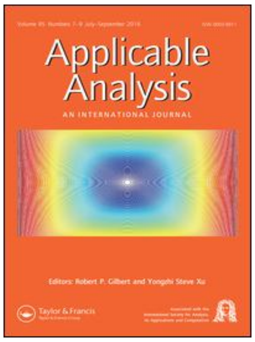

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Method of asymptotic partial decomposition of domain for multistructures

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ABSTRACT

Method of asymptotic partial decomposition of a domain (MAPDD) proposed and justified earlier for thin domains (rod structures, tube structures) is generalized and justified for the multistructures, i.e. domains consisting of a set of thin cylinders connecting some massive 3D domains. In the present paper, the Dirichlet boundary value problem for the steady-state Stokes equations is considered. This problem is reduced to the Stokes equations in the massive domains coupled with the Poiseuille-type flows within the thin cylinders at some distance from the bases (the MAPDD approximation problem). The high-order estimates for the difference of the exact solution to the initial problem and the solution to the MAPDD approximation problem is proved.

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1. Introduction

The Stokes equations in thin tube structures is the simpliest linear model for the viscous flow in pipelines or blood vessels. The method of asymptotic partial decomposition of a domain (MAPDD) allows to reduce essentially the computer resources needed for the numerical solution of such problems. This method combines the three-dimensional description in some neighborhoods of bifurcations and the one-dimensional description out of these small subdomains and it prescribes some special junction conditions at the interface between these 3D and 1D submodels (see [1–3]). This method was generalized for the case of the non-steady Navier–Stokes equations [4]. It is justified via a construction of an asymptotic expansion of the solution. Namely, first for the exact solution and an asymptotic expansion an estimate is proved, then the same estimate is proved for the difference of an asymptotic expansion and the solution of the "partially decomposed" problem, obtained by the MAPDD, and finally, the triangle inequality establishes the estimate for the difference between the exact solution of the original problem and the solution of the MAPDD problem.

However, this justification fails in the case when an asymptotic expansion is not constructed or when such expansion is too bulky. In particular, it is the case of the flows in thin tube structures connected with some massive domains (reservoirs) although this case may have numerous applications. That is why in the present paper a new approach to the justification of the MAPDD is proposed. It doesn't need a complete asymptotic expansion and uses the theorems on the stabilization of a solution of the Stokes equations in a semi-infinite cylinder (see [4–8]) and the a priori estimates for the original problem. This approach is demonstrated below for the Stokes equations with the

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no-slip boundary conditions set in a domain, consisting of several massive domains connected by a tree-like set of thin cylinders (thin tube structure). The ratio of the diameters of cylinders to their heights is a small parameter ε. The Stokes equations in such multistructure correspond to the variational formulation for the unknown velocity sought in the subspace of the Sobolev space H_0^1 of divergence-free vector-valued functions. The MAPDD approximates this problem by its projection on the subspace of vector-valued functions having the Poiseuille flow shape inside the cylinders at the distance greater than δ from the bases of the cylinders. The main theorem says that for any natural number J and for $\delta = \text{const}J\varepsilon|\ln\varepsilon|$ the solution $u_{\varepsilon,\delta}$ of this projected problem is ε^J – close to exact solution u_ε of the original problem: $\|u_{\varepsilon,\delta} - u_\varepsilon\|_{H^1} = O(\varepsilon^J)$.

The main idea of the proof uses the exponential stabilization of a solution of the Stokes equations in a semi-infinite cylinder with no-slip conditions at the boundary to some Poiseuille's flow. First, the proof is given for a simplest multistructure, consisting of two massive domains connected by a thin cylinder. Namely, we consider the traces of the exact solution at the bases of the cylinder and replace at the distance greater than δ the exact solution by a Poiseuille flow with the same flow rate as at the bases. Then we prove that modified in this way exact solution still satisfies the original problem as well as the projected MAPDD problem with small residuals in the right-hand sides. Due to the stabilization theorems these residuals are of the order ε^J if δ is of the order $\text{const}J\varepsilon|\ln\varepsilon|$ and it finalizes the proof. This approach is further generalized for the Stokes equations set in the general multistructure consisting of several massive domains and a thin tube structure the same as in [4]. Mention that the notion of microstructure was introduced by Ciarlet [9] and an asymptotic analysis for various partial differential equations set in these domains was developed as well in [10–13].

2. The Dirichlet's problem for the Stokes equations set in a multistructure

Let us define a multistructure as a union of several "massive" domains and a thin tube structure defined in [4,14,15].

Let $O_1, O_2, \dots, O_N$ be N different points in $\mathbb{R}^n$, $n = 2, 3$, and $e_1, e_2, \dots, e_M$ be M closed segments each connecting two of these points (i.e. each $e_j = \overline{O_{i_j}O_{k_j}}$, where $i_j, k_j \in \{1, \dots, N\}, i_j \neq k_j$). All points O_i are supposed to be the ends of some segments e_j. The segments e_j are called edges of the graph. The points O_i are called nodes. Any two edges e_j and e_i, $i \neq j$, can intersect only at the common node. A node is called vertex if it is an end point of only one edge.

Denote $\mathcal{B} = \bigcup_{j=1}^M e_j$ the union of edges and assume that $\mathcal{B}$ is a connected set. The graph $\mathcal{G}$ is defined as the collection of nodes and edges.

Let e be some edge, $e = \overline{O_iO_j}$. Consider two Cartesian coordinate systems in $\mathbb{R}^n$. The first one has the origin in O_i and the axis $O_ix_1^{(e)}$ has the direction of the ray $[O_iO_j)$; the second one has the origin in O_j and the opposite direction, i.e. $O_j\tilde{x}_1^{(e)}$ is directed over the ray $[O_jO_i)$.

With every edge e_j we associate a bounded domain $\sigma_j \subset \mathbb{R}^{n-1}$ having a Lipschitz boundary $\partial\sigma^j, j = 1, \dots, M$. For every edge $e_j = e$ and associated $\sigma_j = \sigma^{(e)}$ we denote by $B_\varepsilon^{(e)}$ the cylinder

$$B_\varepsilon^{(e)} = \left\{ x^{(e)} \in \mathbb{R}^n : x_1^{(e)} \in (0, |e|), \frac{x^{(e)\,\prime}}{\varepsilon} \in \sigma^{(e)} \right\},$$

where $x^{(e)\prime} = (x_2^{(e)}, \dots, x_n^{(e)})$, $|e|$ is the length of the edge e and $\varepsilon > 0$ is a small parameter. Notice that the edges e_j and Cartesian coordinates of nodes and vertices O_j, as well as the domains σ_j, do not depend on ε.

Let $\omega^1, \dots, \omega^N$ be bounded independent of ε domains in $\mathbb{R}^n$ with Lipschitz boundaries $\partial\omega^j$; introduce the nodal domains $\omega_\varepsilon^j = \left\{ x \in \mathbb{R}^n : \dfrac{x - O_j}{\varepsilon} \in \omega^j \right\}$.

By a tube structure we call the following domain

$$B^{\varepsilon} = \left(\bigcup_{j=1}^{M} B_{\varepsilon}^{(e_j)}\right) \bigcup \left(\bigcup_{j=1}^{N} \omega_{\varepsilon}^{j}\right).$$

Assume that the bounded domains $G_1, \ldots, G_s$ with Lipschitz boundaries are such that $\bar{G}_i \cap \bar{G}_j,\ j \neq i$, each $\bar{G}_j$ has an intersection with the set $\mathcal{B}$ at some subset of the nodes: $O_i,\quad i \in \mathcal{M}_j$, where $\mathcal{M}_j$ are subsets of $\{1, \ldots, N\}$. Denote $G = \cup_{i=1}^{s} G_i$. Assume that the union $G \cup B_\varepsilon$ is a bounded domain (connected open set) with a C^2-smooth boundary. Assume that the vector-valued function $\mathbf{f}$ inependent of ε is defined on G (as an element of $L^2(G)$) and extended to the whole domain $G \cup B_\varepsilon$ by zero values, so that $\mathbf{f} \in \mathbf{L}^2(G \cup B_\varepsilon)$. It vanishes out of G.

Consider the Dirichlet's boundary value problem for the stationary Stokes equation:

$$\begin{cases} -\nu\Delta \mathbf{u}_\varepsilon + \nabla p_\varepsilon = \mathbf{f}(x),\ x \in G \cup B_\varepsilon\ , \\ \mathrm{div}\mathbf{u}_\varepsilon = 0,\ x \in G \cup B_\varepsilon\ , \\ \mathbf{u}_\varepsilon = 0,\ x \in \partial(G \cup B_\varepsilon), \end{cases} \tag{1}$$

Introduce the space $\mathbf{H}^1_{div0}(G \cup B_\varepsilon)$ the space of vector valued functions

$$\mathbf{H}^1_{div0}(G \cup B_\varepsilon) = \left\{\mathbf{v} \in \mathbf{H}^1_0(G \cup B_\varepsilon) | div\mathbf{v} = 0\right\}.$$

The variational formulation is: to find a vector-valued function $\mathbf{u}_\varepsilon \in \mathbf{H}^1_{div0}(G \cup B_\varepsilon)$ such that for any test function $\mathbf{v} \in \mathbf{H}^1_{div0}(G \cup B_\varepsilon)$

$$\nu \int_{G\cup B_\varepsilon} \nabla \mathbf{u}_\varepsilon(x) : \nabla \mathbf{v}(x) \mathrm{d}x = \int_{G\cup B_\varepsilon} \mathbf{f}(x) \cdot \mathbf{v}(x) \mathrm{d}x. \tag{2}$$

It is well known that there exists a unique solution to this problem (see [16]) and that the solution satisfies an a priori estimate (the Poincaré-Friedrichs inequality with a constant independent of ε is proved in a standard way by plunging $G \cup B_\varepsilon$ into a parallelepiped independent of ε and extension of functions by zero):

$$\|\mathbf{u}_\varepsilon\|_{\mathbf{H}^1(G\cup B_\varepsilon)} \leq \bar{C}\|\mathbf{f}\|_{\mathbf{L}^2(G\cup B_\varepsilon)}, \tag{3}$$

where $\bar{C}$ does not depend on $\mathbf{f}$. Denote $C_f = \bar{C}\|\mathbf{f}\|_{\mathbf{L}^2(G\cup B_\varepsilon)}$.

So, the norm $\|\mathbf{u}_\varepsilon\|_{\mathbf{H}^1(G\cup B_\varepsilon)}$ is bounded by a constant independent of ε. Therefore, for any edge e the norm $\|\mathbf{u}_\varepsilon\|_{\mathbf{H}^{1/2}(B_\varepsilon^{(e)} \cap \{x_1^{(e)}=a\})},\ a \in (0, |e|/2)$ is bounded by a constant C_f independent of ε.

Remark: Here, the definition of the norm $\|\mathbf{v}\|_{\mathbf{H}^{1/2}(B_\varepsilon^{(e)} \cap \{x_1^{(e)}=a\})}$, $a \in (0, |e|/2)$ is given as

$$\inf_{\mathbf{u}\in\mathbf{H}^1(G\cup B_\varepsilon), \mathbf{u}|_{B_\varepsilon^{(e)}\cap\{x_1^{(e)}=a\}}=\mathbf{v}} \|\mathbf{u}\|_{\mathbf{H}^1(B_\varepsilon^{(e)} \cap \{x_1^{(e)} \in (a, a+e_{\min}/2\})},$$

where $e_{\min} = \min_{1\leq j\leq M} |e_j|$.

Let δ be a small positive number much greater than ε. For any edge $e = \overline{O_iO_j}$ of the graph introduce two hyperplanes orthogonal to this edge and crossing it at the distance δ from its ends.

Denote the cross sections of the cylinder $B_\varepsilon^{(e)}$ by these two hyperplanes, respectively, by $S_{i,j}$ (the cross section at the distance δ from O_i), and $S_{j,i}$ (the cross-section at the distance δ from O_j), and denote the part of the cylinder between these two cross sections by $B_{ij}^{dec,\varepsilon}$.

Define the subspace $\mathbf{H}^{1,\delta}_{div0}(G \cup B_\varepsilon)$ of the space $\mathbf{H}^1_{div0}(G \cup B_\varepsilon)$ such that on every truncated cylinder $B_{ij}^{dec,\varepsilon}$ its elements (vector-valued functions) coincide with the Poiseuille flows described in local variables. Namely, if the local variables $x^{(e)}$ for the edge e coinside with the global ones x then the Poiseuille flow is defined as

$$\mathbf{V_P}(x) = const\ (\nu_P(x'/\varepsilon), 0, \dots, 0)^T,$$

where $\nu_P(y)$ is a solution to the Dirichlet's problem for the Poisson equation on $\sigma^{(e)}$:

$$-\nu\Delta\nu_P(y) = 1,\ \ y \in \sigma^{(e)},\ \ \nu_P(y) = 0,\ \ y \in \partial\sigma^{(e)}\ \ . \tag{4}$$

If e has the cosines directors $k_{e1}, \dots, k_{en}$ and the local variables $x^{(e)}$ are related to the global ones by equation $x^{(e)} = x^{(e)}(x)$ then the Poiseuille flow is

$$\mathbf{V_P}(x) = const\ (k_{e1}\nu_P((x^{(e)}(x))'/\varepsilon), \dots, k_{en}\nu_P((x^{(e)}(x))'/\varepsilon))^T,$$

$x' = (x_2, \dots, x_n)$. In the case $const = 1$ denote the Poiseuille flow $\mathbf{V_P^0}$.

The method of asymptotic partial domain decomposition (MAPDD) replaces the problem (1) by its projection on $\mathbf{H}^{1,\delta}_{div0}(G \cup B_\varepsilon)$:

Find $\mathbf{u}_{\varepsilon,\delta} \in \mathbf{H}^{1,\delta}_{div0}(G \cup B_\varepsilon)$, such that for any test function $\mathbf{v} \in \mathbf{H}^{1,\delta}_{div0}(G \cup B_\varepsilon)$ the following integral identity holds:

$$\nu \int_{G\cup B_\varepsilon} \nabla\mathbf{u}_{\varepsilon,\delta}(x) : \nabla\mathbf{v}(x)\mathrm{d}x = \int_{G\cup B_\varepsilon} \mathbf{f}(x)\cdot\mathbf{v}(x)\mathrm{d}x. \tag{5}$$

Applying the Lax-Milgram argument one can prove that there exists a unique solution $\mathbf{u}_{\varepsilon,\delta}$ of the partially decomposed problem.

3. Estimate for the difference between the exact solution and the MAPDD solution

Theorem 1: *Given natural number J there exists a constant C (independent of ε and J) such that if $\delta = CJ\varepsilon|\ln\varepsilon|$ then*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon,\delta}\|_{H^1(G\cup B_\varepsilon)} = O(\varepsilon^J)\ \ . \tag{6}$$

***Proof*:** Consider first a simplified tube structure that is the set G, consisting of two massive domains G_1 and G_2 connected by a thin cylinder $B^{(e)}_\varepsilon = B^\varepsilon_{(0,1)} = (0,1)\times\sigma_\varepsilon$ with two smoothing domains $\omega^j_\varepsilon,\ j = 1,2$. Here $\sigma_\varepsilon = \{x' \in \mathbb{R}^{n-1}|x'/\varepsilon \in \sigma\}$, σ is a bounded domain with Lipschitz boundary in $\mathbb{R}^{n-1}$, G_1 and G_2 are bounded domains with Lipschitz boundary in $\mathbb{R}^n$, $n = 2,3$, $\bar{G_1}\cap\bar{G_2} = \emptyset$. The thin tube structure here has two vertices: $O_1 = (0,\dots,0)$ and $O_2 = (1,0,\dots,0)$, $\mathcal{B} = e = \{x_1 \in (0,1)|x_2 = \dots = x_n = 0\}$; domains $\omega^j_\varepsilon,\ j = 1,2$ are defined as in the previous section. As before, $G = G_1\cup G_2$, $B_\varepsilon = \omega^1_\varepsilon \cup B^{(e)}_\varepsilon \cup \omega^2_\varepsilon$, $\partial(G\cup B_\varepsilon) \in C^2$. Denote $B^\varepsilon_{(a,b)} = (a,b)\times\sigma_\varepsilon$.

Asymptotic approximations of the solution to this problem may be constructed as in [17]. However, an asymptotic expansion contains some polynomially decaying boundary layers and this circumstance makes the asymptotic expansion too bulky. Moreover, it is not applicable in the general case. So, let us justify the MAPDD approach.

Denote $c_P = \int_\sigma \nu_P(y')\mathrm{d}y'$.

For simplicity assume that $(G\cup B_\varepsilon)\cap\{0 < x_1 < 1\} = B^\varepsilon_{(0,1)}$ (this assumption can be easily removed).

Consider the traces

$$\mathbf{v}^0(x') = \mathbf{u}_\varepsilon(0,x'),\ \ \mathbf{v}^1(x') = \mathbf{u}_\varepsilon(1,x')\ \ .$$

Denote

$$c_\varepsilon = \frac{\int_{\sigma_\varepsilon} v^0_1(x')\mathrm{d}x'}{\varepsilon^{n-1}c_P} = \frac{\int_{\sigma_\varepsilon} v^1_1(x')\mathrm{d}x'}{\varepsilon^{n-1}c_P} \tag{7}$$

the normalized flow rate of the velocity in the tube.

Evidently, the difference $\mathbf{u}_\varepsilon - c_\varepsilon\mathbf{V_P^0}$ has the vanishing flux:

$$\int_{\sigma_\varepsilon} (\mathbf{u}_\varepsilon - c_\varepsilon\mathbf{V_P^0})\cdot\mathbf{e}\mathrm{d}x' = 0,$$

where $\mathbf{e} = (1, 0, \dots, 0)$.

The boundedness of the traces $\|\mathbf{u}_\varepsilon\|_{\mathbf{H}^{1/2}(B_\varepsilon \cap \{x_1 = i\})}$, $i = 0, 1$ and the Cauchy-Schwarz inequality yield: $c_\varepsilon = O(\varepsilon^{-\frac{n-1}{2}})$.

Applying the triangle inequality and the mentioned boundedness of the traces we get: for $i = 1, 2$

$$\|\mathbf{u}_\varepsilon - c_\varepsilon \mathbf{V}_\mathbf{P}^\mathbf{0}\|_{\mathbf{H}^{1/2}(B_\varepsilon \cap \{x_1 = i\})} \leq \|\mathbf{u}_\varepsilon\|_{\mathbf{H}^{1/2}(B_\varepsilon \cap \{x_1 = i\})} + |c_\varepsilon| \|\mathbf{V}_\mathbf{P}^\mathbf{0}\|_{\mathbf{H}^{1/2}(\sigma_\varepsilon)} = O(\varepsilon^{-1}) \tag{8}$$

because $\|\mathbf{V}_\mathbf{P}^\mathbf{0}\|_{\mathbf{H}^{1/2}(\sigma_\varepsilon)} = O(\varepsilon^{\frac{n-1}{2}-1})$ and $c_\varepsilon = O(\varepsilon^{-\frac{n-1}{2}})$.

Consider the boundary layer problem in $\Omega_0 = (0, \infty) \times \sigma$:

$$\begin{cases} -\nu \Delta \mathbf{u}_{BL0}(y) + \nabla p_{BL0}(y) = 0, \quad y \in \Omega_0, \\ \operatorname{div} \mathbf{u}_{BL0}(y) = 0, \quad y \in \Omega_0, \\ \mathbf{u}_{BL0}(y) = 0, \quad y \in \partial\Omega_0 \backslash \{y_1 = 0\}, \\ \mathbf{u}_{BL0}(y) = \mathbf{v}^0(\varepsilon y') - c_\varepsilon \mathbf{V}_P^0(y'), \quad y_1 = 0. \end{cases} \tag{9}$$

Applying Theorem A.2 [4] we get:

$$\|\mathbf{u}_{BL0}\|_{W_\alpha^{1,2}(\Omega_0)} \leq c \|\mathbf{v}^0 - c_\varepsilon \mathbf{V}_P^0\|_{H^{\frac{1}{2}}(\sigma)} \quad , \tag{10}$$

where $\alpha > 0$ depends on σ only, and constant c depends on σ and ν; $W_\alpha^{1,2}(\Omega_0)$ is the space of functions of $H^1(\Omega_0)$ having the finite norm

$$\|v\|_{W_\alpha^{1,2}(\Omega_0)} = \|v e^{\alpha |y_1|}\|_{H^1(\Omega_0)} \quad .$$

□

Remark: Here the norm $\|v\|_{H^{\frac{1}{2}}(\sigma)}$ is defined as $\inf_{u \in H^1(\Omega_0), u|_{\{0\} \times \sigma} = v} \|u\|_{H^1(\Omega_0)}$. Evidently, this norm is equivalent (with constants independent of ε) to the norm

$$\inf_{u \in H^1((0,b) \times \sigma), u|_{\{0\} \times \sigma} = v} \|u\|_{H^1((0,b) \times \sigma)}$$

for any $b \geq 1$.

Indeed, there exists a constant C independent of ε and b such that any function $u \in H^1((0,b) \times \sigma)$ can be extended as $\tilde{u} \in H^1((0, b+1) \times \sigma)$ such that $\tilde{u}_{y_1 = b+1} = 0$ and $\|\tilde{u}\|_{H^1((0,b+1) \times \sigma)} \leq C \|u\|_{H^1((0,b) \times \sigma)}$. This extension can be continued "further" on $H^1(\Omega_0)$ as equal to zero out of the cylinder $(0, b+1) \times \sigma$. The proof of this assertion is a direct corollary of an even extension with respect to the plane $y_1 = b$ (see [18], Chapter I) and multiplication of the extended part for all $y_1 > b$ by a cut off function $\zeta(y_1 - b)$, where

$$\zeta(s) = \begin{cases} 1, \ 0 \leq |s| \leq \frac{1}{6} \\ 0, \ \frac{1}{3} \leq |s|, \end{cases}, \zeta \in C^2(\mathbb{R}) \quad . \tag{11}$$

So,

$$\inf_{u \in H^1((0,b) \times \sigma), u|_{\{0\} \times \sigma} = v} \|u\|_{H^1((0,b) \times \sigma)} \leq \inf_{u \in H^1(\Omega_0), u|_{\{0\} \times \sigma} = v} \|u\|_{H^1(\Omega_0)}$$
$$\leq C \inf_{u \in H^1((0,b) \times \sigma), u|_{\{0\} \times \sigma} = v} \|u\|_{H^1((0,b) \times \sigma)}.$$

Consider next a similar boundary layer problem in

$$\Omega_1 = (-\infty, 0) \times \sigma$$

for $\mathbf{u}_{BL1}$:

$$\begin{cases} -\nu\Delta\mathbf{u}_{BL1}(y) + \nabla p_{BL1}(y) = 0, & y \in \Omega_1, \\ \mathrm{div}\mathbf{u}_{BL1}(y) = 0, & y \in \Omega_1, \\ \mathbf{u}_{BL1}(y) = 0, & y \in \partial\Omega_1 \backslash \{y_1 = 0\}, \\ \mathbf{u}_{BL1}(y) = \mathbf{v}^1 - c_\varepsilon \mathbf{V}_P^0, & y_1 = 0. \end{cases} \tag{12}$$

As before we get

$$\|\mathbf{u}_{BL1}\|_{W_\alpha^{1,2}(\Omega_1)} \le c\|\mathbf{v}^1 - c_\varepsilon \mathbf{V}_P^0\|_{H^{\frac{1}{2}}(\sigma)} \quad , \tag{13}$$

(as above $W_\alpha^{1,2}(\Omega_1)$ is the space of functions of $H^1(\Omega_1)$ having the finite norm

$$\|v\|_{W_\alpha^{1,2}(\Omega_1)} = \|ve^{\alpha|y_1|}\|_{H^1(\Omega_1)}) \quad .$$

Taking into account the factor $\varepsilon^{-\frac{n-1}{2}}$ appearing after the change $y = x/\varepsilon$ in the norm L^2 of a function and the factor $\varepsilon^{1-\frac{n-1}{2}}$ in the norm L^2 of its gradient we get the inequality $\|\mathbf{v}^i - c_\varepsilon \mathbf{V}_P^0\|_{H^{\frac{1}{2}}(\sigma)} \le \varepsilon^{-\frac{n-1}{2}} \|\mathbf{v}^i - c_\varepsilon \mathbf{V}_P^0\|_{H^{\frac{1}{2}}(\{0\}\times\sigma_\varepsilon)} \le \bar{c}\varepsilon^{-1-\frac{n-1}{2}}$ where $\bar{c}$ is a constant independent of ε (here we used inequality (8)).

Define an auxiliary function, approximation to the solution in $B^\varepsilon_{(0,1)}$:

$$\mathbf{u}_\varepsilon^a = c_\varepsilon \mathbf{V}_P^0\left(\frac{x'}{\varepsilon}\right) + \mathbf{u}_{BL0}\left(\frac{x}{\varepsilon}\right)\zeta\left(\frac{x_1}{\delta}\right) + \mathbf{u}_{BL1}\left(\frac{x_1-1}{\varepsilon}, \frac{x'}{\varepsilon}\right)\zeta\left(\frac{x_1-1}{\delta}\right) \tag{14}$$

where $\delta = C_J\varepsilon|\ln(\varepsilon)|$, and C_J is independent of ε and will be chosen in such a way that

$$\mathcal{F}_{i,\frac{\delta}{6\varepsilon}} \le c\bar{c}\varepsilon^{J+2} \tag{15}$$

where

$$\mathcal{F}_{i,R} = \|\mathbf{u}_{BLi}(y)\|_{H^1(\Omega_{i,R})},$$

and

$$\Omega_{i,R} = \Omega_i \cap \{|y_1| > R\}.$$

Note that

$$e^{\alpha\frac{\delta}{6\varepsilon}}\mathcal{F}_{i,\frac{\delta}{6\varepsilon}} \le \|e^{\alpha|y_1|}\mathbf{u}_{BLi}(y)\|_{H^1(\Omega_{i,\frac{\delta}{6\varepsilon}})} \le \|e^{\alpha|y_1|}\mathbf{u}_{BLi}(y)\|_{H^1(\Omega_i)},$$

while due to estimates (10) and (13) the last norm is evaluated by $c\bar{c}\varepsilon^{-1-\frac{n-1}{2}}$:

$$\|e^{\alpha|y_1|}\mathbf{u}_{BLi}(y)\|_{H^1(\Omega_i)} \le c\bar{c}\varepsilon^{-1-\frac{n-1}{2}}.$$

Then for $\delta = C_J\varepsilon|\ln(\varepsilon)|$ we get

$$e^{\alpha C_J|\ln(\varepsilon)|/6}\mathcal{F}_{i,\frac{\delta}{6\varepsilon}} \le c\bar{c}\varepsilon^{-1-\frac{n-1}{2}},$$

i.e.

$$\mathcal{F}_{i,\frac{\delta}{6\varepsilon}} \le c\bar{c}e^{\alpha C_J\ln(\varepsilon)/6-1-\frac{n-1}{2}}.$$

Let us take $C_J \ge 6(J + 3 + \frac{n-1}{2})/\alpha$, then

$$\mathcal{F}_{i,\frac{\delta}{6\varepsilon}} \le c\bar{c}\varepsilon^{J+2}$$

and so, making the change $x = \varepsilon y$, we get

$$\|\mathbf{u}_{BL0}\left(\frac{x}{\varepsilon}\right)\zeta\left(\frac{x_1}{\delta}\right)+\mathbf{u}_{BL1}\left(\frac{x_1-1}{\varepsilon},\frac{x'}{\varepsilon}\right)\zeta\left(\frac{x_1-1}{\delta}\right)\|_{H^1\left(B^{\varepsilon}_{(\delta/6,1-\delta/6)}\right)}\leq\bar{c}_1\varepsilon^{J+2+\frac{n-1}{2}-1}=O\left(\varepsilon^{J+1}\right).$$

Here $\bar{c}_1$ is independent of ε and J.

Note that for any $y_1>0$, the integral over a cross section of the first component of $\mathbf{u}_{BL0}$ is equal to zero: $\int_\sigma u_{BL0,1}(y)\mathrm{d}y'=0$ because this integral is independent of y_1 and $\mathbf{u}_{BL0}\in W^{1,2}_{\alpha}(\Omega_0)$. So,

$$\int_{B^{\varepsilon}_{(\delta/6,\delta/3)}}\operatorname{div}\left(\mathbf{u}_{BL0}\left(\frac{x}{\varepsilon}\right)\zeta\left(\frac{x_1}{\delta}\right)\right)\mathrm{d}x=\frac{1}{\delta}\int_{B^{\varepsilon}_{(\delta/6,\delta/3)}}u_{BL0,1}\left(\frac{x}{\varepsilon}\right)\zeta'\left(\frac{x_1}{\delta}\right)\mathrm{d}x=0.$$

Applying the estimates of [19] for thin structures (Lemma 3.1, the change of variables $y'=x'/\varepsilon, y_1=x_1/\delta$ and the change of function $\mathbf{W}'=\varepsilon^{-1}\mathbf{w}', W_1=\delta^{-1}w_1$ as in the proof of Lemma 3.6) we prove that there exists $\mathbf{w}\in\mathbf{H}^1_0(B^{\varepsilon}_{(\delta/6,\delta/3)})$ such that

$$\operatorname{div}\mathbf{w}=-\operatorname{div}\left(\mathbf{u}_{BL0}\left(\frac{x}{\varepsilon}\right)\zeta\left(\frac{x_1}{\delta}\right)\right)$$

i.e.

$$\operatorname{div}\mathbf{w}=-\frac{1}{\delta}u_{BL0,1}\left(\frac{x}{\varepsilon}\right)\zeta'\left(\frac{x_1}{\delta}\right)$$

and

$$\|\mathbf{w}\|_{\mathbf{H}^1\left(B^{\varepsilon}_{(\delta/6,\delta/3)}\right)}=O\left(\varepsilon^J\right).$$

Indeed, for a domain independent of ε and δ, cylinder $B^1_{(1/6,1/3)}$ we get: there exists a vector-valued function $\mathbf{W}\in\mathbf{H}^1_0(B^1_{(1/6,1/3)})$ such that

$$\operatorname{div}\mathbf{W}(y)=\hat{h}(y),$$

where

$$\hat{h}(y)=-\frac{1}{\delta}u_{BL0,1}\left(\frac{\delta}{\varepsilon}y_1,y'\right)\zeta'(y_1)$$

and

$$\|\mathbf{W}(y)\|_{\mathbf{H}^1(B^1_{(1/6,1/3)})}\leq C\|\hat{h}\|_{\mathbf{L}^2(B^1_{(1/6,1/3)})}.$$

Then taking into account the change of variables, we get:

$$\operatorname{div}\mathbf{w}(x)=\hat{h}(x_1/\delta,x'/\varepsilon)$$

and for $\varepsilon,\delta<1$,

$$\begin{aligned}\|\nabla\mathbf{w}\|_{L^2(B^{\varepsilon}_{(\delta/6,\delta/3)})}&\leq\sqrt{\varepsilon^{n-1}\delta}\|\nabla\mathbf{W}\|_{L^2(B^1_{(1/6,1/3)})}\\&\leq C\sqrt{\varepsilon^{n-1}\delta}\|\hat{h}(y)\|_{\mathbf{L}^2(B^1_{(1/6,1/3)})}\\&\leq C\|\hat{h}(x_1/\delta,x'/\varepsilon)\|_{\mathbf{L}^2(B^{\varepsilon}_{(\delta/6,\delta/3)})}.\end{aligned}$$

Applying finally the Poincaré-Friedrichs inequality, we get

$$\|\mathbf{w}\|_{\mathbf{H}^1\left(B^{\varepsilon}_{(\delta/6,\delta/3)}\right)}\leq\frac{c}{\delta}\varepsilon^{J+2}=O\left(\varepsilon^J\right).$$

The same function $\mathbf{w}$ is constructed in the domain $B^{\varepsilon}_{(1-\delta/3,1-\delta/6)}$ and satisfies the following conditions:

$$\mathrm{div}\mathbf{w} = -\mathrm{div}\left(\mathbf{u}_{BL1}\left(\frac{x_1-1}{\varepsilon}, \frac{x'}{\varepsilon}\right)\zeta\left(\frac{x_1-1}{\delta}\right)\right)$$

and

$$\|\mathbf{w}\|_{\mathbf{H}^1\left(B^{\varepsilon}_{(1-\delta/3,1-\delta/6)}\right)} = O\left(\varepsilon^J\right).$$

Consider $\mathbf{u}^{(J)}_{\varepsilon} = \mathbf{u}^a_{\varepsilon} + \mathbf{w}$. Evidently the difference $\mathbf{u}^{(J)}_{\varepsilon} - \mathbf{u}_{\varepsilon}$ satisfies the Stokes equations in the cylinder $B^{\varepsilon}_{(0,1)}$ with no-slip condition on the boundary with a residual of order $O(\varepsilon^J)$ in the following sense:

for any test function $\mathbf{v} \in \mathbf{H}^1_{div0}(B^{\varepsilon}_{(0,1)})$

$$\nu \int_{B^{\varepsilon}_{(0,1)}} \nabla\left(\mathbf{u}^{(J)}_{\varepsilon}(x) - \mathbf{u}_{\varepsilon}\right) : \nabla\mathbf{v}(x)\mathrm{d}x = -\nu\int_{B^{\varepsilon}_{(0,1)}} \nabla\mathbf{r}_{\varepsilon}(x) : \nabla\mathbf{v}(x)\mathrm{d}x \tag{16}$$

where

$$\mathbf{r}_{\varepsilon} = w + \mathbf{u}_{BL0}\left(\frac{x}{\varepsilon}\right)\left(\zeta\left(\frac{x_1}{\delta}\right)-1\right) + \mathbf{u}_{BL1}\left(\frac{x_1-1}{\varepsilon}, \frac{x'}{\varepsilon}\right)\left(\zeta\left(\frac{x_1-1}{\delta}\right)-1\right)$$

and

$$\|\nabla\mathbf{r}_{\varepsilon}\|_{\mathbf{L}^2\left(B^{\varepsilon}_{(0,1)}\right)} = O\left(\varepsilon^J\right).$$

So, applying the a priori estimate, we get: the following inequality holds:

$$\|\mathbf{u}_{\varepsilon} - \mathbf{u}^{(J)}_{\varepsilon}\|_{H^1(B^{\varepsilon}_{(0,1)})} = O\left(\varepsilon^J\right) \ . \tag{17}$$

Then $\mathbf{u}^{(J)}_{\varepsilon} = \mathbf{u}^a_{\varepsilon} + \mathbf{w}$ extended as $\mathbf{u}_{\varepsilon}$ out of the cylinder $B^{\varepsilon}_{(0,1)}$ satisfies the estimate

$$\|\mathbf{u}_{\varepsilon} - \mathbf{u}^{(J)}_{\varepsilon}\|_{H^1(G\cup B_{\varepsilon})} = O\left(\varepsilon^J\right) \tag{18}$$

and the Stokes equations with a residual of order $O(\varepsilon^J)$ in Equation (2) in the following sense:

for any test function $\mathbf{v} \in \mathbf{H}^1_{div0}(G \cup B_{\varepsilon})$

$$\nu\int_{G\cup B_{\varepsilon}} \nabla\mathbf{u}^{(J)}_{\varepsilon}(x) : \nabla\mathbf{v}(x)\mathrm{d}x = \int_{G\cup B_{\varepsilon}} \mathbf{f}(x)\cdot\mathbf{v}(x)\mathrm{d}x - \nu\int_{G\cup B_{\varepsilon}} \nabla\mathbf{r}_{\varepsilon}(x) : \nabla\mathbf{v}(x)\mathrm{d}x \tag{19}$$

where

$$\|\nabla\mathbf{r}_{\varepsilon}\|_{\mathbf{L}^2(G\cup B_{\varepsilon})} = O\left(\varepsilon^J\right).$$

Consider now the projection of problem (1) on the subspace $\mathbf{H}^{1,\delta}_{div0}(G \cup B_{\varepsilon})$. By the Lax-Milgram theorem there exists a unique solution $\mathbf{u}_{\varepsilon,\delta}$ to this projection and $\mathbf{u}^{(J)}_{\varepsilon}$ belongs to the space $\mathbf{H}^{1,\delta}_{div0}(G\cup B_{\varepsilon})$ and satisfies as before (see (19)) its variational formulation with a residual of order $O(\varepsilon^J)$. Then applying an a priori estimate we get:

$$\|\mathbf{u}^{(J)}_{\varepsilon} - \mathbf{u}_{\varepsilon,\delta}\|_{H^1(G\cup B_{\varepsilon})} = O(\varepsilon^J). \tag{20}$$

Estimates (18), (20), imply (6).

In the general case the proof is similar: the traces of $\mathbf{u}_\varepsilon$ on $x_1 = 0$ and $x_1 = 1$ for every edge e are replaced by the traces $x_1^{(e)} = a\varepsilon$ and $x_1^{(e)} = |e| - a\varepsilon$, where a is such that all cross-sections of a cylinder $B_\varepsilon^{(e)}$ between $x_1^{(e)} = a\varepsilon$ and $x_1^{(e)} = |e| - a\varepsilon$ do not contain points of other cylinders nor points of smoothing domains ω_ε^j. The construction of an approximate solution $\mathbf{u}_\varepsilon^{(J)}$ and the derivation of the estimates (18), (20) are provided for every cylinder $B_\varepsilon^{(e)}$ and then for the whole domain $G \cup B_\varepsilon^{(e)}$ we get (6).[20]

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