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On Abrikosov Lattice Solutions of the Ginzburg-Landau Equations

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Abstract We prove existence of Abrikosov vortex lattice solutions of the Ginzburg-Landau equations of superconductivity, with multiple magnetic flux quanta per fundamental cell. We also revisit the existence proof for the Abrikosov vortex lattices, streamlining some arguments and providing some essential details missing in earlier proofs for a single magnetic flux quantum per a fundamental cell.

Keywords Magnetic vortices · Superconductivity · Ginzburg-Landau equations · Abrikosov vortex lattices · Bifurcations

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1 Introduction

1.1 The Ginzburg-Landau equations. The Ginzburg-Landau model of superconductivity describes a superconductor contained in $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, in terms of a

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complex order parameter $\Psi : \Omega \to \mathbb{C}$, and a magnetic potential $A : \Omega \to \mathbb{R}^d$.[1] The Ginzburg-Landau theory specifies that the difference between the superconducting and normal free energies[2] in a state (Ψ, A) is

$$E_\Omega(\Psi, A) := \int_\Omega |\nabla_A \Psi|^2 + |\operatorname{curl} A|^2 + \frac{\kappa^2}{2}(1 - |\Psi|^2)^2, \tag{1.1}$$

where ∇_A is the covariant derivative defined as $\nabla - iA$ and κ is a positive constant that depends on the material properties of the superconductor and is called the Ginzburg-Landau parameter. In the case $d = 2$, $\operatorname{curl} A := \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$ is a scalar-valued function. It follows from the Sobolev inequalities that for bounded open sets Ω, the energy E_Ω is well-defined and C^∞ as a functional on the Sobolev space H^1.

The critical points of this functional must satisfy the well-known Ginzburg-Landau equations inside Ω:

$$\Delta_A \Psi = \kappa^2(|\Psi|^2 - 1)\Psi, \tag{1.2a}$$

$$\operatorname{curl}^* \operatorname{curl} A = \operatorname{Im}(\bar{\Psi} \nabla_A \Psi). \tag{1.2b}$$

Here $\Delta_A = -\nabla_A^* \nabla_A$, ∇_A^* and curl^* are the adjoints of ∇_A and curl. Explicitly, $\nabla_A^* F = -\operatorname{div} F + iA \cdot F$, and $\operatorname{curl}^* F = \operatorname{curl} F$ for $d = 3$ and $\operatorname{curl}^* f = \left(\frac{\partial f}{\partial x_2}, -\frac{\partial f}{\partial x_1}\right)$ for $d = 2$.

The key physical quantities for the Ginzburg-Landau theory are

- the density of superconducting pairs of electrons, $n_s := |\Psi|^2$;
- the magnetic field, $B := \operatorname{curl} A$;
- the current density, $J := \operatorname{Im}(\bar{\Psi} \nabla_A \Psi)$.

Let $\kappa_c = \frac{1}{\sqrt{2}}$. All superconductors are divided into two classes with different properties: Type I superconductors, which have $\kappa < \kappa_c$ and exhibit first-order phase transitions from the non-superconducting state to the superconducting state, and Type II superconductors, which have $\kappa > \kappa_c$ and exhibit second-order phase transitions and the formation of vortex lattices. Existence of these vortex lattice solutions is the subject of the present paper.

1.2 Abrikosov lattices. In 1957, Abrikosov [1] discovered solutions of (1.2a) in $d = 2$ whose physical characteristics n_s, B, and J are (non-constant and) periodic with respect to a two-dimensional lattice, $\mathcal{L}$, whilst independent of the third dimension, and which have a single flux per lattice cell.[3] We call such solutions the *($\mathcal{L}-$)Abrikosov vortex lattices* or the *($\mathcal{L}-$)Abrikosov lattice solutions* or an abbreviation of thereof. (In physics literature they are variously called mixed states, Abrikosov mixed states, Abrikosov vortex states.) Due to an error of calculation Abrikosov

[1] The Ginzburg-Landau theory is reviewed in every book on superconductivity and most of the books on solid state or condensed matter physics. For reviews of rigourous results see the papers [7, 9, 16, 24] and the books [14, 17, 22, 23]

[2] In the problem we consider here it is appropriate to deal with Helmholtz free energy at a fixed average magnetic field $b := \frac{1}{|\Omega|} \int_\Omega \operatorname{curl} A$, where $|\Omega|$ is the area or volume of Ω.

[3] Such solutions correspond to cylindrical geometry.

concluded that the lattice which gives the minimum average energy per lattice cell[4] is the square lattice. Abrikosov's error was corrected by Kleiner, Roth, and Autler [18], who showed that it is in fact the triangular lattice which minimises the energy.

Since their discovery, Abrikosov lattice solutions have been studied in numerous experimental and theoretical works. Of more mathematical studies, we mention the articles of Eilenberger [13], Lasher [19], Chapman [6] and Ovchinnikov [21].

The rigourous investigation of Abrikosov solutions began soon after their discovery. Odeh [20] sketched a proof of existence for various lattices using variational and bifurcation techniques. Barany, Golubitsky, and Turski [5] applied equivariant bifurcation theory and filled in a number of details, and Takáč [25] has adapted these results to study the zeros of the bifurcating solutions. Further details and new results, in both, variational and bifurcation, approaches, were provided by [11, 12]. In particular, [12] proved partial results on the relation between the bifurcation parameter and the average magnetic field b (left open by previous works) and on the relation between the Ginzburg-Landau energy and the Abrikosov function, and [11] (see also [12]) found boundaries between superconducting, normal and mixed phases.

Amongst related results, a relation of the Ginzburg-Landau minimisation problem, for a fixed, finite domain and external magnetic field, in the regime of $\kappa \to \infty$, to the Abrikosov lattice variational problem was obtained in [2, 4].

The above investigation was completed and extended in [26, 27]. To formulate the results of these papers, we introduce some notation and definitions. For a lattice $\mathcal{L} \subset \mathbb{R}^2$, we denote by $\Omega^{\mathcal{L}}$ and $|\Omega^{\mathcal{L}}|$ a fundamental lattice cell and its area, respectively. Next, assuming a single flux per fundamental lattice cell, we define the following function on lattices $\mathcal{L} \subset \mathbb{R}^2$:

$$\kappa_c(\mathcal{L}) := \sqrt{\frac{1}{2}\left(1 - \frac{1}{\beta(\mathcal{L})}\right)}, \tag{1.3}$$

where $\beta(\mathcal{L})$ is the Abrikosov parameter, see e.g. [26, 27], defined as

$$\beta(\mathcal{L}) := \frac{\langle |\phi|^4 \rangle_{\Omega^{\mathcal{L}}}}{\langle |\phi|^2 \rangle_{\Omega^{\mathcal{L}}}^2}, \tag{1.4}$$

where $\langle f \rangle_\Omega$ denotes the average, $\langle f \rangle_\Omega = \frac{1}{|\Omega|} \int_\Omega f$, of a function f over $\Omega \subset \mathbb{R}^2$ and ϕ is the solution to the problem

$$(-\Delta_{a^1} - 1)\phi = 0, \quad \phi(x+s) = e^{i(\frac{1}{2}s \cdot Jx + c_s)}\phi(x), \quad \forall s \in \sqrt{\frac{2\pi}{|\Omega^{\mathcal{L}}|}}\mathcal{L}, \tag{1.5}$$

where $Jx = (-x_2, x_1)$ for $x = (x_1, x_2)$, $a^1(x) := -\frac{1}{2}Jx$ and c_s satisfies the condition $c_{s+t} - c_s - c_t - \frac{1}{2}bs \wedge t \in 2\pi\mathbb{Z}$. We will show below (Proposition 5.1) that the problem (1.5) has a unique solution and therefore β is well-defined. The following results were proven in [26, 27] (see these papers and review [24] for references to earlier results):

[4]Since for lattice solutions the energy over $\mathbb{R}^2$ (the total energy) is infinite, one considers the average energy per lattice cell, i.e. energy per lattice cell divided by the area of the cell.

Theorem 1.1 *For every lattice $\mathcal{L}$ satisfying*

$$\left|1 - b/\kappa^2\right| \ll 1 \text{ and } (\kappa - \kappa_c(\mathcal{L}))(\kappa^2 - b) \geq 0, \text{ where } b := \frac{2\pi n}{|\Omega^{\mathcal{L}}|}, \tag{1.6}$$

with $n = 1$, the following holds

(I) *The (1.2a) have an $\mathcal{L}$–Abrikosov lattice solution in a neighbourhood of the branch of normal solutions.*
(II) *The above solution is unique, up to symmetry, in a neighbourhood of the normal branch.*

Due to the flux quantisation (see below), the quantity $b := \frac{2\pi}{|\Omega^{\mathcal{L}}|}$, entering the theorem, is the average magnetic flux per lattice cell, $b := \frac{1}{|\Omega^{\mathcal{L}}|}\int_{\Omega^{\mathcal{L}}} \operatorname{curl} A$. We note that due to the reflection symmetry of the problem we can assume that $b \geq 0$.

All the rigourous results proven so far deal with Abrikosov lattices with one quantum of magnetic flux per lattice cell. Partial results for higher magnetic fluxes were proven in [3, 6].

1.3 Result. In this paper, we prove existence of Abrikosov vortex lattice solutions of the Ginzburg-Landau equations, with multiple magnetic flux quanta per a fundamental cell, for certain lattices and for certain flux quanta numbers.

We also revisit the existence proof for the Abrikosov vortex lattices, streamlining some arguments and providing some essential details missing in earlier proofs for a single magnetic flux quantum per a fundamental cell.

As in the previous works, we consider only bulk superconductors filling all $\mathbb{R}^3$, with no variation along one direction, so that the problem is reduced to one on $\mathbb{R}^2$.

To formulate our results, we need some definitions. Motivated by the idea that most stable (i.e. most physical) solutions are also most symmetric, we look for solutions which are most symmetric amongst vortex lattice solution for a given lattice and given the number of the flux quanta per fundamental cells. Following [10], we denote

1. $G(\mathcal{L})$ to be the group of symmetries of the lattice $\mathcal{L}$,
2. $T(\mathcal{L})$ to be the subgroup of $G(L)$ consisting of lattice translations, and
3. $H(\mathcal{L}) := G(\mathcal{L}) \cap O(2) \approx G(\mathcal{L})/T(\mathcal{L})$, the maximal non-translation subgroup.

Note that the non-$SO(2)$ part of $H(\mathcal{L}) := G(\mathcal{L}) \cap O(2)$ comes from reflections and since all reflections in $G(\mathcal{L}) \cap O(2)$ can be obtained as products of rotations and one fixed reflection, which we take to be $z \mapsto \bar{z}$, it suffices for us to consider the conjugation action. Since the conjugation is not holomorphic, we show in Section A that there is no solutions having this symmetry. This implies that the maximal point symmetry group of the GL equations is

$$SH(\mathcal{L}) := G(\mathcal{L}) \cap SO(2)(= H(\mathcal{L}) \cap SO(2)).$$

Hence, we look for solutions amongst functions whose physical properties are invariant under action of $SH(\mathcal{L})$.

Definition 1.2 (Maximal symmetry) We say a vortex lattice solution on $\mathbb{R}^2$ is *maximally symmetric* iff all related physical quantities (i.e. $n_s := |\Psi|^2$, $B :=$ curl A, $J := \mathrm{Im}(\bar{\Psi}\nabla_A\Psi)$) are invariant under the action of the group $SH(\mathcal{L})$, where $\mathcal{L}$ is the underlying lattice of the solution.

Furthermore, we are interested in vortex lattice solutions with the following natural property

Definition 1.3 ($\mathcal{L}-$ irreducibility) We say that a solution is $\mathcal{L}-$irreducible iff there are no finer lattice for which it is a vortex lattice solution.

Our main result is the following

Theorem 1.4 *Assume the conditions of Theorem 1.1, except for* $n = 1$*, hold. Assume either n is one of 2,4,6,8,10 and* $\mathcal{L}$ *is a hexagonal lattice or* $n = 3$ *and* $\mathcal{L}$ *is arbitrary and let* $\lambda = \kappa^2 n/b$*. Then the GLEs have a unique (in a Sobolev space of index 2),* $\mathcal{L}-$*irreducible, maximally symmetric solution branch* $(\lambda(s), \Psi(s), A(s))$, $s \geq 0$ *small. After rescaling* (4.2)*, this branch is of the form* (8.2).

Theorem 1.4 follows from Theorems 8.1, 10.5 and 10.14 and Corollary 10.12 below. As discussed in [8, 24], this result can be reformulated as a result for line bundles over the complex torus.

As was mentioned above, we revisit the existence proof of [26, 27] streamlining some arguments and providing some essential details either missing or only briefly mentioned ([26, 27]) in earlier proofs of the existence of Abrikosov vortex lattices.

After introducing general properties of (1.2a) in Sections 2–4, we prove an abstract conditional result in Sections 5–8, from which we derive Theorem 1.1 in Section 9 (giving a streamlined proof of this result) and Theorem 1.4, in Section 10.

2 Properties of the Ginzburg-Landau Equations

2.1 Symmetries

The Ginzburg-Landau equations exhibit a number of symmetries, that is, transformations which map solutions to solutions:

The gauge symmetry,

$$(\Psi(x), A(x)) \mapsto (e^{i\eta(x)}\Psi(x), A(x) + \nabla\eta(x)), \qquad \forall \eta \in C^2(\mathbb{R}^2, \mathbb{R}); \tag{2.1}$$

The translation symmetry,

$$(\Psi(x), A(x)) \mapsto (\Psi(x+t), A(x+t)), \qquad \forall t \in \mathbb{R}^2; \tag{2.2}$$

The rotation and reflection symmetry,

$$(\Psi(x), A(x)) \mapsto (\Psi(R^{-1}x), RA(R^{-1}x)), \qquad \forall R \in O(2). \tag{2.3}$$

2.2 Homogeneous Solutions

Since the GL equations have gauge and translation symmetries, one expects they have translationally invariant (up to gauge transformations) solutions. Indeed, there are two such solutions: the perfect superconductor solution where $\Psi_S \equiv 1$ and $A_S \equiv 0$, and the *normal* (or non-superconducting) solution where $\Psi_N = 0$ and A_N is such that $\operatorname{curl} A_N =: b$ is constant. (We see that the perfect superconductor is a solution only when the magnetic field is absent. On the other hand, there is a normal solution, $(\Psi_N = 0,\ A_N,\ \operatorname{curl} A_N = \text{constant})$, for any constant magnetic field.)

These solutions are the primary candidates for the ground state of the model, i.e. a solution with the smallest energy per unit area.

3 Lattice Equivariant States

3.1 Gauge-Periodicity

Our focus in this paper is on states (Ψ, A) defined on all of $\mathbb{R}^2$, but whose physical properties, the density of superconducting pairs of electrons, $n_s := |\Psi|^2$, the magnetic field, $B := \operatorname{curl} A$, and the current density, $J := \operatorname{Im}(\bar{\Psi}\nabla_A\Psi)$, are doubly-periodic with respect to some lattice $\mathcal{L}$. We call such states $\mathcal{L}$*−lattice states*.

One can show that a state $(\Psi, A) \in H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{C}) \times H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ is a $\mathcal{L}$-lattice state if and only if translation by an element of the lattice results in a gauge transformation of the state, that is, for each $t \in \mathcal{L}$, there exists a function $g_t \in H^2_{loc}(\mathbb{R}^2; \mathbb{R})$ such that

$$\Psi(x+t) = e^{ig_t(x)}\Psi(x) \text{ and } A(x+t) = A(x) + \nabla g_t(x),\ \forall t \in \mathcal{L}, \tag{3.1}$$

almost everywhere. States satisfying (3.1) will be called *($\mathcal{L}$−) equivariant (vortex) states*.

It is clear that the gauge, translation, and rotation symmetries of the Ginzburg-Landau equations map lattice states to lattice states. In the case of the gauge and translation symmetries, the lattice with respect to which the solution is periodic does not change, whereas with the rotation symmetry, the lattice is rotated as well. It is a simple calculation to verify that the magnetic flux per cell of solutions is also preserved under the action of these symmetries.

Note that (Ψ, A) is defined by its restriction to a single cell and can be reconstructed from this restriction by lattice translations.

3.2 Flux Quantisation

The important property of lattice states is that the magnetic flux through a lattice cell is quantised,

$$\int_{\Omega^{\mathcal{L}}} \operatorname{curl} A = 2\pi n \tag{3.2}$$

for some integer n, with $\Omega^{\mathcal{L}}$ any fundamental cell of the lattice. This implies that

$$|\Omega^{\mathcal{L}}| = \frac{2\pi n}{b}, \tag{3.3}$$

where b is the average magnetic flux per lattice cell, $b := \frac{1}{|\Omega^{\mathcal{L}}|}\int_{\Omega^{\mathcal{L}}} \operatorname{curl} A$.

Indeed, using the Stokes theorem on the l.h.s. of (3.2) and the second equation in (3.1), we find (3.2).

Equation (3.2) then imposes a condition on the area of a cell, namely, (3.3).

4 Fixing the Gauge and Rescaling

In this section we fix the gauge for solutions, (Ψ, A), of (1.2a) and then rescale them to eliminate the dependence of the size of the lattice on b. Our space will then depend only on the number of quanta of flux and the shape of the lattice. Given a lattice $\mathcal{L}$ with a fundamental domain $\Omega^{\mathcal{L}}$, it is convenient to introduce the normalised lattice

$$\mathcal{L}^{\text{norm}} := \sqrt{\frac{2\pi}{|\Omega^{\mathcal{L}}|}}\mathcal{L} \tag{4.1}$$

4.1 Fixing the Gauge

The gauge symmetry allows one to fix solutions to be of a desired form. Let $A^b(x) = \frac{b}{2}Jx \equiv \frac{b}{2}x^{\perp}$, where $x^{\perp} = Jx = (-x_2, x_1)$ and J is the symplectic matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We will use the following preposition, first used by [20] and proved in [25] (an alternate proof is given in in Appendix A of [27]).

Proposition 4.1 *Let (Ψ', A') be an $\mathcal{L}$-equivariant state, and let b be the average magnetic flux per cell. Then there is a $\mathcal{L}$-equivariant state (Ψ, A), that is gauge-equivalent to (Ψ', A'), such that*

(i) $\Psi(x+s) = e^{i(\frac{b}{2}x\cdot Js + c_s)}\Psi(x)$ *and* $A(x+s) = A(x) + \frac{b}{2}Js$ *for all* $s \in \mathcal{L}$;
(ii) $\operatorname{div} A = 0,\ \int_{\Omega}(A - A^b) = 0.$

Here c_s satisfies the condition $c_{s+t} - c_s - c_t - \frac{1}{2}bs \wedge t \in 2\pi\mathbb{Z}$.

4.2 Rescaling

Let $b = \frac{2\pi n}{|\Omega^{\mathcal{L}}|}$ (see (3.3)). We define the rescaled fields (ψ, a) as

$$(\psi(x), a(x)) := (r\Psi(rx), rA(rx)),\ r = \sqrt{\frac{|\Omega^{\mathcal{L}}|}{2\pi}} = \sqrt{\frac{n}{b}}. \tag{4.2}$$

We summarise the effects of the rescaling above:

(A) Ψ and A solve the Ginzburg-Landau equations if and only if ψ and a solve

$$(-\Delta_a - \lambda)\psi = -\kappa^2|\psi|^2\psi, \ \lambda = \kappa^2 n/b, \tag{4.3a}$$
$$\operatorname{curl}^* \operatorname{curl} a = \operatorname{Im}(\bar{\psi}\nabla_a\psi). \tag{4.3b}$$

(B) (Ψ, A) is a $\mathcal{L}$-equivariant state iff (ψ, a) is a $\mathcal{L}^{\text{norm}}$-equivariant state. Moreover, if (Ψ, A) is of the form described in Proposition 4.1, then (ψ, a) satisfies

$$\psi(x+t) = e^{i\frac{n}{2}x\cdot Jt + ic_t}\psi(x), \ a(x+t) = a(x) + \frac{n}{2}Jt, \ \forall t \in \mathcal{L}^{\text{norm}} \tag{4.4}$$
$$\operatorname{div} a = 0, \ \int_{\Omega^\tau}(a - a^n) = 0, \ \text{ where } a^n(x) := \frac{n}{2}Jx, \tag{4.5}$$

and c_t, which satisfies the condition

$$c_{s+t} - c_s - c_t - \frac{1}{2}ns \wedge t \in 2\pi\mathbb{Z}. \tag{4.6}$$

(C) $\frac{1}{|\Omega^{\mathcal{L}}|}E_{\Omega^{\mathcal{L}}}(\Psi, A) = \mathcal{E}_\lambda(\psi, \alpha)$, where $a = a^n + \alpha$, with $a^n(x) := \frac{n}{2}Jx$, $\lambda = \kappa^2 r^2 = \kappa^2\frac{n}{b}$ and (remember, $|\Omega^{\text{norm}}| = 2\pi$)

$$\mathcal{E}_\lambda(\psi, \alpha) = \frac{1}{2\pi}\int_{\Omega^{\text{norm}}}\left(|\nabla_a\psi|^2 + |\operatorname{curl} a|^2 + \frac{\kappa^2}{2}\left(|\psi|^2 - \frac{\lambda}{\kappa^2}\right)^2\right)dx. \tag{4.7}$$

Our problem then is: for each $n = 1, 2, \ldots$, find (ψ, a), solve the rescaled Ginzburg-Landau (4.2) and satisfying (4.4).

4.3 Lattice Shape

We identify $\mathbb{R}^2$ with $\mathbb{C}$, via the map $(x_1, x_2) \to x_1 + ix_2$, and, applying a rotation, if necessary, bring any lattice $\mathcal{L}$ to the form

$$\mathcal{L} = r(\mathbb{Z} + \tau\mathbb{Z}), \tag{4.8}$$

where $r > 0$, $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$, which we assume from now on. If $\mathcal{L}$ satisfies (3.3), then $r = \sqrt{\frac{2\pi n}{b \operatorname{Im}\tau}}$. For (4.8), the normalised lattice (4.1) becomes

$$\mathcal{L}^{\text{norm}} := \sqrt{\frac{2\pi}{\operatorname{Im}\tau}}(\mathbb{Z} + \tau\mathbb{Z}). \tag{4.9}$$

We note that $|\Omega^{\text{norm}}| = 2\pi$. In what follows, the parameter τ is fixed and, to simplify the notation, we *omit the superindex* 'norm' at $\mathcal{L}^{\text{norm}}$ and Ω^{norm} and write simply $\mathcal{L}$ and Ω.

5 The Linear Problem

In this section we consider the linearization of (4.2) satisfying (4.4) on the normal solution $(0, a^n)$, with, recall, $a^n(x) := \frac{n}{2}Jx$. This leads to the linear problem:

$$-\Delta_{a^n}\psi_0 = \lambda\psi_0, \tag{5.1}$$

for ψ_0 satisfying the gauge - periodic boundary condition (see (4.4))

$$\psi_0(x+t) = e^{i(\frac{n}{2}x\cdot Jt+c_t)}\psi_0(x),\ \forall t \in \mathcal{L}. \tag{5.2}$$

Our goal is to prove the following

Proposition 5.1 *The operator $-\Delta_{a^n}$ is self-adjoint on its natural domain (with the additional condition (5.2)) and its spectrum is given by*

$$\sigma(-\Delta_{a^n}) = \{\,(2m+1)n : m = 0, 1, 2, \dots\}, \tag{5.3}$$

with each eigenvalue is of the multiplicity n. Moreover,

$$\mathrm{Null}(-\Delta_{a^n} - n) = e^{\frac{in}{2}x_2(x_1+ix_2)}V_n, \tag{5.4}$$

where V_n is spanned by functions of the form (below $z = (x_1+ix_2)/\sqrt{\frac{2\pi}{\mathrm{Im}\,\tau}}$)

$$\theta(z,\tau) := \sum_{m=-\infty}^{\infty} c_m e^{i2\pi mz},\ c_{m+n} = e^{-in\pi z}e^{i2m\pi\tau}c_m. \tag{5.5}$$

Such functions are determined entirely by the values of $c_0, \dots, c_{n-1}$ and therefore form an n-dimensional vector space.

Proof The self-adjointness of the operator $-\Delta_{a^n}$ is well-known. To find its spectrum, we introduce the complexified covariant derivatives (harmonic oscillator annihilation and creation operators), $\bar\partial_{a^n}$ and $\bar\partial^*_{a^n} = -\partial_{a^n}$, with

$$\bar\partial_{a^n} := (\nabla_{a^n})_1 + i(\nabla_{a^n})_2 = \partial_{x_1} + i\partial_{x_2} + \frac{1}{2}n(x_1+ix_2). \tag{5.6}$$

One can verify that these operators satisfy the following relations:

$$[\bar\partial_{a^n}, (\bar\partial_{a^n})^*] \;=\; 2\,\mathrm{curl}\, a^n = 2n; \tag{5.7}$$

$$-\Delta_{a^n} - n \;=\; (\bar\partial_{a^n})^*\bar\partial_{a^n}. \tag{5.8}$$

As for the harmonic oscillator (see for example [15]), this gives explicit information about the spectrum of $-\Delta_{a^n}$, namely (5.3), with each eigenvalue is of the same multiplicity. Furthermore, the above properties imply

$$\mathrm{Null}(-\Delta_{a^n} - n) = \mathrm{Null}\,\bar\partial_{a^n}. \tag{5.9}$$

We find $\mathrm{Null}\,\bar\partial_{a^n}$. A simple calculation gives the following operator equation

$$e^{-\frac{n}{2}(ix_1x_2-x_2^2)}\bar\partial_{a^n}e^{\frac{n}{2}(ix_1x_2-x_2^2)} = \partial_{x_1} + i\partial_{x_2}.$$

(The transformation on the l.h.s. is highly non-unique.) This immediately proves that

$$\bar\partial_{a^n}\psi_0 = 0, \tag{5.10}$$

if and only if $\theta = e^{-\frac{n}{2}(ix_1x_2-x_2^2)}\psi$ satisfies $(\partial_{x_1} + i\partial_{x_2})\theta = 0$. We now identify $x \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$ and see that this means that θ is analytic and

$$\psi_0\,(x) = e^{-\frac{\pi n}{2\mathrm{Im}\,\tau}(|z|^2-z^2)}\theta(z,\tau),\ z = (x_1+ix_2)/\sqrt{\frac{2\pi}{\mathrm{Im}\,\tau}}. \tag{5.11}$$

where we display the dependence of θ on τ. The gauge-periodicity of ψ_0 transfers to θ as follows

$$\theta(z+1,\tau)=\theta(z,\tau), \qquad \theta(z+\tau,\tau)=e^{-2\pi inz}e^{-in\pi\tau}\theta(z,\tau).$$

The first relation ensures that θ has a absolutely convergent Fourier expansion of the form $\theta(z,\tau)=\sum_{m=-\infty}^{\infty}c_m e^{2\pi miz}$. The second relation, on the other hand, leads to relation for the coefficients of the expansion: $c_{m+n}=e^{-in\pi z}e^{i2m\pi\tau}c_m$, which together with the previous statement implies (5.5). □

Remark 5.2 Using (5.5) and (5.11), we can show that for $n=1$

$$\psi_0(x)=\psi_0(-x). \tag{5.12}$$

6 Setup of the Bifurcation Problem

In this section we reformulate the Ginzburg-Landau equations as a bifurcation problem. We write $a=a^n+\alpha$ and substitute this into (4.2) to obtain

$$(L^n-\lambda)\psi=-h(\psi,\alpha), \qquad M\alpha=J(\psi,\alpha), \tag{6.1}$$

where $h(\psi,\alpha):=2i\alpha\cdot\nabla_{a^n}\psi+|\alpha|^2\psi+\kappa^2|\psi|^2\psi$ and $J(\psi,\alpha):=\operatorname{Im}(\bar\psi\nabla_{a^n+\alpha}\psi)$ and

$$L^n:=-\Delta_{a^n}\ \text{and} M:=\operatorname{curl}^*\operatorname{curl}. \tag{6.2}$$

The pair (ψ,α) satisfies the conditions (4.4)–(4.4), with $a=a^n+\alpha$, $a^n(x):=\frac{n}{2}Jx$,which we reproduce here

$$\psi(x+t)=e^{i(\frac{n}{2}x\cdot Jt+c_t)}\psi(x), \tag{6.3}$$

$$\alpha(x+t)=\alpha(x)\ \text{and}\ \operatorname{div}\alpha=0, \tag{6.4}$$

where $t\in\mathcal{L}$ and c_t satisfies the condition (4.6). We take $c_t=0$ on the basis vectors $t=\sqrt{\frac{2\pi}{\operatorname{Im}\tau}}, \sqrt{\frac{2\pi}{\operatorname{Im}\tau}}\tau$ (see (4.9)). Then the relation (4.6) gives

$$c_s=\pi npq,\quad \text{for } s=\sqrt{\frac{2\pi}{\operatorname{Im}\tau}}(p+q\tau),\ p,q\in\mathbb{Z}, \tag{6.5}$$

which we assume in what follows.

We consider (6.1) on the space $\mathscr{H}_n^2\times\vec{\mathscr{H}}^2$, where $\mathscr{H}_n^s$ and $\vec{\mathscr{H}}^s$ are the Sobolev spaces of order s associated with the L^2-spaces

$$\mathscr{L}_n^2:=\{\psi\in L^2(\mathbb{R}^2,\mathbb{C}):\psi\ \text{satisfies (6.3)}\}, \tag{6.6}$$

$$\vec{\mathscr{L}}^2:=\{\alpha\in L^2(\mathbb{R}^2,\mathbb{R}^2):\alpha\ \text{satisfies (6.4)}\}, \tag{6.7}$$

where $\operatorname{div}\alpha$ is understood in the distributional sense, with the inner products of $L^2(\Omega,\mathbb{C})$ and $L^2(\Omega,\mathbb{R}^2)$, i.e. $\int\bar\psi\psi'$ and $\int\alpha\cdot\alpha'$.

We define L^n and M on the spaces $\mathscr{L}_n^2$ and $\vec{\mathscr{L}}^2$, with the domains $\mathscr{H}_n^2$ and $\vec{\mathscr{H}}^2$, respectively. The properties of L^n were described in Proposition 5.1. The properties of M are summarised as in the following proposition

Proposition 6.1 *M is a non-negative operator on $\vec{\mathscr{L}}^2$ with the domain $\vec{\mathscr{H}}^2$ and with purely discrete spectrum. Furthermore, 0 is an eigenvalue of M with the one-dimensional eigenspace spanned by* **1**.

The fact that M is positive follows immediately from its definition. We note that its being strictly positive is the result of restricting its domain to elements having the divergence and mean zero.

Let P' be the orthogonal projection onto the divergence free vector fields ($P' = \frac{1}{-\Delta}$ curl* curl). We introduce the new system

$$(L^n - \lambda)\psi + h(\psi, \alpha) = 0, \qquad M\alpha - P'J(\psi, \alpha) = 0, \tag{6.8}$$

where we left the first equation unchanged and in the second equation we introduced the projection P' .

Proposition 6.2 *Assume (λ, ψ, α) is a solution of the system* (6.8) *satisfying* (6.3)–(6.3). *Then* div $J(\psi, \alpha) = 0$ *and therefore (λ, ψ, α) solves the original system* (6.1).

Proof Assume $\chi \in H^1_{\text{loc}}$ and is $\mathcal{L}-$periodic (we say, $\chi \in H^1_{\text{per}}$). Following [26], we differentiate the equation $\mathcal{E}_\lambda(e^{is\chi}\psi, \alpha + s\nabla\chi) = \mathcal{E}_\lambda(\psi, \alpha)$, w.r.to s at $s = 0$, use that curl $\nabla\chi = 0$ and integrate by parts, to obtain

$$\text{Re}\langle -\Delta_{a^n+\alpha}\psi + \kappa^2(|\psi|^2 - 1)\psi, i\chi\psi\rangle + \langle J(\psi, \alpha), \nabla\chi\rangle = 0. \tag{6.9}$$

(Due to conditions (6.3)–(6.3) and the $\mathcal{L}-$periodicity of χ, there are no boundary terms.) This, together with the first equation in (6.8), implies

$$\langle J(\psi, \alpha), \nabla\chi\rangle = 0. \tag{6.10}$$

Since the last equation holds for any $\chi \in H^1_{\text{per}}$, we conclude that div $J(\psi, \alpha) = 0$. □

In Sections 7–10 we solve the system (6.8), subject to the conditions (6.3)–(6.3).

We conclude this section by establishing some general properties of the map $F : \mathbb{R} \times \mathscr{H}^2_n \times \vec{\mathscr{H}}^2 \to \mathscr{L}^2_n \times \vec{\mathscr{L}}^2$ is defined by the l.h.s. of (6.8). For a map $F(\lambda, u)$, $u = (\psi, \alpha)$, we denote by $\partial_\psi F(\lambda, u)/\partial_u F(\lambda, u)$ its Gâteaux derivative in ψ/u. Furthermore, we use the obvious notation $F = (F_1, F_2)$. For $f = (f_1, f_2)$, we introduce the gauge transformation as $T_\delta f = (e^{i\delta} f_1, f_2)$. The following proposition lists some properties of F.

Proposition 6.3

(a) *F is analytic as a map between real Banach spaces,*
(b) *for all λ, $F(\lambda, 0) = 0$,*
(c) *for all λ, $\partial_u F(\lambda, 0) = A_\lambda$,*
(d) *for all $\delta \in \mathbb{R}$, $F(\lambda, T_\delta u) = T_\delta F(\lambda, u)$.*
(e) *for all u (resp. ψ), $\langle u, F(\lambda, u)\rangle \in \mathbb{R}$ (resp. $\langle \psi, F_1(\lambda, u)\rangle \in \mathbb{R}$).*

Proof The first property follows from the definition of F. (b) through (d) are straightforward calculations. For (e), since $\langle u, F\rangle = \langle \psi, F_1\rangle + \langle \alpha, F_2\rangle$ and $\langle \alpha, F_2(\lambda, u)\rangle$ is real, the statements $\langle u, F(\lambda, u)\rangle \in \mathbb{R}$ and $\langle \psi, F_1(\lambda, u)\rangle \in \mathbb{R}$ are equivalent. Now, we calculate that

$$\begin{aligned}\langle \psi, F_1(\lambda, \psi, \alpha)\rangle &= \langle \psi, (L^n - \lambda)\psi\rangle + 2i\int_\Omega \bar{\psi}\alpha\cdot\nabla\psi \\ &\quad +2\int_\Omega (\alpha\cdot a^n)|\psi|^2 + \int_\Omega |\alpha|^2|\psi|^2 + \kappa^2\int_\Omega |\psi|^4.\end{aligned}$$

The final three terms are clearly real and so is the first because $L^n - \lambda$ is self-adjoint. For the second term we integrate by parts and use the fact that the boundary terms vanish due to the periodicity of the integrand to see that

$$\operatorname{Im} 2i\int_\Omega \phi\alpha\cdot\nabla\bar{\psi} = \int_\Omega \alpha\cdot(\bar{\psi}\nabla\psi + \psi\nabla\bar{\psi}) = -\int_\Omega (\operatorname{div}\alpha)|\psi|^2 = 0,$$

where we have used that $\operatorname{div}\alpha = 0$. Thus this term is also real and (e) is established. □

7 Reduction to a Finite-Dimensional Problem

In this section we reduce the problem of solving (6.8) to a finite dimensional problem. We address the latter in the next section. We use the standard method of Lyapunov-Schmidt reduction.

We proceed in the generality we need later on. Let $X = X' \times X''$ and $Y = Y' \times Y''$ be closed subspaces of $\mathscr{H}_n^2 \times \vec{\mathscr{H}}^2$ and $\mathscr{L}_n^2 \times \vec{\mathscr{L}}^2$, respectively, s.t.

$$X \subset Y, \text{ densely, and } F : \mathbb{R}\times X \to Y \text{ and is } C^2. \tag{7.1}$$

We rewrite (6.8) as a single equation

$$F(\lambda, u) = 0, \tag{7.2}$$

where $u := (\psi, \alpha)$ and, recall, the map $F : \mathbb{R}\times\mathscr{H}_n^2\times\vec{\mathscr{H}}^2 \to \mathscr{L}_n^2\times\vec{\mathscr{L}}^2$ is defined by the l.h.s. of (6.8). We can write F as

$$F(\lambda, u) = A_\lambda u + f(u). \tag{7.3}$$

Here $A_\lambda := \operatorname{diag}(L^n - \lambda, M)$ and

$$f(u) := (h(u), -P'J(u)), \tag{7.4}$$

with $h(u)$ and $J(u)$ defined after (6.1).

Recall that the operator $A_\lambda := \operatorname{diag}(L^n - \lambda, M)$ is introduced after the (7.3). Since $A_\lambda = dF(\lambda, 0)$, it maps X into Y. We let $K = \operatorname{Null}_X A_n \subset X$.

We let P be the orthogonal projection in Y onto K and let $\bar{P} := I - P$. Since 0 is an isolated eigenvalue of A_n, P can be explicitly given as the Riesz projection,

$$P := -\frac{1}{2\pi i}\oint_\gamma (A_n - z)^{-1}\,dz, \tag{7.5}$$

where $\gamma \subseteq \mathbb{C}$ is a contour around 0 that contains no other points of the spectrum of A_n.

Writing $u = v + w$, where $v = Pu$ and $w = \bar{P}u$, we see that the equation $F(\lambda, u) = 0$ is therefore equivalent to the pair of equations

$$PF(\lambda, v + w) = 0, \tag{7.6}$$

$$\bar{P}F(\lambda, v + w) = 0. \tag{7.7}$$

We will now solve (7.6) for $w = \bar{P}u$ in terms of λ and $v = Pu$.

Lemma 7.1 *There is a neighbourhood, $U \subset \mathbb{R} \times K$, of $(n, 0)$, such that for any (λ, v) in that neighbourhood, (7.6) has a unique solution $w = w(\lambda, v)$. This solution $w(\lambda, v) = (w_1, w_2)$ satisfies*

$$w(\lambda, v) \text{ is } C^2 \text{ in } (\lambda, v), \tag{7.8}$$

$$\|\partial_\lambda^m \partial_v^n w_i\| = O(\|v\|^{4-i-n}),\ i = 1, 2,\ m + n \leq 1, \tag{7.9}$$

where the norms are in the space $\mathscr{H}_n^2$.

Proof We introduce the map $G : \mathbb{R} \times K \times \bar{X} \to \bar{Y}$, where $\bar{X} := \bar{P}X = X \ominus K$ and $\bar{Y} := \bar{P}Y = Y \ominus K$, defined by

$$G(\lambda, v, w) = \bar{P}F(\lambda, v + w).$$

It has the following properties (a) G is C^2; (b) $G(\lambda, 0, 0) = 0\ \forall\lambda$; (c) $d_w G(\lambda, 0, 0)$ is invertible for $\lambda = n$. Applying the Implicit Function Theorem to $G = 0$, we obtain a function $w : \mathbb{R} \times K \to \bar{X}$, defined on a neighbourhood of $(n, 0)$, such that $w = w(\lambda, v)$ is a unique solution to $G(\lambda, v, w) = 0$, for (λ, v) in that neighbourhood. This proves the first statement.

By the implicit function theorem and the differentiability of F, the solution has the property (7.8).

By (7.3) and the fact that product of $\mathscr{H}_n^2/\vec{\mathscr{H}}^2$ functions is again a $\mathscr{H}_n^2/\vec{\mathscr{H}}^2$ function (and the norms are bounded correspondingly), implies that h and J, entering (7.4) and defined after (6.1), satisfy

$$\|h(u)\|_{H^2} \lesssim \|u\|_{H^2}^3 \text{ and } \|J(u)\|_{H^2} \lesssim \|u\|_{H^2}^2.$$

Using (7.3), we can rewrite (7.6) as

$$A_\lambda w = -\bar{P}f(\lambda, u). \tag{7.10}$$

Since by the definition of $\bar{P}$ and self-adjointness of A_λ, $A_\lambda^\perp := \bar{P}A_\lambda\bar{P}\big|_{\mathrm{Ran}\,\bar{P}}$ is invertible for λ close to n, with the uniformly bounded inverse and since A_λ is diagonal and f is of the form (7.4), we conclude that $\|w_1\| \lesssim \|h(u)\|_{H^2} \lesssim \|u\|_{H^2}^3$ and $\|w_2\| \lesssim \|J(u)\|_{H^2} \lesssim \|u\|_{H^2}^2$. Recalling that $u = v + w$, this gives the second relation in (7.8). The first relation in (7.8) is proven similarly. □

We substitute the solution $w = w(\lambda, v)$ into (7.6) and see that the latter equation in a neighbourhood of $(n, 0)$ is equivalent to the equation (the *bifurcation equation*)

$$\gamma(\lambda, v) := PF(\lambda, v + w(\lambda, v)) = 0. \tag{7.11}$$

Note that $\gamma : \mathbb{R} \times K \to K$. We show that w and γ inherit the symmetry of the original equation:

Lemma 7.2 *For every $\delta \in \mathbb{R}$, $w(\lambda, e^{i\delta}v) = T_\delta w(\lambda, v)$ and $\gamma(\lambda, e^{i\delta}v) = e^{i\delta}\gamma(\lambda, v)$.*

Proof We first check that $w(\lambda, e^{i\delta}v) = T_\delta w(\lambda, v)$. We note that by definition of w,

$$G(\lambda, e^{i\delta}v, w(\lambda, e^{i\delta}v)) = 0,$$

but by the symmetry of F, we also have $G(\lambda, e^{i\delta}v, e^{i\delta}w(\lambda, v)) = T_\delta G(\lambda, v, w(\lambda, v)) = 0$. The uniqueness of w then implies that $w(\lambda, e^{i\delta}v) = T_\delta w(\lambda, v)$. Using that $e^{i\delta}v = T_\delta v$, we can now verify that

$$\begin{aligned}\gamma(\lambda, e^{i\delta}v) = PF(\lambda, e^{i\delta}v + w(\lambda, e^{i\delta}v)) &= PF(\lambda, T_\delta(v + w(\lambda, v))) \\ &= PT_\delta F(\lambda, v + w(\lambda, v)).\end{aligned}$$

Since P is of the form $P = P_1 \oplus 0$, where P_1 acts on the first component, we have $PT_\delta F(\lambda, v + w(\lambda, v)) = e^{i\delta}PF(\lambda, v + w(\lambda, v)) = e^{i\delta}\gamma(\lambda, v)$, which implies the second statement. □

Thus we have shown the following

Corollary 7.3 *In a neighbourhood of $(n, 0)$ in $\mathbb{R} \times X$, (λ, u), where $u = (\psi, \alpha)$, solves (4.2) or (6.1) if and only if (λ, v), with $v = Pu$, solves (7.11). Moreover, the solution u of (6.1) can be reconstructed from the solution v of (7.11) according to the formula*

$$u = v + w(\lambda, v), \tag{7.12}$$

where $w = w(\lambda, v)$ is the unique solution to (7.6) described in Lemma 7.1.

Solving the bifurcation (7.11) is a subtle problem. We do this in the next section assuming $\dim_{\mathbb{C}} \mathrm{Null}_{X'}(L^n - n) = 1$.

8 Existence Result Assuming $\dim_{\mathbb{C}} \mathrm{Null}_{X'}(L^n - n) = 1$

The main result of this section is the following theorem which gives a general, but conditional result.

Theorem 8.1 *Assume (i) $\mathcal{L}^\tau$ satisfies (1.6), (ii) (7.1) holds and (iii) X' be a closed subspace of $\mathscr{H}_n^2$, s.t.*

$$\dim_{\mathbb{C}} \mathrm{Null}_{X'}(L^n - n) = 1. \tag{8.1}$$

Then, for every τ, there exist $\epsilon > 0$ and a branch, $(\lambda_s, \psi_s, \alpha_s)$, $s \in [0, \sqrt{\epsilon})$, of nontrivial solutions of the rescaled Ginzburg-Landau (4.2), unique modulo the global *gauge symmetry (apart from the trivial solution $(n, 0, a^n)$) in a sufficiently small neighbourhood of $(n, 0, a^n)$ in $\mathbb{R} \times X$, and such that*

$$\begin{cases} \lambda_s = n + O(s^2), \\ \psi_s = s\psi_0 + O_{\mathscr{H}^2_n}(s^3), \\ a_s = a^n + O_{\vec{\mathscr{H}}^2}(s^2), \end{cases} \tag{8.2}$$

where ψ_0 is the solution of the problem (5.1)–(5.2), with $\lambda = n$, (normalised as $\langle|\psi_0|^2\rangle = 1$).

Proof Our goal is to solve the (7.11) for λ. By Proposition 6.1, we have

$$\mathrm{Null}_X\, A_n = \mathrm{Null}_{X'}(L^n - n) \times \{\mathrm{const}\}. \tag{8.3}$$

This relation and assumption (8.1) yield that the projection P can be written, for $u = (\psi, \alpha)$, as

$$Pu = (s\psi_0, \langle\alpha\rangle) \text{ with } s := \frac{1}{\|\psi_0\|^2}\langle\psi_0, \psi\rangle,$$
$$\psi_0 \in \mathrm{Null}_{X'}(L^n - n), \ \|\psi_0\| = 1, \tag{8.4}$$

where, recall, $\langle\alpha\rangle := \frac{1}{|\Omega|}\int_\Omega \alpha$. Hence, we can write the map γ in the bifurcation (7.11) as $\gamma = (\psi_0\gamma_1, \gamma_2)$, where $\gamma_i : \mathbb{R} \times \mathbb{C} \times \mathbb{R}^2 \to \mathbb{C}$ are given by

$$\gamma_1(\lambda, s, \mu) := \langle\psi_0, F_1(\lambda, v + w(\lambda, v))\rangle \tag{8.5}$$
$$\gamma_2(\lambda, s, \mu) := \langle F_2(\lambda, v + w(\lambda, v))\rangle, \tag{8.6}$$

with $v \equiv v_{s,\mu} := (s\psi_0, s\mu)$. The bifurcation (7.11) is equivalent to the equations

$$\gamma_1(\lambda, s, \mu) = 0, \qquad \gamma_2(\lambda, s, \mu) = 0. \tag{8.7}$$

First, we consider the equation $\gamma_2(\lambda, s, \mu) = 0$. We solve for μ given real s and real λ. The choice of real s looses no generality as we will see later that $\gamma_1(\lambda, s, \mu)$ is real if s is real; and clearly γ_2 is real valued if s is real. By the 2nd equation in (6.8) and the facts $\langle P'J(\psi, \alpha)\rangle = \langle \mathbf{1}, P'J(\psi, \alpha)\rangle = \langle P'\mathbf{1}, J(\psi, \alpha)\rangle$ and $P'\mathbf{1} = \mathbf{1}$, we have

$$\gamma_2(\lambda, s, \mu) = \langle P'J(v + w(\lambda, v))\rangle = \langle J(v + w(\lambda, v))\rangle. \tag{8.8}$$

By (7.12), (8.4) and (7.8), $\partial_\xi^n w(\lambda, v) = (O_{\mathscr{H}^2_n}(s^3), O_{\vec{\mathscr{H}}^2}(s^2))$, for $n = 0, 1$ and $\xi = \lambda, \mu$. Hence

$$J(v + w(\lambda, v)) = s^2\, \mathrm{Im}(\bar{\psi}_0 \nabla_{a^n}\psi_0) - s^3 s|\psi_0|^2\mu + O_{\mathscr{H}^2_n}(s^4) + O_{\vec{\mathscr{H}}^2}(s^4). \tag{8.9}$$

Recall, that curl* maps scalar functions, f, into vector-fields, $\mathrm{curl}^* f = (\partial_2 f, -\partial_1 f)$. The next lemma, due to [26], shows that the leading term on the r.h.s. of (8.9) drops out under taking the average. (For the reader's convenience, the proof of this lemma is given at the end of this section.)

Lemma 8.2

$$\mathrm{Im}(\bar{\psi}_0 \nabla_{a^n} \psi_0) = -\frac{1}{2}\,\mathrm{curl}^* \, |\psi_0|^2. \tag{8.10}$$

By (8.10), $\langle \mathrm{Im}(\bar{\psi}_0 \nabla_{a^n} \psi_0)\rangle = 0$. This and $\langle |\psi_0|^2\rangle = 1$ give

$$\gamma_2(\lambda, s, \mu) = s^3\mu + O(s^4).$$

Define $\tilde{\gamma}_2(\lambda, s, \mu) := s^{-3}\gamma_2(\lambda, s, \mu)$. Then $\tilde{\gamma}_2(\lambda, 0, 0) = 0$ (for any λ) and it is easy to see that $\partial_\mu \tilde{\gamma}_2(\lambda, s, \mu) = 1 + O(s)$. Thus the equation $\tilde{\gamma}_2(\lambda, s, \mu) = 0$ has a unique solution, $\mu = \mu(\lambda, s)$, provided s is sufficiently small, and this solution is of the form

$$\mu = O(s). \tag{8.11}$$

Next, we address the equation $\gamma_1(\lambda, s, \mu) = 0$, where $\mu = \mu(\lambda, s)$ as above. First, we show that $\gamma_1(\lambda, s, \mu) \in \mathbb{R}$ for $s \in \mathbb{R}$. Using that the projection $\bar{P}$ is self-adjoint, $\bar{P}w(\lambda, v) = w(\lambda, v)$ and that $w(\lambda, v)$ solves $\bar{P}F(\lambda, v + w) = 0$, we find

$$\langle w(\lambda, v), F(\lambda, v + w(\lambda, v))\rangle = \langle w(\lambda, v), \bar{P}F(\lambda, v + w(\lambda, v))\rangle = 0.$$

Therefore, recalling $v \equiv v_{s,\mu} := (s\psi_0, s\mu)$ and using that $\langle v, F\rangle = s\langle \psi_0, F_1\rangle + s\mu\langle F_2\rangle$ and $\langle F_2(\lambda, v + w(\lambda, v))\rangle = -\gamma_2(\lambda, s, \mu) = 0$, we have, for $s \neq 0$,

$$\begin{aligned}\langle \psi_0, F_1(\lambda, v + w(\lambda, v))\rangle &= s^{-1}\langle v, F(\lambda, v + w(\lambda, v))\rangle \\ &= s^{-1}\langle v + w(\lambda, v), F(\lambda, v + w(\lambda, v))\rangle,\end{aligned}$$

and this is real by Proposition 6.3 (6.3) and the fact that the part $\langle w_2(\lambda, v), F_2(\lambda, v + w(\lambda, v))\rangle$ of the inner product on the r.h.s. is real.

Next, for any μ, by Lemma 7.2, $\gamma_1(\lambda, s, \mu) = e^{i \arg s}\gamma_1(\lambda, |s|, \mu)$. Therefore $\gamma_1(\lambda, s, \mu) = 0$ is equivalent to the equation

$$\gamma_1'(\lambda, s, \mu) = 0 \tag{8.12}$$

for the restriction $\gamma_1' : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ of the function γ_1 to $\mathbb{R} \times \mathbb{C} \times \mathbb{R}$, i.e., for real s.

Now, recall that the map $F = (F_1, F_2)$ is defined by the l.h.s. of (6.8) and can be written as (7.3). Using $w(\lambda, v) = O(s^3)$, and recalling $v \equiv v_{s,\mu} := (s\psi_0, s\mu)$, we find

$$F_1(\lambda, v + w(\lambda, v)) = s(-\Delta_{a^n} - \lambda)\psi_0 + O_{\mathscr{H}^2_n}(s^3). \tag{8.13}$$

Using this and $(-\Delta_{a^n} - n)\psi_0 = 0$ and denoting the first component of P by P_1, we obtain

$$P_1F_1(\lambda, v + w(\lambda, v)) = s(n - \lambda)\psi_0 + O_{\mathscr{H}^2_n}(s^3). \tag{8.14}$$

If we write $\gamma_1'(\lambda, s, \mu) = s\tilde{\gamma}_1(\lambda, s, \mu)$, then, since $\langle |\psi_0|^2\rangle = 1$, we have $\tilde{\gamma}_1(\lambda, s, \mu) = n - \lambda + O(s^2)$. Since $\tilde{\gamma}_1(n, 0, \mu) = 0$ (for any μ) and, as easy to see, $\partial_\lambda \tilde{\gamma}_1(\lambda, s, \mu) = -1 + O(s^2)$, the equation $\tilde{\gamma}_1(\lambda, s, \mu) = 0$ has the unique solution, λ_s, and this solution is of the form

$$\lambda_s = n + O(s^2). \tag{8.15}$$

Now, we know that (λ, u) solves $F(\lambda, u) = 0$ if and only if $u = v + w(\lambda, v)$, $v \equiv v_{s,\mu}$, and λ, s, μ solve $\gamma(\lambda, s, \mu) = 0$ (see (7.11)), or $\gamma_i(\lambda, s, \mu) = 0$, $i = 1, 2$ (see (8.7)). By above, near $(n, 0, 0)$, (8.7) has two branches of solutions, $s = 0$ and λ, μ are arbitrary and (λ_s, s, μ), where λ_s and μ are given by (8.15) and (8.11), respectively, and s is sufficiently small, but otherwise is arbitrary. For $s = 0$, we have $v_{s,\mu} = 0$ and therefore $u = v_{s,\mu} + w(\lambda_s, v_{s,\mu}) = 0$, which gives the trivial solution. In the other case, $u = u_s := v_{s,\mu} + w(\lambda_s, v_{s,\mu}) = (s\psi_0 + O_{\mathscr{H}_n^2}(s^3), s\mu + O_{\vec{\mathscr{H}}^2}(s^2)))$. Remembering $a = a^n + \alpha$, we see that the solutions $(\lambda_s, u_s) = (\lambda_s, \psi_s, \alpha_s)$ are of the form (8.2). □

Proof of Lemma 8.2 Multiplying (5.10) by $\bar{\psi}_0$ and taking imaginary and real parts of the resulting equation gives

$$\operatorname{Im} \bar{\psi}_0 (\nabla_{\vec{a}^n})_1 \psi_0 = -\operatorname{Re} \bar{\psi}_0 (\nabla_{\vec{a}^n})_2 \psi_0 = -\frac{1}{2} \partial_{x_2} |\psi_0|^2$$

and

$$\operatorname{Im} \bar{\psi}_0 (\nabla_{\vec{a}^n})_2 \psi_0 = \operatorname{Re} \bar{\psi}_0 (\nabla_{\vec{a}^n})_1 \psi_0 = \frac{1}{2} \partial_{x_1} |\psi_0|^2,$$

which, in turn, gives (8.10). □

Remark 8.3 For $n > 1$, the null space of M is trivial for the space X introduced in Section 10 (a proof is given in Remark 10.13). The statement is true also for $n = 1$, if we use that $(\Psi(x), A(x)) \to (\Psi(-x), -A(-x))$ is a symmetry of the GLEs and impose the parity condition

$$(\Psi(-x), -A(-x)) = (\Psi(x), A(x)) \tag{8.16}$$

(see also Remark 5.2). Taking this into account in (8.3), we see that the proof above for the symmetry reduced spaces simplifies.

9 Bifurcation Theorem for $n = 1$

In this section we let $\mathcal{L}$ be an arbitrary lattice satisfying (1.6) and take $n = 1$. For $n = 1$, we can take $X = \mathscr{H}_1^2 \times \vec{\mathscr{H}}^2$ and $Y = \mathscr{L}_1^2 \times \vec{\mathscr{L}}^2$ (see the paragraph preceding (7.1)). By (8.1) and Proposition 5.1, the space $K = \text{Null}\, A_n$ has the complex dimension 1 and therefore Theorem 8.1 is applicable and gives Theorem 1.1, statements (I) and (II). □

10 Bifurcation Theorem for $n > 1$ and Point Symmetries

As above, n will denote the number of flux quanta through a fundamental cell of $\mathcal{L}$. We want to prove the existence of Abrikosov lattices for $n \geq 2$. The main notions of the section are as follows:

1. Number of flux quanta, n.

2. $\mathcal{L}-$ irreducibility. We are interested in $\mathcal{L}-$equivariant solutions which are not equivariant for any finer lattice. We call such solutions $\mathcal{L}-$irreducible.
3. Multiplicity, which is defined as the dimension of the linear subspace, $\mathrm{Null}_X\, A_n$. The difficulty of bifurcation theory reduces considerably if the multiplicity is one. We call the corresponding solutions *simple*.

The former is achieved by employing the symmetries of the lattice to reduce the dimension of $\mathrm{Null}_X\, A_n$, more precisely, to find X satisfying (7.1) and $\dim_{\mathbb{C}} \mathrm{Null}_X\, A_n = 1$.

In the next two subsections, we outline the general strategy of reducing multiplicity by the group symmetry and choose appropriate subgroups of the point group to impose as symmetry group. Then, in the following three subsections, we give the actual proof.

10.1 Symmetry Reduction

Let $X_n = \mathscr{H}_n^2 \times \vec{\mathscr{H}}^2$ and $Y_n = \mathscr{L}_n^2 \times \vec{\mathscr{L}}^2$. Define the action of $H(\mathcal{L})$ on our spaces by

$$\rho_g(\psi(x), \alpha(x)) = (\psi(gx), g^{-1}\alpha(gx)), \tag{10.1}$$

where $g \in H(\mathcal{L})$. (The groups we deal with are abelian, so (10.1) defines a representation.) We begin with the following

Lemma 10.1 *Let n be even. Then $\rho_g : X_n \to X_n\ \forall g \in SH(\mathcal{L})$.*

Proof By the definition, it suffices to show that if $\psi(gx)$ satisfies (6.3). We check this condition. Recalling (6.3), we have $\psi(g(x+t)) = e^{i(\frac{n}{2}gx\cdot Jgt + c_{gt})}\psi(gx)$. One can compute easily that $gx \cdot Jgt = x \cdot Jt, \forall g \in SH(\mathcal{L})$. Furthermore by (6.5), we have $e^{ic_{gt}} = e^{ic_t}$, for n even, and hence the result follows. $\square$

Let $F(\lambda, u)$, $u = (\psi, \alpha)$, be the map defined by the l.h.s. of (6.8) and $u_0 = (0, 0)$, the normal state. Since the map $F(\lambda, u)$ is rotationally, translationally and gauge covariant and $\rho_g a^n = a^n\ \forall g \in H(\mathcal{L})$ (recall, $a = a^n + \alpha$), we have the following lemma

Lemma 10.2 *Let $\tilde{T}_\chi^{gauge} : (\psi, \alpha) \to (e^{i\chi}\psi, \alpha)$. Then*

$$F(\lambda, \rho_g u) = \rho_g F(\lambda, u),\ \ g \in H(\mathcal{L}), \tag{10.2}$$

$$F(\lambda, \tilde{T}_\chi^{gauge} u) = \tilde{T}_\chi^{gauge} F(\lambda, u),\ \ \forall \chi \in \mathbb{R}. \tag{10.3}$$

Define $\tilde{\rho}_g := \tilde{T}_{-\chi_g}^{\mathrm{gauge}}\rho_g$, where χ_g are some constants. (If χ_g is a representation of G, then so is $\tilde{\rho}_g$. By Proposition C.1, we do not loose any generality by assuming χ_g are constants.) The lemma above leads to the following

Proposition 10.3 *Let n be even. Then for ρ given in (10.1), we have*

$$\tilde{\rho}_g \operatorname{Null}_{X_n} dF(\lambda, u) = \operatorname{Null}_{X_n} dF(\lambda, \tilde{\rho}_g u), \ \forall g \in H(\mathcal{L}). \tag{10.4}$$

Hence, if G is a subgroup of $H(\mathcal{L})$ and $\tilde{\rho}_g u_* = u_*, \forall g \in G$, then the subspace $\operatorname{Null} dF(\lambda, u_*)$ *is invariant under the action of G.*

Proof Indeed, differentiating $F(\lambda, \tilde{\rho}_g u) = \tilde{\rho}_g F(\lambda, u)$ w.r. to u, we obtain

$$dF(\lambda, \tilde{\rho}_g u)\rho_g \xi = \tilde{\rho}_g dF(\lambda, u)\xi,$$

which gives (10.4). □

Clearly, $\tilde{\rho}_g u_0 = u_0, \ \forall g \in H(\mathcal{L})$, for the normal state $u_0 := (0, 0)$, so, by Proposition 10.3,

$$\operatorname{Null}_{X_n} dF(\lambda, u_0) \ \text{ is invariant under } \rho_g, \ \forall g \in H(\mathcal{L}). \tag{10.5}$$

Recall that $dF(\lambda, u_0) = A_\lambda$. By formula (8.3), it suffices to concentrate on $\operatorname{Null}_{\mathscr{H}_n^2}(L^n - n)$. The action ρ_g induces the action, ρ'_g onψ's:

$$\rho'_g \psi(x) = \psi(g^{-1}x), \ \forall g \in SH(\mathcal{L}). \tag{10.6}$$

Since $\operatorname{Null}_{\mathscr{H}_n^2 \times \vec{\mathscr{H}}^2} A_n$ is invariant under $\tilde{\rho}_g$ and due to formula (8.3), we conclude

Corollary 10.4 *Let n be even. Then* $\operatorname{Null}_{\mathscr{H}_n^2}(L^n - n)$ *is invariant under the gauge and* (10.6) *transformations, and therefore under $\tilde{\rho}'_g, \ \forall g \in SH(\mathcal{L})$, where $\tilde{\rho}'_g := e^{-i\chi_g}\rho'_g$, the restriction of $\tilde{\rho}_g$ to ψ's.*

For a subgroup $G \subset G(\mathcal{L})$, we require that a solution in question is $G-$*equivariant* w.r.to this action, in the sense that it satisfies

$$\rho_g u = \tilde{T}^{\text{gauge}}_{\chi_g} u. \tag{10.7}$$

where $u = (\psi, \alpha)$ and $\tilde{T}^{\text{gauge}}_{\chi} : (\psi, \alpha) \to (e^{i\chi}\psi, \alpha)$, for some functions χ_g (satisfying the corresponding co-cycle condition). (It turns out it is sufficient to assume that χ_g are constants, see Proposition C.1.)

Note that, if $u = (\psi, \alpha)$ satisfies (10.7), then ψ obeys the equivariance condition

$$\rho'_g \psi = \xi_g \psi, \quad \xi_g := e^{i\chi_g}, \quad g \in G. \tag{10.8}$$

Now, let G be a subgroup of $H(\mathcal{L})$ with the irreducible representations labelled by σ. We define the subspaces

$$X_{n\sigma} \subset X_n : \tilde{\rho}\big|_{X_{n\sigma}} \ \text{ is multiple of } \tilde{\rho}^\sigma, \tag{10.9}$$

$$Y_{n\sigma} \subset Y_n : \tilde{\rho}\big|_{X_{n\sigma}} \ \text{ is multiple of } \tilde{\rho}^\sigma. \tag{10.10}$$

Then $F : \mathbb{R} \times X_{n\sigma} \to Y_{n\sigma}$. Now, our goal is to choose G and σ such that $\mathrm{Null}_{X_{n\sigma}}\, dF(\lambda, u_0)$ is one-dimensionall at the bifurcation point $\lambda = n$. Then Theorem 8.1, with the spaces X and Y, appearing in (7.1), given by $X = X_{n\sigma}$ and $Y = Y_{n\sigma}$, would be applicable and would give the desired result, Theorem 1.4.

Note that for any G with $\rho_g u_0 = u_0, \forall g \in G$, the bifurcation (7.11) is invariant under ρ_g,

$$\gamma(\lambda, \rho_g v) = \gamma(\lambda, v). \tag{10.11}$$

10.2 Discrete Subgroups of $SO(2)$

As was discussed above the maximal symmetry group of $\mathrm{Null}_{X_n} A_n$ is the group $G(\mathcal{L}) \cap SO(2) = H(\mathcal{L}) \cap SO(2)$. The Crystallographic restriction theorem says that $H(\mathcal{L})$ is either the cyclic, C_k, or dihedral, D_k, group, with $k = 1, 2, 3, 4, 6$. Since the reflection is not a holomorphic map, we can rule out D_k as follows. We identify $z = x_1 + iz_2$. First we note that D_k has the presentation $\langle r, s \mid r^k = s^2 = (sr)^2 = 1\rangle$. The group D_k induces action on ψ's via rotation

$$r \cdot \psi(z) = \psi(Rz) \tag{10.12}$$

where R is rotation by $2\pi/k$ and reflection (we take it to be reflection about x_1-axis for simplicity, by conjugation by powers of r, we can generate all other reflections)

$$s \cdot \psi(z) = \psi(\bar{z}) \tag{10.13}$$

If we take D_k group as our symmetry group, then we would require, in particular,

$$\psi(\bar{z}) = s \cdot \psi(z) = e^{it}\psi(z) \tag{10.14}$$

for some constant phase e^{it}. If ψ is in the null space of the linear operator, we pass it to theta functions by substituting $\psi = e^h\theta$ for $h = e^{C(ix_1x_2 - x_2^2)}$ where C is a real constant and theta holomorphic. This means

$$e^{h(\bar{z}) - h(z)}\theta(\bar{z}) = e^{it}\theta(z). \tag{10.15}$$

We note $h(\bar{z}) - h(z) = C(-ix_1x_2 - x_2^2) - C(ix_1x_2 - x_2^2) = -2Cix_1x_2 = -C(z^2 - (\bar{z})^2)/2$. Hence

$$e^{C(\bar{z})^2/2}\theta(\bar{z}) = e^{it}e^{Cz^2/2}\theta(z). \tag{10.16}$$

Note that the LHS is a function of $\bar{z}$ and the right hand side is a function of z. This is not possible unless each side is a constant. Substitute $z = x \in \mathbb{R}$ to get $e^{Cz^2/2}\theta(z) = 0$ and, thus, $\theta(z) = 0$. Consequently, we consider subgroups C_k only.

The case $k = 1$ is trivial and gives us nothing new. Hence as a symmetry group, G, we take one of the cyclic group of rotations, C_k, of order $k = 2, 3, 4, 6$.

For $k = 3$, the lattice whose symmetry group is C_3 is the hexagonal lattice. So it is to our advantage to consider C_6 instead for a stronger symmetry reduction. The case $k = 4$ corresponds to square lattice, the proof of existence in this case is similar to the case $k = 6$ but requires a smaller selection of flux n's. Thus, we consider only C_2 and C_6.

The group C_k is generated by a rotation $R_k \in SO(2)$ by the angle $2\pi/k$. If we identify $\mathbb{R}^2$ with $\mathbb{C}$, under $(x_1, x_2) \leftrightarrow x_1 + ix_2$, then R_k is identified with the multiplication by

$$\xi_k = e^{2\pi i/k} \in U(1).$$

We can specify the action (10.1) and (10.6) to the present group by defining

$$\rho_k(\psi(x), \alpha(x)) = (\psi(R_k^{-1}x), R_k\alpha(R_k^{-1}x)), \tag{10.17}$$

$$\rho_k'\psi(x) = \psi(R_k^{-1}x), \tag{10.18}$$

where $k \in \mathbb{Z}$. Then the equivariance conditions (10.7) and (10.8) become, respectively,

$$\rho_k u = \tilde{T}^{\text{gauge}}_{r\chi_k}u, \ \xi_k := e^{i\chi_k}, \qquad \rho_k'\psi = \xi_k^r\psi. \tag{10.19}$$

for some $r \in \mathbb{Z}$. Thus the group representation problem is eventually reduced to the eigenvalue problem for the operator ρ_k'.

10.3 Spaces X and Y

Since the groups C_k are finite abelian groups, their irreducible unitary representations are 1-dimensional and, on $\mathcal{H}^2$, coincide with the eigenspaces of the operator ρ_k'. Since ρ_k' is unitary and satisfies

$$(\rho_k')^k = \mathbf{1}, \tag{10.20}$$

it has exactly k eigenvalues, $\xi_k^r = e^{2\pi ir/k}$, $r = 0, \ldots k-1$. In this case, we specify our spaces for (ψ, α)'s as

$$X_{n,k,r} := \{u \in X_n : \rho_k u = \tilde{T}^{\text{gauge}}_{r\chi_k}u\}, \tag{10.21}$$

$$Y_{n,k,r} = \{u \in Y_n : \rho_k u = \tilde{T}^{\text{gauge}}_{r\chi_k}u\}. \tag{10.22}$$

and the corresponding spaces for ψ's as:

$$X'_{n,k,r} := \{\psi \in \mathscr{H}_n^2 : \rho_k'\psi = \xi_k^r\psi\}, \tag{10.23}$$

$$Y'_{n,k,r} = \{\psi \in \mathscr{L}_n^2 : \rho_k'\psi = \xi_k^r\psi\}. \tag{10.24}$$

Then, by Lemma 10.2, $F : \mathbb{R} \times X_{n,k,r} \to Y_{n,k,r}$, so condition (7.1) holds.

These are the X, X', Y and Y' spaces of Section 7 (see (7.1)).

10.4 Multiplicity (Spaces $V_{n,k,r}$)

Let n be the flux quantum number. For $n = 1, 2, \ldots, k = 1, 2, \ldots, r = 0, 1, 2, \ldots, k-1$, we define the spaces

$$\tilde{V}_n := \mathrm{Null}_{X_n}(L^n - n) \text{ and } \tilde{V}_{n,k,r} := \mathrm{Null}_{X'_{n,k,r}}(L^n - n). \tag{10.25}$$

Our first goal is to prove the following

Theorem 10.5 *Let n be even. Then $\tilde{V}_{n,k,r}$ is one dimensional for $k = 6$ and for the pairs*

$$\begin{aligned} (n, r) = {} & (2, 0), (2, 2), (4, 0), (4, 1), (4, 2), (4, 4), && (10.26)\\ & (6, 1), (6, 2), (6, 3), (6, 4) && (10.27)\\ & (8, 1), (8, 3), (8, 4), (8, 5) && (10.28)\\ & (10, 3), (10, 5) && (10.29) \end{aligned}$$

To prove this theorem, we pass to the corresponding spaces of theta functions. The latter are more rigid since they are holomorphic.

By the definition, the space $\tilde{V}_n$ is related to the space V_n, defined in Proposition 5.1 as

$$\tilde{V}_n = f_n V_n, \ f_n(x) := e^{\frac{in}{2} x_2(x_1 + i x_2)} = e^{-\frac{C}{2}(|z|^2 - z^2)}, \tag{10.30}$$

where $C := \frac{\pi n}{\mathrm{Im}\,\tau}$ and $z := (x_1 + i x_2)/\sqrt{\frac{2\pi}{\mathrm{Im}\,\tau}}$, or in terms of the functions,

$$\psi(x) = f_n(z)\theta(z), \ f_n(z) := e^{\frac{in}{2} x_2(x_1 + i x_2)} = e^{-\frac{C}{2}(|z|^2 - z^2)}. \tag{10.31}$$

Elements, θ, of the subspace V_n, will be called *n-theta functions*. Similarly, we define the spaces $V_{n,k,r}$ by

$$\tilde{V}_{n,k,r} = f_n V_{n,k,r}. \tag{10.32}$$

We define the induced action on theta functions via $T_{n,k} := f_n^{-1} \tilde{\rho}'_{k,j} f_n$. We have

Lemma 10.6 *The operator $T_{n,k}$ is unitary and satisfies $(T_{n,k})^k = \mathbf{1}$. Consequently, its spectrum consists of the eigenvalues of the form ξ_k^r for some $r = 0, \ldots, k-1$. Moreover, the eigenfunctions corresponding to the eigenvalue ξ_k^r has zero at $z = 0$ of the order r.*

Proof Equation (10.20) and the definition $T_{n,k} := f_n^{-1} \tilde{\rho}'_{k,j} f_n$ imply the first claim. To show the second claim, let λ be any eigenvalue. Expanding $\theta(z) = a z^m + O(|z|^{m+1})$, where $a \neq 0$ and $m \geq 0$, and $e^x = 1 + O(|x|)$ and writing out the eigenvalue equation, we see that to lowest order in z,

$$\lambda a z^m = a \xi_k^m z^m \tag{10.33}$$

Hence $\lambda = \xi_k^m$. □

Corollary 10.7 *Let n be even. Then $V_{n,k,r}$ are eigenspaces of the operator $T_{n,k}$ corresponding to the eigenvalues ξ_k^r.*

We recall that the Wigner–Seitz cell around a lattice point is defined as the locus of points in space that are closer to that lattice point than to any of the other lattice points. To eliminate the overlap between the Wigner–Seitz cells around different points, we agree on the choice of their boundaries. Say, observing that the Wigner–Seitz cell is a (slanted) hexagon, we set the boundary of a Wigner–Seitz cell to contain the *three left-most edges and the two left-most vertices* (see Fig. 1). Hence Wigner–Seitz cells of a lattice tile $\mathbb{R}^2$ without an intersection.

By a standard result about theta functions (see Theorem B.2 of Appendix B) or line bundles, theta functions are entirely determined by their zeros, z_j, and multiplicities, $m(z_j)$, in a Wigner–Seitz cell, W. By analogy with holomorphic sections of line bundles, we call the collection of zeros and multiplicities of a theta function, θ, its divisor and denote $\text{div}(\theta) = \sum_{z\in W} m(z)z$. The degree of a theta function, θ, is defined as the degree of its divisor, $|div(\theta)| = \sum_{z\in W} m(z)$. Then $\theta \in V_n \iff |div(\theta)| = n$.

Corollary 10.7 and Lemma 10.6 and standard results about theta functions mentioned above imply

Corollary 10.8 $\theta \in V_{n,k,r} \iff$ *the following three conditions hold: (a) $|div(\theta)| = n$ (i.e. θ has n zeros counting their the multiplicities); (b) $m(0) = r$ (i.e. θ has the zero of the multiplicity r at the origin); (c) $div(T_{n,k}(\theta)) = div(\theta)$ (i.e. $div(\theta)$ is invariant under the transformation $T_{n,k}$ (i.e. rotation by $2\pi/k$)).*

10.4.1 C_6

By Corollary 10.8, we want to translate the eigenvalue problem $T_{n,6}\theta = \xi^r\theta$ into the existence of divisors corresponding to the zeros of θ. This would allows us to find 1-1 correspondence between all such θ and simple diagrams for our analysis.

Let div (divisor) denote a finite collection of points in the Wigner-Seitz cell W, centred at the origin, together with their multiplicities, i.e. a map from W to $\mathbb{Z}^+$ with a finite number of non-zero values. We can identify the divisors with the diagrams as in Fig. 1 (the WS cell with the choice of points and multiplicities), the latter provide handy illustrations. Then we obtain a map

$$\text{Div} : \text{theta functions} \to \text{divisors/diagrams}, \tag{10.34}$$

Since we are interested in eigenvectors of $T_{n,6}$, we restrict div to the set of eigenvalues of $T_{n,6}$. In particular, let

$$V^{\text{div}}_{n,k,r} := \{\text{div} : |\,\text{div}\,| = n,\ |\,\text{div}(0)| \equiv r \bmod k,\ T_{n,k}\,\text{div} = \text{div}\}.$$

We have the following result, proven in Section 11:

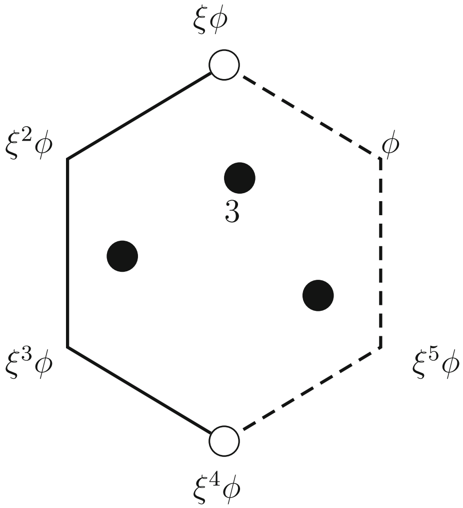


Fig. 1 Typical diagram of a divisor. The black dots denote nonzero point on W. Each black dot is assumed to have multiplicity 1 unless otherwise indicated by a number next to it

Theorem 10.9 (Classification Theorem for C_6-invariant Theta Functions) *The map* $\mathrm{Div} : V_{n,6,r} \to V^{\mathrm{div}}_{n,6,r}$ *is a bijection, and in particular*

$$\dim V_{n,6,r} = \dim V^{\mathrm{div}}_{n,6,r}.$$

To compute $\dim V^{\mathrm{div}}_{n,k,r}$ it is convenient to give each point of W the index which is the number of elements in the orbit under $T_{n,k}$ generated by this point. Thus, for $k = 6$, all interior points of W and all boundary points, besides the vertices and the midpoints of the edges, have the index 6. The boundary vertices and the midpoints of the edges have the indices 2 and 3, respectively, and the origin has the index 1.

By the orbit-stabiliser theorem, there is no divisor with index 4 or 5 where the multiplicity is simple at each point, since 4 and 5 do not divide 6.

We identify orbits with the same index. Denote the multiplicity of points in the orbit of the index i by m_i, so that $m_1 = r$. Then we have the relation

$$\sum_i i m_i \equiv 1 \cdot m_1 + 2m_2 + 2m_3 + 6m_6 = n. \tag{10.35}$$

We use this equation to classify the diagrams to obtain

Theorem 10.10 *Let n be even. Then $V_{n,k,r}$ is one dimensional for $k = 6$ and for the pairs*

$$(n, r) = (2, 0), (2, 2), (4, 0), (4, 1), (4, 2), (4, 4), \tag{10.36}$$
$$(6, 1), (6, 2), (6, 3), (6, 4) \tag{10.37}$$
$$(8, 1), (8, 3), (8, 4), (8, 5) \tag{10.38}$$
$$(10, 3), (10, 5) \tag{10.39}$$

This result implies Theorem 10.5. A table describing the explicit spanning theta functions for $V_{n,6,r}$ can be found in Appendix D

10.4.2 C_2

For $\xi_2 = -1$, the corresponding irreducible representations of C_2 are simply even and odd functions. By the correspondence (10.31), the evenness and oddness of ψ translates to the same property of θ. Hence, we easily see that linear compatibility is satisfied as V_n can be decomposed into odd and even functions

Lemma 10.11 *Let $V_n = V_{n,\text{even}} \oplus V_{n,\text{odd}}$ be the decomposition of V_n into even and odd functions. Then* $\dim V_{n,\text{even/odd}} \geq 1$

Proof Let $\theta_0, ..., \theta_{n-1}$ be the standard basis for the set of theta functions as in Theorem A.2. We recall that

$$\theta_m(-z) = \theta_{n-m \text{ mod } n}(z) \tag{10.40}$$

This shows that the set of odd theta functions are spanned by

$$\sigma_{j,-}(z) = \theta_j(z) - \theta_{n-j}(z) \tag{10.41}$$

Similarly, the even functions are spanned by

$$\sigma_{j,+}(z) = \theta_j(z) + \theta_{n-j}(z) \tag{10.42}$$

□

Corollary 10.12 *If $n = 3$, then* $\dim V_{n,k,1} = \dim V_{n,odd} = 1$ *and consequently* $\dim \tilde{V}_{n,k,1} = \dim \tilde{V}_{n,odd} = 1$.

Remark 10.13 (The null space of M) We claim that null$M = 0$ on $X_{n,k,r}$. Indeed, we see that $(\psi, a) \in X_{n,k,r}$ only if $R_k^{-1} a(R_k x) = a(x)$ where R_k represents rotation by $2\pi/k$ (c.f. equation (10.22) and (10.23)). This means

$$\int_\Omega R^{-1} a(R_k x) dx = R_k^{-1} \int_\Omega a(R_k x) dx = R_k^{-1} \int_\Omega a(x) dx \tag{10.43}$$

if we choose a fundamental cell Ω to be invariant (up to a boundary set) under C_k. This means, if $R_k \neq 1$, then $\int_\Omega a(x)dx = 0$. Since the null space of M, on X, consists of constants only. Hence the statement follows.

10.5 Irreducibility

The main result of this subsection is

Theorem 10.14 *The spanning theta function of $V_{n,k,j}$ is irreducible for the pairs*

$$(n, j) = (4, 0), (6, 3), (8, 5), (10, 5),$$

and for n prime.

Using (10.30), irreducibility of ψ translate to irreducibility of θ. We say that θ is *reducible (to $\mathcal{L}'$)* if there is a finer lattice, $\mathcal{L}'$, containing $\mathcal{L}$ s.t. the corresponding ψ is gauge periodic with respect to $\mathcal{L}'$. Otherwise we say that θ is *irreducible*. We now prove irreducibility of θ below. Theorem 10.14 follows from Propositions 10.15 and 10.17 below.

10.5.1 Irreducibility for C_6 Symmetry

Proposition 10.15 *The spanning theta function of $V_{n,k,j}$ is irreducible for the pairs*

$$(n, j) = (4, 0), (6, 3), (8, 5), (10, 5).$$

Proof To prove irreducibility, we need the following basic lemma: □

Lemma 10.16 *Let $\mathcal{L} \subset \mathcal{L}'$ be lattices. Let $\Omega_{\mathcal{L}}$ be any fundamental cell of $\mathcal{L}$. Then precisely one of the following holds: there is a $v \in \mathcal{L}'$ such that $v \in \Omega_{\mathcal{L}} \backslash \mathcal{L}$ or $\mathcal{L}' = \mathcal{L}$.*

Proof Assume that no such $v \in \mathcal{L}'$ with $v \in \Omega_{\mathcal{L}} \backslash \mathcal{L}$ exists. That is, every $v \in \mathcal{L}'$ such that $v \in \Omega_{\mathcal{L}}$ is contained in $\mathcal{L}$. Since translates of $\Omega_{\mathcal{L}}$ tiles the entire plane and $\mathcal{L} \subset \mathcal{L}'$, we conclude that every element of $\mathcal{L}'$ is in $\mathcal{L}$. That is, $\mathcal{L} = \mathcal{L}'$. □

Now, by choice of theta functions indicated in Table D.1, we see that for vortex number n, the number of zeros of the chosen theta at the origin differs from the number of zeros at any other point in $\Omega_{\mathcal{L}}$. If $\mathcal{L}'$ is any finer lattice containing $\mathcal{L}$ with respect to which our solution is gauge periodic, then Lemma 10.16 implies that the number of flux per fundamental cell of $\mathcal{L}$ for our chosen theta $=$ (number of zero at the origin) $+ n > n$. This is a contradiction.

10.5.2 Irreducibility of Odd Theta Functions with Prime Flux

Proposition 10.17 *Let θ be an odd theta function with prime flux p. Then θ is irreducible.*

Proof Let θ be gauge periodic with respect to $\mathcal{L}$. Let $\mathcal{L} \subset \mathcal{L}'$ be any finer lattice. Let q denote the number of zeros of θ in a fundamental cell of $\mathcal{L}'$. We first claim that $q \mid p$. Let u, v be the generators of $\mathcal{L}'$ and $\Omega_{\mathcal{L}'}$ be the fundamental cell of $\mathcal{L}'$ formed by taking the convex hall of u and v (together with appropriate boundary). Define an equivalence relationship as follows: two translates of $\Omega_{\mathcal{L}'}$, $s + \Omega_{\mathcal{L}'}$ and $s' + \Omega_{\mathcal{L}'}$ for $s, s' \in \mathcal{L}'$, are said to be equivalent if $s - s' \in \mathcal{L}$. Let $s_1 + \Omega_{\mathcal{L}'}, ..., s_k + \Omega_{\mathcal{L}'}$ be maximally inequivalent for $s_1, ..., s_k \in \mathcal{L}'$. Since translates of $\Omega_{\mathcal{L}'}$ tile the entire plane, we conclude that by appropriate translates of the s_j's, $s_1 + \Omega_{\mathcal{L}'} \cup ... \cup s_k + \Omega_{\mathcal{L}'}$ is a fundamental domain of $\mathcal{L}$. In particular, $p = qk$ since θ has the same number of zeros in each fundamental cell of $\mathcal{L}'$. Since p is prime, either $q = p$ or $q = 1$. If $q = p$, then $\mathcal{L} = \mathcal{L}'$. Otherwise $q = 1$. Since $0, 1/2, \tau/2$ are zeros of θ and each fundamental cell $s + \Omega_{\mathcal{L}'}$ has exactly one zero, we conclude that $0, 1/2, \tau/2 \in \mathcal{L}'$.

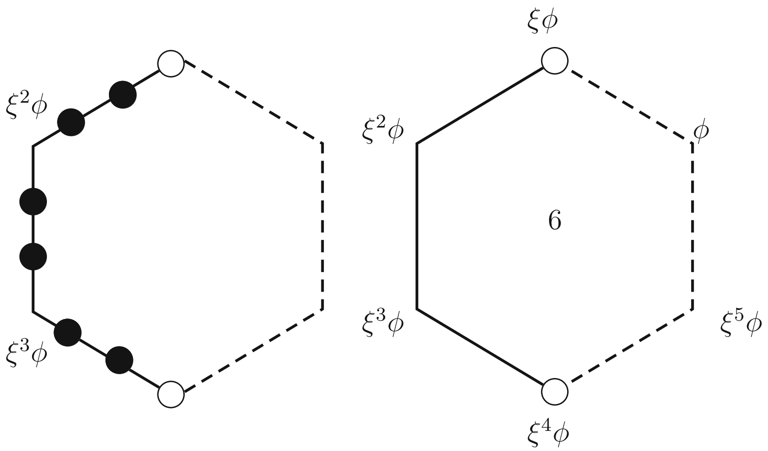


Fig. 2 Index 6 divisors. The figure on the left has six distinct dots on its left most 3 edges, forming an orbit for C_6. The figure on the r.h.s. indicates six distinct dots forming an orbit of C_6 in the interior of the WS cell

So in particular $\mathcal{L} \subset \frac{1}{2}(\mathbb{Z} + \tau\mathbb{Z})$ and ψ is gauge periodic with respect to $\frac{1}{2}(\mathbb{Z} + \tau\mathbb{Z})$. This is clearly not possible, for otherwise $4 \mid p$ is a contradiction. $\square$

11 Proof of Theorem 10.9

We note that V^{div} is the subset of the $\mathbb{Z}$-module of divisors generated by diagrams of the form in Figs. 2–5 such that the multiplicity of each point is non-negative. So we show that there is a theta function corresponding to each such diagram. This would show that $V^{\text{div}} \subset \text{Div}(V^{EV})$. In what follows $\tau = \xi = e^{\pi i/3}$ as before. Existence

Fig. 3 Pictorial discription of θ_2

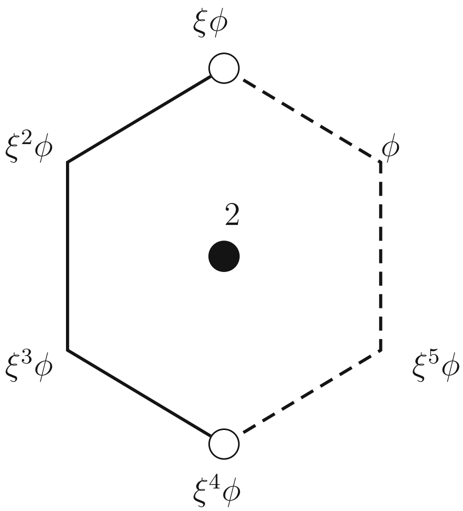
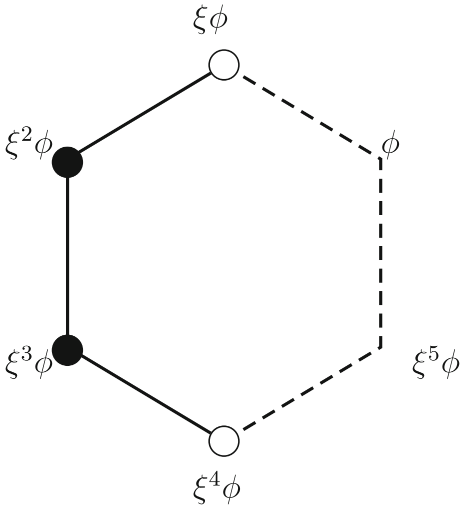


Fig. 4 Pictorial description of θ_0



of θ_2 is a direct result of Theorem A.6 with $n = 2$. Namely, the set of permissible double zeros for theta functions in V_2 is just $\frac{1}{2}(\mathbb{Z} + \tau\mathbb{Z})$.

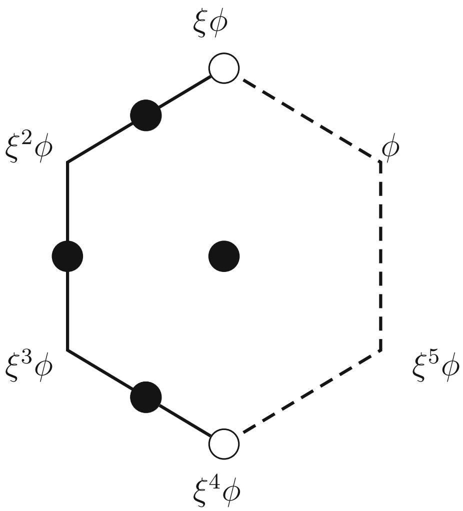
To construct θ_0, let $\theta_{2,0}, \theta_{2,1}$ be a basis for V_2 as in Theorem A.2. Form the function

$$\sigma(z) := \det \Sigma(z) := \det \begin{pmatrix} \theta_{2,0}(z) & \theta_{2,1}(z) \\ \theta_{2,0}(-z) & \theta_{2,1}(-z) \end{pmatrix} \tag{11.1}$$

By Theorem A.2, $\theta_{2,i}(z)$ is symmetric about 0 for $i = 0, 1$ when $n = 2$, thus $\sigma(z) = 0$ identically. In particular, for $z_0 = \frac{1}{4}(\tau + 1)$, there are constants c_0, c_1 such that $c_0\theta_{2,0} + c_1\theta_{2,1}$ has two simple zeros located at $z_0, -z_0$, respectively.

The theta function θ_4 is the Wronskian, Θ, of $\theta_{2,0}$ and $\theta_{2,1}$. For $n = 2$, Theorem A.7 shows that the location of the zeros of Θ are precisely the set of permissible zeros

Fig. 5 Pictorial description of θ_4



for a singular 2-theta function. In this case, it is $\frac{1}{2}(1+\tau)$ by Theorem A.6. It matches the definition of θ_4.

Finally, we show existence of theta functions with 6 distinct zeros on the WS-cell. Let $a_1, a_2, a_3 \in \mathbb{C}$. Consider

$$\sigma(z) := \det \begin{pmatrix} \theta_{6,0}(z+a_1) & \theta_{6,1}(z+a_1) & \cdots & \theta_{6,5}(z+a_1) \\ \theta_{6,0}(z-a_1) & \theta_{6,1}(z-a_1) & \cdots & \theta_{6,5}(z-a_1) \\ \theta_{6,0}(z+a_2) & \theta_{6,1}(z+a_2) & \cdots & \theta_{6,5}(z+a_2) \\ \theta_{6,0}(z-a_2) & \theta_{6,1}(z-a_2) & \cdots & \theta_{6,5}(z-a_2) \\ \theta_{6,0}(z+a_3) & \theta_{6,1}(z+a_3) & \cdots & \theta_{6,5}(z+a_3) \\ \theta_{6,0}(z-a_3) & \theta_{6,1}(z-a_3) & \cdots & \theta_{6,5}(z-a_3) \end{pmatrix} \tag{11.2}$$

Recalling that $\theta_{n,m}(-z) = \theta_{n,n-m \bmod n}(z)$ by Theorem A.2, we see that

$$\sigma(-z) \tag{11.3}$$

$$= \det \begin{pmatrix} \theta_{6,0}(z-a_1) & \theta_{6,5}(z-a_1) & \theta_{6,4}(z-a_1) & \theta_{6,3}(z-a_1) & \theta_{6,2}(z-a_1) & \theta_{6,1}(z-a_1) \\ \theta_{6,0}(z+a_1) & \theta_{6,5}(z+a_1) & & \cdots & & \theta_{6,1}(z+a_1) \\ \theta_{6,0}(z-a_2) & \theta_{6,5}(z-a_2) & & \cdots & & \theta_{6,1}(z-a_2) \\ \theta_{6,0}(z+a_2) & \theta_{6,5}(z+a_2) & & \cdots & & \theta_{6,1}(z+a_2) \\ \theta_{6,0}(z-a_3) & \theta_{6,5}(z-a_3) & & \cdots & & \theta_{6,1}(z-a_3) \\ \theta_{6,0}(z+a_3) & \theta_{6,5}(z+a_3) & & \cdots & & \theta_{6,1}(z+a_3) \end{pmatrix}$$

$$= (-1)^3 \det \begin{pmatrix} \theta_{6,0}(z+a_1) & \theta_{6,5}(z+a_1) & \cdots & \theta_{6,1}(z+a_1) \\ \theta_{6,0}(z-a_1) & \theta_{6,5}(z-a_1) & \cdots & \theta_{6,1}(z-a_1) \\ \theta_{6,0}(z+a_2) & \theta_{6,5}(z+a_2) & \cdots & \theta_{6,1}(z+a_2) \\ \theta_{6,0}(z-a_2) & \theta_{6,5}(z-a_2) & \cdots & \theta_{6,1}(z-a_2) \\ \theta_{6,0}(z+a_3) & \theta_{6,5}(z+a_3) & \cdots & \theta_{6,1}(z+a_3) \\ \theta_{6,0}(z-a_3) & \theta_{6,5}(z-a_3) & \cdots & \theta_{6,1}(z-a_3) \end{pmatrix}$$

$$= (-1)^3(-1)^2\sigma(z) \tag{11.4}$$

$$= -\sigma(z) \tag{11.5}$$

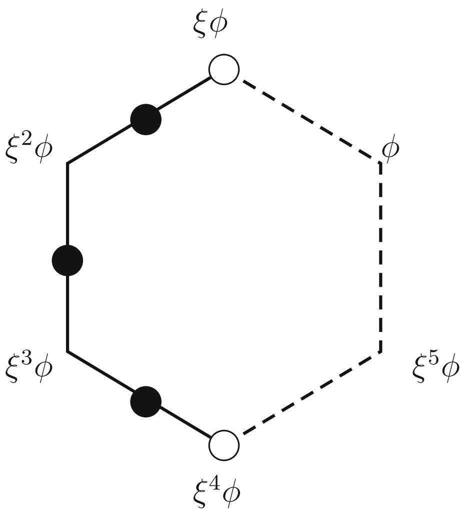
where the factor $(-1)^3$ arises from interchanging the $2i-1$ and $2i$-th row for $i = 1, 2, 3$. The $(-1)^2$ factor occurs after interchanging the second and the 6-th column and interchanging the third and the fourth column. So we have that $\sigma(0) = 0$. This proves the desired claim that $\sigma(z)$ has a kernel.

To prove that $\mathrm{Div}(V^{EV}) = V^{\mathrm{div}}$, we study orbits of C_6 on the WS-cell. We will use the divisor and theta function picture (see (10.34)) interchangeably. We find all possible orbits of the action of C_6 on the WS-cell. The only choice of having index 1 is the case where the origin has index 1. The only index 2 possibility where each point has index one is shown in Fig. 4. Then we have index 3 divisors. The possible location of points on W with index 3 each with multiplicity 1 is as shown in Fig. 6

By the orbit-stablizer theorem, there is no divisor with index 4 or 5 where the multiplicity is simple at each point, since 4 and 5 do not divide 6. Finally, we consider the index 6 case. The possible divisors are shown in Fig. 2.

Now we are ready for the proof of Theorem 10.9. First note that injectivity of the map Div is a direct consequence of Proposition A.3. Now if θ is any C_6-equivariant theta function, its zeros are unions of orbits of C_6. We may divide θ by C_6-equivariant theta functions corresponding to elements of V^{div} to produces new theta functions with fewer zeros in the Wigner-Seitz cell. Note that this division process preserves

Fig. 6 Index 3 divisor

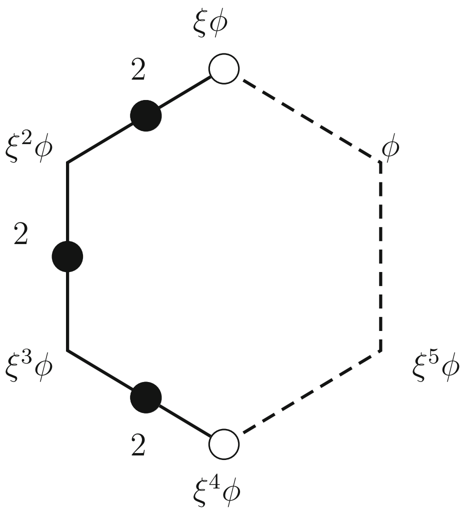


C_6-equivariance. We repeat this process until any further division results in a non-theta-function. We claim that the resulting function, σ, is a complex number. If so, we have completely factor θ by theta functions from V^{div} and the bijection is established.

Now, we study σ. σ cannot have any zeros that form an orbit of C_6 of size 6, otherwise they can be removed by dividing by an element from V^{div}, contradicting the definition of σ. It can neither have zeros that form orbits of size 2 for the same reason. Hence, the zeros of σ can only be in the following two configuration: one zero at the origin, or as shown in Fig. 6. To see this, if there are zeros as in Fig. 6, but with higher multiplicity, we divide σ by $\theta_2^{-1}\theta_4^2$ as shown in Fig. 7 to remove all the multiplicities. Likewise we can divide by θ_2 to remove even multiplicity at the origin.

So, we may assume that σ has a simple zero at the origin. Indeed, if σ has three zeros as in Fig. 6, we divide it by $\theta_2^{-1}\theta_4$ to obtain a theta function with a single zero

Fig. 7 Pictorial description of $\theta_2^{-1}\theta_4^2$



at the origin. But this is not allowed as V_1 is 1-dimensional and whose generator has zero at $\frac{1}{2}(1+\tau)=\frac{1}{2}(1+\xi)$ by Proposition 11.1 below. Thus this case never occurs and the proof is complete.

To prove the assertion above, we pass the problem back to linear solutions ψ from the theta functions via (5.11) and use the complexified co-ordinates $x = x_1 + ix_2$. We have

Proposition 11.1 *Let ψ satisfy the gauge-periodicity condition* (5.2)*. Then it has zero at $\frac{1}{2}(1+\tau)$. In particular, when $n = 1$, ψ does not vanish at 0. Thus by uniqueness of theta functions, we conclude there is no theta function with a single zero at the origin (with our imposed boundary condition).*

Proof Let us denote by $z = \frac{1}{2}(1+\tau)$. Due to the quasiperiodic boundary conditions

$$\psi(y+1) = e^{\frac{ikny_2}{2}}\psi(y) \tag{11.6}$$

$$\psi(y+\tau) = e^{\frac{ikn(\tau_1 y_2-\tau_2 y_1)}{2}}\psi(y) \tag{11.7}$$

Applying these relations at the point $z = (-(\tau_1+1)/2, -\tau_2/2)$, we find that

$$\psi(z+1) = e^{-\frac{ikn\tau_2}{4}}\psi(z) \tag{11.8}$$

$$\psi(z+\tau) = e^{\frac{ikn\tau_2}{4}}\psi(z) \tag{11.9}$$

Now, utilising the symmetry $\psi(-x) = \psi(x)$, we deduce that $\psi(z+1) = \psi(z+\tau)$. Thus

$$e^{\frac{ikn\tau_2}{4}}\psi(z) = e^{\frac{-ikn\tau_2}{4}}\psi(z),$$

or equivalently

$$e^{\frac{ikn\tau_2}{2}}\psi(z) = \psi(z).$$

Since $k\tau_2 = 2\pi$, this relation becomes $(e^{in\pi}-1)\psi(z) = 0$, and implies when n is odd that ψ vanishes at z. □

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Appendix

Recall that x and z are related as in (5.11) and identify $x = (x_1, x_2)$ with $x_1 + ix_2$.

A On Solutions of the Linearised Problem

Lemma A.1 *There is no linear solution ψ, as in* Section 5*, such that*

$$\psi(\bar{x}) = e^{ig_r(z)}\psi(x) \tag{A.1}$$

for some real valued g.

Proof Assume for the sake of contradiction that such ψ exists. Then

$$\psi(x) = e^{\frac{n}{4}(z^2-|z|^2)}\theta(z) \tag{A.2}$$

for some holomorphic θ. Equation (A.1) becomes

$$\theta(\bar{z}) = e^{\frac{n}{4}(z^2-\bar{z}^2)}e^{ig_r(z)}\theta(z) \tag{A.3}$$

Taking ∂_z on both sides, we see that

$$0 = e^{\frac{n}{4}(z^2-\bar{z}^2)}e^{ig_r(z)}\left(-\frac{n}{2}\bar{z}\theta + i\theta\partial_z g_r + \theta'\right) \tag{A.4}$$

This shows that the term in the bracket vanishes identically. In particular,

$$\left(-\frac{n}{2}z + i\partial_z g_r\right)\theta = -\theta' \tag{A.5}$$

Taking $\partial_{\bar{z}}$ again, we see that

$$(+i\partial_{\bar{z}}\partial_z g_r)\theta = 0 \tag{A.6}$$

Since θ has at most finitely many zeros, we conclude

$$-\Delta g_r = 0 \tag{A.7}$$

This shows that g_r is harmonic. Since it is periodic, it is a constant. Equation (A.3) then shows that such solution is impossible. □

B Theta Functions

In this appendix we review basic properties of theta functions, which are likely to be known but which we could not find in the literature. From now on, we fix a lattice shape τ and a lattice

$$\mathcal{L}_\tau = \mathbb{Z} + \tau\mathbb{Z} \tag{B.1}$$

throughout this appendix (unless otherwise stated).

B.1 Basic Properties

In this section, we prove some basic properties of the theta functions. Let n be fixed. Define for $0 \le m \le n-1$,

$$\theta_{n,m}(z) = \sum_{l\in[m]_n} \gamma^{l^2} e^{2\pi i l z} \tag{B.2}$$

where $\gamma := e^{\pi i \tau/n}$ and $[m]_n = \{a \in \mathbb{Z} \ : \ a = m \bmod n\}$.

Theorem B.1 *The $\theta_{n,m}$'s form a basis for V_n that satisfy*

1. $\theta_{n,m}(z + \frac{1}{n}) = e^{2\pi i m/n}\theta_{n,m}(z)$
2. $\theta_{n,m}(-z) = \theta_{n,n-m}(z)$
3. $\theta_{n,m}(z + \tau/n) = \gamma^{-1}e^{2\pi i z}\theta_{n,m+1}(z)$

Theorem B.2 *Any n-theta function has exactly n zeros modulo translation by lattice elements. Moreover, any two theta functions that share the same zeros (counting multiplicity) are linearly dependent.*

Theorem B.3 *$\theta_{1,0}$ has a simple zero at $\frac{1}{2}(1+\tau)$*

Proof See Proposition 11.1. □

Theorem B.4 *Suppose that $\theta \in V_n$ and $\sigma \in V_m$, then $\theta\sigma \in V_{n+m}$.*

Proof Inspection. □

The proof of the theorems consists of the following lemmas:

Proof of Theorem A.2 Expanding in $e^{2\pi ikz}$ for $k \in \mathbb{Z}$, the coefficients of any element of V_n satisfies the recurssion $c_{m+n} = c_m e^{i(2m+n)\pi\tau}$. This recursion implies that for $0 \le m \le n-1$, we have that

$$c_{m+ln} = c_m e^{i\pi\tau(l^2 n + 2lm)} \tag{B.3}$$

where l is an integer. So the functions

$$\sum_{k\in\mathbb{Z}} e^{i\pi\tau(k^2 n+2km)} e^{2\pi i(nk+m)z},\ m = 0, ..., n-1 \tag{B.4}$$

form a basis for the eigenspace. If we let $l = kn+m$, then we can rewrite the above as

$$\sum_{l\in[m]_n} e^{i\pi\tau\frac{l^2-m^2}{n}} e^{2\pi ilz} = e^{-i\pi m^2/n}\theta_m \tag{B.5}$$

Now we prove the three bullet points. We note that

$$\theta_m\left(z+\frac{1}{n}\right) = \sum_{l\in[m]_n} \gamma^{l^2} e^{2\pi ilz} e^{2\pi il/n} \tag{B.6}$$

Since $l \in [m]_n$, we have that $l/n - m/n \in \mathbb{Z}$. Hence

$$\theta_m\left(z+\frac{1}{n}\right) = e^{2\pi im/n}\theta_m(z) \tag{B.7}$$

Now for the second item, we note

$$\theta_m(-z) = \sum_{l\in[m]_n} \gamma^{l^2} e^{-2\pi ilz} \tag{B.8}$$

$$= \sum_{l\in[m]_n} \gamma^{(-l)^2} e^{2\pi i(-l)z} \tag{B.9}$$

$$= \sum_{l\in[n-m]_n} \gamma^{l^2} e^{2\pi ilz} \tag{B.10}$$

$$= \theta_{n-m \bmod n}(z) \tag{B.11}$$

Finally, recalling that $\gamma = e^{\pi i\tau/n}$, we note that

$$\theta_m(z+\tau/n) = \sum_{k\in\mathbb{Z}} \gamma^{(kn+m)^2} e^{2\pi i(kn+m)z+2\pi i(kn+m)\tau/n} \tag{B.12}$$

$$= \gamma^{-1}\sum_{k\in\mathbb{Z}} \gamma^{k^2n^2+2knm+m^2+2kn+2m+1} e^{2\pi i(kn+m)z} \tag{B.13}$$

$$= \gamma^{-1}\sum_{k\in\mathbb{Z}} \gamma^{(kn+m+1)^2} e^{2\pi i(kn+m)z} \tag{B.14}$$

$$= \gamma^{-1} e^{-2\pi i z}\theta_{m+1 \bmod n}(z) \tag{B.15}$$

□

Proof of Theorem A.3 First we prove that elements of V_n has exactly n zeros modulo translation by lattice elements. We compute the winding number of θ. First, since θ is holomorphic, its zeros are discrete. Hence we may assume WLOG that all the zeros are in the interior of the fundamental domain. Let Ω denote the fundamental domain. Then the total number of zeros of θ is given by

$$\frac{1}{2\pi i}\int_{\partial\Omega}\frac{\theta'}{\theta}dz \tag{B.16}$$

Since $\theta(z) = \theta(z+1)$, the integral along the $t\tau$ and $t\tau+1$ for $t \in [0, 1]$ is zero. Let $y(z) = e^{-i\pi\tau}e^{\alpha z}$ where $\alpha = -2\pi i$. Since $\theta(z+\tau) = y^n\theta(z)$ and $y' = \alpha y$, we see that $\theta'(z+\tau) = y^n(z)\theta'(z) + n\alpha y^n(z)\theta(z)$. Hence, only the horizonal segment of the line integral contribute:

$$\frac{1}{2\pi i}\int_{\partial\Omega}\frac{\theta'}{\theta}dz = \frac{1}{2\pi i}\int_0^1 \frac{\theta'(t)}{\theta(t)} - \frac{\theta'(1+\tau-t)}{\theta(1+\tau-t)}dt \tag{B.17}$$

$$= \frac{1}{2\pi i}\int_0^1 \frac{\theta'(t)}{\theta(t)} - \frac{y^n(1-t)\theta'(1-t)+n\alpha y^n(1-t)\theta(1-t)}{y^n(1-t)\theta(1-t)}dt \tag{B.18}$$

$$= \frac{1}{2\pi i}\int_0^1 \frac{\theta'(t)}{\theta(t)} - \frac{\theta'(1-t)+n\alpha\theta(1-t)}{\theta(1-t)}dt \tag{B.19}$$

$$= \frac{1}{2\pi i}\int_0^1 \frac{\theta'(t)}{\theta(t)} - \frac{\theta'(1-t)}{\theta(1-t)}dt+n \tag{B.20}$$

$$= n \tag{B.21}$$

Next, we show that any two theta functions that share the same zeros (counting multiplicity) are linearly dependent. Let θ and φ be the two nonzero zeta functions that shares the same zeros. Set $f(z) = \theta(a)/\varphi(z)$. We show that

1. $f(z)$ can be extended analytically to all of $\mathbb{C}$ and
2. $f(z)$ is doubly periodic.

Certainly f is holomorphic away from zeros of φ. We only need to show that f can be extended analytically to zeros of φ. But this is precisely the requirement that θ and φ share the same zeros (counting multiplicity).

For the second item, we note that

$$f(z+1) = \theta(z+1)/\varphi(z+1) = \theta(z)/\varphi(z) = f(z) \tag{B.22}$$

and

$$f(z+\tau) = \frac{\theta(z+\tau)}{\varphi(z+\tau)} = \frac{e^{-2\pi i n z} e^{-\pi i n \tau} \theta(z)}{e^{-2\pi i n z} e^{-\pi i n \tau} \varphi(z)} = \frac{\theta(z)}{\varphi(z)} = f(z) \tag{B.23}$$

This shows that f is doubly periodic.

Now, Liouville's theorem shows that f must be constant. It follows that θ and φ are collinear. □

B.2 Classification of Singular n-theta Functions

Theorem B.5 *Let X_n be the set of singular n-theta functions mod scaling. Then*

$$X_n = \left\{ \theta_0^n \left(z + \frac{1}{n}(a + b\tau) \right) e^{2\pi i b z} : a, b \in \mathbb{Z} \right\} \tag{B.24}$$

where θ_0 is a basis for V_1. Moreover, $|X_n| = n^2$. The location of zeros of elements in X_n form the set

$$\frac{1}{2}(1+\tau) + \frac{1}{n}(\mathbb{Z} + \tau\mathbb{Z}) \tag{B.25}$$

As before, we establish the theorem through various lemmas. The idea of the proof is as follows: by Theorem A.3, we may identify elements of X_n with the location of their zeros. We attempt to locate the zeros of singular n-theta function first and show that there are only n^2 possible locations in a fundamental cell. So $|X_n| = n^2$. Then we explicitly construct n^2 singular n-theta functions to complete the proof.

To locate the zeros of singular n-theta functions, we study the Wronskian of a particular set of nice basis element: $\Theta(z) := \det(\theta_j^{(i)})$ for $i, j \in \{0, ..., n-1\}$, where $\theta_j^{(i)}$ means the i-th derivative of θ_j (see (B.2) for definition θ_j).

Proposition B.6 *The function Θ is holomorphic and*

1. *The locations of the zeros of Θ are exactly the locations where a singular n-theta function can have zero.*
2. $\Theta(-z) = (-1)^{n+1}\Theta(z)$,
3. $\Theta(z + 1/n) = (-1)^{n+1}\Theta(z)$,
4. $\Theta(z + \tau/n) = (-1)^{n+1}\gamma^{n(n-1)}y^n\Theta(z)$ *where $y = e^{-i\pi\tau}e^{\alpha z}$ and $\alpha = -2\pi i$.*

Proof We recall that the θ_m's form a basis for V_n. If $\theta(z) = \sum_m a^m \theta_m(z)$ has n zeros at z_0, then

$$0 = \theta^{(i)}(z_0) = \sum_m a^m \theta_m^{(i)}(z) \tag{B.26}$$

for $i = 0, ..., n-1$. So the matrix $(\theta_j^{(i)}(z_0))$ has a nonzero vector $(a^0, ..., a^{n-1})$ in its kernel. Hence $\Theta(z_0) = 0$. Conversely, if $\Theta(z_0) = 0$, then we can find a nonzero vector $(a^0, ..., a^{n-1})$ in the kernel of the matrix $(\theta_j^{(i)}(z_0))$. Then $\theta = a^m\theta_m$ has n-zeros at z_0.

Recall from Theorem A.2 that $\theta_m(-z) = \theta_{n-m \bmod n}(z)$. It follows that $\theta_m^{(k)}(-z) = (-1)^k \theta_{n-m \bmod n}^{(k)}(z)$. If n is even, then after $z \mapsto -z$, every even row in the matrix $(\theta_j^{(i)})$ picks up a minus sign, and moreover, we need to interchange the m-th collumn with the $(n-m \bmod n)$-th collumn for $0 < m < n/2$. Together we pick up $n/2 + n/2 - 1$ minus signs for Θ. So $\Theta(-z) = -\Theta(z)$. If n is odd, we pick up $(n-1)/2$ minus signs from the even rows and need to interchange $(n-1)/2$ columns. So $\Theta(-z) = \Theta(z)$.

Recall from Theorem A.2 that $\theta_m(z + 1/n) = \zeta^m \theta_m(z)$ where $\zeta = e^{2\pi i/n}$. It follows after $z \mapsto z + 1/n$, the m-th column of $(\theta_j^{(i)})$ picks up a factor of ζ^{m-1}. Hence $\Theta(z + 1/n) = \zeta^{\sum_{k=0}^{n-1} k} \Theta(z) = (-1)^{n+1} \Theta(z)$.

Finally, we recall from Theorem A.2 and the definition $y = e^{-i\pi\tau} e^{2\pi i z} = \gamma^{-n} e^{2\pi i z}$ that

$$\begin{aligned} \theta_m(z + \tau/n) &= \gamma^{-1} e^{2\pi i z} \theta_{m+1 \bmod n}(z) && \text{(B.27)} \\ &= \gamma^{-1} \gamma^n y \theta_{m+1 \bmod n}(z) && \text{(B.28)} \\ &= \gamma^{n-1} \theta_{m+1 \bmod n}(z) && \text{(B.29)} \end{aligned}$$

Repeated differentiation shows that

$$\theta_m^{(k)}(z + \tau) = \gamma^{n-1} \sum_{i=0}^{k} \binom{k}{i} (y)^{(i)} \theta_{m+1}^{(k-i)}(z) \tag{B.30}$$

Hence

$$\left(\theta_j^{(i)}(z + \tau/n)\right) = \gamma^{n-1} E \begin{pmatrix} y & & & \\ (y)' & y & & \\ (y)'' & 2(y)\prime & y & \\ \vdots & & & \ddots \\ (y)^n & & \cdots & & y \end{pmatrix} \left(\theta_j^{(i)}(z)\right) \tag{B.31}$$

where E is the matrix that corresponds to a permutation of columns$(1, 2, ..., n) \mapsto (2, 3, ..., n, 1)$. It follows that

$$\begin{aligned} &\det\left(\theta_j^{(i)}(z + \tau/n)\right) \\ &= (-1)^{n+1} \det \left[\gamma^{n-1} \begin{pmatrix} y & & & \\ (y)' & y & & \\ (y)'' & 2(y)' & y & \\ & & & \ddots \\ (y)^n & & & & y \end{pmatrix} \left(\theta_j^{(i)}(z)\right) \right] \end{aligned} \tag{B.32}$$

(where $(-1)^{n+1} = \det E$). Hence $\Theta(z + \tau) = (-1)^{n+1} \gamma^{n(n-1)} y^n \Theta(z)$. □

Corollary B.7 $\Theta \in V_{n^2}$

Proof The lemma above shows that

$$\Theta(z+1) = \Theta\left(z + \sum_{i=1}^{n} 1/n\right) = (-1)^{(n+1)n}\Theta(z) = \Theta(z) \tag{B.33}$$

We repeat the above proof with τ/n replaced by τ. Note first that $\theta(z+\tau) = e^{-2\pi inz-\pi in\tau}\theta(z)$ for all $\theta \in V_n$. Set $Y = e^{-2\pi inz-\pi in\tau}$, then we see that

$$\left(\theta_j^{(i)}(z+\tau)\right) = \begin{pmatrix} Y & & & & \\ (Y)' & Y & & & \\ (Y)'' & 2(Y)' & Y & & \\ \vdots & & & \ddots & \\ (Y)^n & & \cdots & & Y \end{pmatrix} \left(\theta_j^{(i)}(z)\right) \tag{B.34}$$

Taking det of both sides, we see that $\Theta(z+\tau) = Y^n\Theta(z) = e^{-2\pi in^2z-\pi in^2}\Theta(z)$, which is precisely the defining conditions of elements of V_{n^2}. □

Corollary B.8 $|X_n| = n^2$.

Proof The uniqueness Theorem A.3 shows us that $|X_n|$ is equal to the number of possible locations of zeros of singular n-theta functions. Proposition A.7 shows that that this is equal to the size of the zero set of Θ mod L_τ. Since $\Theta \in V_{n^2}$. We conclude by Theorem A.3, again, that $|X_n| = n^2$. □

Now, we obtain explicit formulae for elements of X_n. To do this, we need the following lemma

Lemma B.9 *If $\theta \in V_n$, so is*

$$\gamma(z) = \theta\left(z + \frac{1}{n}(a+b\tau)\right)e^{2\pi ibz} \tag{B.35}$$

for $a, b \in \mathbb{Z}$.

Proof We check that

$$\gamma(z+1) = \theta\left(z + \frac{1}{n}(a+b\tau)+1\right)e^{2\pi ibz+2\pi ib} \tag{B.36}$$

$$= \gamma(z) \tag{B.37}$$

since $b \in \mathbb{Z}$. Similarly,

$$\gamma(z+\tau) = \theta\left(z + \frac{1}{n}(a+b\tau)+\tau\right)e^{2\pi ibz+2\pi ib\tau} \tag{B.38}$$

$$= e^{-\pi in\tau-2\pi inz-2\pi i(a+b\tau)}\theta\left(z + \frac{1}{n}(a+b\tau)\right)e^{2\pi ibz+2\pi ib\tau} \tag{B.39}$$

$$= e^{-\pi in\tau-2\pi inz}\gamma(z) \tag{B.40}$$

since $a, b \in \mathbb{Z}$. □

Now, let θ_0 be a basis for V_1. From Theorems A.4 and A.5, we see that that $\theta_0^n \in X_n$, it follows by Lemma A.10 that

$$\theta_{a,b}(z) := \theta_0^n \left(z + \frac{1}{n}(a + b\tau) \right) e^{2\pi i b z} \tag{B.41}$$

are all in X_n for $a, b \in \mathbb{Z}$. But there are exactly $n^2 = |X_n|$ number of distinct such functions (mod scaling). So X_n is contains exactly these elements. Moreover, by Proposition 11.1, the zero of θ_0 is at $\frac{1}{2}(1+\tau)$. So the zeros of $\theta_{a,b}$ are located at $\frac{1}{2}(1+\tau) - \frac{1}{n}(a + b\tau)$.

C Choice of χ_g

The action of point groups is given by

$$\psi(gx) = e^{i\chi_g}\psi(x). \tag{C.1}$$

for some χ_g, which we determine below.

Proposition C.1 *Let $g \in SH(\mathcal{L})$ and ψ is a linear solution satisfying (C.1), then χ_g are constant.*

Proof We identify $SH(\mathcal{L})$ as a subset of $\mathbb{C}$ so that gx is the multiplication of the two complex numbers g and x. Assume that χ_g satisfies (C.1). Since ψ is a linear solution, by (5.11), we can find a holomorphic theta function θ such that $\theta(x) = h(x)\psi(x)$ for some smooth, nonvanishing, h with the property $(\bar{\partial}h)(x) = \frac{b}{2}xh(x)$ where $\bar{\partial} := \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$. Then (C.1) is equivalent to the fact that

$$H_g(x) := h(gx)e^{i\chi_g}h(x)^{-1} \tag{C.2}$$

is holomorphic. Taking $\bar{\partial}$, this requirement is equivalent to

$$0 = \bar{\partial}(h(gx)e^{i\chi_g}h(x)^{-1}) \tag{C.3}$$

$$= \left(i\bar{\partial}\chi_g + \frac{b}{2}\bar{g}gx - \frac{b}{2}x \right) h(gx)e^{i\chi_g}h(x)^{-1}. \tag{C.4}$$

Since $|g| = 1$ and $h(gx)e^{i\chi_g}h(x)^{-1}$ is invertible, we see that

$$\bar{\partial}\chi_g = 0 \tag{C.5}$$

Since χ_g are real valued, it is a constant. □

As a result of the the proposition, it suffices for us to look for gauge invariant (ψ, A) under actions of $H(\mathcal{L})$ whose gauge factor $h_g(x) = e^{i\chi_g}$ is a constant. Hence we consider spaces of the form

$$\left\{ \psi\left(R_\xi^{-1} i x\right) = \eta\psi(x),\ R_\xi A\left(R_\xi^{-1} x\right) = \eta' A(x) \right\} \tag{C.6}$$

where $\eta, \eta' \in \mathbb{C}$. One realises that such space corresponds to irreducible representations of $H(\mathcal{L})$.

D Table of C_6-Equivariant Theta Functions

Vortex number	Value of r	Theta functions that span $V_{n,6,r}$
n = 2	0	θ_0
	2	θ_2
n = 4	0	θ_0^2
	1	θ_4
	2	$\theta_0\theta_4$
	4	θ_2^2
n = 6	0	$\theta_0^3,\ \theta_2^3, \theta_4^2\theta_2^{-1}$
	1	$\theta_0\theta_4$
	2	$\theta_0^2\theta_2$
	3	$\theta_4\theta_2$
	4	$\theta_0\theta_2^2$
n = 8	0	$\theta_0^4,\ \theta_0\theta_2^3$
	1	$\theta_0^2\theta_4$
	2	$\theta_2^4,\ \theta_4^2,\ \theta_0^3\theta_2$
	3	$\theta_0\theta_4\theta_2$
	4	$\theta_0^2\theta_2^2$
	5	$\theta_4\theta_2^2$
n = 10	0	$\theta_0^5,\ \theta_0^2\theta_2^3$
	1	$\theta_0^3\theta_4,\ \theta_4\theta_2^3$
	2	$\theta_0^4\theta_2,\ \theta_0\theta_2^4,\ \theta_0\theta_4^2$
	3	$\theta_0^2\theta_4\theta_2$
	4	$\theta_0^3\theta_2^2,\ \theta_2^5,\ \theta_4^2\theta_2$
	5	$\theta_0\theta_4\theta_2^2$

(D.1)

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