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MONOCHROMATIC CYCLE PARTITIONS

TESIS PARA OPTAR AL GRADO DE
DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA

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MONOCHROMATIC CYCLE PARTITIONS

The first part of this thesis concerns monochromatic cycle partitions. We make the following three contributions.

Our first result is that for any colouring of the edges of the complete bipartite graph $K_{n,n}$ with 3 colours there are 5 disjoint monochromatic cycles which together cover all but $o(n)$ vertices of the graph. In the same situation, 18 disjoint monochromatic cycles together cover all vertices.

Next we show that given any 2-local edge-colouring of the edges of the balanced complete bipartite graph $K_{n,n}$, its vertices can be covered with at most 3 disjoint monochromatic paths. And, we can cover all vertices of any complete or balanced complete bipartite r-locally edge-coloured graph with $O(r^2)$ disjoint monochromatic cycles. We also determine the 2-local bipartite Ramsey number of a path: Every 2-local edge-colouring of the edges of $K_{n,n}$ contains a monochromatic path on n vertices.

Finally, we prove that any edge-colouring in red and blue of a graph on n vertices and of minimum degree $2n/3 + o(n)$ admits a partition into three monochromatic cycles. This confirms a conjecture of Pokrovskiy approximately.

The second part of this thesis contains two independent results about (proper) edge-colouring and parameter estimation respectively.

Regarding edge-colouring, we conjecture that any graph G with treewidth k and maximum degree $\Delta(G) \geq k + \sqrt{k}$ satisfies $\chi'(G) = \Delta(G)$. In support of the conjecture we prove its fractional version.

Concerning parameter estimation we study, for any fixed monotone graph property $\mathcal{P} = \mathrm{Forb}(\mathcal{F})$, the sample complexity of estimating a bounded graph parameter $z_{\mathcal{F}}$ that, for an input graph G, counts the number of spanning subgraphs of G that satisfy $\mathcal{P}$. Using a new notion of vertex partitions, we improve upon previous upper bounds on the sample complexity of estimating $z_{\mathcal{F}}$.

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MONOCHROMATIC CYCLE PARTITIONS

La primera parte de esta tesis es acerca de particiones de ciclos monocromáticos. Hacemos las siguientes tres contribuciones.

Nuestro primero resultado es que por todos coloramientos de las aristas de un grafo completo bipartito $K_{n,n}$ con 3 colores hay 5 ciclos monocromáticos disjuntos que juntos cubren todos menos $o(n)$ vertices del grafo. En la misma situación, 18 ciclos monocromáticos disjuntos juntos cubren todos de los vertices.

A continuación mostramos que dado cualquier coloramiento 2-local de las aristas del grafo completo bipartito balanceado $K_{n,n}$, sus vertices pueden ser cubiertos por 3 caminos monocromáticos disjuntos. Además, podemos cubrir todos los vertices de cualquier grafo completo o bipartido completo r-localmente colorado con $O(r^2)$ ciclos monocromáticos disjuntos. También determinamos el numero de Ramsey 2-local bipartito: Todos coloramientos 2-locales de las aristas de $K_{n,n}$ contienen un camino monocromático de n vertices.

Finalmente, probamos que todos los coloramientos en rojo y azul de un grafo con n vertices y grado mínimo $2n/3 + o(n)$ permite una partición en 3 ciclos monocromáticos. Esto confirma una conjetura de Porkovskiy aproximadamente.

La segunda parte de esta tesis contiene dos resultados independientes sobre coloramientos de aristas (genuino) y estimación de parámetros respectivamente.

Con respeto coloramientos de aristas, conjeturamos que cualquier grafo G con tamaño de arboles k y grado maximo $\Delta(G) \geq k + \sqrt{k}$ satisfecha $\chi'(G) = \Delta(G)$. Probamos la versión fraccional de esta conjetura.

Relacionado a la estimación de parámetros estudiamos, por cualquier propriedad monótona $\mathcal{P} = \mathrm{Forb}(\mathcal{F})$, la complexidad de la muestra de estimar un parámetro limitado $z_{\mathcal{F}}$ que, por un grafo de input G , cuenta el numero de subgrafos generadores de G que satisfacen $\mathcal{P}$. Utilizando un nuevo concepto de particiones de vertices, mejoramos las cotas anteriores de la complexidad de la muestra de estimar $z_{\mathcal{F}}$.

To my parents.

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I would like to thank Maya Stein for guidance and help I have received in while writing this thesis. You have been an excellent PhD supervisor and collaborator.

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Contents

Introduction

The field of monochromatic partitioning aims to combine Ramsey theory with covering problems. For instance, given a complete graph K_n whose edges are coloured in red and blue, how many monochromatic cycles are needed to partition its vertices? Note that here we count edges, single vertices and the empty set as cycles as well to omit some trivial cases. Lehel conjectured that a red and a blue cycle are enough [9]. If we replace the cycles with paths this is quickly seen to be true [45]. Gyárfás showed that in a red and blue edge coloured complete graph one can find a red and a blue cycle which cover all vertices and intersect only in a single vertex [52]. However, it took roughly twenty years to improve on that. Using the regularity lemma and an approach of Łuczak [79], Łuczak and Rödl and Szemerédi gave a proof of Lehel's conjecture for sufficiently large complete graphs [80]. Later Allan showed the same for graphs of smaller size (but still fairly large) [1]. Finally the conjecture was completely resolved by Bessy and Tomassé relying on elemental arguments only [13].

Monochromatic partitioning has since then (and in particular lately) received a fair amount of attention. The base question was generalized in many directions. The host graph K_n has been replaced by bipartite graphs [64], tripartite graphs [97], graphs of bounded minimum degree [10], graphs of fixed independence number [96], infinite graphs [88] and hypergraphs [62]. The monochromatic cycles, have been changed for paths [84], trees [65], k-regular graphs [43] and graphs of bounded degree [51]. The problem of monochromatic partitioning and, in parts, the solutions can be generalized from r-edge-colourings to r-local edge-colourings [25]. It is also worth mentioning that there is a whole branch of research dedicated to studying similar problems in terms of covers instead of partitions. For more details we refer the reader to the recent survey of Gyárfás [54].

0.1 Many colours

A natural generalization of the base problem is to ask what happens if the edges of the complete graph are coloured with $r > 2$ colours. For example is it possible to partition K_n into a number of monochromatic cycles independent of n? This was settled positively by Erdős, Gyárfás and Pyber [31].

Theorem 0.1.1 (Erdős, Gyárfás and Pyber '91)**.** *Any r-edge-coloured K_n admits a partition into $25r^2 \log r$ monochromatic cycles.*

The proof of Theorem 0.1.1 has been fairly influential and foreshadowed what is now called the absorption method (see [89]). It is instructive to have a closer look. To this end let us introduce the following auxiliary results.

Definition 0.1.2 (Crown)**.** For $k \geq 3$, a k-crown consists of a cycle $(v_1, \ldots, v_k)$ together with a set of tips, that is additional vertices $A = \{a_1, \ldots, a_k\}$ such that (v_i, v_{i+1}, a_i) forms a triangle for each $i \in [k] \pmod{k}$.

Note that crowns are Hamiltonian and remain so after the deletion of any subset of tips. Moreover, since the maximum degree is bounded by 4, the Ramsey number of crowns grows linear in their size. This is made numerically precise in the following lemma, whose short but technical proof we omit.

Lemma 0.1.3 ([31]). *For $k \geq 3$, any r-edge-colouring of K_n admits a monochromatic k-crown with $k \geq n/(2r(r!)^3$.*

We will also use a classic result of extremal graph theory.

Theorem 0.1.4 (Erdős and Gallai '59). *For $k \geq 3$, any graph on n vertices and with at least $(n-1)(k-1)/2$ edges contains a cycle of order at least k.*

In particular Theorem 0.1.4 implies that any r-edge-coloured K_n contains a monochromatic cycle of order at least n/k. Finally we need the following lemma about bipartite graphs.

Lemma 0.1.5 ([31]). *Let H be an r-edge-coloured complete bipartite graph with bipartition classes A, B and such that $|A| \geq r^3|B|$. Then B can be covered by r^2 vertex disjoint monochromatic cycles.*

Now we are ready to prove Theorem 0.1.1.

Proof. We use Lemma 0.1.3 to pick a, say red, k-crown C_0 with set of tips A and $k \geq n/(2r(r!)^3)$. Fix a positive integer t. For $1 \leq i \leq t$ we apply Theorem 0.1.4 to pick a monochromatic cycle C_i in the graph $K_n - \bigcup_{0 \leq j \leq i-1} V(C_j)$. Denote by B the vertices of $K_n - \bigcup_{0 \leq j \leq t} V(C_j)$ An elementary calculation shows that $|A| \geq r^3|B|$ provided that $t \geq \lfloor 24r^2 \log r \rfloor$. Hence we can apply Lemma 0.1.5 to cover the vertices of B by r^2 monochromatic cycles. By design the remainder of the k-crown is Hamiltonian. Hence we have obtained a monochromatic cycle partition of size $1 + 24r^2 \log r + r^2$. □

Given Theorem 0.1.1 we can ask for the *minimum* number of monochromatic cycles needed to partition the vertex set of an r-edge-coloured complete graph. It is not hard to see that we need at least r cycles. For instance take a partition $V_1, \ldots, V_r$ of K_{2^r-1} with $|V_i| = 2^i$, and for $i \leq j$ give all V_i–V_j edges colour i. A cycle partition of this colouring contains a cycle of colour i for each $i \in [r]$. (In fact the same holds for a monochromatic path cover.) Erdős, Gyárfás and Pyber believed that this bound is sharp and generalized Lehel's conjecture as follows.

Conjecture 0.1.6 (Erdős, Gyárfás and Pyber '91). *Any r-edge-coloured complete graph admits a partition into r monochromatic cycles.*

Note that unlike Lehel's conjecture the colours of the cycles are allowed to repeat. This is necessary because there are colourings, where a partition into cycles of distinct colours is not possible, e.g. Figure 1.

For $r = 3$ some progress towards Conjecture 0.1.6 has been made. Gyárfás, Ruszinkó, Sárközy, Szemerédi showed that all but $o(n)$ vertices of a 3-edge-coloured K_n can be partitioned into 3 monochromatic cycles [61]. However, somewhat surprisingly Pokrovskiy recently

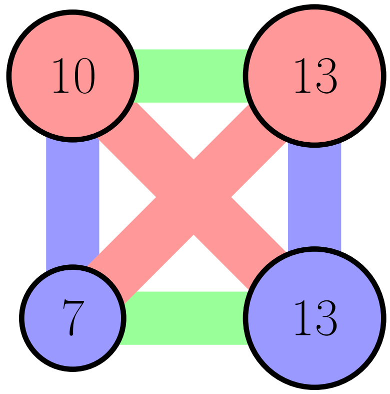

Figure 1: The numbers indicate the size of the vertex set. Coloured like this, K_{43} has no partition into two monochromatic paths or into three monochromatic paths of (pairwise) distinct colours.

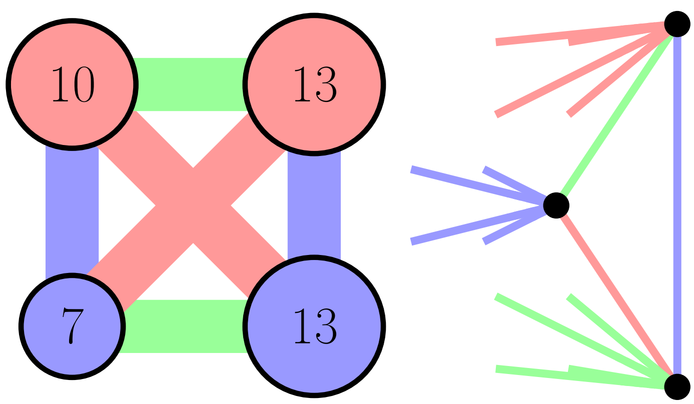

Figure 2: Pokrovskiy's counterexample.

found a counterexample, proving that Conjecture 0.1.6 is wrong [84]. He defined colourings of (arbitrary large) complete graphs for all $r \geq 3$, which require r monochromatic cycles and an additional vertex for a proper partition. Let us give the details:

Example 0.1.7. We focus on the case $r = 3$ and $n = 42$. The examples for larger r, n are very similar. Consider a colouring of $H = K_{43}$ with colours $\{1, 2, 3\}$ as in Figure 1. So in particular H has no partition into two monochromatic paths or into three monochromatic paths of (pairwise) distinct colours. Now we add three additional vertices v_1, v_2, v_3 and colour the edges between v_{i} and $V(H)$ and $v_{\mathrm{i}+1}v_{\mathrm{i}+2}$ with colour i mod 3 (see Figure 2). We claim that the 3-edge-coloured complete graph obtained this way can not be partitioned into 3 monochromatic cycles. Indeed assume otherwise and let C_1, C_2, C_3 be such a partition. If, C_1 say, consists only of edges v_1v_2, then C_2 and C_3 contain a partition of H into two monochromatic paths. This is not possible. So none of cycles C_1, C_2, C_3 contains an edge of type $v_{\mathrm{i}}v_j$. However, since the edges between v_{i} and $V(H)$ have colour i, this implies that the cycles C_1, C_2, C_3 have pairwise distinct colours. Hence they contain a partition of H into three monochromatic paths. A contradiction.

Nevertheless Pokrovskiy (and others) believes that Conjecture 0.1.6 is not too far off the mark.

Conjecture 0.1.8 (Pokrovskiy [84]). *For every r there is a number c_r such that any r-edge-coloured complete graph admits a partition into r monochromatic cycles and c_r vertices.*

In support of his conjecture Pokrovskiy recently proved the case of $r = 3$ with $c_r = 43000$ [85]. Using completely different methods, Letzter independently obtained the same result for large n but with a better constant of $c_r = 60$ [75].

In general, i.e. for all r, the best known upper bound for monochromatic cycle partitioning complete graphs has been obtained by Gyárfás, Ruszinkó, Sárközy, Szemerédi in 2006.

Theorem 0.1.9 ([58]). *Provided n is large enough, every r-edge-colouring of K_n admits a partition into at most $100r \log r$ monochromatic cycles.*

The better bound, in comparison to Theorem 0.1.1, is due to an improvement of Lemma 0.1.5, where under the same conditions only $8r \log r$ monochromatic cycles are needed to cover B, and switching from crowns to dense matchings, which have better Ramsey numbers. Both of these improvements require the Regularity Lemma either in their proof or in their implementation. However, as in the proof of Theorem 0.1.1, the majority of the vertices of the graph is still covered by greedily applying Theorem 0.1.4. As noted in [58], it seems unlikely that the bound can be improved significantly beyond $O(r \log r)$, without replacing this process with a less greedy strategy.

0.2 Bipartite graphs

After proving Theorem 0.1.1 Erdős, Gyárfás and Pyber asked if similar results could be obtained for bipartite graphs. Haxell solved this problem by proving the following [64].

Theorem 0.2.1 (Haxell '97). *Any r-edge-coloured $K_{n,n}$ admits a partition into $O(r^2 \log^2 r)$ monochromatic cycles.*

The main difficulty in adapting the proof of Theorem 0.1.1 to bipartite graphs is that crowns are not bipartite. The other steps, i.e. covering the majority of the graph with Theorem 0.1.4 and absorbing the leftover with Lemma 0.1.5 work as before. Haxell showed that uniform graphs are a suitable replacement for crowns, i.e. they are bipartite and Hamiltonian, even after deleting some vertices. Roughly speaking a bipartite graph is uniform, if it has linear minimum degree and no (balanced) subgraph of linear size is empty (see Chapter 1 Section 1.5 for details). Large uniform graphs can be found in any sufficiently dense graph, which in our case is the subgraph of the colour which has the most edges. Peng, Rödl and Ruciński [83] lowered the bound of Theorem 0.2.1 to $O(r^2 \log r)$, by finding larger uniform graphs under the same density conditions. By replacing Theorem 0.1.4 in the step of covering the majority of the graph, Stein and I recently improved this bound further to $O(r^2)$ (Theorem 2.1.3 in Chapter 2). In the case of $r = 3$, Haxell showed that 1695 cycles are sufficient. (Erdős had proposed 25$ to anyone who could show that 1995 cycles are sufficient.) Schaudt, Stein and I improved this to 18 monochromatic cycles. We also obtained the stronger result that, provided n is large enough, any 3-edge-colouring of $K_{n,n}$ admits a partition of all but $o(n)$ vertices into 5 monochromatic cycles [72]. These results are presented in Chapter 1.

0.3 Local colourings

Local edge-colourings present a generalization of r-edge-colourings. An edge-colouring is r-local if no vertex is adjacent to more than r edges of distinct colours. Local edge-colourings have been studied in the context of Ramsey theory (see [57, 99, 103]). With respect to monochromatic cycle partitions, Conlon and Stein recently generalised Theorem 0.1.1 to r-local colourings [25].

Theorem 0.3.1 (Conlon and Stein 16)**.** *Any r-locally edge-coloured complete graph admits a partition into $O(r^2 \log r)$ monochromatic cycles.*

Additionally they showed that if $r = 2$, then two cycles suffice. The proof of Theorem 0.1.1 follows the approach taken in 0.1.1. Since local Ramsey numbers are bounded (linearly) by ordinary Ramsey numbers, monochromatic crowns and large cycles can be found without difficulty. However Lemma 0.1.5 needs to be proved again in a local setting. Stein and I improved the bound of Theorem 0.3.1 to $O(r^2)$ for sufficiently large n by avoiding the use of Theorem 0.1.4 [73]. We also show that it is possible to partition 3-locally coloured complete bipartite graphs into 3 monochromatic paths and determine the 3-local path Ramsey number. The details are in Chapter 2.

0.4 Non-complete graphs

Motivated by ideas of Schelp, Balogh et al. asked if Lehel's conjecture stays true for graphs of bounded minimum degree [10]. They conjectured the following: given any graph G on n vertices and of maximum degree $3n/4$, for any colouring of the edges in red and blue, there are a red and a blue cycle which together partition the vertices of G. Note that there are graphs of minimum degree $3n/4 - 1$ that do not admit such a partition. In support of their conjecture, Balogh et al. proved an approximate result [10]. They showed that, for every ε there is an n_0 such that for any graph G on $n \geq n_0$ vertices and with minimum degree at least $(3/4+\varepsilon)n$, any colouring of the edges of G in red and blue admits disjoint red and blue cycles which together cover all but εn vertices. DeBiasio and Nelsen were able to improve on this by obtaining a proper partition into a red and a blue cycle under the same condition [27]. Finally Letzter proved the conjecture for sufficiently large n [74]. All three results use the method introduced in [79], which itself relies on the Regularity Lemma. Based on these advances Pokrovskiy conjectured that similar results are true for graphs of lower minimum degree. In particular, he conjectured that red and blue edge coloured graphs of minimum degree $2n/3$ ($n/2$) can be partitioned into 3 (4) monochromatic cycles [85]. There are examples which show that these numbers are essentially tight. In Chapter 3 we confirm the first part of his conjecture approximately. We prove that for every ε there is an n_0 such that for any graph G on $n \geq n_0$ vertices and with minimum degree at least $(2/3+\varepsilon)n$, any colouring of the edges of G in red and blue admits a partition into 3 monochromatic cycles.

0.5 Other results

Besides monochromatic partitioning I have worked on the following topics.

0.6 Edge-colouring of sparse graphs

Let us introduce two concepts. The chromatic index of a graph G is the least integer, such that G admits an edge-colouring where no two adjacent edges receive the same colour. Generally speaking, the tree-width of a graph indicates how similar a graph is to a tree (see Chapter 4 for a proper definition). Here we are interested in the chromatic index of graphs with fixed treewidth and high maximum degree. Nakano, Nishizeki and Zhou [82] showed that a graph of treewidth k and maximum degree $\Delta \geq 2k$ has chromatic index Δ. Bruhn, Gellert and I believe that this can be improved to $\Delta \geq k+\sqrt{k}$. If true this would be best possible as examples show. In support of our conjecture we proved its fractional version. The proofs contain graph decomposition arguments to obtain structural properties and suitable adjacency lemmas that can be applied to these. The details are presented in Chapter 4.

0.7 Parameter estimation

Together with Hoppen, Kohayakawa, Lefmann and Stagni I worked on a project about parameter estimation. For any fixed monotone graph property $\mathcal{P} = Forb(\mathcal{F})$, we studied the sample complexity of estimating a bounded graph parameter $z_{\mathcal{F}}$ that, for an input graph G, counts the number of spanning subgraphs of G that satisfy $\mathcal{P}$. To improve upon previous upper bounds on the sample complexity, we showed that the vertex set of any graph that satisfies a monotone property $\mathcal{P}$ may be partitioned equitably into a constant number of classes in such a way that the cluster graph induced by the partition is not far from satisfying a natural weighted graph generalization of $\mathcal{P}$. Properties for which this holds are said to be *recoverable*, and the study of recoverable properties may be of independent interest. The proofs use (weak) graph regularity and probabilistic arguments. The results are presented in Chapter 5.

Chapter 1

Almost partitioning a 3-edge-coloured $K_{n,n}$ into 5 monochromatic cycles[1]

Richard Lang, Oliver Schaudt and Maya Stein

Abstract

We show that for any colouring of the edges of the complete bipartite graph $K_{n,n}$ with 3 colours there are 5 disjoint monochromatic cycles which together cover all but $o(n)$ of the vertices. In the same situation, 18 disjoint monochromatic cycles together cover all vertices.

1.1 Introduction

The monochromatic cycle partition problem is a Ramsey-type problem that originated in work of Gerencsér and Gyárfás [45] and Gyárfás [52], and lately received a considerable amount of attention from the community. Given a graph G, and a (not necessarily proper) colouring of its edges with r colours, we are interested in covering $V(G)$ with mutually disjoint monochromatic cycles, using as few cycles as possible. (For technical reasons, single vertices, single edges and the empty set count as cycles as well.) To state the problem more precisely, the aim is to determine the smallest number $m = m(r, G)$ such that for any r-edge colouring of G, there are m disjoint monochromatic cycles that cover $V(G)$.

The case $G = K_n$ received the most attention so far. An easy construction shows that at least r cycles are necessary to cover all the vertices, and Erdős, Gyárfás and Pyber [31] showed that the number of cycles needed is a function of r (independent of n). The currently best known upper bound of $100r \log r$ (for large n) for this function is due to Gyárfás, Ruszinkó, Sárközy and Szemerédi [58]. For $r = 2$, Bessy and Thomassé [13] showed that a partition into 2 cycles (even of different colours) always exists, thus proving a conjecture of Lehel [9] and extending earlier work of [80, 1]. (See also [85] for an alternative proof.) Motivated by ideas of Schelp, Balogh et al. [10] suggested a strengthening of Lehel's conjecture: Every 2-coloured n-vertex graph of minimum degree at least $3n/4$ can be partitioned into a red and a blue cycle. As evidence for their conjecture, Balogh et al. [10] proved an asymptotic version: All but $o(n)$ vertices of any 2-coloured n-vertex graph of minimum degree $(3/4 + o(1))n$ can

[1]The results of this chapter have been accepted for publication in SIAM J. Discrete Math [72].

be partitioned into a red and a blue cycle. DeBiasio and Nelsen [27] adapted the absorbing method of [90], to show that under the same conditions, *all* vertices of the graph can be partitioned into a red and a blue cycle. Extending this technique, Letzter [74] proved the conjecture of Balogh et al. for large n.

The conjecture [31] that r monochromatic cycles suffice to partition any r-coloured complete graph for all $r \geq 3$, was disproved by Pokrovskiy [84]. However, his examples allow partitions of all but one vertex. In light of this, it has been proposed to tone down the conjecture, allowing for a constant number of uncovered vertices [10, 84]. On the positive side, for $r = 3$, three monochromatic cycles suffice to partition of all but $o(n)$ vertices of K_n, and, for large enough n, 17 monochromatic cycles partition all of $V(K_n)$; this was shown by Gyárfás, Ruszinkó, Sárközy, and Szemerédi [61]. (Actually, by a slight modification of their method, one can replace the number 17 with 10, see Section 1.5.3). Very recently, Pokrovskiy [85] showed that it is indeed possible to partition all but a constant number of vertices of a 3-coloured complete graph into at most 3 cycles [85]. This was independently confirmed by Letzter with a better constant [75].

For G being the balanced complete bipartite graph $K_{n,n}$, first upper bounds for monochromatic cycle partitions were given by Haxell [64] and by Peng, Rödl and Ruciński [83]. The current best known result is that $4r^2$ monochromatic cycles suffice to partition all vertices of $K_{n,n}$, if n is large [73].

For a lower bound, an easy construction shows we need at least $2r - 1$ cycles to cover all the vertices. For instance, starting out with a properly r-edge-coloured $K_{r,r}$, blow up each vertex in one partition class to a set of size r, while in the other partition class only blow up one vertex to a set of size $r(r - 1) + 1$. A similar construction is given in [84].

We believe that the lower bound of $2r - 1$ might be the correct answer to the monochromatic cycle partition problem in balanced complete bipartite graphs. This suspicion has recently been confirmed for $r = 2$ by Letzter [75], after preliminary work of Schaudt and Stein [97]. (See also [76] for a short proof for a partition into 4 cycles. Our contribution here is that the lower bound of $2r - 1$ is asymptotically correct also for $r = 3$.

Theorem 1.1.1. *For any 3-edge-colouring of $K_{n,n}$,*

(a) *there is a partition of all but $o(n)$ vertices of $K_{n,n}$ into five monochromatic cycles, and*

(b) *if n is large enough, then the vertices of $K_{n,n}$ can be partitioned into 18 monochromatic cycles.*

The second part of our theorem improves the formerly best bound of 1695 disjoint monochromatic cycles for covering any 3-edge coloured $K_{n,n}$ [64]. We remark that in [97] it is shown that 12 monochromatic cycles suffice to partition all the vertices of any two-coloured $K_{n,n}$.

A related result for $r = 2$ and for partitions into *paths*, is due to Pokrovskiy [84]. He showed that a 2-edge-coloured $K_{n,n}$ can be partitioned into two monochromatic paths, unless the colouring is a *split colouring*, that is, an edge-colouring that has a colour-preserving homomorphism to a properly edge-coloured $K_{2,2}$. In a split colouring, three disjoint monochromatic cycles (or paths) are always enough to cover all vertices. Pokrovskiy [84] conjectures $2r - 1$ disjoint monochromatic paths suffice for arbitrary r.

We now briefly sketch the proof of our main result, Theorem 1.1.1, thereby explaining the structure of the paper. The proof of Theorem 1.1.1(a) involves the construction of large monochromatic connected matchings (see below) and an application of the Regularity Lemma [102]. This method has been introduced by Łuczak [79] and became a standard approach.

A *monochromatic connected matching* is a matching in a connected component of the graph spanned by the edges of a single colour, and such a component is called a *monochromatic component*. Slightly abusing notation, we treat matchings as both edge subsets and 1-regular subgraphs. The following is our key lemma. Its proof is given in Section 1.2.

Lemma 1.1.2. *Let the edges of $K_{n,n}$ be coloured with three colours. Then there is a partition of the vertices of $K_{n,n}$ into five or less monochromatic connected matchings.*

Now for the proof of Theorem 1.1.1(a), apply the Regularity Lemma to the given 3-edge-coloured $K_{n,n}$. The reduced graph Γ is almost complete bipartite and inherits a 3-colouring (via majority density of the pairs). A robust version of Lemma 1.1.2, namely Lemma 1.3.1 (see Section 1.3), permits us to partition almost all of R into five monochromatic connected matchings. In the subsequent step, presented in Section 1.4, we apply a specific case of the Blow-up Lemma [60, 70, 79] to get from our matchings to five monochromatic cycles which together partition almost all vertices of $K_{n,n}$.

The proof of Theorem 1.1.1(b) is given in Section 1.5.2. It combines ideas of Haxell [64] and Gyárfás et al. [61] with Theorem 1.1.1(a). First, we fix a large monochromatic subgraph H, which is Hamiltonian and remains so even if some of the vertices are deleted from it. Then, using Theorem 1.1.1(a), we cover almost all vertices of $K_{n,n} - V(H)$ with five vertex-disjoint monochromatic cycles. The amount of still uncovered vertices being much smaller than the order of H, we can apply a Lemma from [58] in order to absorb these vertices using vertices from H, and producing only a few more cycles. We finish by taking one more monochromatic cycle, which covers the remainder of H.

1.2 Covering with connected matchings

In this section we give the proof of the exact version of Lemma 1.1.2. Its proof has been written with the proof of the more technical robust counterpart (Lemma 1.3.1 in Section 1.3) in mind, in order to ease the transition between the two proofs. It may therefore appear to be a bit overly lengthy in some of its parts.

1.2.1 Preliminaries

This subsection contains some preliminary results for the proof of our key lemma, Lemma 1.1.2, which is given in the subsequent subsection. We start with some definitions. The *biparts* of a bipartite graph H are its partition classes, which we denote by $\overline{H}$ and $\underline{H}$. If $X \subseteq \overline{H}$ and $Y \subseteq \underline{H}$, or if $X \subseteq \underline{H}$ and $Y \subseteq \overline{H}$, we write $[X, Y]$ for the bipartite subgraph induced by the edges between X and Y.

Definition 1.2.1 (empty graph, trivial graph)**.** A bipartite graph is *empty* if it has no vertices and *trivial* if one of its biparts has no vertices.

For a colouring of the edges of H with colours red, green and blue, a *red component* R is a connected component in the subgraph obtained by deleting the non-red edges and a *red matching* is a matching whose edges are red. The same terms are defined for colours green and blue. We now introduce two types of colourings for 2-coloured bipartite graphs. We call an edge colouring of a bipartite graph H in red and blue a *V-colouring* if there are monochromatic components R and B of distinct colours such that

1. each of R and B is non-trivial;
2. $R \cup B$ is spanning in H;
3. $|V(\overline{R \cap B})| = |V(\overline{H})|$ or $|V(\underline{R \cap B})| = |V(\underline{H})|$.

A colouring of $E(H)$ in red and blue is *split*, if

1. all monochromatic components are non-trivial;
2. each colour has exactly two monochromatic components.

The following lemma classifies the component structure of a 2-coloured bipartite graph.

Lemma 1.2.2. *If the bipartite 2-edge-coloured graph H is complete, then one of the following holds:*

(a) *There is a spanning monochromatic component,*

(b) *H has a V-colouring, or*

(c) *the edge-colouring is split.*

Proof. Let R be a non-trivial component in colour red, say. Set $X := H - R$ and note that all edges in $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ are blue.

We first assume that $|\overline{X}| = 0$. If also $|\underline{X}| = 0$, we are done, since then R is spanning. Otherwise, $|\underline{X}| > 0$, and thus the colouring is a V-colouring.

So by symmetry we can assume that both $|\overline{X}| > 0$ and $|\underline{X}| > 0$. If there is a blue edge in R or in X, then H is spanned by one blue component. Hence, all edges inside R and X are red and the colouring is split. □

Corollary 1.2.3. *If a bipartite 2-edge-coloured graph H is complete, then*

(a) *there are one or two non-trivial monochromatic components that together span H, and*

(b) *if the colouring is not split, then there is a colour with exactly one non-trivial component.*

Let us now turn to monochromatic matchings.

Lemma 1.2.4. *Let H be a balanced bipartite complete graph whose edges are coloured red and blue. Then either*

(a) *H is spanned by two vertex disjoint monochromatic connected matchings, one of each colour, or*

(b) *the colouring is split and*

- *H is spanned by one red and two blue vertex disjoint connected monochromatic matchings and*

- *H is spanned by one blue and two red vertex disjoint connected monochromatic matchings.*

Proof. First assume that the colouring is split. We take one red maximum matching in each of the two red components. This leaves at least one of the blue components with no vertices on each side. We extract a third maximum matching from the leftover of the other blue component, thus leaving one of its sides with no vertices. Thus the three matchings together span H. Note that we could have switched the roles of red and blue in order to obtain two blue and one red matching that span H.

So by Lemma 1.2.2, we may assume that either there is a colour, say red, with a spanning component R, or H has a V-colouring, with components R in red and B in blue, say. In either case, we take a maximum red matching M in R. Then there is an induced balanced bipartite subgraph of H, whose edges are all blue, which contains all uncovered vertices of each bipart of H. If this subgraph is trivial, we are done. Otherwise, we finish by extracting from it a maximum blue matching $M' \subseteq B$. As H is complete and there are no leftover edges in said subgraph, we obtain that $M \cup M'$ spans H, and we are done. $\square$

We continue with a lemma about the component structure of 3-edge-coloured bipartite graphs.

Lemma 1.2.5. *Let the edges of the complete bipartite graph H be coloured in red, green and blue, such that each colour has at least four non-trivial components; then there are three monochromatic components that together span H.*

Proof. Let R be a red non-trivial component. Since there are three more red non-trivial components, the three graphs $X := H - R$, $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ are each non-trivial. Moreover, the edges of the latter two graphs are green and blue. By Corollary 1.2.3(a) there are one or two non-trivial monochromatic components that together span $[\underline{R}, \overline{X}]$. So, if $[\overline{R}, \underline{X}]$ has a spanning monochromatic component, then we can span H with at most three components, which is as desired. Therefore and by symmetry we may assume from now on that none of $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ has a spanning monochromatic component. Suppose $[\overline{R}, \underline{X}]$ has a split-colouring. By Lemma 1.2.2, either $[\underline{R}, \overline{X}]$ is split or one of $\underline{R}$ and $\overline{X}$ is contained in the intersection of a blue and a green monochromatic component. In the latter case the union of three monochromatic components of the same colour contains one of the biparts of H. But this is impossible as each colour has at least four non-trivial components. On the other hand, if both $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ have a split colouring, then each bipart of H is contained in the union of four green components as well as in the union of four blue components, and thus all edges in X are red. But then there are only two non-trivial red components, R and X, a contradiction.

So by Lemma 1.2.2, and by symmetry, we know that $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ both have green/blue V-edge-colourings. Thus each of $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ has a non-trivial blue component and a non-trivial green component, say these are B_1, G_1 and B_2, G_2 respectively. Furthermore, $\underline{X}$ or $\overline{R}$ is contained in the intersection $B_1 \cap G_1$, and $\overline{X}$ or $\underline{R}$ is spanned by the intersection $B_2 \cap G_2$.

We first look at the case where $\underline{X}$ is contained in $\underline{B_1 \cap G_1}$. If $\underline{R}$ is contained in $\underline{B_2 \cap G_2}$, then both green and blue have at most two spanning components, which is a contradiction. On the other hand, if $\overline{X}$ is contained in $\overline{B_2 \cap G_2}$, then H is spanned by the union of R and the blue components in H that contain B_1 and B_2, and we are done.

Consequently we can assume by symmetry and by Lemma 1.2.2 that $\overline{R}$ is spanned by $\overline{B_1 \cap G_1}$ and $\underline{R}$ is spanned by $\underline{B_2 \cap G_2}$. Observe that $[\underline{G_1}, \overline{G_2}]$ is coloured red and blue and $[\underline{B_1}, \overline{B_2}]$ is coloured red and green, since otherwise, we obtain the desired cover. Suppose there is a red component of $[\underline{G_1}, \overline{G_2}]$ that is spanning in $[\underline{G_1}, \overline{G_2}]$. Such a component, together with B_1 and B_2, spans H. So, we can assume $[\underline{G_1}, \overline{G_2}]$ has no red spanning red component. Moreover, since there are at least four non-trivial blue components, $[\underline{G_1}, \overline{G_2}]$ contains two blue components, which are non-trivial each.

Since these blue components are non-trivial in H, $[\underline{G_1}, \overline{G_2}]$ does not have a V-colouring (in itself). Thus, by Lemma 1.2.2, $[\underline{G_1}, \overline{G_2}]$ is split coloured in red and blue. Similarly we see that $[\underline{B_1}, \overline{B_2}]$ is split coloured in red and green.

Consider the edges in $[\underline{G_1}, \overline{B_2}]$ and $[\underline{B_1}, \overline{G_2}]$. If any of these edges is green or blue, then our graph is spanned by three green or by three blue components. On the other hand, if all edges in $[\underline{G_1}, \overline{B_2}]$ and $[\underline{B_1}, \overline{G_2}]$ are red, then H has only three non-trivial red components, a contradiction. $\square$

1.2.2 Proof of Lemma 1.1.2

We are now ready to prove Lemma 1.1.2. Let H be a balanced bipartite complete graph of order $2n$. Our aim is to show that H can be spanned with five vertex disjoint monochromatic connected matchings. We suppose that this is wrong in order to obtain a contradiction. We prove a series of claims in order to reduce the problem to a specific colouring, which then receives a distinct treatment.

Claim 1.2.6. *Each colour has at least three non-trivial components.*

Proof. Suppose the claim is wrong for colour red, say. By assumption, there are two (possibly trivial) red components R_1 and R_2 in H, such that all other red components are trivial. Let M be a maximum red matching in $R_1 \cup R_2$. Then every edge in the balanced bipartite subgraph $X := H - M$ is green or blue. By Lemma 1.2.4, H can be spanned with three vertex-disjoint monochromatic connected matchings. So in total we found at most five vertex-disjoint monochromatic connected matchings that together span H. $\square$

Claim 1.2.7. *There are no two monochromatic components that together span H.*

Proof. Suppose the claim is wrong and there are monochromatic components R and B that together span H. By Claim 1.2.6 we can assume that they have distinct colours, say R is red and B is blue. Take a red matching M^{red} of maximum size in R and a blue matching M^{blue} of maximum size in $B - V(M^{\text{red}})$. Set $R' := R - V(M^{\text{red}} \cup M^{\text{blue}})$ and $B' := B - V(M^{\text{red}} \cup M^{\text{blue}})$. By maximality, any edge between $\overline{B'}$ and $\underline{R'}$ is green. The same holds for the edges between $\underline{B'}$ and $\overline{R'}$.

If $[\underline{B'}, \overline{R'}]$ is empty, we finish by picking a maximum matching in $[\underline{R'}, \overline{B'}]$. We proceed analogously if $[\underline{R'}, \overline{B'}]$ is empty. Assuming that both are non-empty we now pick now pick a maximum matching in each of the green components of $H - V(M^{\text{red}} \cup M^{\text{blue}})$ that contain $[\overline{B'}, \underline{R'}]$, $[\underline{B'}, \overline{R'}]$. (If this is the same component, we only pick one matching. If R' or B' is empty, we let the matchings be empty.) Call these green matchings M_1^{green} resp. M_2^{green}. Let $B'' := B' - V(M_1^{\text{green}} \cup M_2^{\text{green}})$ and $R'' := R' - V(M_1^{\text{green}} \cup M_2^{\text{green}})$.

Observe that by the maximality of M_1^{green} and M_2^{green}, if one of $\underline{R''}$, $\overline{B''}$ is non-empty, then the other one is empty. The same holds for the sets $\underline{B''}$, $\overline{R''}$. Thus one of the two graphs R'', B'' is empty, say this is B''.

The edges in R'' are green and blue. If R'' contains no green edges, we can pick another blue matching of maximum size and are done. Then again, if R'' contains a green edge, it follows by maximality of M_1^{green} and M_2^{green} that both of them are empty, which implies that there are no green edges in $R' \cup B'$. In this case we ignore M_1^{green} and M_2^{green} and finish as follows: By Lemma 1.2.4, R' can be spanned by at most 3 vertex disjoint monochromatic connected matchings. This proves the claim. $\square$

Claim 1.2.8. *Let Y and Z be monochromatic components of distinct colours such that $Y \cap Z$ is non-trivial. Then $Y - Z$ is not empty.*

Proof. Let Y be a red component, Z be a blue component, and let $X := H - (Y \cup Z)$. Suppose that $Y - Z$ is empty. We first note that all edges in $[\overline{Y \cap Z}, \underline{X}]$ and $[\underline{Y \cap Z}, \overline{X}]$ are green. Moreover, by Claim 1.2.6, there is another non-trivial blue component in H, which implies that X is non-trivial.

The subgraphs $[\overline{Y \cap Z}, \underline{X}]$ and $[\underline{Y \cap Z}, \overline{X}]$ cannot belong to the same green component, since otherwise H is spanned by the union of said green component and Z, which is not possible by Claim 1.2.7. Consequently, X has no green edges. By Claim 1.2.6 there is a green non-trivial component $G \subseteq Y \cup Z$. As $H = Z \cup (Y - Z) \cup X$ and $Y - Z$ is empty, we obtain that $G \cap Z$ is non-trivial in H and $G - Z \subseteq Y - Z$ is empty. Thus G has the same properties as Y with respect to Z and we can repeat the same arguments as above to obtain that all edges in X are blue. But this is a contradiction to Claim 1.2.7, as X and Z together span H. $\square$

Claim 1.2.9. *There is a colour that has exactly three non-trivial components.*

Proof. We show that there is a colour with at most three non-trivial components. This together with Claim 1.2.6 yields the desired result. So suppose otherwise. Then each colour has at least four non-trivial components. By Lemma 1.2.5, there are components X, Y and Z that together span H.

By assumption, not all of X, Y and Z have the same colour. If two of these components, say X and Y, have the same colour, say red, then $H - (X \cup Y)$ contains a red component that is non-trivial, by the assumption that our claim is false. The intersection of this red component with Z is non-trivial. Hence we get a contradiction to Claim 1.2.8.

So assume X is red, Y is blue and Z is green. We claim that (after possibly swapping top and bottom parts)

$$\underline{(Y \cap Z) - X} \text{ is empty.} \tag{1.2.1}$$

Indeed, otherwise $(Y \cap Z) - X$ is non-trivial. Then, as $[\underline{X}, \overline{(Y \cap Z) - X}]$ is non-trivial and its edges are green and blue, we get $\underline{X} \subseteq Y \cup Z$ since every vertex in $\underline{X}$ sees a vertex in $\overline{Y \cap Z}$. In the same way we obtain $\overline{X} \subseteq Y \cup Z$. Thus $Z \cup Y$ is spanning, which is not possible by Claim 1.2.7. This proves (1.2.1).

By assumption, $H - X$ contains three non-trivial red components R_1, R_2 and R_3, say. For $i \neq j$, $[\overline{R_i \cap (Y - Z)}, \underline{R_j \cap (Z - Y)}]$ has no red, blue or green edges and thus is trivial. So for at most one $i \in \{1, 2, 3\}$ the subgraph $R_i \cap [\overline{Y - Z}, \underline{Z - Y}]$ is non-trivial. The same holds for $[\underline{R_i \cap (Y - Z)}, \overline{R_j \cap (Z - Y)}]$. Consequently, and by the pigeonhole principle, we can assume that,

$$R_1 \cap [\overline{Y - Z}, \underline{Z - Y}] \text{ and } R_1 \cap [\underline{Y - Z}, \overline{Z - Y}] \text{ are both trivial.} \tag{1.2.2}$$

As R_1 is non-trivial, we can suppose that without loss of generality $R_1 \cap Y$ is non-trivial. Thus, by (1.2.1) $\underline{R_1 \cap (Y - Z)}$ is non-empty. Hence, by (1.2.2) we get:

$$|\overline{R_1} \cap \overline{Z - Y}| = 0. \tag{1.2.3}$$

Moreover, Claim 1.2.8 (applied to R_1 and Y) implies that R_1 has at least one vertex in $\overline{Z - Y}$ or $\underline{Z - Y}$. By (1.2.3) we have the latter case and hence

$$\underline{R_1 \cap (Z - Y)} \text{ and } \underline{R_1 \cap (Y - Z)} \text{ are each non-empty.} \tag{1.2.4}$$

The fact that $[\overline{Y - (X \cup Z)}, \underline{R_1 \cap (Z - Y)}]$ and $[\overline{Z - (X \cup Y)}, \underline{R_1 \cap (Y - Z)}]$ only have red edges, together with (1.2.2) and (1.2.4), yields that

$$\overline{Y - (X \cup Z)} \text{ and } \overline{Z - (X \cup Y)} \text{ are each empty} \tag{1.2.5}$$

Now by (1.2.5) (and by the existence of R_1, R_2, R_3), we know that $\overline{(Y \cap Z) - X}$ is non-empty. So each vertex of $\underline{X}$ has a neighbour in $\overline{(Y \cap Z) - X}$ and hence $\underline{X} \subseteq \underline{Y \cup Z}$. Since, by Claim 1.2.7, H is not spanned by $Y \cup Z$, we have that $\overline{X - (Y \cup Z)}$ is non-empty. This and (1.2.4) imply that $[\overline{X - (Y \cup Z)}, \underline{Y - (X \cup Z)}]$ and $[\overline{X - (Y \cup Z)}, \underline{Z - (X \cup Y)}]$ are non-trivial each. As the edges of these subgraphs are green and blue respectively, there are green and blue components G and B such that $\underline{H - X - [(G \cap Y) \cup (B \cap Z)]}$ is empty.

Now let G' be another non-trivial green component. Then $\underline{G' - X}$ is empty, while $\underline{G' \cap X}$ is non-empty. By (1.2.5) it follows that $\overline{G' - X}$ is empty, while $\overline{G' \cap X}$ is non-empty. This is not possible by Claim 1.2.8 and completes the proof. □

Using Claim 1.2.9 we assume from now on that without loss of generality, colour red has exactly three non-trivial components R_1, R_2 and R_3. For i $= 1, 2, 3$, let M_i be a red matching of maximum size in R_i.

The remaining graph $Y := H - M_1 - M_2 - M_3$ has no red edges. If Y is trivial, then as $|\overline{Y}| = |\underline{Y}|$, the graph Y is empty, and so we are done. If Y can be spanned by two disjoint monochromatic connected matchings, we are also done, since in that case, we found five matchings which together span H. So we can assume that the colouring of Y is split, by Lemma 1.2.4 and as the edges of Y are green and blue. We denote the blue and green components of Y by B_1', B_2', respectively G_1', G_2', where $\overline{B_1'} = \overline{G_1'}$, $\overline{B_2'} = \overline{G_2'}$, $\underline{B_1'} = \underline{G_2'}$, and $\underline{B_2'} = \underline{G_1'}$. Note that the subgraph

$$B_1' \cup B_2' \cup M_1 \cup M_2 \cup M_3 \text{ is spanning in } H. \tag{1.2.6}$$

By Lemma 1.2.4, Y can be spanned by two blue matchings $M_4 \subseteq B_1'$, $M_5 \subseteq B_2'$ and an additional green matching. If any of the matchings M_i is trivial, we can ignore it and still have a sufficiently large cover of H. Thus we get that

$$B_1',\ B_2',\ G_1',\ G_2',\ M_1,\ M_2, \text{ and } M_3 \text{ are non-trivial.} \tag{1.2.7}$$

Moreover, let B_1 and B_2 be the blue components in H that contain B_1' and B_2', respectively. We define G_1 and G_2 analogously. If $B_1 = B_2$, we are done as $M_4 \cup M_5$ is a connected matching. This and symmetry imply

$$B_1 \neq B_2 \text{ and } G_1 \neq G_2. \tag{1.2.8}$$

The colouring so far is shown in Figure 1.1.

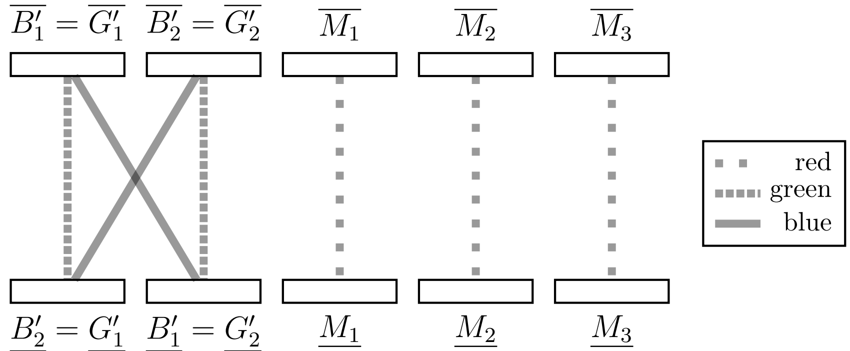

Figure 1.1: The structure of the colouring before Claim 1.2.10

Claim 1.2.10. *For each* $\mathrm{i} = 1, 2, 3$ *we have that*

(a)
- *if* $|\overline{M_{\mathrm{i}}} \setminus \overline{G_1 \cup G_2}| > 0$*, then* $\underline{B_1'} \subseteq \underline{R_{\mathrm{i}}}$ *or* $\underline{B_2'} \subseteq \underline{R_{\mathrm{i}}}$*;*
- *if* $|\overline{M_{\mathrm{i}}} \setminus \overline{B_1 \cup B_2}| > 0$*, then* $\underline{G_1'} \subseteq \underline{R_{\mathrm{i}}}$ *or* $\underline{G_2'} \subseteq \underline{R_{\mathrm{i}}}$*;*

(b)
- *if* $|\underline{M_{\mathrm{i}}} \setminus \underline{G_1 \cup G_2}| > 0$*, then* $\overline{B_1'} \subseteq \overline{R_{\mathrm{i}}}$ *or* $\overline{B_2'} \subseteq \overline{R_{\mathrm{i}}}$*;*
- *if* $|\underline{M_{\mathrm{i}}} \setminus \underline{B_1 \cup B_2}| > 0$*, then* $\overline{G_1'} \subseteq \overline{R_{\mathrm{i}}}$ *or* $\overline{G_2'} \subseteq \overline{R_{\mathrm{i}}}$*;*

(c)
- *if* $|\overline{M_{\mathrm{i}}} \setminus \overline{G_1 \cup G_2 \cup B_1 \cup B_2}| > 0$*, then* $\underline{B_1' \cup B_2'} = \underline{G_1' \cup G_2'} \subseteq \underline{R_{\mathrm{i}}}$*;*
- *if* $|\underline{M_{\mathrm{i}}} \setminus \underline{G_1 \cup G_2 \cup B_1 \cup B_2}| > 0$*, then* $\overline{B_1' \cup B_2'} = \overline{G_1' \cup G_2'} \subseteq \overline{R_{\mathrm{i}}}$*.*

Proof. For the first part of (a), assume $|\overline{M_1} \setminus \overline{G_1 \cup G_2}| > 0$. Note that there is no green edge between $\overline{M_1} \setminus \overline{G_1 \cup G_2}$ and $\underline{G_1'}$. First assume that $\overline{M_1 \cap B_1} \setminus \overline{G_1 \cup G_2}$ is non-empty. Then, by 1.2.8, any edge between $\overline{M_1 \cap B_1} \setminus \overline{G_1 \cup G_2}$ and $\underline{B_2'} = \underline{G_1'}$ is red. So, by (1.2.7) the result follows. So we can assume that this is not true. Similarly the result holds if $|\overline{M_1 \cap B_2} \setminus \overline{G_1 \cup G_2}| > 0$. Therefore we can assume that $\overline{M_1} \setminus \overline{B_1 \cup B_2 \cup G_1 \cup G_2}$ is non-empty. In this case, since all edges between $\overline{M_1} \setminus \overline{G_1 \cup G_2 \cup B_1 \cup B_2}$ and $\underline{B_1'}$ are red, the result follows again by (1.2.7). Statement (b) and the second part of (a) follow similarly.

For the first part of (c), note that any edge between $\overline{M_{\mathrm{i}}} \setminus \overline{G_1 \cup G_2 \cup B_1 \cup B_2}$ and $\underline{B_1' \cup B_2'} = \underline{G_1' \cup G_2'}$ has to be red and use (1.2.7). The second part of (c) is analogous. ☐

By Claim 1.2.6 there are green and blue non-trivial components $G_3 \neq G_1, G_2$ and $B_3 \neq B_1, B_2$ in H.

Claim 1.2.11. *It holds that* $|V(G_3 \cap B_3 \cap (M_1 \cup M_2 \cup M_3))| > 0$*.*

Proof. Assume otherwise. That is, assume

$$|V(G_3 \cap B_3 \cap (M_1 \cup M_2 \cup M_3))| = 0.$$

The components B_3 and G_3 do not meet with $B_1' \cup B_2' = G_1' \cup G_2'$ and by (1.2.6), there are no vertices outside of $B_1' \cup B_2' \cup M_1 \cup M_2 \cup M_3$. We conclude that $B_3 \cap (M_1 \cup M_2 \cup M_3)$ and $G_3 \cap (M_1 \cup M_2 \cup M_3)$ are each non-trivial. Hence there are indices $\mathrm{i}, \mathrm{i}', j, j'$ such that there is a blue non-trivial subgraph $B_3' \subseteq B_3$ and a green non-trivial subgraph $G_3' \subseteq G_3$ such that $\overline{B_3'} \subseteq \overline{M_{\mathrm{i}}}$ and $\underline{B_3'} \subseteq \underline{M_{\mathrm{i}'}}$, and $\overline{G_3'} \subseteq \overline{M_j}$ and $\underline{G_3'} \subseteq \underline{M_{j'}}$. Actually, we can choose these indices such that $\mathrm{i} \neq \mathrm{i}'$ and $j \neq j'$. Since if $\mathrm{i} = \mathrm{i}'$, say, Claim 1.2.8 yields that $(B_3 \cap H) \setminus M_\mathrm{i}$ is not empty and therefore, by (1.2.6), there is some index $k \neq \mathrm{i}$ such that $B_3 \cap M_k$ is not empty, which allows us to swap i' for k.

For an index $k \neq \mathrm{i}$, the edges between $\overline{B_3'} \subseteq \overline{R_1 \cap M_\mathrm{i}}$ and $\underline{G_3' \cap M_k}$ are blue and green. As by our initial assumption $|V(G_3 \cap B_3 \cap (M_1 \cup M_2 \cup M_3))| = 0$, this implies that $|\underline{G_3 \cap M_k}| = 0$. In the same way we obtain that $|\overline{G_3 \cap M_k}| = 0$ for $k \neq \mathrm{i}'$ or $|\overline{B_3' \cap M_\mathrm{i}}| = 0$, but the latter cannot happen by the choice of B_3'. Hence we have $\mathrm{i} = j'$ and $\mathrm{i}' = j$; in other words,

$$|\underline{M_\mathrm{i} \cap G_3}| > 0,\ |\overline{M_j \cap G_3}| > 0,\ |\overline{M_\mathrm{i} \cap B_3}| > 0 \text{ and } |\underline{M_j \cap B_3}| > 0.$$

So by Claim 1.2.10 (a) and (b), either we have $B_1' \subseteq R_\mathrm{i}$ and $B_2' \subseteq R_j$, or we have $G_1' \subseteq R_\mathrm{i}$ and $G_2' \subseteq R_j$. Indeed, the fact that $|\underline{M_\mathrm{i} \cap G_3}| > 0$ together with Claim 1.2.10 (b) implies that one of $\overline{B_1'} = \overline{G_1'} \subseteq \overline{R_\mathrm{i}}$, $\overline{B_2'} = \overline{G_2'} \subseteq \overline{R_\mathrm{i}}$ holds. Without loss of generality, we assume the latter. Next, as $|\overline{M_\mathrm{i} \cap B_3}| > 0$, Claim 1.2.10 (a) implies that $\underline{G_1'} = \underline{B_2'} \subseteq \underline{R_\mathrm{i}}$ or $\underline{G_2'} = \underline{B_1'} \subseteq \underline{R_\mathrm{i}}$. Without loss of generality, we assume the former. We repeat the same with index j, and since we already know that $B_2' \subseteq R_\mathrm{i}$, the output of Claim 1.2.10 has to be $\underline{B_1'} = \underline{G_2'} \subseteq \underline{R_j}$ for $|\overline{M_j \cap G_3}| > 0$ and $\overline{B_1'} = \overline{G_1'} \subseteq \overline{R_j}$ for $|\underline{M_j \cap B_3}| > 0$. For the remainder, let us assume that $B_1' \subseteq R_\mathrm{i}$ and $B_2' \subseteq R_j$.

Then $G_1' \cap R_k = \emptyset = G_2' \cap R_k$, where k is the third index, which together with Claim 1.2.10 (a) and (b), gives that $|R_k \cap (G_3 \cup B_3)| = 0$. The edges between $\underline{B_2'} = \underline{G_1'} \subseteq \underline{G_1 \cap R_j}$ and $\overline{B_3' \cap R_\mathrm{i}}$ have to be green, which implies $\overline{B_3'} \subseteq \overline{G_1}$. As any edge between $\overline{B_3'}$ and $\underline{R_k - B_3}$ has to be green this implies $|\underline{R_k \cap G_1}| > 0$ since R_k is non-trivial and $|\underline{R_k \cap B_3}| = 0$. This also implies that $|\underline{R_k - G_1}| = 0$.

By repeating the same argument with $\overline{B_1'} = \overline{G_1'} \subseteq \overline{G_1}$ and $\underline{B_3'}$, it follows that $|\overline{R_k \cap G_1}| > 0$ and $|\overline{R_k - G_1}| = 0$. So $R_k \cap G_1$ is non-trivial and $R_k - G_1$ is empty, a contradiction to Claim 1.2.8. □

Claim 1.2.11 and the symmetry between the M_i in both biparts allow us to assume that without loss of generality

$$|\overline{M_3 \cap G_3 \cap B_3}| > 0. \tag{1.2.9}$$

This implies $|\overline{M_3} \setminus \overline{G_1 \cup G_2 \cup B_1 \cup B_2}| > 0$ and thus by Claim 1.2.10(c) with $\mathrm{i} = 3$ we obtain

$$\underline{B_1'} \cup \underline{B_2'} = \underline{G_1'} \cup \underline{G_2'} \subseteq \underline{R_3}. \tag{1.2.10}$$

This implies that $(\underline{R_1 \cup R_2}) \cap (\underline{G_1' \cup G_2'}) = \emptyset$. Since the edges between $\overline{M_3 \cap G_3 \cap B_3}$ and $\underline{R_1 \cup R_2}$ are coloured green and blue, we have by 1.2.9 that

$$\underline{M_1 \cup M_2} \subseteq \underline{R_1 \cup R_2} \subseteq \underline{G_3 \cup B_3}. \tag{1.2.11}$$

So, by (1.2.7) and Claim 1.2.10(b) with $\mathrm{i} = 1$, we can assume that without loss of generality

$$\overline{B_1'} = \overline{G_1'} \subseteq \overline{R_1} \tag{1.2.12}$$

and hence by (1.2.7) and Claim 1.2.10(b) with $\mathrm{i} = 2$ it follows that

$$\overline{B_2'} = \overline{G_2'} \subseteq \overline{R_2}. \tag{1.2.13}$$

The structure of the colouring so far is sketched in Figure 1.2. The assertions (1.2.12)

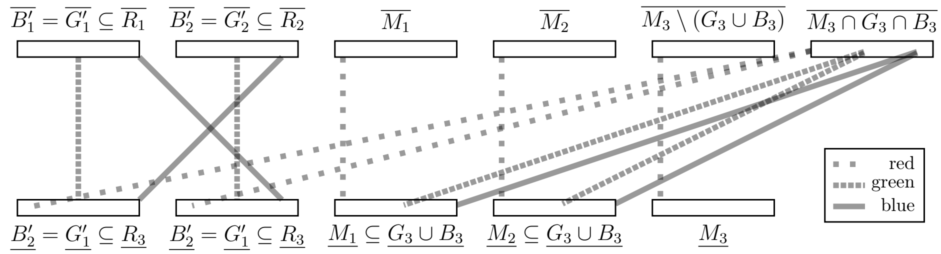

Figure 1.2: The structure of the colouring after (1.2.13).

and (1.2.13) imply that $\overline{R_3} \cap \overline{G_1' \cup G_2'} = \emptyset$. Suppose that there is an $x \in \overline{R_1 \cup R_2} \setminus \overline{G_1 \cup G_2 \cup B_1 \cup B_2}$. By 1.2.10, the edges between x and $\underline{G_1' \cup G_2'} = \underline{B_1' \cup B_2'}$ are not red, and neither green or blue by choice of x. As G_1' and G_2' are both non-trivial in H by (1.2.7) and H is complete, we obtain a contradiction. Hence

$$\overline{M_1 \cup M_2} \setminus \overline{G_1 \cup G_2 \cup B_1 \cup B_2} = \emptyset. \tag{1.2.14}$$

In the same fashion, suppose there is an $x \in (\underline{M_3} \setminus \underline{G_1 \cup G_2}) \cup (\underline{M_3} \setminus \underline{B_1 \cup B_2})$. By (1.2.12) and (1.2.13), the edges between x and $\overline{B_1'} = \overline{G_1'}$ respectively $\overline{B_2'} = \overline{G_2'}$ are neither green nor blue by choice of x. Again, using (1.2.7) and the completeness of H, we obtain a contradiction as

$$\underline{M_3} \setminus \underline{G_1 \cup G_2} = \underline{M_3} \setminus \underline{B_1 \cup B_2} = \emptyset. \tag{1.2.15}$$

Finally, suppose there is an $x \in \overline{B_3 \cup G_3} \cap \overline{M_1 \cup M_2}$. By (1.2.7), x sees vertices in $\underline{M_3}$. This, however, contradicts (1.2.15) and thus

$$\overline{B_3 \cup G_3} \cap \overline{M_1 \cup M_2} = \emptyset. \tag{1.2.16}$$

Next, we restore the symmetry between the colours.

Claim 1.2.12. *Each colour has exactly three components.*

Proof. We already know that R_1, R_2 and R_3 are the only red components in H. Suppose there is a (possibly trivial) green component G_4 distinct from G_1, G_2 and G_3. Assume first that $\underline{G_4} \neq \emptyset$. Note that any edge between $\underline{G_4}$ and $\overline{G_1' \cup G_2'}$ is red or blue. By 1.2.8, no vertex of $\underline{G_4}$ can send blue edges to both $\overline{G_1'}$ and $\overline{G_2'}$. Moreover, by (1.2.12) and (1.2.13), no vertex of $\underline{G_4}$ can send red edges to both $\overline{G_1'}$ and $\overline{G_2'}$. Since H is complete and $\overline{G_1'} = \overline{B_1'}$ and $\overline{G_2'} = \overline{B_2'}$ are non-trivial, we derive $\underline{G_4} \subseteq \underline{R_1 \cup R_2} \cap \underline{B_1 \cup B_2}$. But this contradicts (1.2.9), because H is complete.

Now let us assume that $\underline{G_4} = \emptyset$, and so, $\overline{G_4} \neq \emptyset$. In other words, G_4 consists of a single vertex with no incident green edges. Suppose that $\overline{G_4} \cap \overline{M_3} = \emptyset$. So by (1.2.7) and (1.2.10), the edges between $\overline{G_4}$ and $\underline{G_1' \cup G_2'}$ are blue, which contradicts that B_1' and B_2' lie in distinct blue components, as asserted by 1.2.8. Therefore $\overline{G_4} \subseteq \overline{M_3}$. As $\underline{G_4} = \emptyset$, all edges between $\overline{G_4}$ and $\underline{M_1 \cup M_2}$ are blue. By (1.2.15) and (1.2.16), $B_3 \subseteq [\underline{M_1 \cup M_2}, \overline{M_3}]$. Since H is complete and B_3 is non-trivial, we obtain that $\overline{G_4} \subseteq \overline{B_3}$. We also have that $G_3 \subseteq [\underline{M_1 \cup M_2}, \overline{M_3}]$ by (1.2.15) and (1.2.16). Since G_3 is non-trivial it follows that, $\underline{G_3} \cap \underline{M_1 \cup M_2}$ is non-empty. Since the edges between $\overline{G_4}$ and $\underline{G_3}$ are blue, we obtain that $\underline{M_1 \cup M_2} \cap \underline{G_3 \cap B_3} \neq \emptyset$. But this represents a contradiction to (1.2.12) or (1.2.13), since there is no colour left for the edges between $\underline{G_3 \cap B_3}$ and $\overline{B_1' \cup B_2'}$. Since a fourth blue component would behave the same way as G_4, this finishes the proof of the claim. $\square$

By (1.2.10) it follows that $\underline{R_\mathrm{i}} = \underline{M_\mathrm{i}}$ for $\mathrm{i} = 1, 2$. In the same way (1.2.12) and (1.2.13) imply that

$$\overline{R_3} = \overline{M_3}. \tag{1.2.17}$$

For $1 \le \mathrm{i}, j, k \le 3$ we denote $\overline{\mathrm{i}|j|k} := \overline{R_\mathrm{i} \cap G_j \cap B_k}$ and $\underline{\mathrm{i}|j|k} := \underline{R_\mathrm{i} \cap G_j \cap B_k}$. From 1.2.7, 1.2.9, 1.2.12 and 1.2.13 we obtain that

$$|\overline{1|1|1}|, |\overline{2|2|2}|, |\overline{3|3|3}| > 0. \tag{1.2.18}$$

Note that by definition and completeness it follows that for all $\mathrm{i}, \mathrm{i}', j, j', k, k'$ with $\mathrm{i} \neq \mathrm{i}'$, $j \neq j'$ and $k \neq k'$ we have (modulo switching biparts)

$$\text{if } |\overline{\mathrm{i}|j|k}| > 0, \text{ then } |\underline{\mathrm{i}'|j'|k'}| = 0. \tag{1.2.19}$$

Let us show that $\underline{\mathrm{i}|j|k} = \emptyset$, unless i, j, k are pairwise different. Indeed, otherwise, if say $\underline{1|1|k} \neq \emptyset$ for $k = 1, 2$ or 3, we obtain a contradiction to (1.2.19) as $|\overline{2|2|2}|, |\overline{3|3|3}| > 0$ by (1.2.18).

Hence $\underline{H}$ can be decomposed into sets $\underline{\mathrm{i}|j|k}$, where $1 \le \mathrm{i}, j, k \le 3$ are pairwise different. So we have:

$$\underline{1|3|2} \cup \underline{1|2|3} \cup \underline{2|3|1} \cup \underline{2|1|3} \cup \underline{3|2|1} \cup \underline{3|1|2} = \underline{H}. \tag{1.2.20}$$

Claim 1.2.13. *We have* $\overline{H} = \overline{1|1|1} \cup \overline{2|2|2} \cup \overline{3|3|3} \cup \overline{3|1|2} \cup \overline{3|2|1}$.

Proof. First, we show there is no $\overline{\mathrm{i}|j|k} \neq \emptyset$ such that exactly two of i, j, k are equal. If $\overline{3|1|1} \neq \emptyset$, say, then $|\underline{1|2|3}|, |\underline{1|3|2}| = 0$ by (1.2.19). Together with (1.2.20), this implies that R_1 is trivial, a contradiction. Second, note that 1.2.10 implies that $\overline{3|1|2}$ and $\overline{3|2|1}$ are non-empty. Again, by (1.2.19), it follows that $\overline{\mathrm{i}|j|k} = \emptyset$, if $\mathrm{i} \neq 3$ and $3 \in \{j, k\}$. This proves the claim. □

Claim 1.2.14. *We have* $\overline{H} = \overline{1|1|1} \cup \overline{2|2|2} \cup \overline{3|3|3}$.

Proof. By the previous claim it remains to show that $\overline{3|1|2} = \overline{3|2|1} = \emptyset$. To this end, suppose that $\overline{3|1|2} \neq \emptyset$ and thus $|\underline{1|2|3}|, |\underline{2|3|1}| = 0$ by (1.2.19). If $\overline{3|2|1} \neq \emptyset$ as well, then by (1.2.19) also $|\underline{1|3|2}| = 0$ which, by Claim 1.2.13 and 1.2.20 gives the contradiction that $R_1 \subseteq [\overline{1|1|1}, \underline{1|2|3} \cup \underline{1|3|2}]$ is trivial. So we have

$$\overline{H} = \overline{1|1|1} \cup \overline{2|2|2} \cup \overline{3|3|3} \cup \overline{3|1|2},$$

with $\overline{3|1|2} \neq \emptyset$. This partition is shown in Figure 1.3.

Ignoring from now on the matchings M_1 and M_2, we aim at covering H with M_3 and four other matchings. To this end take a green matching M_1^{green} of maximum size in $G_1 - M_3$ and next a blue matching M_2^{blue} of maximum size in $B_2 - M_3 - M_1^{\text{green}}$. Denote

- $\overline{\mathrm{i}|j|k}' := \overline{\mathrm{i}|j|k} \setminus \overline{M_3 \cup M_1^{\text{green}} \cup M_2^{\text{blue}}}$ and
- $\underline{\mathrm{i}|j|k}' := \underline{\mathrm{i}|j|k} \setminus \underline{M_3 \cup M_1^{\text{green}} \cup M_2^{\text{blue}}}$.

We can assume that $M_3 \cup M_1^{\text{green}} \cup M_2^{\text{blue}}$ is not spanning. Thus, as H is complete, the maximality of the matchings M_3, M_1^{green} and M_2^{blue} implies that $\overline{3|1|2}', \underline{3|1|2}' = \emptyset$.

Moreover it follows that

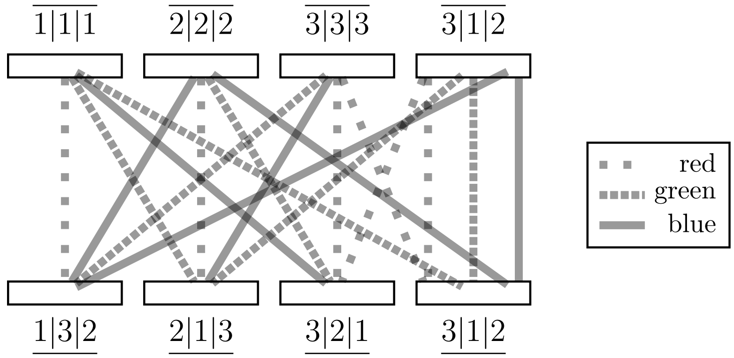

Figure 1.3: The colouring from the proof of Claim 1.2.14

- $|\overline{1|1|1}'| = 0$ or $|\underline{2|1|3}'| = 0$ by maximality of $M_1^{\text{green}} \subseteq G_1$,
- $|\overline{2|2|2}'| = 0$ or $|\underline{1|3|2}'| = 0$ by maximality of $M_2^{\text{blue}} \subseteq B_2$,
- $\overline{3|3|3}' = \emptyset$ as $\overline{R_3} = \overline{M_3}$ by (1.2.17).

If $|\overline{1|1|1}'|, |\overline{2|2|2}'| = 0$, then we have found three disjoint connected matchings that span H, contradicting our assumption. If $|\underline{2|1|3}'|, |\underline{1|3|2}'| = 0$, we take a green matching in G_2 and a blue maximum matching in B_1, among the yet unmatched vertices. After this step, there are no vertices of $\underline{3|2|1}'$ left uncovered and therefore all vertices of $\underline{H}$ are covered. Thus, as H is balanced, we have found five disjoint monochromatic connected matchings which together span H. So, either $|\overline{2|2|2}'|, |\underline{2|1|3}'| = 0$, or $|\overline{1|1|1}'|, |\underline{1|3|2}'| = 0$. In either case we can find two disjoint monochromatic connected matchings that cover all vertices of the two other sets from the previous sentence and all vertices of $\underline{3|2|1}'$. So we have five disjoint monochromatic connected matchings spanning H, a contradiction. $\square$

For ease of notation we set

$$X := |\overline{1|1|1}|,\ Y := |\overline{2|2|2}|,\ Z := |\overline{3|3|3}| \ \text{ and}$$

$$A := |\underline{1|3|2}|,\ B := |\underline{1|2|3}|,\ C := |\underline{2|3|1}|,\ D := |\underline{2|1|3}|,\ E := |\underline{3|2|1}|,\ F := |\underline{3|1|2}|.$$

By Claim 1.2.14 and 1.2.20 we have $|\overline{H}| = X + Y + Z$ and $|\underline{H}| = A + B + C + D + E + F$. Note that the edges between any upper and lower part are monochromatic (see Figure 1.4).

Also note that we reached complete symmetry between the colours and the indices of the components, so we will from now on again treat them as interchangeable.

Observe that for (at least) one index $\mathrm{i} \in \{1, 2, 3\}$ it holds that $|\overline{R_{\mathrm{i}}}| \le |\underline{R_{\mathrm{i}}}|$. We shall call such an index i a *weak* index for the colour red. If furthermore $|\overline{R_{\mathrm{i}}}| < |\underline{R_{\mathrm{i}} \cap B_j}| = |\underline{R_{\mathrm{i}} \cap G_k}|$ and $|\overline{R_{\mathrm{i}}}| < |\underline{R_{\mathrm{i}} \cap B_k}| = |\underline{R_{\mathrm{i}} \cap G_j}|$, where j, k are the other two indices from $\{1, 2, 3\}$, then we call i *very weak* for colour red. Analogously define *(very) weak* indices for colours blue and red.

Claim 1.2.15. *If index* i *is weak for colour* c*, then*

(a) the indices in $\{1, 2, 3\} - \{\mathrm{i}\}$ *are not weak for colour* c*, and*

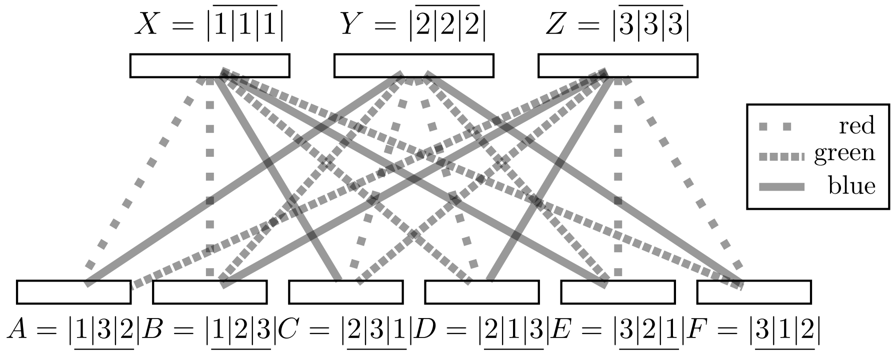

Figure 1.4: The partition of $K_{n,n}$.

(b) index i *is very weak for colour c.*

Proof. Let us show this for i $= 2$ and colour red (the other cases are analogous). By assumption, $Y \leq C + D$. Since $X < A + B$ and $Z < E + F$ cannot both hold, we can assume without loss of generality that $Z \geq E + F$. Now if $X \leq A + B$, then we pick maximal red matchings in $[\overline{1|1|1}, \underline{1|3|2} \cup \underline{1|2|3}]$, $[\overline{2|2|2}, \underline{2|3|1} \cup \underline{2|1|3}]$ and $[\underline{3|2|1} \cup \underline{3|1|2}, \overline{3|3|3}]$, thus covering all vertices of $\overline{1|1|1} \cup \overline{2|2|2} \cup \underline{3|2|1} \cup \underline{3|1|2}$. To finish we cover all of the remaining vertices in $\overline{3|3|3} \cup (\underline{H \setminus R_3})$ with a blue and a green matching, a contradiction. Hence $X > A + B$. Using this fact, $Z > E + F$ follows by symmetry. This proves (a).

In order to show (b), let us first prove that $Y < C$. We pick a maximal red matching in each of R_1 and R_3, thus covering all vertices of $\underline{R_1 \cup R_3}$. Now if $Y \geq C$, then all vertices of $\underline{2|3|1}$ are contained in a maximal red matching that also contains all vertices of $\overline{2|2|2}$. We cover all of the remaining vertices in $\overline{R_1 \cup R_3}$ with a blue and a green matching, a contradiction. The fact that $Y < D$ follows analogously. □

Suppose two of the three indices $1, 2, 3$ are weak for different colours, say 1 is weak for red and 2 is weak for green. Then Claim 1.2.15(b) gives that $X < A$ and $Y < E$. Thus we can match all vertices of $\overline{1|1|1}$ into $\underline{1|3|2}$ and all vertices of $\overline{2|2|2}$ into $\underline{3|2|1}$ with two matchings, one red and one green, and cover all of the remaining vertices with three disjoint matchings, one from each of R_3, G_3, B_3, a contradiction.

Hence, since each colour has a weak index, there is an index i that is weak for all three colours, i $= 2$ say. We match all vertices of $\overline{2|2|2}$ into $\underline{3|1|2}$ with a blue matching M. Let us from now work with the remaining set $\underline{3|1|2}' = \underline{3|1|2} \setminus \overline{V(M)}$ of cardinality $F' = F - Y$. Set $n' = n - Y$. (So instead of five we will have to find four monochromatic connected matchings covering all vertices of $H - M$.) Without loss of generality assume $Z \geq X$. Claim 1.2.15(a) gives that

$$X > A + B, C + E, D + F' \text{ and } Z > A + C, B + D, E + F'. \qquad (1.2.21)$$

Hence $X > n'/3$. So, one of the three sums $A + C, B + D, E + F'$ has to be strictly smaller than X, say $A + C < X$. Consequently, $Z = n' - X < B + D + E + F'$.

If $Z \geq D + E + F'$, then we cover all vertices of $\underline{R_3 - M}$ with a red matching, and cover all vertices of the remains of $\overline{3|3|3}$ with a blue matching that also covers all vertices of $\underline{2|1|3}$. Now all that is left on the top is $\overline{1|1|1}$, which we can match with a red and a blue matching into the remains of $\underline{1|3|2} \cup \underline{1|2|3} \cup \underline{2|3|1}$. Thus we found four connected matchings that cover all vertices of $H - \overline{V(M)}$, and are done.

So we may assume $Z < D + E + F'$ and thus $X > A + B + C$. If $X \leq A + B + C + E$, then we can proceed similarly as in the previous paragraph to find four matchings covering all vertices of H. Hence $X > A + B + C + E$, implying that $Z < D + F'$. But by (1.2.21) we have $D + F' < X$ a contradiction to our assumption that $X \leq Z$. This finishes the proof of Lemma 1.1.2.

1.3 Covering almost all vertices with connected matchings

1.3.1 Preliminaries

The goal of this section is to prove a version of Lemma 1.1.2 for almost complete graphs. This result is given in Lemma 1.3.1.

Let G be a graph with biparts A and B and let H be a subgraph of G. We call H *γ-dense* in G if it has at least $\gamma|A||B|$ edges. If $H = G$, we often simply say G is γ-dense. Let H be a subgraph of G. If H has biparts $X \subseteq A$ and $Y \subseteq B$ such that $|X| \geq \gamma|A|$ and $|Y| \geq \gamma|B|$, then we call H *γ-non-trivial* (in G), or we say G is γ-spanned by H. Usually, we use the term γ-non-trivial when $\gamma \approx 0$ and we use the term γ-spanned when $\gamma \approx 1$.

Lemma 1.3.1. *There is an $\varepsilon_0 > 0$ such that for each $0 < \varepsilon \leq \varepsilon_0$ there are n_0 and $\rho = \rho(\varepsilon)$ such that for all $n \geq n_0$ the following holds.*

Every 3-edge-coloured balanced bipartite $(1-\varepsilon)$-dense graph of size $2n$ is $(1-\rho)$-spanned by at most five disjoint monochromatic connected matchings.

For the proof of Lemma 1.3.1 we need some more notation. Again, let G be a graph with biparts A and B and let H be a subgraph of G. We say H has *γ-complete degree* in G if $\deg_H(y) > \gamma|A|$ for $y \in B \cap V(H)$ and $\deg_H(x) > \gamma|B|$ for $x \in A \cap V(H)$. Clearly, if H has γ-complete degree in G, then in particular, H is γ-dense in G.

The following lemmas are well-known and follow from standard averaging arguments.

Lemma 1.3.2. *For $\varepsilon > 0$ let H be a $(1-\varepsilon)$-dense bipartite graph. Then H has a $(1-\sqrt{\varepsilon})$-spanning subgraph H' with $(1-2\sqrt{\varepsilon})$-complete degree (in H).*

Lemma 1.3.3. *For $1/4 > \varepsilon > 0$ let H be a bipartite graph with biparts A, B, having $(1-\varepsilon)$-complete degree. Then any 2ε-non-trivial subgraph of H is connected.*

We omit the easy proofs of the next two lemmas.

Lemma 1.3.4. *For $\delta, \varepsilon > 0$ let H be a $(1-\varepsilon)$-dense bipartite graph with a δ- subgraph H'. Then H' is $(1-\varepsilon/\delta^2)$-dense in H'.*

Lemma 1.3.5. *For $\delta, \varepsilon > 0$ let H be a bipartite graph of $(1-\varepsilon)$-complete degree and H' be a δ-non-trivial subgraph. Then H' has $(1-\varepsilon/\delta)$-complete degree in itself.*

The proof of the next lemma is given as a warm-up. In the remainder of this section H is a bipartite graph with biparts $\overline{H}$ and $\underline{H}$.

Lemma 1.3.6. *For $1/5 \geq \varepsilon > 0$ let H be a 2-edge-coloured bipartite graph of $(1-\varepsilon)$-complete degree, with bipartition A, B. Then H has a $((1-\varepsilon)/2)$-spanning monochromatic component.*

Proof. Having $(1-\varepsilon)$-complete degree, H has a monochromatic component R with $|\overline{R}| \geq (1-\varepsilon)|\overline{H}|/2$. If R is $((1-\varepsilon)/2)$-spanning we are done. Otherwise the monochromatic subgraph $[\overline{R}, \underline{H-R}]$ is $((1-\varepsilon)/2)$-spanning, and it is connected by Lemma 1.3.3. $\square$

In order to formulate a dense version of Lemma 1.2.2 we need to define dense variants of V-colourings and split colourings. We say a colouring of $E(H)$ in red and blue is an *ε-V-colouring* if there are monochromatic components R and B of distinct colours such that

1. each of R and B is ε-non-trivial in H;
2. $R \cup B$ is $(1-\varepsilon)$-spanning in H;
3. $|V(\overline{R \cap B})| \geq (1-\varepsilon)|V(\overline{H})|$ or $|V(\underline{R \cap B})| \geq (1-\varepsilon)|V(\underline{H})|$.

A colouring of $E(H)$ in red and blue is *ε-split*, if

1. all monochromatic components are ε-non-trivial;
2. each colour has exactly two monochromatic components.

The following is a robust analogue of Lemma 1.2.2.

Lemma 1.3.7. *Let $\varepsilon < 1/6$. If the bipartite 2-edge-coloured graph H has $(1-\varepsilon)$-complete degree, then one of the following holds:*

(a) *There is a $(1-3\varepsilon)$-spanning monochromatic component,*

(b) *H has a 3ε-V-colouring, or*

(c) *the edge-colouring is 2ε-split.*

Proof. Let R be an $((1-\varepsilon)/2)$-spanning component in colour red, say. Such a component exists by Lemma 1.3.6. Set $X := H - R$ and note that all edges in $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ are blue.

We first assume that $|\overline{X}| < 3\varepsilon|V(\overline{H})|$. If also $|\underline{X}| < 3\varepsilon|V(\underline{H})|$, we are done, since then R is $(1-3\varepsilon)$-spanning. Otherwise, $|\underline{X}| \geq 3\varepsilon|V(\underline{H})|$, and thus the blue subgraph $[\underline{X}, \overline{R}]$ is connected by Lemma 1.3.3 and the colouring is a 3ε-V-colouring.

So by symmetry we can assume that both $|\overline{X}| \geq 3\varepsilon|V(\overline{H})|$ and $|\underline{X}| \geq 3\varepsilon|V(\underline{H})|$. If there is a blue edge in R or in X, then H is spanned by one blue component by Lemma 1.3.3. Hence, all edges inside R and X are red and the colouring is 2ε-split, again using Lemma 1.3.3. $\square$

Corollary 1.3.8. *Let $\varepsilon < 1/6$. If a bipartite 2-edge-coloured graph H has $(1-\varepsilon)$-complete degree, then*

(a) *there are one or two 2ε-non-trivial monochromatic components that together $(1-3\varepsilon)$-span H, and*

(b) *if the colouring is not 2ε-split, then there is a colour with exactly one 3ε-non-trivial component.*

Now we prove an analogue of Lemma 1.2.4.

Lemma 1.3.9. *Let $\varepsilon < 1/6$, and let H be a balanced bipartite graph of $(1-\varepsilon)$-complete degree whose edges are coloured red and blue. Then either*

(a) *H is $(1-5\varepsilon)$-spanned by two vertex disjoint monochromatic connected matchings, one of each colour, or*

(b) *the colouring is 2ε-split and*

- *H is $(1-2\varepsilon)$ is spanned by one red and two blue vertex disjoint monochromatic connected matchings and*
- *H is $(1-2\varepsilon)$ is spanned by one blue and two red vertex disjoint monochromatic connected matchings.*

Proof. First assume that the colouring is 2ε-split. We take one red maximum matching in each of the two red components. This leaves at least one of the blue components with less than $\varepsilon|\overline{H}|$ vertices on each side. We extract a third maximum matching from the leftover of the other blue component, thus leaving one of its sides with less than $\varepsilon|\overline{H}|$ vertices. All three matchings are clearly connected (or possibly empty, in case of the third matching) Thus the three matchings together $(1-2\varepsilon)$-span H. Note that we could have switched the roles of red and blue in order to obtain two blue and one red matching that $(1-2\varepsilon)$-span H.

So by Lemma 1.3.7, we may assume that either there is a colour, say red, with an $(1-3\varepsilon)$-spanning component R, or H has a 3ε-V-colouring, with components R in red and B in blue, say. In either case, we take a maximum red matching M in R. Then there is an induced balanced bipartite subgraph of H, whose edges are all blue, which contains all but at most $3\varepsilon|V(H)|$ of the uncovered vertices of each bipart of H. If this subgraph is not 2ε-non-trivial, we are done. Otherwise, we finish by extracting from it a maximum blue matching $M' \subseteq B$, note that M' is connected by Lemma 1.3.3. As H has $(1-\varepsilon)$-complete degree and there are no leftover edges in said subgraph, we obtain that $M \cup M'$ $(1-4\varepsilon)$-span H, and we are done. □

We now prove a robust analogue of Lemma 1.2.5.

Lemma 1.3.10. *Let $1/6^6 > \varepsilon > 0$. Let the edges of the bipartite graph H of $(1-\varepsilon)$-complete degree be coloured in red, green and blue, such that each colour has at least four $\varepsilon^{1/6}$-non-trivial components; then there are three monochromatic components that together $(1-\varepsilon^{1/6})$-span H.*

Proof. Set $\gamma := \varepsilon^{1/6}$ and let R be a red γ-non-trivial component. Throughout the proof we shall make use of Lemma 1.3.3 without mentioning it explicitly. Since there are three more red γ-non-trivial components, the three graphs $X := H - R$, $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ are each γ-non-trivial and by Lemma 1.3.5, each of them has $(1-\gamma^2)$-complete degree (in themselves). Moreover, the edges of the latter two graphs are green and blue. By Corollary 1.3.8(a) there are one or two $2\gamma^2$-non-trivial monochromatic components that together $(1-3\gamma^2)$-span $[\underline{R}, \overline{X}]$. So, if $[\overline{R}, \underline{X}]$ has a $(1-3\gamma^2)$-spanning monochromatic component, then we can $(1-3\gamma^2)$-span H with at most three components, which is as desired. Therefore and by symmetry we may assume from now on that none of $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ has a $(1-3\gamma^2)$-spanning monochromatic component. Suppose $[\overline{R}, \underline{X}]$ has a $2\gamma^2$-split-colouring. By Lemma 1.3.7, either $[\underline{R}, \overline{X}]$ is $2\gamma^2$-split or a fraction of $(1-3\gamma^2)$ of one of $\underline{R}$ and $\overline{X}$ is contained in the intersection of a blue and a green monochromatic component. In the latter case the union of three monochromatic components of the same colour contains a fraction of $(1-3\gamma^2)$ vertices of one of the biparts of H. But this is impossible as each colour has at least four γ-non-trivial components, and $\gamma > 3\gamma^2$. On the other hand, if both $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ have a $2\gamma^2$-split colouring, then each bipart of H is contained in the union of four green components as well as in the union of four blue components, and thus all edges in X are red. But then there are only two γ-non-trivial red components, R and X, a contradiction.

So by Lemma 1.3.7, and by symmetry, we know that $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ both have green/blue $3\gamma^2$-V-edge-colourings. Thus each of $[\overline{R}, \underline{X}]$ and $[\underline{R}, \overline{X}]$ has a $3\gamma^2$-non-trivial blue component and a $3\gamma^2$-non-trivial green component, say these are B_1, G_1 and B_2, G_2 respectively. Furthermore, a fraction of $(1-3\gamma^2)$ of $\underline{X}$ or $\overline{R}$ is contained in the intersection $B_1 \cap G_1$, and a fraction of $\overline{X}$ or $\underline{R}$ is contained in the intersection $B_2 \cap G_2$.

We first look at the case where a fraction of $(1-3\gamma^2)$ of $\underline{X}$ is contained in $\underline{B_1 \cap G_1}$. If a fraction of $(1-3\gamma^2)$ of $\underline{R}$ is contained in $\underline{B_2 \cap G_2}$, then, as $\gamma > 6\gamma^2$, both green and blue have at most two γ-non-trivial components, which is a contradiction. On the other hand, if a fraction of $(1-3\gamma^2)$ of $\overline{X}$ is contained in $\overline{B_2 \cap G_2}$, then H is $(1-3\gamma^2)$-spanned by the union of R and the blue components in H that contain B_1 and B_2, and we are done.

Consequently we can assume by symmetry and by Lemma 1.3.7 that a fraction of $(1-3\gamma^2)$ of $\overline{R}$ is contained in the intersection $\overline{B_1 \cap G_1}$ and a fraction of $(1-3\gamma^2)$ of $\underline{R}$ is contained in the intersection $\underline{B_2 \cap G_2}$. Observe that $[\underline{G_1}, \overline{G_2}]$ is coloured red and blue and $[\underline{B_1}, \overline{B_2}]$ is coloured red and green, since otherwise, we obtain the desired cover. As these two graphs are each $3\gamma^3$-non-trivial subgraphs of H, and as $\varepsilon/(3\gamma^3) = \gamma^3/3$, Lemma 1.3.5 implies they have $(1-\gamma^3/3)$-complete degree (in themselves). Suppose there is a red component of $[\underline{G_1}, \overline{G_2}]$ that is $(1-\gamma)$-spanning in $[\underline{G_1}, \overline{G_2}]$. Such a component, together with B_1 and B_2, $(1-2\gamma)$-spans H as $\gamma < 1/3$. So, we can assume $[\underline{G_1}, \overline{G_2}]$ has no $(1-\gamma)$-spanning red component. Moreover, since there are at least four γ-non-trivial blue components, $[\underline{G_1}, \overline{G_2}]$ contains two blue components, which are $\gamma/2$-non-trivial each as $\gamma/2 > 3\gamma^2$.

Since these blue components are γ-non-trivial in H, $[\underline{G_1}, \overline{G_2}]$ does not have a γ^3-V-colouring (in itself). Thus, by Lemma 1.3.7 with input $\varepsilon_{Lem 1.3.7} = \gamma^3/3$, $[\underline{G_1}, \overline{G_2}]$ is $2\gamma^3/3$-split coloured in red and blue. Similarly we see that $[\underline{B_1}, \overline{B_2}]$ is $2\gamma^3/3$-split coloured in red and green.

Consider the edges in $[\underline{G_1}, \overline{B_2}]$ and $[\underline{B_1}, \overline{G_2}]$. If any of these edges is green or blue, then our graph is $(1-2\gamma^3/3)$-spanned by three green or by three blue components. On the other hand, if all edges in $[\underline{G_1}, \overline{B_2}]$ and $[\underline{B_1}, \overline{G_2}]$ are red, then $[\underline{G_1 \cup B_1}, \overline{B_2 \cup G_2}]$ is connected in red by Lemma 1.3.3, and thus, H has only three γ-non-trivial red components, a contradiction. $\square$

1.3.2 Proof of Lemma 1.3.1

We are now ready to prove Lemma 1.3.1. We will not give specific bounds for $\varepsilon_0 > 0$ and n_0 but assume that they are sufficiently small respectively large as we go through the proof. For $0 < \varepsilon \leq \varepsilon_0$ let $n \geq n_0$ and H be a balanced bipartite $(1-\varepsilon)$-dense graph which has $(1-\varepsilon)$-complete degree and order $2n$, where $n \geq n_0$.

We choose numbers δ, γ, ρ such that

$$\varepsilon \ll \delta \ll \gamma \ll \rho < 1. \tag{1.3.1}$$

Although these numbers could in principle be specified, we refrain from doing so in order to not spoil the neatness of the argumentation. Our aim is to show that H can be $(1-\rho)$-spanned with five vertex disjoint monochromatic connected matchings. We suppose that this is wrong in order to obtain a contradiction. Lemma 1.3.1 then follows by Lemma 1.3.2.

The next claim is the robust analogue of Claim 1.2.6.

Claim 1.3.11. *Each colour has at least three γ-non-trivial components.*

Proof. Suppose the claim is wrong for colour red, say. Let $\mathcal{Y}$ be the set of all red components with top bipart smaller than γn and let $\mathcal{Z}$ be the set of all red components with bottom bipart smaller than γn. The total number of edges in red components that are not γ-non-trivial is less than

$$\sum_{Y \in \mathcal{Y}} \gamma n |\underline{Y}| + \sum_{Z \in \mathcal{Z}} \gamma n |\overline{Z}| < 2\gamma n^2.$$

Thus, deleting the (red) edges of all $Y \in \mathcal{Y} \cup \mathcal{Z}$, we obtain a spanning subgraph H' of H that is $(1-3\gamma)$-dense in itself and in which each red component is either γ-non-trivial or trivial.

By assumption, there are two (possibly trivial) red components R_1 and R_2 in H', such that all other red components are trivial. Let M be a maximum red matching in $R_1 \cup R_2$. Then every edge in the balanced bipartite subgraph $X := H' - M$ is green or blue.

If the (at most) two connected matchings in M together $(1-\rho)$-span H, we are done. Otherwise X is ρ-non-trivial in H', and thus $(1-(\rho/20)^2)$-dense, by Lemma 1.3.4 and since we assume $3\gamma \leq (\rho^2/20)^2$.

We apply Lemma 1.3.2 to obtain a subgraph $H'' \subseteq X$ that $(1-\rho/20)$-spans X and has $(1-\rho/10)$-complete degree. By Lemma 1.3.9, H'' can be $(1-\rho/2)$-spanned with three vertex-disjoint monochromatic connected matchings. So in total we found at most five vertex-disjoint monochromatic connected matchings that together $(1-\rho)$-span H. □

A subgraph $X \subseteq H$ is called *ε-empty*, if both $|\underline{X}| < \varepsilon |\underline{H}|$ and $|\overline{X}| < \varepsilon |\overline{H}|$ hold. The next claim is a robust version of Claim 1.2.7.

Claim 1.3.12. *There are no two monochromatic components that together $(1-\gamma/2)$-span H.*

Proof. Suppose the claim is wrong and there are monochromatic components R and B that together $(1-\gamma/2)$-span H. By Claim 1.3.11 we can assume that they have distinct colours, say R is red and B is blue. Take a red matching M^{red} of maximum size in R and a blue matching M^{blue} of maximum size in $B - V(M^{\text{red}})$. Set $R' := R - V(M^{\text{red}} \cup M^{\text{blue}})$ and $B' := B - V(M^{\text{red}} \cup M^{\text{blue}})$. By maximality, any edge between $\overline{B'}$ and $\underline{R'}$ is green. The same holds for the edges between $\underline{B'}$ and $\overline{R'}$.

If $[\underline{B'}, \overline{R'}]$ is γ-empty, we finish by picking a maximum matching in $[\underline{R'}, \overline{B'}]$. We proceed analogously if $[\underline{R'}, \overline{B'}]$ is γ-empty. So at least one R' or B' is γ-non-trivial. Thus, since H has $(1-\varepsilon)$-complete degree, all edges of $[\underline{B'}, \overline{R'}]$ lie in the same green component. The same holds for $[\underline{R'}, \overline{B'}]$.

Assuming that both are non-empty we now pick now pick a maximum matching in each of the green components of $H - V(M^{\text{red}} \cup M^{\text{blue}})$ that contain $[\overline{B'}, \underline{R'}]$, $[\underline{B'}, \overline{R'}]$. (If this is the same component, we only pick one matching. If R' or B' is γ-empty, we let the matchings be empty.) Call these green matchings M_1^{green} resp. M_2^{green}. Let $B'' := B' - V(M_1^{\text{green}} \cup M_2^{\text{green}})$ and $R'' := R' - V(M_1^{\text{green}} \cup M_2^{\text{green}})$.

Observe that by the maximality of M_1^{green} and M_2^{green}, if one of $\underline{R''}$, $\overline{B''}$ has size at least εn, then the other one is empty. The same holds for the sets $\underline{B''}$, $\overline{R''}$. Thus one of the two graphs R'', B'' is ε-empty, say this is B''. If R'' is 2γ-empty, we are done, so we can assume that R'' is γ-non-trivial.

The edges in R'' are green and blue. If R'' contains no green edges, we can pick another blue matching of maximum size and are done. Then again, if R'' contains a green edge, it follows by maximality of M_1^{green} and M_2^{green} that both of them have a size of less than $2\varepsilon n$. In this case we ignore M_1^{green} and M_2^{green} and finish as follows: By Lemma 1.3.5, R'' has

$(1-\varepsilon/\gamma)$-complete degree in itself. So, by Lemma 1.3.9, R'' can be $(1-5\varepsilon/\gamma)$-spanned by at most 3 vertex disjoint monochromatic connected matchings. This proves the claim. $\square$

Claim 1.3.13. *Let Y and Z be monochromatic components of distinct colours such that $Y \cap Z$ is 2ε-non-trivial. Then $Y - Z$ is not $\gamma/4$-empty.*

Proof. Let Y be a red component, Z be a blue component, and let $X := H - (Y \cup Z)$. Suppose that $Y - Z$ is $\gamma/4$-empty. We first note that all edges in $[\overline{Y \cap Z}, \underline{X}]$ and $[\underline{Y \cap Z}, \overline{X}]$ are green. Moreover, by Claim 1.3.11, there is another γ-non-trivial blue component in H and hence, X is 2ε-non-trivial in H, since $\gamma - \gamma/4 > 2\varepsilon$ by (1.3.1).

Thus the subgraphs $[\overline{Y \cap Z}, \underline{X}]$ and $[\underline{Y \cap Z}, \overline{X}]$ are connected in green by Lemma 1.3.3. But they cannot belong to the same green component, since otherwise H is $(1-\gamma/4)$-spanned by the union of said green component and Z, which is not possible by Claim 1.3.12. Consequently, X has no green edges. By Claim 1.3.11 there is a green γ-non-trivial component $G \subseteq Y \cup Z$. As $H = Z \cup (Y - Z) \cup X$ and $Y - Z$ is $(\gamma/4)$-empty, we obtain that $G \cap Z$ is $(3\gamma/4)$-non-trivial in H and $G - Z \subseteq Y - Z$ is $(\gamma/4)$-empty. Thus G has the same properties as Y with respect to Z and we can repeat the same arguments as above to obtain that all edges in X are blue. Hence X is connected in blue by Lemma 1.3.3. But this is a contradiction to Claim 1.3.12, as X and Z together $(1-\gamma/4)$-span H. $\square$

Claim 1.3.14. *There is a colour that has exactly three δ-non-trivial components.*

Proof. We show that there is a colour with at most three δ-non-trivial components. This together with Claim 1.3.11 yields the desired result. So suppose otherwise. Then each colour has at least four δ-non-trivial components. By Lemma 1.3.10, there are components X, Y and Z that together $(1-\varepsilon^{1/6})$-span H.

By assumption, and as $\delta > \varepsilon^{1/6}$ by (1.3.1), not all of X, Y and Z have the same colour. If two of these components, say X and Y, have the same colour, say red, then $H - (X \cup Y)$ contains a red component that is δ-non-trivial in H, by the assumption that our claim is false. As $\delta \geq \varepsilon^{1/6} + 2\varepsilon$ by (1.3.1), we have that the intersection of this red component with Z is 2ε-non-trivial in H. Hence we get a contradiction to Claim 1.3.13 as $\gamma/4 > \varepsilon^{1/6}$ by (1.3.1).

So assume X is red, Y is blue and Z is green. We claim that (after possibly swapping top and bottom parts)

$$\underline{(Y \cap Z) - X} \text{ has less than } \varepsilon n \text{ vertices.} \tag{1.3.2}$$

Indeed, otherwise $(Y \cap Z) - X$ is ε-non-trivial. Then, as $[\underline{X}, \overline{(Y \cap Z) - X}]$ is ε-non-trivial and its edges are green and blue, we get $\underline{X} \subseteq Y \cup Z$ since every vertex in $\underline{X}$ sees a vertex in $\overline{Y \cap Z}$. In the same way we obtain $\overline{X} \subseteq Y \cup Z$. Thus $Z \cup Y$ is $(1-\varepsilon^{1/6})$-non-trivial, which is not possible by Claim 1.3.12. This proves (1.3.2).

By assumption, $H - X$ contains three δ-non-trivial red components R_1, R_2 and R_3, say. For $i \neq j$, $[\overline{R_i \cap (Y - Z)}, \underline{R_j \cap (Z - Y)}]$ has no red, blue or green edges and thus cannot be ε-non-trivial. So for at most one $i \in \{1, 2, 3\}$ the subgraph $R_i \cap [\overline{Y - Z}, \underline{Z - Y}]$ is ε-non-trivial. The same holds for $[\underline{R_i \cap (Y - Z)}, \overline{R_j \cap (Z - Y)}]$. Consequently, and by the pigeonhole principle we can assume that

$$\text{none of } R_1 \cap [\overline{Y - Z}, \underline{Z - Y}] \text{ and } R_1 \cap [\underline{Y - Z}, \overline{Z - Y}] \text{ is } \varepsilon\text{-non-trivial.} \tag{1.3.3}$$

By (1.3.3) and as R_1 is δ-non-trivial, at least one of $R_1 \cap Z$, $R_1 \cap Y$ is 3ε-non-trivial. We will assume the former. Thus, by (1.3.2) $\underline{R_1 \cap (Y - Z)}$ has a size of at least $2\varepsilon n$. Hence, by (1.3.3) we get:

$$|\overline{R_1 \cap Z - Y}| < \varepsilon n. \tag{1.3.4}$$

Moreover, Claim 1.3.13 (applied to R_1 and Y implies that R_1 has at least $\gamma n/4 - \varepsilon^{1/6} n > 2\varepsilon n$ vertices in $\overline{Z - Y}$ or $\underline{Z - Y}$. By (1.3.4) we have the latter case and hence

$$\underline{R_1 \cap (Z - Y)} \text{ and } \underline{R_1 \cap (Y - Z)} \text{ each have a size of at least } 2\varepsilon n. \tag{1.3.5}$$

The fact that $[\overline{Y - (X \cup Z)}, \underline{R_1 \cap (Z - Y)}]$ and $[\overline{Z - (X \cup Y)}, \underline{R_1 \cap (Y - Z)}]$ only have red edges, together with (1.3.3) and (1.3.5), yields that

$$\overline{Y - (X \cup Z)} \text{ and } \overline{Z - (X \cup Y)} \text{ each have less than } \varepsilon n \text{ vertices.} \tag{1.3.6}$$

Now by (1.3.6) (and by the existence of R_1, R_2, R_3), we know that $\overline{(Y \cap Z) - X}$ has at least εn vertices. So each vertex of $\underline{X}$ has a neighbour in $\overline{(Y \cap Z) - X}$ and hence $\underline{X} \subseteq \underline{Y \cup Z}$. Since, by Claim 1.3.12, H is not $(1 - \varepsilon^{1/6} - 2\varepsilon)$-spanned by $Y \cup Z$, we have that $\overline{X - (Y \cup Z)}$ has a size of at least $2\varepsilon n$. This and (1.3.5) imply that $[\overline{X - (Y \cup Z)}, \underline{Y - (X \cup Z)}]$ and $[\overline{X - (Y \cup Z)}, \underline{Z - (X \cup Y)}]$ are 2ε-non-trivial each. As the edges of these subgraphs are green and blue respectively and as Lemma 1.3.3 applies, there are green and blue components G and B such that $\underline{H - X - [(G \cap Y) \cup (B \cap Z)]}$ has a size of less than $\varepsilon n + \varepsilon^{1/6} n$ by (1.3.2).

Now let G' be another δ-non-trivial green component. Then $\underline{G' - X}$ has at most $\varepsilon^{1/6} n$ vertices, while $\underline{G' \cap X}$ has at least $2\varepsilon n$ vertices. By (1.3.6) it follows that $\overline{G' - X}$ has at most $\varepsilon n + \varepsilon^{1/3} n$ vertices, while $\overline{G' \cap X}$ has at least $2\varepsilon n$ vertices. This is not possible by Claim 1.3.13 and completes the proof. □

Using Claim 1.3.14 we assume from now on that without loss of generality the colour red has exactly three δ-non-trivial components R_1, R_2 and R_3. For i $= 1, 2, 3$ let M_{i} be a red matching of maximum size in R_{i}.

None of the red edges in $Y := H - M_1 - M_2 - M_3$ is in a red δ-non-trivial component. As seen in the proof of Claim 1.3.11, the number of red edges which are not in δ-non-trivial red components sums up to at most $2\delta n^2$. Therefore the number of red edges in Y is at most $2\delta n^2$. Let Y' be the subgraph of Y where these edges have been deleted. Note that the edges of Y' are coloured in blue and green. Moreover, H is still $(1 - 3\delta)$-dense after the removal of the red edges of Y.

If Y' is not $(3\delta)^{1/3}$-non-trivial, then we are done as Y' is balanced. Otherwise Y' is $(1 - (3\delta)^{1/3})$-dense by Lemma 1.3.4 and thus contains a $(1 - (3\delta)^{1/6})$-spanning subgraph Y'' of Y with $(1 - 2(3\delta)^{1/6})$-complete degree, by Lemma 1.3.2. By removing at most $(3\delta)^{1/6} n$ vertices from Y'' we can assure that Y'' is balanced. If Y'' can be $(1 - 10(3\delta)^{1/6})$-spanned by two disjoint monochromatic connected matchings, we are done, since in that case, we found five matchings which together $(1 - 11(3\delta)^{1/6})$-span H. Otherwise, as the edges of Y'' are green and blue the colouring of Y'' is $4(3\delta)^{1/6}$-split in Y'', by Lemma 1.3.9. We denote its blue and green components by B'_1, B'_2, respectively G'_1, G'_2, with $\overline{B'_1} = \overline{G'_1}$, $\overline{B'_2} = \overline{G'_2}$, $\underline{B'_1} = \underline{G'_2}$, and $\underline{B'_2} = \underline{G'_1}$.

Since Y'' is $(1 - (3\delta)^{1/6})$-spanning in Y' it is also $(1 - (3\delta)^{1/6})$-spanning in Y. Therefore the subgraph

$$B'_1 \cup B'_2 \cup M_1 \cup M_2 \cup M_3 \text{ is } (1 - (3\delta)^{1/6})\text{-non-trivial in } H. \tag{1.3.7}$$

By Lemma 1.3.9, Y'' can be $(1-4(3\delta)^{1/6})$-spanned by two blue matchings $M_4 \subseteq B'_1$, $M_5 \subseteq B'_2$ and an additional green matching. If any of the matchings M_i has less than γn edges, we can ignore it and still have a sufficiently large cover of H. Thus we get that

$$B'_1,\ B'_2,\ G'_1,\ G'_2,\ M_1,\ M_2, \text{ and } M_3 \text{ are } \gamma\text{-non-trivial in } H. \tag{1.3.8}$$

Moreover, let B_1 and B_2 be the blue components in H that contain B'_1 and B'_2, respectively. We define G_1 and G_2 analogously. If $B_1 = B_2$, we are done as $M_4 \cup M_5$ is a connected matching. This and symmetry implies

$$B_1 \neq B_2 \text{ and } G_1 \neq G_2. \tag{1.3.9}$$

Claim 1.3.15. *For each* $i = 1, 2, 3$ *we have that*

(a)
- *if* $|\overline{M_i} \setminus \overline{G_1 \cup G_2}| > 6\varepsilon n$, *then* $\underline{B'_1} \subseteq \underline{R_i}$ *or* $\underline{B'_2} \subseteq \underline{R_i}$;
- *if* $|\overline{M_i} \setminus \overline{B_1 \cup B_2}| > 6\varepsilon n$, *then* $\underline{G'_1} \subseteq \underline{R_i}$ *or* $\underline{G'_2} \subseteq \underline{R_i}$;

(b)
- *if* $|\underline{M_i} \setminus \underline{G_1 \cup G_2}| > 6\varepsilon n$, *then* $\overline{B'_1} \subseteq \overline{R_i}$ *or* $\overline{B'_2} \subseteq \overline{R_i}$;
- *if* $|\underline{M_i} \setminus \underline{B_1 \cup B_2}| > 6\varepsilon n$, *then* $\overline{G'_1} \subseteq \overline{R_i}$ *or* $\overline{G'_2} \subseteq \overline{R_i}$;

(c)
- *if* $|\overline{M_i} \setminus \overline{G_1 \cup G_2 \cup B_1 \cup B_2}| > 2\varepsilon n$, *then* $\underline{B'_1 \cup B'_2} = \underline{G'_1 \cup G'_2} \subseteq \underline{R_i}$;
- *if* $|\underline{M_i} \setminus \underline{G_1 \cup G_2 \cup B_1 \cup B_2}| > 2\varepsilon n$, *then* $\overline{B'_1 \cup B'_2} = \overline{G'_1 \cup G'_2} \subseteq \overline{R_i}$.

Proof. For the first part of (a), assume $|\overline{M_1}\setminus\overline{G_1 \cup G_2}| > 6\varepsilon n$. Note that there is no green edge between $\overline{M_1}\setminus\overline{G_1 \cup G_2}$ and $\underline{G'_1}$. First assume that $\overline{M_1 \cap B_1}\setminus\overline{G_1 \cup G_2}$ has a size of at least $2\varepsilon n$. Then, by 1.3.9, any edge between $\overline{M_1 \cap B_1}\setminus\overline{G_1 \cup G_2}$ and $\underline{B'_2} = \underline{G'_1}$ is red. So, by Lemma 1.3.3 and (1.3.8) the result follows. So we can assume that this is not true. Similarly, the result holds if $|\overline{M_1 \cap B_2} \setminus \overline{G_1 \cup G_2}| \geq 2\varepsilon n$. Therefore, we can assume that $\overline{M_1} \setminus \overline{B_1 \cup B_2 \cup G_1 \cup G_2}$ has a size of at least $2\varepsilon n$. In this case, since all edges between $\overline{M_1} \setminus \overline{B_1 \cup B_2 \cup G_1 \cup G_2}$ and $\underline{B'_1}$ are red, the result follows again by Lemma 1.3.3 and (1.3.8). Item (b) and the second part of (a) follow similarly.

For the first part of (c), note that any edge between $\overline{M_i}\setminus\overline{G_1 \cup G_2 \cup B_1 \cup B_2}$ and $\underline{B'_1 \cup B'_2} = \underline{G'_1 \cup G'_2}$ has to be red and use Lemma 1.3.3 with (1.3.8). The second part of (c) is analogous. □

By Claim 1.3.11 there are green and blue γ-non-trivial components $G_3 \neq G_1, G_2$ and $B_3 \neq B_1, B_2$ in H.

Claim 1.3.16. *It holds that* $|V(G_3 \cap B_3 \cap (M_1 \cup M_2 \cup M_3))| > 36\varepsilon n$.

Proof. Assume otherwise. That is, assume

$$|V(G_3 \cap B_3 \cap (M_1 \cup M_2 \cup M_3))| \leq 36\varepsilon n.$$

The components B_3 and G_3 do not meet with $B'_1 \cup B'_2 = G'_1 \cup G'_2$ and by (1.3.7), there are not more than $2(3\delta)^{1/6}n$ vertices outside of $B'_1 \cup B'_2 \cup M_1 \cup M_2 \cup M_3$. As $\gamma > 2(3\delta)^{1/6} + \delta$ by (1.3.1), we conclude that $B_3 \cap (M_1 \cup M_2 \cup M_3)$ and $G_3 \cap (M_1 \cup M_2 \cup M_3)$ are each δ-non-trivial. Hence there are indices i, i', j, j' such that there is a blue 37ε-non-trivial subgraph $B'_3 \subseteq B_3$ and a green 37ε-non-trivial subgraph $G'_3 \subseteq G_3$ such that $\overline{B'_3} \subseteq \overline{M_i}$ and $\underline{B'_3} \subseteq \underline{M_{i'}}$, and $\overline{G'_3} \subseteq \overline{M_j}$

and $\underline{G_3'} \subseteq \underline{M_{j'}}$. Actually, we can choose these indices such that $\mathrm{i} \neq \mathrm{i}'$ and $j \neq j'$. Since if $\mathrm{i} = \mathrm{i}'$, say, Claim 1.3.13 yields that $(B_3 \cap H) \setminus M_\mathrm{i}$ is not $\gamma/4$-emptyand therefore, by (1.3.1) and (1.3.7), there is some index $k \neq \mathrm{i}$ such that $B_3 \cap M_k$ is not 37ε-empty, which allows us to swap i' for k.

For an index $k \neq \mathrm{i}$, the edges between $\overline{B_3' \cap M_\mathrm{i}}$ and $\underline{G_3' \cap M_k}$ are blue and green. As by our initial assumption $|V(G_3 \cap B_3 \cap (M_1 \cup M_2 \cup M_3))| \leq 36\varepsilon n$, this implies that $|\underline{G_3 \cap M_k}| \leq 36\varepsilon n$. In the same way we obtain that $|\overline{G_3 \cap M_k}| \leq 36\varepsilon n$ for $k \neq \mathrm{i}'$ or $|\overline{B_3' \cap M_\mathrm{i}}| \leq 36\varepsilon n$, but the latter cannot happen by the choice of B_3'. Hence we have $\mathrm{i} = j'$ and $\mathrm{i}' = j$; in other words,

$$|\underline{M_\mathrm{i} \cap G_3}| \geq 37\varepsilon n,\ |\overline{M_j \cap G_3}| \geq 37\varepsilon n,\ |\overline{M_\mathrm{i} \cap B_3}| \geq 37\varepsilon n \text{ and } |\underline{M_j \cap B_3}| \geq 37\varepsilon n.$$

So by Claim 1.3.15 (a) and (b), either we have $B_1' \subseteq R_\mathrm{i}$ and $B_2' \subseteq R_j$, or we have $G_1' \subseteq R_\mathrm{i}$ and $G_2' \subseteq R_j$. Indeed, the fact that $|\underline{M_\mathrm{i} \cap G_3}| \geq 37\varepsilon n$ together with Claim 1.3.15 (b) implies that $\overline{B_1'} = \overline{G_1'} \subseteq \overline{R_\mathrm{i}}$ or $\overline{B_2'} = \overline{G_2'} \subseteq \overline{R_\mathrm{i}}$. Without loss of generality, we assume the latter. Next, since $|\overline{M_\mathrm{i} \cap B_3}| \geq 37\varepsilon n$, and by Claim 1.3.15 (a), we get that $\underline{G_1'} = \underline{B_2'} \subseteq \underline{R_\mathrm{i}}$ or $\underline{G_2'} = \underline{B_1'} \subseteq \underline{R_\mathrm{i}}$. Without loss of generality, we assume the former. We repeat the same with index j, but as we already have $B_2' \subseteq R_\mathrm{i}$, the output of Claim 1.3.15 has to be $\underline{B_1'} = \underline{G_2'} \subseteq \underline{R_j}$ for $|\overline{M_j \cap G_3}| \geq 37\varepsilon n$ and $\overline{B_1'} = \overline{G_1'} \subseteq \overline{R_j}$ for $|\underline{M_j \cap B_3}| \geq 37\varepsilon n$. For the remainder of the proof, let us assume that $B_1' \subseteq R_\mathrm{i}$ and $B_2' \subseteq R_j$. Then $G_1' \cap R_k = \emptyset = G_2' \cap R_k$, where k is the third index, which together with Claim 1.3.15 (a) and (b) gives that $R_k \cap (G_3 \cup B_3)$ is 6ε-empty. The edges between $\underline{B_2'} = \underline{G_1'} \subseteq \underline{G_1 \cap R_j}$ and $\overline{B_3' \cap R_\mathrm{i}}$ have to be green, which implies that $\overline{B_3'} \subseteq \overline{G_1}$. As any edge between $\overline{B_3'}$ and $\underline{R_k - B_3}$ has to be green we deduce that $|\underline{R_k \cap G_1}| \geq 2\varepsilon n$ since R_k is γ-non-trivial and $|\underline{R_k \cap B_3}| \leq 6\varepsilon n$. This also implies that $|\underline{R_k - G_1}| \leq 6\varepsilon n$.

By repeating the same argument with $\overline{B_1'} = \overline{G_1'} \subseteq \overline{G_1}$ and $\underline{B_3'}$, it follows that $|\overline{R_k \cap G_1}| \geq 2\varepsilon n$ and $|\overline{R_k - G_1}| \leq 6\varepsilon n$. So $R_k \cap G_1$ is 2ε-non-trivial and $R_k - G_1$ is 6ε-empty, a contradiction to Claim 1.3.13. □

Claim 1.3.16 allows us assume that without loss of generality

$$|\overline{M_3 \cap G_3 \cap B_3}| > 6\varepsilon n. \tag{1.3.10}$$

This implies $|\overline{M_3} \setminus \overline{G_1 \cup G_2 \cup B_1 \cup B_2}| > 2\varepsilon$ and thus by Claim 1.3.15(c) with $\mathrm{i} = 3$ we obtain

$$\underline{B_1' \cup B_2'} = \underline{G_1' \cup G_2'} \subseteq \underline{R_3}. \tag{1.3.11}$$

This implies that $(\underline{R_1 \cup R_2}) \cap (\underline{G_1' \cup G_2'}) = \emptyset$. Since the edges between $\overline{M_3 \cap G_3 \cap B_3}$ and $\underline{R_1 \cup R_2}$ are coloured green and blue, we have by 1.3.10 and Lemma 1.3.3 that

$$\underline{M_1 \cup M_2} \subseteq \underline{R_1 \cup R_2} \subseteq \underline{G_3 \cup B_3}. \tag{1.3.12}$$

So, by (1.3.8) and Claim 1.3.15(b) with $\mathrm{i} = 1$, we can assume that without loss of generality

$$\overline{B_1'} = \overline{G_1'} \subseteq \overline{R_1}, \tag{1.3.13}$$

and hence, by (1.3.8) and Claim 1.3.15(b) with $\mathrm{i} = 2$ it follows that

$$\overline{B_2'} = \overline{G_2'} \subseteq \overline{R_2}. \tag{1.3.14}$$

The last two assertions imply that $\overline{R_3} \cap \overline{G_1' \cup G_2'} = \emptyset$. Suppose that there is an $x \in \overline{R_1 \cup R_2} \setminus \overline{G_1 \cup G_2 \cup B_1 \cup B_2}$. By 1.3.11, the edges between x and $\underline{G_1' \cup G_2'} = \underline{B_1' \cup B_2'}$ are not red, and neither green or blue by choice of x. As G_1' and G_2' are both γ-non-trivial in H by (1.3.8) and H has $(1-\varepsilon)$-complete degree, we obtain a contradiction. Hence

$$\overline{M_1 \cup M_2} \setminus \overline{G_1 \cup G_2 \cup B_1 \cup B_2} = \emptyset. \tag{1.3.15}$$

In the same fashion, suppose there is an $x \in (\underline{M_3} \setminus \underline{G_1 \cup G_2}) \cup (\underline{M_3} \setminus \underline{B_1 \cup B_2})$. By (1.3.13) and (1.3.14), and by the choice of x, the edges between x and $\overline{B_1'} = \overline{G_1'}$ respectively $\overline{B_2'} = \overline{G_2'}$ are neither green nor blue. Again, using (1.3.8) and the $(1-\varepsilon)$-completeness of H, we obtain

$$\underline{M_3} \setminus \underline{G_1 \cup G_2} = \underline{M_3} \setminus \underline{B_1 \cup B_2} = \emptyset. \tag{1.3.16}$$

Finally, suppose there is an $x \in \overline{B_3 \cup G_3} \cap \overline{M_1 \cup M_2}$. By (1.3.8) and the $(1-\varepsilon)$-completeness of H, x sees vertices in $\underline{M_3}$. This, however, contradicts (1.3.16) and thus

$$\overline{B_3 \cup G_3} \cap \overline{M_1 \cup M_2} = \emptyset. \tag{1.3.17}$$

Now let us turn to back the graph H, for reasons that will become clear below. Assume that H has a red edge vw outside of $M_1 \cup M_2 \cup M_3$. By maximality of the matchings M_i, vw is not part of R_1, R_2 or R_3. By (1.3.8), (1.3.13) and (1.3.14) we have $\underline{vw} \in G_1 \cap B_2$ or $\underline{vw} \in G_2 \cap B_1$. However, both cases contradict (1.3.10). This yields

$$V(H) = V(B_1') \cup V(B_2') \cup V(M_1) \cup V(M_2) \cup V(M_3). \tag{1.3.18}$$

Next, we restore the symmetry between the colours.

Claim 1.3.17. *Each colour has exactly three components.*

Proof. By (1.3.18) there are no red edges in $Y = H - V(M_1 \cup M_2 \cup M_3)$ and hence $Y = Y' = Y''$. By (1.3.11), (1.3.13) and (1.3.14) R_1, R_2 and R_3 are the only red components in H.

Suppose there is a (possibly trivial) green component G_4 distinct from G_1, G_2 and G_3. Assume first that $\underline{G_4} \neq \emptyset$. Note that any edge between $\underline{G_4}$ and $\overline{G_1' \cup G_2'}$ is red or blue. By 1.3.9, no vertex of $\underline{G_4}$ can send blue edges to both $\overline{G_1'}$ and $\overline{G_2'}$. Moreover, by (1.3.13) and (1.3.14), no vertex of $\underline{G_4}$ can send red edges to both $\overline{G_1'}$ and $\overline{G_2'}$. Since H has $(1-\varepsilon)$-complete degree and $\overline{G_1'} = \overline{B_1'}$ and $\overline{G_2'} = \overline{B_2'}$ are γ-non-trivial, we derive $\underline{G_4} \subseteq \underline{R_1 \cup R_2} \cap \underline{B_1 \cup B_2}$. But this contradicts (1.3.10), because H is $(1-\varepsilon)$-complete.

Now let us assume that $\underline{G_4} = \emptyset$, and so, $\overline{G_4} \neq \emptyset$. In other words, G_4 consists of a single vertex with no incident green edges. Suppose that $\overline{G_4} \cap \overline{M_3} = \emptyset$. So by (1.3.8) and (1.3.11), the edges between $\overline{G_4}$ and $\underline{G_1' \cup G_2'}$ are blue, which contradicts that B_1' and B_2' lie in distinct blue components, as asserted by 1.3.9. Therefore $\overline{G_4} \subseteq \overline{M_3}$. So as $\underline{G_4} = \emptyset$, all edges between $\overline{G_4}$ and $\underline{M_1 \cup M_2}$ are blue. By (1.3.16), (1.3.17) and (1.3.18), $B_3 \subseteq [\underline{M_1 \cup M_2}, \overline{M_3}]$. Since H is $(1-\varepsilon)$-complete and B_3 is γ-non-trivial, we obtain that $\overline{G_4} \subseteq \overline{B_3}$. We also have that $G_3 \subseteq [\underline{M_1 \cup M_2}, \overline{M_3}]$ by (1.3.16), (1.3.17) and (1.3.18). Since G_3 is γ-non-trivial it follows that, $\underline{G_3} \cap \underline{M_1 \cup M_2}$ has a size of at least γn. Since the edges between $\overline{G_4}$ and $\underline{G_3}$ are blue, we obtain that $\underline{M_1 \cup M_2} \cap \underline{G_3} \cap \underline{B_3} \neq \emptyset$. But this represents a contradiction to (1.3.13) or (1.3.14), since there is no colour left for the edges between $\underline{G_3 \cap B_3}$ and $\overline{B_1' \cup B_2'}$. Since a fourth blue component would behave the same way as G_4, this finishes the proof of the claim. □

By (1.3.11) and (1.3.18) it follows that $\underline{R_{\mathrm{i}}} = \underline{M_{\mathrm{i}}}$ for $\mathrm{i} = 1, 2$. In the same way (1.3.13), (1.3.14) and (1.3.18) imply that

$$\overline{R_3} = \overline{M_3}. \tag{1.3.19}$$

For $1 \le \mathrm{i}, j, k \le 3$ we denote $\overline{\mathrm{i}|j|k} := \overline{R_{\mathrm{i}} \cap G_j \cap B_k}$ and $\underline{\mathrm{i}|j|k} := \underline{R_{\mathrm{i}} \cap G_j \cap B_k}$. From 1.3.8, 1.3.10, 1.3.13 and 1.3.14 we obtain that

$$|\overline{1|1|1}|, |\overline{2|2|2}|, |\overline{3|3|3}| > 6\varepsilon n. \tag{1.3.20}$$

Note that by definition and $(1-\varepsilon)$-completeness it follows that for all $\mathrm{i}, \mathrm{i}', j, j', k, k'$ with $\mathrm{i} \neq \mathrm{i}'$, $j \neq j'$ and $k \neq k'$ we have (modulo switching biparts)

$$\text{if } |\overline{\mathrm{i}|j|k}| \ge \varepsilon n, \text{ then } |\underline{\mathrm{i}'|j'|k'}| = 0. \tag{1.3.21}$$

Let us show that $\underline{\mathrm{i}|j|k} = \emptyset$, unless i, j, k are pairwise different. Indeed, otherwise, if say $\underline{1|1|k} \neq \emptyset$ for $k = 1, 2$ or 3, we obtain a contradiction to (1.3.21) as $|\overline{2|2|2}|, |\overline{3|3|3}| \ge 6\varepsilon n$ by (1.3.20). Then the edges of the graph $[\underline{1|1|k}, \overline{2|2|2} \cup \overline{3|3|3}]$ are all blue as H has $(1-\varepsilon)$-complete degree, implying that $2 = \mathrm{k} = 3$, a contradiction. Hence $\underline{H}$ can be decomposed into sets $\underline{\mathrm{i}|j|k}$, where $1 \le \mathrm{i}, j, k \le 3$ are pairwise different. So we have:

$$\underline{1|3|2} \cup \underline{1|2|3} \cup \underline{2|3|1} \cup \underline{2|1|3} \cup \underline{3|2|1} \cup \underline{3|1|2} = \underline{H}. \tag{1.3.22}$$

Claim 1.3.18. *We have* $\overline{H} = \overline{1|1|1} \cup \overline{2|2|2} \cup \overline{3|3|3} \cup \overline{3|1|2} \cup \overline{3|2|1}$.

Proof. First, we show there is no $\overline{\mathrm{i}|j|k} \neq \emptyset$ such that exactly two of i, j, k are equal. If $3|1|1 \neq \emptyset$, say, then $|\underline{1|2|3}|, |\underline{1|3|2}| \le \varepsilon n$ by (1.3.21). Together with (1.3.22), this implies that R_1 is not γ-non-trivial, a contradiction. Second, note that 1.3.11 implies that $\overline{3|1|2}$ and $\overline{3|2|1}$ have each a size of at least γn. Again, by (1.3.21), it follows that $\underline{\mathrm{i}|j|k} = \emptyset$, if $\mathrm{i} \neq 3$ and $3 \in \{j, k\}$. This proves the claim. □

Claim 1.3.19. *We have* $\overline{H} = \overline{1|1|1} \cup \overline{2|2|2} \cup \overline{3|3|3}$.

Proof. By the previous claim it remains to show that $\overline{3|1|2} = \overline{3|2|1} = \emptyset$. To this end, suppose that $\overline{3|1|2} \neq \emptyset$ and thus $|\underline{1|2|3}|, |\underline{2|3|1}| \le \varepsilon n$ by (1.3.21). If $\overline{3|2|1} \neq \emptyset$ as well, then by (1.3.21) also $|\underline{1|3|2}| \le \varepsilon n$ which, by Claim 1.3.18 and 1.3.22 gives the contradiction that $R_1 \subseteq [\overline{1|1|1}, \underline{1|2|3} \cup \underline{1|3|2}]$ is not γ-non-trivial. So we have

$$\overline{H} = \overline{1|1|1} \cup \overline{2|2|2} \cup \overline{3|3|3} \cup \overline{3|1|2},$$

with $\overline{3|1|2} \neq \emptyset$. This partition is shown in Figure 1.3.

Ignoring from now on the matchings M_1 and M_2, we aim at covering H with M_3 and four other matchings. To this end take a green matching M_1^{green} of maximum size in $G_1 - M_3$ and next a blue matching M_2^{blue} of maximum size in $B_2 - M_3 - M_1^{\text{green}}$. Denote

- $\overline{\mathrm{i}|j|k}' := \overline{\mathrm{i}|j|k} \setminus \overline{M_3 \cup M_1^{\text{green}} \cup M_2^{\text{blue}}}$ and
- $\underline{\mathrm{i}|j|k}' := \underline{\mathrm{i}|j|k} \setminus \underline{M_3 \cup M_1^{\text{green}} \cup M_2^{\text{blue}}}$.

We can assume that $M_3 \cup M_1^{\text{green}} \cup M_2^{\text{blue}}$ is not $(1-\varepsilon)$-spanning. Thus, as H has $(1-\varepsilon)$-complete degree, the maximality of the matchings M_3, M_1^{green} and M_2^{blue} implies that $\overline{3|1|2}', \underline{3|1|2}' = \emptyset$.

Moreover it follows that

- $|\overline{1|1|1}'| \le \varepsilon n$ or $|\underline{2|1|3}'| \le \varepsilon n$ by maximality of $M_1^{\text{green}} \subseteq G_1$,
- $|\overline{2|2|2}'| \le \varepsilon n$ or $|\underline{1|3|2}'| \le \varepsilon n$ by maximality of $M_2^{\text{blue}} \subseteq B_2$,
- $\overline{3|3|3}' = \emptyset$ as $\overline{R_3} = \overline{M_3}$ by (1.3.19).

If $|\overline{1|1|1}'|, |\overline{2|2|2}'| \le \varepsilon n$, then we have found three disjoint connected matchings that $(1-2\varepsilon)$-span H, contradicting our assumption. If $|\underline{2|1|3}'|, |\underline{1|3|2}'| \le \varepsilon n$, we take a green matching in G_2 and a blue maximum matching in B_1, among the yet unmatched vertices. After this step, there are at most εn vertices of $\underline{3|2|1}'$ left uncovered and therefore all but at most $3\varepsilon n$ vertices of $\underline{H}$ are covered. Thus, as H is balanced, we have found five disjoint monochromatic connected matchings which together $(1-3\varepsilon)$-span H. So, either $|\overline{2|2|2}'|, |\underline{2|1|3}'| \le \varepsilon n$, or $|\overline{1|1|1}'|, |\underline{1|3|2}'| \le \varepsilon n$. In either case we can find two disjoint monochromatic connected matchings that cover all but at most $2\varepsilon n$ vertices of the two other sets from the previous sentence and all but at most $2\varepsilon n$ vertices of $\underline{3|2|1}'$. So we have five disjoint monochromatic connected matchings $(1-4\varepsilon)$-spanning H, a contradiction.

□

For ease of notation we set

$$X := |\overline{1|1|1}|,\ Y := |\overline{2|2|2}|,\ Z := |\overline{3|3|3}| \ \text{ and}$$

$$A := |\underline{1|3|2}|,\ B := |\underline{1|2|3}|,\ C := |\underline{2|3|1}|,\ D := |\underline{2|1|3}|,\ E := |\underline{3|2|1}|,\ F := |\underline{3|1|2}|.$$

By Claim 1.3.19 and 1.3.22 we have $|\overline{H}| = X + Y + Z$ and $|\underline{H}| = A + B + C + D + E + F$. Note that the edges between any upper and lower part are monochromatic (see Figure 1.4). Also note that we reached complete symmetry between the colours and the indices of the components, so we will from now on again treat them as interchangeable.

Observe that for (at least) one index $\mathrm{i} \in \{1,2,3\}$ it holds that $|\overline{R_\mathrm{i}}| \le |\underline{R_\mathrm{i}}|$. We shall call such an index i a *weak* index for the colour red. If furthermore $|\overline{R_\mathrm{i}}| < |\underline{R_\mathrm{i} \cap B_j}| = |\underline{R_\mathrm{i} \cap G_k}|$ and $|\overline{R_\mathrm{i}}| < |\underline{R_\mathrm{i} \cap B_k}| = |\underline{R_\mathrm{i} \cap G_j}|$, where j, k are the other two indices from $\{1,2,3\}$, then we call i *very weak* for colour red. Analogously define *(very) weak* indices for colours blue and red.

Claim 1.3.20. *If index* i *is weak for colour c, then*

(a) the indices in $\{1,2,3\} - \{\mathrm{i}\}$ are not weak for colour c, and

(b) index i *is very weak for colour c.*

Proof. Let us show this for $\mathrm{i} = 2$ and colour red (the other cases are analogous). By assumption, $Y \le C + D$. Since $X < A + B$ and $Z < E + F$ cannot both hold, we can assume without loss of generality that $Z \ge E + F$. Now if $X \le A + B$, then we pick maximal red matchings in $[\overline{1|1|1}, \underline{1|3|2} \cup \underline{1|2|3}]$, $[\overline{2|2|2}, \underline{2|3|1} \cup \underline{2|1|3}]$ and $[\underline{3|2|1} \cup \underline{3|1|2}, \overline{3|3|3}]$, thus covering all but at most $3\varepsilon n$ vertices of $\overline{1|1|1} \cup \overline{2|2|2} \cup \underline{3|2|1} \cup \underline{3|1|2}$. To finish we cover all but $4\varepsilon n$ of the remaining vertices in $\overline{3|3|3} \cup (\underline{H \setminus R_3})$ with a blue and a green matching, a contradiction. Hence $X > A + B$. Using this fact, $Z > E + F$ follows by symmetry. This proves (a).

In order to show (b), let us first prove that $Y < C$. We pick a maximal red matching in each of R_1 and R_3, thus covering all but at most $2\varepsilon n$ vertices of $\underline{R_1 \cup R_3}$. Now if $Y \ge C$, then all but at most εn vertices of $\underline{2|3|1}$ are contained in a maximal red matching that also contains all but at most εn vertices of $\overline{2|2|2}$. We cover all but $4\varepsilon n$ of the remaining vertices in $\overline{R_1 \cup R_3}$ with a blue and a green matching, a contradiction. The fact that $Y < D$ follows analogously. □

Suppose two of the three indices $1, 2, 3$ are weak for different colours, say 1 is weak for red and 2 is weak for green. Then Claim 1.3.20(b) gives that $X < A$ and $Y < E$. Thus we can match all but at most εn vertices of $\overline{1|1|1}$ into $\underline{1|3|2}$ and all but at most εn vertices of $\overline{2|2|2}$ into $\underline{3|2|1}$ with two matchings, one red and one green, and cover all but $6\varepsilon n$ of the remaining vertices with three disjoint matchings, one from each of R_3, G_3, B_3, a contradiction.

Hence, since each colour has a weak index, there is an index i that is weak for all three colours, i $= 2$ say. We match all but at most εn vertices of $\overline{2|2|2}$ into $\underline{3|1|2}$ with a blue matching M. Further choose a subset $F \subseteq \underline{3|1|2} \setminus V(M)$ of size $|\overline{2|2|2}| - |V(M)/2| \leq \varepsilon n$, and let us from now work with the remaining set $\underline{3|1|2}' = \underline{3|1|2} \setminus (V(M) \cup F)$ of cardinality $F' = F - Y$. Set $n' = n - Y$. (So instead of five we will have to find four monochromatic connected matchings covering all but few vertices of $H - M$.) Without loss of generality assume $Z \geq X$. Claim 1.3.20(a) gives that

$$X > A + B, C + E, D + F' \text{ and } Z > A + C, B + D, E + F'. \tag{1.3.23}$$

Hence $X > n'/3$. So, one of the three sums $A + C, B + D, E + F'$ has to be strictly smaller than X, say $A + C < X$. Consequently, $Z = n' - X < B + D + E + F'$.

If $Z \geq D + E + F'$, then we cover all but at most εn vertices of $\underline{R_3 - M}$ with a red matching, and cover all but at most εn vertices of the remains of $\overline{3|3|3}$ with a blue matching that also covers all but at most εn vertices of $\underline{2|1|3}$. Now all that is left on the top is $\overline{1|1|1}$, which we can match with a red and a blue matching into the remains of $\underline{1|3|2} \cup \underline{1|2|3} \cup \underline{2|3|1}$ (except for εn vertices). Thus we found four connected matchings that cover all but at most γn vertices of $H - V(M)$, and are done.

So we may assume $Z < D + E + F'$ and thus $X > A + B + C$. If $X \leq A + B + C + E$, then we can proceed similarly as in the previous paragraph to find four matchings covering all vertices of H. Hence $X > A + B + C + E$, implying that $Z < D + F'$. But by (1.3.23) we have $D + F' < X$ a contradiction to our assumption that $X \leq Z$. This proves Lemma 1.3.1.

1.4 From connected matchings to cycles

In this section we prove Theorem 1.1.1(a). We basically follow the approach of Łuczak [79], which has become a standard method in this field. Therefore we present only an outline of the proof, omitting most of the tedious details that have been discussed in earlier works in more general contexts. We refer the interested reader to [10, 27, 58, 61, 63, 80].

For a graph G the bipartite subgraph $H = [A, B] \subseteq G$ is *(ε, G)-regular* if

$$X \subseteq A,\ Y \subseteq B,\ |X| > \varepsilon|A|,\ |Y| > \varepsilon|B| \text{ imply } |\mathrm{d}_G(X, Y) - \mathrm{d}_G(A, B)| < \varepsilon.$$

A vertex-partition $\{V_0, V_1, \ldots, V_l\}$ of $l + 1$ *clusters* of a graph G is called *(ε, G)-regular*, if

(a) $|V_1| = |V_2| = \ldots = |V_l|$;

(b) $|V_0| < \varepsilon n$;

(c) apart from at most $\varepsilon\binom{l}{2}$ exceptional pairs, the graphs $[V_i, V_j]$ are (ε, G)-regular.

Lemma 1.4.1 (Regularity Lemma with prepartition and colours)**.** *For every $\varepsilon > 0$ and positive integers $m, r, s \in \mathbb{N}$ there are $m \le M \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that for $n \ge n_0$ the following holds. For any set of mutually edge-disjoint graphs $G_1, G_2, \dots, G_r$ with $V(G_1) = V(G_2) = \dots = V(G_r) = V$, with $|V| = n$, and any partition $W_1 \cup \dots \cup W_s = V$, there is a partition $V_0 \cup V_1 \cup \dots \cup V_l$ of V into $l+1$ clusters such that*

(a) $m \le l \le M$;

(b) *for each $1 \le i \le l$ there is $1 \le j \le s$ such that $V_i \subseteq W_j$;*

(c) *$V_0 \cup V_1 \cup \dots \cup V_l$ is (ε, G_i)-regular for each $1 \le i \le r$.*

Let us now prove Theorem 1.1.1(a). Let $n \gg 0$ and $0 < \varepsilon \ll 1$. Let the edges of $K_{n,n}$ with biparts W_1 and W_2 be coloured in red, green and blue. We denote by G_1, G_2 and G_3 the graphs induced by the edges of each of the colours.

For $m \gg 0$ and $\varepsilon \ll \mathrm{d} \ll 0$, Lemma 1.4.1 provides a vertex-partion $V_0, V_1, \dots, V_l$ of $K_{n,n}$ satisfying Lemma 1.4.1(a)–(c) for some $M \ge m$. As usual, we define the $(\varepsilon, \mathrm{d})$-reduced graph Γ by identifying a new vertex v_i with each cluster V_i for $1 \le i \le l$. For $1 \le i, j \le l$ and $1 \le q \le 3$ we place an edge of colour q between two vertices v_i, v_j if the subgraph $[V_i, V_j]$ of the respective clusters has edge-density at least d in G_q and is (ε, G_q)-regular. To get a simple graph, we keep an arbitrary edge from each multi-edge.

Since the clusters have the same size, we can, if necessary, remove some of them to obtain a balanced bipartite $(1 - 2\varepsilon)$-complete subgraph of Γ, which we will continue to call Γ. Therefore Lemma 1.3.1 can be used to cover all but at most $\rho|V(\Gamma)|$ vertices of Γ by five vertex-disjoint monochromatic connected matchings $M_1, \dots, M_5$. We finish the proof by turning these five matchings into monochromatic cycles of $K_{n,n}$ using the following lemma from [79].

Lemma 1.4.2. *Let $0 < \varepsilon \ll \rho \ll \mathrm{d} \le 1$ and let Γ be the $(\varepsilon, \mathrm{d})$-reduced graph of $G_1, G_2, \dots, G_r$, obtained from Lemma 1.4.1. Assume that there is a set of disjoint monochromatic connected matchings $\mathcal{M}$ in Γ. Let $U \subseteq V(G)$ be the set of vertices, which are in clusters associated to the vertices of $V(M)$. Then there are $|\mathcal{M}|$ monochromatic cycles in G partitioning all but $(1-\rho)|U|$ vertices of U.*

1.5 Covering all vertices

1.5.1 Preliminaries

We call a balanced bipartite subgraph H of a $2n$-vertex graph $(1 - \varepsilon)$-*Hamiltonian*, if any balanced bipartite subgraph of H with at least $2(1 - \varepsilon)n$ vertices is Hamiltonian. The next lemma is a combination of results from [64, 83].

Lemma 1.5.1. *For any $1 > \gamma > 0$, there is an $n_0 \in \mathbb{N}$ such that any balanced bipartite graph on $2n \ge 2n_0$ vertices and of edge density at least γ has a $(1 - \gamma/4)$-Hamiltonian subgraph of size at least $\gamma^{3024/\gamma} n/3$.*

We make no attempt to optimise the bounds in Lemma 1.5.1. For the proof, we need some definitions and tools. For a graph G, and disjoint $A, B \subseteq V(G)$ let $\mathrm{e}(A, B)$ denote the number of edges in $[A, B]$. For $0 < \varepsilon, \sigma < 1$, $[A, B]$ is called (ε, σ)-*dense* if $\mathrm{e}(X, Y) \geq \sigma|A||B|$ for every $X \subseteq A$, $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and and $|Y| \geq \varepsilon|B|$.

Theorem 1.5.2 (Peng et. al [83]). *Given a bipartite balanced graph of size* $2n$ *and edge density* $0 < \gamma < 1$. *Then for all* $0 < \varepsilon < 1$ *there is an* $(\varepsilon, \gamma/2)$*-dense balanced subgraph on at least* $\gamma^{12/\varepsilon}n/2$ *vertices.*

For $0 < \varepsilon, \delta < 1$, we say that the balanced subgraph $H = [A, B]$ is (ε, δ)-*uniform* in G, if it has minimum degree at least $\delta|A|$, and any ε-non-trivial subgraph of H has at least one edge. The next result, due to Haxell, shows that sufficiently strong uniformity implies hamiltonicity.

Theorem 1.5.3 (Haxell [64]). *Let* $\varepsilon > 0$ *be given, and suppose that* $H = [A, B]$ *is a bipartite graph with* $|A| = |B| \geq \frac{1}{\varepsilon}$ *such that* H *is* (ε, δ)*-uniform for* $\delta > 7\varepsilon$. *Then* H *is Hamiltonian.*

Proof of Lemma 1.5.1. Set $\varepsilon := \gamma/253$ and $n_0 := 2\gamma^{-12/\varepsilon}\varepsilon^{-1}$. Let H be a balanced bipartite graph of density γ and size $2n \geq 2n_0$. Apply Theorem 1.5.2 to obtain a balanced $(\varepsilon, \gamma/2)$-dense subgraph $[A, B] \subseteq H$ of size at least $\gamma^{12/\varepsilon}n/2$. Deleting at most $\varepsilon|A|$ vertices on either side, we arrive at a $(2\varepsilon, \gamma/3)$-uniform subgraph $[X, Y] \subseteq [A, B]$ of size at least $\gamma^{12/\varepsilon}n/3$.

In order to see that $[X, Y]$ is (1-$\gamma/4$)-Hamiltonian, delete an arbitrary fraction of at most $\gamma/4 < 1/4$ vertices from each of X, Y. Clearly, the obtained subgraph $[X', Y']$ is $(3\varepsilon, \frac{\gamma}{12})$-uniform, and has size at least $\gamma^{12/\varepsilon}n_0/4 \geq 1/(3\varepsilon)$. Thus Theorem 1.5.3 applies and we are done. □

Finally, we make use of the following lemma due to Gyárfás et al. It allows us to absorb small vertex sets with few monochromatic cycles.

Lemma 1.5.4 (Gyárfás et al. [59]). *There is a constant* $n_0 \in \mathbb{N}$ *such that for* $n \geq n_0$ *and* $m \leq \frac{n}{(8r)^{8(r+1)}}$, *and for any* r*-colouring of* $K_{n,m}$, *there are* $2r$ *disjoint monochromatic cycles covering all* m *vertices on the smaller side.*

1.5.2 Proof of Theorem 1.1.1(b)

Let A and B be the two partition classes of the 3-edge-coloured $K_{n,n}$. We assume that $n \geq n_0$, where we specify n_0 later. Pick subsets $A_1 \subseteq A$ and $B_1 \subseteq B$ of size $\lceil n/2 \rceil$ each. Say red is the majority colour of $[A_1, B_1]$. Lemma 1.5.1 applied with $\gamma = 1/3$ yields a red $(1 - 1/12)$-Hamiltonian subgraph $[A_2, B_2]$ of $[A_1, B_1]$ with

$$|A_2| = |B_2| \geq 3^{9999}|A_1| \geq 3^{-10^4}n.$$

Set $H := G - (A_2 \cup B_2)$, and note that each bipart of H has order at least $\lfloor n/2 \rfloor$. Let $\delta := 24^{-32} \cdot 3^{-10^4}$. Assuming n_0 is large enough, Theorem 1.1.1(a) yields five monochromatic vertex-disjoint cycles covering all but at most $2\delta n$ vertices of H. Let $X_A \subseteq A$ (resp. $X_B \subseteq B$) be the set of uncovered vertices in A (resp. B). Since we may assume none of the monochromatic cycles is an isolated vertex, we have $|X_A| = |X_B| \leq \delta n$.

By the choice of δ, and since we assume n_0 to be sufficiently large, we can apply Lemma 1.5.4 to the bipartite graphs $[A_2, X_B]$ and $[B_2, X_A]$. We obtain a union $\mathcal{C}$ of twelve vertex-disjoint monochromatic cycles that together cover $X_A \cup X_B$. As $|X_A| = |X_B| \leq \delta n \leq 3^{-10^4}/12$, we know that $[A_2, B_2] - V(\bigcup \mathcal{C})$ contains a red Hamiltonian cycle. Thus, in total, we covered G with at most $5 + 12 + 1 = 18$ vertex-disjoint monochromatic cycles.

1.5.3 A remark on 3-coloured complete graphs

The number of 17 cycles needed to partition a 3-coloured complete graph, obtained by Gyárfás et al. [61], is not expected to be optimal. By a slight modification of their method, one can replace the number 17 with (the still not optimal number) 10.

Erdős et al. [31] have shown that any large enough 3-coloured K_n has a monochromatic *triangle cycle* of linear size. That is, a union of two cycles $(u_1, u_2, \dots, u_k, u_1)$ and $(u_1, v_1, u_2, v_2, \dots, u_k, v_k, u_1)$. Clearly, after the deletion of an arbitrary subset of the *outer vertices*, $\{v_1, \dots, v_k\}$, the triangle cycle still has a Hamiltonian cycle.

Given a sufficiently large 3-coloured K_n, we proceed as follows. First we reserve the vertex set of a linear sized monochromatic triangle cycle T for later use. We cover the remaining graph, except for some small set X, with three vertex-disjoint monochromatic cycles, using the result of Gyárfás et al. [61]. We then use Lemma 1.5.4 to cover all of X with six vertex-disjoint monochromatic cycles, which use some of the outer vertices of T (and X). This can be done since T is of linear size while $|X|$ is a vanishing fraction of n. We finish by covering the remains of T with a monochromatic Hamiltonian cycle.

Chapter 2

Local colourings and monochromatic partitions in complete bipartite graphs[1]

Richard Lang and Maya Stein

Abstract

We show that for any 2-local colouring of the edges of the balanced complete bipartite graph $K_{n,n}$, its vertices can be covered with at most 3 disjoint monochromatic paths. And, we can cover the vertices of any complete or balanced complete bipartite r-locally coloured graph with $O(r^2)$ disjoint monochromatic cycles.
We also determine the 2-local bipartite Ramsey number of a path: Every 2-local colouring of the edges of $K_{n,n}$ contains a monochromatic path on n vertices.

2.1 Introduction

The problem of partitioning a graph into few monochromatic paths or cycles, first formulated explicitly in the beginning of the 80's [**52**], has lately received a fair amount of attention. Its origin lies in Ramsey theory and its subject are complete graphs (later substituted with other types of graphs), whose edges are coloured with r colours. Call such a colouring an r-colouring; note that this need not be a proper edge-colouring. The challenge is now to find a small number of disjoint monochromatic paths, which together cover the vertex set of the underlying graph. Or, instead of disjoint monochromatic paths, we might ask for disjoint monochromatic cycles. Here, single vertices and edges count as cycles. Such a cover is called a monochromatic path partition, or a monochromatic cycle partition, respectively. It is not difficult to construct r-colourings that do not allow for partitions into less than r paths, or cycles.[2]

At first, the problem was studied mostly for $r = 2$, and the complete graph K_n as the host graph. In this situation, a partition into two disjoint paths always exists [**45**], regardless of the size of n. Moreover, we can require these paths to have different colours. An extension of this fact, namely that every 2-colouring of K_n has a partition into two monochromatic cycles

[1]The results of this chapter have been published in the European Journal of Combinatorics [**73**].
[2]For instance, take vertex sets $V_1, \dots, V_r$ with $|V_i| = 2^i$, and for $i \leq j$ give all V_i–V_j edges colour i.

of different colours was conjectured by Lehel, and verified by Bessy and Thomassé [**13**], after preliminary work for large n [**1, 80**].

A generalisation of these two results for other values of r, i.e. that any r-coloured K_n can be partitioned into r monochromatic paths, or into r monochromatic cycles, was conjectured by Gyárfás [**53**] and by Erdős, Gyárfás and Pyber [**31**], respectively. The conjecture for cycles was recently disproved by Pokrovskiy [**84**]. He gave counterexamples for all $r \geq 3$, but he also showed that the conjecture for paths is true for $r = 3$. Gyárfás, Ruszinkó, Sárközy and Szemerédi [**58**] showed that any r-coloured K_n can be partitioned into $O(r \log r)$ monochromatic cycles, improving an earlier bound from [**31**].

Monochromatic path/cycle partitions have also been studied for bipartite graphs, mainly for $r = 2$. A 2-colouring of $K_{n,n}$ is called a split colouring if there is a colour-preserving homomorphism from the edge-coloured $K_{n,n}$ to a properly edge-coloured $K_{2,2}$. Note that any split colouring allows for a partition into three paths, but not always into two. However, split colourings are the only 'problematic' colourings, as the following result shows.

Theorem 2.1.1 (Pokrovskiy [**84**]). *Let the edges of $K_{n,n}$ be coloured with 2 colours; then $K_{n,n}$ can be partitioned into two paths of distinct colours or the colouring is split.*

Split colourings can be generalised to more colours [**84**]. This gives a lower bound of $2r - 1$ on the path/cycle partition number for $K_{n,n}$. For $r = 3$, this bound is asymptotically correct [**72**]. For an upper bound, Peng, Rödl and Ruciński [**83**] showed that any r-coloured $K_{n,n}$ can be partitioned into $O(r^2 \log r)$ monochromatic cycles, improving a result of Haxell [**64**]. We improve this bound to $O(r^2)$.

Theorem 2.1.2. *For every $r \geq 1$ there is an n_0 such that for $n \geq n_0$, for any r-locally coloured $K_{n,n}$, we need at most $4r^2$ disjoint monochromatic cycles to cover all its vertices.*

Theorem 2.1.2 follows immediately from Theorem 2.1.3 (b) below. Let us mention that the monochromatic cycle partition problem has also been studied for multipartite graphs [**97**], and for arbitrary graphs [**10, 96**], or replacing paths or cycles with other graphs [**43, 94, 95**].

Our main focus in this paper is on monochromatic cycle partitions for *local colourings* (Theorem 2.1.2 being only a side-product of our local colouring results). Local colourings are a natural way to generalise r-colourings. A colouring is r-local if no vertex is adjacent to more than r edges of distinct colours. Local colourings have appeared mostly in the context of Ramsey theory [**14, 22, 56, 57, 92, 99, 103, 104**].

With respect to monochromatic path or cycle partitions, Conlon and Stein [**25**] recently generalised some of the above mentioned results to r-local colourings. They show that for any r-local colouring of K_n, there is a partition into $O(r^2 \log r)$ monochromatic cycles, and, if $r = 2$, then two cycles suffice. In this paper we improve their general bound for complete graphs, and give the first bound for monochromatic cycle partitions in bipartite graphs. In both cases, $O(r^2)$ cycles suffice.

Theorem 2.1.3. *For every $r \geq 1$ there is an n_0 such that for $n \geq n_0$ the following holds.*

(a) If K_n is r-locally coloured, then its vertices can be covered with at most $2r^2$ disjoint monochromatic cycles.

(b) If $K_{n,n}$ is r-locally coloured, then its vertices can be covered with at most $4r^2$ disjoint monochromatic cycles.

We do not believe our results are best possible, but suspect that in both cases (K_n and $K_{n,n}$), the number of cycles needed should be linear in r.

Conjecture 2.1.4. *There is a c such that for every r, every r-local colouring of K_n admits a covering with cr disjoint cycles. The same should hold replacing K_n with $K_{n,n}$.*

Our second result is a generalisation of Theorem 2.1.1 to local colourings:

Theorem 2.1.5. *Let the edges of $K_{n,n}$ be coloured 2-locally. Then $K_{n,n}$ can be partitioned into 3 or less monochromatic paths.*

So, in terms of monochromatic path partitions, it does not matter whether our graph is 2-locally coloured, or if the total number of colours is 2. For more colours this might be different, but we have not been able to construct r-local colourings of $K_{n,n}$ which need more than $2r-1$ monochromatic paths for covering the vertices.

We prove Theorem 2.1.3 in Section 2.2 and Theorem 2.1.5 in Section 2.3. These proofs are totally independent of each other.

Theorem 2.1.5 relies on a structural lemma for 2-local colourings, Lemma 2.3.1. This lemma has a second application in local Ramsey theory. As mentioned above, some effort has gone into extending Ramsey theory to local colourings. In particular, in [**57**], Gyárfás et al. determine the 2-local Ramsey number of the path P_n. This number is defined as the smallest number m such that in any 2-local colouring of K_m a monochromatic path of length n is present. In [**57**], it is shown that the 2-local Ramsey number of the path P_n is $\lceil\frac{3}{2}n-1\rceil$. Thus the usual 2-colour Ramsey number of the path, which is $\lfloor\frac{3}{2}n-1\rfloor$ and the 2-local Ramsey number of the path P_n only differ by at most 1 (depending on the parity of n).

The *bipartite* 2-colour Ramsey number of the path P_n is defined as a pair (m_1, m_2), with $m_1 \geq m_2$ such that for any pair m'_1, m'_2 we have that $m'_i \geq m_i$ for both $i = 1, 2$ if and only if every 2-colouring of $K_{m'_1,m'_2}$ contains a monochromatic path P_n. Gyárfás and Lehel [**55**] and, independently, Faudree and Schelp [**35**] determined the bipartite 2-colour Ramsey number of P_{2m} to be $(2m-1, 2m-1)$. The authors of [**35**] also show that for the odd path P_{2m+1} this number is $(2m+1, 2m-1)$. Observe that suitable split colourings can be used to see the sharpness of these Ramsey numbers.

We use our auxiliary structural result, Lemma 2.3.1, and the result of [**55**] to determine the 2-local bipartite Ramsey number for the even path P_{2m}. As for complete host graphs, it turns out this number coincides with its non-local pendant.

Theorem 2.1.6. *Let $K_{2m-1,2m-1}$ be coloured 2-locally. Then there is a monochromatic path on $2m$ vertices.*

It is likely that similar arguments can be applied to obtain an analogous result for odd paths (but such an analogue is not straightforward). Clearly, the result from [**35**] together with Theorem 2.1.6 (for $m+1$) imply that the 2-local bipartite Ramsey number for the odd path P_{2m+1} is one of $(2m+1, 2m-1)$, $(2m+1, 2m)$, $(2m+1, 2m+1)$.

In view of the results from [**25**] and our Theorems 2.1.3, 2.1.5 and 2.1.6, it might seem that in terms of path- or cycle-partitions, r-local colourings are not very different from r-colourings. Let us give an example where they do behave differently, even for $r = 2$.

It is shown in [**97**] that any 2-coloured complete tripartite graph can be partitioned into at most 2 monochromatic paths, provided that no part of the tripartition contains more than half of the vertices. This is not true for 2-local colourings: Let G be a complete tripartite graph with triparts U, V and W such that $|U| = 2|V| = 2|W| \geq 6$. Pick vertices $u \in U$, $v \in V$ and $w \in W$ and write $U' = U \setminus \{u\}$, $V' = V \setminus \{v\}$ and $W' = W \setminus \{w\}$. Now colour the edges of $[W' \cup \{v\}, U']$ red, $[V' \cup \{w\}, U']$ green and the remaining edges blue. This is a 2-local colouring. However, since no monochromatic path can cover all vertices of U', we need at least 3 monochromatic paths to cover all of G.

Note that in our example, the graph G contains a 2-locally coloured balanced complete bipartite graph. This shows that in the situation of Theorem 2.1.5, we might need 3 paths even if the 2-local colouring is not a split colouring (and thus a 2-colouring). Blowing this example up, and adding some smaller sets of vertices seeing new colours, one obtains examples of r-local colourings of balanced complete bipartite graphs requiring $2r - 1$ monochromatic paths.

2.2 Proof of Theorem 2.1.3

In this section we will prove our bounds for monochromatic cycle partitions, given by Theorem 2.1.3. The heart of this section is Lemma 2.2.1. This lemma enables us to use induction on r, in order to prove new bounds for the number of monochromatic matchings needed to cover an r-locally coloured graph. In particular, we find these bounds for the complete and the complete bipartite graph. All of this is the topic of Subsection 2.2.1.

To get from monochromatic cycles to the promised cycle cover, we use a nowadays standard approach, which was first introduced in [**79**]. We find a large robust Hamiltonian graph, regularise the rest, find monochromatic matchings covering almost all, blow them up to cycles, and the absorb the remainder with the robust Hamiltonian graph. The interested reader may find a sketch of this well-known method in Subsection 2.2.2.

2.2.1 Monochromatic matchings

Given a graph G with an edge colouring, a monochromatic connected matching is a matching in a connected component of the subgraph that is induced by the edges of a single colour.

Lemma 2.2.1. *For $k \geq 2$, let the edges of a graph G be coloured k-locally. Suppose there are m monochromatic components that together cover $V(G)$, of colours $c_1, \dots, c_m$. Then there are m vertex-disjoint monochromatic connected matchings $M_1, \dots, M_m$, of colours $c_1, \dots, c_m$, such that the inherited colouring of $G \setminus V(\bigcup_{i=1}^m M_i)$ is a $(k-1)$-local colouring.*

Proof. Let G be covered by m monochromatic components $C_1, \dots, C_m$ of colours $c_1, \dots, c_m$. Let $M_1 \subseteq C_1$ be a maximum matching in colour c_1. For $2 \leq i \leq m$ we iteratively pick maximum matchings $M_i \subseteq C_i \setminus V(\bigcup_{j<i} M_j)$ in colour c_i. Set $M := \bigcup_{j \leq m} M_j$.

Now let v be any vertex in $H := G \setminus V(M)$. Say $v \in V(C_i \setminus V(M))$. In particular, vertex v sees colour c_i in G. However, by maximality of M_i, vertex v does not see the colour c_i in H. Thus in H, vertex v sees at most $k-1$ colours. Hence, the inherited colouring of H is a $(k-1)$-local colouring, which is as desired. □

Corollary 2.2.2. *If K_n is r-locally edge coloured, and H is obtained from K_n by deleting $o(n^2)$ edges, then*

(a) $V(K_n)$ can be covered with at most $r(r+1)/2$ monochromatic connected matchings, and

(b) all but $o(n)$ vertices of H can be covered with at most $r(r+1)/2$ monochromatic connected matchings.

Note that the matchings from (b) are connected in H.

Proof. The proof is based on the following easy observation. In any colouring of K_n, the closed monochromatic neighbourhoods of any vertex v together cover K_n. Since the colouring is a k-local colouring, we can cover all of $V(K_n)$ with k components. Now apply Lemma 2.2.1 successively to obtain the bound from (a).

For (b), it suffices to observe that we can choose at each step a vertex v that has $o(n)$ non-neighbours in the current graph. For, if at some step, there is no such vertex, then a simple calculation shows we have already covered all but $o(n)$ of $V(K_n)$, and can hence abort the procedure. □

Corollary 2.2.3. *If $K_{n,n}$ is r-locally edge coloured, and H is obtained from $K_{n,n}$ by deleting $o(n^2)$ edges, then*

(a) $V(K_{n,n})$ can be covered with at most $(2r-1)r$ monochromatic connected matchings, and

(b) all but $o(n)$ vertices of H can be covered with at most $(2r-1)r$ monochromatic connected matchings.

Note that the matchings from (b) are connected in H.

Proof. The proof very similar to the proof Corollary 2.2.2. We only note that in any colouring of $K_{n,n}$ the two closed monochromatic neighbourhoods of any edge form a vertex cover of size at most $2r-1$. □

2.2.2 From matchings to cycles

2.2.2.1 Regularity

Regularity is the key for expanding our partition of an r-locally coloured K_n or $K_{n,n}$ into monochromatic connected matchings to a partition of almost all vertices into monochromatic cycles. We follow an approach introduced by Łuczak [**79**], which has become a standard method for cycle embeddings in large graphs. We will focus on the parts where our proof differs from other applications of this method (see [**58, 61, 72**]).

The main result of this section is:

Lemma 2.2.4. *If K_n and $K_{n,n}$ are r-locally edge coloured, then*

(a) all but $o(n)$ vertices of K_n can be covered with at most $r(r+1)/2$ monochromatic cycles.

(b) all but $o(n)$ vertices of $K_{n,n}$ can be covered with at most $(2r-1)r$ monochromatic cycles.

Before we start, we need a couple of regularity preliminaries. For a graph G and disjoint subsets of vertices $A, B \subseteq V(G)$ we denote by $[A, B]$ the bipartite subgraph with biparts A and B and edge set $\{ab \in E(G) : a \in A, b \in B\}$. We write $\deg_G(A, B)$ for the number of edges in $[A, B]$. If $A = \{a\}$ we write shorthand $\deg_G(a, B)$.

The subgraph $[A, B]$ is (ε, G)-*regular* if

$$|\deg_G(X, Y) - \deg_G(A, B)| < \varepsilon$$

for all $X \subseteq A$, $Y \subseteq B$ with $|X| > \varepsilon|A|$, $|Y| > \varepsilon|B|$. Moreover, $[A, B]$ is (ε, δ, G)-*super-regular* if it is (ε, G)-regular and

$$\deg_G(a, B) > \delta|B| \text{ for each } a \in A \text{ and } \deg_G(b, A) > \delta|A| \text{ for each } b \in B.$$

A vertex-partition $\{V_0, V_1, \ldots, V_l\}$ of the vertex set of a graph G into $l+1$ *clusters* is called (ε, G)-*regular*, if

(i) $|V_1| = |V_2| = \ldots = |V_l|$;

(ii) $|V_0| < \varepsilon n$;

(iii) apart from at most $\varepsilon\binom{l}{2}$ exceptional pairs, the graphs $[V_i, V_j]$ are (ε, G)-regular.

The following version of Szemerédi's regularity lemma is well-known. The given prepartition will only be used for the bipartition of the graph $K_{n,n}$ in Lemma 2.2.4 (b). The colours on the edges are represented by the graphs G_i.

Lemma 2.2.5 (Regularity lemma with prepartition and colours)**.** *For every $\varepsilon > 0$ and $m, t \in \mathbb{N}$ there are $M, n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds.*
For all graphs $G_0, G_1, G_2, \ldots, G_t$ with $V(G_0) = V(G_1) = \ldots = V(G_t) = V$ and a partition $A_1 \cup \ldots \cup A_s = V$, where $r \geq 2$ and $|V| = n$, there is a partition $V_0 \cup V_1 \cup \ldots \cup V_l$ of V into $l+1$ clusters such that

(a) *$m \leq l \leq M$;*

(b) *for each $1 \leq i \leq l$ there is a $1 \leq j \leq s$ such that $V_i \subseteq A_j$;*

(c) *$V_0 \cup V_1 \cup \ldots \cup V_l$ is (ε, G_i)-regular for each $0 \leq i \leq t$.*

Observe that the regularity lemma provides regularity only for a number of colours bounded by the input parameter t. However, the total number of colours of an r-local colouring is not bounded by any function of r (for an example, see Section 2.3.1). Luckily, it turns out that it suffices to focus on the t colours of largest density, where t depends only on r and ε. This is guaranteed by the following result from [**56**].

Lemma 2.2.6. *Let a graph G with average degree a be r-locally coloured. Then one colour has at least $a^2/2r^2$ edges.*

Corollary 2.2.7. *For all $\varepsilon > 0$ and $r \in \mathbb{N}$ there is a $t = t(\varepsilon, r)$ such that for any r-local colouring of K_n or $K_{n,n}$, there are t colours such that all but at most εn^2 edges use these colours.*

Proof. We only prove the corollary for $K_{n,n}$, as the proof for K_n is very similar. Let $t := \lceil -\frac{2r^2}{\varepsilon} \log \varepsilon \rceil$. We iteratively take out the edges of the colours with the largest number of edges. We stop either after t steps, or before, if we the remaining graph has density less than

ε. At each step Lemma 2.2.6 ensures that at least a fraction of $\frac{\varepsilon}{2r^2}$ of the remaining edges has the same colour.[3] Hence we can bound the number of edges of the remaining graph by

$$\left(1 - \frac{\varepsilon}{2r^2}\right)^t n^2 \leq \mathrm{e}^{-\varepsilon t/2r^2} n^2 \leq \varepsilon n^2.$$

□

2.2.2.2 Proof of Lemma 2.2.4

We only prove part (b) of Lemma 2.2.4, since the proof of part (a) is very similar and actually simpler. For the sake of readability, we assume that $n_0 \gg 0$ is sufficiently large and $0 < \varepsilon \ll 1$ is sufficiently small without calculating exact values.

Let the edges of $K_{n,n}$ with biparts A_1 and A_2 be coloured r-locally and encode the colouring by edge-disjoint graphs $G_1, \ldots, G_s$ on the vertex set of $K_{n,n}$. By Corollary 2.2.7, there is a $t = t(\varepsilon, r)$ such that the union of $G_1, \ldots, G_t$ covers all but at most $\varepsilon n^2/8r^2$ edges of $K_{n,n}$. We merge the remaining edges into $G_0 := \bigcup_{\mathrm{i}=t+1}^{s} G_{\mathrm{i}}$. Note that the colouring remains r-local and by the choice of t, we have

$$|E(G_0)| \leq \varepsilon n^2/8r^2. \tag{2.2.1}$$

For ε, t and $m := 1/\varepsilon$, the regularity lemma (Lemma 2.2.5) provides n_0 and M such for all $n \geq n_0$ there is a vertex-partition $V_0, V_1, \ldots, V_l$ of $K_{n,n}$ satisfying Lemma 2.2.5(a)–(c) for $G_0, G_1, \ldots, G_t$.

As usual, we define the reduced graph R which has a vertex v_{i} for each cluster V_{i} for $1 \leq \mathrm{i} \leq l$. We place an edge between vertices v_{i} and v_j if the subgraph $[V_{\mathrm{i}}, V_j]$ of the respective clusters is non-empty and forms an (ε, G_q)-regular subgraph for all $0 \leq q \leq t$. Thus, R is a balanced bipartite graph with at least $(1-\varepsilon)\binom{l}{2}$ edges.

Finally, the colouring of the edges of $K_{n,n}$, induces a *majority colouring* of the edges of R. More precisely, we colour each edge $v_{\mathrm{i}}v_j$ of R with the colour from $\{0, 1, \ldots, t\}$ that appears most on the edges of the subgraph $[V_{\mathrm{i}}, V_j] \subseteq G$ (in case of a tie, pick any of the densest colours). Note that if $v_{\mathrm{i}}v_j$ is coloured i then by Lemma 2.2.6,

$$[V_{\mathrm{i}}, V_j] \text{ has at least } \tfrac{1}{2r^2}(\tfrac{n}{2l})^2 \text{ edges of colour i.} \tag{2.2.2}$$

Our next step is to verify that the majority colouring is an r-local colouring of R. To this end we need the following easy and well-known fact about regular graphs.

Fact 2.2.8. *Let $[A, B]$ be an ε-regular graph of density $\mathrm{d} > \varepsilon$. Then at most $\varepsilon|A|$ vertices from A have no neighbours in B.*

Claim 2.2.9. *The colouring of the reduced graph R is r-local.*

Proof. Assume otherwise. Then there is a vertex $v_{\mathrm{i}} \in V(R)$ that sees $r+1$ different colours in R. By Fact 2.2.8, all but at most $(r+1)\varepsilon|V_{\mathrm{i}}| < |V_{\mathrm{i}}|$ of the vertices in V_{i} see $r+1$ different colours in $K_{n,n}$, contradicting the r-locality of our colouring. □

[3]Here we use that in a balanced bipartite graph H with $2n$ vertices, m edges, average degree a and density d we have $a^2 = \frac{4m^2}{4n^2} = \mathrm{d}m$.

By (2.2.1), and by (2.2.2), we know that R has at most $|E(G_0)|\frac{4l^2\cdot 2r^2}{n^2} \leq \varepsilon l^2$ edges of colour 0. Delete these edges and use Corollary 2.2.3 to cover all but $o(l)$ vertices of R with $(2r-1)r$ vertex-disjoint monochromatic matchings $M^1, \dots, M^{(2r-1)r}$ of spectrum $1, \dots, t$.

We finish by applying Łuczak's technique for blowing up matching to cycles [**79**]. This is done by using the following (by now well-known) lemma.

Lemma 2.2.10. *Let $t \geq 1$ and $\gamma > 0$ be fixed. Suppose R is the edge-coloured reduced graph of an edge-coloured graph H, for some γ-regular partition, such that each edge vw of R corresponds to a γ-regular pair of density at least $\sqrt{\gamma}$ in the colour of vw.*
If all but at most $\gamma|V(R)|$ vertices of R can be covered with t disjoint connected monochromatic matchings, then there is a set of at most t monochromatic disjoint cycles in H, which together cover all but at most $10\sqrt{\gamma}|V(H)|$ vertices of H.

For completeness, let us give an outline of the proof of Lemma 2.2.10.

Sketch of a proof of Lemma 2.2.10. We start by connecting in H the pairs corresponding to matching edges with monochromatic paths of the respective colour, following their connections in R. We do this in a cyclic manner, that is, if $v_{\mathrm{i}_1}v_{j_1}, \dots, v_{\mathrm{i}_s}v_{j_s}$ forms the matching, then we take paths $P_1, \dots, P_s$ in a way that P_ℓ connects V_{j_ℓ} and $V_{\mathrm{i}_{\ell+1}}$ (modulo ℓ). The end-vertex of each P_ℓ can be taken as a typical vertex of the graph $[V_{\mathrm{i}_\ell}, V_{j_\ell}]$ or $[V_{\mathrm{i}_{\ell+1}}, V_{j_{\ell+1}}]$ (this is important as we later have to 'fill up' the matching edges accordingly). We find the connecting paths simultaneously for all matchings.

Note that, as t is fixed, the paths chosen above together consume only a constant number of vertices of H. So we can we connect the monochromatic paths using the matching edges, blowing up the edges to long paths, where regularity and density ensure that we can fill up all but a small fraction of the corresponding pairs. This gives the desired cycles.

A more detailed explanation of this argument can be found in the proof of the main result of [**59**]. □

2.2.3 The absorbing method

In this subsection we prove Theorem 2.1.3. We apply a well known absorbing argument introduced in [**31**]. To this end we need a few tools.

Call a balanced bipartite subgraph H of a $2n$-vertex graph ε-*Hamiltonian*, if any balanced bipartite subgraph of H with at least $2(1-\varepsilon)n$ vertices is Hamiltonian. The next lemma is a combination of results from [**64, 83**] and can be found in [**72**] in the following explicit form.

Lemma 2.2.11. *For any $1 > \gamma > 0$, there is an $n_0 \in \mathbb{N}$ such that any balanced bipartite graph on $2n \geq 2n_0$ vertices and of edge density at least γ has a $\gamma/4$-Hamiltonian subgraph of size at least $\gamma^{3024/\gamma}n/3$.*

The following lemma is taken from [**25**].

Lemma 2.2.12. *Suppose that A and B are vertex sets with $|B| \leq |A|/r^{r+3}$ and the edges of the complete bipartite graph between A and B are r-locally coloured. Then all vertices of B can be covered with at most r^2 disjoint monochromatic cycles.*

Sketch of a proof of Theorem 2.1.3. Here we only prove part (b) of Theorem 2.1.3, since the proof of (a) is almost identical. The differences are discussed at the end of the section.

Let A and B be the two partition classes of the r-locally edge coloured $K_{n,n}$. We assume that $n \geq n_0$, where we specify n_0 later. Pick subsets $A_1 \subseteq A$ and $B_1 \subseteq B$ of size $\lceil n/2 \rceil$ each. Say red is the majority colour of $[A_1, B_1]$. Then by Lemma 2.2.6, there are at least $n^2/8r^2$ red edges in $[A_1, B_1]$.

Lemma 2.2.11 applied with $\gamma = 1/10r^2$ yields a red $\gamma/4$-Hamiltonian subgraph $[A_2, B_2]$ of $[A_1, B_1]$ with
$$|A_2| = |B_2| \geq \gamma^{3024/\gamma}|A_1|/3 \geq \gamma^{3024/\gamma} n/7.$$
Set $H := G - (A_2 \cup B_2)$, and note that each bipart of H has order at least $\lfloor n/2 \rfloor$. Let $\delta := \gamma^{4000/\gamma}$. Assuming n_0 is large enough, Lemma 2.2.4(b) provides $(2r-1)r$ monochromatic vertex-disjoint cycles covering all but at most $2\delta n$ vertices of H. Let $X_A \subseteq A$ (resp. $X_B \subseteq B$) be the set of uncovered vertices in A (resp. B). Since we may assume none of the monochromatic cycles is an isolated vertex, we have $|X_A| = |X_B| \leq \delta n$.

By the choice of δ, and since we assume n_0 to be sufficiently large, we can apply Lemma 2.2.12 to the bipartite graphs $[A_2, X_B]$ and $[B_2, X_A]$. This gives $2r^2$ vertex-disjoint monochromatic cycles that together cover $X_A \cup X_B$. Again, we assume none of these cycles is trivial. As $|X_A| = |X_B| \leq \delta n$, we know that the remainder of $[A_2, B_2]$ contains a red Hamilton cycle. Thus, in total, we found a cover of G with at most $(2r-1)r + 2r^2 + 1 \leq 4r^2$ vertex-disjoint monochromatic cycles.

As claimed above, the proof of Theorem 2.1.3(a) is very similar. The main difference is that instead of an ε-Hamiltonian subgraph we use a large red triangle cycle. A triangle cycle T_k consists of a cycle on k vertices $\{v_1, \ldots, v_k\}$ and k additional vertices $A = \{a_1, \ldots a_k\}$, where a_i is joined to v_i and v_{i+1} (modulo k). Note that T_k remains Hamiltonian after the deletion of any subset of vertices of A. We use some classic Ramsey theory to find a large monochromatic triangle cycle T_k in an r-locally coloured K_n, as shown in **[25]**. Next, Lemma 2.2.4(a) guarantees we can cover most vertices of $K_n \setminus T_k$ with $r(r+1)/2$ monochromatic cycles. We finish by absorbing the remaining vertices B into A with only one application of Lemma 2.2.12, thus producing r^2 additional cycles. As noted above, the remaining part of T_k is Hamiltonian and so we have partitioned K_n into $r(r+1)/2 + r^2 + 1 \leq 2r^2$ monochromatic cycles. □

2.3 Bipartite graphs with 2-local colourings

In this section we prove Theorem 2.1.5 and Theorem 2.1.6. We start by specifying the structure of 2-local colourings of $K_{n,n}$. Let G be any graph, and let the edges of G be coloured arbitrarily with colours in $\mathbb{N}$. We denote by C_i the subgraph of G induced by vertices that are adjacent to any edge of colour i. Note that C_i can contain edges of colours other than i. If for colours i, j the intersection $V(C_i) \cap V(C_j)$ is empty, we can merge i and j as we are only interested in monochromatic paths. We call an edge colouring *simple*, if $V(C_i) \cap V(C_j) \neq \emptyset$ for all colours i, j that appear on an edge.

In **[57]** it was shown that the number of colours in a simple 2-local colouring of K_n is bounded by 3. In the next lemma we will see that for $K_{n,n}$ the number of colours in a simple 2-local colouring is bounded by 4. For $r \geq 3$, however, simple r-local colourings can have an arbitrary large number of colours: take a $t \times t$ grid G and colour the edges of the column i and row i with colour i for $1 \leq i \leq t$. Then add edges of a new colour $t+1$ until G is complete (or complete bipartite) and observe that G is 3-locally edge coloured and simple, but the total number of colours is $t+1$.

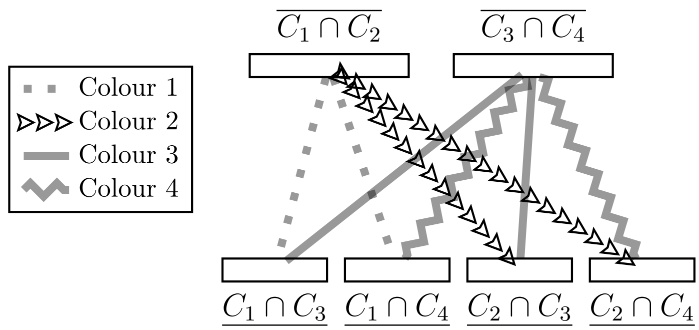

Figure 2.1: The four colour case of Lemma 2.3.1.

In what follows, we denote partition classes of a bipartite graph H (which we imagine as either top or bottom) by $\overline{H}$ and $\underline{H}$.

Lemma 2.3.1. *Let $K_{n,n}$ have a simple 2-local colouring. Then the total number of colours is at most four. In particular, if there are (edges of) colours 1,2,3 and 4, then*

- $\overline{K_{n,n}} = \overline{C_1 \cap C_2} \cup \overline{C_3 \cap C_4}$ *and*
- $\underline{K_{n,n}} = \underline{C_1 \cap C_3} \cup \underline{C_1 \cap C_4} \cup \underline{C_2 \cap C_3} \cup \underline{C_2 \cap C_4}$

as shown in Figure 2.1 (modulo swapping colours and swapping $\overline{K_{n,n}}$ with $\underline{K_{n,n}}$).

Proof of Lemma 2.3.1. We can assume there are at least four colours in total, as otherwise there is nothing to show. We start by observing that for any four distinct colours i, j, k, ℓ, if $v \in V(C_\mathrm{i} \cap C_j)$ and $w \in V(C_k \cap C_\ell)$, then, by 2-locality, v and w cannot lie in opposite classes of $K_{n,n}$. Thus either $V(C_\mathrm{i} \cap C_j) \cup V(C_k \cap C_\ell) \subseteq \underline{K_{n,n}}$ or $V(C_\mathrm{i} \cap C_j) \cup V(C_k \cap C_\ell) \subseteq \overline{K_{n,n}}$. Fixing four colours $1, 2, 3, 4$, and considering their six (by simplicity non-empty) intersections, the pigeon-hole principle gives that (after possibly swapping colours and/or top and bottom class of $K_{n,n}$),

$$V(C_1 \cap C_3) \cup V(C_2 \cap C_4) \cup V(C_1 \cap C_4) \cup V(C_2 \cap C_3) \subseteq \underline{K_{n,n}}. \tag{2.3.1}$$

As every colour must see both top and bottom of $K_{n,n}$, we have that $V(C_1 \cap C_2) \cup V(C_3 \cap C_4) \subseteq \overline{K_{n,n}}$. By 2-locality there are no other colours. □

2.3.1 Partitioning into paths

In this subsection we prove Theorem 2.1.5. For the sake of contradiction, assume that $K_{n,n}$ is 2-locally edge-coloured such that there is no partition into three monochromatic paths. Since we are not interested in the actual colours of the path we can assume the colouring to be simple. Furthermore Theorem 2.1.1 implies that there are at least three colours.

A path is *even* if it has an even number of vertices.

Claim 2.3.2. *There is no even monochromatic path P such that $\overline{K_{n,n} \setminus P}$ is contained in $\overline{C_\mathrm{i} \cap C_j}$ for distinct colours i, j.*

Proof. Suppose the contrary and let P be as described in the claim and of maximum length. Since the colouring is 2-local and $\overline{K_{n,n} \setminus P} \subseteq \overline{C_\mathrm{i} \cap C_j}$, the graph on $K_{n,n} \setminus P$ is 2-coloured. Using Theorem 2.1.1, we are done unless the colouring on $K_{n,n} \setminus P$ is split.

In that case, let p be the endpoint of P in $\underline{K_{n,n}}$. Since $\overline{K_{n,n} \setminus P} \subseteq \overline{C_\mathrm{i} \cap C_j}$, the edges between p and $\overline{K_{n,n} \setminus P}$ have colours i or j. So P has colour $k \notin \{\mathrm{i}, j\}$, as otherwise we could use the splitness of $K_{n,n} \setminus P$ to extend P with two extra vertices. But then, p can only see one more colour apart from k, so we may assume that all the edges between p and $\overline{K_{n,n} \setminus P}$ have colour i. Now cover $K_{n,n} \setminus P$ by two paths P_1 and P_2 of the colour i and one path of the colour j. The paths P_1 and P_2 can be joined using the vertex p to give the three required paths. □

Now the case of four colours of Lemma 2.3.1 is easily solved: without loss of generality suppose that $|\overline{C_1 \cap C_2}| \leq n/2$. By symmetry between colours 1 and 2 we can assume that $|\overline{C_2}| \leq |\underline{C_2}|$. So there exists an even colour 2 path P covering $\overline{C_2} = \overline{C_1 \cap C_2}$ and we are done by Claim 2.3.2. This proves the following claim.

Claim 2.3.3. *The total number of colours is three.*

Our next aim is to show that the colouring looks like in Figure 2.2, that is, that every vertex sees two colours. For this, we need the next claim and the following definition. We say that a subgraph of $H \subseteq K_{n,n}$ is connected in colour i, if every two vertices of H are connected by a path of colour i in H.

Claim 2.3.4. *There is no even monochromatic path P such that $K_{n,n} \setminus P$ is connected in some colour* i.

Proof. Assume the opposite and let P be as described in the claim. Simplify the colouring of $K_{n,n} \setminus V(P)$ to a 2-colouring by merging all colours distinct from i. (Note that since all vertices see i, by 2-locality no vertex can see more than one of the merged colours.) The new colouring is not a split colouring by the assumption on i. Hence Theorem 2.1.1 applies, and we are done. □

Claim 2.3.5. *Each vertex sees two colours.*

Proof. Suppose that there is a vertex in $\overline{K_{n,n}}$ that sees only colour, 1 say. Then by 2-locality $\underline{C_2 \cap C_3} = \emptyset$. Since the colouring is simple we know that $\overline{C_2 \cap C_3} \neq \emptyset$. Therefore $\underline{K_{n,n}} \subseteq \underline{(C_1 \cap C_2) \cup (C_1 \cap C_3)}$. If $|\overline{C_2 \cap C_3}| > |\underline{C_1 \cap C_3}|$, we can choose an even path of colour 3 that contains all vertices of $\underline{C_1 \cap C_3}$ and apply Claim 2.3.2. Otherwise, let P be an even path of colour 3 between $|\overline{C_2 \cap C_3}|$ and $|\underline{C_1 \cap C_3}|$ that covers all vertices of $\overline{C_2 \cap C_3}$. Since all remaining vertices lie in C_1, the subgraph $K_{n,n} \setminus P$ is connected in colour 1 and we are done by Claim 2.3.4. □

Claims 2.3.3 and 2.3.5 ensure that for the rest of the proof we can assume that the colouring is exactly as shown in Figure 2.2 (with some of the sets possibly being empty). Now, let us see how Claim 2.3.2 implies that we easily find the three paths if one of the C_i is complete bipartite in colour i.

Claim 2.3.6. *For* $\mathrm{i} \in \{1, 2, 3\}$, *the graph* C_i *is not complete bipartite in colour* i.

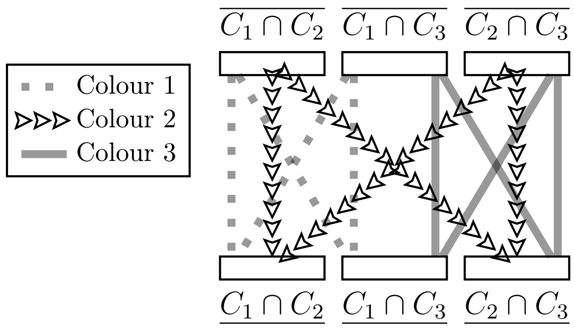

Figure 2.2: There are three colours and each vertex sees exactly two colours.

Proof. Suppose the contrary and let C_{i} contain only edges of colour i. Take out a longest even path of colour i in C_{i}. This leaves us either with only $\underline{C_j \cap C_k}$ in the bottom partition class, or with only $\overline{C_j \cap C_k}$ in the top partition class (where j and k are the other two colours). We may thus finish by applying Claim 2.3.2, after possibly switching top and bottom parts. $\square$

Claim 2.3.7. *For* $\mathrm{i} \in \{1,2,3\}$*, the graph* C_{i} *is connected in colour* i.

Proof. For contradiction, suppose that C_3 is not connected in colour 3 (the other colours are symmetric). Then there are two edges e, f of colour 3 belonging to C_3 that are not joined by a path of colour 3. First assume we can choose e in $E(C_2 \cap C_3)$. Since all edges between $C_1 \cap C_3$ and $C_2 \cap C_3$ have colour 3, we get $f \in E(C_2 \cap C_3)$, and $C_1 \cap C_3$ has no vertices. But this contradicts our assumption that the colouring is simple. Therefore, $C_2 \cap C_3$ and, by symmetry, $C_1 \cap C_3$ contain no edges of colour 3.

By symmetry (between the top and bottom partition) we can assume that $|\overline{C_1 \cap C_3}| \geq |\underline{C_1 \cap C_3}|$. Further, we have $|\overline{C_1 \cap C_3}| < |\underline{C_1 \cap C_2}| + |\underline{C_1 \cap C_3}|$, since otherwise we could find an even path of colour 1 that covers all of $\underline{C_1 \cap C_2} \cup \underline{C_1 \cap C_3}$ and use Claim 2.3.2. So we can choose an even path P of colour 1, alternating between $\overline{C_1 \cap C_3}$ and $\underline{C_1 \cap C_2} \cup \underline{C_1 \cap C_3}$, that contains both $\overline{C_1 \cap C_3}$ and $\underline{C_1 \cap C_3}$. Thus $K_{n,n} \setminus P$ is connected in colour 2 and Claim 2.3.4 applies. $\square$

Let us now show that for pairwise distinct $\mathrm{i}, j, k \in \{1,2,3\}$ we have

$$\text{at least one of } \underline{C_{\mathrm{i}} \cap C_j}, \overline{C_{\mathrm{i}} \cap C_k} \text{ is not empty.} \tag{2.3.2}$$

To see this, note that the edges between $\overline{C_{\mathrm{i}} \cap C_j}$ and $\underline{C_{\mathrm{i}} \cap C_k}$ are of colour i. Thus if (2.3.2) does not hold, we can find a colour i (possibly trivial) path P that covers one of these two sets. Hence either in the top or in the bottom part of $K_{n,n}$, the path P covers all but $C_j \cap C_k$. We can thus finish with Claim 2.3.2.

Together with the fact that every colour must see both top and bottom class, (2.3.2) immediately implies that for pairwise distinct $\mathrm{i}, j, k \in \{1,2,3\}$ we have

$$\text{at least one of } C_{\mathrm{i}} \cap C_j, C_{\mathrm{i}} \cap C_k \text{ meets both } \underline{K_{n,n}} \text{ and } \overline{K_{n,n}}. \tag{2.3.3}$$

So, of the three bipartite graphs $C_{\mathrm{i}} \cap C_j$, two have non-empty tops and bottoms. Hence, after possibly swapping colours, we know that the four sets $\underline{C_1 \cap C_{\mathrm{i}}}$, $\overline{C_1 \cap C_{\mathrm{i}}}$, $\mathrm{i} = 2, 3$, are non-empty. Observe that after possibly swapping colours 2 and 3, and/or switching partition classes of $K_{n,n}$, we have one of the following situations:

(i) $|\overline{C_1 \cap C_2}| \geq |\underline{C_1 \cap C_3}|$ and $|\underline{C_1 \cap C_2}| \geq |\overline{C_1 \cap C_3}|$, or

(ii) $|\overline{C_1 \cap C_2}| \geq |\underline{C_1 \cap C_3}|$ and $|\underline{C_1 \cap C_2}| \leq |\overline{C_1 \cap C_3}|$.

In either of these situations, note that as all involved sets are non-empty, by Claim 2.3.7 there is an edge e_1 of colour 1 in $E(C_1 \cap C_2) \cup E(C_1 \cap C_3)$. So if we are in situation (ii), we can find an even path of colour 1 covering all of $\underline{C_1 \cap C_3} \cup \underline{C_1 \cap C_2}$. Now Claim 2.3.2 applies, and we are done. So assume from now on we are in situation (i).

Similarly as above, by (2.3.3), there is an edge e_2 of colour 2 in $E(C_3 \cap C_2) \cup E(C_1 \cap C_2)$. By Claim 2.3.6, C_3 is not complete bipartite in colour 3. So we can assume that at least one of e_1 or e_2 is chosen in C_3 and hence the two edges are not incident.

Extend e_1 to an even colour 1 path P covering all of $\underline{C_1 \cap C_3}$, using (apart from e_1) only edges from $[\underline{C_1 \cap C_3}, \overline{C_1 \cap C_2}]$ and from $[\underline{C_1 \cap C_2}, \overline{C_1 \cap C_3}]$, while avoiding the endvertices of e_2, if possible. If we had to use one of the endvertices of e_2 in P, then P either covers all of $\underline{C_1 \cap C_2}$ or all of $\overline{C_1 \cap C_2}$. In either case we may apply Claim 2.3.2, and are done. On the other hand, if we could avoid both endvertices of e_2 for P, then Claim 2.3.4 applies and we are done. This finishes the proof of Theorem 2.1.5.

2.3.2 Finding long paths

In this subsection we prove Theorem 2.1.6. We will use the following theorem, which resolves the problem for the case of of 2-colourings.

Theorem 2.3.8 ([**35, 55**])**.** *Every 2-edge-coloured $K_{p+q-1,p+q-1}$ contains a colour 1 path of length $2p$ or a colour 2 path of length $2q$.*

As in the last section, C_i denotes the subgraph induced by the vertices that have an edge of colour i. Recall that the length of a path is the number of its vertices.

Lemma 2.3.9. *Let $K_{2m-1,2m-1}$ be 2-locally coloured with colours $1, 2, 3$. Then for distinct colours i, j there is a monochromatic path of length at least*

$$\min\{2m, 2\max(|\overline{C_\mathrm{i} \cap C_j}|, |\underline{C_\mathrm{i} \cap C_j}|)\}.$$

Proof. By symmetry, we can assume that $|\overline{C_\mathrm{i} \cap C_j}| \geq |\underline{C_\mathrm{i} \cap C_j}|$. Moreover, we can assume that $\overline{C_\mathrm{i} \cap C_j} \neq \emptyset$, as otherwise there is nothing to prove. Then by 2-locality,

$$\underline{C_k \setminus (C_\mathrm{i} \cup C_j)} = \emptyset, \tag{2.3.4}$$

where k denotes the third colour.

We apply Theorem 2.3.8 to a balanced subgraph of $C_\mathrm{i} \cap C_j$ with $p = m - |\underline{C_\mathrm{i} \setminus C_j}|$ and $q = m - |\underline{C_j \setminus C_\mathrm{i}}|$. For this, note that we have

$$p + q - 1 = 2m - 1 - |\underline{C_\mathrm{i} \setminus C_j}| - |\underline{C_j \setminus C_\mathrm{i}}| \overset{(2.3.4)}{=} |\underline{C_\mathrm{i} \cap C_j}| \leq |\overline{C_\mathrm{i} \cap C_j}|.$$

By symmetry between i and j we can assume that the outcome of Theorem 2.3.8 is a colour i path P of length $2(m - |\underline{C_\mathrm{i} \setminus C_j}|)$. Let $R \subseteq [\overline{C_\mathrm{i} \cap C_j} \setminus \overline{P}, \underline{C_\mathrm{i} \setminus C_j}]$ be a path of colour i and length

$$r = \min(2|\overline{C_\mathrm{i} \cap C_j} \setminus \overline{P}|, 2|\underline{C_\mathrm{i} \setminus C_j}|).$$

If $r = 2|\underline{C_\mathrm{i} \setminus C_j}|$, then we can join P and R to a path of length of $2m$. Otherwise $r = 2|\overline{C_\mathrm{i} \cap C_j} \setminus \overline{P}|$ and we can join P and R to a path of length of $2|\overline{C_\mathrm{i} \cap C_j}|$. □

Now let us prove Theorem 2.1.6 by contradiction. To this end, assume that $K_{2m-1,2m-1}$ is coloured 2-locally and has no monochromatic path on $2m$ vertices. Since we are not interested in the actual colours of the path we can assume the colouring to be simple, as in the previous subsection. Furthermore Theorem 2.3.8 implies that there are at least three colours.

We now apply Lemma 2.3.1. The four colour case of Lemma 2.3.1 is quickly resolved: Without loss of generality suppose that $|\overline{C_1 \cap C_2}| \geq m$. By symmetry between colours 1 and 2, we can assume that $|\underline{C_1 \cap C_3} \cup \underline{C_1 \cap C_4}| \geq m$. Thus we easily find a colour 1 path of length $2m$ alternating between these sets. This proves:

Claim 2.3.10. *The total number of colours is three.*

We can now exclude vertices that see only one colour.

Claim 2.3.11. *Each vertex sees two colours.*

Proof. Suppose that there is a vertex in $\overline{K_{2m-1,2m-1}}$ that sees only colour 1, say. Then by 2-locality, $\underline{C_2 \cap C_3} = \emptyset$. Since the colouring is simple we know that $\overline{C_2 \cap C_3} \neq \emptyset$. Therefore $\underline{K_{2m-1,2m-1}} \subseteq \underline{(C_1 \cap C_2) \cup (C_1 \cap C_3)}$. Since one of $\underline{C_1 \cap C_2}$ and $\underline{C_1 \cap C_3}$ must have size at least m, we are done by Lemma 2.3.9. $\square$

Put together, Claims 2.3.10 and 2.3.11 allow us to assume that the colouring is as shown in Figure 2.2. The next claim follows instantly from Lemma 2.3.9.

Claim 2.3.12. *For distinct colours* i, j *we have* $\max(|\overline{C_\mathrm{i} \cap C_j}|, |\underline{C_\mathrm{i} \cap C_j}|) < m$.

As the three top parts sum up to $2m-1$, and so do the three bottom parts, we immediately get:

Claim 2.3.13. $\underline{C_\mathrm{i} \cap C_j}, \overline{C_\mathrm{i} \cap C_j} \neq \emptyset$ *for all distinct* $\mathrm{i}, j \in \{1, 2, 3\}$.

The next claim requires some more work. Recall that a subgraph of $H \subseteq K_{n,n}$ is connected in colour i, if every two vertices of H are connected by a path of colour i in H.

Claim 2.3.14. *If the subgraph* C_i *is connected in colour* i*, then there are distinct* $j, k \in \{1, 2, 3\} \setminus \{\mathrm{i}\}$ *such that* $|\overline{C_\mathrm{i} \cap C_j}| \geq |\underline{C_\mathrm{i} \cap C_k}|$, $|\underline{C_\mathrm{i} \cap C_j}| > |\overline{C_\mathrm{i} \cap C_k}|$ *(modulo swapping top and bottom partition classes) and* $|V(C_\mathrm{i} \cap C_k)| < m$.

Proof. Suppose that C_i is connected in colour i and let $j, k \in \{1, 2, 3\} \setminus \{\mathrm{i}\}$ be such that $|\overline{C_\mathrm{i} \cap C_j}| \geq |\underline{C_\mathrm{i} \cap C_k}|$ (after possible swapping top and bottom partition). By Claim 2.3.13, and as C_i is connected in colour i, we find an edge $\mathrm{e_i} \in E(C_\mathrm{i} \cap C_j) \cup E(C_\mathrm{i} \cap C_k)$ of colour i. Choose an even path $P \subseteq [\overline{C_\mathrm{i} \cap C_j}, \underline{C_\mathrm{i} \cap C_k}]$ which covers $\underline{C_\mathrm{i} \cap C_k}$ and ends in one of the vertices of $\mathrm{e_i}$.

For the first part of the claim, assume to the contrary that $|\underline{C_\mathrm{i} \cap C_j}| \leq |\overline{C_\mathrm{i} \cap C_k}|$. Take an even path $P' \subseteq [\underline{C_\mathrm{i} \cap C_j}, \overline{C_\mathrm{i} \cap C_k}]$ which covers $\underline{C_\mathrm{i} \cap C_j}$ and ends in a vertex of $\mathrm{e_i}$. Since P and P' are joined by $\mathrm{e_i}$ we infer that $|\underline{C_\mathrm{i} \cap C_k}| + |\underline{C_\mathrm{i} \cap C_j}| < m$. But then $|\underline{C_j \cap C_k}| \geq m$ in contradiction to Claim 2.3.12. This shows that $|\underline{C_\mathrm{i} \cap C_j}| > |\overline{C_\mathrm{i} \cap C_k}|$, as desired.

This allows us to pick an even path $P'' \subseteq [\underline{C_\mathrm{i} \cap C_j}, \overline{C_\mathrm{i} \cap C_k}]$ of colour i, which covers $\overline{C_\mathrm{i} \cap C_k}$ and ends in one of the vertices of $\mathrm{e_i}$. Join P and P'' via $\mathrm{e_i}$ to obtain a colour i path of length at least $2|\overline{C_\mathrm{i} \cap C_k}| + 2|\underline{C_\mathrm{i} \cap C_k}| = 2|V(C_\mathrm{i} \cap C_k)|$. So by our assumption that there is no monochromatic path of length $2m$, we obtain $|V(C_\mathrm{i} \cap C_k)| < m$, as desired. $\square$

Claim 2.3.15. *For at most one pair of distinct indices* $\mathrm{i}, j \in \{1,2,3\}$ *it holds that* $|V(C_\mathrm{i} \cap C_j)| < m$.

Proof. Suppose, on the contrary, that $C_1 \cap C_2$ and $C_1 \cap C_3$ each have less than m vertices. Then $C_2 \cap C_3$ has at least $2m$ vertices. Therefore one of its partition classes has size at least m, a contradiction to Claim 2.3.12. □

We are now ready for the last step of the proof of Theorem 2.1.6. We start by observing that if for some $\mathrm{i} \in \{1,2,3\}$, the subgraph C_i is not connected in colour i, then (letting j, k be the other two indices) the edges of the graphs $C_\mathrm{i} \cap C_j$ and $C_\mathrm{i} \cap C_k$ are all of colour j, or colour k, respectively, and thus both C_j and C_k are connected in colour j, or colour k, respectively. So we can assume that there are at least two distinct indices $j, k \in \{1,2,3\}$, such that the subgraphs C_j, C_k are connected in colour j, or in colour k, respectively. Say these indices are 1 and 3.

We use Claim 2.3.14 twice: For C_1 it yields that one of $C_1 \cap C_3$ and $C_1 \cap C_2$ has less than m vertices. For C_3 it yields that one of $C_1 \cap C_3$ and $C_2 \cap C_3$ has less than m vertices. So by Claim 2.3.15 we get that necessarily,

$$|V(C_1 \cap C_3)| < m, \ |V(C_1 \cap C_2)| \geq m, \ |V(C_2 \cap C_3)| \geq m. \tag{2.3.5}$$

Again using Claim 2.3.14, this implies that C_2 is not connected in colour 2. So by Claim 2.3.13 and the fact that the edges between $C_1 \cap C_2$ and $C_2 \cap C_3$ are complete bipartite in colour 2, we have that

$$C_1 \cap C_2 \text{ is complete bipartite in colour 1.} \tag{2.3.6}$$

Also, in light of (2.3.5), Claim 2.3.14 with input $\mathrm{i} = 1$ gives $j = 2$ and $k = 3$ and thus $|\overline{C_1 \cap C_2}| \geq |\underline{C_1 \cap C_3}|$, $|\underline{C_1 \cap C_2}| > |\overline{C_1 \cap C_3}|$ (after possibly swapping top and bottom partition). Choose two balanced paths of colour 1: The first path $P \subseteq [\overline{C_1 \cap C_2}, \underline{C_1 \cap C_3}]$ such that it covers $\underline{C_1 \cap C_3}$. The second path $P' \subseteq [\underline{C_1 \cap C_2}, \overline{C_1 \cap C_3}]$ such that it covers $\overline{C_1 \cap C_3}$. As by (2.3.6) we know that $C_1 \cap C_2$ is complete bipartite in colour 1, we can join P and P' with a path of colour 1 in $C_1 \cap C_2$, such that the resulting path P'' covers one of $\overline{C_1}$, $\underline{C_1}$. Since by assumption, P'' has less than $2m$ vertices, we obtain that $\overline{C_2 \cap C_3}$ or $\underline{C_2 \cap C_3}$ has size at least m, a contradiction to Claim 2.3.12. This finishes the proof of Theorem 2.1.6.

Chapter 3

Partitioning a red and blue edge coloured graph of minimum degree $2n/3 + o(n)$ into three monochromatic cycles

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Abstract

It is proved that any edge colouring in red and blue of a graph on n vertices and of minimum degree $2n/3 + o(n)$ admits a partition into three monochromatic cycles.

3.1 Introduction

In the late 90's Łuczak, Rödl and Szemerédi confirmed a conjecture of Lehel and proved that every edge colouring of K_n in red and blue admits a partition into a red and a blue cycle, provided that n is large enough [80]. This was later generalized to smaller and finally all n by Allen [1] and Bessy and Thomassé [13]. (Note that here we count edges, single vertices and the empty set as cycles as well, to omit some trivial cases.) Motivated by ideas of Schelp, Balogh et al. asked if Lehel's conjecture stays true for graphs of bounded minimum degree [10]. They conjectured the following: given any graph G on n vertices and of maximum degree $3n/4$, for any colouring of the edges in red and blue, there are a red and a blue cycle which together partition the vertices of G. Note that there are graphs of minimum degree $3n/4 - 1$ that do not admit such a partition. In support of their conjecture, Balogh et al. proved an approximate version [10]. They showed that, for every β there is an n_0 such that for any graph G on $n \geq n_0$ vertices and with minimum degree at least $(3/4 + \beta)n$, any colouring of the edges of G in red and blue admits disjoint red and blue cycles which together cover all but βn vertices. DeBiasio and Nelsen were able to improve on this by obtaining a proper partition into a red and a blue cycle under the same degree conditions [27]. Finally, Letzter proved the full conjecture for all sufficiently large n [74].[1] Based on these advances Pokrovskiy conjectured that similar results are true for graphs of even lower minimum degree.

[1]In fact the conjecture is wrong for small graphs. There is a counterexample on 19 vertices [27].

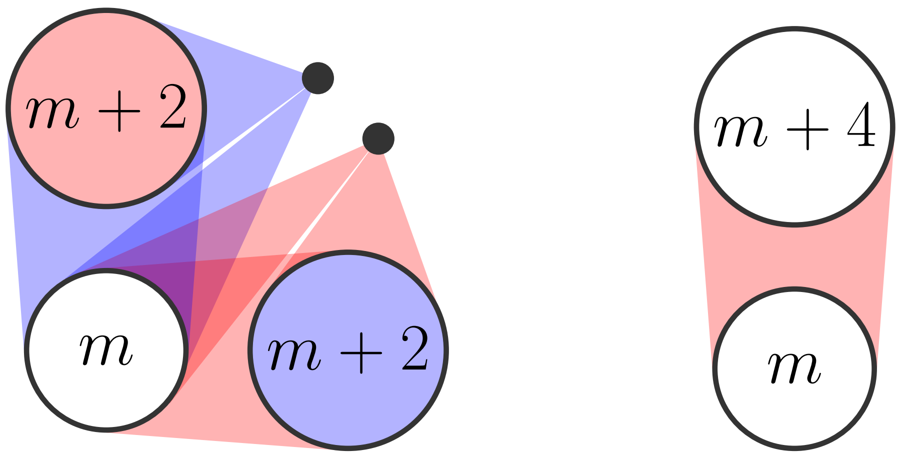

Figure 3.1: Graphs without partitions into three (four) monochromatic cycles.

In particular, he conjectured that red and blue edge coloured graphs of minimum degree $2n/3$ ($n/2$) can be partitioned into 3 (4) monochromatic cycles [85]. There are examples which show that these numbers are essentially tight (see Figure 3.1). Here we verify the first part of Pokrovskiy's conjecture approximately.

Theorem 3.1.1 (Main result). *For each $\beta > 0$, there is an n_0 such that the following holds. Let G be a graph on $n \geq n_0$ vertices and minimum degree at least $(2/3 + \beta)n$. Then any colouring of the edges of G in red and blue admits a partition of $V(G)$ into three monochromatic cycles.*

Let us briefly summarize the approaches that have been taken in this field so far. To prove Lehel's conjecture the authors of [80] used an approach involving the Regularity Lemma. In the following we will assume that the reader is familiar with the regularity method. Definitions and details about regularity are provided in the next section. Suppose that the edges of $G = K_n$ are coloured in red and blue. Given a regular partition of G, we can colour the edges of the reduced graph $\mathcal{G}$ by majority, i.e. by colouring an edge red if most of the edges in the respective regular pair of G are red and blue otherwise. Fix a red (blue) connected component $\mathcal{R}$ ($\mathcal{B}$) in the subgraph of $\mathcal{G}$ induced by the red (blue) edges. By an argument of Łuczak a matching in $\mathcal{R} + \mathcal{B}$ of size $c|V(\mathcal{G})|$ corresponds to (disjoint) red and blue cycles in G, which together cover approximately $2cn$ vertices [79]. This is, roughly speaking, achieved by connecting the clusters of the matching edges by short monochromatic paths and then selecting almost spanning paths between the clusters of the matching edges. In [80] it was shown by elemental arguments that the subgraphs of both red and blue edges of $\mathcal{G}$ are connected (or otherwise we are done). Given that $\mathcal{R}$ and $\mathcal{B}$ are the only components, it is fairly easy to find a perfect matching in $\mathcal{R} + \mathcal{B}$, which leads to a red and a blue cycle covering all but $o(n)$ vertices. To obtain a proper cycle partition, the Łuczak et al. then include the remaining uncovered vertices into these two cycles by a case analysis.

The same approach is taken in [10] for graphs of minimum degree $(3/4 + \beta)n$. Due to Łuczak's argument, the problem of finding two monochromatic cycles, which cover all but βn vertices reduces to finding two monochromatic connected components, which together admit a perfect matching.

To obtain a proper partition under the same degree conditions, DeBiasio and Nelsen did the following. Instead of absorbing the vertices manually as in [80], they identified certain subgraphs before applying the Regularity Lemma, which later allow to absorb the leftover vertices automatically. Note that this was necessary, because the analysis of graphs with minimum degree $(3/4 + \beta)n$ is considerably more complex compared to complete graphs. Let us give some more details. A red subgraph $R \subseteq G$ is robust if any two vertices are connected by a linear number of constant size paths. It is shown in [27] that R admits a (red) path P_R^{Abs} with the following property. For any set $W \subseteq V(R)$ of sublinear size, there is a path P_R^W with vertex set $V(P_R^W) = V(P_R^{\mathrm{Abs}}) \cup W$ and which has the same ends as P_R^{Abs}. Given these definitions the approach of [27] goes as follows. Let G be a graph on n vertices and of minimum degree at least $3n/4 + o(n)$ and suppose its edges are coloured in red and blue. Using elemental arguments we cover G with red and a blue robust subgraphs R and B. We pick absorbing paths P_R^{Abs} and P_B^{Abs} and set their vertex sets aside. Next we apply the Regularity Lemma to $G - P_R^{\mathrm{Abs}} - P_B^{\mathrm{Abs}}$ to obtain a regular partition and an (edge coloured) reduced graph $\mathcal{G}$. By a lemma of [27] the robust subgraph R (B) corresponds to a connected component $\mathcal{R}$ ($\mathcal{B}$) in $\mathcal{G}$. More precisely, the vertices of R (B) are contained in the clusters of the vertices of $\mathcal{R}$ ($\mathcal{B}$). We then have to show that $\mathcal{R} + \mathcal{B}$ contains a perfect matching $\mathcal{M}$. (Note that we can not change the red and blue components at this point.) Once this is settled, we can, as before, use Łuczak's argument to construct a red and a blue cycle in G, which are disjoint and together cover all but $o(n)$ vertices of G. By the correspondence between R and $\mathcal{R}$ (B and $\mathcal{B}$) we can furthermore assume that P_R^{Abs} (P_B^{Abs}) is included as segment on the red (blue) cycle. We then finish by absorbing the remaining vertices into P_R^{Abs} and P_B^{Abs}, which is possible because $R + B$ covers G.

To obtain the same partition for graphs of minimum degree (exactly) $3n/4$ Letzter followed the above approach and developed it further [74]. For instance she proved that connected components in a reduced graph correspond to robust components in the original graph. Thus the argument of covering G with robust monochromatic subgraphs reduces to covering (the reduced graph) with monochromatic connected components, which is less technical. Note that we now use two applications of the Regularity Lemma. One for the robust components and one (with smaller input) for Łuczak's argument. The proof of [74] is fairly involved, due to the analysis of the many extremal cases that have to be considered when dealing with graphs of minimum degree $3n/4$.

Although the method of absorbing paths has been applied successfully, it is not without shortcomings. For instance, it is inconvenient that the choice of the monochromatic components (i.e. the robust subgraphs) and the selection of the matchings is separated by an application of the regularity lemma. This adds further technicalities to the arguments, in particular when it comes to the analysis of extremal cases. Another difficulty arises when one of the robust subgraphs, R say, is close to being bipartite, i.e. when there are disjoint sets $V(R) = X \cup Y$ such that X and Y contain only few red edges. In this case the path P_R^{Abs} can only absorb sets W with $|W \cap X| = |W \cap Y|$. This situation can be handled, if a $V(R)$ has a large intersection with $V(B)$ and B is far from being bipartite, but leads to further technical and repetitive discussions.

Here we implement a new approach, which was developed to solve cycle partitioning problems in hypergraphs by Garbe, Lang, Lo, Mycroft and Sanhueza [44]. Suppose that G is a graph on n vertices, of minimum degree at least $2n/3+o(n)$ and its edges are coloured in red and blue. We start by applying the Regularity Lemma to obtain a regular partition and the correspond reduced graph. Roughly speaking, our strategy is to find three monochromatic components in the reduced graph, which admit a robust perfect matching. This is possible unless the colouring takes an extremal configuration, in which case we have to fall back to ad hoc arguments. Following the argument of Łuczak, we then connect the clusters of the matching edges by short monochromatic paths. Although this is not trivial, we can assume that the exceptional vertices, i.e. vertices which do not behave regularly enough, are contained on these short paths. This will create some imbalances between the clusters of the matching edges. We restore the balance between the matching edges up to a constant by adding some more vertices on the above mentioned short paths. To determine which path receives how many vertices, we solve a weighted matching problem in an auxiliary graph, which is feasible by the robustness of the perfect matching. We then perform another more subtle balancing step, which leaves the clusters of all matching edges equally sized. To finish we apply the Blow Up Lemma to find monochromatic spanning paths in each of the matching edges. Together with the short paths this yields the desired cycle partition.

This approach comes with the advantages that we can isolate the main lemmas (finding the components, distributing exceptional vertices and solving extremal configurations) cleanly in the big picture and there is little repetition in our arguments. Moreover, although it is an involved argument, the implementation is not too technical and requires only the Regularity and Blow Up Lemma.

The rest of the paper is organized as follows. In the next section we introduce some notation and concepts related to the regularity method. In Section 3.3 we present the proof of Theorem 3.1.1. Sections 3.4, 3.5 and 3.6 are dedicated to the Lemmas concerning finding the components, distributing the exceptional vertices and solving the cases of extremal colourings, respectively.

3.2 Preliminaries

In this section we introduce some notation and tools, which we will need for the proof of Theorem 3.1.1.

3.2.1 Notation

Let $G = (V, E)$ be graph. The *order* of G is $|V(G)|$ and the *size* of G is $|E(G)|$. We denote the *neighbourhood* of a vertex v by $N_G(v)$ and $N_G(v, W) = N_G(v) \cap W$ for a set of vertices $W \subseteq V(G)$. We denote the degree of v by $\deg_G(v) = |N_G(v)|$ and $\deg_G(v, W) = |N_G(v) \cap W|$. For a set of vertices $S \subseteq G$ we write $N_G(S) = \bigcup_{s \in S} N(s)$. If it is clear from the context, we often drop the index G.

For another graph H we denote by $G + H$ the graph on vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

Suppose that G is red and blue edge coloured graph. We denote by G_{red} (G_{blue}) the subgraph on $V(G)$ that contains the red (blue) edges. A red (blue) component of G is a connected component of G_{red} (G_{blue}). In particular a vertex v with $\deg_{\text{red}}(v) = 0$ is in a red component of order 1. We will sometimes refer to red (blue) as colour 1 (2) and write $G_1 = G_{\text{red}}$ ($G_2 = G_{\text{blue}}$).

A v-w-path (walk) is a path (walk) that starts in v and ends in w. If we treat an edge, single vertex and the empty set as cycle, we will explicitly mention this.

3.2.2 Regularity

Given a graph G and disjoint vertex sets $V, W \subseteq V(G)$ we denote the number of edges between V and W by $\mathrm{e}(V,W)$ and the density of (V,W) by $\mathrm{d}(V,W) = \mathrm{e}(V,W)/(|V||W|)$. The pair (V,W) is called ε-regular, if $|V| = |W|$ and for all subsets $X \subseteq V$, $Y \subseteq W$ with $|X| \geq \varepsilon|V|$ and $|Y| \geq \varepsilon|W|$ it follows that $|\mathrm{d}(V,W) - \mathrm{d}(X,Y)| \leq \varepsilon$. We say that a vertex $v \in V$ has *typical degree* in (V,W), if $\deg(v,W) \geq (\mathrm{d}(V,W) - \varepsilon)|W|$. It follows directly from the definition of ε-regularity that

$$\text{all but at most } \varepsilon|V| \text{ vertices in } V \text{ have typical degree in } (V,W). \tag{3.2.1}$$

The next lemma allows us to find (spanning) paths in regular pairs. It is a corollary of the mighty Blow Up Lemma [70], but can also be proved independently with not too much effort.

Lemma 3.2.1. *(Large paths in regular pairs) Let n be an integer and let ε, d be numbers with $0 < 1/n \ll \varepsilon \ll \mathrm{d} < 1$. Suppose that (V_1, V_2) is an ε-regular pair of density $\mathrm{d} = \mathrm{d}(V_1, V_2)$ and with $|V_1| = |V_2| = n$. For $\mathrm{i} = 1, 2$ let $W_\mathrm{i} \subseteq V_\mathrm{i}$ be a vertex set of size at most $|V_\mathrm{i}|/2$ and which contains all vertices of V_i that do not have typical degree in (V_1, V_2).*

Then for any two vertices $v_\mathrm{i} \in V_\mathrm{i} \setminus W_\mathrm{i}$, where $\mathrm{i} \in [2]$, and any even integer $4 \leq k \leq \min(|V_1 \setminus W_1|, |V_2 \setminus W_2|)$ there is a v_1-v_2-path of order k alternating between V_1 and V_2.

Szemerédi's Regularity Lemma allows to partition the vertex set of a graph into clusters of vertices, in a way that most pairs of clusters are regular [102]. We will use the regularity lemma in its degree form and with 2 colours (see [71]).

Lemma 3.2.2 (Regularity Lemma). *For every $\varepsilon > 0$ and integer m_0 there exists $M = M(\varepsilon, m_0)$ such that the following holds. Let G be a graph on n vertices whose edges are coloured in red and blue and let $\mathrm{d} > 0$. Then there exists a partition $\{V_0, \ldots, V_m\}$ of $V(G)$ and a subgraph G' of G with vertex set $V(G) \setminus V_0$ such that the following holds:*

(a) $m_0 \leq m \leq M$*;*

(b) $|V_0| \leq \varepsilon n$ *and* $|V_1| = \ldots = |V_m| \leq \lceil \varepsilon n \rceil$*;*

(c) $\deg_{G'}(v) \geq \deg_G(v) - (2\mathrm{d} + \varepsilon)n$ *for each* $v \in V(G) \setminus V_0$*;*

(d) $G'[V_\mathrm{i}]$ *has no edges for* $\mathrm{i} \in [m]$*;*

(e) all pairs (V_i, V_j) are ε-regular and with density either 0 or at least d in each colour in G'.

Let G be a red and blue edge coloured graph with a partition $V_0, \ldots, V_m$ obtained from Lemma 3.2.2 with parameters ε, m_0 and d. We define the $(\varepsilon, \mathrm{d})$-*reduced graph* $\mathcal{G}$ to be a graph with vertex set $V(\mathcal{G}) = \{x_1, \ldots, x_m\}$ and where two vertices x_i and x_j are connected by a red (blue) edge, if (V_i, V_j) is an ε-regular pair of density at least d in red (blue). It is often convenient to refer to a *cluster* V_i via its corresponding vertex in the reduced graph, i.e. $V_\mathrm{i} = V(x_\mathrm{i})$. Note that $\mathcal{G}$ inherits the minimum degree of G. More precisely, if G has minimum degree cn, it follows that

$$\mathcal{G} \text{ has minimum degree at least } (c - 2\mathrm{d} - \varepsilon)m. \tag{3.2.2}$$

The next lemma of Łuczak allows us to connect clusters by short paths, if their counterparts in the reduced graph lie in the same connected component.

Lemma 3.2.3 (Connecting Paths [79])**.** *Let n be an integer and let ε, d be numbers with $0 < 1/n \ll \varepsilon \ll \mathrm{d} \leq 1$. Let G be a graph on n vertices and with an $(\varepsilon, \mathrm{d})$-regular graph $\mathcal{G}$ obtained from Lemma 3.2.2. Suppose that $W \subseteq V(G)$ is a vertex set which contains at most $|V(x)|/2$ vertices of each cluster $V(x)$. Let $x_1y_1, x_2y_2 \in E(\mathcal{G})$ be edges in a connected component $\mathcal{C} \subseteq \mathcal{G}$ of colour c.*

Then for any two vertices $v_{\mathrm{i}} \in V(x_{\mathrm{i}})$ of typical degree in $(V(x_{\mathrm{i}}), V(y_{\mathrm{i}}))$, there is a colour c v_1-v_2-path $P \subseteq G$ of order at most $2m$ and which avoids any vertices of W. In addition, if $\mathcal{C}$ contains an odd cycle, we can choose the parity of $|V(P)|$.

3.3 Proof of Theorem 3.1.1

In this section we present the proof of Theorem 3.1.1.

(A) **Hierarchy:** Given $\beta > 0$ as input of Theorem 3.1.1, we define an integer n_0 and numbers ε, γ obeying the following hierarchy

$$0 \ll 1/n_0 \ll \varepsilon \ll \gamma \ll \mathrm{d} \ll \beta < 1/3. \tag{$\ll$}$$

More precisely, we set

$$\beta = 2^{100}\sqrt{\mathrm{d}} \text{ and } \gamma = \mathrm{d}/2^{100}.$$

Suppose $\varepsilon_{3.2.1}$ satisfies Lemma 3.2.1 with input d and $\varepsilon_{3.2.3}$ satisfies Lemma 3.2.3 with input d. We choose

$$0 < \varepsilon \leq \min\left(\varepsilon_{3.2.1}, \varepsilon_{3.2.3}, \beta^{100}/2^{100}\right)$$

We then apply Lemma 3.2.2 with input ε to obtain an integer M. Suppose $n_{3.2.1}$ satisfies Lemma 3.2.1 with input ε, d and $n_{3.2.3}$ satisfies Lemma 3.2.3 with input ε, d. We choose the integer n_0 such that

$$n_0 \geq \max(M \cdot n_{3.2.1}, n_{3.2.3}, M^{100}/2^{100}\varepsilon).$$

In the following d will indicate the density of the regular clusters. The constant γ has three functions with respect to the monochromatic components, that we will choose in the reduced graph. It serves as lower bound for the robustness of the matching, as lower bound of the component intersections and it keeps track of the distance of the colouring to an extremal configuration. The constant ε indicates the regularity of the pairs. We will bound the number of exceptional vertices and the differences in the cluster sizes in terms of ε.

Functions: n: graph order, ε: regularity, exceptional vertices, differences in cluster sizes γ: matching robustness, overlap, distance to extremal colouring, d: density of regular pairs β: input minimum degree.

(B) **Input graph:** Let G be a red and blue edge coloured graph on $n \geq n_0$ vertices and with $\delta(G) \geq (2/3+\beta)n$. We have to show that G contains three monochromatic vertex disjoint cycles, which together cover all vertices.

(C) **Regularity:** Let $V_0, V_1, \ldots, V_m$ be a (regular) partition of $V(G)$ as guaranteed by Lemma 3.2.2 with $m \leq M$. We define the $(\varepsilon, \mathrm{d})$-reduced graph $\mathcal{G}$ as explained in Section 3.2.2. Thus $\mathcal{G}$ has a minimum degree of at least

$$\delta(\mathcal{G}) \overset{(3.2.2)}{\geq} (2/3+\beta-2\mathrm{d}-\varepsilon)m \overset{(\ll)}{\geq} (2/3+\beta/2)m \tag{3.3.1}$$

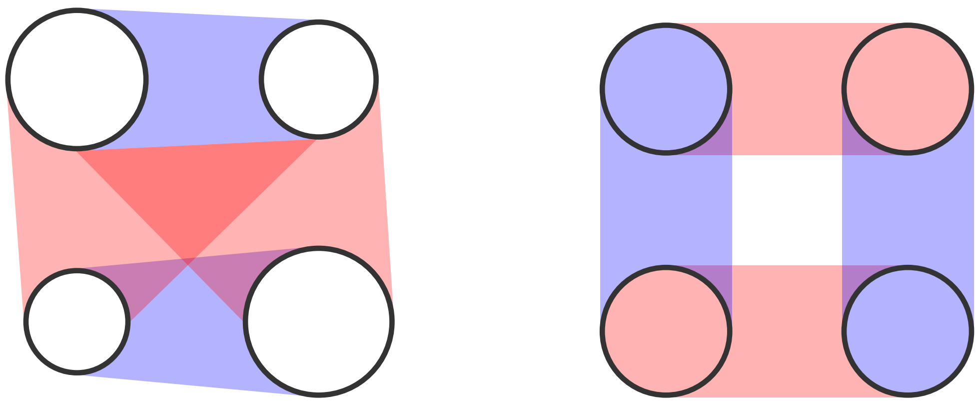

Figure 3.2: The extremal colourings.

for any $x \in V(\mathcal{G})$ we have

$$(1-\varepsilon)n/m \leq |V(x)| \leq n/m \tag{3.3.2}$$

and if an edge $xy \in E(\mathcal{G})$ has not colour i then

$$\mathrm{e_i}(V(x), V(y) \leq \mathrm{d}n^2/m. \tag{3.3.3}$$

(D) **Choose components:** In this step we will choose three monochromatic components in the reduced graph, which support our strategy. This is possible unless the colourings has one of the following two configurations (see also Figure 3.2).

Definition 3.3.1 (Extremal colourings)**.** Let G be red and blue edge coloured graph. We say that the colouring of G is γ-extremal, if one of the following (modulo swapping colours) holds

(a) G has a spanning bipartite red component with bipartition classes X_1, X_2 and such that $\big||X_1| - |X_2|\big| \leq \gamma n$. There are (exactly) two bipartite blue components B_1, B_2 with $V(B_1) \subseteq X_1$ and $V(B_2) \subseteq X_2$.

(b) G has red and blue monochromatic components R_1, R_2, B_1, B_2 such that $V(G) = \bigcup_{\mathrm{i},j\in[2]} V(R_\mathrm{i}) \cap V(B_j)$ and $|V(R_\mathrm{i}) \cap V(B_j)| \leq (1/4+\gamma)$ for $1 \leq \mathrm{i}, j \leq 2$. Moreover the edges of $G[V(R_\mathrm{i}) \cap V(B_j)]$ are blue if $\mathrm{i} = j$ and red otherwise.

We now state our key lemma, which will allow us to choose monochromatic components with the desired properties. We defer its proof to Section 3.4.

Lemma 3.3.2 (Find components)**.** *Let G be a graph on n vertices and with $\delta(G) \geq (2/3+8\gamma)n$, whose edges are coloured in red and blue. Then there are monochromatic connected components $C_1, C_2, C_3 \subseteq G$ such that $C = \bigcup C_\mathrm{i}$ spans G and the following holds.*

(i) ***Robust perfect matching:*** *The colouring of G is (4γ)-extremal, or every stable subset S of $V(C)$ has $|N_C(S)| \geq |S| + \gamma n$.*

(ii) ***Overlap:*** *One of the following holds*

(a) $|\bigcup_{\mathrm{i}\neq j} V(C_\mathrm{i}) \cap V(C_j)| \geq (1/3+\gamma)n$ *or*

(b) C_1 *is spanning,* C_1 *or* C_2 *contains an odd cycle, and* $C_3 = \emptyset$*.*

(iii) ***Odd monochromatic cycle:*** *If* C_1, C_2, C_3 *are each bipartite, then* $C_3 = \emptyset$*.*

(iv) ***Odd cycle:*** C *contains an odd cycle.*

(v) ***Connectivity:*** C *is connected.*

We obtain monochromatic connected components $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \subseteq \mathcal{G}$ by applying Lemma 3.3.2 with input $\mathcal{G}$ and γ. This is possible by (3.3.1) and since $\beta/2 \geq 8\gamma$ by ($\ll$). If the colouring is extremal, we set $\mathcal{C} = R_1 \cup B_1 \cup B_2$ for configuration (a) and $\mathcal{C} = R_1 \cup R_2 \cup B_1 \cup B_2$ for configuration (b). These cases will receive additional attention in step (F).

Further notation: For an edge $xy \in E(\mathcal{G})$ let $V(xy) = V(x) \cup V(y)$. We denote the colour of $\mathcal{C}_\mathrm{i}$ by c_i. (If $\mathcal{C}_\mathrm{i} = \emptyset$, then $c_\mathrm{i} = 1$.) For an edge $xy \in E(\mathcal{C})$ let $\mathrm{Bad}(xy) \subseteq V(xy)$ contain all vertices of non-typical degree in colour $c = 1, 2$ in $(V(x), V(y))$, i.e. all $x \in V(x)$ and $y \in V(y)$ with $\deg_c(x, V(y)) < (\mathrm{d} - \varepsilon)|V(x)|$ and $\deg_c(y, V(x)) \leq (\mathrm{d} - \varepsilon)|V(y)|$. Note that

$$|\mathrm{Bad}(xy)| \overset{(3.2.1)}{\leq} 4\varepsilon|V(x)| \tag{3.3.4}$$

by (3.2.1) and Lemma 3.2.2(e).

(E) **Fix vertex cover:** In this step we choose a vertex cover of $\mathcal{C}$ that has bounded maximum degree. We will work with 2-matchings, which are a relaxation of matchings and closely related to fractional matchings.

Definition 3.3.3 (Perfect 2-matching)**.** A perfect 2-matching of a graph G is a function $\omega : E(G) \to \{0, 1, 2\}$, such that $\sum_{w \in N(v)} \omega(vw) = 2$ for every vertex $v \in V(G)$.

The next theorem is a convenient analogue of Tutte's classical characterization of matchings for 2-matchings.

Theorem 3.3.4 (Corollary 30.1a in [100])**.** *A graph* G *has a perfect 2-matching if and only if every stable set* $S \subseteq V(G)$ *satisfies* $|N(S)| \geq |S|$*.*

Lemma 3.3.2(i) allows us to select a perfect 2-matching $\omega^{\text{Blow Up}}$ of $\mathcal{G}$ for which we set $\mathcal{M}^{\text{Blow Up}} = E(\mathcal{G}) \setminus \ker(\omega^{\text{Blow Up}})$. The important properties of $\mathcal{M}^{\text{Blow Up}}$ (viewed as a graph) are that it is spanning in $\mathcal{G}$ and its maximum degree is bounded by 2. Recall that our goal is to find spanning monochromatic paths between the clusters of edges of $\mathcal{M}^{\text{Blow Up}}$ and then obtain a monochromatic cycle partition of G by connecting these up. However, some of the vertices in the clusters of $\mathcal{M}^{\text{Blow Up}}$ are just not fit for this argument. (More precisely they do not satisfy the conditions of Lemma 3.2.1 and 3.2.3.) We call the set of these vertices $\mathrm{Bad}^{\text{Blow Up}} = \bigcup_{\mathrm{e} \in \mathcal{M}^{\text{Blow Up}}} \mathrm{Bad}(\mathrm{e})$. For any edge $\mathrm{e} \in E(\mathcal{C})$ we call the vertices of $V(\mathrm{e}) \setminus (\mathrm{Bad}(\mathrm{e}) \cup \mathrm{Bad}^{\text{Blow Up}})$ $\mathcal{M}^{\textit{Blow Up}}$*-typical.*

Note that since the maximum degree of $\mathcal{M}^{\text{Blow Up}}$ is bounded by 2 and by (3.3.4), most vertices are $\mathcal{M}^{\text{Blow Up}}$-typical, i.e. $|\mathrm{Bad}(\mathrm{e}) \cup \mathrm{Bad}^{\text{Blow Up}}| \leq 24\varepsilon|V(x)|$.

(F) **Extremal colourings:** In this step we will take care of the extremal colourings, that are possible outcomes of Lemma 3.3.2. We will take advantage of the fact that if some of the components are connected by short monochromatic paths, we can treat them as a single connected component and continue as usual.

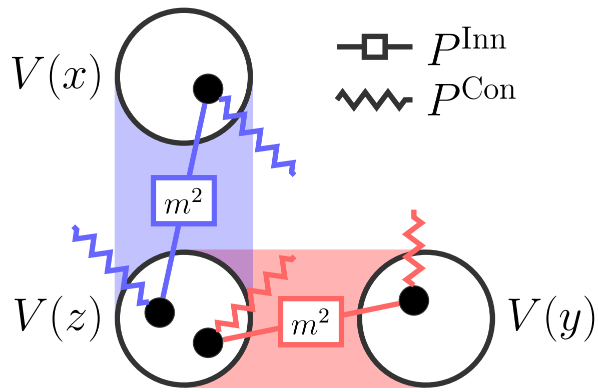

Figure 3.3: Inner paths and connecting paths of the edges $xz, yz \in E(\mathcal{G})$

Suppose that the colouring takes the form of Definition 3.3.1(a). Let $\mathcal{R}$ be the spanning, red say, bipartite component and $\mathcal{B}_1, \mathcal{B}_2$ the bipartite blue components of $\mathcal{G}$. Given edges $\mathrm{e_i} \in E(\mathcal{B}_\mathrm{i})$ for $\mathrm{i} \in [2]$, we call a path that connects any $\mathcal{M}^{\text{Blow Up}}$-typical vertices of e_1 and e_2 a *blue bridge*, if it has order at most three. We say that the colouring of G *admits blue bridges*, if there are (at least) two vertex disjoint blue bridges. Should such bridges not exist, then we have to find a cycle partition via ad hoc arguments. These are contained in the proof of the next claim, which we present in Section 3.6.

Claim 3.3.5. *If the colouring takes the form of Definition 3.3.1(a), then G admits either blue bridges or has a monochromatic cycle partition as desired.*

If the colouring admits blue bridges, we set $\mathcal{C}_1 := \mathcal{R}_1$, $\mathcal{C}_2 := \mathcal{B}_1 \cup \mathcal{B}_2$, $\mathcal{C}_3 := \emptyset$. Note that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ trivially satisfies the conditions of Lemma 3.3.2, in particular (ii)(a). Hence we can continue with step (G).

Now suppose that the colouring takes the form of Definition 3.3.1(b) with red and blue connected components $\mathcal{R}_1, \mathcal{R}_2, \mathcal{B}_1, \mathcal{B}_2$. We define blue (red) bridges between $\mathcal{B}_1, \mathcal{B}_2$ ($\mathcal{R}_1, \mathcal{R}_2$) as above. Similar to before we have to find the desired cycle partition manually, if there are no red or blue bridges. The details are in Section 3.6.

Claim 3.3.6. *If the colouring takes the form of Definition 3.3.1(b), then G admits red bridges, blue bridges or has a monochromatic cycle partition as desired.*

If the colouring admits blue bridges, we set $\mathcal{C}_1 := \mathcal{R}_1$, $\mathcal{C}_2 := \mathcal{B}_1 \cup \mathcal{B}_2$, $\mathcal{C}_3 := \mathcal{R}_2$. If the colouring admits red bridges, we set $\mathcal{C}_1 := \mathcal{B}_1$, $\mathcal{C}_2 := \mathcal{R}_1 \cup \mathcal{R}_2$, $\mathcal{C}_3 := \mathcal{B}_2$. Note that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ in both cases trivially satisfies the conditions of Lemma 3.3.2, in particular (ii)(a). Hence we can continue with step (G).

(G) **Set up cycles:** Here we choose disjoint short cycles of colour c_i that visit the clusters of all edges of $\mathcal{C}_\mathrm{i}$ for $\mathrm{i} = 1, 2, 3$. These cycles will serve as skeleton of the desired cycle partition. We intend to carefully extend some of their segments, until finally all vertices are covered.

Let us first assume the colouring of $\mathcal{G}$ is not (4γ)-extremal and let $E(\mathcal{C}_\mathrm{i}) = \{\mathrm{e}_1, \dots, \mathrm{e}_{|E(\mathcal{C}_\mathrm{i})|}\}$. For each edge $\mathrm{e}_j = xy$, we choose a colour c_i *inner path* $P_\mathrm{i}^{\text{Inn}}(\mathrm{e}_j)$ of order m^2 and with $\mathcal{M}^{\text{Blow Up}}$-typical ends $s_j \in V(x)$, $t_j \in V(y)$. This is possible by Lemma 3.2.1 and our choices in ($\ll$). Next we connect t_j and s_{j+1} with a colour $\mathrm{c_i}$ *connecting path* $P_{\mathrm{i}}^{\mathrm{Con}}(\mathrm{e}_j)$ of order at most $2m$ (where $j, j+1$ are taken modulo $|E(\mathcal{C}_{\mathrm{i}})|$). This is possible by Lemma 3.2.3 and our choices in ($\ll$). We can furthermore assume that all of these paths are internally vertex disjoint. (See Figure 3.3 for an illustration.)

Now let that the colouring be (4γ)-extremal in $\mathcal{G}$ and suppose we have, blue say, bridges. We repeat the same steps as above, except that we fix an ordering of $E(\mathcal{C}_2) = E(\mathcal{B}_1) \cup E(\mathcal{B}_2)$ where the first $|E(\mathcal{B}_1)|$ edges belong to $\mathcal{B}_1$. We then ensure that the blue bridges are subpaths of the two blue paths $P_2^{\mathrm{Con}}(\mathrm{e}_{|E(\mathcal{B}_1)|})$ and $P_2^{\mathrm{Con}}(\mathrm{e}_{|E(\mathcal{B}_1)|+|E(\mathcal{B}_2)|})$ which connect the clusters of edges of $\mathcal{B}_1$ and $\mathcal{B}_2$. From now on extremal and non-extremal colourings are handled the same way.

Set

$$\mathcal{P} = \sum_{\mathrm{i} \in [3]} \sum_{\mathrm{e} \in E(\mathcal{C}_{\mathrm{i}})} P_{\mathrm{c_i}}^{\mathrm{Inn}}(\mathrm{e}) + P_{\mathrm{c_i}}^{\mathrm{Con}}(\mathrm{e})$$

and observe that $\mathcal{P}$, as desired, consists of three disjoint monochromatic cycles. Moreover for each vertex $x \in V(\mathcal{C})$, the intersection $V(x) \cap V(\mathcal{P})$ has a size of at most

$$2m^3 + m^3 \overset{(\ll)}{\leq} \varepsilon |V(x)|. \tag{3.3.5}$$

(There are at most m^2 paths of type P^{Con}, each of which contains at most $2m$ vertices. For each $x \in V(\mathcal{G})$ are at most m paths of type P^{Inn}, each of which contains m^2 vertices of $V(x)$.)

(H) **Distribute exceptional vertices:** In this step we deal with the vertices that do not behave regularly enough to be within the reach of Lemma 3.2.1. Denote these *exceptional vertices* by $V^{\mathrm{Exc}} = \bigcup_{\mathrm{e} \in \mathcal{M}^{\text{Blow Up}}} \mathrm{Bad}(\mathrm{e}) \cup V_0$ and observe that

$$|V^{\mathrm{Exc}}| = |\bigcup_{\mathrm{e} \in \mathcal{M}^{\text{Blow Up}}} \mathrm{Bad}(\mathrm{e})| + |V_0| \overset{(3.3.4)}{\leq} 4\varepsilon |V(x)| m + \varepsilon n \leq 5\varepsilon n \tag{3.3.6}$$

Using the overlap of Lemma 3.3.2(ii), we distribute the exceptional vertices evenly on the cycles of $\mathcal{P}$. This is made precise in the next claim, whose proof we defer to Section 3.5.

Claim 3.3.7 (Distribute exceptional vertices)**.** *For* $\mathrm{i} = 1, 2, 3$ *there are colour* $\mathrm{c_i}$ *paths* P_{i}^{Exc} *such that for* $P^{Exc} = \bigcup_{\mathrm{i}} P_{\mathrm{i}}^{Exc}$

(i) $|V(x) \cap V(P^{Exc})| \leq \sqrt{\varepsilon} |V(x)|$ *for each* $x \in V(\mathcal{G})$*,*

(ii) P_{i}^{Exc} *has the same ends as* $P_{\mathrm{i}}^{Con}(\mathrm{e}_{\mathrm{i}}^{Exc})$ *for some edge* $\mathrm{e}_{\mathrm{i}}^{Exc} \in E(\mathcal{C}_{\mathrm{i}})$ *and*

(iii) $V^{Exc} \subseteq V(P^{Exc})$*.*

In addition, if Lemma 3.3.2 (ii)(b) holds, then P_3^{Exc} *is a monochromatic cycle on its own.*

We replace $P_{\mathrm{i}}^{\mathrm{Con}}(\mathrm{e}_{\mathrm{i}}^{\mathrm{Exc}})$ by $P_{\mathrm{i}}^{\mathrm{Exc}}$ in $\mathcal{P}$, keeping the names for convenience.

(I) **Resizing inner paths:** In what follows we have to overcome two obstacles. Firstly, we need to add exceptional vertices, like those contained in V_0, to these cycles. Secondly, we have to balance the clusters of $\mathcal{M}^{\text{Blow Up}}$ in order to cover all vertices by extending the respective inner paths. We will achieve the latter by resizing some of the inner paths that do not correspond to edges of $\mathcal{M}^{\text{Blow Up}}$. Let us make this operation precise.

Definition 3.3.8 (Extending and reducing inner paths)**.** Let k be non-negative integer and $\mathrm{i} \in [3]$. We *extend* (*reduce*) the inner path of $\mathrm{e} \in E(C_\mathrm{i})$ by $2k$ vertices by performing the following modifications of $\mathcal{P}$. Firstly, select a path $P^{\text{New}}_{c_\mathrm{i}}$ of colour c_i with the following properties

- $|V(P^{\text{New}}_{c_\mathrm{i}})| = |V(P^{\text{Inn}}_{c_\mathrm{i}}(\mathrm{e}))| + 2\sigma k$ where $\sigma = 1$ ($\sigma = -1$),
- $P^{\text{new}}_{c_\mathrm{i}}$ alternates between $\mathcal{M}^{\text{Blow Up}}$-typical vertices of xy,
- $P^{\text{New}}_{c_\mathrm{i}}$ terminates in the ends of $P^{\text{Inn}}_{c_\mathrm{i}}(\mathrm{e})$ and
- $V(P^{\text{New}}_{c_\mathrm{i}}) \subseteq \big(V(G) \setminus V(\mathcal{P})\big) \cup V\big(P^{\text{Inn}}_{c_\mathrm{i}}(\mathrm{e})\big)$.

Secondly, replace $P^{\text{Inn}}_{c_\mathrm{i}}(\mathrm{e})$ with $P^{\text{New}}_{c_\mathrm{i}}$ in $\mathcal{P}$, while keeping the names for convenience.

Let us observe the following properties of path extensions and reductions, which follow from Lemma 3.2.1 and our choices in ($\ll$).

Remark 3.3.9 (Path extensions and reductions)**.**

(1) Since the original inner paths have at least m^2 vertices, we are allowed to reduce each inner path by up to m^2 vertices.

(2) Given any edge $xy \in E(\mathcal{C})$ with $|V(x) \setminus V(\mathcal{P})|, |V(y) \setminus V(\mathcal{P})| \geq |V(x)|/2$, we can extend the vertex set of the inner path of xy by up to

$$2 \min\big(|V(x) \setminus V(\mathcal{P})|, |V(y) \setminus V(\mathcal{P})|\big) - |Bad(xy)|.$$

(3) If additionally $xy \in \mathcal{M}^{\text{Blow Up}}$, then we can extend the vertex set of the inner path of xy even up to

$$2 \min\big(|V(x) \setminus V(\mathcal{P})|, |V(y) \setminus V(\mathcal{P})|\big).$$

(J) **Parity:** From the next step on we will add (remove) vertices to (from) $\mathcal{P}$ exclusively by extending (reducing) inner paths. Since these extensions (reductions) are done in pairs of vertices, we need to ensure that the number of uncovered vertices is even. If $V(G) \setminus V(\mathcal{P})$ has even size we continue with step (K), otherwise we make the following adjustment.

If $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ are each bipartite, we add a single vertex from $V(G) \setminus V(\mathcal{P})$ to $\mathcal{P}$. This vertex will be a monochromatic cycle on its own.[2]

If, on the other hand, $\mathcal{C}_\mathrm{i}$ has an odd cycle, we select an edge $\mathrm{e} \in E(\mathcal{C}_\mathrm{i})$ such that $\mathrm{e} \neq \mathrm{e}^{\text{Exc}}$ and $P^{\text{Con}}_\mathrm{i}(\mathrm{e})$ contains no colour c_i bridges. This is possible since $\mathcal{C}_\mathrm{i}$ contains at least 3 edges. We then replace $P^{\text{Con}}_\mathrm{i}(\mathrm{e})$ by a path $P^{\text{Parity}}_\mathrm{i}(\mathrm{e})$ of order less than $2m$ and with $|P^{\text{Con}}_\mathrm{i}(\mathrm{e})| - |P^{\text{Parity}}_\mathrm{i}(\mathrm{e})| = 1 \mod 2$. This is possible by Lemma 3.2.3 and our choices in ($\ll$).

[2] Note that $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ being each bipartite and Lemma 3.3.2(ii)(b) are mutually exclusive situations. This is important, since in both cases (i.e. in step (H) and (J)) a monochromatic cycle not corresponding to a component $\mathcal{C}_\mathrm{i}$ has been added and in what follows we will obtain another cycle for each $\mathcal{C}_\mathrm{i} \neq \emptyset$.

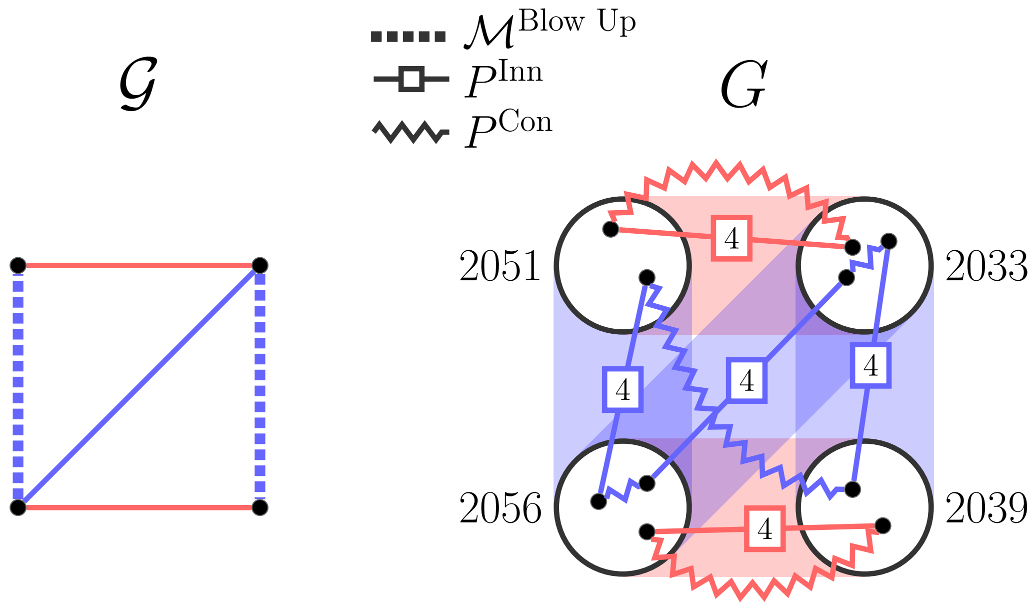

Figure 3.4: An illustration of $\mathcal{G}$ and G after step (J). The numbers indicate the sizes of the clusters $V(x) \setminus V(\mathcal{P})$.

(K) **Balance matching edges up to a constant:** Recall that our intention is to extend the inner paths of the edges of $\mathcal{M}^{\text{Bal}}$, until all vertices are covered. If the clusters have all the same size, this can be done easily as explained in Remark 3.3.9(2). However, the last few steps have possibly created some imbalances in the cluster sizes. So before going on, we need to ensure that the remaining clusters have the same size. To restore balance, we will carefully extend the inner paths. The size of these extensions will be indicated by a 2-matching in an auxiliary graph. Consider the following lemma.

Lemma 3.3.10. *Suppose that G is a graph on n vertices, such that every stable set $S \subseteq V(G)$ satisfies $|N(S)| \geq |S| + \gamma n$. Let H be the blow up graph of G, where each vertex of G is replaced by a cluster $B(x)$ of k vertices and each edge of G is replaced by a complete bipartite graph $K_{k,k}$.*

Then for any set $U \subseteq V(H)$, where U contains at most γk vertices of each cluster $B(x)$, $H - U$ has a perfect 2-matching.

Proof of Lemma 3.3.10. Suppose that $H' = H - U$ has no perfect matching for some $U \subseteq V(H)$, where U contains at most αk vertices of each cluster $B(x)$. We will show that $\alpha > \gamma$. By Theorem 3.3.4 there is a stable set S with $|N_{H'}(S)| < |S|$. Let S_0 be of maximal size among all sets S with this property. For each $x \in V(G)$ let $B'(x) = B(x) \setminus U$ be the cluster of H' corresponding to $x \in V(G)$. A quick case distinction shows that for any two vertices $v, w \in B'(x)$ the maximality of S_0 and the fact that $N_{H'}(v) = N_{H'}(w)$ imply that $\{v, w\}$ is a subset of either S_0, $N_{H'}(S_0)$ or $V(G) \setminus (S_0 \cup N_{H'}(S_0))$. So for $S_1 = \{x \in V(G) : B'(x) \subseteq S_0\}$ we have

$$|S_1|k \geq |S_0| > |N_{H'}(S_0)| \geq (1 - \alpha)k|N_G(S_1)|.$$

In particular $|S_1| + \alpha n > |N_G(S_1)|$. Since S_1 is a stable set in G, this implies $\alpha > \gamma$. $\square$

We plan to apply Lemma 3.3.10 with $k = \lfloor |V(x)|/2 \rfloor$. Let H be the blow up graph of $\mathcal{G}$ as in the statement. Select a set $U(v) \subseteq B(v)$ of $|V(x) \cap V(\mathcal{P})|$ vertices for each $x \in V(\mathcal{G})$ and set $U := \bigcup_{x \in V(\mathcal{G})} U(x)$. By the choices in steps (G)–(J), in particular (3.3.5) and Claim 3.3.7(i) we have

$$|U(x)| = |V(x) \cap V(\mathcal{P})| \leq 2\sqrt{\varepsilon}|V(x)| \overset{(\ll)}{\leq} \gamma \lfloor \frac{|V(x)|}{2} \rfloor = \gamma k.$$

By Lemma 3.3.2(i) and as $\mathcal{C}$ is spanning in $\mathcal{G}$, we can apply Lemma 3.3.10 with input $\mathcal{G}$, U, k and γ. Thus H has a (perfect) 2-matching

$$\omega^{\mathrm{Bal}} : E(H) \to \{0, 1, 2\}$$

where $\sum_{w \in N(v)} \omega^{\mathrm{Bal}}(vw) = 2$ for every vertex $v \in V(H)$. Observe that this implies for each $x \in V(\mathcal{G})$

$$\sum_{v \in B(x)} \sum_{w \in N_H(v)} \omega^{\mathrm{Bal}}(vw) = 2|B(x) \setminus U(x)| = 2(k - |V(x) \cap V(\mathcal{P})|) \qquad \Leftrightarrow$$

$$|V(x) \cap V(\mathcal{P})| = k - \frac{\sum_{v \in B(x)} \sum_{w \in N_H(v)} \omega^{\mathrm{Bal}}(vw)}{2}.$$

and therefore

$$\begin{aligned}
|V(x) \setminus V(\mathcal{P})| &= |V(x)| - |V(x) \cap V(\mathcal{P})| \\
&= |V(x)| - k + \frac{\sum_{v \in B(x)} \sum_{w \in N_H(v)} \omega^{\mathrm{Bal}}(vw)}{2} \\
&= \lceil \frac{|V(x)|}{2} \rceil + \sum_{y \in N_{\mathcal{G}}(x)} \frac{\sum_{v \in B(x)} \sum_{w \in B(y)} \omega^{\mathrm{Bal}}(vw)}{2} \\
&= \lceil \frac{|V(x)|}{2} \rceil + K_x + \sum_{y \in N_{\mathcal{G}}(x)} \lfloor \frac{\sum_{v \in B(x)} \sum_{w \in B(y)} \omega^{\mathrm{Bal}}(vw)}{2} \rfloor,
\end{aligned}$$

where $0 \leq K_x \leq m-1$. With this in mind, we extend the inner paths of each $xy \in E(\mathcal{C})$ by

$$2 \lfloor \frac{\sum_{v \in \mathrm{B}(x), u \in \mathrm{B}(y)} \omega^{\mathrm{Bal}}(vu)}{2} \rfloor$$

vertices. As said in Definition 3.3.8 we keep the name of $\mathcal{P}$ for convenience. Then each $x \in V(\mathcal{G})$ satisfies

$$|V(x) \setminus V(\mathcal{P})| = \lceil \frac{|V(x)|}{2} \rceil + K_x \tag{3.3.7}$$

and hence the cluster sizes are balanced up to a difference of $m - 1$. The procedure of step (K) is illustrated in Figure 3.5.

(L) **Balance matching edges completely:** In this step, we aim to balance the cluster sizes completely. To this end let us define the following.

Definition 3.3.11 (Shift)**.** For vertices $x, y \in V(\mathcal{G})$ an x-y-shift consists of the following modifications of $\mathcal{P}$.

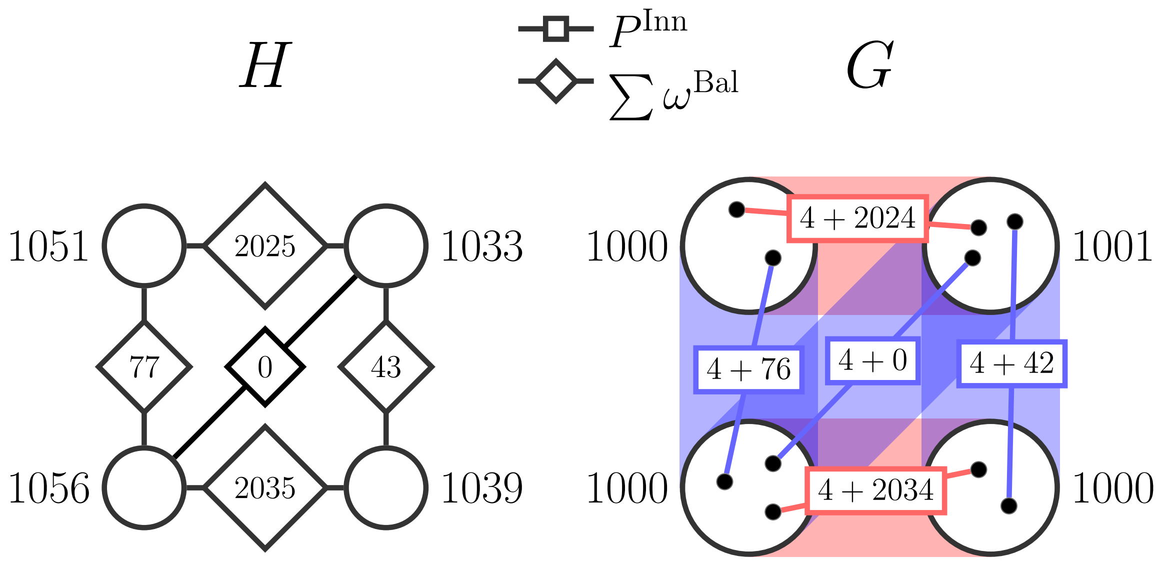

Figure 3.5: Given the example of Figure 3.4, we solve a matching problem in an auxiliary graph to balance the clusters. The numbers indicate the sizes of the clusters of H and G respectively. (The connecting paths are hidden for sakes of clarity.)

(i) Fix an even x-y-walk $P^{\text{Shift}} = (x = w_1, w_2, \ldots, w_{2h-1}, w_{2h} = y)$ with $h \leq m$.

(ii) If $\mathrm{i} \in [2h-1]$ is even, extend the inner path of $w_\mathrm{i} w_{\mathrm{i}+1}$ by 2 vertices, otherwise decrease it by 2 vertices.

Let us make the following remarks about shifts.

- After an x-y-shift, the sizes of $V(x) \setminus V(\mathcal{P})$ and $V(y) \setminus V(\mathcal{P})$ are each reduced by 1 (by 2 if $x = y$) and the size of $V(z) \setminus V(\mathcal{P})$ remains the same for all other $z \in V(\mathcal{G}) \setminus \{x, y\}$.
- By Remark 3.3.9(1), we can perform up to m^2 shifts.
- By Lemma 3.3.2(iv)–(v), $\mathcal{C}$ is connected and contains an odd cycle. Hence any two vertices $x, y \in V(\mathcal{C})$ admit an x-y-shift.

With the above in mind, we can greedily balance the leftover of the clusters. Set $l := \lfloor |V(x)|/4 \rfloor$. Firstly, as long as there is a vertex $x \in V(\mathcal{C})$ with $2l+2 \leq |V(x) \setminus V(\mathcal{P})|$, we perform an x-x-shift. Secondly, as long as there are distinct vertices $x, y \in V(\mathcal{G})$ with $2l < |V(x) \setminus V(\mathcal{P})|$ and $2l < |V(y) \setminus V(\mathcal{P})|$ we perform an x-y-shift. As said in Definition 3.3.8 we keep the name of $\mathcal{P}$ for convenience. Note that by (3.3.7) and the definition of l this procedure stops after at most m^2 shifts. Since the parity of $|V(G) \setminus V(\mathcal{P})|$ remains even under extensions and reductions of inner paths (and we have restricted ourselves to these operations since step (J)), we finish with $|V(x) \setminus V(\mathcal{P})| = 2l$ for each $x \in V(\mathcal{C})$.

(M) **Blow up matching edges:** It remains to extend the inner paths of each edge $\mathrm{e} \in \mathcal{M}^{\text{Blow Up}}$ by $4l$ vertices, if e is an isolated edge in $\mathcal{M}^{\text{Blow Up}}$, and $2l$ vertices otherwise. This is possible as explained in Remark 3.3.9(3).

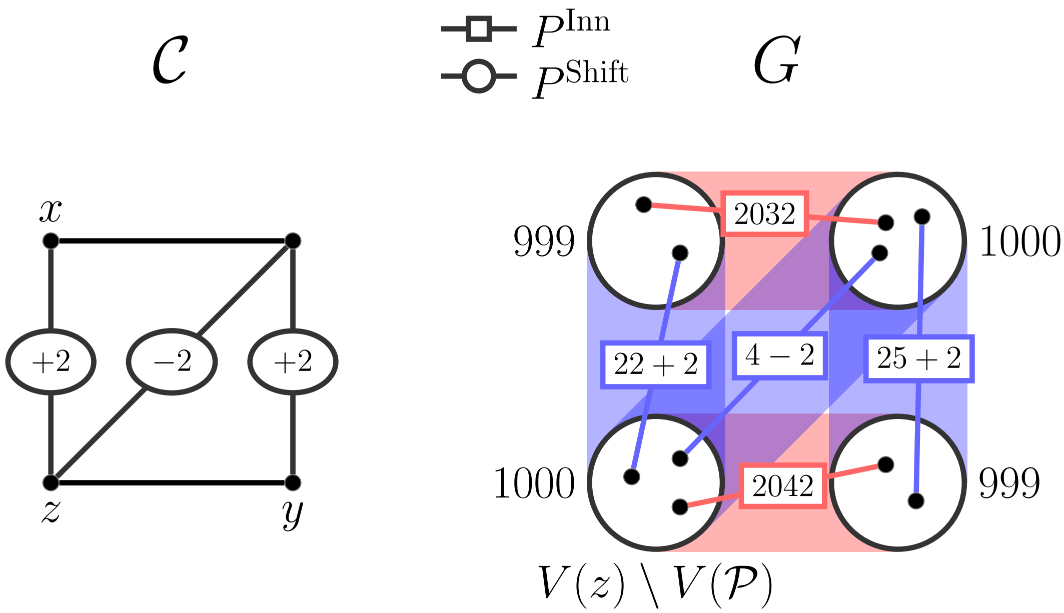

Figure 3.6: Continuing with the example of Figure 3.5, we illustrate an x-y-shift.

3.4 Proof of Lemma 3.3.2

This section is dedicated to the proof of Lemma 3.3.2. Let G be a red and blue edge coloured graph on n vertices and of minimum degree at least $(2/3 + 8\gamma)n$.

We start by showing that G is spanned by two monochromatic components.

Claim 3.4.1. *There are monochromatic components C_1, C_2 that together span G.*

Proof. Denote the monochromatic components of G by $C = \{C_1, C_2, \dots, C_l\}$. Let $v_1, \dots, v_q$ be a maximal number of vertices that are pairwise in distinct monochromatic components, i.e. v_{i} and v_j are in distinct red and in distinct blue components for $\mathrm{i} \neq j$. Now construct a bipartite (multi)-graph H with vertex set C as follows. For each vertex $v \in V(G)$ we place an edge between the red and blue components that contain v. Observe that any matching in H has a size of at most q. By Kőnig's theorem the edges of H are covered by vertices $C_1, \dots, C_q$ say. By the definition of H this implies that $C_1 + \dots + C_q$ spans G. Hence the claim follows from

$$\frac{2}{3}nq \leq \sum_{\mathrm{i}} \deg(v_{\mathrm{i}}) = \sum_{\mathrm{i}} \mathrm{d}_{\mathrm{red}}(v_{\mathrm{i}}) + \sum_{\mathrm{i}} \mathrm{d}_{\mathrm{blue}}(v_{\mathrm{i}}) \leq 2(n-1) \qquad \Leftrightarrow$$
$$q \leq 2.$$

$\square$

Let C_1 and C_2 be the components of Claim 3.4.1. We will treat the cases where one of C_1 and C_2 is spanning, C_1 and C_2 have the same colour and C_1 and C_2 have distinct colours differently.

Definition 3.4.2. For a spanning subgraph $H \subseteq G$, we call a stable set $S \subseteq V(G)$ *bad* in H, if $|N_H(S)| < |S| + \gamma n$.

Remark 3.4.3. *Observe that if $H \subseteq G$ is bipartite (and spanning), then its largest bipartition class is a bad set. In other words, Lemma 3.3.2(i) implies Lemma 3.3.2(iv) (unless the colouring is extremal).*

3.4.1 (Case: One of C_1 and C_2 is spanning)

We assume that $R := C_1$ is spanning and red. Assume $H = R + \sum B_{\rm i}$ for some blue components $B_{\rm i}$ and suppose that S is bad in H. Then

$$\begin{aligned} 2|N_H(S)| - \gamma n < |S| + |N_H(S)| \leq n & \qquad\qquad \Leftrightarrow \\ |N_H(S)| < (1/2 + \gamma/2)n. & \end{aligned} \tag{3.4.1}$$

Moreover, if B is a blue component with $v \in S \cap V(B)$, then

$$|S| + \gamma n > |N_H(S)| \geq \deg_H(v) \geq (2/3 + 8\gamma)n - |V(B)|. \tag{3.4.2}$$

Note that in particular, this implies that $B \neq B_{\rm i}$, since otherwise $\deg_G(v) = \deg_H(v)$ in contradiction to (3.4.1). Hence for any $B_{\rm i}$ of $\sum B_{\rm i}$ we have

$$S \cap V(B_{\rm i}) = \emptyset. \tag{3.4.3}$$

Let us continue with some further observations.

Claim 3.4.4. *Let $H = R + \sum C_{\rm i}$ for blue components $C_{\rm i}$. Suppose S is a bad set in H and let $B_1, \ldots, B_t$ be the blue components that intersect with S, ordered in decreasing vertex order. Then $t \leq 2$ and $|V(\sum_{{\rm i} \in [t]} B_{\rm i})| \geq (1/3 + 3\gamma)n$.*

In addition the following holds.

(a) If $|S| \geq (1/3 - 8\gamma)n$, then $V(G) \setminus N_H(S) \subseteq V(\sum_{{\rm i} \in [t]} B_{\rm i})$.

(b) If $t = 2$, then $|S| \geq (1/3 + 15\gamma)n$.

(c) If $|S| < (1/3 - 8\gamma)n$, then $|V(B_1)| \geq (1/3 + 7\gamma)n$. If in addition, B_1 is bipartite, then $|V(B_1)| \geq (2/3 + 7\gamma)n$.

(d) Finally, if B_1 is bipartite, then $t = 1$.

Before we prove Claim 3.4.4 let us note that by (3.4.3) the sets $V(B_{\rm i})$ are disjoint from the sets $V(C_{\rm i})$.

Proof. First note that for $W := V(G) \setminus (N_H(S) \cup S)$

$$H \text{ has no edges in } S \text{ and no edges between } S \text{ and } W. \tag{3.4.4}$$

Fix $v_{\rm i} \in V(B_{\rm i}) \cap S$ and observe that

$$|(S \cup W) \cap V(B_{\rm i})| \geq \deg_{B_{\rm i}}(v_{\rm i}, S \cup W) \overset{(3.4.4)}{\geq} (2/3 + 8\gamma)n - |N_H(S)|. \tag{3.4.5}$$

To prove $t \leq 2$, let us assume that $t > 2$ and obtain a contradiction. By (3.4.4) we obtain

$$
\begin{aligned}
|S \cup W| &\geq \sum_{\mathrm{i} \in [3]} \deg_G(v_\mathrm{i}, S \cup W) && \Rightarrow \\
3|N_H(S)| + |S \cup W| &\overset{(3.4.5)}{\geq} (6/3 + 24\gamma)n && \Leftrightarrow \\
2|N_H(S)| &\geq (3/3 + 24\gamma)n && \Leftrightarrow \\
|N_H(S)| &\geq (1/2 + 12\gamma)n
\end{aligned}
$$

in contradiction to (3.4.1). This proves $t \leq 2$. It remains to show properties (a)–(d) and that $|V(\sum_{\mathrm{i} \in [t]} B_\mathrm{i})| \geq (1/3 + 3\gamma n)$.

Part (a) follows from the observation that if $|S| \geq (1/3 - 8\gamma)n$, then any vertex in $V(G) \setminus (N_H(S) \cup S)$ has edges to S (and these edges are not in H by (3.4.4)).

Now let us show part (b). Suppose that $t = 2$. Then for $\mathrm{i} \in [2]$ and $v_\mathrm{i} \in S \cap V(B_\mathrm{i})$ we have

$$
\begin{aligned}
|W| &\geq |W \cap V(B_1)| + |W \cap V(B_2)| \\
&\geq \deg_G(v_1, W) + \deg_G(v_2, W) \\
&\geq 2(2/3 + 8\gamma)n - 2|N_H(S)| - |S \cap V(B_1)| - |S \cap V(B_2)| \\
&= (4/3 + 16\gamma)n - 2|N_H(S)| - |S|,
\end{aligned}
$$

where we used that S is stable in the last line. This together with $V(G) = W \cup S \cup N_H(S)$, gives

$$|S| + \gamma n > |N_H(S)| \geq (1/3 + 16\gamma)n$$

as desired. This proves part (b).

To show (c), let us suppose that $|S| < (1/3 - 8\gamma)n$. It follows from (b) that $t = 1$. Since S is bad in H we obtain

$$
\begin{aligned}
|V(B_1)| &\geq |(S \cup W) \cap V(B_1)| \\
&\overset{(3.4.5)}{\geq} (\frac{2}{3} + 8\gamma)n - |N_H(S)| \\
&\geq (\frac{2}{3} + 7\gamma)n - |S| \\
&\geq (1/3 + 15\gamma)n
\end{aligned}
$$

as desired. Now let us assume that B_1 is in addition bipartite. If $G[S]$ contains an edge vw, then bipartiteness and (3.4.4) ensures $N_G(v) \cap N_G(w) = \emptyset$. Hence

$$
\begin{aligned}
|V(B_1)| &\overset{(3.4.4)}{\geq} |S| + \deg_G(v, W) + \deg_G(w, W) \\
&\overset{(3.4.5)}{\geq} |S| + (4/3 + 16\gamma)n - |S| - 2|N_H(S)| \\
&\geq (4/3 + 14\gamma)n - 2|S| \\
&\geq (2/3 + 30\gamma)n.
\end{aligned}
$$

Similarly, if $G[S]$ contains no edge, it follows for any $v \in S$ that

$$
\begin{aligned}
|V(B_1)| &\overset{(3.4.4)}{\geq} |S| + |N_G(v) \setminus (S \cup N_H(S))| \\
&\geq |S| + (2/3 + 8\gamma)n - |N_H(S)| \\
&\geq (2/3 + 7\gamma)n.
\end{aligned}
$$

This proves (c).

To show part (d), let us assume that B_1 and B_2 are both bipartite and non-empty. So $|S| \geq (1/3 + 15\gamma)n$ by (b). This implies that there is an edge in $G[S]$. So we can assume that there are vertices $v_1, v_2 \in S$ that belong to distinct bipartition classes of B_1. By (3.4.4) the neighbourhoods of v_1 and v_2 in $S \cup W$ are disjoint in G. Now (3.4.5) and (3.4.4) imply for any $v_3 \in V(B_2)$ that

$$\begin{aligned} |S \cup W| &\geq \sum_{\mathrm{i} \in [3]} \deg_G(v_\mathrm{i}, S \cup W) && \Leftrightarrow \\ 3|N_H(S)| + |S \cup W| &\geq (\frac{6}{3} + 24\gamma)n && \overset{(3.4.5)}{\Leftrightarrow} \\ 2|N_H(S)| &\geq (\frac{3}{3} + 24\gamma)n && \Leftrightarrow \\ |N_H(S)| &\geq (\frac{1}{2} + 12\gamma)n \end{aligned}$$

in contradiction to (3.4.1). This proves part (d).

Finally we have to show that $|V(\sum_{\mathrm{i} \in [t]} B_\mathrm{i})| \geq (1/3 + 3\gamma)n$. If $|S| \geq (1/3 + 3\gamma)n$ this follows from $S \subseteq V(\sum_{\mathrm{i} \in [t]} B_\mathrm{i})$. So let us assume that $|S| < (1/3 + 3\gamma)n$ and hence $t = 1$ by (b). Then

$$\begin{aligned} (1/3 + 4\gamma)n > |S| + \gamma n &\overset{(3.4.2)}{>} (2/3 + 8\gamma)n - |V(B_1)| && \Rightarrow \\ |V(B_1)| &> (1/3 + 4\gamma n) \end{aligned}$$

as desired. □

Now let us show that we can select two blue components which together with R satisfy the conditions of Lemma 3.3.2.

Remark 3.4.5. *Note that as we always select R, Lemma 3.3.2(v) holds trivially.*

Claim 3.4.6. *There are no two blue components B_1, B_2 with $|V(B_1)|, |V(B_2)| \geq (1/3+3\gamma)n$ or we are done.*

Proof. Suppose otherwise and let B_1, B_2 be blue components with

$$|V(B_1)|, |V(B_2)| > (1/3 + 3\gamma)n. \tag{3.4.6}$$

If $H_0 := R + B_1 + B_2$ has a bad set, then by Claim 3.4.4 there are two more blue components which together contain at least $(1/3 + 3\gamma)n$ vertices. But this contradicts (3.4.6). Hence H_0 satisfies Lemma 3.3.2(i) and (ii)(a). If one of R, B_1, B_2 contains an odd cycle, then H_0 satisfies Lemma 3.3.2(iii) and we are done. Hence we can assume that

$$R, B_1, B_2 \text{ are each bipartite.} \tag{3.4.7}$$

Let us fix $\mathrm{i} \in [2]$ for a moment and set $H_\mathrm{i} = R + B_{3-\mathrm{i}}$. If H_i has no bad sets, then it satisfies the conditions of Lemma 3.3.2, in particular (ii)(a). Thus we can assume that H_i has bad sets S_i and let $Y_\mathrm{i} = V(G) \setminus N_{H_\mathrm{i}}(S_\mathrm{i})$.

Claim 3.4.4 allows us to denote the blue components that intersect with S_i by $B_3^\mathrm{i}, B_4^\mathrm{i}$ (possibly with $B_4^\mathrm{i} = \emptyset$) and implies additionally that $|V(B_3^\mathrm{i} + B_4^\mathrm{i})| \geq (1/3 + 3\gamma)n$.

By (3.4.6) and as $S_{\rm i} \cap V(B_{3-{\rm i}}) = \emptyset$ by (3.4.3), we can assume that $B_{\rm i} = B_3^{\rm i}$ and so $B_4^{\rm i} = \emptyset$ by Claim 3.4.4(d). Note that if $|S_{\rm i}| < (1/3 - 8\gamma)n$, then Claim 3.4.4(c) together with (3.4.7) implies that $|V(B_{\rm i})| \geq (2/3 + 7\gamma)n$. This contradicts (3.4.6) and hence we can assume that

$$|S_{\rm i}| \geq (1/3 - 8\gamma)n \tag{3.4.8}$$

Therefore Claim 3.4.4(a) gives that

$$Y_{\rm i} \subseteq V(B_3^{\rm i} + B_4^{\rm i}) = V(B_{\rm i}). \tag{3.4.9}$$

By (3.4.1) this implies

$$(1/2 - \gamma n) \leq |Y_{\rm i}| \leq |V(B_{\rm i})|$$

and hence for any $v_{\rm i} \in Y_{\rm i}$

$$\deg_{\rm blue}(v_{\rm i}) \geq (2/3 + 8\gamma)n - |V(B_{3-{\rm i}})| \geq (1/6 + 7\gamma)n. \tag{3.4.10}$$

Recall that $B_{\rm i}$ is bipartite and denote its bipartition by $X_1^{\rm i}, X_2^{\rm i}$. Let $j \in [2]$. By (3.4.8) $G[S_{\rm i}]$ contains a (blue) edge $w_1^{\rm i} w_2^{\rm i}$ with $w_j^{\rm i} \in X_j^{\rm i}$. We have $N_G(w_1^{\rm i}, V(B_{\rm i})) \cap N_G(w_2^{\rm i}, V(B_{\rm i})) = \emptyset$ and so (3.4.10) implies

$$|X_j^{\rm i}| \geq (1/6 + 7\gamma)n. \tag{3.4.11}$$

Hence for any $v_j^{\rm i} \in X_j^{\rm i}$

$$\begin{aligned} \deg_{\rm red}(v_j^{\rm i}) &\geq \deg_G(v_j^{\rm i}) - \deg_{\rm blue}(v_j^{\rm i}) \geq (2/3 + 8\gamma)n - |X_{3-j}^{\rm i}| \\ &\geq (2/3 + 8\gamma)n - \big(n - |X_j^{\rm i}| - |X_{3-j}^{3-{\rm i}}| - |X_j^{3-{\rm i}}|\big) \\ &\overset{(3.4.11)}{\geq} (1/3 + 29\gamma)n. \end{aligned} \tag{3.4.12}$$

Now if $G[V(B_{\rm i})]$ contains a red edge, then it follows by (3.4.12) that G contains a red triangle. But this contradicts that R is bipartite. Similarly, if there is a vertex outside of $V(B_1 + B_2)$, then it must send at least $(1/6 + 7\gamma)n$ red edges into both $V(B_1)$ and $V(B_2)$ by (3.4.9) and (3.4.1). Thus we can find cycle of order 5 in R in contradiction to its bipartiteness. Hence the colouring is γ-extremal as in Definition 3.3.1(a). □

Suppose that

$$B_0 \text{ is a blue component of maximum order} \tag{3.4.13}$$

and set $H_0 := R + B_0$.

Claim 3.4.7. *H_0 has a bad set.*

Proof. Suppose that H_0 has no bad sets. If one of R and B_0 has an odd cycle, H_0 satisfies Lemma 3.3.2(ii)(b) and (iii) and we are done. Similarly, if $|V(B_0)| \geq (1/3 + \gamma)n$, then H_0 satisfies Lemma 3.3.2(ii)(a) and (iii), which together with Remark 3.4.3 implies that we are done as well. Thus we assume that R and B_0 are both bipartite and

$$|V(B_0)| < (1/3 + \gamma)n \tag{3.4.14}$$

If there is a non-bipartite blue component B_1 with $|V(B_0 + B_1)| \geq (1/3 + \gamma)n$, we are done again. So we assume that no such component exists. Let $V(R) = P \cup Q$ be a bipartition of R, where $|P| \geq |Q|$. Note that any vertex $p \in P$ has $\deg_{\rm blue}(v, P) \geq (1/6 + 8\gamma n)$. By (3.4.14) and $|P| \geq n/2$ we can choose p such that it belongs to a bipartite blue component $B_1 \neq B_0$. Let $p' \in P$ such that pp' is blue. Since $N_{B_1}(p) \cap N_{B_1}(p') = \emptyset$, we obtain $|V(B_1)| \geq (1/3 + 16\gamma)n$ in contradiction to (3.4.13) and (3.4.14). □

Let S_0 be a bad set of minimum size in H_0. Claim 3.4.4 allows us to denote the blue components that intersect with S_0 by B_1, B_2 (possibly with $B_2 = \emptyset$) and implies additionally that $|V(B_1 + B_2)| \geq (1/3 + 3\gamma)n$. Let us fix $\mathrm{i} \in [2]$ for a moment. Note that $B_\mathrm{i} \neq \emptyset$, since otherwise $|V(B_0)| \geq |V(B_{3-\mathrm{i}})| \geq (1/3 + 3\gamma)n$ by (3.4.13) in contradiction to Claim 3.4.6. Hence B_i contains an odd cycle by Claim 3.4.4(d). It follows by Claim 3.4.4(a) and (b) that for $Y_0 = V(G) \setminus N_{H_\mathrm{i}}(S_\mathrm{i})$ we have

$$Y_0 \subseteq V(B_1 + B_2). \tag{3.4.15}$$

Let and set $H_\mathrm{i} = R + B_0 + B_{3-\mathrm{i}}$. Note that H_i satisfies Lemma 3.3.2(iii) and (ii)(a). The latter follows since $|V(B_0)|$ is maximal and $|V(B_1 + B_2)| \geq (1/3 + 3\gamma)n$.

If H_i has no bad sets, we are done by Remarks 3.4.3 and 3.4.5 and as $B_{3-\mathrm{i}}$ as an odd cycle. So suppose that S_i is a bad set in H_i and set $Y_\mathrm{i} = V(G) \setminus N_{H_\mathrm{i}}(S_\mathrm{i})$. Claim 3.4.4 allows us to denote the blue components that intersect with S_i by $B_3^\mathrm{i}, B_4^\mathrm{i}$ (possibly with $B_4^\mathrm{i} = \emptyset$) and implies additionally that $|V(B_3^\mathrm{i} + B_4^\mathrm{i})| \geq (1/3 + 3\gamma)n$. As above, note that

$$B_j^\mathrm{i} \neq \emptyset \text{ for } j = 3, 4, \tag{3.4.16}$$

since otherwise $|V(B_0)| \geq |V(B_{3-j}^\mathrm{i})| \geq (1/3+3\gamma)n$ by (3.4.13) in contradiction to Claim 3.4.6. It follows by Claim 3.4.4(a) and (b) that

$$Y_\mathrm{i} \subseteq V(B_3^\mathrm{i} + B_4^\mathrm{i}). \tag{3.4.17}$$

By (3.4.3) we have

$$V(B_0 + B_{3-\mathrm{i}}) \cap S_\mathrm{i} = \emptyset. \tag{3.4.18}$$

Before we continue, let us also note that trivially $B_{3-\mathrm{i}}$ equals at most one of B_3^i and B_4^i and assume without loss of generality that

$$B_{3-\mathrm{i}} \neq B_3^\mathrm{i}. \tag{3.4.19}$$

By vertices in S_i have the same neighbours in H_0 and H_i. Thus S_i is bad in H_0 as well and hence

$$|S_\mathrm{i}| \geq |S_0| \tag{3.4.20}$$

by minimality of S_0. Let us continue by proving that

$$S_0 \cap S_\mathrm{i} \neq \emptyset. \tag{3.4.21}$$

Suppose this is not true and $S_0 \cap S_\mathrm{i} = \emptyset$. Together with (3.4.15) and (3.4.18) this implies for $W_0 = Y_0 \setminus S_0$ that

$$S_\mathrm{i} \subseteq \big(N_{H_0}(S_0) \setminus V(B_0)\big) \cup \big(V(B_\mathrm{i}) \cap W_0\big). \tag{3.4.22}$$

Thus

$$\begin{aligned}
&|S_0| + \gamma n - |V(B_0)| + |V(B_\mathrm{i}) \cap W_0| \\
&\geq |N_{H_0}(S_0)| - |V(B_0)| + |V(B_\mathrm{i}) \cap W_0| \\
&\overset{(3.4.15)}{=} |N_{H_0}(S_0) \setminus V(B_0)| + |V(B_\mathrm{i}) \cap W_0| \\
&\overset{(3.4.22)}{\geq} |S_\mathrm{i}| \overset{(3.4.20)}{\geq} |S_0|,
\end{aligned} \tag{3.4.23}$$

which gives

$$
\begin{aligned}
2|V(B_0)| &\overset{(3.4.13),(3.4.15)}{\geq} |Y_0| \geq |S_0| + |V(B_{\mathrm{i}}) \cap W_0| \\
&\overset{(3.4.23)}{\geq} |S_0| + |V(B_0)| - \gamma n
\end{aligned} \tag{3.4.24}
$$

and therefore by Claim 3.4.4(b)

$$
|V(B_{\mathrm{i}})| + \gamma n \overset{(3.4.23)}{\geq} |V(B_0)| \overset{(3.4.24)}{\geq} |S_0| - \gamma n \geq (1/3 + 14\gamma)n,
$$

in contradiction to Claim 3.4.6. This proves (3.4.21).

By (3.4.21) we can fix $v_{\mathrm{i}} \in S_0 \cap S_{\mathrm{i}}$. By definition, v_{i} has no red edges to any vertex of $Y_0 \cup Y_{\mathrm{i}}$. Hence

$$
\begin{aligned}
\deg_{\mathrm{blue}}(v_{\mathrm{i}}) &\geq \deg_G(v_{\mathrm{i}}) - \deg_{\mathrm{red}}(v_{\mathrm{i}}) \\
&\geq \deg_G(v_{\mathrm{i}}) - |V(G) \setminus (Y_0 \cup Y_{\mathrm{i}})| \\
&\geq (2/3 + 8\gamma)n + |Y_0 \cup Y_{\mathrm{i}}| - n \\
&\geq |Y_0| + |Y_{\mathrm{i}} \setminus Y_0| - (1/3 - 8\gamma)n \\
&\overset{(3.4.1),(3.4.15),(3.4.17)}{\geq} \left|\big(Y_{\mathrm{i}} \cap V(B_3^{\mathrm{i}} + B_4^{\mathrm{i}})\big) \setminus \big(Y_0 \cap V(B_1 + B_2)\big)\right| + (1/6 + 7\gamma)n \\
&\overset{(3.4.3),(3.4.19)}{\geq} |Y_{\mathrm{i}} \cap V(B_3^{\mathrm{i}})| + (1/6 + 7\gamma)n,
\end{aligned} \tag{3.4.25}
$$

more precisely, the last line follows because $B^{\mathrm{i}} \neq B_j^{3-\mathrm{i}}$ by (3.4.3) for $j = 3, 4$ and $B^{3-\mathrm{i}} \neq B_3^{3-\mathrm{i}}$ by (3.4.19). Observe that any two blue components, which are both distinct from B_0, have together at most $(2/3 + 6\gamma)n$ vertices. Otherwise, as B_0 is maximal by (3.4.13), the larger of these two blue components, together with B_0, presents a contradiction to Claim (3.4.6). Hence it follows that

$$
(2/3 + 6\gamma)n \geq \sum_{\mathrm{i}} \deg_{\mathrm{blue}}(v_{\mathrm{i}}) \overset{(3.4.25)}{\geq} (1/3 + 14\gamma)n + \sum_{\mathrm{i}} |Y_{\mathrm{i}} \cap V(B_3^{\mathrm{i}})| \tag{3.4.26}
$$

which for $v_3^{\mathrm{i}} \overset{(3.4.16)}{\in} S_{\mathrm{i}} \cap V(B_3^{\mathrm{i}})$ implies

$$
\begin{aligned}
(1/3 - 8\gamma)n &\overset{(3.4.26)}{\geq} \sum_{\mathrm{i}} |Y_{\mathrm{i}} \cap V(B_3^{\mathrm{i}})| \\
&\geq \sum_{\mathrm{i}} \deg_G(v_3^{\mathrm{i}}, Y_{\mathrm{i}}) \\
&\geq \sum_{\mathrm{i}} \big(\deg_G(v_3^{\mathrm{i}}) - |N_{H_{\mathrm{i}}}(S_{\mathrm{i}})|\big) \\
&\overset{(3.4.1)}{\geq} 2(2/3 + 8\gamma)n - 2(1/2 + \gamma)n \\
&\geq (1/3 + 14\gamma)n,
\end{aligned}
$$

a contradiction.

3.4.2 (Case: C_1 and C_2 have distinct colours)

If C_1 or C_2 is spanning, we continue as in Subsection 3.4.1. Thus let us assume that there are vertices $v_i \in V(C_i) \setminus V(C_j)$ for $i, j = 1, 2$ and $j \neq i$. Since

$$|N_G(v_i) \cap N_G(v_j)| \geq (1/3 + 16\gamma)n, \tag{3.4.27}$$

it follows that $|V(C_1) \cap V(C_2)| \geq (1/3 + 16\gamma)n$. Moreover

$$|V(C_j) \setminus V(C_i)| \leq |V(G) \setminus N_G(v_i)| < (1/3 - 8\gamma)n \tag{3.4.28}$$

and hence $|V(C_j)| \geq (2/3 + 8\gamma)n$. By symmetry we can assume that C_1 is red and C_2 is blue. By (3.4.28) any two vertices outside of $V(C_1)$, which are each incident to at least $(n/6 - 4\gamma)n$ red edges, must be in the same red component, which we denote by C_3. (If no such vertex exists, we set $C_3 := \emptyset$.) We claim that $C := C_1 + C_2 + C_3$ satisfies the conditions of Lemma 3.3.2. As by Claim 3.4.1 $C_1 \cup C_2$ spans G, part (v) is trivial and since $|V(C_1) \cap V(C_2)| \geq (1/3 + 8\gamma)n$ part (ii)(a) holds as well. By (3.4.27), $N(v_1) \cap N(v_2)$ contains an edge. So v_1 is on a red triangle or v_2 is on a blue triangle. Hence parts (iii) follows.

It remains to show that C satisfies part (i). Let S be any stable set in C. We have to show that $|N_C(S)| \geq |S| + \gamma n$. Let us make the following observations. Any vertex v in $V(C_1) \cap V(C_2)$ or $V(C_3)$ has degree

$$\deg_C(v) \geq (2/3 + 8\gamma)n.$$

Recall that by definition of C_3, any vertex $v \in V(C_2) \setminus V(C_1 + C_3)$ has $\deg_{\mathrm{red}}(v) < (1/6 - 4\gamma)n$ and so

$$\deg_C(v) \geq (2/3 + 8\gamma)n - |V(C_2) \setminus V(C_1 + C_3)| \geq (1/2 + 12\gamma)n. \tag{3.4.29}$$

Note that since S is bad

$$\begin{aligned} 2|N_C(S)| - \gamma n < |S| + |N_C(S)| \leq n & \qquad \Leftrightarrow \\ |N_C(S)| < (n/2 + \gamma/2)n. & \end{aligned} \tag{3.4.30}$$

Since $|N_C(S)| \geq \deg_C(v)$ for any $v \in S$, (3.4.29) and (3.4.30) imply that $S \subseteq V(C_1) \setminus V(C_2)$. By (3.4.28) this gives $|S| \leq (1/3 - 8\gamma)n$. However vertices $v \in V(C_1) \setminus V(C_2)$ have degree

$$\deg_C(v) \geq (2/3 + 8\gamma)n - |V(C_1) \setminus V(C_2)| \overset{(3.4.28)}{\geq} (1/3 + 16\gamma)n$$

and thus $|N_C(S)| \geq |S| + 24\gamma n$ which is clearly enough.

3.4.3 (Case: C_1 and C_2 the same colour)

Suppose that $R_1 := C_1$ and $R_1 := C_2$ are both red with

$$|V(R_1)| \geq |V(R_2)|. \tag{3.4.31}$$

Let $B_1, \ldots, B_t$ be the blue components intersect that with R_2. If $t = 1$, we continue as in Subsection 3.4.2. Hence we can assume that $t \geq 2$. So for $i \in [t]$ and $v_i \in V(B_i) \cap V(R_2)$ we have

$$\begin{aligned} |V(R_1)| &\geq \sum_{i \in [t]} \deg(v_i, V(R_1)) \\ &\geq t(2/3 + 8\gamma)n - t|V(R_2)|, \end{aligned}$$

which, since $R_1 + R_2$ is spanning, gives

$$\frac{n}{2} \overset{(3.4.31)}{\geq} |V(R_2)| \overset{(3.4.31)}{\geq} \frac{t(2/3 + 8\gamma)n - n}{t - 1}. \tag{3.4.32}$$

This implies $t = 2$ and so $|V(R_2)| \geq (1/3 + 16\gamma)n$ by (3.4.32). Thus every vertex in $V(R_1)$ has a neighbour in $V(R_2)$ and so $V(G) = V(B_1 + B_2)$ as well. Set $I_{\mathrm{i},j} = R_{\mathrm{i}} \cap B_j$ for $\mathrm{i}, j \in [2]$ and note that the sets $I_{\mathrm{i},j}$ are each non-empty and partition $V(G)$. Since vertices in $I_{\mathrm{i},j}$ have no neighbours in $I_{3-\mathrm{i},3-j}$ and as $\delta(G) \geq (2/3 + 8\gamma)m$ we obtain

$$|I_{\mathrm{i},j}| < (1/3 - 8\gamma)n \text{ for } \mathrm{i}, j \in [2]. \tag{3.4.33}$$

In particular, this implies that each

$$R_1, R_2, B_1, B_2 \text{ contain each at least } (1/3 + 16\gamma)n \text{ vertices,} \tag{3.4.34}$$

which in turn gives

$$\begin{aligned} \delta\big(G[I_{\mathrm{i},j}]\big) &\geq \delta(G) - |I_{3-\mathrm{i},j} \cup I_{\mathrm{i},3-j}| \\ &\overset{(3.4.33)}{\geq} (2/3 + 8\gamma)n - 2(1/6 - 8\gamma)n \geq 24\gamma n. \end{aligned} \tag{3.4.35}$$

We will show that the union of three components of $\{R_1, R_2, B_1, B_2\}$ satisfies the conditions of Lemma 3.3.2 or the colouring is (4γ)-extremal as in Definition 3.3.1(b). Note that for any such union, Lemma 3.3.2(v) holds since $R_1 \cup R_2$, $B_1 \cup B_2$ span G and part (ii)(a) follows by (3.4.34). The next claim yields parts (iii) and (iv).

Claim 3.4.8. *If one colour has a bipartite component, then the other colour has no bipartite component.*

Proof. Suppose that, R_1 say, is bipartite and let X and Y be its colour classes. We claim that (after possibly switching X and Y)

$$X = I_{1,1} \text{ and } Y = I_{1,2}. \tag{3.4.36}$$

Suppose otherwise and let $x \in I_{1,1} \cap X$ and $y \in I_{1,1} \cap Y$. Note that x and y have each $(2/3 + 8\gamma)n - |V(B_1)|$ neighbours in $I_{1,2} \cap X$ and $I_{1,2} \cap Y$ respectively. Thus by (3.4.33)

$$\begin{aligned} 2\big((2/3 + 8\gamma)n - |V(B_1)|\big) \leq |I_{1,2}| < (1/3 - 8\gamma)n \qquad &\Leftrightarrow \\ n/2 < |V(B_1)|. \qquad & \end{aligned}$$

However, since by (3.4.33) each vertex in $I_{1,1}$ has neighbours in $I_{1,2}$, we obtain that $I_{1,2}$ has vertices of both X and Y as well. By a symmetric argument it follows that $n/2 < |V(B_2)|$, a contradiction. This proves (3.4.36).

If, B_1 say, is bipartite as well, then the same reasoning shows that one of its colours classes equals $I_{1,1}$. Consequently $I_{1,1}$ contains no edges in contradiction to (3.4.35). This proves the claim. □

It remains to show that the union of any three components of R_1, R_2, B_1, B_2 satisfies Lemma 3.3.2(i) or the colouring is (4γ)-extremal as in Definition 3.3.1(b). To this end, let us assume that the union of any three components of R_1, R_2, B_1, B_2 does not satisfy Lemma 3.3.2(i).

For each $\mathrm{i} \in [2]$ let S_{R_i} be a bad set of minimum size in $G - E(R_\mathrm{i})$. Similarly, let S_{B_i} be a bad set of minimum size in $G - E(B_\mathrm{i})$. (Note that these sets exists, by our assumptions above.) Then

$$\begin{aligned} 2|N_{G-E(R_\mathrm{i})}(S)| - \gamma n < |S| + |N_{G-E(R_\mathrm{i})}(S)| \leq n \qquad & \Leftrightarrow \\ |N_{G-E(R_\mathrm{i})}(S)| < (n/2 + \gamma/2)n. & \end{aligned}$$

Recall that the sets $I_{k,j}$ partition $V(G)$. So any vertex $v \in V(R_{3-\mathrm{i}})$ satisfies $\deg_G(v) = \deg_{G-E(R_\mathrm{i})}(v)$ and hence $v \notin S_{R_\mathrm{i}}$. It follows by symmetry that

$$S_{R_\mathrm{i}} \subseteq V(R_\mathrm{i}) \text{ and } S_{B_\mathrm{i}} \subseteq V(B_\mathrm{i}). \tag{3.4.37}$$

In fact, we can show a bit more:

Claim 3.4.9. *For each* $\mathrm{i} \in [2]$ *there are* $j, k \in [2]$ *such* $S_{R_\mathrm{i}} \subseteq I_{j,k}$. *Similarly, for each* $\mathrm{i} \in [2]$ *there are* $j, k \in [2]$ *such that* $S_{B_\mathrm{i}} \subseteq I_{j,k}$.

Proof. Suppose that the claim does not hold for, say S_{R_2}. So $S_{R_2} \cap V(B_\mathrm{i}) \neq \emptyset$ for $\mathrm{i} = 1, 2$. Then

$$\begin{aligned} & |S_{R_2} \cap V(B_1)| + |S_{R_2} \cap V(B_2)| + \gamma n \\ &= |S_{R_2}| + \gamma n > |N_{G-E(R_S)}(S_{R_2})| \\ &\geq |N_{G-E(R_S)}\left(S_{R_2} \cap V(B_1)\right)| + |N_{G-E(R_S)}\left(S_{R_2} \cap V(B_2)\right)|, \end{aligned}$$

where the last line follows because the edges between $S_{R_2} \cap B_1$ and $S_{R_2} \cap B_2$ are red (and hence in R_2). Thus by Definition 3.4.2, $S_{R_2} \cap V(B_1)$ or $S_{R_2} \cap V(B_2)$ is a bad set as well in contradiction to the minimality of S_{R_2}. □

Claim 3.4.10. *Let* $\mathrm{i}, j \in [2]$. *If* $S_{R_\mathrm{i}} \cap V(B_j) \neq \emptyset$, *then* $I_{3-\mathrm{i},j} \subseteq N_{G-E(R_\mathrm{i})}(S_{R_\mathrm{i}})$. *Similarly, if* $S_{B_\mathrm{i}} \cap V(R_j) \neq \emptyset$, *then* $I_{j,3-\mathrm{i}} \subseteq N_{G-E(B_\mathrm{i})}(S_{B_\mathrm{i}})$.

Proof. Suppose the claim is wrong for S_{R_2} and B_1. Let $v \in S_{R_2} \cap V(B_1)$ and $w \in I_{1,1} \setminus N_{G-E(R_2)}(S_{R_2})$. Note (3.4.37) and Claim 3.4.9 imply

$$S_{R_2} \subseteq I_{2,1}. \tag{3.4.38}$$

By choice of S_{R_2}, we have

$$|S_{R_2}| + \gamma n > |N_{G-E(R_2)}(S_{R_2})| \geq \deg_{G-E(R_2)}(v, I_{1,1}) \geq (2/3 + 8\gamma)n - |V(R_2)|$$

and

$$|I_{2,1} \setminus S_{R_2}| \geq \deg_{G-E(R_2)}(w, I_{2,1}) \geq (2/3 + 8\gamma)n - |V(R_1)|$$

Summing the two inequalities gives

$$\begin{aligned} |I_{2,1}| + |S_{R_2}| - |S_{R_2} \cap I_{2,1}| > (4/3 + 15\gamma)n - |V(R_1)| - |V(R_2)| \qquad & \Leftrightarrow \\ |I_{2,1}| \overset{(3.4.38)}{>} (1/3 + 15\gamma)n & \end{aligned}$$

in contradiction to (3.4.33). □

Claim 3.4.11. *If $S_{R_i} \subseteq I_{i,j}$ for $i, j \in [2]$, then $S_{B_j} \subseteq I_{3-i,j}$. Similarly, if $S_{B_i} \subseteq I_{j,i}$ for $i, j \in [2]$, then $S_{R_j} \subseteq I_{j,3-i}$.*

Proof. Suppose the claim is wrong for S_{R_2} and $I_{2,2}$, i.e. $S_{R_2} \subseteq I_{2,2}$ and $S_{B_2} \cap I_{2,2} \neq \emptyset$. By Claim 3.4.9 this implies

$$S_{R_2}, S_{B_2} \subseteq I_{2,2}. \tag{3.4.39}$$

For any $v \in I_{2,2}$ we have

$$\begin{aligned}
\deg_G(v, I_{2,2}) &\geq (2/3 + 8\gamma)n - |I_{1,2}| - |I_{2,1}| \\
&\geq \big((1/3 + 4\gamma)n - |I_{1,2}|\big) + \big((1/3 + 4\gamma)n - |I_{2,1}|\big) \\
&\overset{(3.4.33)}{\geq} \big(|I_{2,2}| - |I_{1,2}|\big) + \big(|I_{2,2}| - |I_{2,1}|\big) + 24\gamma n.
\end{aligned}$$

Hence v has either

$$\deg_{\text{red}}(v, I_{2,2}) > |I_{2,2}| - |I_{2,1}| + 12\gamma n$$

or

$$\deg_{\text{blue}}(v, I_{2,2}) > |I_{2,2}| - |I_{1,2}| + 12\gamma n.$$

Observe that since $|I_{1,1}| + |I_{2,2}| < (2/3 - 16\gamma)n$ by (3.4.33), it follows that $|I_{1,2}| + |I_{2,1}| \geq (1/3 + 16\gamma)n$. Hence

$$\begin{aligned}
\big(|I_{2,2}| - |I_{1,2}|\big) + \big(|I_{2,2}| - |I_{2,1}|\big) &\overset{(3.4.33)}{\leq} |I_{2,2}| - (1/3 + 16\gamma)n + (1/3 - 8\gamma)n \\
&= |I_{2,2}| - 24\gamma n.
\end{aligned}$$

So without loss of generality there are more than $|I_{2,2}| - |I_{1,2}| + 16\gamma n$ vertices in $I_{2,2}$ which each have $\deg_{\text{red}}(v, I_{2,2}) > |I_{2,2}| - |I_{2,1}| + 12\gamma n$. However, by (3.4.39) and Claim 3.4.10

$$|I_{2,1}| \leq |N_{G-E(B_2)}(S_{B_2})| < |S_{B_2}| + \gamma n.$$

But this contradicts that S_{B_2} is stable set in $G - E(B_2)$ and therefore contains no red edges. □

By Claim 3.4.9 we can assume that without loss of generality $S_{R_2} \subseteq I_{2,1}$. It follows by (3.4.37) and recursively applying Claim 3.4.11 that $S_{B_1} \subseteq I_{1,2}$, $S_{R_1} \subseteq I_{1,2}$ and $S_{B_2} \subseteq I_{2,2}$. Thus Claim 3.4.10 yields the following inequalities

$$\begin{aligned}
|I_{1,1}| &\leq |N_{G-E(R_2)}(S_{R_2})| < |S_{R_2}| + \gamma n \leq |I_{2,1}| + \gamma n, \\
|I_{2,1}| &\leq |N_{G-E(B_2)}(S_{B_2})| < |S_{B_2}| + \gamma n \leq |I_{2,2}| + \gamma n, \\
|I_{2,2}| &\leq |N_{G-E(R_1)}(S_{R_1})| < |S_{R_1}| + \gamma n \leq |I_{1,2}| + \gamma n \text{ and} \\
|I_{1,2}| &\leq |N_{G-E(B_1)}(S_{B_1})| < |S_{B_1}| + \gamma n \leq |I_{1,1}| + \gamma n.
\end{aligned}$$

Hence the colouring is (4γ)-extremal as in Definition 3.3.1(b).

3.5 Distribute exceptional vertices

In this Section we prove Claim 3.3.7. The proof differs slightly depending on the two outcomes of Lemma 3.3.2(ii). Before we start let us explain shortly a hidden difficulty in the proof. Suppose that we want to prove Claim 3.3.7 given Lemma 3.3.2(ii)(a). We set

$$\mathcal{Z} = \bigcup_{1 \leq \mathrm{i} \neq j \leq 3} V(C_\mathrm{i}) \cap V(C_j) \subseteq V(\mathcal{G}), \quad Z = \bigcup_{x \in \mathcal{Z}} V(x) \subseteq V(G).$$

Note that as $\delta(G) \geq (2/3 + \beta)n$, we have $\deg_G(w, Z) \geq \beta n$ for any $w \in V(G)$. Suppose, for sakes of exposition, that $\mathcal{Z}$ is contained in the, red say, connected component C_1 and all edges between V^{Exc} and Z are red. With a little caution we can greedily choose red paths $P_1, \ldots, P_r$, which alternate between Z and V^{Exc}, and such that all vertices of V^{Exc} are covered and no cluster $V(x)$ is hit too many times. However in order to obtain P^{Exc} we need to connect the P_i's with each other. Suppose that P_i ends in $V(x_\mathrm{i})$ and let $x_\mathrm{i} y_\mathrm{i} \in E(\mathcal{G}_{\mathrm{red}})$ be the red edge connecting x_i to the other vertices of C_1. To connect P_i with the other paths we have to use at least 2 vertices of $V(y_\mathrm{i})$. However it may be the case that $y = y_1$ for all $\mathrm{i} \in [r]$ and $\{y, x_1, \ldots, x_r\}$ induces a red star in $\mathcal{G}$. Then we have to use $2r$ vertices of $V(y)$. So in order to connect the paths P_i, we need to bound r (see (3.5.5)), which requires some additional arguments.

Proof of Claim 3.3.7 given Lemma 3.3.2(ii)(a). Let us define

$$\mathcal{Z} = \bigcup_{1 \leq \mathrm{i} \neq j \leq 3} V(C_\mathrm{i}) \cap V(C_j) \subseteq V(\mathcal{G}), \quad Z = \bigcup_{x \in \mathcal{Z}} V(x) \subseteq V(G).$$

By Lemma 3.3.2(ii)(a) we have $|\mathcal{Z}| \geq (1/3 + \gamma)m$ and therefore $|Z| \geq (1 - \varepsilon)n|\mathcal{Z}|/m \geq n/3$ by (3.3.2) and ($\ll$). So as $\delta(G) \geq (2/3 + \beta)n$, it follows that $\deg_G(w, Z) \geq \beta n$ for any $w \in V(G)$. Let us set

$$r = \lceil 2/\beta \rceil. \tag{3.5.1}$$

We can partition V^{Exc} by setting $V_1^{\mathrm{Exc}} := \{v \in V^{\mathrm{Exc}} : \deg_{\mathrm{red}}(v, Z) \geq n/r\}$ and $V_2^{\mathrm{Exc}} := V^{\mathrm{Exc}} \setminus V_1^{\mathrm{Exc}}$. Let $\mathrm{i} \in [2]$ and recall that $G_\mathrm{i} \subseteq G$ is the subgraph on $V(G)$ with edges of colour i. Define an auxiliary graph H_i on vertex set $V_\mathrm{i}^{\mathrm{Exc}}$, by connecting two vertices $v, w \in V_\mathrm{i}^{\mathrm{Exc}}$ if $|N_{G_\mathrm{i}}(v, Z) \cap N_{G_\mathrm{i}}(w, Z)| \geq n/r^3$. It follows that

$$\text{the independence number of } H_\mathrm{i} \text{ is bounded by } r. \tag{3.5.2}$$

Indeed, suppose otherwise and let $w_1, \ldots, w_{r+1}$ be pairwise disjoint non-adjacent vertices in H_i. We obtain a contradiction as follows.

$$\begin{aligned} |Z| &\geq \left| \bigcup_{p \in [r+1]} N_{G_\mathrm{i}}(w_p, Z) \right| \\ &\geq (r+1)\frac{|Z|}{r} - \sum_{1 \leq q < p \leq r+1} |N_{G_\mathrm{i}}(w_q, Z) \cap N_{G_\mathrm{i}}(w_p, Z)| \\ &\geq |Z| \Big(\frac{r+1}{r} - \frac{\binom{r+1}{2}}{r^3} \Big) > |Z|. \end{aligned}$$

This proves (3.5.2). Now a classic result of Pósa (see [77]) guarantees that H_{i} can be partitioned into $r_{\mathrm{i}} \le r$ disjoint cycles $F_{\mathrm{i}}^1, \dots, F_{\mathrm{i}}^{r_{\mathrm{i}}}$ (edges and vertices).

For each $z \in \mathcal{Z}$ we fix a red and a blue edge $\mathrm{e}_{\mathrm{red}}(z), \mathrm{e}_{\mathrm{blue}}(z) \in E(\mathcal{C})$ containing z. We set Z^{Bad} to be $\bigcup_{z\in\mathcal{Z}} \mathrm{Bad}(\mathrm{e}_{\mathrm{red}}(z)) \cup \mathrm{Bad}(\mathrm{e}_{\mathrm{blue}}(z))$. In other words, each vertex in $Z \setminus Z^{\mathrm{Bad}}$ is $\mathcal{M}^{\text{Blow Up}}$-typical with respect to some red and some blue edge of $\mathcal{C}$. By definition of H_{i} any two consecutive vertices $v, w \in V(F_{\mathrm{i}}^j)$ share a colour i neighbourhood of size $|N_{G_{\mathrm{i}}}(v, Z) \cap N_{G_{\mathrm{i}}}(w, Z)| \ge n/r^3$. Hence

$$\begin{aligned} |\big(N_{G_{\mathrm{i}}}(v,Z) \cap N_{G_{\mathrm{i}}}(w,Z)\big) \setminus Z^{\mathrm{Bad}}| &\overset{(3.3.4)}{\ge} n/r^3 - 8\varepsilon|V(x)||\mathcal{Z}| \\ &\ge n/r^3 - 8\varepsilon n \overset{(3.5.1),(\ll)}{\ge} \beta n/4. \end{aligned}$$

Let $Z(v, w)$ consist of all $z \in Z$ for which

$$|\big(N_{G_{\mathrm{i}}}(v,Z) \cap N_{G_{\mathrm{i}}}(w,Z) \cap V(z)\big) \setminus V(Z^{\mathrm{Bad}})| \ge \beta|V(x)|/4. \tag{3.5.3}$$

We can bound the size of $Z(v, w)$ by

$$\frac{|V^{\mathrm{Exc}}|}{\frac{\sqrt{\varepsilon}}{2}|V(x)|} \overset{(3.3.6)}{\le} \frac{5\varepsilon n}{\frac{\sqrt{\varepsilon}}{2}|V(x)|} \overset{(\ll)}{<} \frac{\beta n/4}{|V(z)|} \le |Z(v,w)|. \tag{3.5.4}$$

This allows us to (semi-)greedily choose a colour i path P_{i}^j for each cycle F_{i}^j satisfying the following properties:

(i) each P_{i}^j contains $V(F_{\mathrm{i}}^j)$ and alternates between $V(F_{\mathrm{i}}^j)$ and Z,

(ii) $|V(x) \cap \bigcup_{\mathrm{i},j} V(P_{\mathrm{i}}^j)| \le \sqrt{\varepsilon}|V(x)|/2$,

(iii) the paths P_{i}^j and $P_{\mathrm{i}'}^{j'}$ are vertex disjoint if $j \ne j'$ or $\mathrm{i} \ne \mathrm{i}'$,

(iv) each P_{i}^j ends in $\mathcal{M}^{\text{Blow Up}}$-typical vertices in an edge $xy \in E(\mathcal{C})$ of colour i (and with $x \in \mathcal{Z}$).

More precisely, let $\mathcal{P}'$ be a collection of paths satisfying the above conditions for $\mathrm{i} < \mathrm{i}'$, $j < j'$. When embedding vertices $v, w \in Z$ which are neighbours in the cycle $V(F_{\mathrm{i}'}^{j'})$, inequality (3.5.4) guarantees that there is a vertex $z \in Z(v, w)$ such that $|V(z) \cap \bigcup V(\mathcal{P}')| < \sqrt{\varepsilon}|V(z)|/2$. This allows us to extend the path $P_{\mathrm{i}'}^{j'}$ greedily as

$$\begin{aligned} &|\big(N_{G_{\mathrm{i}'}}(v,Z) \cap N_{G_{\mathrm{i}'}}(w,Z) \cap V(z)\big) \setminus V(Z^{\mathrm{Bad}})| - |V(\mathcal{P}')| \\ &\overset{(3.5.3),(3.3.6)}{\ge} \beta|V(x)|/4 - 5\varepsilon n \overset{(\ll)}{>} 0. \end{aligned}$$

By Lemma 3.2.3 we can join these $r_1 + r_2$ paths P_{i}^j with monochromatic paths, each of size $2m$, to three monochromatic paths $P_{\mathrm{i}}^{\mathrm{Exc}}$ that end in the same vertices of $P_{\mathrm{i}}^{\mathrm{Con}}(\mathrm{e}^{\mathrm{Exc}})$ for some $\mathrm{e}^{\mathrm{Exc}} \in E(\mathcal{C}_{\mathrm{i}})$. As desired, we have for every $x \in V(\mathcal{G})$

$$|V(x) \cap V(P^{\mathrm{Exc}})| \le \sqrt{\varepsilon}|V(x)|/2 + 4rm \overset{(3.5.1),(\ll)}{\le} \sqrt{\varepsilon}|V(x)|. \tag{3.5.5}$$

$\square$

Proof of Claim 3.3.7 given Lemma 3.3.2(ii)(b). The proof goes almost identically. Without loss of generality we can assume that C_1 has colour 1. Set $\mathcal{Z} = V(C_1)$, $Z = \bigcup_{x \in \mathcal{Z}} V(x)$ and define H_i as above. The main difference is that since $\deg_2(v, Z) \geq (2/3 + \beta/4)n$ for any $v \in Z$, H_2 has now a spanning cycle (edge or vertex) F_2^1 or is empty, i.e. $r_2 \leq 1$. So we can proceed as above, with the only difference being that (if $r_2 = 1$) we let P_2^1 end in the same $\mathcal{M}^{\text{Blow Up}}$-typical vertices. Thus P_2^1 is a blue cycle on its own. $\square$

3.6 Extremal colourings

In this section we prove Claim 3.3.5 and Claim 3.3.6. We will use the following result about Hamiltonian cycles.

Theorem 3.6.1 (Chvátal)**.** *Let G be a graph with vertex degree sequence $\mathrm{d}_1 \leq \ldots \leq \mathrm{d}_n$. If for every $1 \leq \mathrm{i} < n/2$ we have $\mathrm{d}_\mathrm{i} \geq \mathrm{i} + 1$ or $\mathrm{d}_{n-\mathrm{i}} \geq n - \mathrm{i}$, then G is Hamiltonian.*

Corollary 3.6.2. *Let H be a balanced bipartite graph with bipartition classes X and Y. Let X have vertex degree sequence $x_1 \leq \ldots \leq x_n$ and Y have vertex degree sequence $y_1 \leq \ldots \leq y_n$. If for every $\mathrm{i} \in [n]$ we have $x_\mathrm{i} \geq \mathrm{i} + 1$ or $y_{n-\mathrm{i}} \geq n - \mathrm{i} + 1$, then H is Hamiltonian.*

Proof. Let H' be obtained from H by adding edges between vertices of Y until $G[Y]$ is complete. So every vertex in y gains $n-1$ new neighbours. Let $\mathrm{d}_1, \ldots, \mathrm{d}_{2n}$ be the degree sequence of H'. Since by assumption $y_\mathrm{i} \geq 1$, we can assume that $\mathrm{d}_\mathrm{i} = x_\mathrm{i}$ and $\mathrm{d}_{2n-\mathrm{i}+1} = y_{n-\mathrm{i}+1}$ for $\mathrm{i} \in [n]$. It follows that $\mathrm{d}_\mathrm{i} \geq \mathrm{i} + 1$ or $\mathrm{d}_{2n-\mathrm{i}} \geq 2n - \mathrm{i}$ for $\mathrm{i} \in [n-1]$. By Theorem 3.6.1 H', has a Hamiltonian cycle C. Since $|X| = |Y|$, C has no edges within Y. Thus C is a Hamiltonian cycle of H as well. $\square$

Proof of Claim 3.3.5. Let $\mathcal{R}$ be the spanning, red say, bipartite component of $\mathcal{G}$ with bipartition classes $\mathcal{X}_R$ and $\mathcal{Y}_R$. As the colouring is (4γ)-extremal we have

$$||\mathcal{X}_R| - |\mathcal{Y}_R|| \leq 4\gamma m. \tag{3.6.1}$$

Set $X_R := \bigcup_{x \in \mathcal{X}_R} V(x)$ and $Y_R := \bigcup_{y \in \mathcal{Y}_R} V(y)$. Let $\mathcal{B}_\mathrm{i}$ be the bipartite blue components of $\mathcal{G}$ with bipartition classes $\mathcal{X}_{B_\mathrm{i}}, \mathcal{Y}_{B_\mathrm{i}}$. Suppose that $V(\mathcal{B}_1) \subseteq \mathcal{X}_R$ and $V(\mathcal{B}_2) \subseteq \mathcal{Y}_R$.

By (3.6.1), it follows that $\deg_{\text{blue}}(x, \mathcal{X}_R), \deg_{\text{blue}}(y, \mathcal{X}_R) \geq (1/6 + 4\gamma)m$ for any $x, y \in \mathcal{X}_R$. If xy is additionally a (blue) edge, then $N_{\text{blue}}(x, \mathcal{X}_R) \cap N_{\text{blue}}(y, \mathcal{X}_R) = \emptyset$ and hence

$$|\mathcal{X}_{B_1}|, |\mathcal{Y}_{B_1}| \leq |\mathcal{X}_R| - (1/6 + 4\gamma)m \overset{(3.6.1)}{\leq} m/3. \tag{3.6.2}$$

Fix a vertex $x \in \mathcal{X}_{B_1}$. Since $\mathcal{G}_{\text{red}}[\mathcal{X}_{B_1}]$ is stable, it follows that

$$\mathrm{e}_{\text{red}}\left(V(x), \bigcup_{y \in \mathcal{X}_{B_1}} V(y)\right) \overset{(3.3.3)}{\leq} \mathrm{d}|\mathcal{X}_{B_1}|\,(n/m)^2.$$

So by (3.6.2), all but at most $\sqrt{\mathrm{d}}n/m$ vertices v of $V(x)$ satisfy

$$\begin{aligned} \deg_G(v, X_R) &\leq \deg_G(v, \bigcup_{y \in \mathcal{X}_{B_2}} V(y)) + \deg_G(v, \bigcup_{y \in \mathcal{X}_{B_1}} V(y)) \\ &\leq n/3 + \sqrt{\mathrm{d}}|\mathcal{X}_{B_1}|n/m \\ &\overset{(\ll)}{\leq} (1/3 + \beta/4)n. \end{aligned} \tag{3.6.3}$$

This motivates the following definition:

$$X_R^{1/3} = \{v \in X_R : \ \deg_{\text{red}}(v, Y_R) \le (1/3 + \beta/2)n\},$$
$$Y_R^{1/3} = \{v \in Y_R : \ \deg_{\text{red}}(v, X_R) \le (1/3 + \beta/2)n\}.$$

By (3.6.3) and by symmetry we have

$$|X_R^{1/3}|, |Y_R^{1/3}| \le \sqrt{\mathrm{d}} n \overset{(\ll)}{\le} \frac{\beta}{4} n. \tag{3.6.4}$$

Next, let

$$X'_R = \{v \in V(G) \setminus Y_R : \ \deg_{\text{blue}}(v, Y_R) \le \beta n\},$$
$$Y'_R = \{v \in V(G) \setminus X_R : \ \deg_{\text{blue}}(v, X_R) \le \beta n\}.$$

Note that, by (3.3.4), for each $x \in V(\mathcal{B}_1)$ the cluster $V(x)$ contains at most $4\varepsilon|V(x)|$ vertices which are not $\mathcal{M}^{\text{Blow Up}}$-typical in $xy \in E(B_1)$ for all $y \in N_{\mathcal{G}}(x)$. Since there are no blue bridges, this implies that $V(G) \setminus (X'_R \cup Y'_R)$ contains at most one vertex z^* (if this set is empty, we set $z^* = \emptyset$). Without loss of generality suppose that $|X'_R| \ge |Y'_R|$. By definition, any $v \in X'_R$ has

$$\begin{aligned} \deg_{\text{red}}(v, Y_R) &\ge (2/3 + \beta)n - |X_R| - |V_0| \\ &\ge (2/3 + \beta)n - (1/2 + 4\gamma)n - \varepsilon n \\ &\ge (1/6 + \beta/2)n. \end{aligned} \tag{3.6.5}$$

Similarly $\deg_{\text{red}}(y, X_R) \ge (1/6 + \beta/2)n$ for each $y \in Y'_R$.

Before we find a large red cycle in R we need to balance the sizes of sets X'_R and Y'_R. To this end, let us construct a blue cycle P_{blue} with $V(P_{\text{blue}}) \subseteq X_R$, of size $2\lceil (|X'_R| - |Y'_R|)/2 \rceil$. This is possible because $\mathcal{X}_R$ is stable in $\mathcal{R}$ and so the definition of the reduced graph implies that X_R contains at most $\mathrm{d}n^2$ red edges and $\delta(G) \ge (2/3 + \beta)n$. If $|X'_R \setminus V(P_{\text{blue}})| = |Y'_R \setminus V(P_{\text{blue}})|$, we set $C^* := z^*$. If $|X'_R \setminus V(P_{\text{blue}})| = |Y'_R \setminus V(P_{\text{blue}})| + 1$, we choose a path C^* of order at most 2 which contains z^* and some $y \in X'_R$.

To finish we claim that $H = R[V(G) \setminus V(P_{\text{blue}} + C^*)]$ has a Hamiltonian cycle. This follows from Corollary 3.6.2 together with (3.6.5) and (3.6.4). Note that the degree conditions are fulfilled, because $\delta(H) \ge (1/6 + \beta/2)n$ by (3.6.5). Moreover, by (3.6.4) all but $\beta n/2$ vertices of H have a degree of at least $(1/3 + \beta/2)n$. □

We will use the following fact in the proof of Claim 3.3.6.

Fact 3.6.3. *Let G be a graph with a cycle C and a vertex $v \in V(G) \setminus V(C)$. If $\deg_G(v, V(C)) \ge |C|/2$, then there is a cycle C' with $V(C) \cup \{v\}$.*

Proof. By the pigeonhole principle there is an edge $uw \in E(C)$ such that v is adjacent to both u and w. Thus we can take $C' = C - uw + vu + vw$. □

Proof of Claim 3.3.6. Denote the red and blue components by $\mathcal{R}_{\mathrm{i}}$ and $\mathcal{B}_{\mathrm{i}}$ and suppose that there are no red or blue bridges. Set $\mathcal{I}_{\mathrm{i},j} = V(\mathcal{R}_{\mathrm{i}}) \cap V(\mathcal{B}_j)$ for $\mathrm{i}, j \in [2]$. We assume that $G[\mathcal{I}_{\mathrm{i},j}]$ is blue if $\mathrm{i} = j$ and red otherwise. Let $R_{\mathrm{i}} = \bigcup_{x \in V(\mathcal{R}_{\mathrm{i}})} V(x)$, $B_{\mathrm{i}} = \bigcup_{x \in V(\mathcal{B}_{\mathrm{i}})} V(x)$ and $I_{\mathrm{i},j} = \bigcup_{x \in \mathcal{I}_{\mathrm{i},j}} V(x)$ for $\mathrm{i}, j \in [2]$. Recall that by (3.3.4) for any $x \in V(\mathcal{G})$ there are at most

$4\varepsilon n$ vertices $v \in V(x)$ such that v is not $\mathcal{M}^{\text{Blow Up}}$-typical in $xy \in E(\mathcal{G})$ for all $y \in N_{\mathcal{G}}(x)$. This and the assumption that there are no red bridges yields that there is at most one vertex z_{red} with both $\deg_{\text{red}}(z_{\text{red}}, R_1) \geq \beta n/8$ and $\deg_{\text{red}}(z_{\text{red}}, R_2) \geq \beta n/8$. Similarly, as there are no blue bridges, there is at most one vertex z_{blue} with both $\deg_{\text{blue}}(z_{\text{blue}}, B_1) \geq \beta n/8$ and $\deg_{\text{blue}}(z_{\text{blue}}, B_2) \geq \beta n/8$.

Let us handle these extra vertices first. If at most one of z_{red} and z_{blue} exists, we let this vertex be a cycle on its own. If both z_{red} and z_{blue} exist we proceed as follows. If z_{blue} has at least $\beta n/8$ red edges, we can construct a red cycle C^* of order at most 42 containing z_{red} and z_{blue}. Otherwise, we set $C^* = \emptyset$ and note that by symmetry

$$z_{\text{red}}\ (z_{\text{blue}}) \text{ has red (blue) degree at least } (2/3 + \beta/2)n. \tag{3.6.6}$$

In this case we can integrate z_{red} or z_{blue} into one of the large cycles that we find in the next step.

Let $\mathrm{i}, j \in [2]$. By (3.3.3) all but at most $2\sqrt{\mathrm{d}}n$ vertices in $I_{\mathrm{i},j}$ have more than $2\sqrt{\mathrm{d}}n$ neighbours in $I_{3-\mathrm{i},3-j}$. Thus for

$$I'_{\mathrm{i},j} := \{v \in V(G) \setminus V(C^*) :\ \deg_G(v, I_{3-\mathrm{i},3-j}) \leq 2\sqrt{\mathrm{d}}n\}$$

it holds that

$$|I_{\mathrm{i},j} \cap I'_{\mathrm{i},j}| \geq |I_{\mathrm{i},j}| - 2\sqrt{\mathrm{d}}n. \tag{3.6.7}$$

Note that since $\delta(G) \geq (2/3 + \beta)n$ we have

$$(1/4 - 5\gamma)n \overset{(\ll)}{\leq} (1-\varepsilon)|\mathcal{I}_{\mathrm{i},j}|n/m \overset{(3.3.2)}{\leq} |I_{\mathrm{i},j}| \overset{(3.3.2)}{\leq} |\mathcal{I}_{\mathrm{i},j}|n/m \leq (1/4 + 4\gamma)n \tag{3.6.8}$$

for $\mathrm{i}, j \in [2]$. Hence the sets $I'_{\mathrm{i},j}$ are pairwise disjoint. Moreover, since there are no red or blue bridges, all vertices but z_{red} and z_{blue} (should they exist) are in some $I'_{\mathrm{i},j}$. Finally, (3.6.7) and (3.6.8) imply that

$$(1/4 - 3\sqrt{\mathrm{d}})n \overset{(\ll)}{\leq} |I'_{\mathrm{i},j}| \overset{(\ll)}{\leq} (1/4 + 3\sqrt{\mathrm{d}})n. \tag{3.6.9}$$

Claim 3.6.4. *For any* $\mathrm{i} \in [2]$ *the subgraph* $G[I'_{1,\mathrm{i}} \cup I'_{2,\mathrm{i}}]$ *contains a blue spanning cycle and the subgraph* $G[I'_{\mathrm{i},1} \cup I'_{\mathrm{i},2}]$ *contains a red spanning cycle.*

Proof. We will show that $G[I'_{1,1} \cup I'_{2,1}]$ contains a blue spanning cycle. The other cases follow identically. Without loss of generality suppose that $|I'_{1,1}| \geq |I'_{2,1}|$. Let us first make two observations. By (3.3.3), there are at most $\mathrm{d}n^2$ red edges between $I_{1,1}$ and $I_{2,1}$. Similarly, there are at most $\mathrm{d}n^2$ edges between $I_{1,1}$ and $I_{2,2}$. Hence all but $\sqrt{\mathrm{d}}n$ vertices of $w \in I_{1,1}$ satisfy

$$\deg_{\text{blue}}(w, B_1) \geq (2/3 + \beta)n - |I_{1,2}| - 2\sqrt{\mathrm{d}}n \overset{(\ll)}{\geq} (5/12 + \beta/2)n.$$

So by (3.6.9) and the order of C^* at most $2\sqrt{\mathrm{d}}n$ vertices of $I'_{1,1}$ do not satisfy

$$\deg_{\text{blue}}(w, I'_{1,1} \cup I'_{1,2}) \geq 5n/12. \tag{3.6.10}$$

If $v \in I'_{1,1}$, then

$$\begin{aligned}
\deg_{\text{blue}}(v, I'_{2,1}) &\overset{(3.6.7),(3.6.9)}{\geq} \deg_{\text{blue}}(v, I_{2,1}) - \beta n/4 - |V(C^*)| & (3.6.11)\\
&\geq (2/3 + \beta)n - |V_0| - |R_2| - \beta n/4 - \beta n/4 - 42\\
&\geq (2/3 + \beta)n - \varepsilon n - (1/2 + 8\gamma)n - \beta n/2 - 42\\
&\overset{(\ll)}{\geq} (1/6 + \beta/4)n. & (3.6.12)
\end{aligned}$$

Similarly, if $v \in I'_{2,1}$, then

$$\deg_{\text{blue}}(v, I_{1,1}) \geq (1/6 + \beta/4)n. \tag{3.6.13}$$

Now let us show that $H = G_{\text{blue}}[I'_{1,1} \cup I'_{2,1}]$ is Hamiltonian. To this end we select a set $X \subseteq I'_{1,1}$ of size $|I'_{2,1}|$ and such that X contains all vertices of $I'_{1,1}$ that do not satisfy (3.6.10). Note that this is possible, since there are at most $2\sqrt{\mathrm{d}}n$ such vertices and $|I'_{2,2}| \geq (1/4-3\sqrt{\mathrm{d}})n$ by (3.6.9). Let H' be the bipartite subgraph of G_{blue} with bipartition classes $I'_{2,1}$ and X, which contains all edges $xy \in E(G_{\text{blue}})$ of type $x \in X$ and $y \in I'_{2,1}$. Note that by (3.6.9) the partition classes of H' have a size of at most $(1/4 + 3\sqrt{\mathrm{d}})n$ and $\delta(H') \geq n/6$ by (3.6.11), (3.6.13). Hence H' contains a Hamiltonian cycle H by Corollary 3.6.2. By choice of X, the remaining vertices $V(H) \setminus V(C)$ have degree at least $5n/12$ in H. We also have $|V(H) \setminus V(C)| \leq 2\beta n$ and $|V(H)| \leq (1/2 + 6\sqrt{\mathrm{d}})n$ by (3.6.7) and (3.6.9). Hence Fact 3.6.3 allows us to extend C to a Hamiltonian cycle of H. □

Now it is not hard to finish the proof. We apply Claim 3.6.4 to obtain a blue cycle C_1 that covers $I'_{1,1} \cup I'_{2,1}$. Next we apply Claim 3.6.4 to obtain another blue cycle C_2 that covers $I'_{1,2} \cup I'_{2,2}$. If z_{red} is not yet on some cycle, then z_{blue} has blue degree at least $(2/3 + \beta/2)n$ by (3.6.6) and we can add z_{blue} to C_1 or C_2 by by Fact 3.6.3. Finally z_{red} is monochromatic cycle on its own. □

Chapter 4

Chromatic index, treewidth and maximum degree

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Abstract

We conjecture that any graph G with treewidth k and maximum degree $\Delta(G) \geq k + \sqrt{k}$ satisfies $\chi'(G) = \Delta(G)$. In support of the conjecture we prove its fractional version.

4.1 Introduction

The least number $\chi'(G)$ of colours necessary to properly colour the edges of a (simple) graph G is either the maximum degree $\Delta(G)$ or $\Delta(G)+1$. But to decide whether $\Delta(G)$ or $\Delta(G)+1$ colours suffice is a difficult algorithmic problem [66].

Often, graphs with a relatively simple structure can be edge-coloured with only $\Delta(G)$ colours. This is the case for bipartite graphs (König's theorem) and for cubic Hamiltonian graphs. Arguably, one measure of simplicity is *treewidth*, how closely a graph resembles a tree. (See next section for a definition.)

Vizing [105] (see also Zhou et al. [107]) observed a consequence of his adjacency lemma: any graph with treewidth k and maximum degree at least $2k$ has chromatic index $\chi'(G) = \Delta(G)$. Is this tight? No, it turns out. Using two recent adjacency lemmas the requirement on the maximum degree can be dropped to $\Delta(G) \geq 2k-1$ whenever $k \geq 4$; see Section 4.4. This immediately suggests the question: how much further can the maximum degree be lowered? We conjecture:

Conjecture 4.1.1. *Any graph of treewidth k and maximum degree $\Delta \geq k+\sqrt{k}$ has chromatic index Δ.*

The bound is close to best possible: in Section 4.5 we construct, for infinitely many k, graphs with treewidth k, maximum degree $\Delta = k + \lfloor\sqrt{k}\rfloor < k + \sqrt{k}$, and chromatic index $\Delta + 1$. For other values k the conjecture (if true) might be off by 1 from the best bound on Δ. This is, for instance, the case for $k = 2$, where the conjecture is known to hold. Indeed, Juvan et al. [69] show that series-parallel graphs with maximum degree $\Delta \geq 3$ are even Δ-edge-choosable.

In support of the conjecture we prove its fractional version:

Theorem 4.1.2. *Any simple graph of treewidth k and maximum degree $\Delta \geq k + \sqrt{k}$ has fractional chromatic index Δ.*

The theorem follows from a new upper bound on the number of edges:

$$2|E(G)| \leq \Delta|V(G)| - (\Delta - k)(\Delta - k + 1)$$

The bound is proved in Proposition 4.3.1. It implies quite directly that no graph with treewidth k and maximum degree $\Delta \geq k + \sqrt{k}$ can be overfull. (A graph G is *overfull* if it has an odd number n of vertices and strictly more than $\Delta(G)\frac{n-1}{2}$ edges; a subgraph H of G is an *overfull subgraph* if it is overfull and satisfies $\Delta(H) = \Delta(G)$.)

Thus, for certain parameters our conjecture coincides with the overfull conjecture of Chetwynd and Hilton [23]:

Overfull conjecture. Every graph G on less than $3\Delta(G)$ vertices can be edge-coloured with $\Delta(G)$ colours unless it contains an overfull subgraph.

Because we can exclude that graphs with treewidth k and maximum degree $\Delta \geq k + \sqrt{k}$ are overfull, the overfull conjecture (as well as our conjecture) implies that such graphs on less than 3Δ vertices can always be edge-coloured with Δ colours.

Graphs of treewidth k are in particular k-degenerate (see Section 4.2 for the definition and Section 4.6 for a discussion). Indeed, Vizing [105] originally showed that k-degenerate graphs, rather than treewidth k graphs, of maximum degree $\Delta \geq 2k$ have an edge-colouring with Δ colours. We briefly list some related work on edge-colourings and their variants in k-degenerate graphs. Isobe et al. [67] show that any k-degenerate graph of maximum degree $\Delta \geq 4k+3$ has a total colouring with only $\Delta+1$ colours. For graphs that are not only k-degenerate but also of treewidth k, a maximum degree of $\Delta \geq 3k - 3$ already suffices [21]. Noting that they are 5-degenerate, we include some results on planar graphs as well. Borodin, Kostochka and Woodall [19, 20] showed that planar graphs have list-chromatic index $\Delta(G)$ and total chromatic number $\chi''(G) = \Delta(G) + 1$ if $\Delta(G) \geq 11$ or if the maximum degree and the girth are at least 5. Vizing [105] proved that a planar graph G has a $\Delta(G)$-edge-colouring if $\Delta(G) \geq 8$. Sanders and Zhao [93] and independently Zhang [106] extended this to $\Delta(G) \geq 7$.

4.2 Definitions

All graphs in this article are finite and simple. We use standard graph theory notation as found in the book of Diestel [28].

For a graph G a *tree-decomposition* $(T, \mathcal{B})$ consists of a tree T and a collection $\mathcal{B} = \{B_t : t \in V(T)\}$ of *bags* $B_t \subseteq V(G)$ such that

(i) $V(G) = \bigcup_{t \in V(T)} B_t$,

(ii) for each edge $vw \in E(G)$ there exists a vertex $t \in V(T)$ such that $v, w \in B_t$, and

(iii) if $v \in B_s \cap B_t$, then $v \in B_r$ for each vertex r on the path connecting s and t in T.

A tree-decomposition $(T, \mathcal{B})$ has *width* k if each bag has a size of at most $k+1$. The *treewidth* of G is the smallest integer k for which there is a width k tree-decomposition of G.
A tree-decomposition $(T, \mathcal{B})$ of width k is *smooth* if

(iv) $|B_t| = k+1$ for all $t \in V(T)$ and

(v) $|B_s \cap B_t| = k$ for all $st \in E(T)$.

All tree decompositions considered in this paper will be smooth. This is possible as a graph of treewidth at most k always has a smooth tree-decomposition of width k; see Lemma 8 in [15].

The fractional chromatic index of a graph G is defined as

$$\chi'_f(G) = \min\left\{\sum_{M\in\mathcal{M}} \lambda_M \ : \ \lambda_M \in \mathbb{R}_+, \sum_{M\in\mathcal{M}} \lambda_M \mathbb{1}_M(\mathrm{e}) = 1 \quad \forall \mathrm{e} \in E(G)\right\},$$

where $\mathcal{M}$ denotes the collection of all matchings in G and $\mathbb{1}_M$ the characteristic vector of M. For more details on the fractional chromatic index, see for instance Scheinerman and Ullman [98].

4.3 A bound on the number of edges

Theorem 4.1.2 follows quickly from a bound on the number of edges:

Proposition 4.3.1. *A graph G of treewidth k and maximum degree $\Delta(G) \geq k$ satisfies*

$$2|E(G)| \leq \Delta(G)|V(G)| - (\Delta(G) - k)(\Delta(G) - k + 1). \tag{4.3.1}$$

Before proving Proposition 4.3.1 we present one of its consequences:

Lemma 4.3.2. *Let G be a graph of treewidth at most k and maximum degree $\Delta \geq k + \sqrt{k}$. Then G is not overfull.*

Proof. Proposition 4.3.1 implies

$$\frac{2|E(G)|}{|V(G)| - 1} \leq \frac{\Delta|V(G)| - (\Delta - k)(\Delta - k + 1)}{|V(G)| - 1} = \frac{\Delta|V(G)| - (\Delta - k)^2 - \Delta + k}{|V(G)| - 1}$$

and as $\Delta \geq k + \sqrt{k}$ we obtain

$$\frac{2|E(G)|}{|V(G)| - 1} \leq \frac{\Delta|V(G)| - k - \Delta + k}{|V(G)| - 1} = \Delta.$$

This finishes the proof. $\square$

It follows from Edmonds' matching polytope theorem that $\chi'_f(G) = \Delta(G)$, if the graph G does not contain any overfull subgraph of maximum degree Δ; see [101, Ch. 28.5]. As the treewidth of a subgraph is never larger than the treewidth of the original graph, Theorem 4.1.2 is a consequence of Lemma 4.3.2.

The proof of Proposition 4.3.1 rests on two lemmas. We defer their proofs to the end of the section. For a tree T we write $|T|$ to denote the number of its vertices. If $st \in E(T)$ is an edge of T then we let $T_{(s,t)}$ be the component of $T - st$ containing s. For any number k we set $[k]^+ = \max(k, 0)$.

Lemma 4.3.3. *For a tree T and a positive integer* $\mathrm{d} \le |T|$ *it holds that*

$$\sum_{(s,t):st\in E(T)} [\mathrm{d} - |T_{(s,t)}|]^+ \ge \mathrm{d}(\mathrm{d}-1).$$

If T^* is a subtree of T then let $\delta^+(T^*)$ be the set of (s,t) so that st is an edge of T with $s \in V(T^*)$ but $t \notin V(T^*)$. (That is, $\delta^+(T^*)$ may be seen as the set of oriented edges leaving T^*.)

Lemma 4.3.4. *Let T be a tree and let* $\mathrm{d} \le |T|$ *be a positive integer. Then for any subtree $T^* \subseteq T$ it holds that*

$$\sum_{(s,t)\in\delta^+(T^*)} [\mathrm{d} - |T_{(s,t)}|]^+ \le [\mathrm{d} - |T^*|]^+. \tag{4.3.2}$$

We introduce one more piece of notation. If $(T, \mathcal{B})$ is a tree decomposition of the graph G, then for any vertex v of G we denote by $T(v)$ the subtree of T that consists of those vertices corresponding to bags that contain v.

Proof of Proposition 4.3.1. Let $(T, \mathcal{B})$ be a smooth tree decomposition of G of width k. First note that for any vertex v of G, the number of vertices in the union of all bags containing v is at most $|T(v)| + k$ since the tree decomposition is smooth. Thus $\deg(v) \le |T(v)| + k - 1$.

Set $\mathrm{d} \coloneqq \Delta - k + 1 \ge 1$, and observe that $\mathrm{d} \le |V(G)| - k = |T|$ as the tree decomposition is smooth. We calculate

$$\begin{aligned}\Delta - \deg(v) &\ge [\Delta - k + 1 - |T(v)|]^+ \\ &= [\mathrm{d} - |T(v)|]^+ \ge \sum_{(s,t)\in\delta^+(T(v))} [\mathrm{d} - |T_{(s,t)}|]^+,\end{aligned}$$

where the last inequality follows from Lemma 4.3.4.

Consider an edge $st \in E(T)$. Since the tree decomposition is smooth there is exactly one vertex $v \in V(G)$ with $v \in B_s$ and $v \notin B_t$. Setting $\varphi((s,t)) = v$ then defines a function from the set of all (s,t) with $st \in E(T)$ into $V(G)$. Note that $\varphi((s,t)) = v$ if and only if $(s,t) \in \delta^+(T(v))$. Summing the previous inequality over all vertices, we get

$$\begin{aligned}\sum_{v\in V(G)} (\Delta - \deg(v)) &\ge \sum_{v\in V(G)} \sum_{(s,t)\in\varphi^{-1}(v)} [\mathrm{d} - |T_{(s,t)}|]^+ \\ &= \sum_{(s,t):st\in E(T)} [\mathrm{d} - |T_{(s,t)}|]^+ \ge \mathrm{d}(\mathrm{d}-1),\end{aligned}$$

where the last inequality is due to Lemma 4.3.3. This directly implies (4.3.1). $\square$

It remains to prove Lemma 4.3.3 and 4.3.4.

Proof of Lemma 4.3.3. We proceed by induction on $|T| - \mathrm{d}$. The induction starts when $\mathrm{d} = |T|$. Then $[\mathrm{d} - |T_{(s,t)}|]^+ = \mathrm{d} - |T_{(s,t)}|$ and thus

$$\begin{aligned}\sum_{(s,t):st\in E(T)} [\mathrm{d} - |T_{(s,t)}|]^+ &= \sum_{st\in E(T)} \big(|T| - |T_{(s,t)}| + |T| - |T_{(t,s)}|\big) \\ &= \sum_{st\in E(T)} |T_{(t,s)}| + |T_{(s,t)}| = (|T|-1)|T|.\end{aligned}$$

Now, let $\mathrm{d} \le |T|-1$, which implies in particular $|T| \ge 2$. Then T has a leaf ℓ. We set $T' := T - \ell$ and note that $\mathrm{d} \le |T|-1 = |T'|$.

Observe that for any edge $st \in E(T')$ we get

$$|T_{(s,t)}| = \begin{cases} |T'_{(s,t)}| + 1 & \text{if } \ell \in V(T_{(s,t)}), \\ |T'_{(s,t)}| & \text{if } \ell \notin V(T_{(s,t)}). \end{cases}$$

We denote by F the set of all (s,t) for which st is an edge in T' with $\ell \in V(T_{(s,t)})$ and with $|T'_{(s,t)}| \le \mathrm{d}-1$. Then

$$[\mathrm{d} - |T_{(s,t)}|]^+ = \begin{cases} [\mathrm{d} - |T'_{(s,t)}|]^+ - 1 & \text{if } (s,t) \in F, \\ [\mathrm{d} - |T'_{(s,t)}|]^+ & \text{if } (s,t) \notin F. \end{cases} \tag{4.3.3}$$

Among the $(s,t) \in F$ choose (x,y) such that y maximises the distance to ℓ. This means, that $st \in E(T'_{(x,y)})$ for any $(s,t) \in F \setminus \{(x,y)\}$. Consequently,

$$|T'_{(x,y)}| = |E(T'_{(x,y)})| + 1 \ge |F| - 1 + 1|F|.$$

Let r be the unique neighbour of the leaf ℓ. Then $|T_{(\ell,r)}| = 1$, and we obtain

$$[\mathrm{d} - |T_{(\ell,r)}|]^+ = \mathrm{d} - 1 \ge |T'_{(x,y)}| \ge |F|. \tag{4.3.4}$$

We conclude

$$\begin{aligned} \sum_{(s,t):st\in E(T)} [\mathrm{d} - |T_{(s,t)}|]^+ &= [\mathrm{d} - |T_{(\ell,r)}|]^+ + [\mathrm{d} - |T_{(r,\ell)}|]^+ \\ &\quad + \sum_{(s,t):st\in E(T')} [\mathrm{d} - |T_{(s,t)}|]^+ \\ &\overset{(4.3.4)}{\ge} |F| + 0 + \sum_{(s,t):st\in E(T')} [\mathrm{d} - |T_{(s,t)}|]^+ \\ &\overset{(4.3.3)}{=} \sum_{(s,t):st\in E(T')} [\mathrm{d} - |T'_{(s,t)}|]^+ \\ &\ge \mathrm{d}(\mathrm{d}-1), \end{aligned}$$

where the last inequality follows by induction. □

Proof of Lemma 4.3.4. We proceed by induction on $|T| - \mathrm{d}$. For the induction start, consider the case when $\mathrm{d} = |T|$. Then

$$[\mathrm{d} - |T_{(s,t)}|]^+ = [|T| - |T_{(s,t)}|]^+ = |T_{(t,s)}|,$$

which yields

$$\sum_{(s,t)\in\delta^+(T^*)} [\mathrm{d} - |T_{(s,t)}|]^+ = \sum_{(s,t)\in\delta^+(T^*)} |T_{(t,s)}| = |T| - |T^*| = [\mathrm{d} - |T^*|]^+.$$

Now assume $|T| - \mathrm{d} \ge 1$. If every vertex in $T - V(T^*)$ is a leaf of T then t is a leaf for every $(s,t) \in \delta^+(T^*)$. This implies $|T_{(s,t)}| = |T| - 1 \ge \mathrm{d}$ and the left hand side of (4.3.2) vanishes.

Therefore we may assume that there is a leaf $\ell \notin T^*$ of T such that neither l nor its unique neighbour belongs to $V(T^*)$. Set $T' = T - \ell$, and observe that, by choice of ℓ, the set $\delta^+(T^*)$ of edges leaving T^* is the same in T and in T'. Moreover, $|T_{(s,t)}| \geq |T'_{(s,t)}|$ holds for every $(s,t) \in \delta^+(T^*)$. The desired inequality

$$\sum_{(s,t)\in\delta^+(T^*)} [\mathrm{d} - |T_{(s,t)}|]^+ \leq \sum_{(s,t)\in\delta^+(T^*)} [\mathrm{d} - |T'_{(s,t)}|]^+ \leq [\mathrm{d} - |T^*|]^+$$

now follows by induction. □

4.4 A lower bound on the maximum degree

Vizing [105] (see also Zhou et al. [107]) proved that every graph of treewidth k and maximum degree $\Delta \geq 2k$ has an edge-colouring with Δ colours. We show that this is not tight.

Proposition 4.4.1. *For any graph G of treewidth $k \geq 4$ and maximum degree $\Delta(G) \geq 2k-1$ it holds that $\chi'(G) = \Delta(G)$.*

A graph G of maximum degree Δ is Δ-*critical*, if all proper subgraphs can be edge-coloured using not more than Δ colours and $\chi(G) = \Delta + 1$. For the proof of Proposition 4.4.1 we use Vizing's adjacency lemma, as well as two adjacency lemmas that involve the second neighbourhood.

Vizing's adjacency lemma. Let uv be an edge in a Δ-critical graph. Then v has at least $\Delta - \deg(u) + 1$ neighbours of degree Δ.

Theorem 4.4.2 (Zhang [106])**.** *Let G be a Δ-critical graph, and let uwv be a path in G. If $\deg(u) + \deg(w) = \Delta + 2$ then all neighbours of v but u and w have degree Δ.*

Theorem 4.4.3 (Sanders and Zhao [93])**.** *Let G be a Δ-critical graph, and let v be a common neighbour of u and w such that $\deg(u) + \deg(v) + \deg(w) \leq 2\Delta + 1$. Then there are at most $\deg(u) + \deg(v) - \Delta - 3$ common neighbours $x \neq u$ of v and w.*

The rest of this subsection is dedicated to the proof of Proposition 4.4.1. To this end, let us assume Proposition 4.4.1 to be wrong. Then there is a Δ-critical graph G of treewidth at most k for $\Delta = 2k - 1$. (Note that the case $\Delta \geq 2k$ is covered by the above mentioned result of Vizing.) Let $(T, \mathcal{B})$ be a smooth tree-decomposition of G of width $\leq k$. By picking an arbitrary root, we may consider T as a rooted tree. For any $s \in V(T)$, we denote by $\lceil s \rceil$ the subtree of T rooted at s, that is, the subtree of T consisting of the vertices $t \in V(T)$ for which s is contained in the path between t and the root of T.

Recall the definition of $T(v)$ after Lemma 4.3.4. Set $L := \{v \in V(G) : \deg(v) \geq k+2\}$, and choose a vertex $v^* \in L$ that maximises the distance of $T(v^*)$ to the root (among the vertices in L). Let q be the vertex of $T(v^*)$ that achieves this distance. For $S := N(q) \cap T(v^*)$ and any $s \in S$, define $X_s := \bigcup_{t \in V(\lceil s \rceil)} B_t$, and let $X := B_q \cup \bigcup_{s \in S} X_s$. Note that by the definition of v^* and q

$$N(v^*) \subseteq X \text{ and } X \cap L \subseteq B_q. \tag{4.4.1}$$

Claim 4.4.4. *All vertices of $X \setminus B_q$ have degree at most k.*

Proof of Claim 4.4.4. Suppose the statement to be false. Then there is an $s \in S$ for which $X_s \setminus B_q$ contains a vertex of degree at least $k+1$. Fix a vertex $w^* \in \{w \in X_s \setminus B_q : \deg(w) \geq k+1\} =: L'$ that maximises the distance of $T(w^*)$ to s. Let p be the vertex of $T(w^*)$ that achieves this distance. Set $Y = \bigcup_{t \in V(\lceil p \rceil)} B_t$. As in (4.4.1) we have $N(w^*) \subseteq Y$ and $Y \cap L' \subseteq B_p$. Since, moreover, w^* has degree at least $k+1$, it has a neighbour u^* outside B_p, which then has degree at most k (by choice of w^*).

Vizing's adjacency lemma implies that w^* has at least $\Delta - \deg(u^*) + 1 \geq 2k-1-k+1 = k$ neighbours of degree Δ. By (4.4.1), all vertices of degree Δ of Y have to be in $B_q \cap B_s$. Since by smoothness of the tree decomposition $B_q \cap B_s$ is a cutset of size at most k, the vertex w^* is adjacent to all vertices in $B_q \cap B_s$. As w^* is therefore adjacent to at most k vertices of degree Δ it holds $\deg(u^*) = k$. By definition of S, the set B_s contains v^*, which implies that v^* is adjacent to w^* and of degree Δ. As $k \geq 4$, it follows that v^* has degree $\Delta = 2k-1 \geq k+3$, which means by (4.4.1) that v^* has at least three neighbours of degree $\leq k+1$. Thus, v^* has a neighbour of degree $\leq k+1$, which is neither u^* nor w^*. This, however, contradicts Theorem 4.4.2 (applied to v^*, w^*, u^*). ☐

By (4.4.1) and since v^* has degree at least $k+2$, the vertex v^* has a neighbour $u \notin B_q$. (In fact, v^* has at least two such neighbours.) By Vizing's adjacency lemma, applied to uv^*, it follows that v^* has at least $\Delta - \deg(u) + 1 \geq k$ neighbours of degree Δ. In particular, by an analogue of (4.4.1) for s and w^*.

$$v^* \textit{ is adjacent to every vertex in } B_q\textit{, each of which has degree } \Delta. \tag{4.4.2}$$

Claim 4.4.5. *Every $u \in N(v^*) \setminus B_q$ has exactly k neighbours, all of which are contained in B_q.*

Proof of Claim 4.4.5. By Vizing's adjaceny lemma (applied to uv^*), u is of degree at least k. Otherwise v^* has too many high degree neighbours. By (4.4.2), every vertex in B_q has degree Δ and thus $u \notin B_q$. The set B_q is a cutset. This implies that u has all its neighbours in X. However, u cannot be adjacent to any vertex w of degree $\leq k$; otherwise we could extend any Δ-edge-colouring of $G - uw$ to G. It follows from Claim 4.4.4 that all of the k neighbours of u are in B_q. ☐

Since the vertex v^* has degree at least $k+2$, it has two neighbours u, w of degree at most $k+1$ (again by (4.4.1)). By Claim 4.4.5, the degree of u and w is k. Thus, $\deg(u) + \deg(v^*) + \deg(w) \leq k + \Delta + k = 2\Delta + 1$. Moreover, by Claim 4.4.5 and (4.4.2), the vertices v^* and w have $k-1$ common neighbours in B_q. As $k - 1 > \deg(u) + \deg(v^*) - \Delta - 3$, we obtain a contradiction to Theorem 4.4.3. This finishes the proof of Proposition 4.4.1.

4.5 Discussion

Proposition 4.3.1 bounds the number of edges in a graph G of fixed treewidth and maximum degree. A simpler bound – only considering the treewidth – is easily shown by induction (see Rose [91]):

$$2|E(G)| \leq 2k|V(G)| - k(k+1) \tag{4.5.1}$$

For $\Delta < 2k$ and $|V(G)| > \Delta + 1$ a straightforward computation shows that the bound of Proposition 4.3.1 is strictly better than (4.5.1). The bounds are the same if $\Delta = 2k$ or if $|V(G)| = \Delta + 1$. For $\Delta = 2k$ this is illustrated by the kth power P^k of a long path P.

The bound in Proposition 4.3.1 is tight. There are simple examples that show this: Take the complete graph K_k on k vertices and add $r \geq 1$ further vertices each adjacent to each vertex of K_k. These graphs also demonstrate that Conjecture 4.1.1 (if true) would be tight or almost tight. Indeed, if $k + \lfloor\sqrt{k}\rfloor$ is even, and k not a square, then we obtain for $r = \lfloor\sqrt{k}\rfloor + 1$ an overfull graph with maximum degree $\Delta = k + \lfloor\sqrt{k}\rfloor$. If $k + \lfloor\sqrt{k}\rfloor$ is odd, then, by setting $r = \lfloor\sqrt{k}\rfloor$, we obtain an overfull graph with $\Delta = k + \lfloor\sqrt{k}\rfloor - 1$.

These tight graphs, however, have a very special structure. In particular, they all satisfy $|V(G)| = \Delta(G)+1$. Both, Conjecture 4.1.1 and Proposition 4.3.1, stay tight for an arbitrarily large number of vertices compared to Δ:

Proposition 4.5.1. *For every $k_0 \geq 4$ there is a $k \in \{k_0, k_0+1, \ldots, k_0+8\}$ such that for every $n \geq 4k$ there exists a graph G on n vertices with treewidth at most k and maximum degree $\Delta = k + \lfloor\sqrt{k}\rfloor < k + \sqrt{k}$ such that*

$$2|E(G)| = \Delta n - (\Delta - k)(\Delta - k + 1).$$

In particular, the graph G is overfull whenever n is odd.

We need the following lemma.

Lemma 4.5.2. *Let $c, r \in \mathbb{N}$. Then there is a graph with degree sequence*

$$\mathbf{d} = \big(\underbrace{c\ldots,c}_{r+1}, c-1, c-2, \ldots, 1\big) \in \mathbb{Z}^{c+r}$$

if and only if 4 divides $c(2r+c+1)$ and if $r^2 \geq c$.

We defer the proof of Lemma 4.5.2 until the end of the section and only show sufficiency. A closer look at the arguments in the proof yields necessity.

Proof of Proposition 4.5.1. We start by showing with a case distinction that there is a $k \in \{k_0, k_0+1, \ldots, k_0+8\}$ such that

$$k \equiv \left\lfloor\sqrt{k}\right\rfloor \pmod 8 \text{ and } \left\lfloor\sqrt{k}\right\rfloor < \sqrt{k}. \tag{4.5.2}$$

To this end, let i such that $\lfloor\sqrt{k_0}\rfloor \equiv k_0 + \mathrm{i} \pmod 8$ and $0 \leq \mathrm{i} \leq 7$.

Firstly, let us assume that $\mathrm{i} = 0$. If k_0 is not a square, then $k = k_0$ satisfies (4.5.2). Otherwise $k = k_0 + 8$ satisfies (4.5.2) as $k_0 \geq 4 > 1$, and consequently $\lfloor\sqrt{k_0+8}\rfloor = \sqrt{k_0}$.

Secondly, we consider the case that $\mathrm{i} \neq 0$. If $\lfloor\sqrt{k_0+\mathrm{i}}\rfloor = \lfloor\sqrt{k_0}\rfloor$, then $k = k_0 + \mathrm{i}$ satisfies $\lfloor\sqrt{k_0}\rfloor \equiv k \pmod 8$ and $\sqrt{k} > \sqrt{k_0} \geq \lfloor\sqrt{k_0}\rfloor = \lfloor\sqrt{k}\rfloor$, which shows (4.5.2). If, on the other hand, $\lfloor\sqrt{k_0+\mathrm{i}}\rfloor > \lfloor\sqrt{k_0}\rfloor$, then $\lfloor\sqrt{k_0+\mathrm{i}}\rfloor = \lfloor\sqrt{k_0}\rfloor + 1 = \lfloor\sqrt{k_0+\mathrm{i}+1}\rfloor$ as $k_0 \geq 4$. Set $k = k_0 + \mathrm{i} + 1$. By choice of i, we have $\lfloor\sqrt{k_0} + 1\rfloor \equiv k \pmod 8$. Thus, we obtain $\lfloor\sqrt{k}\rfloor \equiv k \pmod 8$ as desired. Moreover, $\sqrt{k} > \sqrt{k_0+\mathrm{i}} \geq \lfloor\sqrt{k_0+\mathrm{i}}\rfloor = \lfloor\sqrt{k_0}\rfloor + 1 = \lfloor\sqrt{k}\rfloor$.

In all cases an element of $\{k_0, k_0+1, \ldots, k_0+8\}$ satisfies (4.5.2).

Next we show that for any $n \geq 4k$, there is a graph G of treewidth k whose degree sequence $(\deg_G(v_1), \deg_G(v_2), \ldots, \deg_G(v_n))$ equals

$$(k, k+1, \ldots, \Delta-1, \Delta, \ldots, \Delta, \Delta-1, \ldots, k+1, k) \tag{4.5.3}$$

with $\Delta = k + \lfloor\sqrt{k}\rfloor$. A computation similar to Lemma 4.3.2 shows that G is overfull if $|V(G)|$ is odd.

We construct G in three steps. First we take a power of a path, where all but the outer vertices have the right degree. We increase the degree of the outer vertices by connecting them to vertices towards the middle of the path. This will create some degree excess for the used vertices. We balance this by deleting a subgraph H provided by Lemma 4.5.2. The construction is illustrated in Figure 4.1. Note that for ease of exposition the parameters k and Δ are not as in this proof.

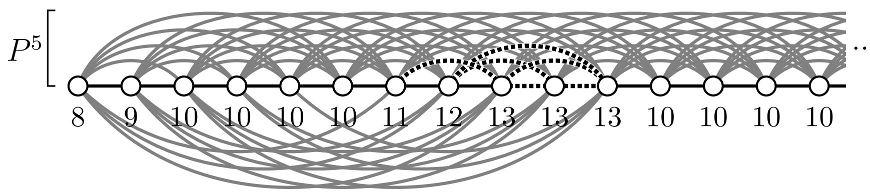

Figure 4.1: Extreme example for $k = 8$ and $\Delta = 10$. The graph H is dotted.

Let P be a $\Delta/2$-th power of a path on vertices $v_1, \ldots, v_n$. This means, v_i and v_j are adjacent if and only if $0 < |i - j| \leq \Delta/2$. As P is symmetric, and as G will be symmetric as well, we concentrate on the part of P on the vertices $v_1, \ldots, v_{\lceil n/2 \rceil}$. We tacitly agree that any additions and deletions of edges are also applied to the other half of P.

Comparing the degrees of P to (4.5.3) we see that all vertices have the target degree except for the initial vertices $v_1, \ldots, v_{\Delta/2}$, whose degree is too small. For $i = 1, \ldots, \Delta - k$ the vertex v_i has degree $\Delta/2 - 1 + i$ but should have degree $k - 1 + i$. We fix this by connecting v_i to $v_{i+\Delta/2+1}, \ldots, v_{i+k+1}$. For $i = \Delta - k + 1, \ldots, \Delta/2$, the vertex v_i should have degree Δ but has degree $\Delta/2 - 1 + i$. We make v_i adjacent to each of $v_{i+\Delta/2+1}, \ldots, v_{\Delta+1}$.

Denote the obtained graph by P' and observe that its vertices in the range of $1, \ldots, \lceil n/2 \rceil$ have the following degrees

$$\underbrace{k, k+1, \ldots, \Delta}_{1,\ldots,\Delta-k+1}, \underbrace{\Delta, \ldots, \Delta}_{\Delta-k+2,\ldots,\Delta/2+1}, \underbrace{\Delta+1, \ldots, k+\tfrac{\Delta}{2}}_{\frac{\Delta}{2}+2,\ldots,k+1}, \underbrace{k+\tfrac{\Delta}{2}, \ldots, k+\tfrac{\Delta}{2}}_{k+2,\ldots,\Delta+1}, \underbrace{\Delta, \ldots, \Delta}_{\Delta+2,\ldots,\lceil n/2 \rceil}$$

Hence all but the vertices v_i with index i between $\Delta/2 + 2$ and $\Delta + 1$ have the correct degree. The difference between their degree in P' and the desired degree is

$$\mathbf{d} = \big(1, 2, \ldots, k - \tfrac{\Delta}{2} - 1, \underbrace{k - \tfrac{\Delta}{2}, \ldots, k - \tfrac{\Delta}{2}}_{\Delta-k+1}\big). \tag{4.5.4}$$

Set $c = k - \frac{\Delta}{2} = \frac{1}{2}\left(k - \lfloor\sqrt{k}\rfloor\right)$ and $r = \Delta - k$. Note that k is chosen in such a way (see (4.5.2)) that c is divisible by 4. As furthermore $r^2 = (\Delta - k)^2 = \lfloor\sqrt{k}\rfloor^2 \geq \frac{1}{2}\left(k - \lfloor\sqrt{k}\rfloor\right) = c$, Lemma 4.5.2 yields that there is a graph H with degree sequence $\mathbf{d}$. Since the vertices $v_{\Delta/2+2}, \ldots, v_{\Delta+1}$ induce a complete graph in P' there is a copy of H in P', such that deleting its edges results in a graph G of the desired degree sequence. Note that for any two adjacent vertices v_i, v_j in P' it holds that $|i - j| \leq k$. This implies that P' is a subgraph of a k-th power of a path. Thus the subgraph G of P' has treewidth at most k. This finishes the proof. □

To prove Lemma 4.5.2 we use the Erdős-Gallai-criterion:

Theorem 4.5.3 (Erdős and Gallai [30]). *There is a graph with degree sequence* $\mathrm{d}_1 \geq \cdots \geq \mathrm{d}_n$ *if and only if* $\sum_{\mathrm{i}=1}^{n} \mathrm{d}_\mathrm{i}$ *is even and if for all* $\ell = 1, \ldots, n$

$$\sum_{\mathrm{i}=1}^{\ell} \mathrm{d}_\mathrm{i} \leq \ell(\ell-1) + \sum_{\mathrm{i}=\ell+1}^{n} \min(\mathrm{d}_\mathrm{i}, \ell). \tag{4.5.5}$$

Proof of Lemma 4.5.2. We check the conditions of Theorem 4.5.3 for the degree sequence $\mathbf{d}$. The parity condition holds as 4 divides $c(2r+c+1)$ and

$$\sum_{\mathrm{i}=1}^{c+r} \mathrm{d}_\mathrm{i} = cr + \frac{c(c+1)}{2} = \frac{c}{2}(2r+c+1).$$

Let of us now verify (4.5.5). If $\ell > c$, then

$$\sum_{\mathrm{i}=1}^{\ell} \mathrm{d}_\mathrm{i} \leq c\ell \leq \ell(\ell-1) \leq \ell(\ell-1) + \sum_{\mathrm{i}=\ell+1}^{c+r} \min(\mathrm{d}_\mathrm{i}, \ell).$$

Thus we can assume that $\ell \leq c$. Two remarks: Firstly, $\min(\mathrm{d}_\mathrm{i}, \ell) = \ell$ for $\mathrm{i} = 1, \ldots, \leq c+r-\ell+1$. Consequently, if $2\ell \leq c+r$ then

$$\begin{aligned}\ell(\ell-1) + \sum_{\mathrm{i}=\ell+1}^{c+r} \min(\mathrm{d}_\mathrm{i}, \ell) &= \ell(\ell-1) + (c+r-2\ell+1)\ell + \frac{\ell(\ell-1)}{2} \\ &= \tfrac{\ell}{2}(2r-1-\ell) + c\ell. \end{aligned} \tag{4.5.6}$$

Secondly, if $\ell > r$, then

$$\sum_{\mathrm{i}=1}^{\ell} \mathrm{d}_\mathrm{i} = c\ell - \frac{(\ell-r-1)(\ell-r)}{2} = c\ell + \tfrac{\ell}{2}(2r+1-\ell) - \tfrac{1}{2}(r^2+r). \tag{4.5.7}$$

Now suppose that $2\ell \leq c+r$. For $\ell \leq r$, we have $\sum_{\mathrm{i}=1}^{\ell} \mathrm{d}_\mathrm{i} = c\ell$ and hence (4.5.5) is easily seen to be satisfied in light of (4.5.6). On the other hand, for $\ell > r$ the assumption of $r^2 \geq c$ together with a comparison of (4.5.6) and (4.5.7) gives (4.5.5).

So let $2\ell > c+r$. This implies that $\ell > r$. Consequently, the right hand side of (4.5.5) is

$$\begin{aligned}\ell(\ell-1) + \sum_{\mathrm{i}=\ell+1}^{c+r} \min(\mathrm{d}_\mathrm{i}, \ell) &= \ell(\ell-1) + \sum_{\mathrm{i}=\ell+1}^{c+r} \mathrm{d}_\mathrm{i} \\ &= \ell(\ell-1) + \tfrac{1}{2}(c+r-\ell)(c+r-\ell+1).\end{aligned}$$

It follows from equation (4.5.7) that (4.5.5) is satisfied if the following expression is non-negative.

$$\begin{aligned}&2\ell(\ell-1) + (c+r-\ell)(c+r-\ell+1) - (2c\ell + \ell(2r+1-\ell) - (r^2+r)) \\ &= 4\ell^2 - 4\ell(c+r) + (c+r)^2 + (c+2r+r^2) - 4\ell \\ &= (2\ell-(c+r))^2 + (c+r) + (r+r^2) - 4\ell \\ &= (2\ell-(c+r))^2 - 2\Big(2\ell - \frac{(c+r)+(r+r^2)}{2}\Big)\end{aligned} \tag{4.5.8}$$

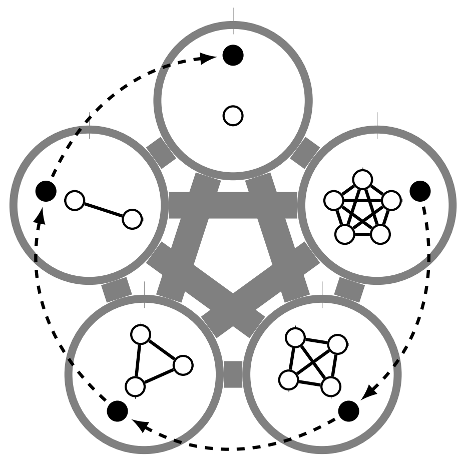

Figure 4.2: The graph G_5 with the vertices v_i drawn in black; thick gray edges indicate that two vertex sets are complete to each other; elimination order of the v_i is shown in dashed lines

First, let $r^2 = c$. Then (4.5.8) equals

$$(2\ell - (c+r))^2 - 2(2\ell - (c+r)) \tag{4.5.9}$$

The term (4.5.9) is negative only if $2\ell - (c+r) = 1$. As $c+r = r^2 + r$ is even (for any integer r), (4.5.9) and thus (4.5.8) is non-negative.

Now let $r^2 > c$. Then (4.5.8) is strictly greater than (4.5.9) and hence non-negative. This shows that (4.5.5) is satisfied.

As (4.5.5) holds for all ℓ, there is a graph with degree sequence $\mathbf{d}$.

□

4.6 Degenerate graphs

Recall that a graph G is *k-degenerate* if there is an enumeration $v_n, \dots, v_1$ of the vertices such that v_{i+1} has degree at most k in $G - \{v_n, \dots, v_i\}$ for every i. By simple induction following the elimination order (or recalling the formula formula $1 + 2 + \ldots + n = n(n+1)/2$), we can obtain a bound with half the degree loss of (4.3.1):

$$2|E(G)| \leq \Delta|V(G)| - \tfrac{1}{2}(\Delta - k)(\Delta - k + 1). \tag{4.6.1}$$

The bound in (4.6.1) turns out to be tight for some Δ, k as the construction below shows. Moreover, by (4.6.1) Theorem 4.1.2 can easily be transferred: Any simple k-degenerate graph of maximum degree $\Delta \geq k + 1/2 + \sqrt{2k + 1/4}$ is not overfull and therefore has fractional chromatic index $\chi'_f(G) = \Delta$.

Consider a positive integer p and let G_p be the complement of the disjoint union of p stars $K_{1,1}, K_{1,2}, \dots, K_{1,p}$; see Figure 4.2. Denote the centre of the ith star by v_i, and let W be the union of all leaves. The graph G_p has $n = p(p+1)/2 + p$ vertices and satisfies $\deg(v_i) = n - 1 - i$ for $i = 1, \dots, p$ and $\deg(w) = n - 2$ for $w \in W$. In particular, the maximum degree of G_p is $\Delta = n - 2$. Setting $k = n - 1 - p$, we note that G_p is k-degenerate as $v_p, v_{p-1}, \dots, v_1$ followed by an arbitrary enumeration of W is an elimination order. Finally, we observe that G_p satisfies (4.6.1) with equality.

Chapter 5

Estimating parameters associated with monotone properties

Carlos Hoppen, Yoshiharu Kohayakawa, Richard Lang, Hanno Lefmann and Henrique Stagni

Abstract

There has been substantial interest in estimating the value of a graph parameter, i.e., of a real-valued function defined on the set of finite graphs, by querying a randomly sampled substructure whose size is independent of the size of the input. Graph parameters that may be successfully estimated in this way are said to be *testable* or *estimable*, and the *sample complexity* $q_z = q_z(\varepsilon)$ of an estimable parameter z is the size of a random sample of a graph G required to ensure that the value of $z(G)$ may be estimated within an error of ε with probability at least 2/3. In this paper, for any fixed monotone graph property $\mathcal{P} = \mathit{Forb}(\mathcal{F})$, we study the sample complexity of estimating a bounded graph parameter $z_{\mathcal{F}}$ that, for an input graph G, counts the number of *spanning* subgraphs of G that satisfy $\mathcal{P}$. To improve upon previous upper bounds on the sample complexity, we show that the vertex set of any graph that satisfies a monotone property $\mathcal{P}$ may be partitioned equitably into a constant number of classes in such a way that the cluster graph induced by the partition is not far from satisfying a natural weighted graph generalization of $\mathcal{P}$. Properties for which this holds are said to be *recoverable*, and the study of recoverable properties may be of independent interest.

5.1 Introduction and main results

In the last two decades, a lot of effort has been put into finding constant-time randomized algorithms (conditional on sampling) to gauge whether a combinatorial structure satisfies some property, or to estimate the value of some numerical function associated with this structure. In this paper, we focus on the graph case and, as usual, we consider algorithms that have the ability to query whether any desired pair of vertices in the input graph is adjacent or not. Let $\mathcal{G}$ be the set of finite simple graphs and let $\mathcal{G}(V)$ be the set of such graphs with vertex set V. We shall consider subsets $\mathcal{P}$ of $\mathcal{G}$ that are closed under isomorphism, which we call *graph properties*. To avoid technicalities, we restrict ourselves to graph properties $\mathcal{P}$ such that $\mathcal{P} \cap \mathcal{G}(V) \neq \emptyset$ whenever $V \neq \emptyset$. For instance, this includes all nontrivial *monotone* and *hereditary* graph properties, which are graph properties that are inherited by subgraphs and by induced subgraphs, respectively. Here, we will focus on monotone properties. The prototypical example of a monotone property is $Forb(F)$, the class of all graphs that do not contain a copy of a fixed graph F. More generally, if $\mathcal{P}$ is a monotone property and $\mathcal{F}$ contains all minimal graphs that are not in $\mathcal{P}$, then the graphs that lie in $\mathcal{P}$ are precisely those that do not contain a copy of an element of $\mathcal{F}$. This class of graphs will be denoted by $\mathcal{P} = Forb(\mathcal{F})$. The elements of $Forb(\mathcal{F})$ are said to be *$\mathcal{F}$-free*.

A graph property $\mathcal{P}$ is said to be *testable* if, for every $\varepsilon > 0$, there exist a positive integer $q_{\mathcal{P}} = q_{\mathcal{P}}(\varepsilon)$, called the *query complexity*, and a randomized algorithm $\mathcal{T}_{\mathcal{P}}$, called a *tester*, which may perform at most $q_{\mathcal{P}}$ queries in the input graph, satisfying the following property. For an n-vertex input graph Γ, the algorithm $\mathcal{T}_{\mathcal{P}}$ distinguishes with probability at least $2/3$ between the cases in which Γ satisfies $\mathcal{P}$ and in which Γ is ε-far from satisfying $\mathcal{P}$, that is, in which no graph obtained from Γ by the addition or removal of at most $\varepsilon n^2/2$ edges satisfies $\mathcal{P}$. This may be stated in terms of graph distances: given two graphs Γ and Γ' on the same vertex set $V(\Gamma) = V(\Gamma')$, we may define the *normalized edit distance* between Γ and Γ' by $d_1(\Gamma, \Gamma') = \frac{2}{|V|^2} |E(\Gamma) \triangle E(\Gamma')|$, where $E(\Gamma) \triangle E(\Gamma')$ denotes the symmetric difference of their edge sets. If $\mathcal{P}$ is a graph property, we let the distance between a graph Γ and $\mathcal{P}$ be

$$d_1(\Gamma, \mathcal{P}) = \min\{d_1(\Gamma, \Gamma') \colon V(\Gamma') = V(\Gamma) \text{ and } \Gamma' \in \mathcal{P}\}.$$

For instance, if $\Gamma = K_n$ and $\mathcal{P} = Forb(K_3)$, Turán's Theorem ensures that $\binom{n}{2} - \lfloor n^2/4 \rfloor$ edges need to be removed to produce a graph that is K_3-free. In particular, $d_1(K_n, Forb(K_3)) \to 1/2$. Thus a graph property $\mathcal{P}$ is testable if there is a tester with bounded query complexity that distinguishes with probability at least $2/3$ between the cases $d_1(\Gamma, \mathcal{P}) = 0$ and $d_1(\Gamma, \mathcal{P}) > \varepsilon$.

The systematic study of property testing was initiated by Goldreich, Goldwasser and Ron [46], and there is a very rich literature on this topic. For instance, regarding testers, Goldreich and Trevisan [49] showed that it is sufficient to consider simpler canonical testers, namely those that randomly choose a subset X of vertices in Γ and then verify whether the induced subgraph $\Gamma[X]$ satisfies some related property $\mathcal{P}'$. For example, if the property being tested is having edge density $1/2$, then the algorithm will choose a random subset X of appropriate size and check whether the edge density of $\Gamma[X]$ is within, say, $\varepsilon/2$ of $1/2$. Regarding testable properties, Alon and Shapira [7] proved that every monotone graph property is testable, and, more generally, that the same holds for hereditary graph properties [6]. For more information about property testing, we refer the reader to [48] and the references therein.

In a similar vein, a function $z \colon \mathcal{G} \to \mathbb{R}$ from the set $\mathcal{G}$ of finite graphs into the real numbers is called a *graph parameter* if it is invariant under relabeling of vertices. A graph parameter $z \colon \mathcal{G} \to \mathbb{R}$ is *estimable* if for every $\varepsilon > 0$ and every large enough graph Γ with probability at least $2/3$, the value of $z(\Gamma)$ can be approximated up to an additive error of ε by an algorithm that only has access to a subgraph of Γ induced by a set of vertices of size $q_z = q_z(\varepsilon)$, chosen uniformly at random. The query complexity of such an algorithm is $\binom{q_z}{2}$ and the size q_z is called its *sample complexity*. Estimable parameters have been considered in [38] and were defined in the above level of generality in [18]. They are often called *testable parameters*. Borgs et al. [18, Theorem 6.1] gave a complete characterization of the estimable graph parameters which, in particular, also implies that the distance from monotone graph properties is estimable. Their work uses the concept of graph limits and does not give explicit bounds on the query complexity required for this estimation.

We obtain results for the bounded graph parameter, which, for a graph family $\mathcal{F}$, counts the number of $\mathcal{F}$-free spanning subgraphs of the input graph Γ. Recall that $G' = (V', E')$ is a *spanning subgraph* of a graph $G = (V, E)$ if $V' = V$ and $E' \subseteq E$.

Formally, given a graph $\Gamma \in \mathcal{G}$ and a family $\mathcal{F}$ of graphs, we denote the set of all $\mathcal{F}$-free spanning subgraphs of Γ by $Forb(\Gamma, \mathcal{F}) = \{G \text{ is a spanning subgraph of } \Gamma\colon G \in Forb(\mathcal{F})\}$, and we consider the parameter

$$z_{\mathcal{F}}(\Gamma) = \frac{1}{|V(\Gamma)|^2} \log_2 |\, Forb(\Gamma, \mathcal{F})|. \tag{5.1.1}$$

For example, if $\mathcal{F} = \{K_3\}$ and $\Gamma = K_n$, computing $z_{\mathcal{F}}$ requires estimating the number of K_3-free subgraphs of K_n up to a multiplicative error of $2^{o(n^2)}$:

$$z_{\mathcal{F}}(K_n) = \frac{1}{n^2} \log_2 |\, Forb(\Gamma, \mathcal{F})| = \frac{1}{n^2} \log_2 2^{\frac{1}{2}\binom{n}{2} + o(n^2)} \to \frac{1}{4}.$$

This was done by Erdős, Kleitman and Rothschild for $\mathcal{F} = \{K_k\}$ [33], see also Erdős, Frankl and Rödl [32] for F-free subgraphs. Counting problems of this type were studied by several people. Consider for instance, the work of Prömel and Steger [86, 87], the logarithmic density in Bollobás [16], and some more recent results about the number of n-vertex graphs avoiding copies of some fixed forbidden graphs [11, 12]. Algorithmic aspects have been investigated by Duke, Lefmann and Rödl [29] and, quite recently, by Fox, Lovász and Zhao [40].

As it turns out, estimating graph parameters $z_{\mathcal{F}}(\Gamma)$ is related to estimating distances of graphs from the corresponding graph property $\mathcal{P} = Forb(\mathcal{F})$. Alon, Shapira and Sudakov [8, Theorem 1.2] proved that the distance to every monotone graph property $\mathcal{P}$ is estimable using a natural algorithm, which simply computes the distance from the induced sampled graph to $\mathcal{P}$. However, one disadvantage of this approach is that the accuracy of the estimate relies heavily on stronger versions of Szemerédi's Regularity Lemma [102, 4]. Therefore, the query complexity is at least of the order $\texttt{TOWER}(\texttt{poly}(1/\varepsilon))$, by which we mean a tower of twos of height that is polynomial in $1/\varepsilon$. Moreover, it follows from a result of Gowers [50] that any approach based on Szemerédi's Regularity Lemma cannot lead to a bound that is better than $\texttt{TOWER}(\texttt{poly}(1/\varepsilon))$.

In this paper, we introduce the concept of *recoverable* graph properties. Roughly speaking, given a function $f\colon (0,1] \to \mathbb{R}$, we say that a graph property $\mathcal{P}$ is f-recoverable if every large graph $G \in \mathcal{P}$ is ε-close to admitting a partition $\mathcal{V}$ of its vertex set into at most $f(\varepsilon)$ classes that witnesses membership in $\mathcal{P}$, i.e., such that any graph that can be partitioned in the same way must be in $\mathcal{P}$.

Theorem 5.1.1. *Let Forb$(\mathcal{F})$ be an f-recoverable graph property, for some function $f\colon (0,1] \to \mathbb{R}$. Then, for all $\varepsilon > 0$ there is n_0 such that, for any graph Γ with $|V(\Gamma)| \geq n_0$, the graph parameter $z_{\mathcal{F}}$ defined in* (5.1.1) *can be estimated within an additive error of ε with sample complexity* $\texttt{poly}(f(\varepsilon/6)/\varepsilon)$.

Although one could apply strong versions of regularity to show that every monotone property $Forb(\mathcal{F})$ is f-recoverable, this approach would provide an upper bound of at least $\texttt{TOWER}(\texttt{poly}(\varepsilon^{-1}))$ for the function f. We find a connection between this notion of recoverability and the graph Removal Lemma, which can lead to better bounds for the function

$f(\varepsilon)$. The Removal Lemma was first stated explicitly in the literature by Alon *et al.* [3] and by Füredi [42]. The following version, which holds for arbitrary families of graphs was first proven in [7].

Lemma 5.1.2 (Removal Lemma). *For every $\varepsilon > 0$ and every (possibly infinite) family $\mathcal{F}$ of graphs, there exist $M = M(\varepsilon, \mathcal{F})$, $\delta = \delta(\varepsilon, \mathcal{F}) > 0$ and $n_0 = n_0(\varepsilon, \mathcal{F})$ such that the following holds. If a graph G on $n \geq n_0$ vertices satisfies $\mathrm{d}_1(G, Forb(\mathcal{F})) \geq \varepsilon$, then there is $F \in \mathcal{F}$ with $|V(F)| \leq M$ such that G has at least $\delta n^{|V(F)|}$ copies of F.* □

Conlon and Fox [26] showed that Lemma 5.1.2 holds with $\delta^{-1}, n_0 \leq \texttt{TOWER}(\texttt{poly}(\varepsilon^{-1}))$. Although this remains the best known bound for the general case, there are many families $\mathcal{F}$ for which Lemma 5.1.2 holds with a significantly better dependency on ε. For families $\mathcal{F} = \{F\}$ where F is an arbitrary graph, Fox [39] (see also [81]) showed that Lemma 5.1.2 holds with both δ^{-1} and n_0 bounded by $\texttt{TOWER}(O(\log(\varepsilon^{-1})))$ — as a consequence, this same bound holds for every *finite* family $\mathcal{F}$. Moreover if F is bipartite, than δ^{-1} and n_0 are polynomial in ε^{-1} and, though it is not possible to get polynomial bounds when F is not bipartite (see [2]), the best known lower bound for δ^{-1} is only quasi-polynomial in ε^{-1}. Lemma 5.1.2 also holds with $\delta^{-1}, M, n_0 \leq \texttt{poly}(\varepsilon^{-1})$ for certain *infinite* families $\mathcal{F}$. For instance, results from [46] provide such polynomial bounds when *Forb*($\mathcal{F}$) is the property of "being k-colorable" (for every positive integer k) or the property of "having a bisection of size at most ρn^2" (for every $\rho > 0$) or many other properties that can be expressed as "partition problems".

We show that every monotone graph property *Forb*($\mathcal{F}$) is f-recoverable for some function f that is only exponential in the bounds given by the Removal Lemma for the family $\mathcal{F}$. In fact, we use a weighted version of this lemma (see Lemma 5.3.6).

Theorem 5.1.3. *For every family $\mathcal{F}$ of graphs, the property Forb($\mathcal{F}$) is f-recoverable for $f(\varepsilon) = n_0 2^{\texttt{poly}(M/\delta)}$, where δ, M and n_0 are as in Lemma 5.3.6 with input $\mathcal{F}$ and ε.*

The case of $\mathcal{F}$ finite is an instance where the bounds given by Lemma 5.3.6 relate polynomially with the bounds of Lemma 5.1.2. In particular, Theorem 5.1.3, together with the abovementioned bounds for Lemma 5.1.2 obtained by Fox [39] for finite families $\mathcal{F}$, implies that *Forb*($\mathcal{F}$) is f-recoverable with $f(\varepsilon) = \texttt{TOWER}(\texttt{poly}(\log(1/\varepsilon)))$.

The remainder of the paper is structured as follows. In Section 5.2 we introduce notation and describe some tools that we use in our arguments. In Section 5.3, we introduce the concept of recoverable graph properties and prove Theorem 5.1.3. Theorem 5.1.1 is a consequence of Theorem 5.4.1, which is the main result in Section 5.4. In Section 5.5 we prove Theorem 5.3.2, which is the technical tool for establishing Theorem 5.4.1. We finish the paper with some concluding remarks in Section 5.6.

5.2 Notation and tools

A *weighted graph* R over a (finite) set of vertices V is a symmetric function from $V \times V$ to $[0, 1]$. A weighted graph R may be viewed as a complete graph (with loops) in which a weight $R(\mathrm{i}, j)$ is given to each edge $(\mathrm{i}, j) \in V(R) \times V(R)$, where $V(R)$ denotes the vertex set of R. The set of all weighted graphs with vertex set V is denoted by $\mathcal{G}^*(V)$ and we define $\mathcal{G}^*$ as the union of all $\mathcal{G}^*(V)$ for V finite. In particular, a *graph* G is a weighted graph such that $G(\mathrm{i}, \mathrm{i}) = 0$, for every $\mathrm{i} \in V(G)$, and either $G(\mathrm{i}, j) = 1$ or $G(\mathrm{i}, j) = 0$

for every $(\mathrm{i}, j) \in V(G) \times V(G)$, $\mathrm{i} \neq j$. For a weighted graph $R \in \mathcal{G}^*(V)$ and for sets $A, B \subset V$, we denote $\mathrm{e}_R(A, B) = \sum_{(\mathrm{i},j)\in A\times B} R(\mathrm{i}, j)$ and $\mathrm{e}(R) = \mathrm{e}(V, V)/2$. Given a graph $G = (V, E)$ and vertex sets $U, W \subseteq V(G)$, let $E_G(U, W) = \{(u, w) \in E \colon u \in U, w \in W\}$ and $\mathrm{e}_G(U, W) = |E_G(U, W)|$.

An *equipartition* $\mathcal{V} = \{V_\mathrm{i}\}_{\mathrm{i}=1}^k$ of a weighted graph R is a partition of its vertex set $V(R)$ such that $|V_\mathrm{i}| \le |V_j| + 1$ for all $(\mathrm{i}, j) \in [k] \times [k]$. We sometimes abuse terminology and say that $\mathcal{V}$ is a partition of R.

Let $\mathcal{V} = \{V_1, \ldots, V_k\}$ be an equipartition into k classes of a graph $G = (V, E)$. The *cluster graph* of G by $\mathcal{V}$ is a weighted graph $G/_{\mathcal{V}} \in \mathcal{G}^*([k])$ such that $G/_{\mathcal{V}}\,(\mathrm{i}, j) = \mathrm{e}_G(V_\mathrm{i}, V_j)/(|V_\mathrm{i}||V_j|)$ for all $(\mathrm{i}, j) \in [k] \times [k]$. For a fixed integer $K > 0$, the set of all equipartitions of a vertex set V into at most K classes will be denoted by $\Pi_K(V)$. We also define the set $G/_{\Pi_K} = \{G/_{\mathcal{V}} : \mathcal{V} \in \Pi_K(V(G))\}$ of all cluster graphs of G whose vertex set has size at most K.

The *distance* between two weighted graphs $R, R' \in \mathcal{G}^*(V)$ on the same vertex set V is given by

$$\mathrm{d}_1(R, R') = \frac{1}{|V|^2} \sum_{(\mathrm{i},j)\in V\times V} |R(\mathrm{i}, j) - R'(\mathrm{i}, j)|.$$

For a property $\mathcal{H} \subseteq \mathcal{G}^*$ of weighted graphs, i.e., for a subset of the set of weighted graphs which is closed under isomorphisms, we define

$$\mathrm{d}_1(R, \mathcal{H}) = \min_{\substack{R'\in\mathcal{H}:\\ V(R')=V(R)}} \mathrm{d}_1(R, R').$$

Unless said otherwise, we will assume that $\mathcal{H}$ contains weighted graphs with vertex sets of all possible sizes.

Next, to set up the version of regularity (or Regularity Lemma) that we use in this work, we use a second well-known distance between weighted graphs. Let $R_1, R_2 \in \mathcal{G}^*(V)$ be weighted graphs on the same vertex set. The *cut-distance* between R_1 and R_2 is defined as

$$\mathrm{d}_\square(R_1, R_2) = \frac{1}{|V|^2} \max_{S,T\subseteq V} |\mathrm{e}_{R_1}(S, T) - \mathrm{e}_{R_2}(S, T)|.$$

Let $\Gamma \in \mathcal{G}(V)$ and $\mathcal{V} = \{V_\mathrm{i}\}_{\mathrm{i}=1}^k$ be a partition of V. We define the weighted graph $\Gamma_\mathcal{V} \in \mathcal{G}^*(V)$ as the weighted graph such that $\Gamma_\mathcal{V}(u, v) = \Gamma/_{\mathcal{V}}\,(\mathrm{i}, j)$ if $u \in V_\mathrm{i}$ and $v \in V_j$. Graph regularity lemmas ensure that, for any large graph Γ, there exists an equipartition $\mathcal{V}$ into a constant number of classes such that $\Gamma_\mathcal{V}$ is a faithful approximation of Γ. Here, we use the regularity introduced by Frieze and Kannan [41]. Henceforth we write $b = a \pm x$ for $a - x \le b \le a + x$.

Definition 5.2.1. A partition $\mathcal{V} = \{V_\mathrm{i}\}_{\mathrm{i}=1}^k$ of a graph Γ is *γ-FK-regular* if $\mathrm{d}_\square(\Gamma, \Gamma_\mathcal{V}) \le \gamma$, or, equivalently if for all $S, T \subseteq V(\Gamma)$ it holds that

$$\mathrm{e}(S, T) = \sum_{(\mathrm{i},j)\in[k]\times[k]} |S \cap V_\mathrm{i}||T \cap V_j|\; \Gamma/_{\mathcal{V}}\,(\mathrm{i}, j) \pm \gamma |V(\Gamma)|^2.$$

The concept of FK-regularity is also known as *weak regularity.*

Lemma 5.2.2 (Frieze-Kannan Regularity Lemma)**.** *For every $\gamma > 0$ and every $k_0 > 0$, there is $K = k_0 \cdot 2^{\mathtt{poly}(1/\gamma)}$ such that every graph Γ on $n \ge K$ vertices admits a γ-FK-regular equipartition into k classes, where $k_0 \le k \le K$.* $\square$

We remark that Conlon and Fox [24] found graphs where the number of classes in any γ-FK-regular equipartition is at least $2^{1/(2^{60}\gamma^2)}$ (for an earlier result, see Lovász and Szegedy [78]).

5.3 Recoverable parameters

The main objective of this section is to introduce the concept of ε-recoverability and to state our main results in terms of it.

5.3.1 Estimation over cluster graphs

For a weighted graph $R \in \mathcal{G}^*(V)$ and a subset $Q \subseteq V$ of vertices, let $R[Q]$ denote the induced weighted subgraph of R with vertex set Q. Let us now define estimable parameters in the context of weighted graphs.

Definition 5.3.1. We say that a function $z\colon \mathcal{G}^* \to \mathbb{R}$ (also called a *weighted graph parameter*) is *estimable* with *sample complexity* $q\colon (0,1) \to \mathbb{N}$ if, for every $\varepsilon > 0$ and every weighted graph $\Gamma^* \in \mathcal{G}^*(V)$ with $|V| \geq q(\varepsilon)$, we have $z(\Gamma^*) = z(\Gamma^*[Q]) \pm \varepsilon$ with probability at least $2/3$, where Q is chosen uniformly from all subsets of V of size q.

The following result states that graph parameters, that can be expressed as the optimal value of some optimization problem over the set $G/_{\Pi_K}$ of all cluster graphs of G of vertex size at most K, can be estimated with a query complexity that is only exponential in a polynomial in K and in the error parameter.

Theorem 5.3.2. *Let $z\colon \mathcal{G} \to \mathbb{R}$ be a graph parameter and suppose that there is a weighted graph parameter $z^*\colon \mathcal{G}^* \to \mathbb{R}$ and constants $K > 0$ and $c > 0$ such that*

1. *$z(\Gamma) = \max_{R \in \Gamma/_{\Pi_K}} z^*(R)$ for every $\Gamma \in \mathcal{G}$, and*
2. *$|z^*(R) - z^*(R')| \leq c \cdot \mathrm{d}_1(R, R')$ for all weighted graphs $R, R' \in \mathcal{G}^*$ on the same vertex set.*

Then z is estimable with sample complexity $\varepsilon \mapsto \mathtt{poly}(K, c/\varepsilon)$.

The proof of Theorem 5.3.2 is rather technical and is therefore deferred to Section 5.5. Moreover, in Section 5.4 we show how to express the parameter $z_{\mathcal{F}}$ introduced in (5.1.1), in terms of the solution of a suitable optimization problem over the set $\Gamma/_{\Pi_K}$ of cluster graphs of Γ of vertex size at most K.

5.3.2 Recovering partitions

We are interested in the property of graphs that are free of *copies* of members of a (possibly infinite) family $\mathcal{F}$ of graphs. To relate this property to a property of cluster graphs, we introduce some definitions. Let $\varphi\colon V(F) \to V(R)$ be a mapping from the set of vertices of a graph $F \in \mathcal{G}$ to the set of vertices of a weighted graph $R \in \mathcal{G}^*$. The *homomorphism weight* $hom_\varphi(F, R)$ of φ is defined as

$$hom_\varphi(F, R) = \prod_{(\mathrm{i},j) \in E(F)} R(\varphi(\mathrm{i}), \varphi(j)).$$

The *homomorphism density* $t(F,R)$ of $F \in \mathcal{G}$ in $R \in \mathcal{G}^*$ is defined as the average homomorphism weight of a mapping in $\Phi := \{\varphi\colon V(F) \to V(R)\}$, that is,

$$t(F,R) = \frac{1}{|\Phi|}\sum_{\varphi\in\Phi} hom_\varphi(F,R).$$

Note that, if F and R are graphs, then $t(F,R)$ is approximately the density of copies of F in R (and converges to this quantity when the vertex size of R tends to infinity). Since weighted graphs will represent cluster graphs associated with a partition of the vertex set of the input graph, it will be convenient to work with the following property of weighted graphs:

$$Forb^*_{hom}(\mathcal{F}) = \{R \in \mathcal{G}^* : t(F,R) = 0 \text{ for every } F \in \mathcal{F}\}.$$

Let $R, S \in \mathcal{G}^*(V)$ be weighted graphs on the same vertex set V. We say that S is a *spanning subgraph* of R, which will be denoted by $S \le R$, if $S(\mathrm{i}, j) \le R(\mathrm{i}, j)$ for every $(\mathrm{i}, j) \in V \times V$. When there is no ambiguity, we will just say that S is a *subgraph* of R. We also define $Forb^*_{hom}(R,\mathcal{F}) = \{S \in Forb^*_{hom}(\mathcal{F}) : S \le R\}$.

The following result shows that having a cluster graph in $Forb^*_{hom}(\mathcal{F})$ witnesses membership in $Forb(\mathcal{F})$.

Proposition 5.3.3. *Let $\mathcal{F}$ be a family of graphs and let $\mathcal{V}$ be an equipartition of a graph G. If $G/\mathcal{V} \in Forb^*_{hom}(\mathcal{F})$, then $G \in Forb(\mathcal{F})$.*

Proof. Let $\mathcal{V} = \{V_\mathrm{i}\}_{\mathrm{i}=1}^k$ be an equipartition of G and let $R = G/\mathcal{V}$. Fix an arbitrary element $F \in \mathcal{F}$ and an arbitrary injective mapping $\varphi\colon V(F) \hookrightarrow V(G)$. Define the function $\psi\colon V(F) \to V(R)$ by $\psi(v) = \mathrm{i}$ if $\varphi(v) \in V_\mathrm{i}$. Now, if $t(F,R) = 0$, there must be some edge $(u,w) \in E(F)$ such that $R(\psi(u),\psi(w)) = 0$, thus $G(\varphi(u),\varphi(v)) = 0$ and hence $hom_\varphi(F,G) = 0$. Since φ and F were taken arbitrarily, we must have $G \in Forb(\mathcal{F})$. $\square$

It is easy to see that the converse of Proposition 5.3.3 does not hold in general. Indeed, there exist graph families $\mathcal{F}$ and graphs $G \in Forb(\mathcal{F})$ such that $G/\mathcal{V}$ is actually very *far* from being in $Forb^*_{hom}(\mathcal{F})$ for some equipartition $\mathcal{V}$ of G. As an example, let G be the n-vertex bipartite Turán graph $T_2(n)$ for the triangle K_3 with partition $V(G) = A \cup B$ and consider $\mathcal{V} = \{V_\mathrm{i}\}_{\mathrm{i}=1}^t$ with $V_\mathrm{i} = A_\mathrm{i} \cup B_\mathrm{i}$, $\mathrm{i} = 1,\dots,t$, where $\{A_\mathrm{i}\}_{\mathrm{i}=1}^t$ and $\{B_\mathrm{i}\}_{\mathrm{i}=1}^t$ are equipartitions of A and B respectively. Then $G/\mathcal{V}$ has weight $1/2$ on every edge, so that the distance of $G/\mathcal{V}$ to the family $Forb^*_{hom}(\{K_3\})$ tends to $1/4$ for t large by Turán's Theorem. More generally, if $\mathcal{V}$ is a random equipartition of a triangle-free graph $G \in Forb(\{K_3\})$ with large edge density, then with high probability the cluster graph $G/\mathcal{V}$ is still approximately $1/4$-far from being in $Forb^*_{hom}(\{K_3\})$.

On the other hand, we will prove that there exist partitions for graphs in $Forb(\mathcal{F})$ with respect to which an approximate version of the converse of Proposition 5.3.3 does hold, that is, we will prove that every graph in $Forb(\mathcal{F})$ is not too far from having a partition into a constant number of classes that witnesses membership in $Forb(\mathcal{F})$. We say that such a partition is *recovering* with respect to $Forb(\mathcal{F})$. Let us make this more precise.

Definition 5.3.4. Let $\mathcal{P} = Forb(\mathcal{F})$ be a monotone graph property. An equipartition $\mathcal{V}$ of a graph $G \in \mathcal{P}$ is *ε-recovering* for $\mathcal{P}$ if $d_1(G/\mathcal{V}, Forb^*_{hom}(\mathcal{F})) \le \varepsilon$.

Definition 5.3.5. Let $\mathcal{P}$ be a graph property. For a fixed function $f\colon (0,1] \to \mathbb{R}$, we say that the class $\mathcal{P}$ is *f-recoverable* if, for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that the following holds. For every graph $G \in \mathcal{P}$ on $n \geq n_0$ vertices, there is an equipartition $\mathcal{V}$ of G into at most $f(\varepsilon)$ classes which is ε-recovering for $\mathcal{P}$.

As a simple example, one can verify that the graph property $\mathcal{P} = \mathit{Forb}(\mathcal{F})$ of being r-colourable is f-recoverable for $f(\varepsilon) = r/\varepsilon$; here and in what follows, for simplicity, we ignore divisibility conditions and drop floor and ceiling signs. Let G be a graph in $\mathcal{P}$, with colour classes $C_1, \ldots, C_r$. Let $k = r/\varepsilon$. Start by fixing parts $V_1, \ldots, V_t$ of size n/k each, with each V_{i} contained in some C_j ($j = j(\mathrm{i})$), and leaving out fewer than n/k vertices from each C_j, $1 \leq j \leq r$. The sets V_{i}, $1 \leq \mathrm{i} \leq t$, cover a subset C'_j of C_j and $X_j = C_j \setminus C'_j$ is left over. We then complete the partition by taking arbitrary parts $U_1, \ldots, U_{k-t}$ of size n/k each, forming a partition of $\bigcup_{1\leq j\leq r} X_j$. The cluster graph $G/\mathcal{V}$ can be made r-partite by giving weight zero to every edge incident to vertices corresponding to $U_1, \ldots, U_{k-t}$. Therefore $G/\mathcal{V}$ is at distance at most $r/k \leq \varepsilon$ from being r-partite. Thus, $d_1(G/\mathcal{V}, \mathit{Forb}^*_{hom}(\mathcal{F})) \leq \varepsilon$ as required.

We finish this section by noting that the definition of f-recoverable properties has some similarity with the notion of *regular-reducible* properties $\mathcal{P}$ defined by Alon, Fischer, Newman and Shapira [5]. When dealing with monotone properties $\mathcal{P} = \mathit{Forb}(\mathcal{F})$, the main difference is that the notion of being regular-reducible requires that every graph $G \in \mathcal{P}$ should have a *regular* partition such that $G/\mathcal{V}$ is close to some property $\mathcal{H}^*$ of weighted graphs, while the definition of f-recoverable properties requires only that every G has a partition $\mathcal{V}$ (regular or not) such that $G/\mathcal{V}$ is close to $\mathit{Forb}^*_{hom}(\mathcal{F})$. Another difference is that $\mathcal{H}^*$ must be such that having a (regular) cluster graph in $\mathcal{H}^*$ witnesses only *closeness* to $\mathcal{P}$, while having a (regular or not) cluster graph in $\mathit{Forb}^*_{hom}(\mathcal{F})$ witnesses membership in $\mathcal{P}$.

5.3.3 Monotone graph properties are recoverable

Szemerédi's Regularity Lemma [102] can be used to show that every monotone (and actually every hereditary) graph property is f-recoverable, for $f(\varepsilon) = \mathtt{TOWER}(\mathtt{poly}(1/\varepsilon))$. In the remainder of this section, we prove that monotone properties $\mathcal{P} = \mathit{Forb}(\mathcal{F})$ are recoverable using a weaker version of regularity along with the Removal Lemma, which leads to an improvement on the growth of f for families $\mathcal{F}$ where the Removal Lemma is known to hold with better bounds than the Regularity Lemma.

We first derive a version of the Removal Lemma stated in the introduction (Lemma 5.1.2) that applies to weighted graphs and homomorphic copies.

Lemma 5.3.6. *For every $\varepsilon > 0$ and every (possibly infinite) family $\mathcal{F}$ of graphs, there exist $\delta = \delta(\varepsilon, \mathcal{F})$, $M = M(\varepsilon, \mathcal{F})$ and $n_0 = n_0(\varepsilon, \mathcal{F})$ such that the following holds. If a weighted graph R such that $|V(R)| > n_0$ satisfies $\mathrm{d}_1(R, \mathrm{Forb}^*_{hom}(\mathcal{F})) \geq \varepsilon$, then there is a graph $F \in \mathcal{F}$ with $|V(F)| \leq M$ such that $t(F, R) \geq \delta$.*

To prove Lemma 5.3.6, we use the following auxiliary result, which follows from work of Erdős and Simonovits [34]. For completeness we include its proof.

Proposition 5.3.7. *Let $\widehat{F}$ and F be graphs in $\mathcal{G}$ such that there is a surjective homomorphism $\zeta\colon V(F) \to V(\widehat{F})$. Then, for every graph H such that $t(\widehat{F}, H) \geq \widehat{\delta}$, we must have $t(F, H) \geq \widehat{\delta}^{\ell}$, where $\ell = (|V(F)| + 1)^{|V(\widehat{F})|}$.*

Proof. We will consider the particular case in which F is obtained from blowing up a single vertex v of $\widehat{F}$ into r distinct vertices $v_1, \ldots, v_r$ with the same adjacency as v, hence we assume that $\zeta(v_j) = v$ for every $j = 1, \ldots, r$ and $\zeta(u) = u$ for every $u \notin \{v_1, \ldots, v_r\}$.

Let $n = |V(H)|$, $\widehat{a} = |V(\widehat{F})|$, $a = |V(F)| = \widehat{a} + r - 1$ and $\widehat{F}_- = \widehat{F} - v$ be the graph on $\widehat{a} - 1$ vertices obtained from $\widehat{F}$ by deleting v. Let $N = t(\widehat{F}_-, H)n^{\widehat{a}-1}$ be the number of homomorphisms from $\widehat{F}_-$ to H and $\varphi_1, \ldots, \varphi_N \in V(H)^{V(\widehat{F}_-)}$ be an enumeration of such homomorphisms. Note that $N \leq n^{\widehat{a}-1}$ and $N \geq t(\widehat{F}, H)/n \geq \widehat{\delta} n^{\widehat{a}-1}$.

For every $\mathrm{i} \in [N]$ and $u \in V(H)$, we consider the function φ_i^u that extends φ_i by mapping v to u. Define $Z_\mathrm{i} = \{u \in V(H)\colon \mathit{hom}_{\varphi_\mathrm{i}^u}(\widehat{F}, H) = 1\}$ and $z_\mathrm{i} = |Z_\mathrm{i}|$. We claim there are z_i^r ways of extending φ_i to a homomorphism from F to H. Indeed, every possible extension $\varphi'_\mathrm{i}\colon V(F) \to V(H)$ of φ_i, such that $\varphi'_\mathrm{i}(v_j) \in Z_\mathrm{i}$, for every $j = 1, \ldots, r$, satisfies $\mathit{hom}_{\varphi'_\mathrm{i}}(F, H) = 1$. Therefore we have $t(F, H)n^a \geq \sum_{\mathrm{i}=1}^N z_\mathrm{i}^r$. Since $g(x) = x^r$ is a convex function for $x \geq 0$ and $r \geq 1$, we get

$$t(F, H)n^a \geq N \left(\frac{\sum_{\mathrm{i}=1}^N z_\mathrm{i}}{N} \right)^r.$$

Now we use the fact that $\sum_{\mathrm{i}=1}^N z_\mathrm{i} = t(\widehat{F}, H)n^{\widehat{a}} \geq \widehat{\delta} n^{\widehat{a}}$ and our previous bounds on N to obtain that

$$t(F, H)n^a \geq \widehat{\delta} n^{\widehat{a}-1} \left(\frac{\widehat{\delta} n^{\widehat{a}}}{n^{\widehat{a}-1}} \right)^r = \widehat{\delta}^{r+1} n^{\widehat{a}+r-1} = \widehat{\delta}^{r+1} n^a.$$

Therefore, $t(F, H) \geq \widehat{\delta}^{r+1} \geq \widehat{\delta}^{a+1}$. The general case may be easily obtained by induction on the number of vertices of $\widehat{F}$. □

Proof of Lemma 5.3.6. Denote by $\widehat{\mathcal{F}}$ the set of all *homomorphic images* of members of $\mathcal{F}$, that is, the set of all graphs $\widehat{F} \in \mathcal{G}$ such that there is a surjective homomorphism $F \to \widehat{F}$, for some $F \in \mathcal{F}$. Let $\widehat{M}, \widehat{\delta}$ and $\widehat{n}_0$ be as in Lemma 5.1.2 with input $\widehat{\mathcal{F}}$ and $\varepsilon/2$. We take

$$M = \max_{\substack{\widehat{F} \in \widehat{\mathcal{F}}: \\ |V(\widehat{F})| \leq \widehat{M}}} \min_{\substack{F \in \mathcal{F}: \\ F \to \widehat{F}}} |V(F)|,$$

$n_0 = \widehat{n}_0$, $\delta = (\varepsilon/2)^{M^2} \widehat{\delta}^\ell$, where $\ell = (M+1)^{\widehat{M}}$.

Let R be a weighted graph such that $|V(R)| > n_0$ and $d_1(R, \mathit{Forb}^*_{\mathit{hom}}(\mathcal{F})) \geq \varepsilon$. We first define a graph $H \in \mathcal{G}(V(R))$ such that $H(\mathrm{i}, j) = 1$ if and only if $R(\mathrm{i}, j) \geq \varepsilon/2$. It follows from $d_1(R, \mathit{Forb}^*_{\mathit{hom}}(\mathcal{F})) \geq \varepsilon$ that $d_1(H, \mathit{Forb}(\widehat{\mathcal{F}})) \geq \varepsilon/2$. Indeed, suppose to the contrary that there exists $H' \in \mathit{Forb}(\widehat{\mathcal{F}})$ such that $d_1(H, H') < \varepsilon/2$. Define R' such that $R'(\mathrm{i}, j) = R(\mathrm{i}, j)$ if $H'(\mathrm{i}, j) = 1$ and $R'(\mathrm{i}, j) = 0$ otherwise. By construction, $R' \in \mathit{Forb}^*_{\mathit{hom}}(\mathcal{F})$, and we get a contradiction from

$$\begin{aligned}
d_1(R,R') &= \frac{1}{|V(R)|^2} \sum_{\substack{\mathrm{i}\in V(R),\\ j\in V(R)}} |R(\mathrm{i},j) - R'(\mathrm{i},j)| \\
&= \frac{1}{|V(R)|^2} \left(\sum_{\substack{(\mathrm{i},j):\\ H(\mathrm{i},j)=1,\\ H'(\mathrm{i},j)=0}} |R(\mathrm{i},j) - R'(\mathrm{i},j)| + \sum_{\substack{(\mathrm{i},j):\\ H(\mathrm{i},j)=0,\\ H'(\mathrm{i},j)=0}} |R(\mathrm{i},j) - R'(\mathrm{i},j)| \right) \\
&\leq \frac{1}{|V(H)|^2} \sum_{\substack{\mathrm{i}\in V(H)\\ j\in V(H)}} |H(\mathrm{i},j) - H'(\mathrm{i},j)| + \frac{1}{|V(R)|^2} \sum_{\substack{\mathrm{i}\in V(R)\\ j\in V(R)}} \frac{\varepsilon}{2} \\
&= d_1(H,H') + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

By Lemma 5.1.2 there must be $\widehat{F} \in \widehat{\mathcal{F}}$, with $|V(\widehat{F})| \leq \widehat{M}$, such that $t(\widehat{F},H) \geq \widehat{\delta}$. By definition of M, there must be $F \in \mathcal{F}$ such that $|V(F)| \leq M$ and there is a surjective homomorphism $F \to \widehat{F}$. It follows from Proposition 5.3.7 that $t(F,H) \geq \widehat{\delta}^{\ell}$. Since

$$hom_{\varphi}(F,R) \geq (\varepsilon/2)^{|E(F)|}\, hom_{\varphi}(F,H) \geq (\varepsilon/2)^{M^2}\, hom_{\varphi}(F,H)$$

for each $\varphi\colon V(F) \to V(R)$, we must have

$$t(F,R) = \frac{\sum_{\varphi} hom_{\varphi}(F,R)}{|V(R)|^{|V(F)|}} \geq \left(\frac{\varepsilon}{2}\right)^{M^2} \cdot \frac{\sum_{\varphi} hom_{\varphi}(F,H)}{|V(H)|^{|V(F)|}} \geq \left(\frac{\varepsilon}{2}\right)^{M^2} \cdot \widehat{\delta}^{\ell} = \delta.$$

□

We will use the next result, which states that a graph has homomorphism densities close to the ones of the cluster graphs with respect to FK-regular partitions.

Lemma 5.3.8 ([18, Theorem 2.7(a)]). *Let $\mathcal{V}$ be a γ-FK-regular equipartition of a graph $G \in \mathcal{G}$. Then, for any graph $F \in \mathcal{G}$ it holds that $t(F,G) = t(F,G_{\mathcal{V}}) \pm 4\mathrm{e}(F)\gamma = t(F, G/_{\mathcal{V}}) \pm 4\mathrm{e}(F)\gamma$.* □

We are now ready to prove Theorem 5.1.3, which establishes that every monotone graph property is f-recoverable.

Proof of Theorem 5.1.3. Let δ, M and n_0 be as in Lemma 5.3.6 with input $\mathcal{F}$ and ε and let $\gamma = \delta/(3M)^2$. By Lemma 5.2.2, it suffices to show that any γ-FK-regular equipartition $\mathcal{V} = \{V_{\mathrm{i}}\}_{\mathrm{i}=1}^{k}$ of a graph $G \in Forb(\mathcal{F})$ into $k \geq n_0$ classes is ε-recovering.

Let $R = G/_{\mathcal{V}}$ and suppose for contradiction that $d_1(R, Forb^{*}_{hom}(\mathcal{F})) \geq \varepsilon$. Then, by Lemma 5.3.6, we have $t(F,R) \geq \delta$ for some graph $F \in \mathcal{F}$ such that $|F| \leq M$. By Lemma 5.3.8, we have $t(F,G) \geq \delta - 2\gamma M^2 > 0$, a contradiction to $G \in Forb(\mathcal{F})$. □

5.4 Estimation of $|\,\boldsymbol{Forb}(\Gamma, \mathcal{F})|$

The objective of this section is to prove Theorem 5.1.1. To do this, we shall approximate the parameter $z_{\mathcal{F}}$ by the solution of an optimization problem as in Theorem 5.3.2. Recall that $\mathit{Forb}^*_{hom}(R, \mathcal{F}) = \{S \leq R : t(F, S) = 0 \text{ for every } F \in \mathcal{F}\}$, and set

$$\mathrm{ex}^*(R, \mathcal{F}) = \frac{1}{|V(R)|^2} \max_{S \in \mathit{Forb}^*_{hom}(R,\mathcal{F})} \mathrm{e}(S),$$

which measures the largest edge density of a subgraph of R not containing a copy of any $F \in \mathcal{F}$ up to a multiplicative constant.

We shall derive Theorem 5.1.1 from the following auxiliary result.

Theorem 5.4.1. *Let $\mathcal{F}$ be a family of graphs such that $\mathrm{Forb}(\mathcal{F})$ is f-recoverable for some function $f\colon (0, 1] \to \mathbb{R}$. Then, for any $\varepsilon > 0$, there exists $K = f(\mathtt{poly}(\varepsilon))$ and $N = \mathtt{poly}(K)$ such that for any graph Γ of vertex size $n \geq N$ it holds that*

$$\frac{\log_2 |\,\mathit{Forb}(\Gamma, \mathcal{F})|}{n^2} = \max_{R \in \Gamma/\Pi_K} \mathit{ex}^*(R, \mathcal{F}) \pm \varepsilon.$$

We define the following subsets of edges of a weighted graph R:

$$E_0(R) = \{(\mathrm{i}, j) \in V(R) \times V(R) : R(\mathrm{i}, j) = 0\}$$
$$E_1(R) = \{(\mathrm{i}, j) \in V(R) \times V(R) : R(\mathrm{i}, j) > 0\}.$$

We will also make use of the *binary entropy function*, defined by $H(x) = -x \log_2(x) - (1-x) \log_2(1-x)$ for $0 < x < 1$. Note that $H(x) \leq -2x \log_2 x$ for $x \leq 1/8$. This function has the property (cf. [68, Corollary 22.2]) that the following inequality holds for $\varepsilon = k/n < 1/2$:

$$\sum_{\mathrm{i}=0}^{k} \binom{n}{\mathrm{i}} \leq 2^{H(\varepsilon)n}. \tag{5.4.1}$$

Proof of Theorem 5.4.1. Let $\mathcal{F}$ be a family of graphs such that $\mathit{Forb}(\mathcal{F})$ is f-recoverable, and fix $\varepsilon > 0$, without loss of generality $\varepsilon < 1$. Let $\varepsilon' = \varepsilon/18$. Using that $\log_2 x \leq x - 1$ for $x < 1$ and $H(y) \leq -2y \log_2 y$ for $0 < y \leq 1/8$, we infer $H(\varepsilon') + \varepsilon' \leq -2\varepsilon' \log_2 \varepsilon' + \varepsilon' \leq 2\varepsilon'(1 - \varepsilon') + \varepsilon' \leq 3\varepsilon' \leq \varepsilon/6$. We set $K = f(\varepsilon'^2)$ and $N \geq 2K^2/\varepsilon$ big enough so that $\log_2 N/N < \varepsilon/3$.

Let Γ be an n-vertex graph, $n \geq N$. We first show that

$$\frac{\log_2 |\,\mathit{Forb}(\Gamma, \mathcal{F})|}{n^2} \geq \max_{R \in \Gamma/\Pi_K} \mathit{ex}^*(R, \mathcal{F}) - \varepsilon.$$

Let $R = \Gamma/\mathcal{V}$ be an arbitrary cluster graph in Γ/Π_K with $\mathcal{V} = \{V_\mathrm{i}\}_{\mathrm{i}=1}^k$ for $k \leq K$. Choose $S \in \mathit{Forb}^*_{hom}(R, \mathcal{F})$ such that $\mathrm{e}(S) = k^2\, \mathit{ex}^*(R, \mathcal{F})$. Further let $G \leq \Gamma$ be the subgraph of Γ such that $G(r, s) = 0$ if there is a pair $(\mathrm{i}, j) \in E_0(S)$ such that $r \in V_\mathrm{i}$ and $s \in V_j$ and $G(r, s) = \Gamma(r, s)$ otherwise. Thus, we obtain G by deleting all edges from Γ between V_i and V_j if $(\mathrm{i}, j) \in E_0(S)$.

Since $\mathrm{e}(S)$ maximizes $ex^*(R,\mathcal{F})$ it follows that $G/\mathcal{V} = S$, which implies, by Proposition 5.3.3, that $G \in Forb(\Gamma,\mathcal{F})$. Since every subgraph of G also lies in $Forb(\Gamma,\mathcal{F})$, we obtain

$$\log_2 |\,Forb(\Gamma,\mathcal{F})| \geq |\mathrm{e}(G)| = \frac{1}{2} \sum_{(\mathrm{i},j)\in[k]\times[k]} S(\mathrm{i},j)|V_\mathrm{i}||V_j| \geq \frac{(n-k)^2}{k^2}\mathrm{e}(S)$$
$$\geq ex^*(R,\mathcal{F})n^2 - kn \geq (ex^*(R,\mathcal{F}) - \varepsilon)n^2.$$

Note that we used the facts that $\mathrm{e}(S) \leq k^2/2$ and $n > k/\varepsilon$, as well as $|V_\mathrm{i}| \geq n/k - 1$ for all i.

Now let us prove the other direction

$$\frac{\log_2 |\,Forb(\Gamma,\mathcal{F})|}{n^2} \leq \max_{R\in\Gamma/\Pi_K} ex^*(R,\mathcal{F}) + \varepsilon.$$

We first define $\mathcal{U} = \bigcup_{G\in Forb(\Gamma,\mathcal{F})} G/\Pi_K$ to be the set of all possible cluster graphs of vertex size at most K of graphs in $Forb(\Gamma,\mathcal{F})$. Since $Forb(\mathcal{F})$ is f-recoverable we can define a function

$$\eta\colon\ Forb(\Gamma,\mathcal{F}) \to \Pi_K \times \mathcal{U}$$
$$G \mapsto (\mathcal{V},T)$$

where $\mathcal{V}$ is an (ε'^2)-recovering partition of G into $k \leq K$ classes and $T = G/\mathcal{V}$. Clearly

$$|\,Forb(\Gamma,\mathcal{F})| \leq |\Pi_K \times \mathcal{U}| \cdot \max_{(\mathcal{V},T)} |\eta^{-1}(\mathcal{V},T)|. \tag{5.4.2}$$

Since each mapping from $V(\Gamma) \to [K]$ gives a partition of $V(\Gamma)$ into at most K classes, we have $|\Pi_K| \leq K^n \leq n^n$. Moreover, given an arbitrary graph $G \in \mathcal{G}(V)$ and any partition $\mathcal{V}$ of V, an edge $G/\mathcal{V}\,(\mathrm{i},j)$ may assume n^2 different values. Hence, we have $|\mathcal{U}| \leq n^{2K^2} \leq n^n$.

Finally we make the following claim, whose proof is deferred for a moment:

$$\log_2\left(\max_{(\mathcal{V},T)} |\eta^{-1}(\mathcal{V},T)|\right) \leq \left(\max_{R\in\Gamma/\Pi_K} ex^*(R,\mathcal{F}) + \frac{\varepsilon}{3}\right)n^2. \tag{5.4.3}$$

Combining this we can take the logarithm of (5.4.2) to get as desired:

$$\log_2 |\,Forb(\Gamma,\mathcal{F})| \leq \log_2(n^n) + \log_2(n^n) + \left(\max_{R\in\Gamma/\Pi_K} ex^*(R,\mathcal{F}) + \frac{\varepsilon}{3}\right)n^2$$
$$\leq \left(\max_{R\in\Gamma/\Pi_K} ex^*(R,\mathcal{F}) + \varepsilon\right)n^2 \qquad (\text{as } \log_2 n/n \leq \varepsilon/3).$$

It remains to prove (5.4.3). To this end, fix $(\mathcal{V},T)$ in the image of η and let $R = \Gamma/\mathcal{V}$. Choose $S' \in Forb^*(R,\mathcal{F})$ such that $d_1(T,S') \leq \varepsilon'^2$. This is possible because $\mathcal{V}$ is an (ε'^2)-recovering partition. Set $E_1 = E_1(S')$ and partition $E_0(S')$ into $E_0^+ := \{(\mathrm{i},j) \in E_0(S') : T(\mathrm{i},j) > \varepsilon'\}$ and $E_0^- = E_0(S') \setminus E_0^+$. Since there are $b(\mathrm{i},j) := \binom{|V_\mathrm{i}||V_j|R(\mathrm{i},j)}{|V_\mathrm{i}||V_j|T(\mathrm{i},j)}$ ways to choose $|V_\mathrm{i}||V_j|T(\mathrm{i},j)$ edges out of the $|V_\mathrm{i}||V_j|R(\mathrm{i},j)$ edges between V_i and V_j in Γ, we obtain

$$|\eta^{-1}(\mathcal{V},T)| \leq \prod_{1\leq \mathrm{i}<j\leq k} b(\mathrm{i},j) \leq \prod_{(\mathrm{i},j)\in E_1} \sqrt{b(\mathrm{i},j)} \prod_{(\mathrm{i},j)\in E_0^+} b(\mathrm{i},j) \prod_{(\mathrm{i},j)\in E_0^-} b(\mathrm{i},j). \tag{5.4.4}$$

Let us estimate the factors of (5.4.4):

We can bound each of the factors $b(\mathrm{i},j)$ of E_1 by $2^{R(\mathrm{i},j)|V_\mathrm{i}||V_j|}$. Since $d_1(T,S') \le \varepsilon'^2$ we have $|E_0^+| \le \varepsilon' k^2$, as otherwise it would be the case that

$$d_1(T,S') \ge \sum_{(\mathrm{i},j)\in E_0^+} |T(\mathrm{i},j) - S'(\mathrm{i},j)| > |E_0^+|\varepsilon' \ge \varepsilon'^2 k^2,$$

which is a contradiction. Clearly, we have $|E_0^-| \le k^2$. This allows us to upper bound each of the factors of E_0^+ trivially by $2^{|V_\mathrm{i}||V_j|}$, and each of the factors of E_0^- by $2^{H(\varepsilon')|V_\mathrm{i}||V_j|}$ using (5.4.1).

Now let $S \in \mathit{Forb}^*(R,\mathcal{F})$ be such that

$$S(\mathrm{i},j) = \begin{cases} 0 & \text{if } (\mathrm{i},j) \in E_0(S') \\ R(\mathrm{i},j) & \text{otherwise.} \end{cases}$$

Taking the logarithm of (5.4.4) and using $|V_\mathrm{i}||V_j| \le (n+k)^2/k^2$ we get

$$\begin{aligned} \log_2 |\eta^{-1}(\mathcal{V},T)| &\le \sum_{(\mathrm{i},j)\in E_1} \frac{R(\mathrm{i},j)}{2}|V_\mathrm{i}||V_j| + \sum_{(\mathrm{i},j)\in E_0^-} H(\varepsilon')|V_\mathrm{i}||V_j| + \sum_{(\mathrm{i},j)\in E_0^+} |V_\mathrm{i}||V_j| \\ &\le \Big(\sum_{(\mathrm{i},j)\in E_1} \frac{R(\mathrm{i},j)}{2k^2} + \sum_{(\mathrm{i},j)\in E_0^-} \frac{H(\varepsilon')}{k^2} + \sum_{(\mathrm{i},j)\in E_0^+} \frac{1}{k^2} \Big)(n+k)^2 \\ &\le \Big(\frac{1}{2k^2} \sum_{(\mathrm{i},j)\in E_1} S(\mathrm{i},j) + H(\varepsilon') + \varepsilon' \Big)(n+k)^2. \end{aligned}$$

Now by using the fact that $S \in \mathit{Forb}^*(R,\mathcal{F})$ and that $H(\varepsilon') + \varepsilon' \le \varepsilon/6$ we infer

$$\log_2 |\eta^{-1}(\mathcal{V},T)| \le \Big(ex^*(R,\mathcal{F}) + \frac{\varepsilon}{6} \Big)(n+k)^2 \le \Big(ex^*(R,\mathcal{F}) + \frac{\varepsilon}{3} \Big) n^2,$$

which implies (5.4.3). □

Proof of Theorem 5.1.1. Let $\mathcal{F}$ be a family of graphs such that $\mathit{Forb}(\mathcal{F})$ is f-recoverable. Set $K = f(\mathtt{poly}(\varepsilon))$ and $N = \mathtt{poly}(K)$ given by Theorem 5.4.1 applied to $\varepsilon/3$. Theorem 5.4.1 ensures that, whenever Γ is a graph on $n \ge N$ vertices, we have

$$\left| \frac{\log_2 |\mathit{Forb}(\Gamma,\mathcal{F})|}{n^2} - \max_{R\in\Gamma/\Pi_K} ex^*(R,\mathcal{F}) \right| \le \frac{\varepsilon}{3}. \tag{5.4.5}$$

Let $\widehat{z}\colon \mathcal{G} \to \mathbb{R}$ be the graph parameter defined by $\widehat{z}(\Gamma) = \max_{R\in\Gamma/\Pi_K} z^*(R)$, where $z^*(R) = ex^*(R,\mathcal{F})$. We claim that, given R and R' in $\mathcal{G}^*(V)$, we have $|z^*(R) - z^*(R')| \le d_1(R,R')$. Indeed, assume without loss of generality that $z^*(R) \ge z^*(R')$ and fix a subgraph $S \le R$ such that $S \in \mathit{Forb}^*_{hom}(R,\mathcal{F})$ and $z^*(R) = \mathrm{e}(S)/|V|^2$. If $S \in \mathit{Forb}^*_{hom}(R',\mathcal{F})$, we are done, so assume that this is not the case. Let S' be a subgraph of S and R' maximizing $\mathrm{e}(S')$, that is, $S'(\mathrm{i},j) = \min\{S(\mathrm{i},j), R'(\mathrm{i},j)\}$. Clearly,

$$\mathrm{e}(S') \ge \mathrm{e}(S) - \frac{1}{2} \sum_{(\mathrm{i},j)\in V\times V} |R(\mathrm{i},j) - R'(\mathrm{i},j)| = \mathrm{e}(S) - \frac{|V|^2}{2}\, d_1(R,R'),$$

so that $0 \le z^*(R) - z^*(R') \le (\mathrm{e}(S) - \mathrm{e}(S'))/|V|^2 \le \frac{1}{2}\,\mathrm{d}_1(R, R')$.

This allows us to apply Theorem 5.3.2 to conclude that $\widehat{z}$ is estimable with sample complexity $q(\varepsilon) = \mathtt{poly}(K, 1/\varepsilon)$. Let Q be chosen uniformly from all subsets of V of size $q' = \max\{q(\varepsilon/3), N\}$ and set $\overline{\Gamma} = \Gamma[Q]$. It follows that, with probability at least 2/3, we have $|\widehat{z}(\overline{\Gamma}) - \widehat{z}(\Gamma)| \le \varepsilon/3$. By (5.4.5) we have $|n^{-2}\log_2|\,Forb(\Gamma, \mathcal{F})| - \widehat{z}(\Gamma)| \le \varepsilon/3$. On the other hand, we can also apply (5.4.5) to $\overline{\Gamma}$ to obtain $\left|\widehat{z}(\overline{\Gamma}) - q'^{-2}\log_2|\,Forb(\overline{\Gamma}, \mathcal{F})|\right| \le \varepsilon/3$. By adding the last three inequalities, we get that

$$\left|\frac{1}{n^2}\log_2|\,Forb(\Gamma, \mathcal{F})| - \frac{1}{q'^2}\log_2|\,Forb(\overline{\Gamma}, \mathcal{F})|\right| \le \varepsilon,$$

as required. □

5.5 Proof of Theorem 5.3.2

Here we will prove Theorem 5.3.2. Its proof is based on the following lemma, which asserts that the set of cluster graphs of a graph Γ is very 'similar' to the set of cluster graphs of 'large enough' samples of Γ.

Lemma 5.5.1. *Given $K > 0$, $\varepsilon > 0$ there is $q = \mathtt{poly}(K, 1/\varepsilon)$ such that the following holds. Consider a graph Γ whose vertex set V has cardinality $n \ge q$ and a random subgraph $\overline{\Gamma} = \Gamma[\overline{V}]$, where $\overline{V}$ is chosen uniformly from all subsets of V of size q. Then, with probability at least 2/3, we have*

1. *for each $\mathcal{V} \in \Pi_K(V)$, there is a $\overline{\mathcal{V}} \in \Pi_K(\overline{V})$ with $\mathrm{d}_1(\Gamma/\mathcal{V}, \overline{\Gamma}/\overline{\mathcal{V}}) \le \varepsilon$, and*
2. *for each $\overline{\mathcal{V}} \in \Pi_K(\overline{V})$, there is a $\mathcal{V} \in \Pi_K(V)$ with $\mathrm{d}_1(\Gamma/\mathcal{V}, \overline{\Gamma}/\overline{\mathcal{V}}) \le \varepsilon$.*

For a set of vertices V and an integer k, define $\Pi_{=k}(V)$ as the set of all equipartitions of V of size *exactly* k. For every $R \in \mathcal{G}^*([k])$ and $\varepsilon \ge 0$, we define the property

$$\mathcal{G}_R^{(\varepsilon)} = \{G \in \mathcal{G} : \exists \mathcal{V} \in \Pi_{=k}(V(G)) \text{ such that } \mathrm{d}_1(G/\mathcal{V}, R) \le \varepsilon\}$$

of all graphs admitting a reduced graph which is ε-close to R. Note that $\mathcal{G}_R^{(\varepsilon)}$ contains no graphs of size less than k. In particular, if $|V(G)| < k$, then $\mathrm{d}_1(G, \mathcal{G}_R^{(\varepsilon)}) = \infty$. Since we will compare $\mathcal{G}_R^{(\varepsilon)}$ only with large graphs, this is not a problem.

The following theorem is a consequence of a more general result of [37, Theorem 2.7]. For our application it suffices to state this result in the case of simple graphs ($r = 2$, $s = 1$) with *density tensor* $\Psi = \{S \in \mathcal{G}^*([k]) : \mathrm{d}_1(R, S) \le \varepsilon\}$.

Theorem 5.5.2 ([37, Theorem 2.7])**.** *For every positive integer k, and every $\varepsilon > 0$ and $\delta > 0$, there is $q' = q'(k, \varepsilon, \delta) = \log^3(\delta^{-1}) \cdot \mathtt{poly}(k, \varepsilon^{-1})$ such that the following holds. For every $R \in \mathcal{G}^*([k])$ there is a randomized algorithm $\mathcal{T}$ which takes as input an oracle access to a graph G of size at least k and satisfies the following properties:*

1. *If $G \in \mathcal{G}_R^{(\varepsilon)}$, then $\mathcal{T}$ ACCEPTS G with probability at least $1 - \delta$.*
2. *If $\mathrm{d}_1(G, \mathcal{G}_R^{(\varepsilon)}) > \varepsilon$, then $\mathcal{T}$ REJECTS G with probability at least $1 - \delta$.*

The query complexity of $\mathcal{T}$ is bounded by q'. □

Corollary 5.5.3. *For every positive integer k, and any $\varepsilon > 0$ and $\delta > 0$, there is an integer $q = q_{5.5.3}(k, \varepsilon, \delta) = \texttt{poly}(k, 1/\varepsilon, \log^3(1/\delta))$ such that for every $R \in \mathcal{G}^*([k])$ and every graph $G \in \mathcal{G}(V)$, with $|V| \geq q$, we have*

1. *If $G \in \mathcal{G}_R^{(\varepsilon)}$, then $\mathbb{P}(\mathrm{d}_1(G[Q], \mathcal{G}_R^{(\varepsilon)}) > \varepsilon) < \delta$.*
2. *If $\mathrm{d}_1(G, \mathcal{G}_R^{(\varepsilon)}) > \varepsilon$, then $\mathbb{P}(G[Q] \in \mathcal{G}_R^{(\varepsilon)}) < \delta$.*

Proof. Fix $R \in \mathcal{G}^*([k])$ and let $\mathcal{T}$ be a tester for the property $\mathcal{G}_R^{(\varepsilon)}$ as in the statement of Theorem 5.5.2, with query complexity $q'(k, \varepsilon, \delta)$.

It follows from a result of Goldreich and Trevisan [49, Theorem 2] (see also [47]), that there is *canonical tester* $\mathcal{T}'$ for $\mathcal{G}_R^{(\varepsilon)}$ with sample complexity $q(k, \varepsilon, \delta) = \texttt{poly}(q'(k, \varepsilon, \delta))$, i.e., a tester that simply chooses a set $Q \in \binom{V}{q}$ uniformly at random and then accepts the input if and only if $G[Q]$ satisfies a certain property $\textsc{Acc}$ of graphs of size q.

To prove (1), if $G \in \mathcal{G}_R^{(\varepsilon)}$ then we get $\mathbb{P}(G[Q] \notin \textsc{Acc}) < \delta$. Moreover, if Q is a set of size q such that $d_1(G[Q], \mathcal{G}_R^{(\varepsilon)}) > \varepsilon$, then $G[Q] \notin \textsc{Acc}$ — because $G[Q]$ must be rejected (with probability 1) when given as *input* to $\mathcal{T}$. So $\mathbb{P}(d_1(G[Q], \mathcal{G}_R^{(\varepsilon)}) > \varepsilon) < \delta$.

Analogously, if $d_1(G, \mathcal{G}_R^{(\varepsilon)}) > \varepsilon$, then $\mathbb{P}(G[Q] \in \textsc{Acc}) < \delta$. Moreover, if Q is a set of size q such that $G[Q] \in \mathcal{G}_R^{(\varepsilon)}$, then $G[Q] \in \textsc{Acc}$ — because $G[Q]$ must be accepted (with probability 1) when given as *input* to $\mathcal{T}$. So $\mathbb{P}(G[Q] \in \mathcal{G}_R^{(\varepsilon)}) < \delta$. □

Lemma 5.5.4. *For $n > 2k$, let $G_1, G_2 \in \mathcal{G}(V)$ with $|V| = n$ and let $\mathcal{V} \in \Pi_{=k}(V)$. Then $\mathrm{d}_1(G_1/_{\mathcal{V}}, G_2/_{\mathcal{V}}) \leq \mathrm{d}_1(G_1, G_2) + 2k/(n - 2k)$.* □

Proof.

$$\begin{aligned}
\mathrm{d}_1(G_1/_{\mathcal{V}}, G_2/_{\mathcal{V}}) &= \frac{1}{k^2} \sum_{(\mathrm{i},j)\in[k]^2} |G_1/_{\mathcal{V}}(\mathrm{i}, j) - G_2/_{\mathcal{V}}(\mathrm{i}, j)| \\
&\leq \frac{1}{k^2} \sum_{(\mathrm{i},j)\in[k]^2} \frac{|\mathrm{e}_{G_1}(V_\mathrm{i}, V_j) - \mathrm{e}_{G_2}(V_\mathrm{i}, V_j)|}{\frac{(n-k)^2}{k^2}} \\
&\leq \frac{1}{(n-k)^2} \sum_{(\mathrm{i},j)\in[k]^2} \sum_{\substack{u\in V_\mathrm{i} \\ v\in V_j}} |G_1(u, v) - G_2(u, v)| \\
&\leq \left(1 + \frac{2k}{n - 2k}\right) \frac{1}{n^2} \sum_{(u,v)\in V^2} |G_1(u, v) - G_2(u, v)| \\
&= \left(1 + \frac{2k}{n - 2k}\right) \mathrm{d}_1(G_1, G_2).
\end{aligned}$$

□

Proof of Lemma 5.5.1. Fix $\varepsilon > 0$ and K as in the statement of the lemma. Let $\delta = \frac{1}{6K} \cdot (\varepsilon/4)^{K^2}$ and take

$$q = \max\left\{\frac{8K}{\varepsilon} + 2K, q_{5.5.3}\left(K, \frac{1}{4}\varepsilon, \delta\right)\right\} = \texttt{poly}(K, \varepsilon).$$

Let $1 \leq k \leq K$. Fix a family $\mathcal{R} \subseteq \mathcal{G}^*([k])$ such that, for every $S \in \mathcal{G}^*([k])$, there is $R \in \mathcal{R}$ such that $d_1(R, S) \leq \varepsilon/4$. There is one such family with cardinality at most $(4/\varepsilon)^{k^2}$.

Let Γ be a graph with vertex set V, where $|V| \geq q$. Let $Q \in \binom{V}{q}$ be chosen uniformly at random and consider the following events

1. $E' = \big[\exists \mathcal{V} \in \Pi_{=k}(V)$ satisfying $d_1(\Gamma/_{\mathcal{V}}\,, \Gamma[Q]/_{\mathcal{V}'}) > \varepsilon$ for *every* $\mathcal{V}' \in \Pi_{=k}(Q)\big]$,

2. $E = \big[\exists \mathcal{V}' \in \Pi_{=k}(Q)$ satisfying $d_1(\Gamma/_{\mathcal{V}}\,, \Gamma[Q]/_{\mathcal{V}'}) > \varepsilon$ for *every* $\mathcal{V} \in \Pi_{=k}(V)\big]$.

We claim that these two events occur each with probability less than $1/(6K)$. It then follows by taking the union bound over $k = 1, \dots K$, that Q satisfies both (1) and (2) of the statement of Lemma 5.5.1 with probability at least $1 - 1/6 - 1/6 = 2/3$.

We only prove that $\mathbb{P}(E) \leq 1/(6K)$. An analogous argument shows that $\mathbb{P}(E') \leq 1/(6K)$. Suppose that event E happens and let $\mathcal{V}' \in \Pi_{=k}(Q)$ be as in (2). Define $S = \Gamma[Q]/_{\mathcal{V}'}$ and let $R \in \mathcal{R}$ be such that $d_1(R, S) \leq \varepsilon/4$. Since $\Gamma[Q] \in \mathcal{G}_S^{(0)}$ and $\Gamma \notin \mathcal{G}_S^{(\varepsilon)}$, the triangle inequality implies that $\Gamma[Q] \in \mathcal{G}_R^{(\varepsilon/4)}$ and $\Gamma \notin \mathcal{G}_R^{(3\varepsilon/4)}$. Therefore

$$\mathbb{P}(E) \leq \mathbb{P}\left(\exists R \in \mathcal{R} : \Gamma[Q] \in \mathcal{G}_R^{(\varepsilon/4)} \text{ and } \Gamma \notin \mathcal{G}_R^{(3\varepsilon/4)}\right).$$

We claim that if $\Gamma \notin \mathcal{G}_R^{(3\varepsilon/4)}$, then $d_1(\Gamma, \mathcal{G}_R^{(\varepsilon/4)}) > \varepsilon/4$. To show this, consider the contrapositive statement and let $\Gamma' \in \mathcal{G}_R^{(\varepsilon/4)}$ such that $d_1(\Gamma, \Gamma') \leq \varepsilon/4$. By definition there is an equipartition $\mathcal{V}' \in \Pi_{=k}(V)$ such that $d_1(\Gamma'/_{\mathcal{V}'}\,, R) \leq \varepsilon/4$. In addition Lemma 5.5.4 implies that

$$d_1(\Gamma/_{\mathcal{V}'}\,, \Gamma'/_{\mathcal{V}'}) \leq d_1(\Gamma, \Gamma') + 2k/(|V| - 2k) \leq \varepsilon/4 + 2K/(q - 2K) \leq \varepsilon/2.$$

It follows from the triangle inequality that $d_1(\Gamma/_{\mathcal{V}'}\,, R) \leq 3\varepsilon/4$. Therefore

$$\begin{aligned}
\mathbb{P}(E) &\leq \mathbb{P}\left(\exists R \in \mathcal{R} : \Gamma[Q] \in \mathcal{G}_R^{(\varepsilon/4)} \text{ and } d_1(\Gamma, \mathcal{G}_R^{(\varepsilon/4)}) > \varepsilon/4\right) \\
&\leq \sum_{R \in \mathcal{R}} \mathbb{P}\left(\Gamma[Q] \in \mathcal{G}_R^{(\varepsilon/4)} \text{ and } d_1(\Gamma, \mathcal{G}_R^{(\varepsilon/4)}) > \varepsilon/4\right) \\
&\leq \delta|\mathcal{R}| \leq 1/(6K)
\end{aligned}$$

where the last line comes from Corollary 5.5.3(2) with $d_1(\Gamma, \mathcal{G}_R^{(\varepsilon/4)}) > \varepsilon/4$.

□

We now deduce Theorem 5.3.2 from Lemma 5.5.1.

Proof of Theorem 5.3.2. Fix $\varepsilon > 0$ and an input graph $\Gamma \in \mathcal{G}(V)$. Let q be as in Lemma 5.5.1 with input K and ε/c. Choose Q uniformly from all subsets of V of size q and set $\overline{\Gamma} = \Gamma[Q]$. We will show that $z(\Gamma) = z(\overline{\Gamma}) \pm \varepsilon$ with probability at least $2/3$.

Let $\mathcal{V} \in \Pi_K(V)$ be an equipartition of Γ such that $z(\Gamma) = z^*(\Gamma/_{\mathcal{V}})$. By Lemma 5.5.1(1), with probability at least $2/3$, there is a partition $\overline{\mathcal{V}}$ of $\overline{\Gamma}$ such that $d_1(\Gamma/_{\mathcal{V}}\,, \overline{\Gamma}/_{\overline{\mathcal{V}}}) < \varepsilon/c$. By the second condition on z^* in the statement of Theorem 5.3.2, we have $|z^*(\overline{\Gamma}/_{\overline{\mathcal{V}}}) - z^*(\Gamma/_{\mathcal{V}})| \leq \varepsilon$, and therefore $z(\overline{\Gamma}) \leq z^*(\overline{\Gamma}/_{\overline{\mathcal{V}}}) \leq z^*(\Gamma/_{\mathcal{V}}) + \varepsilon = z(\Gamma) + \varepsilon$.

A symmetric argument relying on Lemma 5.5.1(2) shows that $z(\Gamma) \leq z(\overline{\Gamma}) + \varepsilon$. □

5.6 Concluding remarks

In this paper, we introduced the concept of f-recoverability of a graph property $\mathcal{P}$. Using this concept, and the fact that any monotone property $\mathcal{P} = \mathit{Forb}(\mathcal{F})$ is recoverable for a function f whose size is given by the Graph Removal Lemma, we found a probabilistic algorithm to estimate the number of $\mathcal{F}$-free subgraphs of a large graph G whose sample complexity does not depend on regularity.

Being a new concept, little is known about f-recoverability itself, and we believe that it would be interesting to investigate this notion in more detail. For instance, in our proof that any monotone property $\mathit{Forb}(\mathcal{F})$ is f-recoverable, we found ε-recovering partitions $\mathcal{V}$ that were γ-FK-regular (in fact, we showed that *any* such partition is ε-recovering), where $\gamma(\varepsilon)$ is chosen in such a way that the Removal Lemma applies. On the other hand, our discussion after Definition 3.5 implies that the property of being r-colourable is ε-recoverable with sample complexity r/ε, and thus we may find an ε-recovering partition whose size is less than the size required to ensure the existence of an FK-regular partition. It is natural to ask for properties that can be recovered by small partitions; more precisely, one could ask for a characterization of properties that are $f(\varepsilon)$-recoverable for $f(\varepsilon)$ polynomial in $1/\varepsilon$.

Here, we restricted ourselves to monotone graph properties. We should mention that the parameter

$$\widehat{z_{\mathcal{P}}}(\Gamma) = \frac{1}{|V(\Gamma)|^2} \log_2 |\{G \leq \Gamma : G \in \mathcal{P}\}|$$

might not even be estimable for arbitrary (non-monotone) properties $\mathcal{P}$. For instance, if $\mathcal{P}$ is the hereditary property of graphs having no independent sets of size three, then the complete graph K_n and the graph $K_n - E(K_3)$, which is obtained from K_n by removing the edges of a triangle, have quite a different number of spanning subgraphs satisfying $\mathcal{P}$, namely $2^{n^2/4}$ and 0, respectively, although their edit distance is negligible. It follows from [18, Theorem 6.1] that $\widehat{z_{\mathcal{P}}}$ is not estimable.

Nevertheless, the definition of f-recoverable can be extended to cope with general hereditary properties, which, along with Theorem 5.3.2, provides a way of estimating other interesting hereditary properties. In particular, this framework is used in a follow-up paper to estimate the edit distance to any fixed hereditary property with a sample complexity similar to the one obtained here. We should mention here that, given a monotone property $\mathit{Forb}(\mathcal{F})$, the parameter $z_{\mathcal{F}}$ is actually closely related to the parameter $\mathrm{d}_{\mathcal{F}} \colon \Gamma \mapsto d_1(\Gamma, \mathit{Forb}(\mathcal{F}))$. In fact, ε-recovering partitions along with techniques analogous to the ones used in [17] can be used to show that, for any graph $\Gamma = (V, E)$, we have

$$\mathrm{d}_{\mathcal{F}}(\Gamma) = \frac{2|E|}{|V|^2} - 2z_{\mathcal{F}}(\Gamma) \pm o(1),$$

which implies that estimating $z_{\mathcal{F}}$ provides an indirect way for estimating $\mathrm{d}_{\mathcal{F}}$.

Chapter 6

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