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# A numerical solution and evaluation of dynamic stiffness of pile groups and comparison to experimental results

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ABSTRACT

A number of solutions and computer programs are already available to determine the dynamic stiffness of complete pile foundations, assuming linear elastic soil behavior and perfect bonding between the piles and the surrounding soil. These are assumptions that would be generally valid for properly designed machine foundations where very small strains should be expected. A number of approximate formulations have also been developed. Among these the most commonly used one is that proposed by Poulos (1971) [1,2] for the static case, computing interaction coefficients between the heads of two piles considered by themselves, then forming a matrix of these coefficients to obtain the interaction between the heads of all the piles in the group. Additional approximations have been suggested, particularly for the computation of the interaction coefficients, using closed form expressions. In this paper, a semi-analytical-semi-numerical formulation has been adopted to calculate the static and dynamic stiffness of pile foundations in the frequency domain, and some approximate expressions are suggested. They are intended for pile groups with pile spacing of the order of two to four diameters, typical range of the modulus of elasticity of the piles over that of the soil between 100 and 1000, and very small amplitude vibrations.



## 1. Introduction

The dynamic stiffness of pile groups was studied by Gomez [3], Kaynia [4], Dai [5,8] and Dai and Roesset [6,7] using an Elasto-Dynamics formulation and assuming linear elastic behavior of piles & soil and perfect bonding between them. They studied groups of 2 by 2, 3 by 3 and 4 by 4 piles, accounting for the complete interaction between all piles and enforcing perfect bonding between the piles and the surrounding soil over the complete lengths of all piles, considering only the pile head interactions. Some other researchers, e.g. Kouroussis [9], instead, employed a three-dimensional finite element method to calculate dynamic stiffness of pile groups in time domain considering nonlinear effect.

In this study, a numerical formulation has been adopted to calculate static and dynamic stiffness of the pile foundations, which is a semi-analytical solution in the frequency domain and also assumes linear elastic behavior and perfect soil-pile bonding, while incorporating the Poulos' approximation instead of complete interaction between all piles. Results show very little differences to those with complete interaction. The analysis consists then of the following steps:

1. Determination of the dynamic stiffness matrix of one cylindrical cavity (to be filled by a pile) in a horizontally layered soil deposit extending to infinity in the horizontal directions for any frequency of interest. This step is carried out using a semi-analytical-semi-numerical formulation developed by Kausel [10]. The formulation uses an analytical solution in the horizontal directions extending to infinity, while the soil deposit is discretized vertically enabling a numerical method. Below the horizontally layered soil deposit, bed rock or very stiff soils are assumed in this study. One can also consider a uniform half space underneath the layered soil deposit with small modifications to the computer program. The dynamic stiffness matrix of the cylindrical cavity is then combined with that of a single pile, which is modeled using beam theory, to obtain the dynamic

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stiffness matrix of the pile-surrounding soil system, $[K_F]$. This matrix would provide the solution to the dynamic analysis of a single pile.

2. Using $[K_F]$ and Poulos' assumption, which considers only one pile at a time and neglects the existence of all other piles, and applying a unit horizontal load at the pile head, one can calculate the horizontal dynamic displacement at the pile head, $u_{1,1}$, and the displacement at the position supposed to be occupied by the other pile, $u_{2,1}$, in the frequency domain, although Poulos' assumption was originally used only for static cases. Using the same approach, one can calculate $u_{1,2}$ and $u_{2,2}$. The resulting expressions for displacements of the two pile heads with forces $P_1$ and $P_2$ applied at each one would be

$$\begin{cases} u_1 = u_{1,1}P_1 + u_{1,2}P_2 = u_{1,1}(P_1 + \alpha_{1,2}P_2) \\ u_2 = u_{2,1}P_1 + u_{2,2}P_2 = u_{2,2}(\alpha_{2,1}P_1 + P_2) \end{cases}$$

$\alpha 1,2 = u1,2/u1,1$ and $\alpha 2,2 = u2,1/u2,2$ are the pile head interaction coefficients. If the heads of the two piles are connected by a rigid cap and a total force $P$ is applied to the cap, writing $\boldsymbol{I} = \pi r^2\{1,1\}^T$, $P = P_1 + P_2 = \boldsymbol{I}^T\{P_1, P_2\}^T$, $\boldsymbol{U} = \{u_1, u_2\}^T$, and defining $\boldsymbol{A} = \begin{Bmatrix} 1 & \alpha_{1,2} \\ \alpha_{2,1} & 1 \end{Bmatrix}$, $\boldsymbol{K} = \begin{Bmatrix} 1/u_{1,1} & 1 \\ 1 & 1/u_{2,2} \end{Bmatrix}$, $\boldsymbol{U} = \boldsymbol{K}^{-1}\boldsymbol{A}\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$ or $\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \boldsymbol{A}^{-1}\boldsymbol{KU}$, and $P = \boldsymbol{I}^T\boldsymbol{A}^{-1}\boldsymbol{KU}$, the dynamic stiffness of the group of two piles is $K_G = \boldsymbol{I}^T\boldsymbol{A}^{-1}\boldsymbol{KI}$. All matrices and vectors are denoted in bold throughout this paper unless notified otherwise. It is also assumed that the two piles have the same horizontal pile head displacements due to the existence of a rigid cap or $u_1 = u_2$, and the cap is fixed against rotation or rocking. This would provide the solution for the case of two piles.

3. For a group of $N$ by $N$ piles considering every combination of two piles $i, j$ from the complete pile foundation, and obtaining the corresponding interaction factors $\alpha_{i,j}$ and $\alpha_{j,i}$, an interaction matrix $A$ of size $N$ by $N$ can be formed in a similar manner. Defining $K_{i,j} = diagonal(1/u_{i,j})$, the pile group stiffness would still be given by $K_G = \boldsymbol{I}^T\boldsymbol{A}^{-1}\boldsymbol{KI}$.

Computer programs were developed implementing the above formulation. Results were then obtained for pile groups of a single pile, 2 by 2, 4 by 4, 6 by 6, 8 by 8 and 10 by 10 piles. The soil used for the study had a shear wave velocity of 100 m/s, a Poisson's ratio of 0.25, a mass density of 2000 kg/m$^3$ and internal (material) damping of 5%. The piles were assumed to have a radius of 0.5 m, pile spacing of 3 m as the base case, a mass density of 2500 kg/m$^3$ and 5% material damping. The modulus of elasticity of the piles was changed to investigate the effect of the $E_P/E_S$ ratio and sensitivity studies were also conducted for pile spacing of 2–4 m. The depth of the soil deposit was assumed to be 50 m for the base case unless specified otherwise. End bearing and floating piles were considered. The end bearing piles had a length of 50 m, the same as the soil deposit, while the floating piles were 25 m long. Sensitivity studies were also conducted for depth of soil deposit, pile spacing, Poisson's ratio and soil material damping.

## 2. Results

The horizontal stiffness of the pile groups was calculated accounting for the full interaction coefficients computed from the elastic analyses. A limited number of field tests have suggested that no interaction takes place beyond a certain distance. As a result, some authors have recommended using a limiting distance of 10 or even 5 diameters, beyond which the interaction between piles is ignored. The differences in results using a limiting distance or not were discussed in an earlier paper (Dai and Roesset [6]) and can be significant. Unfortunately, there is a scarcity of experimental data for very small amplitude vibrations to ascertain which of the two approaches is more realistic. Among the experimental studies, the best one is probably the one carried by Sharnouby and Novak [11]. The results of this study will be compared to theirs.

The horizontal dynamic stiffness of the foundations can be written as

$$K_{dynamic} = K_{real} + iK_{imaginary} = K_{real} + i\Omega C_{eq} = K_{static}\left(k_1 + i\frac{\Omega R_{eq}}{c_S}c_1\right)$$

in which $\Omega$ is the frequency of vibration, $C_{eq}$ is the constant of an equivalent viscous dashpot, $R_{eq} = \sqrt{Ar/\pi} = Ns/\sqrt{\pi}$ is an equivalent radius of the pile group, $Ar = N^2s^2$ is the foundation area as defined by Fig. 1, $s$ is the pile spacing, $c_s$ is the shear wave velocity of the soil deposit, and $k_1$ and $c_1$ are dynamic stiffness coefficients.

The objective of this study was to compute values of the stiffness at zero frequency (static stiffness) and the dynamic coefficients $k_1$ and $c_1$, or the equivalent dashpot, for the frequencies of interest.

### 2.1. Static stiffness

Blaney et al. [12] expressed the lateral static stiffness of a single solid circular pile with the head fixed against rotation as

$$K = \alpha\frac{E_P I_P}{R^3}\left(\frac{E_S}{E_P}\right)^{\beta},$$

where $E_P$, $E_S$ are the Young's moduli of the pile and the soil, respectively, $R$ is the radius of the pile and $I_P$ its moment of inertia. Sanchez Salinero [13] conducted an extensive set of comparisons for the static stiffness of a single pile using the formulations and results presented by Poulos [1,2,14], Kuhlemeyer [15], Novak and Nogami [16] and Blaney et al. [12]. The values of the coefficients varied from 2.38 and 0.80, to 4.6 and 0.83. Using Blaney's formulation [12], Sanchez-Salinero [13] recommended values of $\alpha = 3.34$, $\beta = 0.81$. Results from this study recommend $\alpha = 6.09$ and $\beta = 0.786$. It is worth nothing that static stiffness of a single pile obtained in this study is only a result of Kausel formulation before introducing Poulos' assumption.

The static group factors for end bearing piles with spacing of 3 diameters are presented in Fig. 2. The group factor is defined as the ratio of the group stiffness to that of a single pile multiplied by the

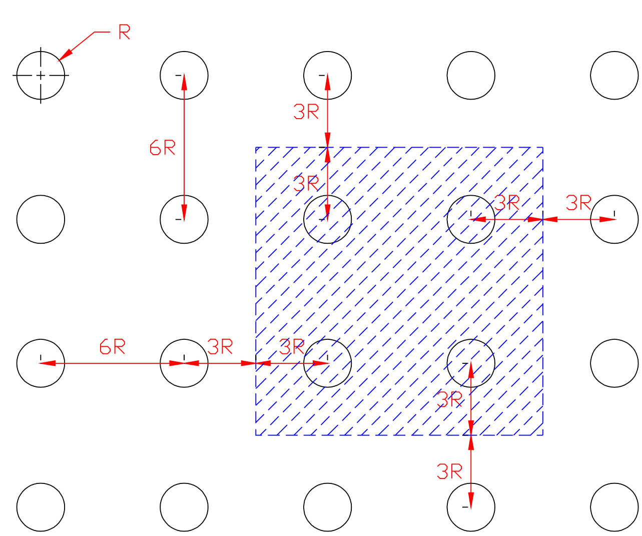


**Fig. 1.** Definition of equivalent area (shaded area) for pile groups.

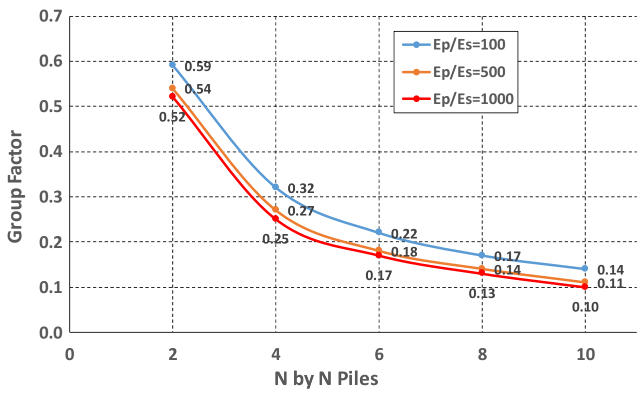


**Fig. 2.** Effect of $E_P/E_S$ on static group factors.

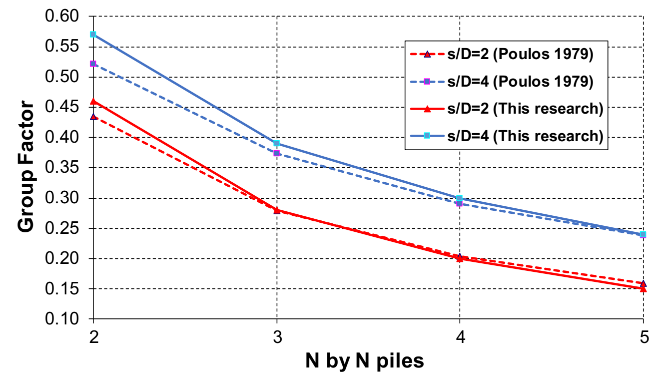


**Fig. 4.** Group factor comparison with poulos 1979.

total number of piles, $GF = \frac{K_G}{n \cdot K_S}$ with $n = N^2$ for the case of $N$ by $N$ piles while $K_G$ and $K_S$ are the static group stiffness and static single pile stiffness, respectively. It should be noticed that in computing the group factors one assumes the same soil properties for the pile group and for a single pile.

The value of the group factor will depend on the ratio of the modulus of elasticity of the pile $E_P$ to that of the soil $E_S$ and on the spacing between piles or its ratio to the pile diameter $s/D$. Ratio of pile length to the depth of soil deposit has little effect on the value of group factors. Fig. 2 shows the effect of the $E_P/E_S$ ratio on the static group factor. From this figure one can conclude that the group factor is approximately inversely proportional to $N$ for an $E_P/E_S$ ratio of 1000, which implies that the static stiffness is proportional to $N$ (the denominator tends to be closer to $N-1$ than to $N$ as the number of pile increases). This is very similar to the case of a rigid mat foundation, whose horizontal stiffness is proportional to the radius and not to the area. With the $E_P/E_S$ ratio decreasing from 1000 to 100, the static group factor will increase by 15–35% depending on the number of piles (the relative increase is larger as the number of piles increases).

The effect of the $s/D$ ratio on the group factor is illustrated in Fig. 3. It can be seen that as the spacing decreases the group factor decreases as well, which implies more interaction effects between piles. For floating piles 25 m long, the group factors and the dynamic stiffness coefficients are almost exactly the same as those of 50 m long end bearing piles.

Fig. 4 shows a comparison of the group factors of this work with those presented by Poulos [17]. To make the results comparable, the depth of the soil deposit and the length of the piles used for this case were 50 m and 25 m, respectively. It can be concluded that, except for a small discrepancy for 2 by 2 pile groups, the static group factors from this research match Poulos' results very well.

From these results the group factor can be expressed approximately as:

$$GF = \left(\frac{0.8}{N}\right)\left(\frac{s}{D}\right)\left(\frac{E_P}{E_S}\right)^{-0.125}$$

and

$$K_{Static} = K_G = GF \cdot K_S = \left(\frac{0.8}{N}\right)\left(\frac{s}{D}\right)\left(\frac{E_P}{E_S}\right)^{-0.125} \cdot 5N^2 G_S\left(\frac{D}{2}\right)\left(\frac{E_P}{E_S}\right)^{0.25}$$

$$\text{or } K_{static} \approx 3.5 G_S R_{eq}\left(\frac{E_P}{E_S}\right)^{0.125} \tag{1}$$

For hollow piles the expression $\frac{E_P}{E_S}$ would have to be replaced in expression (1) and all the following ones by $\left(\frac{E_P}{E_S}\right)_{eff} = \frac{4E_P I_P}{E_S \pi R^4}$, where $R$ is the radius of the individual piles.

To illustrate further the similarity between the horizontal static stiffness of a pile group when accounting for the full interaction coefficients and that of a rigid mat foundation with the same area (as defined in Fig. 1), results were obtained for circular mats with areas corresponding to the 2 by 2, 4 by 4, 6 by 6, 8 by 8 and 10 by 10 pile groups. The mat stiffness was computed using the formula proposed by Elsabee and Morray [18], $K_{mat} = \frac{8G_S R_m}{2-\nu}\left(1 + \frac{2}{3}\frac{B}{R_m}\right)$, where $G_S$ is the shear modulus of the soil deposit, and $R_m$ and $B$ are the radius of the mat and the depth of embedment. As shown in Fig. 5, for small values of $E_P/E_S$, say 100, the static group stiffness is very similar to that of a rigid circular surface mat with the same

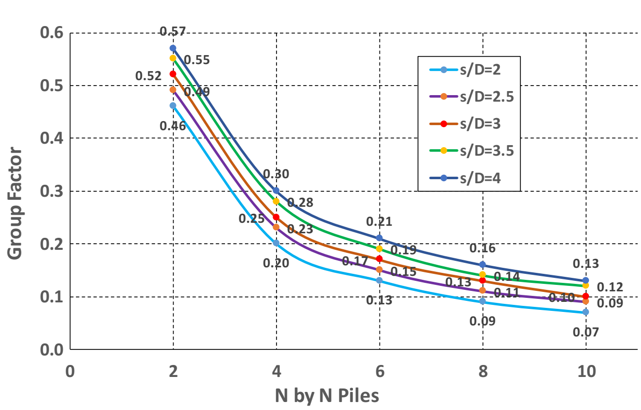


**Fig. 3.** Effect of s/D on static group factor ($E_P = 1000E_S$).

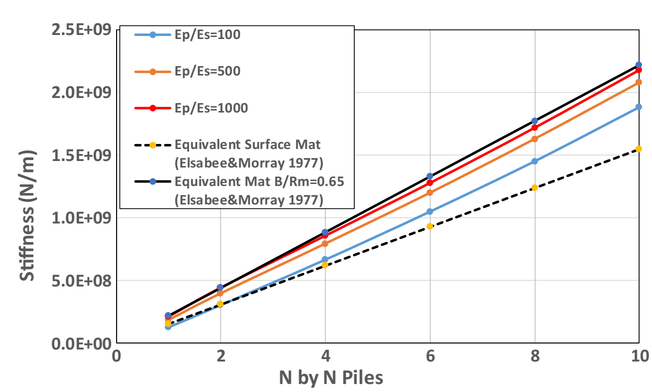


**Fig. 5.** Static stiffness comparison ($E_S$ = constant = 5E7 N/m$^2$, 40 m deep soil deposit).

radius for smaller pile groups, like 2 by 2 and 4 by 4, while the static stiffness of larger pile groups is 10–20% larger than that of an equivalent mat foundation. For $E_P/E_S$ = 500, the static pile group stiffness is about 25–30% higher than that of the rigid mat; and for $E_P/E_S$ = 1000 it is about 30–35% higher. For embedded rigid mats, the static stiffness increases substantially even for a small value of the embedment. If the $B/R_m$ ratio is 0.65 its static stiffness is similar to that of a pile group with $E_P/E_S$ = 1000. A $B/R_m$ ratio of 0.5 seems appropriate for $E_P/E_S$ = 500. From the above, for the pile groups one has approximately $1+\frac{2}{3}\frac{B}{R_{eq}} \approx \frac{1}{\sqrt{2}}\left(\frac{E_P}{E_S}\right)^{0.125}$ leading for a value of $\nu$ between 0.35 and 0.4 to: $K_G \approx 3.5 G_S R_{eq}\left(\frac{E_P}{E_S}\right)^{0.125}$, the same expression obtained above.

### 2.2. Real stiffness coefficient

The real stiffness coefficient $k_1$ is nearly independent of frequency for a single pile over the range of frequencies normally considered, say from 0 to 10 Hz, with a dip at the natural frequency of the soil deposit in shear (0.5 Hz in this case), some small fluctuations around a horizontal line (with a value of 1) for somewhat higher frequencies and a slight decrease for very high frequencies. As the number of piles in the group increases the variation of $k_1$ with frequency becomes more pronounced, as illustrated in Fig. 6. The real stiffness coefficients decrease with increasing frequency exhibiting a parabolic variation, which would suggest that there is a soil mass entrapped between the piles vibrating in-phase with all the piles. The larger the number of piles, the bigger the fluctuations. These fluctuations are associated with the natural frequencies of the soil deposit. For a half space, the variation would be a much smoother second degree parabola. The situation is similar to that of a surface foundation on a soil deposit of finite depth with respect to the fluctuations. To illustrate this point Fig. 7 shows the variation of the real stiffness coefficients $k_1$ for a circular mat foundation with radius $R_m$ on the surface of the same 50 m deep soil deposit considered previously. It can be seen that the fluctuations increase as the radius of the foundation and thus the ratio $R_m/H$ increases. Fig. 8 shows the variation of the real stiffness coefficients for the pile groups when the depth of the stratum is doubled to 100 m (decreasing therefore the $R_m/H$ ratio). The fluctuations decrease in amplitude and the frequencies at which peaks and valleys occur are much smaller than those for the 50 m deep soil layer. The fluctuation also depends on the $E_P/E_S$ and $s/D$ ratio. As $E_P/E_S$ and $s/D$ ratio decrease, the fluctuations become less pronounced, as shown in Figs. 9 and 10, respectively. The amplitude of the fluctuations will also depend of course on the amount of internal soil damping.

Using a second degree parabola to fit each curve in Figs. 6, 8–10, an equivalent mass can be determined to simulate the pile-soil

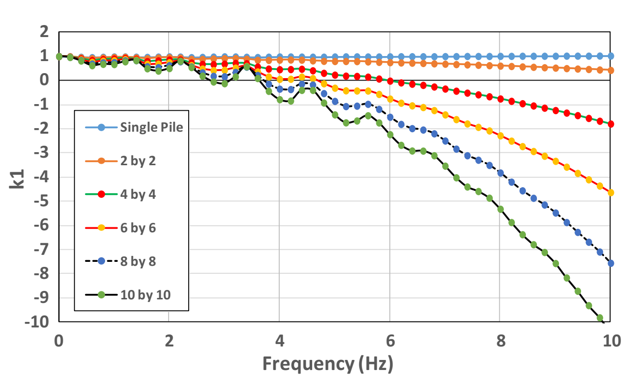


**Fig. 6.** Effect of number of pile on normalized real coefficients $(k_1)$ ($E_P$ = 1000$E_S$ = 5E10 N/m$^2$, 5% material damping, 50 m deep soil deposit, 25 m long piles).

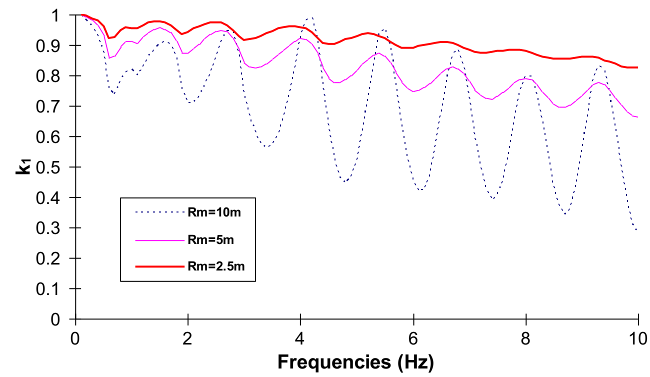


**Fig. 7.** Effect of surface mat foundation radius $R_m$ on real coefficient $(k_1)$ ($E_P$ = 1000$E_S$ = 5E10 N/m$^2$, 5% material damping, 40 m deep soil deposit).

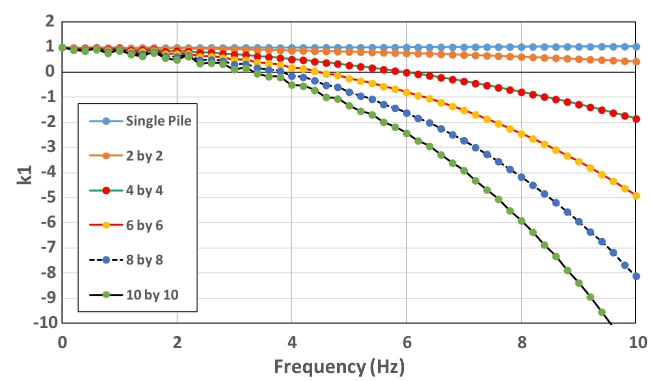


**Fig. 8.** Effect of number of pile on normalized real coefficients $(k_1)$ ($E_P$ = 1000$E_S$ = 5E10 N/m$^2$, 5% material damping, **100 m deep soil deposit**, 25 m long piles).

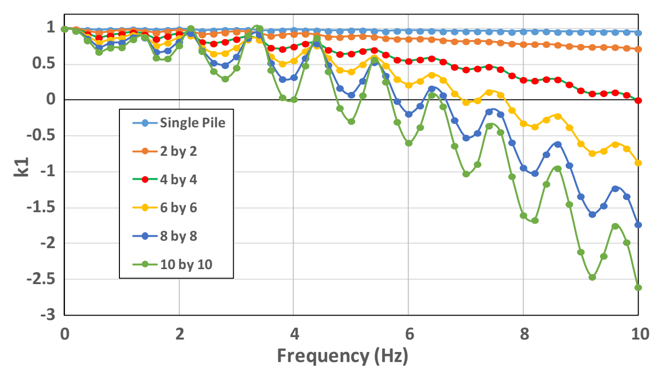


**Fig. 9.** Effect of number of pile on normalized real coefficients $(k_1)$ (**$E_P$ = 100$E_S$ = 5E9 N/m$^2$**, 5% material damping, 50 m deep soil deposit, 25 m long piles).

system in the frequency domain as a single degree of freedom system. For a single degree of freedom system, the dynamic stiffness can be expressed as $K_{dynamic} = k - \Omega^2 m + i\Omega c$, where $k$ is the static stiffness of the spring, $m$ is the mass, $c$ is the viscous damping constant of the dashpot and $\Omega$ is the driving circular frequency. One can fit the real part stiffness coefficients $k_1$ of a pile group by an expression of the form$1 - \Omega^2 b$.. The equivalent mass is then $M_{eq} = K_{static} \cdot b$.

The equivalent mass resulting from a least squares fit varies as a function of the area covered by all the piles as presented in Fig. 13. It is a second degree parabola proportional to $(N-1)^2$, which implies that the equivalent mass is almost proportional to the area

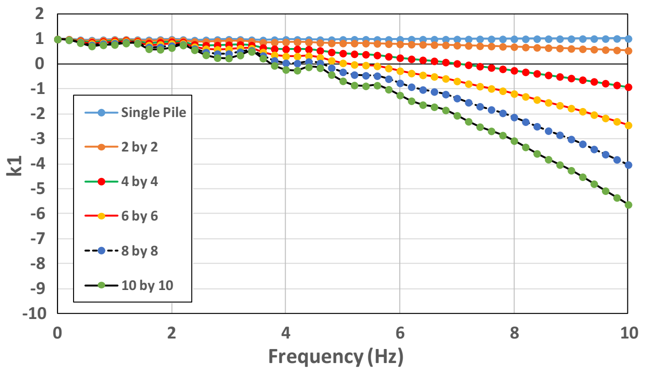


**Fig. 10.** Effect of number of pile on normalized real coefficients ($k_1$) (**s = 2D**, $E_P = 1000E_S$ = 5E10 N/m$^2$, 5% material damping, 50 m deep soil deposit, 25 m long piles).

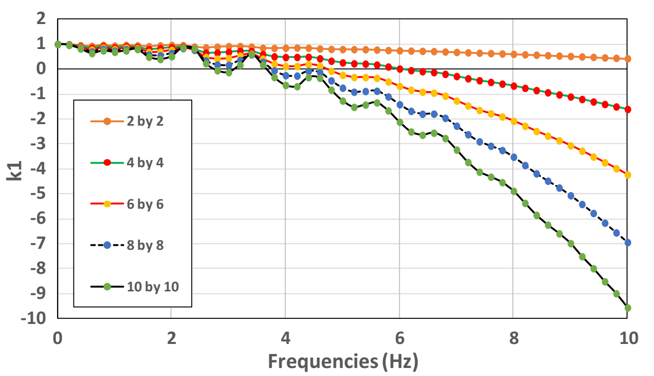


**Fig. 11.** Effect of number of pile on normalized real coefficients ($k_1$) ($E_P = 1000E_S$ = 5E10 N/m$^2$, 5% material damping, 50 m deep soil deposit, 25 m long piles, **Poisson's ratio = 0.35**).

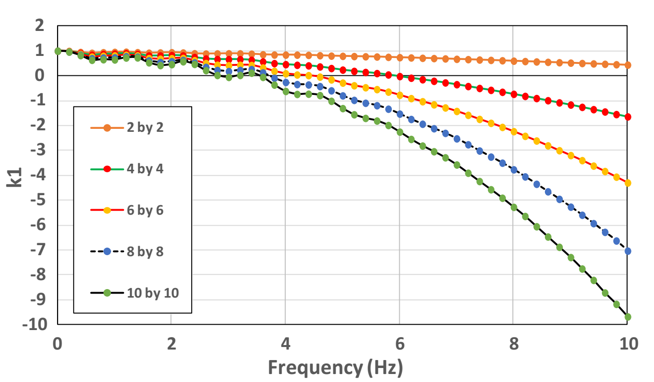


**Fig. 12.** Effect of number of pile on normalized real coefficients ($k_1$) ($E_P = 1000E_S$ = 5E10 N/m$^2$, **10% material damping**, 50 m deep soil deposit, 25 m long piles).

occupied by the cap of the pile group. The equivalent mass for a single pile is almost zero. Its real stiffness is normally assumed to be constant over the range of the frequencies of practical interest. For a pile spacing of 2 diameters, the real coefficients, shown in Fig. 10, follow the same parabolic trend with frequency and the equivalent mass can be calculated in the same way. Equivalent mass also depends on $E_P/E_S$ ratio. Results also show that changes in Poisson's ratio from 0.25 to 0.45 will only decrease the equivalent mass by less than 5%, as Fig. 11 shows the real stiffness coefficients for a Poisson's ratio of 0.35. Fig. 12 shows the real coefficients for 10% soil material damping. It can be seen that when

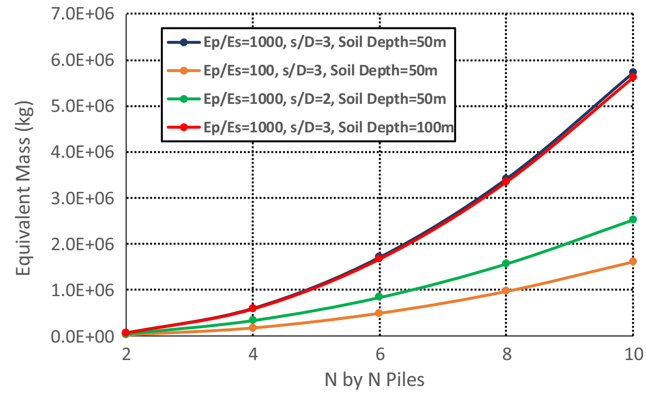


**Fig. 13.** Equivalent mass (5% material damping, 25 m long piles).

soil damping increases, the real coefficients show less fluctuation against exiting frequency but very little effect on the equivalent mass.

### 2.3. Imaginary stiffness coefficients

The imaginary stiffness coefficient $c_1$, representing the radiation damping, after subtracting the effect of the internal soil damping, should be zero below the fundamental shear frequency of the soil layer (0.5 Hz in this case), then jump suddenly and oscillate around a constant value. In reality, if there is some internal soil damping the jump is not sudden but there is a small amount of leakage of energy before the fundamental frequency.

Fig. 14 shows the equivalent dashpot (coefficient $C_{eq}$). It can be seen that as the number of piles increases so do the values of the equivalent dashpot. Fig. 15 shows, however, that the $c_1$ coefficients are about the same for different sizes of the pile groups except for the fluctuations, associated with the natural frequencies of the soil layer, which increase in amplitude with increasing number of piles (as in the case of the real coefficient). The $c_1$ coefficients seem to oscillate approximately around a constant average value. Effect of $E_P/E_S$ ratio on the $c_1$ coefficients is shown in Fig. 16, while effect of $s/D$ in Fig. 17. It can be seen that $c_1$ coefficients are generally increasing when increasing $E_P/E_S$ ratio. It can also be seen that when decreasing $s/D$ ratio it becomes clear that values of $c_1$ coefficients are much larger for smaller pile groups than those for larger ones, as shown in Fig. 17. Fig. 18 shows that the fluctuations associated with the value of $c_1$ coefficients become less prominent when the depth of the soil deposit increases from 50 m to 100 m, the same conclusion obtained for real coefficients. Figs. 19 and 20 show results of sensitivity studies of the effect of Poisson's ratio and soil material damping, respectively, and the same observation

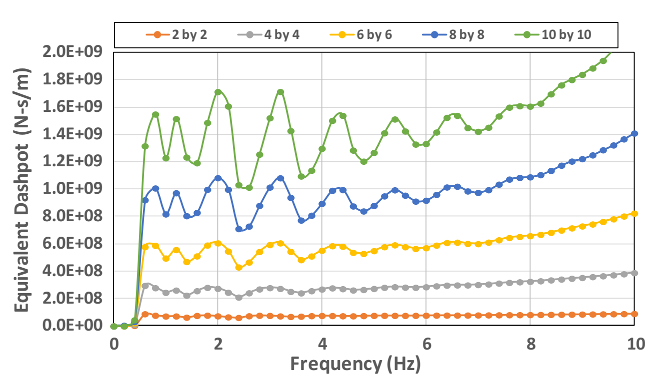


**Fig. 14.** Equivalent dashpot.

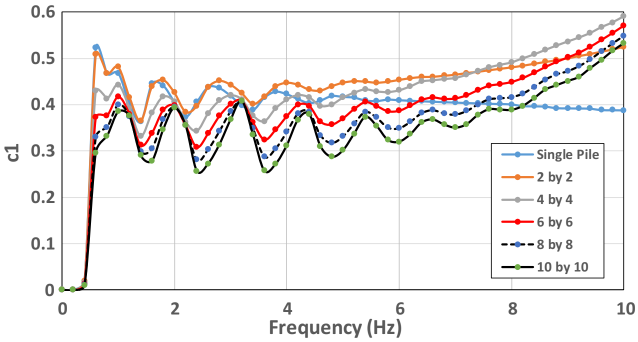


**Fig. 15.** Imaginary stiffness coefficients ($c_1$) ($E_P = 1000E_S = 5E10\ N/m^2$, 5% material damping, 50 m deep soil deposit, 25 m long piles).

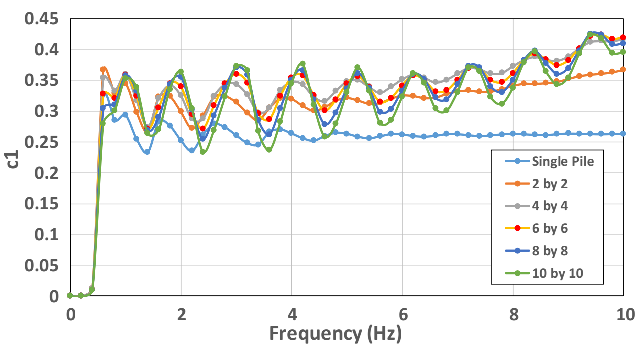


**Fig. 16.** Imaginary stiffness coefficients ($c_1$) (**$E_P = 100E_S = 5E9\ N/m^2$**, 5% material damping, 50 m deep soil deposit, 25 m long piles).

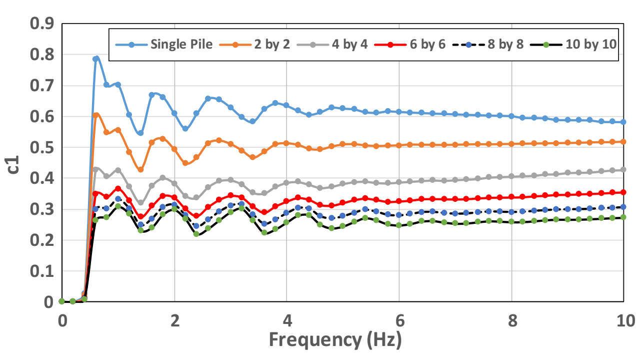


**Fig. 17.** Imaginary stiffness coefficients ($c_1$) (**s = 2D**, $E_P = 1000E_S = 5E10\ N/m^2$, 5% material damping, 50 m deep soil deposit, 25 m long piles).

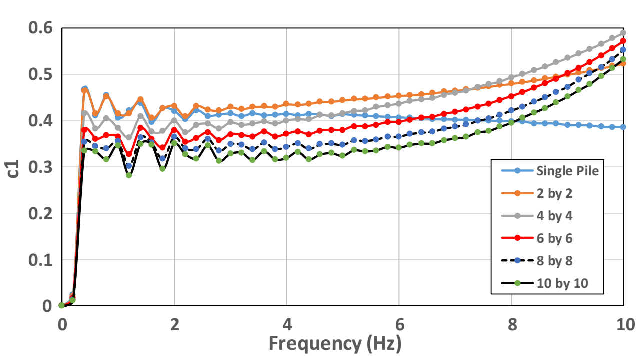


**Fig. 18.** Imaginary stiffness coefficients ($c_1$) ($E_P = 1000E_S = 5E10\ N/m^2$, 5% material damping, **100 m deep soil deposit**, 20 m long piles).

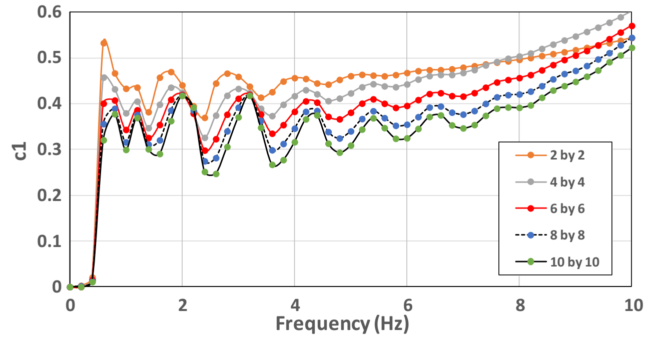


**Fig. 19.** Imaginary stiffness coefficients ($c_1$) ($E_P = 100E_S = 5E9\ N/m^2$, 5% material damping, 50 m deep soil deposit, 25 m long piles, **Poisson's ratio = 0.35**).

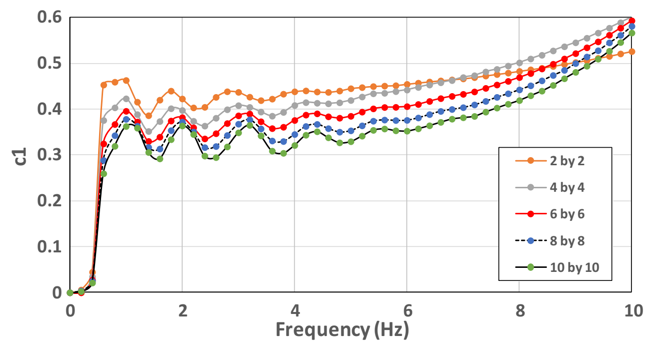


**Fig. 20.** Imaginary stiffness coefficients ($c_1$) ($E_P = 1000E_S = 5E10\ N/m^2$, **10% material damping**, 50 m deep soil deposit, 25 m long piles).

can be made as on the real coefficients that more damping will decrease fluctuations while small changes on Poisson's ratio have little effects. It can be concluded that the $c_1$ coefficients for small pile groups (up to 10 by 10 piles) are in the range of 0.25–0.55, when $E_P/E_S$ ratio is in the range of 100–1000 and $s/D$ ratio is between 2 and 4.

## 3. Comparison with experimental data

As pointed out earlier there is a scarcity of reliable experimental data on the dynamic stiffness of pile groups. Sharnouby and Novak [11] published the results of carefully conducted dynamic tests on a group of 102 small piles and, they compared these data to various numerical predictions. The piles had a radius of 0.0133 m, a Young's modulus of $2 \times 10^{11}\ N/m^2$ and a length of approximately 1 m. The pile spacing was three pile diameters. The soil over the length of the piles, particularly over the top half of the piles, was very soft, with values of $E_P/E_S$ much higher than those that would be expected for typical pile foundations and beyond the range considered in the present study. Even so these experimental results were still compared to the results from this study. In this study, a 10 by 10 pile group was used in the calculation of static pile stiffness and dynamic stiffness coefficients to compare with the results from the pile group of 102 piles originally tested by Sharnouby and Novak.

Sharnouby and Novak reported one set of soil properties (parabolic curve, varying with depth) that they used and another set used supposedly by Waas. The measured value of the static lateral stiffness was $22.8 \times 10^6\ N/m$. Using the first set of soil properties (parabolic curve) the computer program employed for the present study gives a static stiffness of $21.5 \times 10^6\ N/m$. Using the second set (Wass curve) the result is $22.4 \times 10^6\ N/m$. Considering

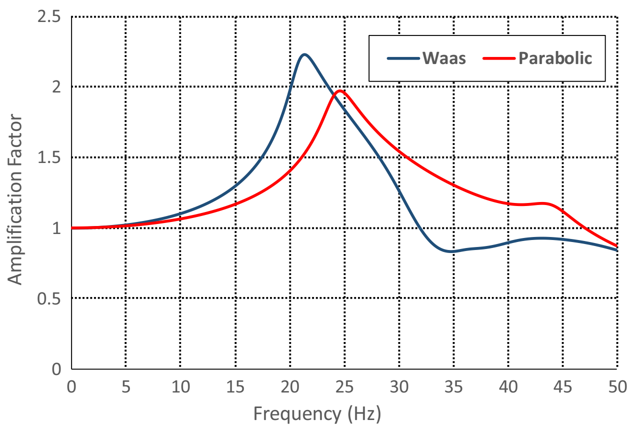


**Fig. 21.** Amplification factor (5% material damping).

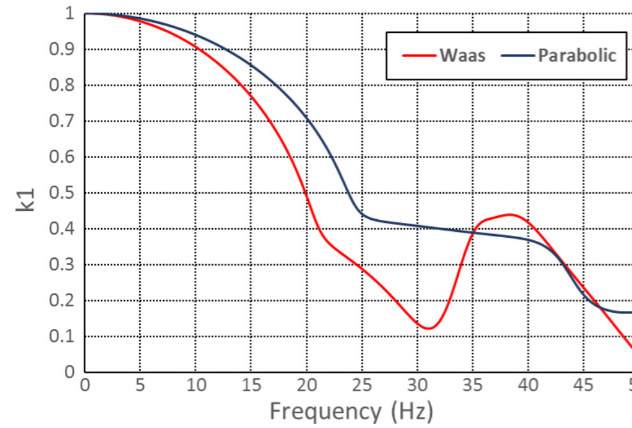


**Fig. 22.** $k_1$ Coefficients for waas and parabolic soil properties.

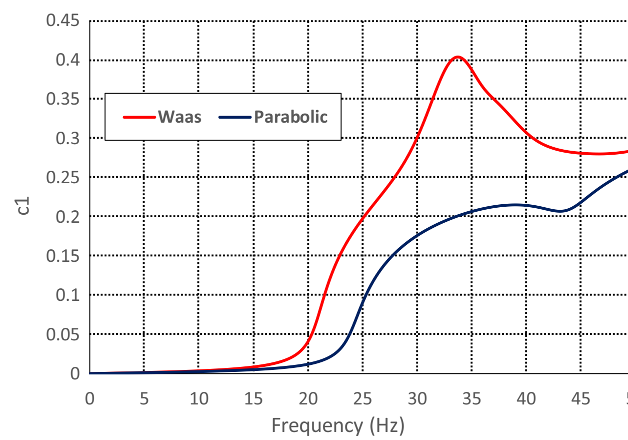


**Fig. 23.** $c_1$ Coefficients for waas and parabolic soil properties.

only 100 piles (10 by 10) were modeled in the computer program from this study while the tested pile group had 102 piles, increasing the computed static stiffness by 2% would result in $22.0 \sim 22.9 \times 10^6$ N/m, which are in very good agreement with the experimental data.

The amplification curve, shown by Novak and Sharnouby [19] for the case of twelve loading plates, exhibits a peak at about 23 Hz with an amplification ratio (ratio of the peak value to the values at high frequencies where the curve is nearly horizontal) of about 2.33 (values of 2.8 at the peak and 1.2 at high frequencies). This would correspond to an effective damping of about 21.4%. Fig. 21 shows the amplification curves (normalized by static amplitude) from the present study with two different soil properties with 5% internal soil damping. It can be seen that the peak is between 22–25 Hz, and the peak amplification factors are about 2.23 and 2.00 for Waas and parabolic soil properties, respectively, which corresponds to an effective damping ratio of about 22% and 25%, respectively.

The formula to calculate of $c_1$ coefficients in this paper was obtained from a uniform soil deposit. When the soil properties vary with depth it is necessary to estimate an equivalent modulus in order to use the formula. The average soil properties from the surface to a depth of 0.5–0.66 radii, which is commonly used for mat foundations, were first used to estimate shear wave velocity. For the present case, since the equivalent radius of the foundation, as defined earlier, is 0.452 m, the resulting equivalent depth is about 0.23 to 0.3 m and calculated equivalent shear wave velocity is between 50 and 60 m/s. Alternatively, if taking a weighted average of the properties over the length of the pile with a quarter sine wave as weighting functions, one would obtain a value of 55 m/s for the first soil profile (used by Sharnouby and Novak) and 57 m/s for the second (used by Waas). The corresponding values of the shear modulus and the Young's modulus are:

$$\text{for } c_S = 55\ \text{m/s}, G_S = 5.00 \times 10^6\ \text{N/m}^2 \text{ and } E_S = 13.00 \times 10^6\ \text{N/m}^2;$$

$$\text{for } c_S = 57\ \text{m/s}, G_S = 5.36 \times 10^6\ \text{N/m}^2 \text{ and } E_S = 13.94 \times 10^6\ \text{N/m}^2.$$

These numbers lead to effective $E_P/E_S$ ratios of about 8900–9500 when accounting for the factor $\frac{4I_P}{\pi R^4}$. Figs. 22 and 23 show the real and imaginary dynamic stiffness coefficients from the present study based on the above estimated soil shear wave velocities.

The predictions of peak frequency, peak amplification and effective damping from the present study are very reasonable and match the experimental data very well. Due to the need to estimate an equivalent shear wave velocity of the soil, to extrapolate the formulae to a value of the effective $E_P/E_S$ well beyond the range considered in the study and limitation of the frequency range, it would not appear that it is appropriate to use least square fit method to calculate equivalent added mass or dashpot for this case.

## 4. Concluding remarks

The formulation and approximate formula presented above provide estimations for the static stiffness and the dynamic coefficients needed to define the dynamic stiffness of regular pile groups as a complex function of frequency in the frequency domain. Alternatively, one can use these values and coefficients ($K_{static}$, $k_1$ and $c_1$) to define an equivalent spring, mass and viscous dashpot to model the foundation in the time domain. In this case, however, some care must be exercised. The use of a traditional single degree of freedom model to reproduce the foundation (the pile group and the surrounding soil) when a structure is placed on top will be valid if there are dynamic forces applied to the structure as in the case of machine foundations. This is indeed the case for which these studies were intended since linear elastic soil behavior was assumed. When dealing with an earthquake excitation the formulation and approximate expressions presented here are questionable because nonlinear effects are being neglected. In addition, one must take into account that the inertia force associated with the equivalent mass is not the result of the absolute acceleration of the mass but of the relative acceleration between the foundation and the free field (the difference between the acceleration at the foundation level when including the structure and the one that would be experienced at the same point without structure).

Consider for instance a simple model of a structure consisting of two masses ($m_1$ on top, $m_2$ at the base) connected by a spring $k$ and a viscous dashpot $c$ and attached to a pile foundation represented by a spring $K_{stat}$, a dashpot $C_{eq}$ and a mass $M_{eq}$. If the earthquake motion that would be experienced on top of the foundation without a structure (but including kinematic interaction effect) is characterized by a displacement $u_G$, a velocity $\dot{u}_G$ and an acceleration $\ddot{u}_G$, using the subscript 1 to refer to the top structure mass, and 2 for the bottom one (connection with the foundation) calling $\boldsymbol{M} = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 + M_{eq} \end{bmatrix}$, $\boldsymbol{C} = \begin{bmatrix} c & -c \\ -c & c + C_{eq} \end{bmatrix}$, $\boldsymbol{K} = \begin{bmatrix} k & -k \\ -k & k + K_{stat} \end{bmatrix}$, $\boldsymbol{U} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$, $\mathbf{U} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$ and, $\dot{\mathbf{U}} = \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix}$ and $\ddot{\mathbf{U}} = \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}$, the equations of motion in the time domain would be $\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{C}\dot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{P}$ with $\boldsymbol{P} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$, $p_1 = 0$ and $p_2 = M_{eq}\ddot{u}_G + C_{eq}\dot{u}_G + K_{stat}u_G$. In terms of relative displacements $\boldsymbol{Y} = \boldsymbol{U} - \boldsymbol{I}u_G$ the vector $\boldsymbol{P}$ would have components $p_1 = -M_1\ddot{u}_G$ and $p_2 = -M_2\ddot{u}_G$.

The formulation present in this study provides a very reliable method to calculation dynamic stiffness of pile foundations, especially for small number of piles (smaller than 10 by 10 pile groups), with reasonable approximations (e.g. Poulos' assumption), while the expressions derived in this paper are approximations intended only for preliminary design estimates in addition to the simplifying assumptions already mentioned. Their validity for large pile groups depends on the validity of using the full interaction coefficients in the linear range, irrespective of the distance between piles. This is a question that needs further research. If one imposes a threshold distance beyond which the interaction is neglected, the static stiffness of the pile groups would be larger than those obtained here and the equivalent mass would be smaller. Research work is also needed on the investigation of nonlinearity of the soil behavior on the dynamic stiffness coefficients of pile groups.

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