

Contents

1	Introduction	1
1.1	Objectives and main results	2
2	Theoretical background	4
2.1	Elementary concepts	4
2.2	Bifurcation theory	5
2.2.1	Linear analysis	5
2.2.2	Nonlinear analysis	7
2.2.3	Pitchfork bifurcation	7
2.2.4	Saddle-Node bifurcation	8
2.3	Spatial instabilities	9
2.4	Fredholm solvability condition	10
2.5	Amplitude equations	11
2.6	Time-dependent forcing	12
2.6.1	Resonance	12
2.6.2	Parametric forcing	13
2.6.3	Kapitza effect	18
2.7	Particle-like solutions in coupled oscillators	19
2.7.1	Sine-Gordon equation	20
2.7.2	ϕ^4 scalar field equation	24
2.7.3	Kink-antikink interaction in dissipative ϕ^4 model	25
2.8	Localized structures in vegetation	29
2.8.1	Mean-field model of vegetation evolution	29
2.8.2	Periodic vegetation patterns	31
2.8.3	Localized structures	32
2.9	Diffraction theory	33
2.9.1	Fraunhofer diffraction	34
2.9.2	Multiple slits diffraction	35
2.10	Liquid crystals	36
2.10.1	Zigzag instability	37
2.11	Numerical methods for integrating partial differential equations	38
2.11.1	Fourth-order Runge-Kutta method	38
2.11.2	Interactive simulations	39
3	Flaming kinks	41
3.1	Flaming kinks in the parametrically driven and damped sine-Gordon equation	41

3.1.1	Main properties	41
3.1.2	Resonant properties	43
3.1.3	Localized structures	44
3.2	Flaming kinks in a ferromagnetic wire	45
3.3	Flaming kinks in the parametrically driven and damped ϕ^4 equation	48
3.3.1	Flaming kink-antikink interaction	48
3.4	Perturbative analysis using Inverse Scattering Transform method	49
3.4.1	General framework of the IST method	49
3.4.2	IST method applied to the sine-Gordon equation	50
3.5	Flaming kinks instabilities	52
3.6	Flaming kinks in two dimensions	54
4	Localized structures lattices in vegetation	59
4.1	Lattices in one dimension	59
4.1.1	Properties of one dimensional localized structures	59
4.1.2	Interaction between two localized structures	61
4.1.3	Multiple localized structures	67
4.2	Lattices in two dimensions	68
4.2.1	Asymptotic behavior of two dimensional localized structures	69
4.2.2	Interaction dynamical equations	69
4.2.3	Lattices of multiple two dimensional localized structures	72
5	Diffraction grating in a zigzag lattice in nematic liquid crystals	73
5.1	Diffraction grating in an empty in-plane switching cell	73
5.1.1	Diffraction orders	75
5.2	Diffraction grating in a perfect zigzag liquid crystal lattice	75
5.3	Diffraction grating temporal evolution	78
6	Conclusions	79
	Bibliography	81
A	Flaming 2π kinks in parametrically driven systems	88
B	Flaming kink-antikink interaction in parametrically driven systems	95
C	Pattern formation mediated by repulsive interaction between localized structures	105
D	Harnessing diffraction grating in an in-plane switching cell submitted to zigzag lattice	112

List of Figures

1.1	(a) Oscillon formed by vibrating vertically granular layers [93]. (b) Two-dimensional localized structures on the surface of a ferrofluid, under a uniform magnetic field normal to the surface [84].	1
2.1	Phase portrait of Lotka-Volterra model (Eq. (2.2)) for $a = 3, b = e = f = 1$ and $c = d = 2$	5
2.2	Two possible generic behaviors of how the eigenvalues of the system evolves when the control parameter r crosses its critical point r_c . (a) Stationary bifurcation. (b) Andronov-Hopf bifurcation.	6
2.3	Bifurcation diagram for different kinds of Pitchfork bifurcations: (a) Supercritical. (b) Subcritical. (c) Subcritical with a quintic term (Eq. (2.6)). The plots show the evolution of the steady stable and unstable states in function of the bifurcation parameter ε . The full (dotted) lines represent the stable (unstable) states. The arrows in (c) account for the path that the equilibrium solution follows when the bifurcation parameter ε is increased or decreased. Notice that there is a jump in $\varepsilon = 0$ and in ε_s	8
2.4	λ_k profile in function of the wavenumber k for different values of ε . The maximum value is ε and is obtained evaluating in $\pm q$. The critical case $\varepsilon = 0$ is usually referred as the <i>marginal case</i>	10
2.5	Amplitude of the steady-state solution (Eq. (2.12)) in function of the forcing frequency ω for different values of the damping coefficient μ , for $\omega_0 = 1$. For the blue line we have used $a = 1$ and $\mu = 0.1$, for the red $a = 5$ and $\mu = 0.7$, and for the green $a = 6$ and $\mu = 1.2$	13
2.6	Driven simple pendulum with a vertical oscillatory pivot at frequency ω and amplitude a	14
2.7	Stability chart of the damped Mathieu equation, known as Ince-Strutt diagram. The yellow part represents the unstable regions, while the blue part the stable regions. For $\mu = 0$ the boundary curves reach the horizontal axis in $\omega = 2\omega_0/n$. The unstable regions around these frequencies are usually called the Arnold Tongues. As μ grows these tongues rise.	15
2.8	2 : 1 tongue for $\mu = 0.1$, with its analytical approximations using 1 and 2 modes as ansatz, expressed in Eqs. (2.16) and (2.17), respectively.	17
2.9	Temporal evolution obtained by integrating numerically the Eq. (2.18), with $\omega_0 = 1, \gamma = 40, \omega = 20$ and $\mu = 0$	18
2.10	Schematic representation of a chain of coupled pendulums by springs.	20

2.11	(a) Spatial profile of kink and antikink solution in function of the variable $s = x - vt$ (see Eq. (2.21)). The full (dotted) line corresponds to the kink (antikink) solution. (b) Representation of the kink solution in the chain of pendulums.	21
2.12	Initial and final instant of the kink-kink collision. (a) Two kinks travel with the same velocity in opposite directions. (b) After the collision, the each kink changes its velocity direction. Then, the kink-kink interaction is repulsive.	22
2.13	Initial and final instant of the kink-antikink collision. (a) One kink and an antikink travel with the same velocity in opposite directions. (b) After the collision, the each kink passes through each other. Then, the kink-antikink interaction is permeable.	23
2.14	Temporal evolution of breather solution. (a) Spatial profile evolution for different times. (b) Corresponding spacetime diagram. We have used $\Omega = 0.5$	24
2.15	Spatial profile of the analytical kink solution of the ϕ^4 equation expressed in Eq. (2.23), with $x_0 = 0$, $v = 0$ and $\varepsilon = 1$	25
2.16	Spatial profile of the ansatz from Eq. (2.16).	26
2.17	Profile of translation and interaction modes τ and χ defined in Eq. (2.27) and (2.28). We have used $\varepsilon = 0.5$	27
2.18	Examples of vegetation patterns in nature: a) tiger bush [1] and b) fairly circles [90].	30
2.19	a) One dimensional stability diagram of steady states for parameters $\kappa = 0.6, \Delta = 0.02, \Gamma = 0.5, \alpha = 0.125$. The dotted lines represent unstable states. In η_c the periodic patterns emerge with wavelength λ , with maximum and minimum values u_{\max} and u_{\min} , respectively. The yellow stripe stands for the region where the localized patches can be found. b) Periodic pattern with wavelength given by Eq. (2.34), using $\eta = 0.05$	32
2.20	a) Localized structure profile in 2 dimensions, using $\kappa = 0.6, \Delta = 0.02, \Gamma = 0.5, \alpha = 0.125$ and $\eta = 0.05$. b) Spatial profile of a LP from the two dimensional Eq (2.32). The dashed line stands for the spatial profile shown in c). d) Spinifex grassland, Yakabindi station, Western Australia (courtesy of Vilis Nams, Dalhousie University, Canada) [2].	33
2.21	Wave diffracted by an aperture A . S and P are the source and receiving points, respectively. (a) Fraunhofer diffraction: Incident and diffracted waves are planes. (b) Fresnel diffraction: The waves curvature is significant.	34
2.22	Schematic representation of a the Fraunhofer diffraction using positive lens. Here A is the aperture plane and f the focal distance.	35
2.23	(a) Schematic representation of N slits of width b , separated by h . (b) Corresponding diffraction grating expressed in Eq. (2.36), with $h = 2.5$, $b = 1$ and $N = 10$. The red line corresponds to the factor $(\sin \beta / \beta)^2$	36
2.24	Different liquid crystal phases: nematic, smectic and cholesteric.	37
2.25	(a) Schematic representation of the liquid crystal IPS cell, connected to a generator. (b) Zig-zag instability exhibited by a nematic liquid crystal filled in this cell.	38
2.26	Screenshot of the software <i>DimX</i> , showing a real time simulation of the parametrically driven and damped sine-Gordon equation.	39

3.1	Flaming kinks obtained from numerical integration of Eq. (3.1) with $\omega_0 = 1.0, \gamma = 0.3, \omega = 1.4, \mu = 0.1, \kappa = 1.0, dx = 0.5$ and $dt = 0.1$. (a) Schematic representation of a flaming kink. (b) Spatiotemporal evolution. (c) Spatial profile of the solution at a certain instant marked with a dashed line in (b).	42
3.2	Waves amplitude in function of the forcing frequency ω for: (a) constant $\gamma = 0.1$ and different values of the damping coefficient μ . (b) constant $\mu = 0.1$ and different values of the forcing amplitude γ . (c) Phase space as a function of frequency and amplitude of the forcing with $\mu = 0.1$. The green zone accounts for the region where flaming kinks are observed.	43
3.3	Localized structures formed due to the flaming kink-antikink interaction. (a) Schematic representation of the solution. (b) Spatiotemporal evolution. (c) Spatial profile of the solution at a certain instant marked with a dashed line in (b). The parameters δ and Δ account for the width and position of the localized structure, respectively.	44
3.4	(a) Temporal width evolution under a perturbation. Notice that it reaches a new equilibrium. (b) Phase space $\{\Delta, \delta\}$ of the localized structures, obtained monitoring periodically the width evolution. The lower panels show the respective profiles of equilibrium widths.	45
3.5	Schematic representation of the magnetization vector.	46
3.6	Schematic representation of a kink solution in the ferromagnetic wire, obtained by integrating numerically Eq. (3.3), using $h = 0.8, \beta = 10$ and $\alpha = 0.02$	47
3.7	Flaming kinks in the parametrically and damped ϕ^4 scalar field equation. In (a) we show the spatial profile at one instant, marked with a dashed line in the spatiotemporal diagram in (b). We have used $\varepsilon = 1, \gamma = 0.5, \omega = 1$ and $\mu = 0.1$	48
3.8	Phase diagram of the flaming kinks instabilities in the Ince-Stutt diagram for $\mu = 0.1$. The colors account for the different instabilities observed. In the region marked as unstable, we do not observe kink solutions.	51
3.9	(a) Spatial profile of the solutions from yellow region. (b) Spatiotemporal evolution. The dashed line corresponds to the profile showed in (a).	52
3.10	(a) Spatial profile of the solutions from blue region. (b) Spatiotemporal evolution. The dashed line corresponds to the profile showed in (a).	53
3.11	(a) Spatial profile of the solutions from purple region. (b) Spatiotemporal evolution. The dashed line corresponds to the profile showed in (a).	53
3.12	(a) Spatial profile of the solutions from green region. (b) Spatiotemporal evolution. The dashed line corresponds to the profile showed in (a).	54
3.13	(a) Spatial profile of two dimensional sine-Gordon kinks. The dashed line corresponds to the one dimensional profile shown in (b).	55
3.14	(a) Schematic representation of a Josephson junction. The green layers represent superconductors and the red one an insulator material. The terms Ψ_1 and Ψ_2 account for the macroscopic wavefunctions in every layer. (b) Kink (or fluxon) solution of the phase difference $\psi = \theta_1 - \theta_2$	55
3.15	(a) Flaming kink spatial profile in two dimensions. (b) The upper figure corresponds to the dotted line showed in (a). The lower figure is the spacetime evolution of this profile, illustrating a typical stationary waves profile.	56
3.16	Interface amplitude in function of the forcing frequency. The blue (red) line corresponds to the amplitude when the forcing frequency is decreased (increased).	57

3.17	(a) Localized structure profile, resulting from the two dimensional flaming kink-antikink interaction. (b) Flaming kink spatial profile for the two dimensional driven and damped ϕ^4 equation. We have set $\varepsilon = 1, \mu = 0.1, \omega = 0.5$ and $\gamma = 0.8$	57
3.18	Fronts between domains with opposite phase in a 7-particle deep layer for different values of the forcing amplitude and frequency. This picture was taken from [94].	58
4.1	One dimensional localized structure profile. The parameters used are $\eta = 0.17, \kappa = 0.8, \Delta = 0.02, \Gamma = 0.5$ and $\alpha = 0.13$. The red line corresponds to the exponential fitting, using Eq. (4.3). For this case the theoretical value of γ is 2.91 and from the fitting 3.06. The coefficient of determination results $R^2 = 0.9978$	60
4.2	(a) Two localized structures separated by a distance r . (b) Numerical data of r in function of time, showing the repulsion between LSs. The parameters used are $\eta = 0.12, \kappa = 0.6, \Delta = 0.02, \Gamma = 0.5, \alpha = 0.125$ and $dx = 0.4$	60
4.3	(a) Eigenvalues spectrum of matrix M , defined in Eq. 4.19, using the same parameters from Fig. 4.1, $dx = 0.1$ and $N = 30$. (b) Zoom of the dashed region marked in (a). (c) Null eigenvector τ . (c) Null eigenvector χ	64
4.4	Curve fitting of numerical data of $r(t)$. The red curve is the fitting obtained using Eq. (4.38). The parameters used were $\eta = 0.12, \kappa = 0.6, \Delta = 0.02, \Gamma = 0.5, \alpha = 0.125, dt = 0.01$ and $dx = 0.4$. The γ obtained was 2.40, which is close to the theoretical gamma $\gamma = \sqrt{\eta/\Delta} = 2.45$. The R^2 of the fitting is 0.9977.	66
4.5	(a) Spatiotemporal diagram of the evolution of multiple LSs at different distances. After some time they reach an equilibrium with constant distance of separation between them. The boundary conditions have been set as periodic. The dashed lines correspond to the instant t_i and t_f , showed in (b) and (c), respectively. The parameters used were $\eta = 0.13, \kappa = 0.7, \Delta = 0.01, \Gamma = 0.5, \alpha = 0.1, dx = 0.3$ and $dt = 0.01$	67
4.6	Evolution of periodic one-dimensional configurations, after removing one localized structure. The figures (a), (b) and (c) show the evolution of a seven, six, and five LSs periodic profile evolution, after removing one LS. The upper and lower profiles show the initial and final profile of each case, respectively. In all cases the LSs rearrange, reaching a new periodic profile with a larger wavelength. The parameters used were $\eta = 0.13, \kappa = 0.7, \Delta = 0.01, \Gamma = 0.5, \alpha = 0.1, dx = 0.26$ and $dt = 0.01$	68
4.7	Two-dimensional structures located at a distance r . The dashed line passes through the centers and will be the axis where we restrict our calculations.	70
4.8	(a) Numerical data of the distance of separation r in function of time, in units of the width w . (b) Curve fitting of numerical data of \dot{r} in function of r , using first term of Eq. (4.54). The distance r is normalized with the LSs width. The parameters used in simulations were $\eta = 0.12, \kappa = 0.6, \Delta = 0.02, \Gamma = 0.5, \alpha = 0.125, dx = dy = 0.3$ and $dt = 0.001$. The R^2 obtained was 0.9924.	71

4.9	Different equilibrium configuration of two dimensional localized structures. We have used periodic boundary conditions. The parameters used were $\eta = 0.12, \kappa = 0.6, \Delta = 0.02, \Gamma = 0.5, \alpha = 0.125, dx = dy = 0.3$ and $dt = 0.001$, with periodic boundary conditions.	72
5.1	(a) Schematic representation of the different zones of the empty IPS cell. (b) One dimensional transmission coefficient model in x direction. (c) Cell picture taken from a microscope.	74
5.2	(a) Sketch of a perfect zigzag lattice, with constant amplitude and wavelength. (b) Transmission coefficient $t(x, y)$ values for some $\{n, m\}$ values.	76
5.3	Spatiotemporal diagram of the diffraction grating for a square signal of $16 V_{pp}$ and 1 kHz. Around $t = 1.5$ seconds the generator is turned on. The distance is measured from the order 0. The intensity variation is notorious in order 0 and 6.	77
5.4	Plots (a) and (b) show the light intensity of diffraction orders 0 and 3 in function of time, for a applied signal of $16 V_{pp}$ and $18 V_{pp}$, respectively. In both cases the signal is sinusoidal, with a frequency of 1 kHz.	78