

ALMOST REDUCIBILITY OF LINEAR DIFFERENCE SYSTEMS FROM A SPECTRAL POINT OF VIEW

ÁLVARO CASTAÑEDA AND GONZALO ROBLEDO

ABSTRACT. We prove that, under some conditions, a linear nonautonomous difference system is Bylov's almost reducible to a diagonal one whose terms are contained in the Sacker and Sell spectrum of the original system.

We also provide an example of the concept of diagonally significant system, recently introduced by Pötzche. This example plays an essential role in the demonstration of our results.

1. INTRODUCTION

Let us consider the non autonomous system of linear difference equations

$$(1.1) \quad x(n+1) = A(n)x(n),$$

where $x(n)$ is a column vector of \mathbb{R}^d and the matrix function $n \mapsto A(n) \in \mathbb{R}^{d \times d}$ satisfies the following properties:

- (P1) $A(n)$ is invertible for any $n \in \mathbb{Z}$,
- (P2) $\sup_{n \in \mathbb{Z}} \|A(n)\| < +\infty$ and $\sup_{n \in \mathbb{Z}} \|A^{-1}(n)\| < +\infty$,

where $\|\cdot\|$ denotes a matrix norm.

The purpose of this article is to study the contractibility or almost reducibility to a diagonal system. Namely, the δ -kinematical similarity of (1.1) to

$$(1.2) \quad y(n+1) = U(n)y(n),$$

where $U(n) = U_D(n)\{I + U_R(n)\}$, where $U_D(n)$ is a diagonal matrix and $U_R(n)$ has some smallness properties which will be explained later.

Definition 1 ([8]). *The system (1.1) is kinematically similar (resp. δ -kinematically similar with a fixed $\delta > 0$) to (1.2) if there exists an invertible transformation $F(n)$ (resp. $F(\delta, n)$) verifying*

$$\sup_{n \in \mathbb{Z}} \|F(n)\| < +\infty \quad \text{and} \quad \sup_{n \in \mathbb{Z}} \|F^{-1}(n)\| < +\infty$$

or respectively

$$\sup_{n \in \mathbb{Z}} \|F(\delta, n)\| < +\infty \quad \text{and} \quad \sup_{n \in \mathbb{Z}} \|F^{-1}(\delta, n)\| < +\infty,$$

such that the change of coordinates $y(n) = F^{-1}(n)x(n)$ (resp. $y(n) = F^{-1}(\delta, n)x(n)$) transforms (1.1) into (1.2).

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The concept of almost reducibility was introduced by Bylov [5] in the continuous context and the following definition is a discrete version.

Definition 2. *The system (1.1) is almost reducible to*

$$y(n+1) = V(n)y(n),$$

if for any $\delta > 0$, is δ -kinematically similar to

$$y(n+1) = V(n)\{I + B(n)\}y(n),$$

with

$$\|B(n)\| \leq \delta \quad \text{for any } n \in \mathbb{Z}.$$

In the case when $V(n)$ is a diagonal matrix it is said that (1.1) is almost reducible to a diagonal system and it was proved in [5] that any continuous linear system satisfy this property and the components of $V(n)$ are real numbers.

The concept of almost reducibility to a diagonal system was rediscovered and improved by F. Lin in [11], who introduces the concept of contractibility. In this paper, we are introducing its discrete version.

Definition 3. *The system (1.1) is contracted to the compact subset $E \subset \mathbb{R}^+$ if is almost reducible to a diagonal system*

$$y(n+1) = \text{Diag}(C_1(n), \dots, C_d(n))y(n),$$

where $C_i(n) \in E$ for any $n \in \mathbb{Z}$.

It is worth to emphasize that while Bylov's result only says that the diagonal components are real numbers, Lin provides explicit localization properties. This arises the following definition.

Definition 4. *The compact set $E \subset \mathbb{R}^+$ is said to be the contractible set of (1.1) if E is the minimal compact set such that the system (1.1) can be contracted.*

In the continuous case, the concept of contractibility has been applied in some results of topological equivalence [12]. The major contribution of Lin's article [11] is to prove that the Sacker and Sell spectrum of $A(n)$ (a formal definition will be given later) is the contractible set of (1.1). To the best of our knowledge, there are no results in the discrete case and the purpose of this article is to obtain conditions for the contractibility of (1.1) by following some lines of Lin's work.

1.1. Notation and terminology. The fundamental matrix of (1.1) is defined by

$$(1.3) \quad X(n) = \begin{cases} A(n-1) \dots A(0) & \text{if } n \geq 1 \\ I & \text{if } n = 0 \\ A^{-1}(n) \dots A^{-1}(-1) & \text{if } n \leq -1. \end{cases}$$

The transition matrix $X(n, k) = X(n)X^{-1}(k)$ is defined as follows:

$$(1.4) \quad X(n, k) = \begin{cases} A(n-1) \dots A(k) & \text{if } n > k \\ I & \text{if } n = k \\ A^{-1}(n) \dots A^{-1}(k-1) & \text{if } n < k. \end{cases}$$

Vector and matrix norms will be respectively denoted by $|\cdot|$ and $\|\cdot\|$. As usual the infinite norm $\|\cdot\|_\infty$ is

$$\|A\|_\infty = \max_{1 \leq i \leq d} \left(\sum_{j=1}^d |a_{ij}| \right).$$

We will also assume the following convention for sums and products [7, Pag.3]:

$$\sum_{j=n_0}^{n_0-1} a_j = 0 \quad \text{and} \quad \prod_{j=n_0}^{n_0-1} a_j = 1.$$

1.2. Outline. The section 2 is devoted to recall the spectral theory for linear difference systems and state the main results. The Section 3 introduces some technical preparatory results. Furthermore, Proposition 4 will provide an example of diagonal significance in the Pötzsche's sense. The main results are proved in the Section 4.

2. MAIN RESULT: ALMOST REDUCIBILITY TO DIAGONAL SYSTEMS AND SPECTRAL THEORY

2.1. Dichotomy and Sacker & Sell's Spectrum.

Definition 5 ([14],[17],[18],[19]). *The system (1.1) has an exponential dichotomy on \mathbb{Z} if there exist numbers $K \geq 1$, $\rho \in (0, 1)$ and a projector $P^2 = P$ such that*

$$(2.1) \quad \begin{cases} \|X(n)PX(k)^{-1}\| \leq K\rho^{n-k} & \text{if } n \geq k, \\ \|X(n)(I-P)X(k)^{-1}\| \leq K\rho^{k-n} & \text{if } n \leq k. \end{cases}$$

Definition 6. *The Sacker–Sell spectrum (also called exponential dichotomy spectrum) of (1.1) is the set $\Sigma(A)$ of $\lambda > 0$ such that the systems*

$$(2.2) \quad x(n+1) = \lambda^{-1}A(n)x(n)$$

have not an exponential dichotomy on \mathbb{Z} .

Remark 1. The fundamental matrix of (2.2) is $X_\lambda(n) = \lambda^{-n}X(n)$. Now, if $\lambda \notin \Sigma(A)$, then there exist numbers $K \geq 1$, $\theta \in (0, 1)$ and a projector $P_\lambda^2 = P_\lambda$ such that $X_\lambda(n)$ satisfies

$$(2.3) \quad \begin{cases} \|X(n)\lambda^{-n}P_\lambda X(k)^{-1}\lambda^k\| \leq K\theta^{n-k} & \text{if } n \geq k, \\ \|X(n)\lambda^{-n}(I-P_\lambda)X(k)^{-1}\lambda^k\| \leq K\theta^{k-n} & \text{if } n \leq k. \end{cases}$$

Proposition 1 (Spectral Theorem). *The Sacker–Sell spectrum $\Sigma(A)$ of (1.1) when $\|A(n)\| < +\infty$ for any $n \in \mathbb{Z}$ is either $\Sigma(A) = \emptyset$ or the union of ℓ closed intervals (called spectral intervals) where $0 < \ell \leq d$:*

$$(2.4) \quad \Sigma(A) = [a_1, b_1] \cup [a_2, b_2] \cup \dots \cup [a_{\ell-1}, b_{\ell-1}] \cup [a_\ell, b_\ell],$$

where $0 < a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_\ell \leq b_\ell$.

Remark 2. If $\lambda \notin \Sigma(A)$, it follow from definition that (2.2) has an exponential dichotomy on \mathbb{Z} with projector P_λ . Nevertheless, it is interesting to note that:

- a) $\text{Rank}(P_\lambda)$ is constant for any $\lambda \in (b_{i-1}, a_i)$ ($i = 2, \dots, \ell$),
- b) If $\lambda_i \in (b_{i-1}, a_i)$ and $\lambda_{i+1} \in (b_i, a_{i+1})$, then $\text{Rank}(P_{\lambda_i}) < \text{Rank}(P_{\lambda_{i+1}})$,

c) $\text{Rank}(P_\lambda) = 0$ for any $\lambda \in (0, a_1)$ and $\text{Rank}(P_\lambda) = d$ for any $\lambda \in (b_\ell, \infty)$.

It is interesting to point out that Siegmund and Aulbach [2],[3] developed an spectral theory directly from (1.1) avoiding the technicalities from linear skew-product flows, which are used in the original work of Sacker and Sell [20]. This approach is widely used in the current research and will simplify our work.

Definition 7. *The system (1.1) has the full spectrum condition if*

$$\Sigma(A) = [a_1, b_1] \cup [a_2, b_2] \cup \dots \cup [a_d, b_d].$$

2.2. Main results. In order to contextualize our main results, let us consider the scalar difference equation studied in [2]:

$$(2.5) \quad x_{n+1} = a_n x_n, \quad \text{with} \quad a_n = \begin{cases} a & \text{if } n \leq -1 \\ b & \text{if } n \geq 0 \end{cases} \quad \text{and} \quad 0 < a \leq b.$$

We claim that (2.5) is contracted to any interval $[c, d]$ with $0 < c \leq a \leq b \leq d$. Indeed, given a fixed $\delta > 0$, we consider

$$F(n, \delta) = \begin{cases} \prod_{j=0}^{n-1} \left(1 + \frac{\delta \cos^2(j)}{j^2+1}\right)^{-1} F_0 & \text{if } n \geq 1 \\ \prod_{j=n}^{-1} \left(1 + \frac{\delta \cos^2(j)}{j^2+1}\right) F_0 & \text{if } n \leq -1, \end{cases}$$

where $F_0 \neq 0$ and we can verify that (2.5) is δ -kinematically similar to

$$y_{n+1} = a_n \{1 + b_n\} y_n \quad \text{with} \quad b_n = \frac{\delta \cos^2(n)}{1 + n^2}.$$

The claim follows since $a_n \in [a, b] \subseteq [c, d]$ and $|b_n| \leq \delta$. Moreover, it is shown in [2] that the linear difference equation (2.5) has $\Sigma(A) = [a, b]$ and we verified that (2.5) can be contracted to any closed interval containing $[a, b]$. In addition, $[a, b]$ is the minimal interval where the system can be contracted. This fact illustrates our main result.

Theorem 1. *If the spectrum $\Sigma(A)$ is a bounded set of \mathbb{R}^+ , then is a contractible set of (1.1).*

Theorem 2. *If (1.1) has the full spectrum condition, then is diagonalizable.*

Remark 3. Note that

- a) A careful reading of the proof of Theorem 1 will show that the coefficient $\delta > 0$ of Definition 3 has an upper bound dependent of $\Sigma(A)$.
- b) In spite that Theorem 2 is not new, we point out the remarkable simplicity of our proof which follows the lines of the proof of Theorem 1.

3. PREPARATORY RESULTS

The kinematically similar between (1.1) and (1.2) will be denoted by $A \simeq U$. Let us recall that kinematical similarity is an equivalence relation and have several properties described in the following lemmatas:

Lemma 1. *If $A \simeq B$, then $\mu A \simeq \mu B$ for any $\mu \in \mathbb{R}^+$.*

Lemma 2. *If $A \simeq B$ and (1.1) has an exponential dichotomy on \mathbb{Z} , then (1.2) has also an exponential dichotomy with the same projector and constant ρ .*

The proof of these results is a straightforward exercise and can be done similarly as in the continuous case.

Lemma 3. *If $A \simeq B$, then $\Sigma(A) = \Sigma(B)$.*

Proof. Firstly, if $A \simeq B$, Lemma 1 implies $\lambda^{-1}A \simeq \lambda^{-1}B$. Secondly if $\lambda \notin \Sigma(A)$, Lemma 2 implies that $\lambda \notin \Sigma(B)$, which is equivalent to $\Sigma(B) \subseteq \Sigma(A)$. The inverse contention can be proved analogously. \square

The following results give us useful properties of the spectrum $\Sigma(A)$:

Lemma 4. *If $\|A(n)\| < \infty$ for any $n \in \mathbb{Z}$ and $\lambda \notin \Sigma(A)$, then*

$$(3.1) \quad \Sigma(\lambda^{-1}A) = \left[\frac{a_1}{\lambda}, \frac{b_1}{\lambda}\right] \cup \left[\frac{a_2}{\lambda}, \frac{b_2}{\lambda}\right] \cup \dots \cup \left[\frac{a_{\ell-1}}{\lambda}, \frac{b_{\ell-1}}{\lambda}\right] \cup \left[\frac{a_{\ell}}{\lambda}, \frac{b_{\ell}}{\lambda}\right]$$

Proof. By definition of spectrum, we know that

$$\Sigma(\lambda^{-1}A) = \left\{ \mu \in \mathbb{R}^+ : x(n+1) = \frac{1}{\mu\lambda}A(n)x(n) \text{ has not an exponential dichotomy on } \mathbb{Z} \right\}$$

Notice that

$$\begin{aligned} \mu \in \Sigma(\lambda^{-1}A) &\Leftrightarrow \frac{1}{\mu\lambda} \in \Sigma(A) \\ &\Leftrightarrow a_j \leq \mu\lambda \leq b_j \text{ for some } j = 1, \dots, \ell \\ &\Leftrightarrow \frac{a_j}{\lambda} \leq \mu \leq \frac{b_j}{\lambda} \end{aligned}$$

and the result follows. \square

Proposition 2. *If $\Sigma(A) \subseteq [a, b]$, and $\lambda > b$ (resp. or $\lambda < a$), the system*

$$x(n+1) = \lambda^{-1}A(n)x(n)$$

has an exponential dichotomy on \mathbb{Z} with projector $P = I$ (resp. with projector $P = 0$) and constants $K = 1$ and $\theta \in (0, 1)$.

Proof. We will consider the case $\lambda > b$, the other one can be proved by the reader in a similar way. By using the discrete Gronwall's inequality [7, Ch.4], it is easy to deduce

$$\|X(n)X^{-1}(k)\| < (1+L)^{n-k} \text{ where } n \geq k \text{ and } L = \sup_{n \in \mathbb{Z}} \|A_n - I\|.$$

Let $h = \max\{L+2, \lambda\}$ and note that $(1+L)/h \in (0, 1)$ and

$$(3.2) \quad \|X(n)\left(\frac{1}{h}\right)^n X^{-1}(k)\left(\frac{1}{h}\right)^{-k}\| < \left(\frac{1+L}{h}\right)^{n-k} \quad n \geq k,$$

which implies that the system

$$x(n+1) = h^{-1}A(n)x(n)$$

has an exponential dichotomy on \mathbb{Z} with projector $P_h = I$. By using Remark 2, we know that (2.2) also has an exponential dichotomy on \mathbb{Z} with $P_\lambda = P_h = I$ for any $\lambda > b$. \square

Remark 4. The result above gives explicit constants for the exponential dichotomy as $K = 1$ and $\theta = (1+L)/h \in (0, 1)$. This fact will be useful in some future steps.

The following result has been proved by Siegmund in [21] in a more general case with conditions less restrictive than **(P1)**–**(P2)**. Moreover, the exponential dichotomy considers variable projectors. Nevertheless, we provide a proof in order to make the article the most self-contained possible.

Proposition 3. *If (1.1) verifies properties **(P1)**–**(P2)** and its spectrum is of type (2.4), then there exist ℓ matrices $B_i(n)$ where $\Sigma(B_i) = [a_i, b_i]$ with $i = 1, \dots, \ell$, such that (1.1) is kinematically similar to the system*

$$(3.3) \quad x(n+1) = \text{Diag}(B_1(n), \dots, B_\ell(n))x(n).$$

Proof. Let us choose $\lambda_\ell \in (b_{\ell-1}, a_\ell)$ then the linear system

$$(3.4) \quad x(n+1) = \lambda_\ell^{-1} A(n)x(n)$$

has an exponential dichotomy with projector P_{λ_ℓ} with $\text{Rank}(P_{\lambda_\ell}) = m_\ell < d$ (see Remark 2 for details).

By using Lemma 1 from [15], we know that (3.4) is kinematically similar to

$$(3.5) \quad y(n+1) = \begin{pmatrix} A_1(n) & 0 \\ 0 & A_2(n) \end{pmatrix} y(n),$$

where $A_1 \in \mathbb{R}^{m_\ell \times m_\ell}$ and $A_2 \in \mathbb{R}^{(d-m_\ell) \times (d-m_\ell)}$. In addition (see *e.g.*, [10, p.281]), the subsystem

$$(3.6) \quad y_1(n+1) = A_1(n)y_1(n)$$

has an exponential dichotomy on \mathbb{Z} with the identity as a projector and

$$(3.7) \quad y_2(n+1) = A_2(n)y_2(n)$$

has an exponential dichotomy on \mathbb{Z} with the null projector.

As a consequence of Lemmas 3 and 4, we know that

$$\Sigma(\lambda_\ell^{-1} A) = \bigcup_{i=1}^{\ell} \left[\frac{a_i}{\lambda_\ell}, \frac{b_i}{\lambda_\ell} \right] = \Sigma(A_1) \cup \Sigma(A_2),$$

where

$$\Sigma(A_1) = \bigcup_{i=1}^{j-1} \left[\frac{a_i}{\lambda_\ell}, \frac{b_i}{\lambda_\ell} \right] \quad \text{and} \quad \Sigma(A_2) = \bigcup_{i=j}^{\ell} \left[\frac{a_i}{\lambda_\ell}, \frac{b_i}{\lambda_\ell} \right]$$

for some $j \in \{1, \dots, \ell\}$.

Now, we will verify that $j = \ell$ and $\Sigma(A_2) = [\frac{a_\ell}{\lambda_\ell}, \frac{b_\ell}{\lambda_\ell}]$. Indeed, let us recall that

$$\Sigma(A_2) = \left\{ \mu \in \mathbb{R}^+ : u(n+1) = \mu^{-1} A_2(n)u(n) \quad \text{has not an exponential dichotomy on } \mathbb{Z} \right\}$$

and note that $1 \notin \Sigma(A_2)$ since (3.7) has an exponential dichotomy on \mathbb{Z} with the null projector. In addition, let us recall that $1 \in (\frac{b_{\ell-1}}{\lambda_\ell}, \frac{a_\ell}{\lambda_\ell})$. Now, if $j < \ell$ and $\mu \in (\frac{b_{\ell-2}}{\lambda_\ell}, \frac{a_{\ell-1}}{\lambda_\ell})$, the system

$$y_2(n+1) = \mu^{-1} A_2(n)y_2(n)$$

will have an exponential dichotomy with a projector Q and statements (a)–(c) from Remark 2 says that $\text{Rank}(Q)$ must be lower than zero, obtaining a contradiction.

As (3.4) and (3.5) are kinematically similar, Lemma 1 implies that (1.1) is kinematically similar to

$$\begin{aligned} y(n+1) &= \begin{pmatrix} \lambda_\ell A_1(n) & 0 \\ 0 & \lambda_\ell A_2(n) \end{pmatrix} y(n) \\ &= \begin{pmatrix} B_0(n) & 0 \\ 0 & B_\ell(n) \end{pmatrix} y(n) \end{aligned}$$

and note that Lemma 4 implies

$$\Sigma(B_0) = \bigcup_{i=1}^{\ell-1} [a_i, b_i] \quad \text{and} \quad \Sigma(B_\ell) = [a_\ell, b_\ell].$$

Now, let us consider the system

$$z(n+1) = B_0(n)z(n)$$

and take $\lambda_{\ell-1} \in (b_{\ell-2}, a_{\ell-1})$. Then, the system

$$(3.8) \quad w(n+1) = \frac{1}{\lambda_{\ell-1}} B_0(n)w(n)$$

has exponential dichotomy with projector $P_{\lambda_{\ell-1}}$ with $\text{Rank}(P_{\lambda_{\ell-1}}) = m_{\ell-1} < m_\ell$.

The system (3.8) can be studied similarly as (3.4) in the paragraphs above and the proof can be achieved recursively. \square

An important matter of spectral theory for differential [4] and difference [19] nonautonomous systems is to determine sufficient conditions ensuring *diagonal significance* in the Pötzche's sense [19], namely that the spectrum $\Sigma(C)$ of an upper triangular system $u(n+1) = C(n)u(n)$ coincides with the union of the diagonal spectra $\Sigma(c_{ii})$. The following result provides an example of diagonal significance and will play a fundamental role in the proof of our main result.

Proposition 4. *Let $C(n) = \{c_{ij}(n)\}$ be a bounded and upper triangular $d \times d$ -matrix function such that $\Sigma(C) = [a, b]$. Then*

$$\Sigma(C) = \bigcup_{i=1}^d \Sigma(c_{ii}).$$

Proof. Firstly, we will prove that $\Sigma(C) \subseteq \bigcup_{i=1}^d \Sigma(c_{ii})$. Let $\lambda \notin \bigcup_{i=1}^d \Sigma(c_{ii})$, then the diagonal system

$$x(n+1) = \lambda^{-1} \text{Diag}(c_{11}(n), \dots, c_{dd}(n))x(n)$$

has an exponential dichotomy. Now, let us consider the upper triangular system

$$(3.9) \quad x(n+1) = \lambda^{-1} C(n)x(n),$$

where

$$(3.10) \quad C^+ = \sup_{n \in \mathbb{Z}} \|C(n)\|.$$

Let us make the change of variables (usually known as β -transformation):

$$y(n) = D_\beta x(n) = \text{Diag}(1, \beta, \dots, \beta^{d-1})^{-1} x(n) \quad \text{where} \quad 0 < \beta < \frac{\delta}{\delta + C^+}$$

and notice that

$$\begin{aligned}
y(n+1) &= D_\beta^{-1}x(n+1) \\
&= D_\beta^{-1} \begin{pmatrix} \frac{1}{\lambda}c_{11}(n) & c_{12}(n) & \cdots & c_{1d}(n) \\ & \ddots & & \vdots \\ & & \ddots & \vdots \\ & & & \frac{1}{\lambda}c_{dd}(n) \end{pmatrix} D_\beta y(n) \\
&= \left[\frac{1}{\lambda} \begin{pmatrix} c_{11}(n) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & c_{dd}(n) \end{pmatrix} + \begin{pmatrix} 0 & \beta c_{12}(n) & \cdots & \beta^{d-1}c_{1d}(n) \\ & \ddots & & \vdots \\ & \ddots & \ddots & \beta c_{d-1d}(n) \\ & & & 0 \end{pmatrix} \right] y(n).
\end{aligned}$$

By using $\|\cdot\|_\infty$ norm combined with $\beta < 1$, we can see that

$$\left\| \begin{pmatrix} 0 & \beta c_{12}(n) & \cdots & \beta^{d-1}c_{1d}(n) \\ & \ddots & & \vdots \\ & \ddots & \ddots & \beta c_{d-1d}(n) \\ & & & 0 \end{pmatrix} \right\|_\infty \leq \delta.$$

Due to roughness results for difference equations [1, Corollary 3] (see also [13, p.276],[16, Proposition 1]), we know that if δ is small enough, the system

$$y(n+1) = \left[\frac{1}{\lambda} \begin{pmatrix} c_{11}(n) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & c_{dd}(n) \end{pmatrix} + \begin{pmatrix} 0 & \beta c_{12}(n) & \cdots & \beta^{d-1}c_{1d}(n) \\ & \ddots & & \vdots \\ & \ddots & \ddots & \beta c_{d-1d}(n) \\ & & & 0 \end{pmatrix} \right] y(n)$$

has an exponential dichotomy. By construction, this system is kinematically similar to (3.9) and Lemma 2 says that (3.9) has also an exponential dichotomy, which implies that $\lambda \notin \Sigma(C)$ and $\Sigma(C) \subseteq \bigcup_{i=1}^d \Sigma(C_{ii})$ follows.

Secondly, we will prove that $\bigcup_{i=1}^d \Sigma(C_{ii}) \subseteq \Sigma(C)$. Let $\lambda \notin \Sigma(C) = [a, b]$ such that $\lambda > b$. By Proposition 2, we have that the system (3.9) has exponential dichotomy with projection $P = I$. That is, the fundamental matrix of (3.9), namely X_λ , satisfies

$$\|X_\lambda(n)X_\lambda^{-1}(k)\|_\infty \leq \rho^{n-k} \quad \text{with } n \geq k.$$

Let us recall that $X_\lambda(n) = X(n)\lambda^{-n}$, where $X(n)$ is the fundamental matrix of the system $x(n+1) = C(n)x(n)$. Now, as $C(n)$ is upper triangular, we can see with the help of (1.4) that

$$X_\lambda(n)X_\lambda^{-1}(k) = \begin{cases} \lambda^{k-n}C(n-1)C(n-2)\cdots C(k) & \text{if } n > k \\ I & \text{if } n = k, \end{cases}$$

which implies that

$$\lambda^{k-n} \prod_{j=k}^{n-1} |c_{ii}(j)| \leq \|X_\lambda(n)X_\lambda^{-1}(k)\|_\infty \leq \rho^{n-k} \quad \text{for any } i = 1, \dots, d,$$

and we conclude that each scalar difference equation $x_i(n+1) = \lambda^{-1}c_{ii}(n)x_i(n)$ ($i = 1, \dots, d$) has an exponential dichotomy with projection 1, which implies that $\lambda \notin \bigcup_{i=1}^n \Sigma(c_{ii})$.

The case $\lambda < a$ can be proved analogously, thus $\bigcup_{i=1}^n \Sigma(c_{ii}) \subseteq \Sigma(C)$ and the Proposition follows. \square

Proposition 5. *If the linear system (1.1) satisfies (P1)–(P2) and can be contracted to a compact set $E \subset (0, +\infty)$, then $\Sigma(A) \subseteq E$.*

Proof. Let us choose $\lambda \notin E$ and notice that the compactness of E allow to define $\alpha = \inf_{x \in E} |\lambda - x| > 0$. By using Definition 4, we have that (1.1) is kinematically similar to

$$y(n+1) = \text{Diag}(C_1(n), \dots, C_d(n))\{I + B(n)\}y(n),$$

where $C_i(n) \in E$ for any $n \in \mathbb{Z}$ and $\sup_{n \in \mathbb{Z}} \|B(n)\| < \delta/\|C\|$. Now, by Lemma 1 we know that (2.2) is δ -kinematically similar to

$$(3.11) \quad y(n+1) = \frac{1}{\lambda} \text{Diag}(C_1(n), \dots, C_d(n))\{I + B(n)\}y(n).$$

Since $C_i(n) \in E$ for any $n \in \mathbb{Z}$ and $i = 1, \dots, d$, without loss of generality, we can assume that

$$\begin{aligned} C_i(n) &< \lambda & \text{if } i = 1, \dots, m \\ C_i(n) &> \lambda & \text{if } i = m+1, \dots, d. \end{aligned}$$

By definition of α , we have that

$$\begin{aligned} C_i(n) &< \lambda - \alpha & \text{if } i = 1, \dots, m \\ C_i(n) &> \lambda + \alpha & \text{if } i = m+1, \dots, d. \end{aligned}$$

We can verify that the system

$$y(n+1) = \frac{1}{\lambda} \text{Diag}(C_1(n), \dots, C_d(n))y(n),$$

has an exponential dichotomy on \mathbb{Z} since

$$\left| \left(\frac{1}{\lambda}\right)^{n-k} \prod_{j=k}^{n-1} C_i(j) \right| \leq \max\left\{ \frac{\lambda-\alpha}{\lambda}, \frac{\lambda}{\lambda+\alpha} \right\}^{n-k} \quad \text{if } n \geq k, \quad i = 1, \dots, m$$

$$\left| \left(\frac{1}{\lambda}\right)^{n-k} \prod_{j=n}^{k-1} C_i^{-1}(j) \right| \leq \max\left\{ \frac{\lambda-\alpha}{\lambda}, \frac{\lambda}{\lambda+\alpha} \right\}^{k-n} \quad \text{if } n \leq k, \quad (i = m+1, \dots, d).$$

By using roughness results, we have that (3.11) has an exponential dichotomy on \mathbb{Z} . Now, due to kinematical similarity and Lemma 2, the system (2.2) has exponential dichotomy. In consequence, $\lambda \notin \Sigma(A)$ and the Proposition follows. \square

4. PROOF OF MAIN RESULTS

4.1. Proof of Theorem 1. The proof will be made in several steps:

Step 1): (1.1) is kinematically similar to an upper triangular system: By Proposition 1 and hypothesis, there exists a positive integer $\ell \leq d$ such that:

$$\Sigma(A) = \bigcup_{i=1}^{\ell} [a_i, b_i], \quad \text{with } 0 < a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_{\ell} \leq b_{\ell} < +\infty.$$

By Proposition 3, we know that (1.1) is kinematically similar to (3.3), where $B_i(n)$ are bounded $m_i \times m_i$ matrix functions and $\Sigma(B_i) = [a_i, b_i]$ ($i = 1, \dots, \ell$).

By using the method of QR factorization, we know that, for each $i \in \{1, \dots, \ell\}$, the systems

$$(4.1) \quad x_i(n+1) = B_i(n)x_i(n)$$

are kinematically similar to the $m_i \times m_i$ upper triangular systems

$$(4.2) \quad y_i(n+1) = C_i(n)y_i(n),$$

where $\Sigma(C_i) = [a_i, b_i]$.

Step 2) Exponential dichotomy of scalar difference systems: From now on, the diagonal terms of the upper triangular matrices C_i described in (4.2) will be denoted by $\{c_{i_{rr}}\}_{r=1}^{m_i}$, where i is a fixed element of $\{1, \dots, \ell\}$. Now, by Proposition 4, we know that the upper triangular system (4.2) has the property of diagonal significance, that is

$$\bigcup_{r=1}^{m_i} \Sigma(c_{i_{rr}}) = [a_i, b_i].$$

By Proposition 2, for any $\delta \in (0, 2a_1)$, the scalar difference equation

$$(4.3) \quad u(n+1) = \frac{c_{i_{rr}}(n)}{\left(a_i - \frac{\delta}{2}\right)} u(n)$$

has exponential dichotomy with projector $P = 0$ and

$$(4.4) \quad s(n+1) = \frac{c_{i_{rr}}(n)}{\left(b_i + \frac{\delta}{2}\right)} s(n)$$

has exponential dichotomy with constant $K = 1$ and projector $P = 1$. In consequence, there exists $\theta \in (0, 1)$ such that

$$(4.5) \quad \begin{cases} |U(n)U^{-1}(k)| \leq \theta^{k-n} & \text{if } k \geq n \\ |S(n)S^{-1}(k)| \leq \theta^{n-k} & \text{if } n \geq k. \end{cases}$$

Where

$$U(n) = \begin{cases} \prod_{j=0}^{n-1} \left(\frac{c_{i_{rr}}(j)}{a_i - \frac{\delta}{2}} \right) & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \\ \prod_{j=n}^{-1} \left(\frac{c_{i_{rr}}(j)}{a_i - \frac{\delta}{2}} \right)^{-1} & \text{if } n < 0 \end{cases} \quad \text{and} \quad S(n) = \begin{cases} \prod_{j=0}^{n-1} \left(\frac{c_{i_{rr}}(j)}{b_i + \frac{\delta}{2}} \right) & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \\ \prod_{j=n}^{-1} \left(\frac{c_{i_{rr}}(j)}{b_i + \frac{\delta}{2}} \right)^{-1} & \text{if } n < 0 \end{cases}$$

are the fundamental matrices of (4.3) and (4.4) respectively.

Step 3) A technical result:

Lemma 5. *For any fixed $i \in \{1, \dots, \ell\}$, there exist two sequences $h_i(n)$ and $\Delta_i(n)$ such that*

$$(4.6) \quad a_i \leq h_i(n) \leq b_i \quad \text{and} \quad |\Delta_i(n)| \leq \frac{\delta}{2} \quad \text{for any } n \in \mathbb{Z}$$

and there exists $0 < M_2 < 1 < M_1$ verifying

$$(4.7) \quad M_2 \leq \prod_{j=0}^{n-1} \left| \frac{c_{i_{rr}}(j)}{h_i(j) + \Delta_i(j)} \right| \leq M_1, \quad \text{if } n \geq 1$$

and

$$(4.8) \quad M_2 \leq \prod_{j=n}^{-1} \left| \left(\frac{c_{i_{rr}}(j)}{h_i(j) + \Delta_i(j)} \right)^{-1} \right| \leq M_1, \quad \text{if } n \leq -1$$

for any $r \in \{1, \dots, m_i\}$.

Proof. See Appendix A. □

As a consequence of this result, we construct the $m_i \times m_i$ matrix:

$$L_i(n) = \text{Diag} \left(\mu_1(n), \dots, \mu_{m_i}(n) \right),$$

where for any $r \in \{1, \dots, m_i\}$, it follows that

$$\mu_r(n) = \begin{cases} \prod_{j=0}^{n-1} \frac{c_{i_{rr}}(j)}{h_i(j) + \Delta_i(j)} & \text{if } n \geq 0 \\ \prod_{j=n}^{-1} \left(\frac{c_{i_{rr}}(j)}{h_i(j) + \Delta_i(j)} \right)^{-1} & \text{if } n \leq -1 \end{cases} \quad \text{for any } n \in \mathbb{Z}.$$

By Lemma 5, we have that

$$(4.9) \quad \|L_i(n)\|_\infty \leq M_2 \quad \text{and} \quad \|L_i^{-1}(n)\|_\infty \leq M_1 \quad \text{for any } n \in \mathbb{Z}.$$

Step 4) The systems (4.1) can be contracted to $[a_i, b_i]$ for any $i = 1, \dots, \ell$: The system (4.2) is kinematically similar to

$$(4.10) \quad z_i(n+1) = \Lambda_i(n) z_i(n),$$

with $y_i(n) = L_i(n) z_i(n)$, where $\Lambda_i(n) = L_i^{-1}(n+1) C_i(n) L_i(n)$ is a $m_i \times m_i$ matrix whose rs -coefficient is defined by

$$\{\Lambda_i(n)\}_{rs} = \begin{cases} h_i(n) + \Delta_i(n) & \text{if } r = s \\ \frac{\mu_s(n)}{\mu_r(n+1)} c_{rs}(n) & \text{if } 1 \leq r < s \leq m_i \\ 0 & \text{if } 1 \leq s < r \leq m_i. \end{cases}$$

Let us define the β -transformation

$$z_i(n) = \text{Diag} \left(1, \beta, \dots, \beta^{m_i-1} \right) w_i(n),$$

where

$$(4.11) \quad 0 < \beta < \frac{\delta}{\delta + \frac{2C^+M_1}{M_2}},$$

and C^+ is defined as in (3.10).

Now, we can see that (4.10) and (4.2) are δ -kinematically similar to

$$w_i(n+1) = \Gamma_i(n)w_i(n),$$

where the rs -coefficient of $\Gamma_i(n)$ is

$$\{\Gamma_i(n)\}_{rs} = \begin{cases} \{\Lambda_i(n)\}_{rs} & \text{if } r = s \\ \beta^{s-r} \{\Lambda_i(n)\}_{rs} & \text{if } 1 \leq r < s \leq m_i \\ 0 & \text{if } 1 \leq s < r \leq m_i. \end{cases}$$

Let us observe that $\Gamma_i(n)$ can be written as follows:

$$\Gamma_i(n) = h_i(n)I + R_i(n),$$

where the rs -coefficient of $R_i(n)$ is defined by

$$\{R_i(n)\}_{rs} = \begin{cases} \Delta_i(n) & \text{if } r = s \\ \beta^{s-r} \frac{\mu_s(n)}{\mu_r(n+1)} c_{rs}(n) & \text{if } 1 \leq r < s \leq m_i \\ 0 & \text{if } 1 \leq s < r \leq m_i. \end{cases}$$

By (4.6), (4.7) and (4.8), we can verify that

$$\begin{aligned} \|R_i(n)\| &\leq \frac{\delta}{2} + \frac{M_1 C^+}{M_2} (\beta + \beta^2 + \dots + \beta^{m_i}) \\ &\leq \frac{\delta}{2} + \frac{M_1 C^+}{M_2} \frac{\beta}{1-\beta} \end{aligned}$$

and by using (4.11) it follows that $\|R_i(n)\|_\infty < \delta$ for any $n \in \mathbb{Z}$.

Summarizing, for any $i = 1, \dots, \ell$, the system (4.1) is δ -kinematically similar to

$$w_i(n+1) = \text{Diag}(h_i(n), \dots, h_i(n)) \{I + h_i^{-1}(n)R_i(n)\} w_i(n),$$

where $h_i(n) \subseteq [a_i, b_i] = \Sigma(B_i)$ and $\|\frac{1}{h_i(n)}R_i(n)\|_\infty < \frac{\delta}{|b_i|}$ and the contractibility of (4.1) follows.

Step 5) The system (1.1) can be contracted to $\Sigma(A)$: By using the previous result, we can see that (1.1) is $\tilde{\delta}$ -kinematically similar (with $\tilde{\delta} = \delta/|b_1|$) to

$$z(n+1) = \text{Diag}(H_1(n), \dots, H_\ell(n)) \{I + R(n)\} z(n),$$

where

$$H_i(n) = \text{Diag}(h_i(n), \dots, h_i(n)) \quad \text{and} \quad R(n) = \text{Diag}(h_1^{-1}(n)R_1(n), \dots, h_\ell^{-1}(n)R_\ell(n))$$

for any $i = 1, \dots, \ell$. Finally, note that $H(n) \subseteq \bigcup_{i=1}^{\ell} [a_i, b_i] = \Sigma(A)$ and $\|R(n)\|_\infty < \tilde{\delta}$ for any $n \in \mathbb{Z}$.

Final step: Let E a compact subset of \mathbb{R}^+ such that (1.1) can be contracted to E . By Proposition 5, we have that $\Sigma(A) \subseteq E$ and the minimality of $\Sigma(A)$ follows.

4.2. Proof of Theorem 2. If (1.1) has the full spectrum condition, Proposition 3 says that (1.1) is kinematically similar to

$$y(n+1) = \text{Diag}(B_1(n), \dots, B_d(n))y(n).$$

As the matrix above has order $d \times d$, the results follows since is a diagonal matrix.

APPENDIX A. PROOF OF LEMMA 5

We will construct a strictly increasing and unbounded sequence $\{N_p\}_{p=0}^{+\infty}$ satisfying $N_0 = 0$ such that the sequences $h_i, \Delta_i: \mathbb{Z}^+ \cup \{0\} \rightarrow \mathbb{R}$ defined by:

$$h_i(j) = \begin{cases} a_j & \text{if } j \in \{N_l, \dots, N_{l+1} - 1\}, \quad (l = 0, 2, 4, \dots) \\ b_j & \text{if } j \in \{N_{l+1}, \dots, N_{l+2} - 1\} \end{cases}$$

and

$$\Delta_i(j) = \begin{cases} -\frac{\delta}{2l} & \text{if } j \in \{N_l, \dots, N_{l+1} - 1\}, \quad (l = 0, 2, 4, \dots) \\ \frac{\delta}{2l} & \text{if } j \in \{N_{l+1}, \dots, N_{l+2} - 1\}, \end{cases}$$

satisfy properties (4.6) and (4.7) on $\mathbb{Z}^+ \cup \{0\}$. The case for \mathbb{Z}^- is similar and be done by the reader.

It is straightforward to see that (4.6) is always satisfied. In order to verify (4.7), we interchange n by k in the first inequality of (4.5), we have:

$$(A.1) \quad \begin{cases} |U(n)U^{-1}(k)| \geq \theta^{k-n} & \text{if } n \geq k \\ |S(n)S^{-1}(k)| \leq \theta^{n-k} & \text{if } n \geq k. \end{cases}$$

By using induction, we will verify that for any $\mu > 1$, there exists a sequence $\{N_p\}_p$ such that

$$(A.2) \quad \frac{1}{\mu} \leq \prod_{j=0}^{n-1} \left| \frac{c_{irr}(j)}{h_i(j) + \Delta_i(j)} \right| \leq \mu \quad \text{for any } n \geq 0.$$

If $k = 0$ in the first inequality of (A.1) we obtain

$$|U(n)U^{-1}(0)| \geq \theta^{-n}, \quad n \geq 0,$$

which implies that $|U(n)|$ is unbounded in \mathbb{Z}^+ since $\theta \in (0, 1)$. In consequence, given $\mu > 1$, there exists $N_1 > 0$ such that $|U(n)U^{-1}(0)| < \mu$ for any $n \in \{0, \dots, N_1\}$. In addition, by using the first inequality of (A.1), have that

$$(A.3) \quad \frac{1}{\mu} < 1 \leq |U(n)U^{-1}(0)| < \mu, \quad n \in \{0, \dots, N_1\}.$$

On the other hand, by using the second inequality of (A.1) and now considering $k = N_1$, we obtain that the sequence $|U(N_1)U^{-1}(0)||S(n)S^{-1}(N_1)|$ is convergent to zero when $n \rightarrow +\infty$. Then, there exists $N_2 > N_1$ such that $|U(N_1)U^{-1}(0)||S(n)S^{-1}(N_1)| \geq 1/\mu$ for any $n \in \{N_1, \dots, N_2\}$. As before, by using again the second inequality of (A.1) combined with (A.3), we have that

$$(A.4) \quad \frac{1}{\mu} \leq |U(N_1)U^{-1}(0)||S(n)S^{-1}(N_1)| < \mu \quad n \in \{N_1, \dots, N_2\}.$$

If $n \in \{N_1, \dots, N_2\}$, by using our definition of $h_i(j)$ and $\lambda_i(j)$, we have that,

$$\begin{aligned} \prod_{j=0}^{n-1} \left| \frac{c_{i_{rr}}(j)}{h_i(j) + \Delta_i(j)} \right| &= \prod_{j=0}^{N_1-1} \left| \frac{c_{i_{rr}}(j)}{a_i - \frac{\delta}{2}} \right| \prod_{j=N_1}^{n-1} \left| \frac{c_{i_{rr}}(j)}{b_i + \frac{\delta}{2}} \right| \\ &= |U(N_1)U^{-1}(0)||S(n)S^{-1}(N_1)| \end{aligned}$$

and (A.2) is verified for any $n \in \{0, \dots, N_2\}$.

As Inductive hypothesis, we will assume that there exists $2m + 1$ numbers $0 = N_0 < N_1 < N_2 < \dots < N_{2m-1} < N_{2m}$ such that

$$\frac{1}{\mu} \leq \prod_{j=0}^{2m-1} |U(N_{j+1})U^{-1}(N_j)||S(n)S^{-1}(N_{j+1})| \leq \mu \quad \text{for any } n \in \{N_{2m-1}, \dots, N_{2m}\}.$$

By using the first inequality of (A.1) and considering $n = 2m$ in the inequality above, we have that

$$(A.5) \quad \left(\prod_{j=0}^{2m-2} |U(N_{j+1})U^{-1}(N_j)||S(N_{j+2})S^{-1}(N_{j+1})| \right) |U(n)U^{-1}(N_{2m})|$$

is unbounded for any $n > N_{2m}$. Then, there exists N_{2m+1} such that this product is lower than μ for any $n \in \{N_{2m}, \dots, N_{2m+1}\}$. In addition, by inductive hypothesis combined with $|U(n)U^{-1}(N_{2m})| \geq 1$ (ensured by A.1), we can see that the product above is lowerly bounded by $1/\mu$.

Finally, we have that

$$\left(\prod_{j=0}^{2m-1} |U(N_{j+1})U^{-1}(N_j)||S(N_{j+2})S^{-1}(N_{j+1})| \right) |S(n)S^{-1}(N_{2m+1})|$$

converges to zero when $n \rightarrow +\infty$. Then, there exists N_{2m+2} such that the product above is bigger than $1/\mu$. As before, by (A.5) combined with $|S(n)S^{-1}(N_{2m+1})| \leq 1$ for any $n \geq N_{2m+1}$, we can deduce that the product above is lower than μ and (A.2) is proved.

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DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE CIENCIAS, UNIVERSIDAD DE CHILE, CASILLA 653, SANTIAGO, CHILE

E-mail address: `castaneda@uchile.cl,grobledo@uchile.cl`