1^V$) $V\colon [T_0, T] \times H \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous function.

($\mathcal{H}_2^V$) $W\colon [T_0, T] \times H \to \mathbb{R}$ is a proper lower semicontinuous function with

$$|W(t, x)| \leq \beta(t)\,(1 + \|x\|) \quad \text{for all } (t, x) \in [T_0, T] \times H,$$

for some $\beta \in C(T_0, T)$.

Lemma 9.1 *Assume, in addition to ($\mathcal{H}_1$), ($\mathcal{H}_2$) and ($\mathcal{H}_3$), that $\alpha \in C(T_0)$ and $h\colon H \to H$ is a Lipschitz function. Then, the set-valued map*

$$\tilde{F}(t, x) := -(\kappa + \alpha(t)h(x))\partial d_{C(t)}(x) \quad (t, x) \in [T_0, T],$$

is upper semicontinuous from $[T_0, T] \times H$ into H_w.

Proof. It is enough to prove that $(t,x) \mapsto \partial d_{C(t)}(x)$ is upper semicontinuous from $[T_0,T] \times H$ into H_w.
Fix $t \in [T_0,T]$ and $x \in H$.
I) Assume that $x \in C(t)$: Let $t_n \to t$, $x_n \to x$ and $x_n^* \rightharpoonup x^*$ with $x_n^* \in \partial d_{C(t_n)}(x_n)$. We have to prove that $x^* \in \partial d_{C(t)}(x)$. For every $n \in \mathbb{N}$ where $x_n \notin C(t_n)$ we have (see [90, Lemma 2.1])

$$x_n^* = \frac{x_n - y_n}{d_{C(t_n)}(x_n)} \in \partial d_{C(t_n)}(y_n), \tag{9.2}$$

for some $y_n \in \text{Proj}_{C(t_n)}(x_n)$ (the projection is nonempty because $(\mathcal{H}_3)$). Then, for each $n \in \mathbb{N}$, we define

$$\hat{x}_n = \begin{cases} x_n, & \text{if } x_n \in C(t_n), \\ y_n, & \text{if } x_n \notin C(t_n), \end{cases}$$

where $y_n \in H$ is given by (9.2). Thus, $\hat{x}_n \to x$, $x_n^* \rightharpoonup x^*$, $\hat{x}_n \in C(t_n)$ and $x_n^* \in \partial d_{C(t_n)}(\hat{x}_n)$. Therefore, using $(\mathcal{H}_3)$ and [118, Lemma 2.2.2], we obtain that $x^* \in \partial d_{C(t)}(x)$.
II) Assume that $x \notin C(t)$: Due to Lemma [90, Lemma 4.2] and [9, Theorem 17.35], it is enough to prove that $(t,x) \rightrightarrows \text{Proj}_{C(t)}(x)$ is sequentially upper semicontinuous from $[T_0,T] \times H$ into H_w at (t,x). Let $x_n \to x$ and $x_n^* \rightharpoonup x^*$ with $x_n^* \in \text{Proj}_{C(t_n)}(x_n)$. We have to prove that $x^* \in \text{Proj}_{C(t)}(x)$. Indeed, due to $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, $(x_n^*)_n$ is relatively compact and thus, $x_n^* \to x^*$ up to a subsequence. Moreover,

$$\begin{aligned} \|x - x^*\| &\le \|x - x_n\| + d_{C(t_n)}(x_n) + \|x_n^* - x^*\| \\ &\le \|x - x_n\| + \kappa|t - t_n| + d_{C(t)}(x_n) + \|x_n^* - x^*\|, \end{aligned}$$

which implies that $\|x - x^*\| \le d_{C(t)}(x)$. Furthermore,

$$d_{C(t)}(x^*) = d_{C(t)}(x^*) - d_{C(t_n)}(x_n^*) \le \kappa|t - t_n| + \|x^* - x_n^*\|,$$

which shows that $x^* \in C(t)$. Therefore, $x^* \in \text{Proj}_{C(t)}(x)$. □

9.1 Lyapunov pairs and invariance

In this section we give an explicit criterion for weak Lyapunov pairs for perturbed sweeping processes (9.1). Throughout this section we assume that $F\colon [T_0,T] \times H \rightrightarrows H$ is a set-valued map with nonempty, closed and convex values. Moreover, we will consider the following conditions:

$(\mathcal{H}_1^F)$ F is scalarly $\mathcal{L} \otimes \mathcal{B}$ measurable on $[T_0,T] \times H$.

$(\mathcal{H}_2^F)$ F is upper semicontinuous from $[T_0,T] \times H$ into H_w.

$(\mathcal{H}_3^F)$ There exist $\alpha \in C(T_0,T)$ and $h\colon H \to \mathbb{R}^+$ Lipschitz such that

$$\|F(t,x)\| := \sup\{\|w\|\colon w \in F(t,x)\} \le \alpha(t)h(x),$$

for all $x \in H$ and a.e. $t \in [T_0,T]$.

The following result, which is the main result of this section, gives a fully characterization of the weak Lyapunov pairs for the perturbed sweeping process (9.1).

Theorem 9.2 *Assume, in addition to* $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ *and* $(\mathcal{H}_3^F)$, *that* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, $(\mathcal{H}_1^V)$ *and* $(\mathcal{H}_2^V)$ *hold. If* $\operatorname{dom} V(t,\cdot) \subseteq C(t)$ *for all* $t \in [T_0, T]$, *then the following conditions are equivalent:*

(i) For all $(t,x) \in \operatorname{dom} V$ *and* $(\theta,\zeta) \in \partial^P V(t,x)$

$$\theta + \inf\{\langle v, \zeta\rangle : v \in -(\kappa + \alpha(t)h(x))\,\partial d_{C(t)}(x) + F(t,x)\} \le -W(t,x).$$

(ii) (V,W) *forms a weak Lyapunov pair for the sweeping process* (9.1).

Proof. Let $G\colon [T_0,T] \times H \times \mathbb{R} \rightrightarrows \mathbb{R} \times H \times \mathbb{R}$ defined by

$$G(t,x,y) = \{1\} \times (-(\kappa + \alpha(t)h(x))\,\partial d_{C(t)}(x) + F(t,x)) \times [-\beta(t)(1+\|x\|), -W(t,x)]$$

Then, due to Lemma 9.1 and $(\mathcal{H}_2^V)$, G is upper semicontinuous from $[T_0,T] \times H \times \mathbb{R}$ into $\mathbb{R} \times H_w \times \mathbb{R}$ with nonempty, closed and convex values. Moreover, for all $(t,x,y) \in [T_0,T] \times H \times \mathbb{R}$

$$\begin{aligned}\|G(t,x,y)\| &:= \sup\{\|v\| \colon v \in G(t,x,y)\}\\ &\le 1 + \kappa + \alpha(t)h(x) + \|F(t,x)\| + \beta(t)\,(1+\|x\|)\\ &\le 1 + \kappa + 2\alpha(t)h(x) + \beta(t)\,(1+\|x\|)\\ &\le (2\alpha(t)L_h + \beta)\|x\| + (2\alpha(t)L_h h(0) + \beta + 1 + \kappa),\end{aligned} \tag{9.3}$$

where L_h is the Lipschitz constant of h.

Furthermore, due to $(\mathcal{H}_1)$, $(\mathcal{H}_3)$, $\operatorname{epi} V$ is ball compact. Indeed, on the one hand,

$$\begin{aligned}\operatorname{epi} V &= \{(t,x,\lambda) \in [T_0,T] \times H \times \mathbb{R} \colon V(t,x) \le \lambda\}\\ &\subseteq \{(t,x,\lambda) \in [T_0,T] \times H \times \mathbb{R} \colon x \in C(t)\}\\ &= \operatorname{graph} C \times \mathbb{R}.\end{aligned}$$

On the other hand, if $((t_n,x_n))_n \subseteq \operatorname{graph} C$ is a bounded sequence, then, after taking a subsequence, $t_n \to \bar{t}$ (without relabeling) for some $t \in [T_0,T]$ and

$$x_n \in C(t_n) \subseteq C(\bar{t}) + \kappa|\bar{t} - t_n|\mathbb{B},$$

which, as a result of $(\mathcal{H}_3)$, proves that $\operatorname{graph} C$ is ball compact.

Hence, due to [41, Theorem 3.3], the following conditions are equivalent.

(a) For all $(t,x,r) \in \operatorname{epi} V$

$$G(t,x,r) \cap T^w_{\operatorname{epi} V}(t,x,r) \neq \emptyset.$$

(b) For all $(t,x,r) \in \operatorname{epi} V$

$$G(t,x,r) \cap \overline{\operatorname{co}}\, T^w_{\operatorname{epi} V}(t,x,r) \neq \emptyset.$$

(c) For all $(\theta,\zeta,\mu) \in N^P(\operatorname{epi} V;(t,x,r))$

$$\theta + \inf\{\langle v,\zeta\rangle + s\mu \colon (1,v,s) \in G(t,x,r)\} \leq 0.$$

(d) $(\operatorname{epi} V, G)$ is weakly invariant, that is, for any $(T_0,x_0,r_0) \in \operatorname{epi} V$ there exists a solution (τ,x,r) of the differential inclusion

$$(\dot\tau(t),\dot x(t),\dot r(t)) \in G(\tau(t),x(t),r(t)) \quad \text{for a.e. } [T_0,T],$$

with $(\tau(T_0),x(T_0),r(T_0)) = (T_0,x_0,r_0)$ such that $(\tau(t),x(t),r(t)) \in \operatorname{epi} V$ for all $t \in [T_0,T]$.

Therefore, to finish the proof, it suffices to show that (c) is equivalent to (i).

(c) $\Rightarrow$ (i): Let $(\theta,\zeta) \in \partial^P V(t,x)$. Then, by virtue of (1.3),

$$(\theta,\zeta,-1) \in N^P(\operatorname{epi} V;(t,x,V(t,x))).$$

Therefore, by using (c),

$$\begin{aligned}\theta + \inf\{\langle v,\zeta\rangle : v \in -(\kappa+\alpha(t)h(x))\partial d_{C(t)}(x) + F(t,x)\} + W(t,x) \\ \leq \theta + \inf\{\langle v,\zeta\rangle - s \colon (1,v,s) \in G(t,x,V(t,x))\} \\ \leq 0,\end{aligned}$$

which implies (i).

(i) $\Rightarrow$ (c): Let $(\theta,\zeta,\mu) \in N^P(\operatorname{epi} V;(t,x,r))$. Then, $\mu \leq 0$ and

$$(\theta,\zeta,\mu) \in N^P(\operatorname{epi} V;(t,x,V(t,x))).$$

<u>First case</u>: $\mu < 0$:
It is not difficult to prove that $r = V(t,x)$. Then, due to (1.3) and (i), we obtain

$$\begin{aligned}&\theta + \inf\{\langle v,\zeta\rangle + s\mu \colon (1,v,s) \in G(t,x,V(t,x))\} \\ &\leq \frac{\theta}{|\mu|}|\mu| + \inf\{\langle v,\frac{\zeta}{|\mu|}\rangle \colon v \in -(\kappa+\alpha(t)h(x))\,\partial d_{C(t)}(x) + F(t,x)\}|\mu| - \mu W(t,x) \\ &\leq -W(x)|\theta| - \theta W(x) \\ &= 0,\end{aligned}$$

which proves (c).
<u>Second case</u> $\mu = 0$:
According to Proposition 1.1, for all $n \in \mathbb{N}$ there exist

$$(\theta_n,\zeta_n,\mu_n) \in N^P(\operatorname{epi} V;(t_n,x_n,V(t_n,x_n)))$$

with $t_n \to t$, $x_n \to x$, $V(t_n, x_n) \to V(t, x)$, $\theta_n \to \theta$, $\zeta_n \to \zeta$, $\mu_n \to 0$ and $\mu_n < 0$. Thus, by the argument given in the first case, for all $n \in \mathbb{N}$

$$\theta_n + \inf\{\langle v, \zeta_n \rangle + s\mu_n \colon (1, v, s) \in G(t_n, x_n, V(t_n, x_n))\} \leq 0. \tag{9.4}$$

Moreover, since $G(t_n, x_n, V(t_n, x_n))$ is closed, convex and bounded (see (9.3)), the infimum in (9.4) is attained at some points $(1, v_n, s_n)$ with

$$s_n \in [-\beta(t_n)(1 + \|x_n\|), -W(t_n, x_n)]$$

and $v_n \in -(\kappa + \alpha(t_n)h(x_n))\,\partial d_{C(t_n)}(x_n) + F(t_n, x_n)$. This implies that

$$v_n \in (\kappa + 2\alpha(t_n)h(x_n))\,\mathbb{B}.$$

Hence, since $(t_n)_n$ and $(x_n)_n$ are bounded, $(v_n)_n$ is bounded and we can assume that $v_n \rightharpoonup \bar{v}$. The upper semicontinuity from $[T_0, T] \times H$ into H_w of F and $\partial d_{C(\cdot)}(:)$, shows that $\bar{v} \in -(\kappa + \alpha(t)h(x))\,\partial d_{C(t)}(x) + F(t, x)$. Therefore, by using (9.4), we get

$$\begin{aligned}
\theta + \inf\{\langle v, \zeta \rangle : (1, v, s) \in G(t, x, r)\} &= \theta + \inf\{\langle v, \zeta \rangle : (1, v, s) \in G(t, x, V(t, x))\} \\
&\leq \theta + \langle \bar{v}, \zeta \rangle \\
&= \lim_{n\to\infty} (\theta_n + \langle v_n, \zeta_n \rangle + s_n\mu_n) \\
&= \lim_{n\to\infty} \inf\{\theta_n + \langle v, \zeta_n \rangle + s\mu_n \colon (1, v, s) \in G(t, x_n, V(t_n, x_n))\} \\
&\leq 0,
\end{aligned}$$

which proves (c). □

As an immediate consequence of Theorem 9.2, by taking V as the indicator function of $C(\cdot)$ and W equals to 0, we obtain the existence of solutions for the sweeping process (9.1).

Theorem 9.3 *Assume that C satisfies $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ and that $F\colon [T_0, T] \times H \rightrightarrows H$ satisfies $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$. Then, for any $x_0 \in C(T_0)$, there exists at least one Lipschitz solution x of the sweeping process* (9.1)*. Moreover,*

$$\|\dot{x}(t)\| \leq \kappa + 2\alpha(t)h(x(t)) \quad a.e.\ t \in [T_0, T].$$

Proof. Let $V\colon [T_0, T] \times H \to \mathbb{R} \cup \{+\infty\}$ be defined by $V(t, x) = I_{C(t)}(x)$ and $W \equiv 0$. Then $\operatorname{dom} V(t, \cdot) = C(t)$ for all $t \in [T_0, T]$. Let $(\theta, \zeta) \in \partial^P V(t, x)$. Then, there exist $\delta, \sigma > 0$ such that for all $(s, x) \in \bar{B}((t, x); \delta)$

$$I_{C(s)}(y) \geq I_{C(t)}(x) + \theta(s - t) + \langle \zeta, y - x \rangle - \sigma\left(|s - t|^2 + \|y - x\|^2\right). \tag{9.5}$$

Hence, if $s = t$, we obtain that $\zeta \in N^P(C(t); x)$. Moreover, due to $(\mathcal{H}_1)$,

$$x \in C(t) \subseteq C(s) + \kappa|t - s|\mathbb{B}.$$

Thus, there exists $b \in \mathbb{B}$ such that $y := x - \kappa|t-s|b \in C(s)$. Then, by virtue of (9.5), for all $|t-s| \leq \max\{\delta, \delta/\kappa\}$

$$\theta(s-t) \leq \langle \zeta, \kappa|t-s|b \rangle + \sigma|s-t|^2 \left(1 + \kappa^2\|b\|^2\right).$$

Therefore, dividing by $|s-t|$ and taking $s \to t$, we obtain that $|\theta| \leq \kappa\|\zeta\|$.

Then, for all $(\theta, \zeta) \in \partial^P V(t,x)$ with $\zeta \neq 0$,

$$\begin{aligned} \theta + \inf\{\langle v, \zeta\rangle : v \in -(\kappa + \alpha(t)h(x))\,\partial d_{C(t)}(x) + F(t,x)\} \\ &\leq \kappa\|\zeta\| - (\kappa + \alpha(t)h(x))\,\langle \frac{\zeta}{\|\zeta\|}, \zeta\rangle + \alpha(t)h(x)\|\zeta\| \\ &\leq 0. \end{aligned}$$

Therefore, all the conditions of Theorem 9.2 hold. Thus, for all $x_0 \in C(t)$, there exists at least one solution x of the sweeping process (9.1). □

When H is a finite-dimensional Hilbert space and $C \equiv H$ we obtain, as a consequence of Theorem 9.2, the well known criteria for Lyapunov pairs for differential inclusions with convex, upper semicontinuous right-hand side (see for example [55]).

Corollary 9.4 *Let H be a finite-dimensional Hilbert space. Assume, in addition to $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$, that $(\mathcal{H}_1^V)$ and $(\mathcal{H}_2^V)$ hold. Then the following conditions are equivalent:*

(i) For all $(t,x) \in \operatorname{dom} V$ and $(\theta,\zeta) \in \partial^P V(t,x)$

$$\theta + \inf\{\langle v,\zeta\rangle : v \in F(t,x)\} \leq -W(t,x).$$

(ii) (V,W) is weak Lyapunov pair for the differential inclusion $\dot{x}(t) \in F(t,x(t))$.

Example 9.1 Let $V\colon H \to \mathbb{R}$ be a C^2 function, α be a positive $C^1(T_0,T)$ function and C be a closed, subsmooth and ball-compact set. Consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -\alpha(t)\nabla V(x(t)) - N\left(C; x(t)\right) & \text{a.e. } t \in [T_0,T], \\ x(T_0) = x_0 \in C(T_0). \end{cases} \tag{9.6}$$

Define the functions $\tilde{V}(t,x) := \alpha(t)V(x) + I_C(x)$ and $W(t,x) := -\dot{\alpha}(t)V(x)$.
Fix $(\theta,\zeta) \in \partial^P \tilde{V}(t,x)$. Then,

$$\theta + \inf\{\langle v,\zeta\rangle : v \in -\alpha(t)\|\nabla V(x)\|\partial d_C(x) - \alpha(t)\nabla V(x)\} \leq \dot{\alpha}(t)V(x),$$

which, by virtue of Theorem 9.2, shows that $(\tilde{V}, W)$ is a Lyapunov pair for (9.6).

9.1.1 Weak invariance

In this section, as a consequence of Theorem 9.2, we give a characterization of weak invariance for the sweeping process.

Definition 9.5 (weak invariance) *We say that K is weakly invariant with respect to the sweeping process* (9.1) *if for all $(T_0, x_0) \in \operatorname{graph}(C) \cap \operatorname{graph}(K)$ there exists a solution of* (9.1) *with $x(T_0) = x_0$ and $x(t) \in K(t)$ for all $t \in [T_0, T]$.*

The following result is an improvement of [58, Theorem 4.3].

Theorem 9.6 *Assume, in addition to $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$, that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and $K(t) \subseteq C(t)$ for all $t \in [T_0, T]$. Then the following conditions are equivalent:*

(i) For all $(\theta, \zeta) \in N^P(\operatorname{graph} K; (t,x))$

$$\theta + \inf\{\langle v, \zeta\rangle : v \in -(\kappa + \alpha(t)h(x))\, \partial d_{C(t)}(x) + F(t,x)\} \leq 0.$$

(ii) For all $x_0 \in K$ there exists a solution of the sweeping process (9.1) *with $x(T_0) = x_0$ and $x(t) \in K(t)$ for all $t \in [T_0, T]$.*

9.2 An application to gradient complementarity dynamical systems

In this section, we illustrate our results with an application to Gradient Complementarity Dynamical Systems (GCDS). A GCDS consists of an ordinary differential equation coupled with complementarity conditions. More explicitly, given $F\colon [T_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $h\colon \mathbb{R}^n \to \mathbb{R}^m$ and $d\colon [T_0, T] \to \mathbb{R}^m$, the defining equations for the GCDS corresponding to F, h and d are

$$\begin{cases} \dot{x}(t) = F(t, x(t)) + Dh^*(x(t))u(t), \\ y(t) = h(x(t)) + d(t), \\ K \ni y(t) \perp u(t) \in K^*, \end{cases} \tag{9.7}$$

where $K \subseteq \mathbb{R}^m$ is a closed convex cone and $K^* = \{y \in \mathbb{R}^m \colon \langle v, y\rangle \geq 0 \text{ for all } v \in K\}$ denotes the dual cone of K. A typical example of GCDS are the Linear Complementarity Dynamical Systems (LCDS) which corresponds to the particular case

$$\begin{cases} \dot{x}(t) = Ax(t) + H^T x(t)u(t), \\ y(t) = Hx(t) + d(t), \\ \mathbb{R}^m_+ \ni y(t) \perp u(t) \in \mathbb{R}^m_+. \end{cases}$$

GCDS, in particular LCDS, is an important class of dynamical systems with several applications such as electrical circuits, dynamic traffic assignment problem, differential Nash games, etc. (see [39, 121, 134] and the references therein). GCDS has been studied by several authors. An usual approach to GCDS is to transform the system into a perturbed sweeping process (see [39, 87]). Indeed, the third line in (9.7) is a complementarity relation between $y(t)$ and $u(t)$ which are forced to remain always orthogonal one to each other. This fact can be expressed in an equivalent way as

$$K \ni y(t) \perp u(t) \in K^* \quad \Leftrightarrow \quad -u(t) \in N\left(K; y(t)\right).$$

Therefore, by using this equivalence, the gradient complementarity dynamical system is formally equivalent (see [39, 87] for more details) to the following perturbed sweeping process:

$$\dot{x}(t) \in -N\left(C(t); x(t)\right) + F(t, x(t)) \quad \text{a.e. } t \in [T_0, T], \tag{9.8}$$

where $C(t) := h^{-1}\left(K - d(t)\right)$ for all $t \in [T_0, T]$. In order to prove the existence for (9.8), some qualifications conditions must be imposed. In fact, in [39, 134] the authors give sufficient conditions to assure the uniformly prox-regularity of the sets $C(t)$. The following result, proved in [87], provide sufficient conditions to assure uniformly subsmoothness and Lipschitz continuity of the set-valued map C.

Proposition 9.7 *Assume that $h\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable with uniformly continuous derivative, $d\colon [T_0, T] \to \mathbb{R}^m$ is Lipschitz continuous and there exists $k > 0$ such that*

$$\mathbb{B}_{\mathbb{R}^m} \subseteq Dh(x)k\mathbb{B}_{\mathbb{R}^n} - K \quad \textit{for all } x \in h^{-1}(K).$$

Then, the set valued map $C\colon [T_0, T] \rightrightarrows \mathbb{R}^n$ defined by $C(t) := h^{-1}\left(K - d(t)\right)$ satisfies $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$.

Hence, we can use Theorem 9.2 to characterize Lyapunov functions of GCDS.

Example 9.2 Let us consider a circuit with an ideal diode, an inductor and a current source (see Figure 9.1), where x is the current through the inductance and a current κ Lipschitz source $i(t)$. The dynamics is given by

$$\begin{cases} \dot{x}(t) = u(t) \\ y(t) = x(t) - i(t) \\ \mathbb{R}_+ \ni y(t) \perp u(t) \geq 0. \end{cases} \tag{9.9}$$

Hence, the system (9.9) is equivalent to

$$\dot{x}(t) \in -N\left(\mathbb{R}_+ + i(t); x(t)\right).$$

Let $V\colon [T_0, T] \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function satisfying

$$\operatorname{dom} V(t, \cdot) \subseteq \mathbb{R}_+ + i(t) \quad \text{for all } t \in [T_0, T].$$

Then, according to Theorem 9.2, V is a Lyapunov function for (9.9) if and only

$$\theta + \kappa\zeta \cdot \mathbb{1}_{\mathbb{R}_-}(\zeta)\, \mathbb{1}_{\{x = i(t)\}}(t, x) \leq 0 \text{ for all } (\theta, \zeta) \in \partial^P V(t, x).$$

where $\mathbb{1}_S$ is the characteristic function of a set S.

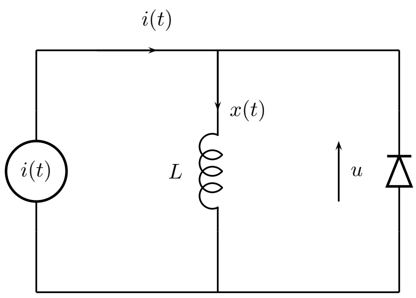


Figure 9.1: A circuit with an ideal diode, an inductor and a current source.

Conclusions and Perspectives

Conclusions

In this thesis, by using tools from nonsmooth and variational analysis, we have studied differential inclusions involving normal cones of nonregular sets in Hilbert spaces. Although the main focus of this thesis has been the sweeping process, the developed methods have allowed us to address several differential inclusions involving normal cones.

In Chapter 2 we have studied the class of positively α-far. This class is broader enough to include convex, uniformly prox-regular, uniformly subsmooth sets, among others. Several characterizations and properties of positively α-far sets are given. Moreover, we have made it clear through this thesis that this class of sets is suitable to deal with differential inclusions involving normal cones. We believe that this class plays and important role in constrained differential inclusions which must be exploited.

In Chapter 3 we have introduced the Galerkin-like method to solve differential inclusions. This new method consists in approximate the original differential inclusion by projecting the state into a n dimensional Hilbert space, but not the velocity. We showed that approximate problems always have a solution and that, under some compactness conditions, the approached problems converges strongly pointwisely (up to a subsequence) to a solution of the original problem. We have showed that this method is well adapted to deal with constrained differential inclusions by providing existence of solutions for the Generalized Sweeping Process. As a result, we obtained existence results for perturbed state-dependent sweeping processes, perturbed sweeping processes and second-order sweeping process, among others. Moreover, in Chapter 7 the Galerkin-like method is used to deal with differential inclusions with nonlocal initial conditions.

In Chapter 4 we have studied existence of solutions of some variant of the perturbed state-dependent sweeping process in finite dimensions. We obtain existence for this variant through the reduction technique for uniformly subsmooth sets. This variant includes two important applications: Complementarity Dynamical Systems and a control systems which arises in the study of vector hysteresis. The existence result for the system control which arises in vector hysteresis, was complemented with the introduction of a catching-up like numerical algorithm and some numerical simulations.

In Chapters 5 and 6 we have established existence results, respectively, for continuous bounded variation and perturbed state-dependent sweeping processes with equi-uniformly subsmooth sets. Our results shows that Moreau-Yosida regularization can be used even for nonregular sets. This achievement opens the door toward possible new developments in nonsmooth analysis because several problems are approximated through this kind of regularization.

In Chapter 7, by using the Galerkin-like method, we have investigated existence for differential inclusions with nonlocal initial conditions in Hilbert spaces. This includes periodic, anti-periodic, mean value or multi-point initial conditions. In particular, we have showed existence for perturbed sweeping processes with nonlocal initial conditions. Then, we have considered abstract differential inclusions when the ambient Hilbert space is compactly embedded in a separable Banach space. We used the concept of bounding functions which allowed us to overcome the lack of a priori bounds of the abstract differential inclusions. Finally, we gave some applications to nonlocal differential complementarity systems and vector hysteresis. Since our result include periodic initial conditions, we believe that these results can be used for further developments in the theory of periodic perturbations and stability of sweeping processes.

In Chapters 8 and 9 we have obtained some criteria for nonsmooth Lyapunov pairs and invariance for perturbed sweeping process with nonregular sets. Moreover, these results were exemplified with an application to gradient complementarity dynamical systems. We believe that these characterizations can be used for further studies of stability of the sweeping process.

Finally, we would like to emphasize that the sweeping process and its variants are an interesting and very rich topic for research. Moreover, there are still many open questions related to this problem and its applications.

Perspectives

The perspectives from the present work are numerous. Here we list the most straightforward with respect to the results and techniques developed in this thesis.

Preservation of positively α-far

In Section 2.2 we have given sufficient conditions for the preservation of uniformly subsmooth sets under intersection and inverse images. It will be very interesting to know, probably under some constraint qualification, when the intersection of two positively α-far sets remains positively α-far.

The Galerkin-like method

In Chapter 3 we have introduced the Galerkin-like method. This method allowed us to deal with constrained differential inclusions in Hilbert spaces. As shown in this thesis, this method has much potential to study dynamical systems. We believe that this method to deal with, for example, integral equations, differential equations as well other differential inclusions. A related problem that can be addressed with this method is the controllability of perturbed sweeping process. We will pursue this in future research.

Moreau-Yosida regularization

In Chapters 5 and 6 we have proved the convergence of the Moreau-Yosida regularization for the sweeping process with nonregular moving sets. This achievement opens the door toward possible new developments in nonsmooth analysis which must to be exploited. Related to the state-dependent sweeping process, in Chapters 5 and 6, the moving sets were supposed to have, respectively, continuous bounded variation and Lipschitz variation with respect to the Hausdorff distance. Thus, it would be interesting to study the convergence of the Moreau-Yosida regularization when the sets have bounded variation with respect to the Hausdorff distance or the truncated distance Hausdorff.

Sweeping process with limiting normal cones

In this work we have considered sweeping process with nonregular sets. In all our results we have prove existence results with the Clarke normal cone. This is due to its properties, in particular, its convexity. Nevertheless, it is well known that this cone can be very large. Therefore, it would be interesting to obtain existence results for sweeping processes with smaller cones than the Clarke normal cone, for example, the limiting normal cone. One step in this direction was given in [23], where the authors consider the perturbed sweeping process with definable moving sets. Definable sets are the main concepts of the theory of o-minimal structures. These sets are become very popular because, on the one hand, are general enough to include important applications and, on the other hand, are "patothology free". It would be interesting to get existence results for the sweeping process and its variant within this context. Moreover, it would be interesting to understand the implications of definable sets in applications such as complementarity dynamical systems, hysteresis, electrical circuits, etc.

Lyapunov pairs for gradient dynamical systems

In Chapters 9 and 8 we obtained some criteria for Lyapunov pairs and invariance for perturbed sweeping process with nonregular sets. Moreover, these characterizations were exemplified with an application to gradient complementarity dynamical systems. Nevertheless, this example is only illustrative and would be very interesting to obtain explicit criteria for Lyapunov pairs for gradient complementarity systems with uniformly subsmooth sets. This would improves the results of [134] where the moving sets was assumed to be uniformly prox-regular. A related problem is the asymptotic stability or existence of equilibria for the sweeping process.

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