Mechanics Research Communications 84 (2017) 60–64

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

Rate type constitutive equations for fiber reinforced nonlinearly vicoelastic solids using spectral invariants

M.H.B.M. Shariff[a], R. Bustamante[b,*], J. Merodio[c]

[a] Department of Applied Mathematics and Science, Khalifa University of Science and Technology, Abu Dhabi, United Arab Emirates
[b] Departamento de Ingeniería Mecánica, Universidad de Chile, Beauchef 851, Santiago Centro, Santiago, Chile
[c] Department of Continuum Mechanics and Structures, Escuela deIngenieros de Caminos, Universidad Politecnica de Madrid, 28040 Madrid, Spain

ARTICLE INFO

Article history:
Received 17 April 2017
Received in revised form 5 June 2017
Accepted 11 June 2017
Available online 16 June 2017

Keywords:
Nonlinear viscoelasticity
Fiber reinforcement
Spectral physical invariants
Deformation indicators
Rate of deformation indicators

ABSTRACT

In this paper we are interested in developing constitutive equations for fiber-reinforced nonlinearly viscoelastic solids. It has been shown that constitutive equations for such bodies can be expressed in terms of a complete minimal set of 18 classical invariants associated with deformation and fiber orientation. In this paper, we give an alternative formulation using a set of spectral invariants. It is shown via the use of spectral invariants that only 11 of the 18 classical invariants are independent. We analyze the spectral invariants for two illustrative deformation gradients: (i) simple tension, and (ii) simple shear.

1. Introduction

In this paper the study initiated in [12], regarding the use of spectral invariants in solid mechanics, is extended to the case of tensor representations for nonlinearly viscoelastic solids (see, for example, [3–5]). Such isotropic bodies are characterized by 10 classical invariants. The presence of a fiber reinforcement, characterized in the reference configuration by a continuous unit vector field, introduces 8 additional classical invariants and generates the anisotropic part of the macroscopic response of the material model, i.e., its transversely isotropic character. The anisotropy of the body, together with finite deformations and viscous effects, has rarely been taken into consideration in the literature. On the other hand the anisotropy of the body has been given considerable attention within the context of nonlinear elasticity (see, for instance, [2] that deal with instabilities associated with composite materials, and [1] for the anisotropy associated with biological materials). Many materials such as biological, geological and synthetic materials exhibit time dependent stress–strain behavior. This is referred to as viscoelastic behavior. In the literature, there exist a few investigations in the context of isotropic nonlinear viscoelastic deformations (see, [14] that reviews the classical nonlinear viscoelastic models within the continuum mechanics framework).

The development of an experimental program with a systematic methodology to determine the numerous material moduli of the model is unlikely. However, having a general model could allow us to propose some simplified models in a rigorous way, which could yet capture the essential features of the original general model. For instance, understanding the physical significance of the two classical fiber reinforced nonlinear elastic invariants has established the connections between the existence of fiber instability and loss of convexity or monotonicity of the mechanical response curves (see [2] and references therein). If we include viscous effects there are additionally six invariants, and it is clear that there is a need to reduce the number of these invariants to one that is amenable to analysis. We dedicate our effort here to that endeavor and, to facilitate our effort, we develop an alternative formulation using spectral invariants to formulate constitutive equations for fiber-reinforced nonlinearly viscoelastic bodies.

Since the 1940's classical invariants have played an important role in the development of constitutive laws in continuum mechanics. Rivlin and others developed trace based (the classical) invariants, because they are convenient and easy to evaluate [13]. However, in many theoretical works, where such invariants are used, there is no interest about fitting with experimental data, the issue of propagation of error, and nor being consistent with physics and the infinitesimal theory. Problems arise because most

* Corresponding author.
E-mail address: rogbusta@ing.uchile.cl (R. Bustamante).

http://dx.doi.org/10.1016/j.mechrescom.2017.06.010
0093-6413/

of the classical invariants do not have an immediate physical meaning and, hence, they are not attractive in seeking to design a rational program of experiments. For example, it is extremely difficult to design an experiment to construct (rigorously) a specific functional form of the strain energy, where it is required to vary a classical invariant while the remaining (classical) invariants are kept fixed [7]. Because of their unclear physical meanings, researchers are not sure which invariants are needed for a given problem, and for simplicity a reduced number of invariants is commonly considered, which may create problems in order to capture the response of the material [11,6]. In addition, it is commonly assumed in the literature that all the invariants in a minimal integrity basis are independent, however, Shariff [8] and Shariff and Bustamante [9] have shown that this is not the case. Furthermore, to evaluate the number of independent classical invariants in a minimal integrity basis is not straightforward [9], due to the difficulty in constructing relations (syzygies) among classical invariants. However, in the case of spectral invariants, the relations among the spectral variables are easily constructed, as shown, for example, by Shariff [10]. Hence, the number of independent spectral invariants can be easily obtained and this is illustrated in this paper for a fiber-reinforced nonlinearly viscoelastic body.

The paper is organized as follows. In Section 2, the relevant kinematic variables and the material model are introduced. The analysis of the invariants is carried out in Section 3 while in Section 4 spectral stress components are given. Finally, in Section 5, spectral results of two specific homogeneous deformations, simple tension and simple shear, are analyzed.

2. Preliminaries

Let $\boldsymbol{X}$ denotes the typical position vector of a material particle in the reference configuration $\mathcal{B}_r$ of the body, and let $\boldsymbol{x}$ denotes the corresponding position vector of the same particle in the deformed configuration $\mathcal{B}_t$ at time t. It follows that there exists a one-to-one mapping $\boldsymbol{\chi}$ such that it assigns to each point $\boldsymbol{X}$ just one point $\boldsymbol{x}$ at each instant t, i.e., $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$ and $\boldsymbol{X} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}, t)$. The deformation gradient tensor is denoted by $\boldsymbol{F}$. The left and right Cauchy–Green stretch tensors, respectively $\boldsymbol{B}$ and $\boldsymbol{C}$, are given by $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$ and $\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}$.

The particle velocity $\boldsymbol{v}$ is defined as $\partial\boldsymbol{\chi}(\boldsymbol{X}, t)/\partial t$. The velocity gradient tensor, denoted $\boldsymbol{L}$, is the gradient of the velocity. It follows immediately that $\dot{\boldsymbol{F}} = \boldsymbol{L}\boldsymbol{F}$, where the superimposed dot designates the time derivative.

We consider a fiber reinforcement defined in the undeformed configuration by the unit vector $\boldsymbol{a}$, which may depend on $\boldsymbol{X}$, i.e. the transverse anisotropy may be with respect to different directions at different points, thus, the material could be inhomogeneous. In order to describe the response of transversely isotropic viscoelastic solids, we need to develop invariants that depend on $\boldsymbol{a}$, $\boldsymbol{C}$ and $\dot{\boldsymbol{C}}$. This was the subject of [4] and here we just summarize the results. Following standard notation in nonlinear elasticity we give the list of classical invariants for a problem, where we have a constitutive equation that depends on $\boldsymbol{C}$, $\dot{\boldsymbol{C}}$ and $\boldsymbol{a}$.

$$I_1 = \mathrm{tr}\boldsymbol{C},\ I_2 = \frac{1}{2}((\mathrm{tr}\boldsymbol{C})^2 - \mathrm{tr}(\boldsymbol{C}^2)),$$

3. Viscoelastic potential

In this paper we only consider a transversely isotropic viscoelastic material with *short-term memory response* and assume there exist a viscoelastic potential

$$W_{(v)} = W_{(v)}(\boldsymbol{C}, \dot{\boldsymbol{C}}, \boldsymbol{a}\otimes\boldsymbol{a}) \tag{11}$$

that is responsible for internal dissipation due to the viscous effects in the sense that

$$\mathrm{tr}\left(\frac{\partial W_{(v)}}{\partial\dot{\boldsymbol{C}}}\dot{\boldsymbol{C}}\right) \geq 0. \tag{12}$$

If $W_{(e)} = W_{(e)}(\boldsymbol{C}, \boldsymbol{a}\otimes\boldsymbol{a})$ is a strain energy function for a transversely isotropic elastic solid, then on removing the internal dissipation part in the Clausius–Duhem inequality we get

$$\begin{aligned}&\frac{\mathrm{tr}(\boldsymbol{S}\dot{\boldsymbol{C}})}{2} - \mathrm{tr}\left(\frac{\partial W_{(v)}}{\partial\dot{\boldsymbol{C}}}\dot{\boldsymbol{C}}\right) - \dot{W}_{(e)}\\&= \frac{1}{2}\mathrm{tr}\left[\left(\boldsymbol{S} - 2\frac{\partial W_{(v)}}{\partial\dot{\boldsymbol{C}}} - 2\frac{\partial W_{(e)}}{\partial\boldsymbol{C}}\right)\dot{\boldsymbol{C}}\right] \geq 0,\end{aligned} \tag{13}$$

where $\boldsymbol{S}$ is the second Piola–Kirchhoff stress tensor. Since $\dot{\boldsymbol{C}}$ is arbitrary in (13), we have

$$\boldsymbol{S} = \boldsymbol{S}_{(e)} + \boldsymbol{S}_{(v)}, \tag{14}$$

where

$$\boldsymbol{S}_{(e)} = 2\frac{\partial W_{(e)}}{\partial\boldsymbol{C}},\ \boldsymbol{S}_{(v)} = 2\frac{\partial W_{(v)}}{\partial\dot{\boldsymbol{C}}}. \tag{15}$$

3.1. Spectral representation

Let us consider the spectral representation

$$\begin{aligned}\boldsymbol{C} &= \sum_{i=1}^{3}\lambda_i^2\boldsymbol{E}_i,\ \dot{\boldsymbol{C}} = \sum_{i=1}^{3}g_i\boldsymbol{D}_i = \sum_{i=1}^{3}2\lambda_i\dot{\lambda}_i\boldsymbol{E}_i\\&\quad + \sum_{i\neq j}^{3}\Omega_{ij}(\lambda_j^2 - \lambda_i^2)\boldsymbol{e}_i\otimes\boldsymbol{e}_j,\end{aligned} \tag{16}$$

where

$$\boldsymbol{E}_i = \boldsymbol{e}_i\otimes\boldsymbol{e}_i,\ \boldsymbol{D}_i = \boldsymbol{d}_i\otimes\boldsymbol{d}_i, \tag{17}$$

and λ_i is an eigenvalue (principal stretch) of the right stretch tensor $\boldsymbol{U}$, $\Omega_{ij} = -\Omega_{ji} = \boldsymbol{e}_i\bullet\dot{\boldsymbol{e}}_j$, g_i is an eigenvalue of $\dot{\boldsymbol{C}}$, and $\boldsymbol{e}_i$ and $\boldsymbol{d}_i$ are eigenvectors of $\boldsymbol{C}$ and $\dot{\boldsymbol{C}}$, respectively. In view of (16), one can write

$$W_{(e)}(\boldsymbol{C}, \boldsymbol{a}\otimes\boldsymbol{a}) = W_{(E)}(\lambda_{1,2,3}, \boldsymbol{E}_{1,2,3}, \boldsymbol{a}\otimes\boldsymbol{a}) \tag{18}$$

and

$$\begin{aligned}&W_{(v)}(\boldsymbol{C}, \dot{\boldsymbol{C}}, \boldsymbol{a}\otimes\boldsymbol{a})\\&= W_{(V)}(\lambda_{1,2,3}, g_{1,2,3}, \boldsymbol{E}_{1,2,3}, \boldsymbol{D}_{1,2,3}, \boldsymbol{a}\otimes\boldsymbol{a}),\end{aligned} \tag{19}$$

where we use the notation $\lambda_{1,2,3}$, for the set λ_1, λ_2, λ_3, and similarly for the others. Both (18) and (19) must satisfy the symmetry property as described in Shariff [11]. Following the work of Shariff [7] we can express (18) in terms of the spectral invariants

$$W_{(E)}(\lambda_{1,2,3}, \boldsymbol{E}_{1,2,3}, \boldsymbol{a}\otimes\boldsymbol{a}) = W_{(a)}(\lambda_{1,2,3}, \zeta_{1,2,3}), \tag{20}$$

where

$$\zeta_i = (\boldsymbol{a}\bullet\boldsymbol{e}_i)^2 \text{ where } \zeta_3 = 1 - \zeta_1 - \zeta_2. \tag{21}$$

It follows that only five spectral invariants in (20) are independent. Dealing with (19) as a material with four preferred directions $\boldsymbol{d}_{1,2,3}$ and $\boldsymbol{a}$ and following the work of Shariff [10] we can express (19) as

$$\begin{aligned}&W_{(V)}(\lambda_{1,2,3}, g_{1,2,3}, \boldsymbol{E}_{1,2,3}, \boldsymbol{D}_{1,2,3}, \boldsymbol{a}\otimes\boldsymbol{a})\\&= W_{(b)}(\lambda_{1,2,3}, g_{1,2,3}, \zeta_{1,2,3}, \alpha^{(1)}_{1,2,3}, \alpha^{(2)}_{1,2,3}, \alpha^{(3)}_{1,2,3}, \beta_{1,2,3}),\end{aligned} \tag{22}$$

where

$$\alpha_i^{(k)} = (\boldsymbol{e}_i\bullet\boldsymbol{d}_k)^2,\ \ \beta_i = (\boldsymbol{d}_i\bullet\boldsymbol{a})^2. \tag{23}$$

Following the work of Shariff [10] for four preferred directions, and considering that $\boldsymbol{d}_i$ are orthonormal, only 11 of the 21 invariants in (22) are independent. It is obvious that $\lambda_{1,2,3}$ and $g_{1,2,3}$ are independent and hence the remaining 5 independent invariants can be chosen from the remaining spectral invariants. It follows that only 11 of the 18 classical invariants proposed by Merodio [3] (see (Appendix)) are independent. Furthermore, all, except g_1, g_2 and g_3, of the spectral invariants are non-negative. Unlike the classical invariants, it is clear that all of the spectral invariants have an immediate physical interpretation: g_i is the value of $\dot{\boldsymbol{C}}$ in the $\boldsymbol{d}_i$ direction, $\alpha_j^{(i)}$, ζ_i and β_i are the square of the cosine between the corresponding directions and the physical meaning for λ_i is obvious.

4. Spectral stress

In order to obtain expressions for the stress tensor, we need to evaluate the spectral components of $\frac{\partial W_{(a)}}{\partial \boldsymbol{C}}$ and $\frac{\partial W_{(b)}}{\partial \dot{\boldsymbol{C}}}$. Following the work of Shariff [7], we have

$$\frac{\partial W_{(e)}}{\partial \boldsymbol{C}} = \sum_{i,j=1}^{3} \left(\frac{\partial W_{(e)}}{\partial \boldsymbol{C}} \right)_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j, \tag{24}$$

where

$$\left(\frac{\partial W_{(e)}}{\partial \boldsymbol{C}} \right)_{ii} = \frac{1}{2\lambda_i} \frac{\partial W_{(a)}}{\partial \lambda_i} \, (\text{no sum in } i),$$

in which $\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{a}$. It is assumed that $W_{(a)}$ has sufficient regularity to ensure that, as λ_i approaches λ_j, (26) has a limit. Similarly

$$\frac{\partial W_{(v)}}{\partial \dot{\boldsymbol{C}}} = \sum_{i,j=1}^{3} \left(\frac{\partial W_{(v)}}{\partial \dot{\boldsymbol{C}}} \right)_{ij} \boldsymbol{d}_i \otimes \boldsymbol{d}_j, \tag{27}$$

where

$$\left(\frac{\partial W_{(v)}}{\partial \dot{\boldsymbol{C}}} \right)_{ii} = \frac{\partial W_{(b)}}{\partial g_i},$$

It follows that

$$\begin{aligned} \boldsymbol{S} &= 2\sum_{i,j=1}^{3} \left(\frac{\partial W_{(e)}}{\partial \boldsymbol{C}} \right)_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \\ &\quad + 2\sum_{i,j=1}^{3} \left(\frac{\partial W_{(v)}}{\partial \dot{\boldsymbol{C}}} \right)_{ij} \boldsymbol{d}_i \otimes \boldsymbol{d}_j. \end{aligned} \tag{30}$$

5. Pure homogeneous deformations

In this section, we study briefly the expressions for some of the spectral invariants presented in the previous section, for the particular case of some homogeneous deformations for a slab.

5.1. Simple tension of a slab

The simple tension of a slab is defined by

$$\boldsymbol{F} \equiv \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix} \tag{31}$$

with respect to a fixed Cartesian system, where $\lambda > 0$ is the uniaxial stretch that depends on time t, we have:

$$\boldsymbol{C} \equiv \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{pmatrix}, \quad \dot{\boldsymbol{C}} \equiv \begin{pmatrix} 2\lambda\dot{\lambda} & 0 & 0 \\ 0 & -\frac{\dot{\lambda}}{\lambda} & 0 \\ 0 & 0 & -\frac{\dot{\lambda}}{\lambda} \end{pmatrix}, \tag{32}$$

i.e., $g_1 = 2\lambda\dot{\lambda}$, $g_2 = -\frac{\dot{\lambda}}{\lambda}$ and $g_3 = -\frac{\dot{\lambda}}{\lambda}$. From (32), the physical interpretation of the invariant g_i is clear. The non-negative invariants ζ_i, β_i and $\alpha_j^{(i)}$ have a constant value.

5.2. Simple shear of a slab

Here, all components of vectors and tensors are relative to a fixed Cartesian system. Consider a simple shear for a slab, where the deformation gradient is of the form

$$\boldsymbol{F} \equiv \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{33}$$

where $\gamma > 0$ is the amount of shear that depends on time t. We then have,

$$\boldsymbol{C} \equiv \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \dot{\boldsymbol{C}} \equiv \begin{pmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 2\gamma\dot{\gamma} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{34}$$

The spectral invariants of $\boldsymbol{C}$ are obtained using the following methodology. Let θ denote the orientation (in the anticlockwise sense relative to the 1-axis) of the in plane Lagrangean principal axes. The angle θ is restricted accordingly by the following (see [7])

$$\frac{\pi}{4} \le \theta < \frac{\pi}{2}. \tag{35}$$

The principal directions are $\boldsymbol{e}_1 \equiv [c, s, 0]^T$, $\boldsymbol{e}_2 \equiv [-s, c, 0]^T$ and $\boldsymbol{e}_3 \equiv [0, 0, 1]^T$, where $c = \cos(\theta)$ and $s = \sin(\theta)$. It can be easily shown (see [7]) that the principal stretches take the values

$$\begin{aligned} \lambda_1 &= \frac{\gamma + \sqrt{\gamma^2+4}}{2} \ge 1, \\ \lambda_2 &= \frac{1}{\lambda_1} = \frac{\sqrt{\gamma^2+4} - \gamma}{2} \le 1, \\ \lambda_3 &= 1, \end{aligned} \tag{36}$$

and

$$c = \frac{1}{\sqrt{1+\lambda_1^2}}, \quad s = \frac{\lambda_1}{\sqrt{1+\lambda_1^2}}, \quad c^2 - s^2 = -\gamma cs. \tag{37}$$

For any preferred direction $\boldsymbol{a}$, the non-negative invariants ζ_i can be easily obtained using (37).

In the case of $\dot{\boldsymbol{C}}$, we have

$$\begin{aligned} g_1 &= \dot{\gamma}(\gamma + \sqrt{\gamma^2 + \dot{\gamma}^2}), \\ g_2 &= \dot{\gamma}(\gamma - \sqrt{\gamma^2 + \dot{\gamma}^2}), \\ g_3 &= 0. \end{aligned} \tag{38}$$

Let ϕ denote the orientation (in the anticlockwise sense relative to the X_1 axis) of the in plane Lagrangean principal axes. The principal directions are $\boldsymbol{d}_1 \equiv [\hat{c}, \hat{s}, 0]^T$, $\boldsymbol{d}_2 \equiv [-\hat{s}, \hat{c}, 0]^T$ and $\boldsymbol{d}_3 \equiv [0, 0, 1]^T$, where $\hat{c} = \cos(\phi)$ and $\hat{s} = \sin(\phi)$. We then have,

$$
\begin{aligned}
&d_1\hat{c}^2 + d_2\hat{s}^2 = 0,\\
&d_1\hat{s}^2 + d_2\hat{c}^2 = 2\gamma\dot{\gamma},\\
&(d_1 - d_2)\hat{c}\hat{s} = \dot{\gamma}.
\end{aligned} \tag{39}
$$

From (39) we obtain, an interesting result,

$$
\tan(2\phi) = -\frac{1}{\gamma} < 0 \tag{40}
$$

which is independent of $\dot{\gamma}$. In the reference configuration $\gamma = 0$ and taking into account the known condition as $\gamma \to \infty$, we deduce that

$$
\frac{\pi}{4} \leq \phi < \frac{\pi}{2}. \tag{41}
$$

In view of (40) and (41), we obtain

$$
\begin{aligned}
&\hat{c} = \frac{1}{\sqrt{1+\alpha^2}},\\
&\hat{s} = \frac{\alpha}{\sqrt{1+\alpha^2}},\\
&\alpha = \gamma + \sqrt{\gamma^2 + 1}.
\end{aligned} \tag{42}
$$

Interestingly, the above suggests that all of $\boldsymbol{d}_i$ are independent of $\dot{\gamma}$ and hence the non-negative invariants $\alpha_j^{(i)}$ and β_i are independent of $\dot{\gamma}$.

Appendix.

Consider the variables

$$
\bar{\zeta}_i = \boldsymbol{a} \bullet \boldsymbol{e}_i,\ \bar{\beta}_i = \boldsymbol{a} \bullet \boldsymbol{d}_i,\ \bar{\alpha}_i^{(j)} = \boldsymbol{e}_i \bullet \boldsymbol{d}_j. \tag{43}
$$

Following the work of Shariff [10] it can be easily shown that only 5 of these 15 variables are independent. Note that $\zeta_i = \bar{\zeta}_i^2$, $\beta_i = \bar{\beta}_i^2$ and $\alpha_i^{(j)} = (\bar{\alpha}_i^{(j)})^2$. The classical invariants [3,13] for a fiber reinforced viscous solid are:

$$
\begin{aligned}
&I_1 = \mathrm{tr}(\boldsymbol{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,\\
&I_2 = \frac{1}{2}[(\mathrm{tr}\boldsymbol{C})^2 - \mathrm{tr}(\boldsymbol{C}^2)] = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2,\\
&I_3 = \det(\boldsymbol{C}) = (\lambda_1\lambda_2\lambda_3)^2,\\
&I_4 = \boldsymbol{a} \bullet (\boldsymbol{C}\boldsymbol{a}) = \zeta_1\lambda_1^2 + \zeta_2\lambda_2^2 + \zeta_3\lambda_3^2,\\
&I_5 = \boldsymbol{a} \bullet (\boldsymbol{C}^2\boldsymbol{a}) = \zeta_1\lambda_1^4 + \zeta_2\lambda_2^4 + \zeta_3\lambda_3^4,
\end{aligned}
$$

$$
\begin{aligned}
&J_1 = \mathrm{tr}(\dot{\boldsymbol{C}}) = \sum_{i,j=1}^{3} g_i,\\
&J_2 = \mathrm{tr}(\dot{\overline{\boldsymbol{C}^2}}) = \sum_{i=1}^{3} g_i^2,\\
&J_3 = \mathrm{tr}(\dot{\overline{\boldsymbol{C}^3}}) = \sum_{i=1}^{3} g_i^3,\\
&J_4 = \mathrm{tr}(\boldsymbol{C}\dot{\boldsymbol{C}}) = \sum_{i,j=1}^{3} \alpha_i^{(j)}\lambda_i^2 g_j,\\
&J_5 = \mathrm{tr}(\boldsymbol{C}^2\dot{\boldsymbol{C}}) = \sum_{i,j=1}^{3} \alpha_i^{(j)}\lambda_i^4 g_j,\\
&J_6 = \mathrm{tr}(\boldsymbol{C}\dot{\overline{\boldsymbol{C}^2}}) = \sum_{i,j=1}^{3} \alpha_i^{(j)}\lambda_i^2 g_j^2,\\
&J_7 = \mathrm{tr}(\boldsymbol{C}^2\dot{\overline{\boldsymbol{C}^2}}) = \sum_{i,j=1}^{3} \alpha_i^{(j)}\lambda_i^4 g_j^2,\\
&J_8 = \mathrm{tr}(\boldsymbol{a} \bullet (\dot{\boldsymbol{C}}\boldsymbol{a})) = \sum_{i=1}^{3} \beta_i g_i,\\
&J_9 = \mathrm{tr}(\boldsymbol{a} \bullet \dot{\overline{\boldsymbol{C}^2}}\boldsymbol{a}) = \sum_{i=1}^{3} \beta_i g_i^2,\\
&J_{10} = \boldsymbol{a} \bullet (\boldsymbol{C}\dot{\boldsymbol{C}}\boldsymbol{a}) = \sum_{i,j=1}^{3} g_i\lambda_j^2\bar{\zeta}_j\bar{\beta}_i\bar{\alpha}_j^{(i)},\\
&J_{11} = \boldsymbol{a} \bullet (\boldsymbol{C}\dot{\overline{\boldsymbol{C}^2}}\boldsymbol{a}) = \sum_{i,j=1}^{3} g_i^2\lambda_j^2\bar{\zeta}_j\bar{\beta}_i\bar{\alpha}_j^{(i)},\\
&J_{12} = \boldsymbol{a} \bullet (\boldsymbol{C}^2\dot{\boldsymbol{C}}\boldsymbol{a}) = \sum_{i,j=1}^{3} g_i\lambda_j^4\bar{\zeta}_j\bar{\beta}_i\bar{\alpha}_j^{(i)},\\
&J_{13} = \boldsymbol{a} \bullet (\boldsymbol{C}^2\dot{\overline{\boldsymbol{C}^2}}\boldsymbol{a}) = \sum_{i,j=1}^{3} g_i^2\lambda_j^4\bar{\zeta}_j\bar{\beta}_i\bar{\alpha}_j^{(i)}.
\end{aligned}
$$

Since, λ_i and d_i are independent, there are a total of 11 spectral variables that are independent. In turn, since in the above expressions the classical invariants are explicitly expressed in terms of the spectral variables, it follows that only 11 of the 18 classical invariants are independent.

References

[1] A. Fallah, M.T. Ahmadian, K. Firozbakhsh, M.M. Aghdam, Micromechanical modeling of rate-dependent behavior of connective tissues, J. Theor. Biol. 416 (2017) 119–128.

[2] J. Merodio, R.W. Ogden, Remarks on instabilities and ellipticity for a fiber-reinforced compressible nonlinearly elastic solid under plane deformation Q, Appl. Math. 63 (2005) 325–333.

[3] J. Merodio, On constitutive equations for fiber-reinforced nonlinearly viscoelastic solids, Mech. Res. Commun. 33 (2006) 764–770.

[4] J. Merodio, K.R. Rajagopal, On constitutive equations for anisotropic nonlinearly viscoelastic solids, Math. Mech. Solids 12 (2007) 131–147.

[5] J. Merodio, J.M. Goicolea, On thermodynamically consistent constitutive equations for fiber-reinforced nonlinearly viscoelastic solids with application to biomechanics, Mech. Res. Commun. 33 (2007) 561–571.

[6] J.G. Murphy, S. Biwa, The counterintuitive mechanical response in simple tension of arterial models that are separable functions of the I1, I4, I6 invariants, Int. J. Nonlinear Mech. 90 (2017) 72–81.

[7] M.H.B.M. Shariff, Nonlinear transversely isotropic elastic solids: an alternative representation, Q. J. Mech. Appl. Math. 61 (2008) 129–149.

[8] M.H.B.M. Shariff, Nonlinear orthotropic elasticity: only six invariants are independent, J. Elast. 110 (2013) 237–241.

[9] M.H.B.M. Shariff, R. Bustamante, On the independence of strain invariants of two preferred direction nonlinear elasticity, Int. J. Eng. Sci. 97 (2015) 18–25.
[10] M.H.B.M. Shariff, The number of independent invariants of an n-preferred direction anisotropic solids, Math. Mech. Solids (2016), http://dx.doi.org/10.1177/1081286516653196.
[11] M.H.B.M. Shariff, Anisotropic separable free energy functions for elastic and non-elastic solids, Acta Mech. (2016) 1–25.
[12] M.H.B.M. Shariff, R. Bustamante, J. Merodio, On the spectral analysis of residual stress in finite elasticity, IMA J. Appl. Math. 82 (2017) 656–680.
[13] A.J.M. Spencer, Theory of invariants, in: A.C. Eringen (Ed.), Continuum Physics, vol. 1, Academic Press, New York, 1971, pp. 239–353.
[14] A. Wineman, Nonlinear viscoelastic solids a review, Math. Mech. Solids 14 (2009) 300–366.