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INFIMAL CONVOLUTION AND OPTIMAL TIME CONTROL PROBLEM III: MINIMAL TIME PROJECTION SET*

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Abstract. We continue the investigation of a general minimal time problem with a convex constant dynamics and a lower semicontinuous extended real-valued target function defined on a Banach space. In this paper we obtain an explicit description of the minimal time projection set in terms of the Legendre–Fenchel conjugate function and provide some sufficient conditions for this set to be nonempty. The single valuedness and Lipschitz property of the minimal time projection are also investigated.

Key words. Fréchet subdifferential, minimal time function, minimal time projection, infimal convolution, Legendre–Fenchel conjugate

AMS subject classifications. Primary, 49J52, 46N10, 58C20; Secondary, 28B20

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1. Introduction. We proceed with the study of the optimal control problem with constant dynamics which was started in [8] and continued in [9]. Given a nonempty closed convex set G of a Banach space $(X, \|\cdot\|)$ with $0 \in G$ and an extended real-valued function f defined on X (called the *target function*), the problem is

$$\text{Minimize } t + f\big(\zeta(t;x)\big) \tag{1.1}$$

over all $t \geq 0$ and all solutions $\zeta(\cdot) = \zeta(\cdot;x)$ of the differential inclusion

$$\frac{d\zeta}{dt}(t) \in -G, \quad t \geq 0, \tag{1.2}$$

with the initial condition

$$\zeta(0) = x. \tag{1.3}$$

Since the right-hand side of the differential inclusion (1.2) is constant and convex closed valued, problem (1.1)–(1.3) can be reformulated as

$$\begin{cases} \text{Minimize} \quad t + f(x - tu), \\ \text{over } t \geq 0 \text{ and } u \in G. \end{cases} \tag{1.4}$$

As shown in [8] the value function T_f of the problem (1.1)–(1.3) can be expressed by the following infimal convolution formula (see [11, 12] for infimal convolution operation):

$$T_f(x) := \inf_{y \in X} \big(f(y) + \rho_G(x - y)\big), \tag{1.5}$$

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where ρ_G denotes the Minkowski functional of G. The function T_f is called the *(generalized) infimum time function* of the problem. The set of minimizers $\Pi_f(x)$ of (1.5), given by

$$\Pi_f(x) := \operatorname{Argmin}\,(f + \rho_G(x - \cdot)) := \{y \in X : \ f(y) + \rho_G(x - y) = T_f(x)\}, \tag{1.6}$$

is called the *(generalized) minimal time projection set*. As the problem (1.5) is a reformulation of the optimal control problem (1.1)–(1.3), the nonemptiness of the set $\Pi_f(x)$ is equivalent to the existence of a solution of the problem (1.1)–(1.3). Moreover, the nonemptiness of $\Pi_f(x)$ is a crucial assumption for various results (e.g., Theorems 3.1, 3.2, 4.2) in [8].

The outline of this third paper of the series on the subject is the following. In section 2 we introduce notation and discuss the relation of the problem (1.1)–(1.3) with a minimal time control problem for a target set, studied previously in Colombo, Goncharov, and Mordukhovich [5]. The nonemptiness of the minimal time projection set $\Pi_f(x)$ is proved in section 3 under diverse conditions on the data, in particular, on Fréchet subdifferentiability of the infimum time function T_f. In section 4 we obtain an explicit description of the minimal time projection set in terms of the Legendre–Fenchel conjugate function. The last section investigates the single valuedness and Lipschitz property of the minimal time projection.

2. Preliminaries. As in [8] and [9] we consider and fix a *closed bounded convex* subset G of X with $0 \in \operatorname{int} G$. We follow usual notation in using $\operatorname{int} S$ (resp., $\operatorname{cl} S$ and $\operatorname{bd} S$) to denote the interior (resp., the closure and the boundary) of a set S in X.

An important special case of the optimal control problem (1.1)–(1.3) is the problem

$$\begin{aligned} &\text{Minimize} \quad t \geq 0 \text{ such that } \zeta(t) \in C \text{ and} \\ &\text{subject to} \quad \dot{\zeta}(\tau) \in -G \text{ a.e. } \tau \in [0, t] \text{ and } \zeta(0) = x \end{aligned} \tag{2.1}$$

with a target set $C \subset X$, which is assumed to be nonempty and closed. Indeed, if we take the target function f as the *indicator function* ψ_C of $C \subset X$ (i.e., $\psi_C(x) = 0$ if $x \in C$ and $\psi_C(x) = +\infty$ otherwise), then the optimal control problem (1.1)–(1.3) reduces to the problem (2.1). In this case the infimum time function and the minimal time projection set, correspondingly, are

$$T_C(x) = \inf_{y \in C} \rho_G(x - y), \tag{2.2}$$

$$\Pi_C(x) = \operatorname*{Argmin}_{y \in C}\, \rho_G(x - y) := \{y \in C : \ \rho_G(x - y) = T_C(x)\}.$$

If G is, in addition, symmetric, then the Minkowski function ρ_G is a norm $\|\cdot\|_G$ on X and $T_C(x) = d_C(x) := \inf_{y \in C} \|x - y\|_G$ is the distance from the point x to the set C associated with the norm $\|\cdot\|_G$. So, if $G = \mathbb{B}_X$ (the closed unit ball with respect to the initial norm $\|\cdot\|$), the function T_C is reduced to the distance function $d_C(\cdot)$ with respect to the norm $\|\cdot\|$, and $\Pi_C(x)$ coincides with the projection set $\operatorname{Proj}_C(x)$, that is,

$$\Pi_C(x) = \operatorname{Proj}_C(x) := \{y \in C : \ \|x - y\| = d_C(x)\}.$$

We will also write sometimes $\operatorname{dist}(\cdot, C)$ in place of d_C.

In the case when $G = \mathbb{B}_X$, the Fréchet subdifferential of d_C at any point in X has been studied in Kruger [10]; see also [1, 2, 4, 13] for other subdifferentials of d_C. The

Fréchet and proximal subdifferentials of the minimal time function T_C in (2.2) have been considered later, when X is finite dimensional, in Wolenski and Zhuang [14], and then in the Hilbert setting in Colombo and Wolenski [6, 7]. Colombo, Goncharov, and Mordukhovich [5] obtained some further strong results concerning T_C and Π_C in (2.2) provided that G is an arbitrary closed bounded convex subset of X. In [8] we investigate properties of the Fréchet and proximal subdifferentials of the infimum time function and in [9] we studied the properties of the limiting subdifferential of this function in the general setting of the problem (1.1)–(1.3).

Let us recall that the Minkowski gauge function (Minkowski functional) of a nonempty closed convex subset $K \subset X$ is

$$\rho_K(x) := \inf\{r \geq 0 : \ x \in rK\} \quad \forall x \in X \tag{2.3}$$

with the usual convention $\inf \emptyset = +\infty$.

Note that the Minkowski gauge function of any closed convex set is sublinear and because of the assumption $0 \in \operatorname{int} G$, the function ρ_G is finite on X and Lipschitz continuous on X. We denote by σ_G the *support function* of G defined on the topological dual X^* and by G^o the *polar set* of G, i.e.,

$$\sigma_G(x^*) := \sup_{x \in G} \langle x^*, x\rangle \quad \text{and} \quad G^o := \{x^* \in X^* : \sigma_G(x^*) \leq 1\}.$$

One can easily see that

$$\rho_G(x) = \sigma_{G^o}(x) := \sup_{x^* \in G^o} \langle x^*, x\rangle \quad \forall x \in X. \tag{2.4}$$

In view of the sublinearity of ρ_G, it is an exercise to verify that

$$x^* \in \partial \rho_G(x) \ \Leftrightarrow \ \Big(x^* \in G^o \text{ and } \langle x^*, x\rangle = \rho_G(x)\Big). \tag{2.5}$$

We recall that, for a convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ and $x \in X$ with $f(x)$ finite, the Fenchel subdifferential $\partial f(x)$ of f at x is defined by

$$\partial f(x) := \{x^* \in X^* : \langle x^*, x' - x\rangle \leq f(x') - f(x) \ \forall x' \in X\};$$

similarly, given a real $\eta \geq 0$, the Fenchel η-subdifferential of f at x is given by

$$\partial_\eta f(x) := \{x^* \in X^* : \langle x^*, x' - x\rangle \leq f(x') - f(x) + \eta \ \forall x' \in X\}.$$

With $x = 0$ we then obtain

$$\partial \rho_G(0) = \partial \sigma_{G^o}(0) = G^o.$$

We emphasize that the assumption of boundedness of G and the assumption $0 \in \operatorname{int} G$ yield two reals $\alpha > 0$ and $\beta > 0$ (which we fix in the whole paper) such that $\frac{1}{\beta}\mathbb{B}_X \subset G \subset \frac{1}{\alpha}\mathbb{B}_X$, which gives

$$\alpha\|x\| \leq \rho_G(x) \leq \beta\|x\| \quad \forall x \in X. \tag{2.6}$$

Throughout the paper, $f : X \to \mathbb{R} \cup \{+\infty\}$ is a *proper lower semicontinuous* (lsc) extended real-valued function and we will assume that for some real constant γ

$$f(y) \geq -\rho_G(-y) + \gamma \quad \forall y \in X; \tag{2.7}$$

recall that such a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is proper when it is finite at some point. Then for all $y \in X$ according to the sublinearity of ρ_G

$$f(y) + \rho_G(x - y) \geq \rho_G(x - y) - \rho_G(-y) + \gamma \geq -\rho_G(-x) + \gamma,$$

and this entails that

$$T_f(x) \geq -\rho_G(-x) + \gamma, \text{ hence in particular, } T_f(x) \in \mathbb{R}\ \forall x \in X. \tag{2.8}$$

Recall that for $x \in \operatorname{dom} f := \{u \in X : f(u) < +\infty\}$ the *Fréchet subdifferential* $\partial^F f(x)$ of f at x is the set of $x^* \in X^*$ such that for any $\eta > 0$ there exists a neighborhood U of x such that for all $x' \in U$

$$\langle x^*, x' - x\rangle \leq f(x') - f(x) + \eta\|x' - x\|.$$

If f is, in addition, convex, then all the above subdifferentials coincide with the Fenchel subdifferential $\partial f(x)$ of f in convex analysis, and further (see, e.g., [11])

$$x^* \in \partial f(x) \Longleftrightarrow x \in \partial f^*(x^*), \tag{2.9}$$

where $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ denotes the *Legendre–Fenchel conjugate* of f, that is, for all $x^* \in X^*$

$$f^*(x^*) = \sup_{y \in X}[f(y) - \langle x^*, y\rangle].$$

3. On the nonemptiness of the minimal time projection set $\Pi_f(\bar{x})$. Let us begin with the case when the proper lsc function f is convex. Under the convexity of f (as already noticed in [8]) we see through (1.5) that the function T_f is convex too. Assuming the convexity of f and invoking a finiteness condition about f^*, the following theorem provides a complete description of the subdifferential of T_f in terms of the functions f and ρ_G.

THEOREM 3.1. *Assume that the Banach space X is reflexive and the proper lsc function f is convex. Asssume also that the Legendre–Fenchel conjugate f^* of f is finite at some point $x^* \in X^*$ with $\|x^*\|_* < \alpha$, where α is given by* (2.6). *Then for every $\bar{x} \in X$ the (generalized) minimal time projection set $\Pi_f(\bar{x})$ is nonempty. Further, for any $\bar{y} \in \Pi_f(\bar{x})$ one has*

$$\partial T_f(\bar{x}) = \partial f(\bar{y}) \cap \partial \rho_G(\bar{x} - \bar{y}).$$

Proof. The space X being reflexive and the functions f and ρ_G being proper, lsc, and convex, we know that $f^{**} := (f^*)^* = f$ and $\rho_G^{**} = \rho_G$. On the other hand, we have that $\rho_G = \sigma_{G^o}$ by (2.4) and hence the Legendre–Fenchel conjugate ρ_G^* of ρ_G is equal to the indicator function ψ_{G^o} of G^o. Further, the inclusion $G \subset \frac{1}{\alpha}\mathbb{B}_X$ (see (2.6)) ensures that $\alpha\mathbb{B}_{X^*} \subset G^o$. Therefore, ρ_G^* is continuous at $x^* \in \operatorname{dom} f^*$, that is, the Moreau–Rockafellar qualification condition is satisfied for f^* and ρ_G^*. This assures us (see, e.g., [11, Proposition 9.2]) that the infimum convolution between f^{**} and ρ_G^{**} is achieved (or exact) and that the equality of the theorem holds true because of the equalities $f^{**} = f$ and $\rho_G^{**} = \rho_G$. ∎

The case when the convex function f is bounded from below is of special interest. It will allow us, in particular, in Corollary 3.3 below to consider for f the indicator function of a closed convex set.

COROLLARY 3.2. *Assume that X is a reflexive Banach space and that the proper lsc function f is convex and bounded from below. Then the (generalized) minimal time projection set $\Pi_f(\bar{x})$ is nonempty for all $\bar{x} \in X$. Further, for any $\bar{y} \in \Pi_f(\bar{x})$ one has*

$$\partial T_f(\bar{x}) = \partial f(\bar{y}) \cap \partial \rho_G(\bar{x} - \bar{y}).$$

Proof. By the boundedness from below of f we have that $f^*(0) = \inf_X f$ is finite. The corollary then follows from the theorem above. ∎

The following improvement of Theorem 4.2 and Corollary 4.4 in [7] (established therein for Hilbert spaces) are obtained from Corollary 3.2 with the indicator function $f = \psi_C$ of a closed convex set C. Recall that $\partial \psi_C(x)$ is the *normal cone* of C at x.

COROLLARY 3.3. *Assume that X is a reflexive Banach space and that C is a nonempty closed convex set of X. Then for any $\bar{x} \in X$ the minimal time projection set $\Pi_C(\bar{x})$ is nonempty and for every $\bar{y} \in \Pi_C(\bar{x})$ one has*

$$\partial T_C(\bar{x}) = N_C(\bar{y}) \cap \partial \rho_G(\bar{x} - \bar{y}).$$

In the general case where the function f is not required to be convex, following (2.8) for each $\varepsilon > 0$ we will denote by $\Pi_f(x, \varepsilon)$ the ε-minimal time projection or ε-Argmin of the function $f + \rho_G(x - \cdot)$, i.e.,

$$\Pi_f(x, \varepsilon) := \{y \in X : f(y) + \rho_G(x - y) \le T_f(x) + \varepsilon\}. \tag{3.1}$$

With the set-valued mapping $\Pi_f(x, \cdot)$ we associate its $w(X, X^*)$-sequential upper limit

$$^{\mathrm{w-seq}}\operatorname*{Limsup}_{\varepsilon \downarrow 0} \Pi_f(x, \varepsilon) := \{{}^{\mathrm{w}}\lim_k y_k : \exists \varepsilon_k \downarrow 0,\ y_k \in \Pi_f(x, \varepsilon_k)\},$$

and also the set of limits through f given by

$$\Lambda_f(x; y) := \{\lim_k f(y_k) : y_k \xrightarrow{w} y,\ y_k \in \Pi_f(x, \varepsilon_k),\ \varepsilon_k \downarrow 0\}.$$

Above the notation ${}^{\mathrm{w}}\lim_k y_k$ stands for the weak limit of the sequence $(y_k)_k$.

The following proposition, which has its own interest, prepares the next theorem.

PROPOSITION 3.4. *Let $x^* \in \partial^F T_f(\bar{x})$ and $y \in {}^{\mathrm{w-seq}}\operatorname{Limsup}_{\varepsilon \downarrow 0} \Pi_f(\bar{x}, \varepsilon)$. Then the following hold.*

(a) *For each sequence $y_k \xrightarrow{w} y$ with $y_k \in \Pi_f(\bar{x}, \varepsilon_k)$ and $\varepsilon_k \downarrow 0$, one has*

$$\lim_k f(y_k) = T_f(\bar{x}) - \rho_G(\bar{x} - y),$$

and hence $\Lambda_f(\bar{x}; y)$ is a singleton, $\Lambda_f(\bar{x}; y) = \{\lambda_f(\bar{x}; y)\}$ with $\lambda_f(\bar{x}; y) = T_f(\bar{x}) - \rho_G(\bar{x} - y)$.

(b) *One also has the inclusion*

$$y \in \bar{x} - \big(T_f(\bar{x}) - \lambda_f(\bar{x}; y)\big)\partial \sigma_G(x^*) = \bar{x} - \rho_G(\bar{x} - y)\partial \sigma_G(x^*).$$

Proof. Observe first that the inclusion $x^* \in \partial^F T_f(\bar{x})$ assures us that there exists some function $\varepsilon(x) \to 0$ as $x \to \bar{x}$ such that for all x in some neighborhood V of $\bar{x}$ we have

$$\langle x^*, x - \bar{x} \rangle \le T_f(x) - T_f(\bar{x}) + \varepsilon(x)\|x - \bar{x}\|. \tag{3.2}$$

Fix any sequence $y_k \xrightarrow{w} y$ with $y_k \in \Pi_f(\bar{x}, \varepsilon_k)$ and $\varepsilon_k \downarrow 0$ with $\varepsilon_k < 1$. Putting $t_k := \sqrt{\varepsilon_k}$ we can translate the inclusion $y_k \in \Pi_f(\bar{x}, \varepsilon_k)$ as

$$T_f(\bar{x}) \le f(y_k) + \rho_G(\bar{x} - y_k) \le T_f(\bar{x}) + t_k^2. \tag{3.3}$$

Setting $x_k := \bar{x} + t_k(y_k - \bar{x})$ and using the boundedness of the sequence $(y_k)_k$ we see that for some integer k_0 we have $x_k \in V$ for all $k \ge k_0$. Combining this with (3.2) and (3.3) gives for all $k \ge k_0$

$$\begin{aligned} \langle x^*, y_k - \bar{x}\rangle - \varepsilon(x_k)\|y_k - \bar{x}\| &\le t_k^{-1}[f(y_k) + \rho_G(x_k - y_k) - f(y_k) - \rho_G(\bar{x} - y_k) + t_k^2] \\ &= t_k^{-1}[(1 - t_k)\rho_G(\bar{x} - y_k) - \rho_G(\bar{x} - y_k) + t_k^2] \\ &= -\rho_G(\bar{x} - y_k) + t_k. \end{aligned} \tag{3.4}$$

On the other hand, we know by [8, Theorem 4.1(a)] that $x^* \in G^o$ and this entails

$$\langle x^*, \bar{x} - y_k\rangle \le \sigma_{G^o}(\bar{x} - y_k) = \rho_G(\bar{x} - y_k),$$

the equality between the second and third members being due to (2.4). Therefore, it follows from (3.4) that

$$-\varepsilon(x_k)\|y_k - \bar{x}\| - t_k + \rho_G(\bar{x} - y_k) \le \langle x^*, \bar{x} - y_k\rangle \le \rho_G(\bar{x} - y_k).$$

Since $\lim_k \langle x^*, \bar{x} - y_k\rangle = \langle x^*, \bar{x} - y\rangle$, we obtain

$$\lim_k \rho_G(\bar{x} - y_k) = \langle x^*, \bar{x} - y\rangle.$$

The continuous convex function ρ_G being lower $w(X, X^*)$-semicontinuous, we can write

$$\rho_G(\bar{x} - y) \le \lim_k \rho_G(\bar{x} - y_k) = \langle x^*, \bar{x} - y\rangle \le \sigma_{G^o}(\bar{x} - y) = \rho_G(\bar{x} - y),$$

and hence

$$\lim_k \rho_G(\bar{x} - y_k) = \langle x^*, \bar{x} - y\rangle = \rho_G(\bar{x} - y).$$

Consequently, (3.3) entails that $\lim_k f(y_k)$ exists and

$$\lim_k f(y_k) = T_f(\bar{x}) - \rho_G(\bar{x} - y),$$

which implies, in particular, the singleton property of $\Lambda_f(\bar{x}; y)$.

Finally, observe that the equality $\langle x^*, \bar{x} - y\rangle = \rho_G(\bar{x} - y)$ ensures by (2.5) and the duality relation (2.9) that

$$\bar{x} - y \in \rho_G(\bar{x} - y)\partial\sigma_G(x^*), \text{ i.e., } y \in \bar{x} - \rho_G(\bar{x} - y)\partial\sigma_G(x^*).$$

The proof is then complete. ∎

Even for $f = \psi_C$, examples in [5] show that the nonvacuity of the Fréchet subdifferential of T_C at $\bar{x} \notin C$ does not entail the nonvacuity of $\Pi_C(\bar{x})$. However, from [8, Theorem 4.1] and Proposition 3.4 we can derive a nonemptiness result for the generalized projection set $\Pi_f(\bar{x})$ through the Fréchet differentiability of the support function σ_G. It reproduces the main ideas of Theorem 4.1 in [5] to the context of the set $\Pi_f(\bar{x})$.

THEOREM 3.5. *Assume that X is a reflexive Banach space and that the lsc proper function f is bounded from below on X with $\bar{x} \notin \mathrm{cl}\,(\mathrm{dom}\, f)$. Assume also that there exists some $x^* \in \partial^F T_f(\bar{x})$ at which the support function σ_G is Fréchet differentiable. Then the set $\Pi_f(\bar{x})$ is nonempty.*

Proof. Fix a sequence $(y_k)_k$ in X such that

$$f(y_k) + \rho_G(\bar{x} - y_k) \longrightarrow T_f(\bar{x}) \tag{3.5}$$

and put $\varepsilon_k := \frac{1}{k} + \big(f(y_k) + \rho_G(\bar{x} - y_k) - T_f(\bar{x})\big)$. Without loss of generality, we may suppose that the sequence $(f(y_k) + \rho_G(\bar{x} - y_k))_k$ is bounded from above. Since the function f is bounded from below, the sequence $(\rho_G(\bar{x} - y_k))_k$ is bounded from above, which ensures the boundedness of $(y_k)_k$. The reflexivity of X allows us by the Banach–Alaoglu theorem to extract a subsequence (that we do not relabel) converging weakly to a point $\bar{y} \in X$. Obviously the sequence $(f(y_k))_k$ is bounded. Extracting a subsequence we may also suppose that $(f(y_k))_k$ converges to some λ and hence by (3.5)

$$\rho_G(\bar{x} - y_k) \longrightarrow T_f(\bar{x}) - \lambda. \tag{3.6}$$

We claim that $T_f(\bar{x}) - \lambda > 0$. Otherwise, (3.6) entails first $T_f(\bar{x}) - \lambda = 0$ and then $\rho_G(\bar{x} - y_k) \longrightarrow 0$, which yields by (2.6) that $y_k \longrightarrow \bar{x}$ strongly. The lower semicontinuity assumption of f and (3.5) give

$$f(\bar{x}) \le \liminf_k f(y_k) \le \liminf_k \big(f(y_k) + \rho_G(\bar{x} - y_k)\big) = T_f(\bar{x}),$$

which is a contradiction because $f(\bar{x}) = +\infty$ (since $\bar{x} \notin \mathrm{dom}\, f$ by assumption) and $T_f(\bar{x})$ is finite.

Since $T_f(\bar{x}) - \lambda > 0$, by (3.6) we may suppose (dropping a finite number of integers k if necessary) that $\rho_G(\bar{x} - y_k) > 0$ for all k. Therefore

$$G \ni u_k := \frac{\bar{x} - y_k}{\rho_G(\bar{x} - y_k)} \longrightarrow \frac{\bar{x} - \bar{y}}{T_f(\bar{x}) - \lambda} \text{ weakly.} \tag{3.7}$$

Observe now that the assumption $x^* \in \partial^F T_f(\bar{x})$ ensures by [8, Theorem 4.1] that $x^* \in G^o$ and $\sigma_G(x^*) = 1$. Further, by definition of ε_k we have $y_k \in \Pi_f(\bar{x}, \varepsilon_k)$ and $\varepsilon_k \downarrow 0$. Consequently, Proposition 3.4 and the inclusion $\lambda \in \Lambda_f(\bar{x}; \bar{y})$ tell us that $\lambda = \lambda_f(\bar{x}; \bar{y}) = T_f(\bar{x}) - \rho_G(\bar{x} - \bar{y})$ and

$$\bar{y} = \bar{x} - \rho_G(\bar{x} - \bar{y}) \nabla^F \sigma_G(x^*),$$

where $\nabla^F \sigma_G(x^*)$ is the Fréchet derivative of σ_G at x^*. Using the latter equality into (3.7) we obtain

$$G \ni u_k \longrightarrow \nabla^F \sigma_G(x^*) \text{ weakly} \quad \text{and} \quad \sigma_G(x^*) = 1.$$

Setting $\eta_k := \sigma_G(x^*) - \langle x^*, u_k \rangle \ge 0$, we derive that for all $y^* \in X^*$

$$\langle y^* - x^*, u_k \rangle \le \sigma_G(y^*) - \langle x^*, u_k \rangle = \sigma_G(y^*) - \sigma_G(x^*) + \eta_k.$$

It ensues that $u_k \in \partial_{\eta_k} \sigma_G(x^*)$. In addition to the latter inclusion, we also have $\eta_k \downarrow 0$ as $k \to \infty$ according to the fact that

$$\langle x^*, u_k \rangle \to \langle x^*, \nabla^F \sigma_G(x^*) \rangle = \sigma_G(x^*) + \psi_G(\nabla^F \sigma_G(x^*)) = \sigma_G(x^*).$$

Since the continuous convex function σ_G is Fréchet differentiable at x^*, the Šmulian theorem (see [3, Theorem 4.2.10]) ensures that $u_k \longrightarrow \nabla^F \sigma_G(x^*)$ strongly, that is,

$$\frac{\bar{x} - y_k}{\rho_G(\bar{x} - y_k)} \longrightarrow \frac{\bar{x} - \bar{y}}{\rho_G(\bar{x} - \bar{y})} \text{ strongly,}$$

which gives $y_k \longrightarrow \bar{y}$ strongly because $\rho_G(\bar{x} - y_k) \longrightarrow T_f(\bar{x}) - \lambda = \rho_G(\bar{x} - \bar{y})$. Combining this with the lower semicontinuity of f assures us thanks to (3.5) that

$$f(\bar{y}) + \rho_G(\bar{x} - \bar{y}) \leq \liminf_k \big(f(y_k) + \rho_G(\bar{x} - y_k)\big) = T_f(\bar{x}),$$

and hence $\bar{y} \in \Pi_f(\bar{x})$, which gives $\Pi_f(\bar{x}) \neq \emptyset$ and completes the proof of the theorem. ∎

Corollary 3.6. *Let X be a reflexive Banach space and C be a closed subset of X. Let $\bar{x} \notin C$ and assume that there exists some $x^* \in \partial^F T_C(\bar{x})$ at which the support function σ_G is Fréchet differentiable. Then the minimal time projection set $\Pi_C(\bar{x})$ is the (nonempty) singleton*

$$\Pi_C(\bar{x}) = \{\bar{x} - T_C(\bar{x}) \nabla^F \sigma_G(x^*)\}.$$

Proof. The nonemptiness of $\Pi_C(\bar{x})$ follows from Theorem 3.5 with the indicator function ψ_C of C in place of f. For any $y \in \Pi_C(\bar{x})$ one has $y \in C$ and $\Lambda_f(\bar{x}; y) = \{0\}$. So, Proposition 3.4(b) implies that $y \in \bar{x} - T_C(\bar{x}) \nabla^F \sigma_G(x^*)$. This yields the singleton property of $\Pi_C(\bar{x})$ and its expression. ∎

Corollary 3.6 can be obtained also as a consequence of results of Colombo, Goncharov, and Mordukhovich (see Corrolary 3.2 and Theorem 4.1 in [5]).

The second corollary considers the case of the usual projection concept. It corresponds to the result of Borwein and Giles [2].

Corollary 3.7. *Assume that X is a reflexive Banach space and that the dual norm $\|\cdot\|_*$ of $\|\cdot\|$ is Fréchet differentiable outside the origin of X^* (one knows that such a norm exists according to the reflexivity of X). Let C be a closed subset of X and $\bar{x} \in X \setminus C$. Then $\mathrm{Proj}_C(\bar{x})$ is a (nonempty) singleton whenever $\partial^F d_C(\bar{x}) \neq \emptyset$, where $d_C(\cdot)$ and $\mathrm{Proj}_C(\cdot)$ are the usual distance function and projection set-valued mapping associated with the norm $\|\cdot\|$. In fact, for any $x^* \in \partial^F d_C(\bar{x})$ one has*

$$\mathrm{Proj}_C(\bar{x}) = \{\bar{x} - d_C(\bar{x}) \nabla^F \|\cdot\|_*(x^*)\}.$$

Proof. Take $G = \mathbb{B}_X$ (the closed unit ball of X with respect to the norm $\|\cdot\|$) and observe that $\sigma_G(\cdot) = \|\cdot\|_*$, $T_C(\cdot) = d_C(\cdot)$, and $\Pi_C(\cdot) = \mathrm{Proj}_C(\cdot)$. The corollary is then a direct consequence of Corollary 3.6. ∎

4. The minimal time projection set through the Legendre–Fenchel conjugate. Let us fix $x \in X$ and $x^* \in \partial^F T_f(x)$. We consider the following minimal time problem

$$\begin{cases} \text{Minimize} \quad t + f(x - tu), \\ \text{over } t \geq 0 \text{ and } u \in \partial \sigma_G(x^*). \end{cases}$$

This problem is a modification of the problem (1.4) with the face $G(x^*) := \partial \sigma_G(x^*)$ of the set G in place of G.

As in the deduction of (1.5), we see that the infimum value of the problem is

$$V_{f,x^*}(x) = \inf_{y \in X} \Big(f(y) + \rho_{G(x^*)}(x - y)\Big),$$

where according to (2.3)

$$\rho_{G(x^*)}(u) = \inf\{r \geq 0 : \; u \in rG(x^*)\}, \quad u \in X.$$

The set of the solutions of this problem will be denoted by

$$M_{f,x^*}(x) = \operatorname*{Argmin}_{y\in X} \Big(f(y) + \rho_{G(x^*)}(x-y)\Big). \tag{4.1}$$

Following Colombo, Goncharov, and Mordukhovich [5], we can say that V_{f,x^*} is a modified (generalized) infimum time function and the set M_{f,x^*} is a modified (generalized) minimal time projection set.

PROPOSITION 4.1. *Let $x \in X$ and $x^* \in \partial^F T_f(x)$. The following statements hold.*
(a) $V_{f,x^*}(x) \geq T_f(x)$;

(b) $\Pi_f(x) \neq \emptyset \quad \Leftrightarrow \quad \begin{cases} V_{f,x^*}(x) = T_f(x), \\ M_{f,x^*}(x) \neq \emptyset; \end{cases}$

(c) *if* $\Pi_f(x) \neq \emptyset$, *then* $\Pi_f(x) = M_{f,x^*}(x)$.

Proof. Since $G(x^*) \subset G$, it follows that $\rho_{G(x^*)}(x-y) \geq \rho_G(x-y)$. This yields (a).

Let $y \in \Pi_f(x)$. Note that $T_f(x) - f(y) = \rho_G(x-y) \geq 0$. By Theorem 4.2 we have $y \in x - (T_f(x) - f(y))G(x^*)$.

Consequently, $\rho_{G(x^*)}(x-y) \leq T_f(x) - f(y)$. Thus, using (a), we get

$$f(y) + \rho_{G(x^*)}(x-y) \leq T_f(x) \leq V_{f,x^*}(x).$$

This gives the assertion $\Rightarrow$ in (b) and the inclusion $\Pi_f(x) \subset M_{f,x^*}(x)$ in (c).

Now let $y \in M_{f,x^*}(x)$ and $V_{f,x^*}(x) = T_f(x)$. We have

$$T_f(x) = V_{f,x^*}(x) = f(y) + \rho_{G(x^*)}(x-y) \geq f(y) + \rho_G(x-y).$$

It means that $y \in \Pi_f(x)$. So the assertion $\Leftarrow$ in (b) and the inclusion $\Pi_f(x) \supset M_{f,x^*}(x)$ in (c) are proved. $\square$

Given a functional $x^* \in X^*$, we consider the following convex cone

$$K_{x^*} = \bigcup_{\lambda \geq 0} \lambda G(x^*).$$

For $x^* = 0$ we have $G(x^*) = G$, and hence $K(x^*) = X$ since $0 \in \operatorname{int} G$. On the other hand, for $x^* \neq 0$, the closed set $G(x^*)$ is bounded (since $G(x^*) \subset G$) and does not contain 0; so, it is easily seen that the cone $K(x^*)$ is closed. Consequently, $K(x^*)$ is a closed convex cone in X for every $x^* \in X^*$.

LEMMA 4.2. *For any $x^* \in X^* \setminus \{0\}$ one has*

$$\rho_{G(x^*)}(u) = \frac{\langle x^*, u\rangle}{\sigma_G(x^*)} + \psi_{K_{x^*}}(u) \quad \forall u \in X.$$

Proof. If $u \notin K_{x^*}$, then $\rho_{G(x^*)}(u) = +\infty$ and $\psi_{K_{x^*}}(u) = +\infty$, so the desired equality holds. Assume now that $u \in K_{x^*}$. It means that there exist $t \geq 0$ and $v \in G(x^*)$ such that $u = tv$. By the definition of the set $G(x^*)$ we obtain that $\langle x^*, v\rangle = \sigma_G(x^*)$ and therefore $v \in \operatorname{bd} G$. Thus $\rho_{G(x^*)}(v) = 1 = \frac{\langle x^*, v\rangle}{\sigma_G(x^*)}$. Consequently, $\rho_{G(x^*)}(u) = t = \frac{\langle x^*, u\rangle}{\sigma_G(x^*)}$ and $\psi_{K_{x^*}}(u) = 0$. This yields the desired equality again. $\square$

LEMMA 4.3. *Assume that X is a Banach space. Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function such that its conjugate function φ^* is Fréchet differentiable at a point $x^* \in \operatorname{dom} \varphi^*$. Then one has $\nabla^F \varphi^*(x^*) \in X$ and*

$$\operatorname*{Argmin}_{y \in X} \Big(\varphi(y) - \langle x^*, y \rangle \Big) = \left\{ \nabla^F \varphi^*(x^*) \right\}.$$

Proof. Denote $\zeta := \nabla^F \varphi^*(x^*) \in X^{**}$ and $g(x) = \varphi(x) - \langle x^*, x \rangle$ for all $x \in X$. Note that $\inf_X g = -\varphi^*(x^*)$. Consider minimizing sequences $(x_k)_k \subset X$ in the sense that

$$\lim_k g(x_k) = \inf_X g. \tag{4.2}$$

Let us show that any such a sequence $(x_k)_k$ converges to ζ in X^{**}. Assume the contrary: there exist $\varepsilon > 0$ and a sequence $(x_k)_k$ with property (4.2) such that $\|x_k - \zeta\| > 2\varepsilon$ for all $k \in \mathbb{N}$. Choose functionals $\xi_k \in X^*$ with $\|\xi_k\| = 1$ and $\langle x_k - \zeta, \xi_k \rangle > 2\varepsilon$ and denote $t_k := \frac{1}{\varepsilon} (g(x_k) - \inf_X g) + \frac{1}{k}$. We have $t_k \to 0$ and

$$\begin{aligned} \varphi^*(x^* + t_k \xi_k) &= \sup_{x \in X} \left(\langle x^* + t_k \xi_k, x \rangle - \varphi(x) \right) \\ &\geq \langle x^* + t_k \xi_k, x_k \rangle - \varphi(x_k) \\ &= t_k \langle \xi_k, x_k \rangle - g(x_k) \\ &> t_k (\langle \zeta, \xi_k \rangle + 2\varepsilon) - \inf_X g - \varepsilon t_k + \frac{\varepsilon}{k} \\ &> t_k \langle \zeta, \xi_k \rangle + \varepsilon t_k + \varphi^*(x^*). \end{aligned}$$

Consequently,

$$\frac{\varphi^*(x^* + t_k \xi_k) - \varphi^*(x^*) - t_k \langle \zeta, \xi_k \rangle}{t_k \|\xi_k\|} > \varepsilon \quad \forall k \in \mathbb{N}.$$

This contradicts the fact that ζ is the Fréchet derivative of φ^* at x^*, so $(x_k)_k$ converges to ζ in X^{**}. Since X is complete, we derive that there is some $x_0 \in X$ such that $\zeta = x_0$.

Let $(x_k)_k$ be a minimizing sequence in the sense of (4.2). As we have shown, $x_k \to x_0$. So, by the lower semicontinuity of g we get $x_0 \in \operatorname{Argmin}_{y \in X} g(y)$. Suppose that $\tilde{x}_0 \in \operatorname{Argmin}_{y \in X} g(y)$. Then the constant sequence $x_k = \tilde{x}_0$ satisfies the minimizing condition (4.2) and as we showed above $x_k \to x_0$, i.e., $\tilde{x}_0 = x_0$. Thus, $\operatorname{Argmin}_{y \in X} g(y) = \{x_0\}$. ∎

We are now ready to state and prove the theorem of this section relying the minimal time projection to the Legendre–Fenchel conjugate of a function involving both f and the set K_{x^*}.

THEOREM 4.4. *Assume that X is a Banach space, f is lower semicontinuous, $\bar{x} \notin \operatorname{cl}(\operatorname{dom} f)$, $V_{f,x^*}(\bar{x}) = T_f(\bar{x})$, and $x^* \in \partial^F T_f(\bar{x})$. Assume also that the conjugate of $(f + \psi_{\bar{x} - K_{x^*}})$ is Fréchet differentiable at x^*. Then one has $\nabla^F (f + \psi_{\bar{x} - K_{x^*}})^*(x^*) \in X$ and*

$$\Pi_f(\bar{x}) = \left\{ \nabla^F (f + \psi_{\bar{x} - K_{x^*}})^*(x^*) \right\}.$$

Proof. By [8, Theorem 4.1] we deduce that $\sigma_G(x^*) = 1$. So, Lemma 4.2 implies that

$$\rho_{G(x^*)}(u) = \langle x^*, u \rangle + \psi_{K_{x^*}}(u) \quad \forall u \in X.$$

It follows by (4.1) that

$$M_{f,x^*}(\bar{x}) = \operatorname*{Argmin}_{x\in X} \Big(f(x) + \langle x^*, \bar{x} - x\rangle + \psi_{K_{x^*}}(\bar{x} - x) \Big).$$

Let us denote

$$\varphi(x) := f(x) + \psi_{K_{x^*}}(\bar{x} - x) = f(x) + \psi_{\bar{x} - K_{x^*}}(x), \quad x \in X.$$

We have

$$M_{f,x^*}(\bar{x}) = \operatorname*{Argmin}_{x\in X} \Big(\varphi(x) - \langle x^*, x\rangle \Big).$$

Lemma 4.3 yields $M_{f,x^*}(\bar{x}) = \{\nabla^F \varphi^*(x^*)\}$. According to (b) and (c) of Proposition 4.1 we have $\Pi_f(\bar{x}) = M_{f,x^*}(\bar{x}) = \{\nabla^F \varphi^*(x^*)\}$. ∎

5. Lipstchitz property of the minimal time projection. This section is devoted to the single valuedness and Lipschitz property of Π_f. We need first the following lemma from [9].

LEMMA 5.1. *Let X be a Hilbert space and $G = \mathbb{B}_X$, and let $f : X \to \mathbb{R}\cup\{+\infty\}$ be Lipschitz continuous on* $\operatorname{dom} f$ *with a constant $L \le 1/2$. Assume that $f + (\gamma/2)\|\cdot\|^2$ is convex for some real constant $\gamma \ge 0$. Let $\bar{x} \in X$, $\varepsilon_0 > 0$ and $\gamma_1 > \gamma$ be such that $2\gamma_1(\operatorname{dist}(\bar{x}, \operatorname{dom} f) + \varepsilon_0) < \frac{1-L}{1+L}$. Then*

$$2\gamma_1\|\bar{x} - y\| < 1 \qquad \forall y \in \Pi_f(\bar{x}, \varepsilon_0), \tag{5.1}$$

where $\Pi_f(\bar{x}, \varepsilon_0)$ is given by (3.1)*. Further, for all positive $\varepsilon \le \varepsilon_0$ one has*

$$\|y_1 - y_2\| \le \max\left\{16\varepsilon,\ \sqrt{\frac{8\varepsilon}{\gamma_1 - \gamma}}\ \right\} \qquad \forall y_1, y_2 \in \Pi_f(\bar{x}, \varepsilon). \tag{5.2}$$

With the above lemma at hand we prove our following theorem.

THEOREM 5.2. *Let X be a Hilbert space and $G = \mathbb{B}_X$. Assume that the function $f : X \to \mathbb{R}\cup\{+\infty\}$ is Lipschitz continuous on* $\operatorname{dom} f$ *with some constant $L \le 1/2$. Assume also that there is some constant $\gamma \ge 0$ such that f is γ-semiconvex in the sense that $f + (\gamma/2)\|\cdot\|^2$ is convex (and consequently,* $\operatorname{dom} f$ *is convex). Then, the following hold.*

(a) *On the open convex set*

$$U := \left\{ x \in X : 2\gamma\operatorname{dist}(x, \operatorname{dom} f) < \frac{1-L}{1+L} \right\}$$

the minimal time projection Π_f is single-valued, i.e., $\Pi_f(x)$ is a singleton for all $x \in U$.

(b) *On the open set $U \setminus \operatorname{cl}(\operatorname{dom} f)$ the (single-valued) mapping Π_f is locally Lipschitz continuous with $K = 16$ as a Lipschitz constant.*

Proof. Fix any $\bar{x} \in U$. Due to the inequality $2\gamma\operatorname{dist}(\bar{x}, \operatorname{dom} f) < \frac{1-L}{1+L}$ there exist a neighborhood U_0 of $\bar{x}$, $\varepsilon_0 > 0$ and $\gamma_1 > \gamma$ such that

$$2\gamma_1\Big(\operatorname{dist}(x, \operatorname{dom} f) + \varepsilon_0\Big) < \frac{1-L}{1+L} \quad \text{for all } x \in U_0.$$

According to Lemma 5.1

$$2\gamma_1\|x - y\| < 1 \qquad \forall x \in U_0, \quad \forall y \in \Pi_f(x, \varepsilon_0), \tag{5.3}$$

and for any positive $\varepsilon \le \varepsilon_0$ and any $x \in U_0$ one has

$$\|y_1 - y_2\| \le \max\left\{16\varepsilon,\ \sqrt{\frac{8\varepsilon}{\gamma_1 - \gamma}}\right\} \qquad \forall y_1, y_2 \in \Pi_f(x, \varepsilon). \tag{5.4}$$

(a) Let $x \in U_0$ and $(y_k)_k$ be a sequence satisfying the condition $f(y_k) + \rho_G(x - y_k) \to T_f(x)$. It means that $y_k \in \Pi_f(x, \varepsilon_k)$ with some $\varepsilon_k \downarrow 0$. The inequality (5.4) implies that $(y_k)_k$ is a Cauchy sequence and hence converges to some $y \in X$. By lower semicontinuity of f and continuity of ρ_G we get $f(y) + \rho_G(x - y) \le T_f(x)$, i.e., $y \in \Pi_f(x)$. It ensues that $\Pi_f(x)$ is a singleton, and hence Π_f is single-valued on U. We identify below the singleton $\Pi_f(x)$ with its element.

(b) Now let $\bar{x} \in U \setminus \mathrm{cl}(\mathrm{dom}\, f)$, and hence $\mathrm{dist}(\bar{x}, \mathrm{dom}\, f) > 0$. Choose a positive real $\delta \le \frac{\varepsilon_0}{2}$ such that

$$\max\left\{16\delta,\ \sqrt{\frac{4\delta}{\gamma_1 - \gamma}}\right\} < \frac{\mathrm{dist}(\bar{x}, \mathrm{dom}\, f)}{33}. \tag{5.5}$$

Consider the neighborhood $W := U_0 \cap B(\bar{x}, \delta)$ of $\bar{x}$. To complete the proof, it suffices to show that Π_f is Lipschitz continuous on W with constant $K = 16$. Let us proceed by contradiction in supposing that there are $x_1, x_2 \in W$ such that, for $y_i = \Pi_f(x_i)$, $i = 1, 2$, we have

$$\|y_1 - y_2\| > 16\|x_1 - x_2\|. \tag{5.6}$$

Let us put

$$x := \frac{x_1 + x_2}{2}, \quad y := \frac{y_1 + y_2}{2}, \quad u := \frac{y_1 - y_2}{2}, \quad v := \frac{x_1 - x_2}{2}, \quad z := x - y.$$

Note that (5.6) is equivalent to the inequality

$$\|v\| < \frac{\|u\|}{16}. \tag{5.7}$$

Since $y_i = \Pi_f(x_i) \in \mathrm{dom}\, f$, by the convexity of $\mathrm{dom}\, f$ if follows that $y \in \mathrm{dom}\, f$, and hence $\|z\| \ge \mathrm{dist}(x, \mathrm{dom}\, f) \ge \mathrm{dist}(\bar{x}, \mathrm{dom}\, f) - \|x - \bar{x}\|$. Bearing in mind that $x_i \in B(\bar{x}, \delta)$, we also have $\|x - \bar{x}\| < \delta$. It follows from (5.5) that $\mathrm{dist}(\bar{x}, \mathrm{dom}\, f) > 33 \cdot 8\delta = 264\delta$. Consequently, $\|z\| \ge \mathrm{dist}(\bar{x}, \mathrm{dom}\, f) - \delta \ge 263\delta$. Using once more the inclusions $x_i \in B(\bar{x}, \delta)$, $i = 1, 2$, we also see that $\|v\| \le \delta$, and thus

$$\|v\| \le \frac{\|z\|}{263}. \tag{5.8}$$

Due to the equalities $y_i = \Pi_f(x_i)$ and the inclusions $x_i \in B(\bar{x}, \delta)$, $i = 1, 2$, and using the 1-Lipschitz property of T_f (which is easily seen and noticed in [8]) we also have

$$f(y_i) + \|\bar{x} - y_i\| \le f(y_i) + \|x_i - y_i\| + \delta = T_f(x_i) + \delta \le T_f(\bar{x}) + 2\delta,$$

and hence $y_i \in \Pi_f(\bar{x}, 2\delta)$. According to (5.4) with $\varepsilon = 2\delta$ we get

$$\|u\| = \frac{\|y_1 - y_2\|}{2} \le \max\left\{16\delta,\ \sqrt{\frac{4\delta}{\gamma_1 - \gamma}}\right\}.$$

Combining this with (5.5) yields $\|u\| \le \operatorname{dist}(\bar{x}, \operatorname{dom} f)/33$. Taking into account that $\|z\| \ge \operatorname{dist}(\bar{x}, \operatorname{dom} f) - \delta \ge \left(1 - \frac{1}{264}\right) \operatorname{dist}(\bar{x}, \operatorname{dom} f)$, we arrive at $\|u\| \le \frac{264\cdot\|z\|}{263\cdot 33}$. So,

$$\|u\| \le \frac{\|z\|}{32}. \tag{5.9}$$

Due to the Lipschitz continuity of f on $\operatorname{dom} f$ we have

$$|f(y \pm u) - f(y)| \le L\|u\|. \tag{5.10}$$

Observe that $f(y_i) + \|x_i - y_i\| = T_f(x_i) \le f(y) + \|x_i - y\|$, and hence

$$\|z+v-u\| - \|z+v\| \le f(y) - f(y+u), \qquad \|z-v+u\| - \|z-v\| \le f(y) - f(y-u). \tag{5.11}$$

In view of (5.10) the first inequality in (5.11) implies

$$\|z+v-u\| - \|z+v\| \le L\|u\| \le \frac{\|u\|}{2}.$$

Therefore,

$$\begin{aligned} &\|z+v\|^2 - 2\langle z+v, u\rangle + \|u\|^2 = \|z+v-u\|^2 \\ &\le \left(\|z+v\| + \frac{\|u\|}{2}\right)^2 \le \|z+v\|^2 + \|z+v\| \cdot \|u\| + \frac{\|u\|^2}{4}, \end{aligned}$$

and hence $-\langle z+v, u\rangle \le \frac{1}{2}\|z+v\| \cdot \|u\|$. So, $-\langle z, u\rangle \le \frac{\|u\|}{2}(\|z\| + 3\|v\|)$. Similarly, the second inequality in (5.11) entails $\langle z, u\rangle \le \frac{\|u\|}{2}(\|z\| + 3\|v\|)$. Using (5.8), we get

$$|\langle z, u\rangle| \le \frac{\|u\|}{2}(\|z\| + 3\|v\|) \le \frac{1}{2}\left(1 + \frac{3}{263}\right)\|z\| \cdot \|u\|.$$

This and (5.7) yields

$$|\langle z, u-v\rangle| \le \frac{1}{2}\left(1 + \frac{3}{263} + \frac{1}{8}\right)\|z\| \cdot \|u\|.$$

In view of (5.7) and (5.9) we also have

$$\|u-v\|^2 \le (\|u\| + \|v\|)^2 \le \left(1 + \frac{1}{16}\right)^2 \|u\|^2 \le \left(1 + \frac{1}{16}\right)^2 \frac{\|z\| \cdot \|u\|}{32}.$$

Consequently,

$$2|\langle z, u-v\rangle| + \|u-v\|^2 \le \left(1 + \frac{3}{263} + \frac{1}{8} + \frac{1}{32}\left(1 + \frac{1}{16}\right)^2\right)\|z\| \cdot \|u\| < \frac{6}{5}\|z\| \cdot \|u\|. \tag{5.12}$$

Putting

$$t_1 = \frac{-2\langle z, u-v\rangle + \|u-v\|^2}{\|z\|^2}, \qquad t_2 = \frac{2\langle z, u-v\rangle + \|u-v\|^2}{\|z\|^2},$$

we get by (5.9), (5.12)

$$|t_i| \le \frac{6\|u\|}{5\|z\|} \le \frac{3}{80} \tag{5.13}$$

and

$$\begin{aligned}
&\|z+v-u\|+\|z-v+u\| \\
&\quad=\sqrt{\|z\|^2-2\langle z,u-v\rangle+\|u-v\|^2}+\sqrt{\|z\|^2+2\langle z,u-v\rangle+\|u-v\|^2} \\
&\quad=\|z\|\left(\sqrt{1-\frac{2\langle z,u-v\rangle}{\|z\|^2}+\frac{\|u-v\|^2}{\|z\|^2}}+\sqrt{1+\frac{2\langle z,u-v\rangle}{\|z\|^2}+\frac{\|u-v\|^2}{\|z\|^2}}\right) \\
&\quad=\|z\|\left(\sqrt{1+t_1}+\sqrt{1+t_2}\right).
\end{aligned}$$

Since $\sqrt{1+t}\geq 1+\frac{t}{2}-\frac{13}{100}t^2$ for all t with $|t|\leq\frac{3}{80}$, it follows by (5.13) that

$$\begin{aligned}
\|z+v-u\|+\|z-v+u\| &\geq \|z\|\left(2+\frac{t_1+t_2}{2}-\frac{13}{100}(t_1^2+t_2^2)\right) \\
=\|z\|\left(2+\frac{\|u-v\|^2}{\|z\|^2}-\frac{13}{100}(t_1^2+t_2^2)\right) &\geq \|z\|\left(2+\frac{\|u-v\|^2}{\|z\|^2}-\frac{13}{50}\left(\frac{6\|u\|}{5\|z\|}\right)^2\right).
\end{aligned}$$

Using (5.7), we arrive at

$$\|z+v-u\|+\|z-v+u\|>2\|z\|+\left(\left(1-\frac{1}{16}\right)^2-\frac{234}{625}\right)\frac{\|u\|^2}{\|z\|}. \tag{5.14}$$

On other hand, since $\sqrt{1+t}\leq 1+\frac{t}{2}$ for all $t\geq -1$ it follows that

$$\begin{aligned}
\|z+v\|+\|z-v\| &= \|z\|\left(\sqrt{1+\frac{2\langle z,v\rangle}{\|z\|^2}+\frac{\|v\|^2}{\|z\|^2}}+\sqrt{1-\frac{2\langle z,v\rangle}{\|z\|^2}+\frac{\|v\|^2}{\|z\|^2}}\right) \\
&\leq 2\|z\|+\frac{\|v\|^2}{\|z\|}\leq 2\|z\|+\frac{\|u\|^2}{16^2\|z\|},
\end{aligned}$$

and by (5.14) we obtain

$$\begin{aligned}
&\|z+v-u\|+\|z-v+u\|-\|z+v\|-\|z-v\| \\
&\quad>\left(\left(1-\frac{1}{16}\right)^2-\frac{234}{625}-\frac{1}{16^2}\right)\frac{\|u\|^2}{\|z\|}>\frac{\|u\|^2}{2\|z\|}.
\end{aligned}$$

Now note from (5.3) that $2\gamma\|x_i-y_i\|<1$, $i=1,2$, and hence $2\gamma\|z\|<1$. Consequently,

$$\|z+v-u\|+\|z-v+u\|-\|z+v\|-\|z-v\|>\gamma\|u\|^2.$$

This and (5.11) yield $2f(y)-f(y+u)-f(y-u)>\gamma\|u\|^2$, which contradicts the convexity of the function $\varphi(x)=f(x)+\frac{\gamma}{2}\|x\|^2$. Then (5.7) cannot hold, and hence the desired Lipschitz property of Π_f is valid on W. ∎

Observe that under the assumptions of Theorem 5.2 the minimal time projection can be multivalued and discontinuous away from dom f outside the open set U in the theorem. Indeed, let $X=\mathbb{R}^2$ be the usual Euclidean plane and $G=\mathbb{B}_X$. Consider the lsc function f satisfying (2.7) and defined on X by

$$f(x,y)=\begin{cases}-\sqrt{1+y^2} & \text{if } 2|y|\leq 1,\ x=0,\\ +\infty & \text{if } 2|y|>1 \text{ or } x\neq 0.\end{cases}$$

Then the assumptions of Theorem 5.2 hold true with $L = \frac{1}{\sqrt{5}}$ and $\gamma = 1$. Since $\Pi_f(x,y) \subset \operatorname{dom} f = \left\{(0,v) \,:\, |v| \le \frac{1}{2}\right\}$, it follows that

$$\Pi_f(x,y) = \left\{(0,\bar v) \,:\, \bar v \in \operatorname*{Argmin}_{|v|\le \frac{1}{2}} \left(\sqrt{x^2+(y-v)^2} - \sqrt{1+v^2}\right)\right\}$$

and hence, identifying a singleton with its element, we have

$$\Pi_f(x,y) = \begin{cases} \left(0, \frac{y}{1-|x|}\right), & |x|+2|y|<1, \\ \left(0, \frac{1}{2}\operatorname{sign} y\right), & |x|+2|y| \ge 1,\ y \ne 0, \\ \left\{\left(0,-\frac{1}{2}\right), \left(0,\frac{1}{2}\right)\right\}, & |x| \ge 1,\ y=0. \end{cases}$$

So, the minimal time projection $\Pi_f(\cdot,\cdot)$ is multivalued and discontinuous at any point $(x,0)$ with $|x| \ge 1$.

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