IMA Journal of Applied Mathematics (2017) **82**, 656–680
doi:10.1093/imamat/hxx007
Advance Access Publication on 3 May 2017

On the spectral analysis of residual stress in finite elasticity

M. H. B. M. Shariff
Department of Applied Mathematics and Science, Khalifa University of Science and Technology, Abu Dhabi, UAE

R. Bustamante*
Departmento de Ingeniería Mecánica, Universidad de Chile, Beauchef 851, Santiago, Chile
*Corresponding author: rogbusta@ing.uchile.cl

and

J. Merodio
Department of Continuum Mechanics and Structures, Escuela de Ingenieros de Caminos, Universidad Politecnica de Madrid, 28040 Madrid, Spain

[Received on 1 November 2016; revised on 10 March 2017; accepted on 30 March 2017]

In the literature, residual stress problems are generally formulated using classical invariants despite most of them having an unclear physical meaning and not having experimental advantages. In this article, we give an alternative formulation for residual stress problems using a set of spectral invariants. These invariants have a clear physical meaning which may facilitate the design of a residual stress experiment. For the case of an energy function that depends on the right Cauchy tensor and the residual stress tensor, we show that only nine of the classical invariants are independent, not 10 as commonly assumed. Details of the spectral formulation are given and several boundary value problems are illustrated.

Keywords: residual stresses; invariants; spectral invariants.

2000 Math Subject Classification: 74A20, 74B20.

1. Introduction

A residual stress is a stress that exists in a reference configuration of an equilibrium body with zero surface traction. Such stresses can appear in a body, for example, due to some manufacturing process for metals (Withers & Bhadeshia, 2001; Brinksmeier *et al.*, 1982), and due to some changes in the mass distribution or remodelling in soft tissue (see, e.g., Fung, 1991 and Chapter 11 of Fung, 1990). The presence of such residual stresses can be detected, for example, by cutting the body and by observing during that process the release of elastic energy, which is shown by some displacement field associated with such cutting (see, e.g., Fung, 1990; Chuong & Fung, 1986.) In biomechanics such stresses are considered to be important in order to reduce the stress concentration, when an organ is under the influence of some external loads (Chuong & Fung, 1986). In the case of arteries, the presence of such stresses can be detected by cutting a piece of artery in the radial direction and an angular opening of the artery is observed (see Figs 11.2:1–11.2:3 of Fung, 1990). Analyses of residual stress problems can be found in Hoger (1985, 1986); Merodio *et al.* (2013) and some of the references mentioned therein.

The determination of residual stresses and the influence of such stresses in the behaviour of elastic bodies is a very important topic due to its many implications in mechanical design (see, e.g., Brinksmeier *et al.*, 1982; Withers & Bhadeshia, 2001), and in the understanding of some processes in biomechanics (see Fung, 1990, 1991), among some of the applications that we can mention. Different theories and methods have been proposed in order to determine such residual stresses (see, e.g., Hoger, 1986; Merodio *et al.*, 2013; Merodio & Ogden, 2014) and the references cited therein. In the present work, we consider as a basis the theory developed recently in Shams *et al.* (2011) and Merodio & Ogden (2014), where the residual stress $\boldsymbol{\tau}$ is considered as a variable in the elastic energy of the body along with the deformation gradient, where now in the reference configuration the body is not stress free. The aim of this article is to present an alternative representation for such energy function considering a new set of invariants defined in terms of the principal variables of the right Cauchy-Green stretch tensor (spectral representation) (see, e.g., Shariff, 2008, 2013; Shariff & Bustamante, 2015; Shariff, 2016b). The proposed formulation uses a set of spectral invariants, where most of the invariant have a clear physical meaning, and hence have an experimental advantage over other types of invariants with no physical interpretation such as most of the classical invariants by Spencer & Rivlin (1962) (or their variants). The advantages of spectral invariants over classical invariants have been discussed in Shariff (2008, 2016a); hence, we will not discuss them here.

This communication is divided in the following sections: In Section 2, some basic equations and relations are shown, while in Section 3, the spectral formulation for residually stressed bodies is presented, where some restrictions are also shown for the constitutive equations. In Section 5, some boundary value problems are solved. Finally, in Section 6 some final remarks are given.

2. Basic equations

2.1. *Kinematics*

In this article, all subscripts i,j and k take the values $1, 2, 3$, unless stated otherwise.

Let $\mathscr{B}$ denotes the elastic body, $\mathbf{x} \in \mathscr{B}_t$ denotes the position of a particle $X \in \mathscr{B}$ in the current configuration $\mathscr{B}_t$. The position of the same particle in the reference configuration is denoted as $\mathbf{X} \in \mathscr{B}_r$, where $\mathscr{B}_r$ is the reference configuration of the body. It is assumed that there exists a one-to-one mapping $\boldsymbol{\chi}$ such that $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ for any time $t > 0$. The deformation gradient, the left Cauchy-Green $\mathbf{B}$ and right Cauchy-Green $\mathbf{C}$ stretch tensors are defined, respectively, as:

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{V}^2, \quad \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^2, \tag{2.1}$$

where $\boldsymbol{\chi}$ is assumed such that $J = \det \mathbf{F} > 0$. More details about the kinematics of deforming bodies can be found in Truesdell & Toupin (1960).

2.2. *Residual stresses*

The residual stress $\boldsymbol{\tau}$ has to satisfy the equilibrium equation

$$\mathrm{Div}\boldsymbol{\tau} = \mathbf{0} \quad \text{in} \quad \mathscr{B}_r \tag{2.2}$$

and the boundary condition

$$\boldsymbol{\tau}\mathbf{N} = \mathbf{0} \quad \text{in} \quad \partial\mathscr{B}_r\,, \tag{2.3}$$

where $\partial\mathscr{B}_r$ is the boundary of $\mathscr{B}_r$ and $\mathbf{N}$ is unit outward normal to $\partial\mathscr{B}_r$. More information about residual stresses can be found, for example, in Hoger (1985) and the references mentioned therein.

3. Constitutive equation and spectral formulation

3.1. *Spectral formulation for anisotropic elastic bodies*

In this section, we present a brief summary of the most important elements of the theory of spectral invariants developed by Shariff and co-workers (see Shariff, 2008, 2013, 2016b,a; Shariff & Bustamante, 2015).

If λ_i and $\mathbf{e}_i$ is an eigenvalue and an eigenvector of the right stretch tensor $\mathbf{U}$, respectively, a general anisotropic elastic strain energy function W_e can be written in the form

$$W_e = \hat{W}(\mathbf{C}) = \tilde{W}(\lambda_1, \lambda_2, \lambda_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3), \tag{3.1}$$

with the symmetrical property

$$\tilde{W}(\lambda_1, \lambda_2, \lambda_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \tilde{W}(\lambda_2, \lambda_1, \lambda_3, \mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3) = \tilde{W}(\lambda_3, \lambda_2, \lambda_1, \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1), \tag{3.2}$$

In view of the non-unique values of $\mathbf{e}_i$ and $\mathbf{e}_j$ when $\lambda_i = \lambda_j$, a function $\tilde{W}$ should be independent of $\mathbf{e}_i$ and $\mathbf{e}_j$ when $\lambda_i = \lambda_j$, and $\tilde{W}$ should be independent of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ when $\lambda_1 = \lambda_2 = \lambda_3$. Hence, when two or three of the principal stretches have equal values the strain energy must have any of the following forms

$$W_e = \begin{cases} W_{(a)}(\lambda, \lambda_k, \mathbf{e}_k)\,, & \text{when } \lambda_i = \lambda_j = \lambda\,, i \neq j \neq k \neq i \\ W_{(b)}(\lambda)\,, & \text{when } \lambda_1 = \lambda_2 = \lambda_3 = \lambda \end{cases} \tag{3.3}$$

As an example of (3.3), consider $W_e = \mathbf{a} \cdot (\mathbf{C}\mathbf{a}) = \sum_{i=1}^{3} \lambda_i^2 (\mathbf{a} \cdot \mathbf{e}_i)^2$, where $\mathbf{a}$ is a fixed unit vector and $\sum_{i=1} (\mathbf{a} \cdot \mathbf{e}_i)^2 = 1$. If $\lambda_1 = \lambda_2 = \lambda$, we have $W_e = W_{(a)}(\lambda, \lambda_3, \mathbf{e}_3) = \lambda^2 + (\lambda_3^2 - \lambda^2)(\mathbf{a} \cdot \mathbf{e}_3)^2$ and in the case of $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, $W_e = W_{(b)}(\lambda) = \lambda^2$. Note that $\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{e}_i \otimes \mathbf{e}_i$ (or $\mathbf{U}$) also has the same property as described in (3.3), where $\otimes$ denotes the dyadic product.

The spectral formulation requires the components of $\frac{\partial W_e}{\partial \mathbf{C}}$ relative to the basis $\{\mathbf{e}_i\}$. Following the work of Shariff (2008), we have

$$\left(\frac{\partial W_e}{\partial \mathbf{C}}\right)_{ii} = \frac{1}{2\lambda_i} \frac{\partial \tilde{W}}{\partial \lambda_i}, \quad i \quad \text{not summed}, \tag{3.4}$$

$$\left(\frac{\partial W_e}{\partial \mathbf{C}}\right)_{ij} = \frac{\frac{\partial \tilde{W}}{\partial \mathbf{e}_i} \cdot \mathbf{e}_j - \frac{\partial \tilde{W}}{\partial \mathbf{e}_j} \cdot \mathbf{e}_i}{2(\lambda_i^2 - \lambda_j^2)}, \quad i \neq j. \tag{3.5}$$

It is assumed that $\tilde{W}$ has sufficient regularity to ensure that, as the value of λ_i approaches λ_j, (3.5) has a limit. The Cauchy stress tensor $\boldsymbol{\sigma}$ for a compressible and an incompressible solid is given, respectively by

$$\boldsymbol{\sigma} = 2J^{-1}\mathbf{F}\frac{\partial W_e}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}}, \quad \boldsymbol{\sigma} = 2\mathbf{F}\frac{\partial W_e}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}} - p\mathbf{I}, \tag{3.6}$$

where p is the Lagrange multiplier associated with the incompressibility constraint $\lambda_1\lambda_2\lambda_3 = 1$, and $\mathbf{F}$ is the deformation gradient tensor. The Eulerian spectral Cauchy stress components σ_{ij} take the form, respectively as

$$\sigma_{ii} = \frac{\lambda_i}{J}\frac{\partial \tilde{W}}{\partial \lambda_i}, \tag{3.7}$$

$$\sigma_{ij} = \frac{J^{-1}\lambda_i\lambda_j}{\lambda_i^2 - \lambda_j^2}\left(\frac{\partial \tilde{W}}{\partial \mathbf{e}_i}\cdot\mathbf{e}_j - \frac{\partial \tilde{W}}{\partial \mathbf{e}_j}\cdot\mathbf{e}_i\right), \quad i \neq j. \tag{3.8}$$

and

$$\sigma_{ii} = \lambda_i\frac{\partial \tilde{W}}{\partial \lambda_i} - p, \quad i \quad \text{not summed}, \tag{3.9}$$

$$\sigma_{ij} = \frac{\lambda_i\lambda_j}{\lambda_i^2 - \lambda_j^2}\left(\frac{\partial \tilde{W}}{\partial \mathbf{e}_i}\cdot\mathbf{e}_j - \frac{\partial \tilde{W}}{\partial \mathbf{e}_j}\cdot\mathbf{e}_i\right), \quad i \neq j. \tag{3.10}$$

In the rest of this work, we only consider the case of incompressible bodies

3.2. *Spectral formulation for residually stressed elastic bodies*

In this section, we extend the results presented in the previous section, for the case of a residually stressed body. The key point of our model is to assume that the energy function depends on the deformation and the residual stress (see Shams *et al.*, 2011), that is, $W_e = W_c(\mathbf{C}, \boldsymbol{\tau})$. Treating the residual stress as a structural tensor, the strain energy W_e can be written as

$$W_e = W_c(\mathbf{C}, \boldsymbol{\tau}) = W_s(\lambda_{1,2,3}, \mathbf{E}_{1,2,3}, \boldsymbol{\tau}), \tag{3.11}$$

where we have defined $\mathbf{E}_i = \mathbf{e}_i \otimes \mathbf{e}_i$ and where we have used the notation

$$\lambda_{1,2,3} \equiv \lambda_1, \lambda_2, \lambda_3 \quad \mathbf{E}_{1,2,3} \equiv \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3. \tag{3.12}$$

The spectral decomposition of the Cauchy-Green stretch tensor is

$$\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{E}_i\,. \tag{3.13}$$

Considering the work of Spencer (1971) and Shariff (2008, 2013, 2016b,a) the function W_e can be expressed as

$$W_e = W(\lambda_{1,2,3}, \zeta_{1,2,3}, \xi_{1,2,3}, \tau_{1,2,3})\,, \tag{3.14}$$

where the new invariants ζ_i and ξ_i are defined as

$$\zeta_i = \mathbf{e}_i \cdot (\boldsymbol{\tau}\mathbf{e}_i), \quad \xi_i = \mathbf{e}_i \cdot (\boldsymbol{\tau}^2\mathbf{e}_i), \quad \text{(there is no sum in } i\text{)}, \tag{3.15}$$

and τ_i, is an eigenvalue of $\boldsymbol{\tau}$. In spectral terms,

$$\boldsymbol{\tau} = \sum_{i=1}^{3} \tau_i \mathbf{d}_i \otimes \mathbf{d}_i, \tag{3.16}$$

where $\mathbf{d}_i$ is an eigenvector of $\boldsymbol{\tau}$.

The function W above must satisfy the symmetry conditions

$$W(\lambda_{1,2,3}, \zeta_{1,2,3}, \xi_{1,2,3}, \tau_{1,2,3}) = W(\lambda_{2,1,3}, \zeta_{2,1,3}, \xi_{2,1,3}, \tau_{1,2,3}) = W(\lambda_{3,1,2}, \zeta_{3,1,2}, \xi_{3,1,2}, \tau_{1,2,3}), \tag{3.17}$$

$$W(\lambda_{1,2,3}, \zeta_{1,2,3}, \xi_{1,2,3}, \tau_{1,2,3}) = W(\lambda_{1,2,3}, \zeta_{1,2,3}, \xi_{1,2,3}, \tau_{2,1,3}) = W(\lambda_{1,2,3}, \zeta_{1,2,3}, \xi_{1,2,3}, \tau_{3,1,2}). \tag{3.18}$$

In (3.14), there are twelve invariants to describe W_e, however, in Appendix A it is shown that there are only nine independent invariants from the list λ_i, ζ_i, ξ_i and τ_i. The corresponding 10 classical invariants (see Spencer, 1971) are expressed explicitly in terms of the spectral invariants as

$$I_1 = \mathrm{tr}(\mathbf{C}) = \sum_{i=1}^{3} \lambda_i^2, \quad I_2 = \frac{1}{2}\left[(\mathrm{tr}\mathbf{C})^2 - \mathrm{tr}\mathbf{C}^2\right] = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2,$$
$$I_3 = \det(\mathbf{C}) = \lambda_1^2\lambda_2^2\lambda_3^2, \tag{3.19}$$

$$I_4 = \mathrm{tr}(\boldsymbol{\tau}) = \sum_{i=1}^{3} \tau_i, \quad I_5 = \mathrm{tr}(\mathbf{C}\boldsymbol{\tau}) = \sum_{i=1}^{3} \zeta_i\lambda_i^2, \quad I_6 = \mathrm{tr}(\mathbf{C}^2\boldsymbol{\tau}) = \sum_{i=1}^{3} \zeta_i\lambda_i^4, \tag{3.20}$$

$$I_7 = \mathrm{tr}(\mathbf{C}\boldsymbol{\tau}^2) = \sum_{i=1}^{3} \xi_i\lambda_i^2, \quad I_8 = \mathrm{tr}(\mathbf{C}^2\boldsymbol{\tau}^2) = \sum_{i=1}^{3} \xi_i\lambda_i^4, \tag{3.21}$$

$$I_9 = \mathrm{tr}(\boldsymbol{\tau}^2) = \sum_{i=1}^{3} \tau_i^2, \quad I_{10} = \mathrm{tr}(\boldsymbol{\tau}^3) = \sum_{i=1}^{3} \tau_i^3. \tag{3.22}$$

Hence, since the classical invariants can be expressed explicitly in terms of spectral invariants and there are only nine independent spectral invariants, only nine of the 10 classical invariants are independent. One might be interested to see if we can construct a relation between the classical invariants only, that is, without the appearance of spectral invariants in the relation. In Appendix B, we show this relation (B16) (not single-valued) between the classical invariants, which proves that only nine classical invariants are independent. Since $\xi_i = \sum_{k=1}^{3} \tau_k^2(\mathbf{e}_i \cdot \mathbf{d}_k)^2$ is non-negative, this implies that I_8 (or I_7) is non-negative. If we consider I_8 to be the dependent invariant, then the relation (B16) can be considered as I_8 to be an implicit function of the remainder invariants. However, we must emphasized that we are only interested in evaluating the number of independent invariants, not the number of invariants that should be in the strain energy function. A discussion on the importance of knowing the number of independent invariants and the number of invariants required in a strain energy function can be found in Shariff (2016b); Shariff & Bustamante (2015).

In view of (3.9) and (3.10), the Eulerian spectral components of the Cauchy stress with respect to the basis $\{\mathbf{R}\mathbf{e}_1, \mathbf{R}\mathbf{e}_2, \mathbf{R}\mathbf{e}_2\}$ ($\mathbf{R}$ is a proper orthogonal tensor, where $\mathbf{F} = \mathbf{R}\mathbf{U}$), take the form

$$\sigma_{ii} = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \quad i \quad \text{not summed}, \tag{3.23}$$

$$\sigma_{ij} = \frac{2\lambda_i\lambda_j}{\lambda_i^2 - \lambda_j^2}\left[\left(\frac{\partial W}{\partial \zeta_i} - \frac{\partial W}{\partial \zeta_j}\right)\mathbf{e}_i \cdot (\boldsymbol{\tau}\mathbf{e}_j) + \left(\frac{\partial W}{\partial \xi_i} - \frac{\partial W}{\partial \xi_j}\right)\mathbf{e}_i \cdot (\boldsymbol{\tau}^2\mathbf{e}_j)\right], \quad i \neq j, \tag{3.24}$$

where $\lambda_1\lambda_2\lambda_3 = 1$. Its important to note that, in this article, our residual stress $\boldsymbol{\tau}$ for an incompressible solid is the Cauchy stress in the reference configuration in the absence of hydrostatic pressure.

3.3. *On the reference configuration for a residually stressed body*

In the reference configuration $\lambda_i = 1$, and we let $\mathbf{e}_i = \mathbf{d}_i$, where $\mathbf{d}_i$ is an eigenvector of $\boldsymbol{\tau}$ and (3.23) and (3.24) simply reduce to

$$\tau_i - p = \frac{\partial W}{\partial \lambda_i}(1_{1,2,3}, \tau_{1,2,3}, \tau^2_{1,2,3}, \tau_{1,2,3}) - p, \tag{3.25}$$

which gives the relation

$$\tau_i = \frac{\partial W}{\partial \lambda_i}(1_{1,2,3}, \tau_{1,2,3}, \tau^2_{1,2,3}, \tau_{1,2,3}), \tag{3.26}$$

where we have used the notation $1_{1,2,3}$ to symbolize the triad 1, 1, 1 that results when we replace $\lambda_i = 1$ in the expression for W.

We note that the residual stress is commonly defined in the literature as the Cauchy stress in the reference configuration in the presence of hydrostatic pressure (see, e.g., Merodio *et al.*, 2013), and hence their relation have the form (considering the new set of invariants presented in this article):

$$\tau_i = \frac{\partial W}{\partial \lambda_i}(1_{1,2,3}, \tau_{1,2,3}, \tau^2_{1,2,3}, \tau_{1,2,3}) - p. \tag{3.27}$$

If we use the classical invariants in the energy function, that is,

$$W_e = W_I(I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9, I_{10}), \tag{3.28}$$

then (3.27) is equivalent to (see Merodio *et al.*, 2013)

$$2\frac{\partial W_I}{\partial I_1} + 4\frac{\partial W_I}{\partial I_2} - p_0 = 0, \quad 2\left(\frac{\partial W_I}{\partial I_5} + 2\frac{\partial W_I}{\partial I_6}\right) = 1, \quad \frac{\partial W_I}{\partial I_7} + 2\frac{\partial W_I}{\partial I_8} = 0. \tag{3.29}$$

The first equation of (3.29) suggests that a hydrostatic stress p_0 is needed to maintain the residual stress in the reference configuration.

3.4. *The strain energy for the case of small strains with arbitrary rotations*

Since, our formulation satisfies objectivity, it is only consistent with the constitutive equation for small strains with arbitrary rotations. For a large strain residual stress formulation to be consistent with infinitesimal elasticity (small displacement gradients), it may require a non-objective finite strain energy function (Hoger, 1993).

The strain energy function for small strains can be expressed as

$$W_e = W_E(\mathbf{U}, \boldsymbol{\tau}), \tag{3.30}$$

and let us define

$$\nu_i = \lambda_i - 1, \tag{3.31}$$

where for this section, we further assume that

$$\mid \nu_i \mid << 1. \tag{3.32}$$

Considering the results presented in Section 3, we have

$$W_E(\mathbf{U}, \boldsymbol{\tau}) = W_s(\nu_{1,2,3}, \zeta_{1,2,3}, \xi_{1,2,3}, \tau_{1,2,3}), \tag{3.33}$$

where from (3.32) we have that $\sum_{i=1}^{3} \nu_i = 0 + O(\nu_i^2)$.

In the case of (3.26) that relation specializes to

$$\tau_i = \frac{\partial W_s}{\partial \nu_i}(0_{1,2,3}, \tau_{1,2,3}, \tau^2_{1,2,3}, \tau_{1,2,3}). \tag{3.34}$$

In view of (3.34), the most general quadratic strain energy function for incompressible small strain elasticity is

$$W_e = \sum_{i=1}^{3} \left(\mu \nu_i^2 + \mu_1 \zeta_i \nu_i^2 + \mu_2 \xi_i \nu_i^2 + \zeta_i \nu_i\right) + \frac{\beta_1}{2} \left(\sum_{i=1}^{3} \zeta_i \nu_i\right)^2 + \frac{\beta_2}{2} \left(\sum_{i=1}^{3} \xi_i \nu_i\right)^2 + \beta_3 \left(\sum_{i=1}^{3} \zeta_i \nu_i\right)\left(\sum_{i=1}^{3} \xi_i \nu_i\right), \tag{3.35}$$

$$= \mu \mathrm{tr}(\boldsymbol{\epsilon}^2) + \mu_1 \mathrm{tr}(\boldsymbol{\epsilon}^2 \boldsymbol{\tau}) + \mu_2 \mathrm{tr}(\boldsymbol{\epsilon}^2 \boldsymbol{\tau}^2) + \mathrm{tr}(\boldsymbol{\epsilon}\boldsymbol{\tau}) + \frac{\beta_1}{2} \mathrm{tr}(\boldsymbol{\epsilon}\boldsymbol{\tau})^2 + \frac{\beta_2}{2} \mathrm{tr}(\boldsymbol{\epsilon}\boldsymbol{\tau}^2)^2 + \beta_3 \mathrm{tr}(\boldsymbol{\epsilon}\boldsymbol{\tau}) \mathrm{tr}(\boldsymbol{\epsilon}\boldsymbol{\tau}^2), \tag{3.36}$$

where the material constant μ has a stress dimension and may depend on $\boldsymbol{\tau}$, and μ_1, μ_2, β_1, β_2 and β_3 are dimensionless constants. We have defined $\boldsymbol{\epsilon} = \mathbf{U} - \mathbf{I}$. Following the work of Hoger (1993), the Cauchy stress for small strain deformation is

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mathbf{R}\left(\boldsymbol{\tau} + \mathbf{L}[\boldsymbol{\epsilon}] + \frac{1}{2}(\boldsymbol{\epsilon}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\epsilon})\right)\mathbf{R}^{\mathrm{T}}, \tag{3.37}$$

where

$$\mathbf{L}[\boldsymbol{\epsilon}] = 2\mu\boldsymbol{\epsilon} + \mu_1(\boldsymbol{\epsilon}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\epsilon}) + \mu_2(\boldsymbol{\epsilon}\boldsymbol{\tau}^2 + \boldsymbol{\tau}^2\boldsymbol{\epsilon}) + \beta_1 \mathrm{tr}(\boldsymbol{\epsilon}\boldsymbol{\tau})\boldsymbol{\tau} + \beta_2 \mathrm{tr}(\boldsymbol{\epsilon}\boldsymbol{\tau}^2)\boldsymbol{\tau}^2 + \beta_3\left(\boldsymbol{\tau}\mathrm{tr}(\boldsymbol{\epsilon}\boldsymbol{\tau}^2) + \boldsymbol{\tau}^2 \mathrm{tr}(\boldsymbol{\epsilon}\boldsymbol{\tau})\right). \tag{3.38}$$

For infinitesimal strain (small displacement gradient) $\mathbf{R} = \mathbf{I}+\mathbf{W}$ and $\boldsymbol{\epsilon} = \mathbf{E}$, where $\mathbf{W}$ is the infinitesimal rotation tensor and $\mathbf{E}$ is the infinitesimal strain. The infinitesimal Cauchy stress then takes the form

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} + (\mathbf{W}\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{W}) + \frac{1}{2}(\mathbf{E}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{E}) + \mathbf{L}[\mathbf{E}]\,. \tag{3.39}$$

The constitutive equation (3.39) is a generalization of some constitutive equations proposed in the literature (see Hoger, 1986; Robertson, 1997; Man, 1998) when specialized to incompressible elastic solids. The modelling of realistic materials often require that the Hartig's law (see Bell, 1973) should be satisfied. Hartig's law states that in uniaxial tension/compression (Man, 1998)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\epsilon} = E_0 - b\sigma_0\,, \tag{3.40}$$

where σ is the Cauchy uniaxial stress, ϵ is the engineering strain, $E_0 = 3\mu_0$ is the Young's modulus at zero stress, μ_0 is the value of μ at zero stress, σ_0 is the initial true stress and b is a dimensionless constant that depends on the material. Since, b must depend on the type of material, six material constants in (3.35) ensure that b does not have a value that is independent of material constants and hence Hartig's law is satisfied. For example, Man (1998) proposed a constitutive equation with three material constants μ_0, k_1 and k_2 (when specialized to incompressible materials) that satisfies Hartig's law. His constitutive equation is a special case of our form (3.39), where their constants are related to ours via the relations $\mu = \mu_0 + \frac{k_1 \mathrm{tr}(\boldsymbol{\tau})}{2}$, $\mu_1 = k_2 + 0.5$, $\mu_2 = \beta_1 = \beta_2 = \beta_3 = 0$, and k_1 and k_2 are dimensionless constants. In this case, following the work of Man (1998), we have

$$b = -(2 + 1.5k_1 + 2k_2) \tag{3.41}$$

that depends on the material constants, k_1 and k_2.

3.5. *Ground state conditions*

Due to the incompressibility condition $\lambda_1\lambda_2\lambda_3 = 1$, we can write

$$W_e = \hat{W}(\lambda_{1,2}, \zeta_{1,2,3}, \xi_{1,2,3}, \tau_{1,2,3}). \tag{3.42}$$

When $\mathbf{F} = \mathbf{I}$ we let $\mathbf{e}_i = \mathbf{d}_i$ and we have the non-zero ground state conditions:

$$\begin{aligned}\frac{\partial^2 \hat{W}}{\partial \lambda_1^2}(1_{1,2}, \tau_{1,2,3}, \tau^2_{1,2,3}, \tau_{1,2,3}) &= 4\mu + 2\mu_1(\tau_1 + \tau_3) + 2\mu_2(\tau_1^2 + \tau_3^2) + 2\tau_3 \\ &\quad + \beta_1(\tau_1 + \tau_3) + \beta_2(\tau_1^2 + \tau_3^2) + 2\beta_3(\tau_1 - \tau_3)(\tau_1^2 - \tau_3^2), \end{aligned} \tag{3.43}$$

$$\begin{aligned}\frac{\partial^2 \hat{W}}{\partial \lambda_1^2}(1_{1,2}, \tau_{1,2,3}, \tau^2_{1,2,3}, \tau_{1,2,3}) &= 4\mu + 2\mu_1(\tau_2 + \tau_3) + 2\mu_2(\tau_2^2 + \tau_3^2) + 2\tau_3 \\ &\quad + \beta_1(\tau_2 + \tau_3) + \beta_2(\tau_2^2 + \tau_3^2) + 2\beta_3(\tau_2 - \tau_3)(\tau_2^2 - \tau_3^2), \end{aligned} \tag{3.44}$$

$$\frac{\partial^2 \hat{W}}{\partial\lambda_1\partial\lambda_2}(1_{1,2},\tau_{1,2,3},\tau_{1,2,3}^2,\tau_{1,2,3}) = 2\mu + 2\tau_3\mu_1 + 2\tau_3^2\mu_2 + \tau_3 + \beta_1(\tau_1-\tau_3)(\tau_2-\tau_3) + \beta_2(\tau_1^2-\tau_3^2)(\tau_2^2-\tau_3^3) + \beta_3\left[(\tau_1-\tau_3)(\tau_2^2-\tau_3^2) + (\tau_2-\tau_3)(\tau_1^2-\tau_3^2)\right]. \tag{3.45}$$

3.6. *A specific constitutive equation*

In this section, we propose a specific constitutive equation for the case of large deformations, which is of the form

$$W_e = \sum_{i=1}^{3}[\mu r_1(\lambda_i) + \mu_1\zeta_i r_2(\lambda_i) + \mu_2\xi_i r_3(\lambda_i) + \zeta_i(\lambda_i-1)] + \frac{\beta_1}{2}\left(\sum_{i=1}^{3}\zeta_i r_4(\lambda_i)\right)^2 + \frac{\beta_2}{2}\left(\sum_{i=1}^{3}\xi_i r_5(\lambda_i)\right)^2 + \beta_3\left(\sum_{i=1}^{3}\zeta_i r_6(\lambda_i)\right)\left(\sum_{i=1}^{3}\xi_i r_7(\lambda_i)\right), \tag{3.46}$$

where, in order to be consistent with the small strain formulation (3.35), we let $r_m(1)=0, m=1,2,...,7$, $r_t'(1) = 1$, $t = 4,5,6,7$, $r_n'(1) = 0$ and $r_n''(1) = 2$, $n = 1,2,3$ (see Shariff, 2016b,a). Also these conditions ensure that the residual stress is the referential Cauchy stress in the absence of hydrostatic pressure. However, if a residual stress is defined as the referential Cauchy stress in the presence of hydrostatic pressure as given in (3.27), the conditions $r_1'(1) = r_1''(1) = 1$ must be imposed. Note that, as mentioned before, in this article, we are only concerned with a residual stress that is defined to be the referential Cauchy stress in the absence of hydrostatic pressure, although some models proposed in the literature that treat the residual stress as the referential Cauchy stress in the presence of hydrostatic pressure are discussed below.

The above form (3.46) is a generalization of some expressions proposed in the literature (Merodio *et al.*, 2013; Merodio & Ogden, 2014). For example, in the case where $r_1'(1) = r_1''(1) = 1$, Merodio & Ogden (2014) proposed

$$W_e = \frac{\mu}{2}(I_1-3) + \frac{1}{2}[I_5 - \mathrm{tr}(\boldsymbol{\tau})] \tag{3.47}$$

and

$$W_e = \frac{\mu}{2}(I_1-3) + \frac{1}{4}[I_6 - \mathrm{tr}(\boldsymbol{\tau})]. \tag{3.48}$$

The expression (3.47) can be obtained from (3.46) if $\mu_1 = \frac{1}{2}$, $\mu_2 = \beta_1 = \beta_2 = \beta_3 = 0$

$$r_1(\lambda) = \frac{\lambda^2-1}{2}, \quad r_2(\lambda) = (\lambda-1)^2, \tag{3.49}$$

whereas (3.48) can be obtained from (3.46) by letting $\mu_1 = \frac{3}{2}$, $\mu_2 = \beta_1 = \beta_2 = \beta_3 = 0$

$$r_1(\lambda) = \frac{\lambda^2-1}{2}, \quad r_2(\lambda) = \frac{1}{4}\left[6(\lambda-1)^2 + 4(\lambda-1)^3 + (\lambda-1)^4\right]. \tag{3.50}$$

3.7. *Restrictions on the ground-state constants*

The strong ellipticity condition is a mathematical restriction on a constitutive function, where it guarantees that the governing partial differential equations of equilibrium are elliptic in character, and hence, in particular, certain types of non-physical singularity, which could otherwise occur and lead to serious numerical problems, are absent. In addition to this, strong ellipticity ensures that the speeds of infinitesimal plane waves propagating through the material are real. The material constants in (3.35) can be restricted using strong ellipticity condition in the reference configuration ($\mathbf{F} = \mathbf{I}$). Mathematically, strong ellipticity condition for an incompressible materials (see Shams *et al.*, 2011) requires

$$\mathbf{m} \cdot [\mathbf{Q}(\mathbf{n})\mathbf{m}] > 0, \quad \mathbf{m} \cdot \mathbf{n} = 0 \tag{3.51}$$

where $\mathbf{m}$ and $\mathbf{n}$ are unit vectors, and in Cartesian components we have

$$(\mathbf{Q}(\mathbf{n}))_{ij} = \left(\frac{\partial^2 W_e}{\partial \mathbf{F}^2}\right)_{piqj} n_p n_q\,, \tag{3.52}$$

where n_i is a Cartesian component of $\mathbf{n}$. In view of (3.35) and (3.52), we need to calculate the derivative $\frac{\partial^2 W_e}{\partial \mathbf{U}^2}$ at $\mathbf{F} = \mathbf{I}$, where the relevant Cartesian components are given below:

$$\left(\frac{\partial^2 W_e}{\partial \mathbf{F}^2}\right)_{nmsr} = \frac{\partial^2 W_e}{\partial F_{mn} \partial F_{rs}} = \mathscr{A}_{nmsr} + \mathscr{C}_{nmsr}, \quad n, m, s, r = 1, 2, 3, \tag{3.53}$$

where

$$\mathscr{A}_{nmsr} = \frac{1}{4}\left(\frac{\partial^2 W_e}{\partial U_{mn}\partial U_{rs}} + \frac{\partial^2 W_e}{\partial U_{nm}\partial U_{rs}} + \frac{\partial^2 W_e}{\partial U_{mn}\partial U_{sr}} + \frac{\partial^2 W_e}{\partial U_{nm}\partial U_{sr}}\right), \tag{3.54}$$

$$\mathscr{C}_{nmsr} = \frac{1}{2}\left[\frac{3}{2}\delta_{rm}\frac{\partial W_e}{\partial U_{ns}} - \frac{1}{2}\left(\frac{\partial W_e}{\partial U_{nr}}\delta_{sm} + \frac{\partial W_e}{\partial U_{mr}}\delta_{ns} + \frac{\partial W_e}{\partial U_{ms}}\delta_{rn}\right)\right], \tag{3.55}$$

where F_{ij} and U_{ij} are the Cartesian components of $\mathbf{F}$ and $\mathbf{U}$, respectively, and it is assumed $\frac{\partial W_e}{\partial U_{ij}} = \frac{\partial W_e}{\partial U_{ji}}$. To obtain the ellipticity condition in the reference configuration, we differentiate (3.36) and get

$$\mathbf{Q}(\mathbf{n}) = \mathbf{Q}_1(\mathbf{n}) + \mathbf{Q}_2(\mathbf{n}) + \mathbf{Q}_3(\mathbf{n}) + \mathbf{Q}_4(\mathbf{n}) + \mathbf{Q}_5(\mathbf{n}) + \mathbf{Q}_6(\mathbf{n}) + \mathbf{Q}_7(\mathbf{n}), \tag{3.56}$$

where

$$\mathbf{Q}_1(\mathbf{n}) = \mu(\mathbf{I} + \mathbf{n} \otimes \mathbf{n}), \quad \mathbf{Q}_2(\mathbf{n}) = \frac{\mu_1}{2}[(\boldsymbol{\tau}\mathbf{n}) \otimes \mathbf{n} + \mathbf{n} \otimes (\boldsymbol{\tau}\mathbf{n}) + (\mathbf{n} \cdot (\boldsymbol{\tau}\mathbf{n}))\mathbf{I} + \boldsymbol{\tau}], \tag{3.57}$$

$$\mathbf{Q}_3(\mathbf{n}) = \frac{\mu_2}{2}[(\boldsymbol{\tau}^2\mathbf{n}) \otimes \mathbf{n} + \mathbf{n} \otimes (\boldsymbol{\tau}^2\mathbf{n}) + (\mathbf{n} \cdot (\boldsymbol{\tau}^2\mathbf{n}))\mathbf{I} + \boldsymbol{\tau}^2], \tag{3.58}$$

$$\mathbf{Q}_4(\mathbf{n}) = \frac{3}{4}((\boldsymbol{\tau}\mathbf{n}) \cdot \mathbf{n})\mathbf{I} - \frac{1}{4}[\mathbf{n} \otimes (\boldsymbol{\tau}\mathbf{n}) + (\boldsymbol{\tau}\mathbf{n}) \otimes \mathbf{n} + \boldsymbol{\tau}], \tag{3.59}$$

$$\mathbf{Q}_5(\mathbf{n}) = \beta_1(\boldsymbol{\tau}\mathbf{n}) \otimes (\boldsymbol{\tau}\mathbf{n}), \quad \mathbf{Q}_6(\mathbf{n}) = \beta_2(\boldsymbol{\tau}^2\mathbf{n}) \otimes (\boldsymbol{\tau}^2\mathbf{n}), \tag{3.60}$$

$$\mathbf{Q}_7(\mathbf{n}) = \beta_3[(\boldsymbol{\tau}^2\mathbf{n}) \otimes (\boldsymbol{\tau}\mathbf{n}) + (\boldsymbol{\tau}\mathbf{n}) \otimes (\boldsymbol{\tau}^2\mathbf{n})]. \tag{3.61}$$

In this article, we will not derive the general inequalities required for the material constants given in (3.35). Since, in Section 5 below, we deal with problems that can be considered as two dimensional, we will give some inequality results for $\mathbf{m}$ and $\mathbf{n}$ in a plane and assume $\boldsymbol{\tau} = \tau_1 \mathbf{d}_1 \otimes \mathbf{d}_1 + \tau_2 \mathbf{d}_2 \otimes \mathbf{d}_2$ in that plane. In the case of a material with $\beta_1 = \beta_2 = \beta_3 = 0$, the necessary and sufficient conditions for (3.51) is

$$\mu + \frac{\mu_1}{2}(\tau_1 + \tau_2) + \frac{\mu_2}{2}(\tau_1^2 + \tau_2^2) + \frac{3}{4}\tau_1 - \frac{1}{4}\tau_2 > 0, \tag{3.62}$$

$$\mu + \frac{\mu_1}{2}(\tau_1 + \tau_2) + \frac{\mu_2}{2}(\tau_1^2 + \tau_2^2) + \frac{3}{4}\tau_2 - \frac{1}{4}\tau_1 > 0. \tag{3.63}$$

In the case where at least one of the β_i's is not zero, then the inequalities (3.62), (3.63)

$$\beta_1 > 0, \quad \beta_2 > 0, \quad \beta_3(\tau_2^2 - \tau_1^2)(\tau_2 - \tau_1) > 0 \tag{3.64}$$

are sufficient for (3.51). For example, the necessary and sufficient condition for the strain energy (3.47) is

$$\mu + \tau_1 > 0, \quad \mu + \tau_2 > 0 \tag{3.65}$$

and for (3.48)

$$\mu + \frac{1}{2}(3\tau_1 + \tau_2) > 0, \quad \mu + \frac{1}{2}(3\tau_2 + \tau_1) > 0\,. \tag{3.66}$$

3.8. *Initial stress symmetry (ISS)*

Recently, Gower *et al.* (2017) developed an initial stress symmetry model (ISS) to deal with residual stressed bodies. For our proposed model, it is beyond the scope of this article to construct general conditions (if possible) that are required so that the ISS constraint is satisfied. However, in this section we give, as an example, a simple spectral constitutive equation that satisfies the ISS constraint. As a first step let us give a brief summary of the ISS constraint. The initial stress symmetry condition states that:

$$\boldsymbol{\sigma} = \boldsymbol{H}(\mathbf{F}, \hat{\boldsymbol{\tau}}, p)\,, \quad \hat{\boldsymbol{\tau}} = \boldsymbol{H}(\mathbf{F}^{-1}, \boldsymbol{\sigma}, p_0) \tag{3.67}$$

for every $\mathbf{F}$ such that $\det \mathbf{F} = 1$ and $\hat{\boldsymbol{\tau}}$ and for some scalar p_0, where $\hat{\boldsymbol{\tau}} = \boldsymbol{\tau} - p_0 \mathbf{I}$ is the residual stress in the presence of hydrostatic pressure. Equivalently, the ISS constraint independent of hydrostatic terms has the form

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{G}(\mathbf{F}, \boldsymbol{\tau})\,, \quad \boldsymbol{\tau} = \boldsymbol{G}(\mathbf{F}^{-1}, \bar{\boldsymbol{\sigma}})\,, \tag{3.68}$$

where $\boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}} - p\mathbf{I}$ and $\bar{\boldsymbol{\sigma}}$ is the Cauchy stress in the absence of hydrostatic pressure. The ISS constraint is based on the assumption that $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\tau}}$ are due to the elastic deformation of a virtual stress-free configuration of an isotropic elastic solid. Following the work of Gower *et al.* (2017), we assume there exist a stress free configuration $\tilde{\mathscr{B}}$ and we propose a simple strain energy function for the current configuration $\mathscr{B}_t$

$$W_e = \mu \sum_{i=1}^{3} r(\bar{\lambda}_i) = \frac{\mu}{2}(\mathrm{tr}(\bar{\mathbf{C}}) - \mathrm{tr}(\ln(\bar{\mathbf{C}})) - 3\,, \tag{3.69}$$

where $r(x) = \frac{1}{2}(x^2 - 2\ln(x) - 1)$, $\bar{\lambda}_i^2$ is an eigenvalue of $\bar{\mathbf{C}} = \bar{\mathbf{F}}^{\mathrm{T}}\bar{\mathbf{F}}$ and $\bar{\mathbf{F}}$ is the deformation gradient tensor of configuration $\mathscr{B}_t$ relative to configuration $\tilde{\mathscr{B}}$. Take note that r satisfies the conditions $r(1) = r'(1) = 0$ and $r''(1) = 2$, imposed in Section 3.6. The Cauchy stress in the configuration $\mathscr{B}_t$ is

$$\boldsymbol{\sigma} = \mu(\bar{\mathbf{F}}\bar{\mathbf{F}}^{\mathrm{T}} - \mathbf{I}) - p\mathbf{I} = \bar{\boldsymbol{\sigma}} - p\mathbf{I}\,, \tag{3.70}$$

where $\bar{\boldsymbol{\sigma}}$ is the Cauchy stress in $\mathscr{B}_t$ in the absence of hydrostatic pressure. Let $\mathbf{F}_0$ be the deformation gradient of $\mathscr{B}_r$ relative to $\tilde{\mathscr{B}}$, we have $\bar{\mathbf{F}} = \mathbf{F}\mathbf{F}_0$. In view that our residual stress is the Cauchy stress in $\mathscr{B}_r$ in the absence of hydrostatic pressure, we have

$$\boldsymbol{\tau} = \mu(\mathbf{B}_0 - \mathbf{I})\,, \tag{3.71}$$

where $\mathbf{B}_0 = \mathbf{F}_0\mathbf{F}_0^{\mathrm{T}}$. It is clear from (3.71) that $\mathbf{B}_0$ can be expressed in terms of $\boldsymbol{\tau}$ independently of the hydrostatic pressure and $\boldsymbol{\tau} = \mathbf{0}$ when $\mathbf{B}_0 = \mathbf{I}$, as expected. If $\boldsymbol{\tau}$ is the Cauchy stress in $\mathscr{B}_r$ in the presence of hydrostatic pressure, as commonly defined in the literature (see e.g., equation (3.4) of Gower *et al.*, 2017), then $\mathbf{B}_0$ depends on $\boldsymbol{\tau}$ and a hydrostatic term, where the values of the hydrostatic term have to be obtained inconveniently via the cubic equation $\det(\mathbf{B}_0) = 1$. In view of (3.70) and (3.71), we have

$$\bar{\boldsymbol{\sigma}} = \mu(\mathbf{F}\mathbf{B}_0\mathbf{F}^{\mathrm{T}} - \mathbf{I}) = \mu\left[\mathbf{F}\left(\frac{\boldsymbol{\tau}}{\mu} + \mathbf{I}\right)\mathbf{F}^{\mathrm{T}} - \mathbf{I}\right] = \mu(\mathbf{F}\mathbf{F}^{\mathrm{T}} - \mathbf{I}) + \mathbf{F}\boldsymbol{\tau}\mathbf{F}^{\mathrm{T}} \tag{3.72}$$

and this gives

$$\boldsymbol{\tau} = \mu(\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}} - \mathbf{I}) + \mathbf{F}^{-1}\bar{\boldsymbol{\sigma}}\mathbf{F}^{-\mathrm{T}}\,. \tag{3.73}$$

Equations (3.72) and (3.73) indicate ISS symmetry between $\boldsymbol{\tau}$ and $\bar{\boldsymbol{\sigma}}$. Let $\hat{\boldsymbol{\tau}} = \boldsymbol{\tau} - p_0\mathbf{I}$ be the Cauchy stress in $\mathscr{B}_r$, hence

$$\hat{\boldsymbol{\tau}} = \mu(\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}} - \mathbf{I}) + \mathbf{F}^{-1}\bar{\boldsymbol{\sigma}}\mathbf{F}^{-\mathrm{T}} - p_0\mathbf{I}\,. \tag{3.74}$$

From (3.70) and (3.72), we get

$$\boldsymbol{\sigma} = \mu(\mathbf{F}\mathbf{F}^{\mathrm{T}} - \mathbf{I}) + \mathbf{F}\boldsymbol{\tau}\mathbf{F}^{\mathrm{T}} - p\mathbf{I}\,. \tag{3.75}$$

Equations (3.74) and (3.75) indicate ISS symmetry between $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$. From (3.69) and using $\mathrm{tr}(\bar{\mathbf{C}}) = \mathrm{tr}(\mathbf{B}_0\mathbf{C})$ we have

$$W_e = \frac{\mu}{2}[\mathrm{tr}(\mathbf{C}) - \mathrm{tr}(\ln(\mathbf{C})] - 3] + \frac{1}{2}\mathrm{tr}[\ln(\mathbf{F}_0^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{F}_0)]\,. \tag{3.76}$$

Clearly from (3.76), as expected, $W_e \neq 0$ when $\mathbf{F} = \mathbf{I}$, since we have set $W_e = 0$ in $\tilde{\mathscr{B}}$. Hence, W_e in (3.76) does not satisfy our property that $W_e = 0$ in $\tilde{\mathscr{B}}_r$. However, if we propose

$$W_e = \frac{\mu}{2}[\mathrm{tr}(\mathbf{C}) - \mathrm{tr}(\ln(\mathbf{C})) - 3] + \frac{1}{2}[\mathrm{tr}(\boldsymbol{\tau}\mathbf{C}) - \mathrm{tr}(\boldsymbol{\tau})], \tag{3.77}$$

where (3.77) is (3.46), when $\mu_1 = 0.5$, $\mu_2 = \beta_1 = \beta_2 = \beta_3 = 0$, $r_1(x) = \frac{1}{2}(x^2 - 2\ln(x) - 1)$ and $r_2(x) = (x-1)^2$, we have , $W_e = 0$ at $\mathbf{F} = \mathbf{I}$. It is straightforward to show that, for the strain energy function (3.77), the stress $\boldsymbol{\sigma}$, $\bar{\boldsymbol{\sigma}}$, $\boldsymbol{\tau}$ and $\hat{\boldsymbol{\tau}}$ satisfy the ISS relations (3.72), (3.73), (3.74) and (3.75).

4. Some preliminary considerations about the simple shear deformation

In this section, we study briefly the simple shear deformation, since for some cylindrical deformations to be studied later on can have non-homogeneous local simple shear which can support residual stress. Let the axes of $\mathbf{x}$ and $\mathbf{X}$ to coincide and the deformation can be described by the equations

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3, \tag{4.1}$$

where $\gamma \geq 0$ is the amount of shear. The deformation gradient with respect to the Cartesian basis takes the form

$$\mathbf{F} \equiv \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.2}$$

Let θ denotes the orientation (in the anticlockwise sense relative to the X_1 axis) of the in plane Lagrangian principal axes. The angle θ is restricted according by the following (Shariff, 2008)

$$\frac{\pi}{4} \leq \theta < \frac{\pi}{2}. \tag{4.3}$$

The principal directions are

$$\mathbf{e}_1 \equiv \begin{bmatrix} c \\ s \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 \equiv \begin{bmatrix} -s \\ c \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{4.4}$$

where $c = \cos(\theta)$ and $s = \sin(\theta)$. It can be easily shown (see, e.g., Shariff, 2008) that the principal stretches take the values

$$\lambda_1 = \frac{s}{c} = \frac{\gamma + \sqrt{\gamma^2 + 4}}{2} \geq 1, \quad \lambda_2 = \frac{1}{\lambda_1} = \frac{c}{s} = \frac{\sqrt{\gamma^2 + 4} - \gamma}{2} \leq 1, \quad \lambda_3 = 1, \tag{4.5}$$

where in this case

$$c = \frac{1}{\sqrt{1 + \lambda_1^2}}, \quad s = \frac{\lambda_1}{\sqrt{1 + \lambda_1^2}}, \quad c^2 - s^2 = -\gamma cs. \tag{4.6}$$

Without loss of generality, we consider $\sigma_{33} = 0$ because the incompressibility allows the superposition of an arbitrary hydrostatic stress without affecting the deformation.

The Cartesian components of stress tensor take the form

$$\sigma_{11} = 2\left[l_1\left(s^2(1+\gamma^2)+\gamma cs\right) + l_2\left(c^2(1+\gamma^2)-\gamma cs\right) - 2l_4cs\right] - p, \tag{4.7}$$

$$\sigma_{12} = 2\left[l_1(\gamma s^2 + cs) + l_2(\gamma c^2 - cs) + l_4\gamma cs\right], \quad \sigma_{22} = 2\left(l_1s^2 + l_2c^2 + 2l_4cs\right) - p, \tag{4.8}$$

$$\sigma_{13} = 2\left(l_5(c+\gamma s) + l_6(-s+\gamma c)\right), \sigma_{23} = 2\left(l_5 s + l_6 c\right), \quad \sigma_{33} = 2l_3 - p, \tag{4.9}$$

where we have defined

$$l_1 = \frac{1}{2\lambda_1}\frac{\partial W}{\partial \lambda_1}, \quad l_2 = \frac{1}{2\lambda_2}\frac{\partial W}{\partial \lambda_2}, \quad l_3 = \frac{1}{2\lambda_3}\frac{\partial W}{\partial \lambda_3}, \tag{4.10}$$

$$l_4 = \frac{1}{\lambda_1^2 - \lambda_2^2}\left[\left(\frac{\partial W}{\partial \zeta_1} - \frac{\partial W}{\partial \zeta_2}\right)\mathbf{e}_1\cdot(\boldsymbol{\tau}\mathbf{e}_2) + \left(\frac{\partial W}{\partial \xi_1} - \frac{\partial W}{\partial \xi_2}\right)\mathbf{e}_1\cdot(\boldsymbol{\tau}^2\mathbf{e}_2)\right], \tag{4.11}$$

$$l_5 = \frac{1}{\lambda_1^2 - \lambda_3^2}\left[\left(\frac{\partial W}{\partial \zeta_1} - \frac{\partial W}{\partial \zeta_3}\right)\mathbf{e}_1\cdot(\boldsymbol{\tau}\mathbf{e}_3) + \left(\frac{\partial W}{\partial \xi_1} - \frac{\partial W}{\partial \xi_3}\right)\mathbf{e}_1\cdot(\boldsymbol{\tau}^2\mathbf{e}_3)\right], \tag{4.12}$$

$$l_6 = \frac{1}{\lambda_2^2 - \lambda_3^2}\left[\left(\frac{\partial W}{\partial \zeta_2} - \frac{\partial W}{\partial \zeta_3}\right)\mathbf{e}_2\cdot(\boldsymbol{\tau}\mathbf{e}_3) + \left(\frac{\partial W}{\partial \xi_2} - \frac{\partial W}{\partial \xi_3}\right)\mathbf{e}_2\cdot(\boldsymbol{\tau}^2\mathbf{e}_3)\right]. \tag{4.13}$$

As mentioned previously we consider $\sigma_{33} = 0$, hence $p = 2l_3$.

In general, the Poynting relation $\sigma_{11} - \sigma_{22} = \gamma\sigma_{12}$ (generally associated with isotropic theory) does not hold. The Poynting relation is a relation between stress components and the deformation, which is independent of the choice of (isotropic) constitutive equation. It is interesting to see if this *universal* relation holds for residual materials under certain conditions. From (4.7) and (4.8)

$$\sigma_{11} - \sigma_{22} = \gamma\sigma_{12} - 2l_4cs(4+\gamma^2), \tag{4.14}$$

Hence, from (4.14), we see that the Poynting relation holds if and only if $l_4 = 0$; no conditions are required for l_5 or l_6. An example of a case when $l_4 = 0$ for an *arbitrary* strain energy function is when one of the directions $\mathbf{d}_1$ or $\mathbf{d}_2$ is parallel to $\mathbf{e}_1$ or $\mathbf{e}_2$, taking note that $\mathbf{d}_3 \otimes \mathbf{d}_3 = \mathbf{I} - (\mathbf{d}_1 \otimes \mathbf{d}_1 + \mathbf{d}_2 \otimes \mathbf{d}_2)$. For example, if the components of $\mathbf{d}_1$ and $\mathbf{d}_2$ are

$$\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ b_3 \end{pmatrix}, \tag{4.15}$$

respectively, where b_3 is any third component of $\mathbf{d}_2$, then the Poynting relation holds at the particular strain when $\gamma = \frac{2}{\sqrt{3}}$.

We note that l_5 and l_6 appear in σ_{13} and σ_{23} only, in view of this, we have the relations

$$s\sigma_{13} + c\sigma_{23} = 2l_5(2cs + \gamma s^2), \quad c\sigma_{13} - s\sigma_{23} = 2l_6(\gamma c^2 - 2cs). \tag{4.16}$$

Since $2cs + \gamma s^2$ and $\gamma c^2 - 2cs = \frac{3\lambda_1^2 - 1}{\lambda_1^2 + 1}$ are both positive, then

$$\lambda_1\sigma_{13} + \sigma_{23} = 0, \tag{4.17}$$

if and only if $l_5 = 0$ and

$$\sigma_{13} - \lambda_1 \sigma_{23} = 0, \tag{4.18}$$

if and only $l_6 = 0$. Note that σ_{13} and σ_{23} in (4.17) and (4.18) have different values. Since l_5 and l_6 can be zero for an arbitrary strain energy, the relations (4.17) and (4.18) are 'deformation-dependent' *universal* relations, that is, they are *independent* of the choice of residual constitutive equation. We use the term 'deformation-dependent' since the relations hold at particular strains and at particular directions of $\mathbf{d}_1$ and $\mathbf{d}_2$. An example of $l_5 = 0$ and $l_6 \neq 0$ at a particular strain is when both $\mathbf{d}_1$ and $\mathbf{d}_2$ are perpendicular to $\mathbf{e}_1$ but not perpendicular to $\mathbf{e}_2$ and $\mathbf{e}_3$. An example of $l_6 = 0$ and $l_5 \neq 0$ is when both $\mathbf{d}_1$ and $\mathbf{d}_2$ are perpendicular to $\mathbf{e}_2$ but not perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_3$. Since in the above two examples, either $l_6 \neq 0$ or $l_5 \neq 0$, and since W is arbitrary, the shear stresses σ_{13} and σ_{23} are generally non-zero. The relations (4.17) and (4.18) may not be straightforward to derive using the classical invariants.

In the case when the directions $\mathbf{d}_1$ and $\mathbf{d}_2$ are perpendicular to the direction $\mathbf{e}_3$, the shear components $\sigma_{13} = \sigma_{23} = 0$ and we have the relations

$$\frac{\partial \lambda_1}{\partial \gamma} = s^2, \quad \frac{\partial \lambda_2}{\partial \gamma} = -c^2, \quad \frac{\partial \zeta_1}{\partial \gamma} = 2\lambda_1 s c^3 \mathbf{e}_1 \cdot (\boldsymbol{\tau} \mathbf{e}_2), \quad \frac{\partial \zeta_2}{\partial \gamma} = -\frac{\partial \zeta_1}{\partial \gamma}, \tag{4.19}$$

$$\frac{\partial \xi_1}{\partial \gamma} = 2\lambda_1 s c^3 \mathbf{e}_1 \cdot (\boldsymbol{\tau}^2 \mathbf{e}_2), \quad \frac{\partial \xi_2}{\partial \gamma} = -\frac{\partial \xi_1}{\partial \gamma}. \tag{4.20}$$

In general, the Poynting relation does not hold. Since a simple shear deformation depends on γ, the strain energy function can be considered as a function of γ, that is, $W_e = \hat{W}(\gamma)$. Using equation (4.20), we can easily deduce (after some algebra) that, for $\mathbf{d}_1$ and $\mathbf{d}_2$ perpendicular to $\mathbf{e}_3$

$$\sigma_{12} = \hat{W}'(\gamma). \tag{4.21}$$

This relation has been studied in Bustamante & Merodio (2010) for other types of elastic bodies. It is important to indicate that (4.21) can be used to study stability

5. Problems with cylindrical symmetry

Here we consider the constitutive equation presented in the previous section to study some boundary value problems with cylindrical symmetry, which could be important from the experimental point of view. First, we introduce a particular residual stress in a circular cylindrical tube and uses it in subsequent sections. Since, some of the cylindrical tube problems discussed in this section have local simple shear deformations, we use the detailed results of Section 4 in the cylindrical problems presented in the subsequent sections.

Since there is no adequate quantitative data available in the literature to justify the proposed constitutive equation (3.46), its pointless to give qualitative and quantitative results for specific strain energy forms based on (3.46). However, the constitutive form (3.46) can be specialised to particular forms proposed in the literature (see Merodio *et al.*, 2013; Merodio & Ogden, 2014) and, in view of this, we only consider some problems, where their solutions are given and analysed in the literature. We emphasize that our spectral formulation can deal with classical invariant formulation but not vice versa; in general, the spectral constitutive equation (3.14) cannot be converted to classical form. The interested reader can see Nam *et al.* (2016) for the solution of some boundary value problem for bodies with initial stresses where the formulation is based on the classical invariants by Rivlin and Spencer.

5.1. *Residual stresses*

In the following section we solve some boundary value problems for a circular cylindrical tube, considering different controllable deformations. The reference configuration is defined by

$$A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq Z \leq L. \tag{5.1}$$

Following Merodio *et al.* (2013), here we consider a residual stress of the form

$$\boldsymbol{\tau} = \tau_1(R)\mathbf{d}_1 \otimes \mathbf{d}_1 + \tau_2(R)\mathbf{d}_2 \otimes \mathbf{d}_2, \tag{5.2}$$

where $\mathbf{d}_1 = \mathbf{E}_r$ and $\mathbf{d}_2 = \mathbf{E}_\theta$ are cylindrical polar coordinate vectors in the reference configuration.

The stress tensor (5.2) must satisfy the equilibrium equation

$$\frac{\mathrm{d}(R\tau_1)}{\mathrm{d}R} = \tau_2, \tag{5.3}$$

The component τ_1 must satisfy the boundary conditions

$$\tau_1(A) = 0, \quad \tau_1(B) = 0. \tag{5.4}$$

In Merodio *et al.* (2013), it has been shown that for above $\boldsymbol{\tau}$, $\tau_2(R)$ has both positive and negative values for $A \leq R \leq B$. For illustration purposes, we choose the simple expression (see Merodio *et al.*, 2013)

$$\tau_1 = \alpha(R - A)(R - B), \tag{5.5}$$

where α is a constant. We shall use the form (5.5) in this article.

5.2. *Pure azimuthal shear*

Consider the problem of pure azimuthal shear of a circular cylindrical tube with cross section in the reference configuration defined by (5.1) The deformation is described by

$$r = R, \quad \theta = \Theta + g(R). \tag{5.6}$$

The deformation gradient in cylindrical polar coordinate system is

$$\mathbf{F} \equiv \begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.7}$$

where $\gamma = rg'(r) = Rg'(R)$. In view of (4.2) this is a local simple shear deformation and the constitutive equation is the same as (4.9), with $\sigma_{r\theta} = \sigma_{12}$, $\sigma_{rr} = \sigma_{22}$ and $\sigma_{\theta\theta} = \sigma_{11}$, where $\sigma_{r\theta}$, σ_{rr} and $\sigma_{\theta\theta}$ are cylindrical polar components of the Cauchy stress. The principal residual stresses are

$$\tau_1 = \tau_{rr}, \quad \tau_2 = \tau_{\theta\theta}. \tag{5.8}$$

The conditions for the universal relations are similar to that described in Section 4. The principal directions of **U** takes the form

$$\mathbf{e}_1 \equiv \begin{pmatrix} s \\ c \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 \equiv \begin{pmatrix} c \\ -s \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 \equiv \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}. \tag{5.9}$$

In view of (4.14), we have

$$\sigma_{\theta\theta} - \sigma_{rr} = \gamma\sigma_{r\theta} - 2l_4 cs(4+\gamma^2)\,, \tag{5.10}$$

If we use the strain energy (3.47), $l_1 = \frac{\mu+\zeta_1}{2}$, $l_2 = \frac{\mu+\zeta_2}{2}$, $l_4 = \frac{\mathbf{e}_1\cdot(\boldsymbol{\tau}\mathbf{e}_2)}{2}$ and the shear stress take the form

$$\sigma_{r\theta} = (\mu+\tau_1)\gamma. \tag{5.11}$$

To obtain (5.11) we used of the relations

$$\mathbf{e}_1\cdot(\boldsymbol{\tau}\mathbf{e}_2) = (\tau_1-\tau_2)cs, \quad \zeta_1 = \tau_1 s^2 + \tau_2 c^2, \quad \zeta_2 = \tau_1 c^2 + \tau_2 s^2. \tag{5.12}$$

The ellipticity condition (3.51) gives

$$\mu+\tau_1 > 0, \quad \mu+\tau_2 > 0. \tag{5.13}$$

A discussion on the influence of residual stress (5.5) on a material with a strain energy of the form (3.47) can be found in Merodio *et al.* (2013).

5.3. *Extension and torsion of a solid cylinder*

The deformation is described by

$$r = \lambda^{-\frac{1}{2}}R, \quad \theta = \Theta + \lambda\tau Z, \quad z = \lambda Z, \tag{5.14}$$

where τ is the amount of torsional twist per unit deformed length and λ is the axial stretch. In the above formulation, r, θ and z are cylindrical polar coordinates in the deformed configuration. The components of the deformation gradient are

$$\mathbf{F} \equiv \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 & 0 \\ 0 & \lambda^{-\frac{1}{2}} & \lambda\gamma \\ 0 & 0 & \lambda \end{pmatrix}, \tag{5.15}$$

where $\gamma = r\tau$ and, in this article, we only consider $\lambda \geq 1$. The Lagrangian principal directions have cylindrical components:

$$\mathbf{e}_1 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 \equiv \begin{pmatrix} 0 \\ c \\ s \end{pmatrix}, \quad \mathbf{e}_3 \equiv \begin{pmatrix} 0 \\ -s \\ c \end{pmatrix}, \tag{5.16}$$

where

$$c = \cos(\phi) = \frac{2}{\sqrt{2(\hat{\gamma}^2+4) + 2\hat{\gamma}\sqrt{\hat{\gamma}^2+4}}}, \quad s = \sin(\phi) = \frac{\hat{\gamma} + \sqrt{\hat{\gamma}^2+4}}{\sqrt{2(\hat{\gamma}^2+4) + 2\hat{\gamma}\sqrt{\hat{\gamma}^2+4}}}, \tag{5.17}$$

with

$$\frac{\pi}{4} \leq \frac{\pi - \tan^{-1}\left(\frac{1}{\sqrt{\lambda^3-1}}\right)}{2} \leq \phi < \frac{\pi}{2} \tag{5.18}$$

and

$$\hat{\gamma} = \frac{\lambda^3\gamma^2 + \lambda^3 - 1}{\lambda^{\frac{3}{2}}\gamma} \geq 0. \tag{5.19}$$

We also have the relation

$$c^2 - s^2 = -\hat{\gamma} cs. \tag{5.20}$$

In the case of simple torsion $\lambda = 1$ and we have $\hat{\gamma} = \gamma$. The principal stretches for a combined extension and torsion deformation have the forms

$$\lambda_1 = \frac{1}{\lambda^{\frac{1}{2}}}, \quad \lambda_2 = \sqrt{\frac{\lambda^3(1+\gamma^2) + 1 + \lambda^{\frac{3}{2}}\gamma\sqrt{\hat{\gamma}^2+4}}{2\lambda}} = \sqrt{\frac{1}{\lambda} + \frac{s\gamma\sqrt{\lambda}}{c}}, \tag{5.21}$$

$$\lambda_3 = \sqrt{\frac{\lambda^3(1+\gamma^2) + 1 - \lambda^{\frac{3}{2}}\gamma\sqrt{\hat{\gamma}^2+4}}{2\lambda}} = \sqrt{\frac{1}{\lambda} - \frac{c\gamma\sqrt{\lambda}}{s}}. \tag{5.22}$$

The cylindrical components of the Cauchy stress take the form:

$$\sigma_{\theta\theta} = -p + 2\left[\frac{l_2c^2 + l_3s^2 - 2l_6cs}{\lambda} + 2\sqrt{\lambda}\gamma((l_2-l_3)cs + l_6(c^2-s^2)) + \lambda^2\gamma^2(l_2s^2 + l_3c^2 + 2l_6cs)\right], \tag{5.23}$$

$$\sigma_{z\theta} = 2\left[\sqrt{\lambda}((l_2-l_3)cs + l_6(c^2-s^2)) + \lambda^2\gamma(l_2s^2 + l_3c^2 + 2l_6cs)\right], \tag{5.24}$$

$$\sigma_{zz} = -p + 2\lambda^2\left(l_2s^2 + l_3c^2 + 2l_6cs\right), \quad \sigma_{r\theta} = 2\left[\frac{l_4c - l_5s}{\lambda} + \sqrt{\lambda}\gamma(l_4s + l_5c)\right], \tag{5.25}$$

$$\sigma_{rr} = -p + \frac{2l_1}{\lambda}, \quad \sigma_{zr} = 2\sqrt{\lambda}\,(l_4s + l_5c)\ . \tag{5.26}$$

In the case of $\boldsymbol{\tau}$ taking the form (5.2), it can be easily shown that $l_4 = l_5 = 0$, which implies that $\sigma_{rz} = \sigma_{r\theta} = 0$. Since the deformation depends on γ and λ, the energy function can be considered as a function of γ and λ, that is, $W_e = \bar{\Omega}(\lambda, \gamma)$. We can easily deduce that,

$$\sigma_{z\theta} = \frac{\partial \bar{\Omega}}{\partial \gamma}. \tag{5.27}$$

In order to obtain the above relation (5.27), we require the following formulae

$$\frac{\partial \lambda_2}{\partial \gamma} = \frac{1}{\lambda_2}\left(sc\sqrt{\lambda} + \lambda^2 \gamma s^2\right), \quad \frac{\partial \lambda_3}{\partial \gamma} = \frac{1}{\lambda_3}\left(-sc\sqrt{\lambda} + \lambda^2 \gamma c^2\right) \tag{5.28}$$

and

$$\frac{\partial s^2}{\partial \gamma} = -\frac{\partial c^2}{\partial \gamma} = \frac{sc}{\lambda_2^2 - \lambda_3^2}\left(2\sqrt{\lambda}(c^2 - s^2) + 4\lambda^2 \gamma sc\right). \tag{5.29}$$

For this problem, we have

$$\zeta_1 = \tau_1, \quad \zeta_2 = \tau_2 c, \quad \zeta_3 = \xi_3 = 0, \quad \xi_1 = \tau_1^2, \quad \xi_2 = \tau_2^2 c, \tag{5.30}$$

$$\mathbf{e}_1 \cdot (\boldsymbol{\tau}\mathbf{e}_2) = \mathbf{e}_1 \cdot (\boldsymbol{\tau}^2\mathbf{e}_3) = \mathbf{e}_1 \cdot (\boldsymbol{\tau}\mathbf{e}_3) = \mathbf{e}_1 \cdot (\boldsymbol{\tau}^2\mathbf{e}_3) = 0, \quad \mathbf{e}_2 \cdot (\boldsymbol{\tau}\mathbf{e}_3) = -\tau_2 cs. \tag{5.31}$$

If we assume that no stress is applied at the surface $r = b$, the mechanical traction N and the torque M applied at the ends of the cylinder are as follows:

$$N = 2\pi \int_0^b \sigma_{zz} r \, \mathrm{d}r, \quad M = 2\pi \int_0^b \frac{\partial \bar{\Omega}}{\partial \gamma} r^2 \, \mathrm{d}r. \tag{5.32}$$

To remove the hydrostatic pressure term in (5.32), we reformulate (5.32) in the form

$$N = \pi \int_0^a (2\sigma_{zz} - \sigma_{rr} - \sigma_{\theta\theta}) r \, \mathrm{d}r, \tag{5.33}$$

using the relation

$$\sigma_{rr} + \sigma_{\theta\theta} = \frac{1}{r}\frac{\mathrm{d}(r^2\sigma_{rr})}{\mathrm{d}r}. \tag{5.34}$$

If the strain energy take the form (3.47) then

$$l_1 = \frac{\mu + \tau_1}{2}, \quad l_2 = \frac{\mu + \tau_2 c}{2}, \quad l_3 = \mu, \quad l_4 = l_5 = 0, \quad l_6 = -\frac{\tau_2 cs}{2}. \tag{5.35}$$

In the case of (3.48), we have,

$$l_1 = \frac{\mu + \lambda_1^2 \tau_1}{2}, \quad l_2 = \frac{\mu + \lambda_2^2 \tau_2 c}{2}, \quad l_3 = \mu, \quad l_4 = l_5 = 0, \quad l_6 = -\frac{(\lambda_2^2 + \lambda_3^2)\tau_2 cs}{4}. \tag{5.36}$$

Some numerical results for the above strain energy functions can be found in Merodio & Ogden (2014).

5.4. *Inflation of a tube*

We consider a cylindrical tube inflated by a pressure P, where the deformation described by

$$r = \sqrt{a^2 + R^2 - A^2}, \quad \theta = \Theta, \quad z = Z, \quad A \leq R \leq B, \tag{5.37}$$

where a and A are the deformed and un-deformed inner radius, respectively. In this case, we have the cylindrical components

$$\mathbf{e}_1 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{5.38}$$

and the principal stretches

$$\lambda_1 = \frac{R}{r}, \quad \lambda_2 = \lambda = \frac{r}{R}, \quad \lambda_3 = 1. \tag{5.39}$$

The non-zero components of the Cauchy stress are

$$\sigma_{rr} = \frac{R}{r}\frac{\partial W_e}{\partial \lambda_1} - p, \quad \sigma_{\theta\theta} = \frac{r}{R}\frac{\partial W_e}{\partial \lambda_2} - p, \quad \sigma_{zz} = \frac{\partial W_e}{\partial \lambda_3} - p. \tag{5.40}$$

The only variable in the strain energy function is λ, hence we can write $W_e = \tilde{W}(\lambda)$ and have the relation

$$\lambda \frac{\partial \tilde{W}}{\partial \lambda} = \sigma_{\theta\theta} - \sigma_{rr}. \tag{5.41}$$

By integrating the radial component of the equilibrium equations and in view of $\sigma_{rr} = 0$ at $r = b$, the pressure

$$P = \int_a^b \lambda \frac{\partial \tilde{W}}{\partial \lambda}\frac{\mathrm{d}r}{r} = \int_A^B \frac{1}{\lambda}\frac{\partial \tilde{W}}{\partial \lambda}\frac{\mathrm{d}R}{R}. \tag{5.42}$$

For the strain energy function (3.47), the pressure (5.42), after using the boundary conditions (5.4), takes the form

$$P = \int_A^B (\mu + \tau_1)\left(1 - \frac{1}{\lambda^4}\right)\frac{\mathrm{d}r}{R}. \tag{5.43}$$

Numerical discussion relating to (5.43) can be found in Merodio *et al.* (2013).

6. Final remarks

In the present communication, we have revisited the theory for residually stressed bodies developed in Shams *et al.* (2011); Merodio *et al.* (2013); Merodio & Ogden (2014), reformulating the constitutive equations considering some new classes of invariants proposed by Shariff (2008, 2013, 2016b,a); Shariff & Bustamante (2015). Such new invariants have a clearer physical meaning in comparison with the classical invariants formulated by Spencer (1971). Hence, they are useful in facilitating the design of residual stress experiment. It has been shown that there are only nine independent spectral invariants from the original list of 12.

The results presented in this article will serve as a stepping stone to study the more complex problem, wherein we will consider the case of a residually stressed body that can react to the presence of an electric field. Such a theory would be particularly interesting for the case of modelling, for example, the behaviour of myocardium, which is a material that can react to electromagnetic fields (see, e.g., Göktepe & Kuhl, 2010 and the reference mentioned therein) and also shows the presence of residual stresses (see, e.g., Figs 11.4:1, 11.4:2 and 11.4:4 of Fung, 1990). For such a problem, for simplicity, we initially assume that the myocardium behaves as an elastic body, the energy function would be of the form $W_e = W_e(\mathbf{C}, \boldsymbol{\tau}, \boldsymbol{E}_l, \mathbf{a}_0, \mathbf{b}_0)$, where $\boldsymbol{E}_l$ would be the electric field in the reference configuration (see, e.g., Dorfmann & Ogden, 2005 and the references mentioned therein) and $\mathbf{a}_0$, $\mathbf{b}_0$ vector fields used to incorporate the anisotropic behaviour of such material (Holzapfel & Ogden, 2009). It is clear that for such a problem, where the constitutive equation depends on many different fields, it is very important to have invariants with a clear physical meaning.

References

Bell, J. F. (1973) The experimental foundations of solids mechanics. *Mechanics of Solids vol. 1, Handbuch der Physik* (C. Truesdell ed). Berlin: Springer, 1–813.

Brinksmeier, E., Cammett, J. T., Köning, W., Leskovar, P., Peters, J. & Tönshoff, H. K. (1982) Residual stresses: measurements and causes in machining processes. *Ann. CIRP,* **31**, 491–510.

Bustamante, R. & Merodio, J. (2010) On simple constitutive restrictions for transversely isotropic nonlinearly elastic materials and isotropic magneto-sensitive elastomers. *J. Eng. Math.,* **68**, 15–26.

Chuong, C. J. & Fung, Y. C. (1986) On residual stresses in arteries. *J. Biomech. Eng.,* **108**, 189–192.

Dorfmann, A., & Ogden, R. W. (2005) Nonlinear electroelasticity. *Acta Mech.* **174**, 167–183.

Fung, Y. C. (1990) *Biomechanics: Motion, Flow, Stress and Growth*. New York, USA: Springer-Verlag New York Inc.

Fung, Y. C. (1991) What are residual stresses doing in our blood vessels? *Ann. Biomed. Eng.,* **19**, 237–249.

Göktepe, S. & Kuhl, E. (2010) Electromechanics of the heart: a unified approach to the strongly coupled excitation-contraction problem. *Comput. Mech.,* **45**, 227–243.

Gower, A. L., Ciarletta, P. & Destrade, M. (2017) Initial stress symmetry and its application in elasticity. *Proc. R. Soc. A,* **471**, 20150448.

Hoger, A. (1985) On the residual stress possible in an elastic body with material symmetry. *Arch. Rat. Mech. Anal.* **88**, 271–290.

Hoger, A. (1986) On the determination of residual stress in an elastic body. *J. Elast.,* **16**, 303–324.

Hoger, A. (1993) The elasticity tensors of a residually stressed material. *J. Elast.,* **31**, 219–237.

Holzapfel, G. A. & Ogden, R. W. (2009) Constitutive modelling of passive myocardium: a structurally based framework for material characterization. *Phil. Trans. R. Soc. A,* **367**, 3445–3475.

Man, C. S. (1998) Hartin's law and linear elasticity with initial stress. *Inverse Prob.,* **14**, 313–319.

Merodio, J. Ogden, R. W. & Rodriguez, J. (2013) The influence of residual stress on finite deformation elastic response. *Int. J. Nonlinear Mech.* **56**, 43–49.

Merodio, J. & Ogden, R. W. (2014) Extension, inflation and torsion of a residually stressed circular cylindrical tube. *Continuum Mech. Thermodyn.* **28**, 157–174.

Nam, N. T., Merodio, J., Ogden, R. W. & Vinh, P. C. (2016) The effect of initial stress on the propagation of surface waves in a layered half-space. *Int. J. Solids Struct.* **88**, 88–100.

Robertson, R. L. (1997) Boundary identifiability of residual stress via Dirichlet to Newumann map. *Inverse Prob.,* **13**, 11077–11019.

Sawyers, K. N. & Rivlin, R. S. (1976) The strain-energy function for elastomers. *Trans. Soc. Rheology* **20**, 545–557.

Sawyers, K. N. (1986) Comments on the paper 'Determination of the stretch and rotation factors in the polar decomposition of the deformation gradient', by A. Hoger and D.E. Carlson. *Quart. Appl. Math.* **44**, 309–311.

Shams, M., Destrade, M. & Ogden, R. W. (2011) Initial stresses in elastic solids: constitutive laws and acoustoelasticity. *Wave Motion* **48**, 552–567.

Shariff, M. H. B. M. (2008) Nonlinear transversely isotropic elastic solids: an alternative representation. *Quat. J. Mech. Appl. Math.* **61**, 129–149.

Shariff, M. H. B. M. (2013) Nonlinear orthotropic elasticity: only six invariants are independent. *J. Elast.* **110**, 237–241.

Shariff, M. H. B. M. & Bustamante, R. (2015) On the independence of strain invariants of two preferred direction nonlinear elasticity. *Int. J. Eng. Sci.* **97**, 18–25.

Shariff, M. H. B. M. (2016a) Anisotropic separable free energy functions for elastic and non-elastic solids. *Acta Mech.,* doi:10.1007/s00707-015-1534-9.

Shariff, M. H. B. M. (2016b) The number of independent invariants of an *n*-preferred direction anisotropic solid. *Math. Mech. Solids,* doi:10.1177/1081286516653196.

Spencer, A. J. M. & Rivlin, R. S. (1962) Isotropic integrity bases for vectors and second-order tensors I. *Arch. Rational Mech. Anal.* **9**, 45–63.

Spencer, A. J. M. (1971) Theory of invariants. *Continuum Physics I* (A. C. Eringen ed). New York: Academic Press, pp. 239–253.

Truesdell, C. A. & Toupin, R. (1960) The classical field theories. *Handbuch der Physik, vol.III/1* (S. Flügge ed). Berlin, Germany: Springer, pp. 226–902.

Withers, P. J. & Bhadeshia, H. K. D. H. (2001) Residual stress part 2: nature and origins. *Mater. Sci. Technol.,* **17**, 366–375.

Appendix A

In this section, we study the number of independent invariants proposed in Section 3.2. From the definitions presented in that section it is possible to see that

$$\sum_{i=1}^{3} \zeta_i = \sum_{i=1}^{3} \tau_i, \quad \sum_{i=1}^{3} \xi_i = \sum_{i=1}^{3} \tau_i^2. \tag{A1}$$

Additionally, following the work of Shariff (2013), we have

$$[\mathbf{e}_1 \cdot (\boldsymbol{\tau}\mathbf{e}_3)]^2 = \xi_1 - \zeta_1^2 - [\mathbf{e}_1 \cdot (\boldsymbol{\tau}\mathbf{e}_2)]^2, \quad [\mathbf{e}_2 \cdot (\boldsymbol{\tau}\mathbf{e}_3)]^2 = \xi_2 - \zeta_2^2 - [\mathbf{e}_1 \cdot (\boldsymbol{\tau}\mathbf{e}_2)]^2, \tag{A2}$$

$$[\mathbf{e}_1 \cdot (\boldsymbol{\tau}\mathbf{e}_2)]^2 = \tau_1\tau_2 + \tau_1\tau_3 + \tau_3\tau_3 + \zeta_1\zeta_2 + \xi_1 + \xi_2 - (\tau_1 + \tau_2 + \tau_3)(\zeta_1 + \zeta_2), \tag{A3}$$

from where we obtain

$$\{\tau_1\tau_2\tau_3 - \zeta_1\zeta_2\zeta_3 + \zeta_1[\mathbf{e}_2 \cdot (\boldsymbol{\tau}\mathbf{e}_3)]^2 + \zeta_2[\mathbf{e}_1 \cdot (\boldsymbol{\tau}\mathbf{e}_3)]^2 + \zeta_3[\mathbf{e}_1 \cdot (\boldsymbol{\tau}\mathbf{e}_2)]^2\}^2$$
$$= 4[\mathbf{e}_1 \cdot (\boldsymbol{\tau}\mathbf{e}_2)]^2[\mathbf{e}_1 \cdot (\boldsymbol{\tau}\mathbf{e}_3)]^2[\mathbf{e}_2 \cdot (\boldsymbol{\tau}\mathbf{e}_3)]^2. \tag{A4}$$

The three constraints in (A1) and (A4) implies that only nine of the twelve spectral invariants are independent.

Appendix B

Here, we show there exists a relation between the ten classical invariants; spectral invariants do not appear in this relation. The existence of this relation proves that only nine of the 10 invariants are independent. The prove of this relation requires relations obtained in Shariff (2013). In order for the reader to easily connect relations in this Appendix to those in Shariff (2013), most of the notations used here are different from the main body of this article, taking note that some of the notations may be equivalent to the spectral variables introduced in the main body.

Initially, we assumed that the values of the eigenvalues are not the same, that is, $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$. The ten classical invariants take the forms (see, e.g., Spencer & Rivlin, 1962)

$$\mathrm{tr}(\mathbf{C}), \quad \mathrm{tr}(\mathbf{C}^2), \quad \mathrm{tr}(\mathbf{C}^3), \quad \mathrm{tr}(\boldsymbol{\tau}), \quad \mathrm{tr}(\boldsymbol{\tau}^2), \quad \mathrm{tr}(\boldsymbol{\tau}^3), \tag{B1}$$

$$\mathrm{tr}(\mathbf{C}\boldsymbol{\tau}), \quad \mathrm{tr}(\mathbf{C}^2\boldsymbol{\tau}), \quad \mathrm{tr}(\mathbf{C}\boldsymbol{\tau}^2), \quad \mathrm{tr}(\mathbf{C}^2\boldsymbol{\tau}^2). \tag{B2}$$

Using the work of Sawyers & Rivlin (1976) and Sawyers (1986), we can show that

$$\lambda_i = \frac{1}{\sqrt{3}}\left[I_1 + 2A\cos\left(\frac{\psi + 2\pi i}{3}\right)\right], \tag{B3}$$

where

$$A = (I_1^2 - 3I_2)^{\frac{1}{2}}, \quad \psi = \cos^{-1}\frac{1}{2A^3}(2I_1^3 - 9I_1I_2 + 27I_3). \tag{B4}$$

Let

$$b_i = \mathbf{e}_i \cdot (\boldsymbol{\tau}\mathbf{e}_i), \quad d_i = \mathbf{e}_i \cdot (\boldsymbol{\tau}^2\mathbf{e}_i), \quad i \quad \text{not summed.} \tag{B5}$$

Hence

$$\mathrm{tr}(\boldsymbol{\tau}) = \sum_{i=1}^{3} b_i, \quad \mathrm{tr}(\mathbf{C}\boldsymbol{\tau}) = \sum_{i=1}^{3} \lambda_i b_i, \quad \mathrm{tr}(\mathbf{C}^2\boldsymbol{\tau}) = \sum_{i=1}^{3} \lambda_i^2 b_i, \tag{B6}$$

and

$$\mathrm{tr}(\boldsymbol{\tau}^2) = \sum_{i=1}^{3} d_i, \quad \mathrm{tr}(\mathbf{C}\boldsymbol{\tau}^2) = \sum_{i=1}^{3} \lambda_i d_i, \quad \mathrm{tr}(\mathbf{C}^2\boldsymbol{\tau}^2) = \sum_{i=1}^{3} \lambda_i^2 d_i. \tag{B7}$$

In view of (B6) and (B7) we have the matrix equations

$$\mathbf{Mb} = \mathbf{f}, \quad \mathbf{Md} = \mathbf{g}, \tag{B8}$$

where

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} \mathrm{tr}(\boldsymbol{\tau}) \\ \mathrm{tr}(\mathbf{C}\boldsymbol{\tau}) \\ \mathrm{tr}(\mathbf{C}^2\boldsymbol{\tau}) \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} \mathrm{tr}(\boldsymbol{\tau}^2) \\ \mathrm{tr}(\mathbf{C}\boldsymbol{\tau}^2) \\ \mathrm{tr}(\mathbf{C}^2\boldsymbol{\tau}^2) \end{pmatrix}. \tag{B9}$$

We note that $\det \mathbf{M} = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3) \neq 0$, hence $\mathbf{M}$ is invertible. In view of (B3) and (B8), we can express $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ explicitly in terms of the classical invariants.

In the following calculations, we use the result by Shariff (2013), let us define

$$I_{ab} = \mathbf{e}_1 \cdot (\boldsymbol{\tau}\mathbf{e}_2), \quad I_{an} = \mathbf{e}_1 \cdot (\boldsymbol{\tau}\mathbf{e}_3), \quad I_{bn} = \mathbf{e}_2 \cdot (\boldsymbol{\tau}\mathbf{e}_3), \tag{B10}$$

$$\bar{I}_1 = \mathrm{tr}(\boldsymbol{\tau}), \quad \bar{I}_2 = \frac{\bar{I}_1^2 - \mathrm{tr}(\boldsymbol{\tau}^2)}{2}, \tag{B11}$$

$$\bar{I}_3 = \det(\boldsymbol{\tau}) = \frac{\mathrm{tr}(\boldsymbol{\tau}^3) - \bar{I}_1 \mathrm{tr}(\boldsymbol{\tau}^2) + \bar{I}_2 \bar{I}_1}{3}, \tag{B12}$$

$$\bar{I}_4 = b_1, \quad \bar{I}_6 = b_2, \quad \bar{I}_5 = d_1, \quad \bar{I}_7 = d_2. \tag{B13}$$

Following the work of Shariff (2013), we have

$$I_{an}^2 = \bar{I}_5 - \bar{I}_4^2 - I_{ab}^2, \quad I_{bn}^2 = \bar{I}_7 - \bar{I}_6^2 - I_{ab}^2, \tag{B14}$$

where

$$I_{ab}^2 = \bar{I}_2 + \bar{I}_4\bar{I}_6 + \bar{I}_5 + \bar{I}_7 - \bar{I}_1(\bar{I}_4 + \bar{I}_6). \tag{B15}$$

Shariff (2013) also proved the relation

$$(\bar{I}_3 - \bar{I}_4\bar{I}_6\bar{I}_n + \bar{I}_4 I_{bn}^2 + \bar{I}_6 I_{an}^2 + \bar{I}_n I_{ab}^2)^2 = 4 I_{ab}^2 I_{an}^2 I_{bn}^2\,. \tag{B16}$$

Since $\mathbf{b}$ and $\mathbf{d}$ can be explicitly expressed in terms of the classical invariants, and in view of (B14) and (B15), the relation (B16) proves that only 9 of the 10 classical invariants are independent.

In the case when two or more of the eigenvalues of $\mathbf{C}$ are equal, the number of independent classical invariants is further reduced. For example, consider the case $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, then the matrix $\mathbf{C} = \lambda\mathbf{I}$ and we have

$$\mathrm{tr}(\mathbf{C}) = 3\lambda, \quad \mathrm{tr}(\mathbf{C}^2) = 3\lambda^2, \quad \mathrm{tr}(\mathbf{C}^3) = 3\lambda^3, \quad \mathrm{tr}(\mathbf{C}\boldsymbol{\tau}) = \lambda \mathrm{tr}(\boldsymbol{\tau}), \tag{B17}$$

$$\mathrm{tr}(\mathbf{C}^2\boldsymbol{\tau}) = \lambda^2 \mathrm{tr}(\boldsymbol{\tau}), \quad \mathrm{tr}(\mathbf{C}\boldsymbol{\tau}^2) = \lambda \mathrm{tr}(\boldsymbol{\tau}^2), \quad \mathrm{tr}(\mathbf{C}^2\boldsymbol{\tau}^2) = \lambda^2 \mathrm{tr}(\boldsymbol{\tau}^2). \tag{B18}$$

It is clear from (B17) and (B18) that only four classical invariants,

$$\mathrm{tr}(\mathbf{C}), \quad \mathrm{tr}(\boldsymbol{\tau}), \quad \mathrm{tr}(\boldsymbol{\tau}^2), \quad \mathrm{tr}(\boldsymbol{\tau}^3) \tag{B19}$$

out of the total 10 are independent. Note that in this case, the eigenvectors of $\mathbf{C}$ are not unique, hence, we can let $\mathbf{e}_i = \mathbf{d}_i$, where $\mathbf{d}_i$ is an eigenvector of $\boldsymbol{\tau}$ and we have $\zeta_i = \tau_i$. In view of this, they are only 6 independent spectral invariants in (3.14).

The number of independent invariants for the case when 2 of the eigenvalues can also be obtained but we shall not derive them here.