Journal of Functional Analysis 274 (2018) 1155–1176

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

The spherical p-harmonic eigenvalue problem in non-smooth domains

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A R T I C L E   I N F O

Article history:
Received 18 April 2017
Accepted 24 July 2017
Available online 29 July 2017
Communicated by H. Brezis

MSC:
35J72
35J92

Keywords:
p-Laplacian operator
Polar sets
Boundary Harnack inequality
p-Martin boundary

A B S T R A C T

We prove the existence of p-harmonic functions under the form $u(r,\sigma) = r^{-\beta}\omega(\sigma)$ in any cone C_S generated by a spherical domain S and vanishing on ∂C_S. We prove the uniqueness of the exponent β and of the normalized function ω under a Lipschitz condition on S.

Contents

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http://dx.doi.org/10.1016/j.jfa.2017.07.012
0022-1236/

1. Introduction

Let $p > 1$, S a domain of the unit sphere S^{N-1} of $\mathbb{R}^N$ and $C_S := \{(r, \sigma) : r > 0, \sigma \in S\}$ the positive cone generated by S. If one looks for p-harmonic functions in C_S under the form $u(x) = u(r, \sigma) = r^{-\beta}\omega(\sigma)$ vanishing on $\partial C_S \setminus \{0\}$, then ω satisfies the *spherical p-harmonic eigenvalue problem* on S

$$
\begin{aligned}
-div'\left(\left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}} \nabla'\omega\right) &= (p-1)\beta(\beta - \beta_0)\left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}} \omega \quad && \text{in } S \\
\omega &= 0 && \text{in } \partial S
\end{aligned}
\tag{1.1}
$$

with $\beta_0 = \frac{N-p}{p-1}$ and were div' and ∇' denote the divergence operator and the covariant gradient on S^{N-1} endowed with the metric induced by its isometric imbedding into $\mathbb{R}^N$. Separable solutions play a key role for describing the boundary behaviour and the singularities of solutions of a large variety of quasilinear equations. When $N = 2$ the equation is completely integrable and has been solved by Kroll [5] in the regular case $\beta < 0$ and Kichenassamy and Véron [4] in the singular case $\beta > 0$. In higher dimension, Tolksdorff [13] proved the following:

Theorem A. *If S is a smooth spherical domain, there exist two couples (β_S, ω_S) and (β'_S, ω'_S) where $\beta_S > 0$ and $\beta'_S < 0$, ω_S and ω'_S are positive $C^2(\overline{S})$-functions vanishing on ∂S which solve (1.1) with $(\beta, \omega) = (\beta_S, \omega_S)$ or $(\beta, \omega) = (\beta'_S, \omega'_S)$. Furthermore β_S and β'_S are unique, and ω_S and ω'_S are unique up to an homothety.*

A more general and transparent proof has been obtained by Porretta and Véron [11], but always in the case of a smooth spherical domain. The aim of this article is to extend Theorem A to a general spherical domain. If we consider an increasing sequence of smooth domains $\{S_k\}$ such that $S_k \subset \overline{S}_k \subset S_{k+1}$ and $\cup_k S_k = S$ we prove the following:

Theorem B. *Assume that S^c is not polar. Then the sequence of the $\beta_{S_k} > 0$ from Theorem A is decreasing and converges to $\beta_S > 0$. There exists $\omega_S \in W_0^{1,p}(S) \cap L^\infty(S)$ weak solution of (1.1) with $\beta = \beta_S$. Furthermore $\beta_S > 0$ is the largest exponent β such that (1.1) admits a positive solution $\omega_S \in W_0^{1,p}(S)$.*

Under a mild assumption on S it is possible to approximate it by a decreasing sequence of smooth domains S'_k such that $S'_k \subset \overline{S}'_k \subset S'_{k-1}$ and $\cap_k S'_k = \overline{S}$

Theorem C. *Assume that $S = \overset{o}{\overline{S}}$. Then the sequence $\beta_{S'_k} > 0$ is increasing and converges to $\hat{\beta}_S > 0$ and there exists $\hat{\omega}_S \in W_0^{1,p}(S) \cap L^\infty(S)$ weak solution of (1.1) with $\beta = \hat{\beta}_S$. Furthermore $\hat{\beta}_S$ is the smallest exponent β such that (1.1) admits a positive solution $\omega_S \in W_0^{1,p}(S)$.*

We prove the uniqueness of the exponent β, under a Lipschitz assumption on S.

Theorem D. *Assume that S is a Lipschitz domain, then $\beta_S = \hat{\beta}_S$ and if ω and ω' are two positive solutions of (1.1) in $W_0^{1,p}(S)$, there exists a constant $c > 0$ such that $c^{-1}\omega' \leq \omega \leq c\omega'$.*

The proof of Theorem C is based upon a sharp form of boundary Harnack inequality proved in [8],

$$\left| \ln \frac{\omega(\sigma_1)}{\omega'(\sigma_1)} - \ln \frac{\omega(\sigma_2)}{\omega'(\sigma_2)} \right| \leq c_1 \left| \sigma_1 - \sigma_2 \right|^\alpha \quad \forall\, \sigma_1, \sigma_2 \in S, \tag{1.2}$$

for some $c_1 = c_1(N, p, S) > 0$ and $\alpha \in (0, 1)$. Actually we have a stronger result, much more delicate to obtain.

Theorem E. *Let S be a Lipschitz subdomain of S^{N-1}. Then two positive solutions of (1.1) in $W_0^{1,p}(S)$ are proportional.*

The proof is based upon a non-trivial adaptation of a series of deep results of Lewis and Nyström [6], [7], [8], [9] concerning the p-Martin boundary of domains. General references for the p-Laplace operator can be found in [10] and applications in [14].

Acknowledgments This article has been prepared with the support of the collaboration programs ECOS C14E08 between Chile and France, FONDECYT grant 3140567 and Millenium Nucleus CAPDE NC130017.

2. Existence

2.1. Estimates

Through this article we assume that S^c is not polar, or equivalently that it has positive $c_{1,p}^{S^{N-1}}$-capacity.

Lemma 2.1. *Assume $p > 1$. Then any solution $\omega \in W_0^{1,p}(S)$ of (1.1) satisfies*

$$\|\omega\|_{C^\gamma(S)} \leq c_1 \|\omega\|_{L^p(S)}, \tag{2.1}$$

if $p > N-1$ *where* $\gamma = 1 - \frac{N-1}{p}$ *if* $p > N-1$ *and*

$$\|\omega\|_{L^\infty(S)} \leq c_1 \|\omega\|_{L^p(S)}, \tag{2.2}$$

if $1 < p \leq N-1$*, where* $c_1 > 0$ *depends on* p*,* N*,* β*.*

Proof. Multiplying the equation by ω and using Hölder's inequality, we derive

$$\begin{aligned} &(i) \quad \int_S \left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p}{2}} dS \leq (\beta(p\beta - (p-1)\beta_0))^{\frac{p}{2}} \int_S |\omega|^p \, dS \quad \text{if } p \geq 2, \\ &(ii) \quad \int_S \left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p}{2}} dS \leq \beta^{p-1}(p\beta - (p-1)\beta_0) \int_S |\omega|^p \, dS \quad \text{if } 1 < p < 2. \end{aligned} \tag{2.3}$$

Notice that these inequalities hold for all $p > 1$. If $p > N-1$ (2.1) follows by Morrey' inequality. Here after we assume $1 < p \leq N-1$. Let $\alpha \geq 1$ and $k > 0$. Then $\zeta = \min\{|\omega|, k\}^{\alpha-1}\omega$ is an admissible test function, hence

1- If $p \geq 2$,

$$\begin{aligned} \int_S \left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}} \langle \nabla'\omega . \nabla'\zeta \rangle dS &= (p-1)\beta(\beta - \beta_0) \int_S \left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}} \omega\zeta dS \\ &\leq c_2 \int_S |\nabla'\omega|^{p-2} \omega^2 \min\{|\omega|, k\}^{\alpha-1} dS + c_2\beta^p \int_S |\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS \\ &\leq c_2 \left(\int_S |\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS\right)^{\frac{p-2}{p}} \left(\int_S |\nabla'\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS\right)^{\frac{2}{p}} \\ &\quad + c_2\beta^p \int_S |\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS, \end{aligned} \tag{2.4}$$

where $c_2 = c_2(N, p, \beta) > 0$. Since

$$\int_S \left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}} \langle \nabla'\omega . \nabla'\zeta \rangle dS \geq c_3(p) \int_S |\nabla'\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS,$$

it implies that there exists $c_4 = c_4(N, p, \beta)$ such that

$$\int_S |\nabla'\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS \leq c_4 \int_S |\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS, \tag{2.5}$$

which yields

$$\int\limits_S |\nabla' j(\omega)|^p \, dS \leq c_4 \int\limits_S |j(\omega)|^p \, dS, \tag{2.6}$$

where $j(\omega) = \min\{|\omega|, k\}^{\frac{\alpha-1}{p}} \omega$.

2- If $1 < p < 2$, then

$$\begin{aligned} &\int\limits_S \left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}} \langle \nabla'\omega . \nabla'\zeta \rangle dS \\ &= \int\limits_S \left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}} |\nabla'\omega|^2 \min\{|\omega|, k\}^{\alpha-1} dS \\ &\quad + (\alpha - 1) \int\limits_{S\cap\{|\omega|<k\}} \left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}} |\nabla'\omega|^2 |\omega|^{\alpha-1} \, dS. \end{aligned} \tag{2.7}$$

Since

$$\begin{aligned} &\int\limits_S \left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}} |\nabla'\omega|^2 \min\{|\omega|, k\}^{\alpha-1} dS \\ &= \int\limits_S \left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p}{2}} \min\{|\omega|, k\}^{\alpha-1} dS \\ &\quad - \beta^2 \int\limits_S \left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}} \min\{|\omega|, k\}^{\alpha-1} \omega^2 dS \\ &\geq \int\limits_S |\nabla'\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS - \beta^2 \int\limits_S \left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}} \min\{|\omega|, k\}^{\alpha-1} \omega^2 dS, \end{aligned}$$

we derive

$$\int\limits_S |\nabla'\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS \leq \beta^{p-1}(p\beta - (p-1)\beta_0) \int\limits_S |\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS, \tag{2.8}$$

which leads to (2.6). Letting $k \to \infty$ we infer by Fatou's lemma,

$$\int\limits_S \left|\nabla' |\omega|^{\frac{\alpha-1}{p}+1}\right|^p dS \leq c_4 \int\limits_S |\omega|^{\alpha-1+p} \, dS. \tag{2.9}$$

If $p < N - 1$ we derive from Sobolev inequality and putting $q = \alpha - 1 + p$ and $s = \frac{N-1}{N-1-p} > 1$

$$\left(\int\limits_S |\omega|^{sq} \, dS\right)^{\frac{1}{s}} \leq c_5 \int\limits_S |\omega|^q \, dS, \tag{2.10}$$

and $c_5 > 0$ depends on N, p and β. Iterating this estimate by Moser's method we derive (2.10).

If $p = N-1$ we have for $1 \leq m < p-1$ and $m^* = \frac{m(N-1)}{N-1-m}$

$$c_6\left(\int_S |\omega|^{(\frac{\alpha-1}{p}+1)m^*}\,dS\right)^{\frac{pm}{m^*}} \leq \left(\int_S \left|\nabla' |\omega|^{\frac{\alpha-1}{p}+1}\right|^m dS\right)^{\frac{p}{m}} \leq |S|^{\frac{p}{m}-1} c_4 \int_S |\omega|^{\alpha-1+p}\,dS,$$

and $c_6 = c_6(N,p)$, hence

$$\left(\int_S |\omega|^{tq}\,dS\right)^{\frac{1}{t}} \leq c_5 \int_S |\omega|^q\,dS, \tag{2.11}$$

with $t = \frac{m(N-1)}{p(N-1-m)} = \frac{m}{N-1-m}$. The proof follows again by Moser's iterative scheme. □

Proposition 2.2. *Let S_1 and S_2 be two subdomains of S^{N-1} such that $S_1 \subset \overline{S}_1 \subset S_2$ and S_2 not polar. Let $\beta_j > 0$, j=1,2, such that there exist positive solutions $\omega_j \in W_0^{1,p}(S_j)$ solutions of*

$$\begin{aligned} -div'\left(\left(\beta_j^2\omega_j^2 + |\nabla'\omega_j|^2\right)^{\frac{p-2}{2}}\nabla'\omega_j\right) &= (p-1)\beta_j(\beta_j-\beta_0)\left(\beta_j^2\omega_j^2+|\nabla'\omega_j|^2\right)^{\frac{p-2}{2}}\omega_j \\ &\quad \text{in } S_j \\ \omega_j &= 0 \\ &\quad \text{in } \partial S_j. \end{aligned} \tag{2.12}$$

Then $\beta_1 \geq \beta_2$.

Proof. Set $u_j(r,\sigma) = r^{-\beta_j}\omega_j(\sigma)$, $C_{S_j} = (0,\infty)\times S_J$ and assume $\beta_1 < \beta_2$. By Harnack inequality $\omega_2 \geq c > 0$ on S_1, thus

$$u_2(r,\sigma) \geq c r^{-\beta_2} \qquad \text{a.e. in } C_{S_1}.$$

For $\epsilon > 0$ there exist $r_\epsilon > 0$ such that

$$\epsilon u_2(x) \geq u_1(x) \qquad \forall\, x \in C_{S_1} \cap \overline{B}_{r_\epsilon}.$$

Let $\delta > 0$, there exists $R_\delta > 0$ such that

$$u_1(x) \leq \delta \qquad \forall\, x \in C_{S_1} \cap B^c_{R_\delta}.$$

Hence $\zeta = (u_1 - \epsilon u_2 - \delta)_+ \in W_0^{1,p}(Q_{S_1}^{r_\epsilon,R_\delta})$, where $Q_{S_1}^{r_\epsilon,R_\delta} = \{x \in C_{S_1} : r_\epsilon < |x| < R_\delta\}$. This implies

$$0 = \int\limits_{Q_{S_1}^{r_\epsilon,R_\delta}} \left\langle |\nabla u_1|^{p-2} \nabla u_1 - |\nabla(\epsilon u_1)|^{p-2} \nabla(\epsilon u_1).\nabla\zeta \right\rangle dx$$

$$= \int\limits_{Q_{S_1}^{r_\epsilon,R_\delta} \cap \{u_1 - \epsilon u_2 \geq \delta\}} \left\langle |\nabla u_1|^{p-2} \nabla u_1 - |\nabla(\epsilon u_1)|^{p-2} \nabla(\epsilon u_1).\nabla(u_1 - u_2) \right\rangle dx.$$

Therefore $\nabla(u_1 - \epsilon u_2 - \delta)_+ = 0$ a.e. in $Q_{S_1}^{r_\epsilon,R\delta}$, which leads to $u_1 - \epsilon u_2 \leq \delta$ in the same set. Letting $\delta \to 0$ yields $R_\delta \to \infty$, thus we obtain $u_1 \leq \epsilon u_2$ in $C_{S_1} \cap \overline{B}^c_{r_\epsilon}$ hence $u_1 \leq 0$ in C_{S_1}, contradiction. □

2.2. Approximations from inside

Proof of Theorem B. Let $\{S_k\}$ be an increasing sequence of smooth domains such that $S_k \subset \overline{S}_k \subset S_{k+1}$. We denote by $\{(\beta_{S_k}, \omega_k)\}$ the corresponding sequence of solutions of (1.1) with $\beta = \beta_{S_k}$ and $\omega = \omega_k$. The sequence $\{\beta_{S_k}\}$ is uniquely determined by [13], it admits a limit $\beta := \beta_S$, and the ω_k are the unique positive solutions such that

$$\int\limits_{S_k} |\omega_k| \, dS = 1.$$

If $p \geq 2$, we have

$$\begin{aligned}
\int\limits_{S_k} |\nabla'\omega_k|^p \, dS &\leq \int\limits_{S_k} \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2\right)^{\frac{p-2}{2}} |\nabla'\omega_k|^2 \, dS \\
&= (p-1)\beta_{S_k}(\beta_{S_k} - \beta_0) \int\limits_{S_k} \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2\right)^{\frac{p-2}{2}} \omega_k^2 dS \\
&\leq 2^{\frac{(p-4)_+}{2}} (p-1)\beta_{S_k}(\beta_{S_k} - \beta_0) \int\limits_{S_k} \left(\beta_{S_k}^{p-2} \omega_k^p + |\nabla'\omega_k|^{p-2} \omega_k^2\right) dS \\
&\leq c_7(N, p, \beta_{S_k}) \int\limits_{S_k} \omega_k^p dS + \frac{1}{2} \int\limits_{S_k} |\nabla'\omega_k|^p \, dS.
\end{aligned}$$

Since $\beta_{S_k} \leq \beta_1$, we derive

$$\int\limits_{S_k} |\nabla'\omega_k|^p \, dS \leq c_8, \tag{2.13}$$

from the normalization assumption with $c_8 = 2c_7(N, p, \beta_1)$.

If $1 < p < 2$, we have

$$\int_{S_k} |\nabla'\omega_k|^p \, dS \leq \int_{S_k} \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2\right)^{\frac{p}{2}} dS$$
$$\leq \beta_{S_k}(p\beta_{S_k} + (p-1)\beta_0) \int_{S_k} \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2\right)^{\frac{p-2}{2}} \omega_k^2 dS$$
$$\leq \beta_k^{p-1}(p\beta_{S_k} + (p-1)\beta_0) \int_{S_k} \omega_k^p dS,$$

and we obtain (2.13) with $c_8 = \beta_1^{p-1}(p\beta_1 + (p-1)\beta_0)$.

Next we extend ω_k by 0 in S_k^c. Then there exists $\omega \in W_0^{1,p}(S)$ such that $\omega_k \rightharpoonup \omega$ weakly in $W_0^{1,p}(S)$, up to subsequence that we still denote $\{\omega_k\}$, and $\omega_k \to \omega$ in $L^p(S)$.

Step 1: We claim that $\nabla'\omega_k$ converges to $\nabla'\omega$ locally in $L^p(S)$.

Let $a \in S$ and $r > 0$ such that $B_{4r}(a) \subset S$. Then for $k \geq k_0$, $\overline{B}_{2r}(a) \subset S_k$. Let $\zeta \in C_0^\infty(B_{2r}(a))$ such that $0 \leq \zeta \leq 1$, $\zeta = 1$ in $B_r(a)$. For test function we choose $\eta_k = \zeta(\omega - \omega_k)$, then

$$\int_{S_k} \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2\right)^{\frac{p-2}{2}} \langle \nabla'\omega_k . \nabla'\eta_k \rangle dS$$
$$= (p-1)\beta_{S_k}(\beta_{S_k} - \beta_0) \int_{S_k} \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2\right)^{\frac{p-2}{2}} \omega_k \eta_k dS.$$

By the above inequality, we have

$$\int_{B_{2r}(a)} \left\langle \left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}} \nabla'\omega - \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2\right)^{\frac{p-2}{2}} \nabla'\omega_k . \nabla'\eta_k \right\rangle dS$$
$$= \int_{B_{2r}(a)} \left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}} \langle \nabla'\omega . \nabla'\eta_k \rangle dS$$
$$- (p-1)\beta_{S_k}(\beta_{S_k} - \beta_0) \int_{S_k} \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2\right)^{\frac{p-2}{2}} \omega_k \eta_k dS.$$

Using the weak convergence of the gradient, we have

$$\lim_{k\to\infty} \int_{B_{2r}(a)} \left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}} \langle \nabla'\omega . \nabla'\eta_k \rangle dS = 0.$$

Since ω_k is uniformly bounded in $W_0^{1,p}(S)$ and $\omega_k \to \omega$ in $L^p(S)$, we have

$$\lim_{k\to\infty}\int\limits_{B_{2r}(a)}\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\omega_k\eta_k dS=0,$$

and

$$\lim_{k\to\infty}\int\limits_{B_{2r}(a)}(\omega-\omega_k)\Big\langle\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}\nabla'\omega-\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\nabla'\omega_k.\nabla'\zeta\Big\rangle dS$$
$$=0.$$

Combining the above relations we infer

$$\lim_{k\to\infty}\int\limits_{B_{2r}(a)}\zeta\Big\langle\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}\nabla'\omega-\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\nabla'\omega_k.\nabla'(\omega-\omega_k)\Big\rangle dS$$
$$=0. \tag{2.14}$$

Next we write

$$\begin{aligned}&\int\limits_{B_{2r}(a)}\zeta\Big\langle\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}\nabla'\omega-\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\nabla'\omega_k.\nabla'(\omega-\omega_k)\Big\rangle dS\\
&=\frac{1}{2}\int\limits_{B_{2r}(a)}\zeta\left(\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}+\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\right)|\nabla'(\omega-\omega_k)|^2 dS\\
&\quad+\frac{1}{2}\int\limits_{B_{2r}(a)}\zeta\left(\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}-\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\right)\\
&\quad\times\left(|\nabla'\omega|^2+\beta^2\omega^2-\beta_{S_k}^2\omega_k^2-|\nabla'\omega_k|^2\right)dS\\
&\quad-\frac{1}{2}\int\limits_{B_{2r}(a)}\zeta\left(\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}-\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\right)\left(\beta^2\omega^2-\beta_{S_k}^2\omega_k^2\right)dS.\end{aligned} \tag{2.15}$$

If $p\geq 2$, we have from (2.4),

$$\begin{aligned}&\int\limits_{B_{2r}(a)}\zeta\Big\langle\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}\nabla'\omega-\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\nabla'\omega_k.\nabla'(\omega-\omega_k)\Big\rangle dS\\
&\geq\frac{1}{2}\int\limits_{B_{2r}(a)}\zeta\left(|\nabla'\omega|^{p-2}+|\nabla'\omega_k|^{p-2}\right)|\nabla'(\omega-\omega_k)|^2 dS\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\int\limits_{B_{2r}(a)}\zeta\left(\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}-\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\right)\left(\beta^2\omega^2-\beta_{S_k}^2\omega_k^2\right)dS\\
&\geq \min\{2^{-1},2^{2-p}\}\int\limits_{B_{2r}(a)}\zeta\,|\nabla'(\omega-\omega_k)|^p\,dS\\
&\quad-\frac{1}{2}\int\limits_{B_{2r}(a)}\zeta\left(\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}-\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\right)\left(\beta^2\omega^2-\beta_{S_k}^2\omega_k^2\right)dS.
\end{aligned}
\tag{2.16}
$$

Since $\omega_k\to\omega$ in $L^p(S)$, $\beta_{S_k}\to\beta$ and ω_k,ω are uniformly bounded in $W_0^{1,p}(S)$, we derive

$$
\int\limits_{B_{2r}(a)}\zeta\left(\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}-\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\right)\left(\beta^2\omega^2-\beta_{S_k}^2\omega_k^2\right)dS\to 0
$$

as $k\to\infty$. Jointly with (2.14) we infer that

$$
\lim_{k\to\infty}\int\limits_{B_r(a)}|\nabla'(\omega-\omega_k)|^p\,dS=0. \tag{2.17}
$$

If $1<p<2$, then

$$
\begin{aligned}
&\int\limits_{B_{2r}(a)}\zeta\Big\langle\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}\nabla'\omega-\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\nabla'\omega_k.\nabla'(\omega-\omega_k)\Big\rangle dS\\
&=\int\limits_{B_{2r}(a)}\zeta\Big\langle\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}\nabla'\omega-\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\nabla'\omega_k.\nabla'(\omega-\omega_k)\Big\rangle dS\\
&\quad+\int\limits_{B_{2r}(a)}\zeta\Big\langle\left(\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}-\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}\right)\nabla'\omega.\nabla'(\omega-\omega_k)\Big\rangle dS.
\end{aligned}
\tag{2.18}
$$

Up to extracting a subsequence, we have that $\omega_k\to\omega$ a.e. in S and that there exists $\Phi\in L^1(S)$ such that

$$
|\omega_k|^p+|\omega|^p\leq\Phi\qquad\text{a.e. in } S\quad\text{and}\quad\forall\,k\geq 1. \tag{2.19}
$$

Since

$$
\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}|\nabla\omega|\leq\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega|^2\right)^{\frac{p-1}{2}}\leq\beta_{S_k}^{p-1}\omega_k^{p-1}+|\nabla'\omega|^{p-1},
$$

and

$$\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}|\nabla\omega|\leq\beta^{p-1}\omega^{p-1}+|\nabla'\omega|^{p-1},$$

we derive that

$$\left|\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}-\left(\beta^2_{S_k}\omega_k^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}\right||\nabla'\omega|\leq 2\left(\beta^{p-1}\Phi^{p-1}+|\nabla'\omega|^{p-1}\right),$$

which implies that

$$\zeta\left(\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}-\left(\beta^2_{S_k}\omega_k^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}\right)\nabla'\omega\to 0\quad\text{ in } L^{p'}(S)$$

where p' is the conjugate of p, and finally

$$\int\limits_{B_{2r}(a)}\zeta\langle\left(\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}-\left(\beta^2_{S_k}\omega_k^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}\right)\nabla'\omega.\nabla'(\omega-\omega_k)\rangle dS\to 0$$
$$\text{as } k\to\infty. \tag{2.20}$$

For the last term on the right-hand side of (2.18), we have, for $\gamma\in\mathbb{R}_+$ and $\mathbf{A},\mathbf{B}\in\mathbb{R}^N$,

$$\begin{aligned}
&\left(\gamma+|\mathbf{B}|^2\right)^{\frac{p-2}{2}}\mathbf{B}-\left(\gamma+|\mathbf{A}|^2\right)^{\frac{p-2}{2}}\mathbf{A}\\
&=\int\limits_0^1\frac{d}{dt}\left(\left(\gamma+|t\mathbf{B}+(1-t)\mathbf{A}|^2\right)^{\frac{p-2}{2}}(t\mathbf{B}+(1-t)\mathbf{A})\right)dt\\
&=\left(\int\limits_0^1\left(\gamma+|t\mathbf{B}+(1-t)\mathbf{A}|^2\right)^{\frac{p-2}{2}}dt\right)(\mathbf{B}-\mathbf{A})\\
&\quad+(p-2)\int\limits_0^1\left(\gamma+|t\mathbf{B}+(1-t)\mathbf{A}|^2\right)^{\frac{p-4}{2}}\langle t\mathbf{B}+(1-t)\mathbf{A}\,.\,\mathbf{B}-\mathbf{A}\rangle(t\mathbf{B}+(1-t)\mathbf{A})dt.
\end{aligned}$$

This implies

$$\begin{aligned}
&\langle\left(\gamma+|\mathbf{B}|^2\right)^{\frac{p-2}{2}}\mathbf{B}-\left(\gamma+|\mathbf{A}|^2\right)^{\frac{p-2}{2}}\mathbf{A}\,.\,\mathbf{B}-\mathbf{A}\rangle\\
&=\left(\int\limits_0^1\left(\gamma+|t\mathbf{B}+(1-t)\mathbf{A}|^2\right)^{\frac{p-2}{2}}dt\right)|\mathbf{B}-\mathbf{A}|^2\\
&\quad+(p-2)\int\limits_0^1\left(\gamma+|t\mathbf{B}+(1-t)\mathbf{A}|^2\right)^{\frac{p-4}{2}}\langle t\mathbf{B}+(1-t)\mathbf{A}\,.\,\mathbf{B}-\mathbf{A}\rangle^2dt.
\end{aligned}$$

We observe that

$$\int_0^1 \left(\gamma+|t\mathbf{B}+(1-t)\mathbf{A}|^2\right)^{\frac{p-4}{2}} \langle t\mathbf{B}+(1-t)\mathbf{A}\,.\,\mathbf{B}-\mathbf{A}\rangle^2 dt$$

$$\leq |\mathbf{B}-\mathbf{A}|^2 \int_0^1 \left(\gamma+|t\mathbf{B}+(1-t)\mathbf{A}|^2\right)^{\frac{p-2}{2}} dt,$$

and since $1<p<2$, we finally obtain

$$\begin{aligned}
&\langle\left(\gamma+|\mathbf{B}|^2\right)^{\frac{p-2}{2}}\mathbf{B}-\left(\gamma+|\mathbf{A}|^2\right)^{\frac{p-2}{2}}\mathbf{A}\,.\,\mathbf{B}-\mathbf{A}\rangle\\
&\geq (p-1)\left(\int_0^1\left(\gamma+|t\mathbf{B}+(1-t)\mathbf{A}|^2\right)^{\frac{p-2}{2}}dt\right)|\mathbf{B}-\mathbf{A}|^2 \qquad (2.21)\\
&\geq (p-1)\,|\mathbf{B}-\mathbf{A}|^2\left(\gamma+1+|\mathbf{B}|^2+|\mathbf{A}|^2\right)^{\frac{p-2}{2}}.
\end{aligned}$$

We plug this estimate into (2.18) with $\gamma=\beta_k^2\omega_k^2$, $\mathbf{A}=\nabla'\omega$ and $\mathbf{B}=\nabla'\omega_k$, then

$$\begin{aligned}
&\int_{B_{2r}(a)}\zeta\langle\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}\nabla'\omega-\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\nabla'\omega_k.\nabla'(\omega-\omega_k)\rangle dS\\
&\geq\int_{B_{2r}(a)}\zeta\,|\nabla'(\omega-\omega_k)|^2\left(\beta_k^2\omega_k^2+1+|\nabla'\omega_k|^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}dS. \qquad (2.22)
\end{aligned}$$

Set $\phi(.)=\beta_k^2\omega_k^2+1+|\nabla'\omega_k|^2+|\nabla'\omega|^2$, then

$$\begin{aligned}
\int_{B_r(a)}|\nabla'\omega-\nabla'\omega_k|^p\,dS&=\int_{B_r(a)}|\nabla'\omega-\nabla'\omega_k|^p\,\phi^{\frac{p(p-2)}{4}}\phi^{-\frac{p(p-2)}{4}}dS\\
&\leq\left(\int_{B_r(a)}|\nabla'\omega-\nabla'\omega_k|^2\,\phi^{\frac{p-2}{2}}dS\right)^{\frac{p}{2}}\left(\int_{B_r(a)}\phi^{\frac{p}{2}}dS\right)^{\frac{2-p}{2}}. \qquad (2.23)
\end{aligned}$$

Jointly with (2.14) and (2.22) we conclude that (2.17). Step 1 follows by a standard covering argument.

Step 2: We claim that ω_k converges to ω in $W_0^{1,p}(S)$.

Up to a subsequence that we denote again by $\{k\}$, we can assume that $\omega_k\to\omega$ and $\nabla'\omega_k\to\nabla'\omega$ a.e. in S. Let $\zeta\in C_0^\infty(S)$, then there exists $k_\epsilon\in\mathbb{N}$ such that the support K of ζ is a compact subset of S_k for all $k\geq k_\epsilon$. If $1<p<2$,

$$\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}|\nabla'\omega_k|\leq|\nabla'\omega_k|^{p-1},$$

which bounded in $L^{p'}(K)$, then uniformly integrable in K and by Vitali's convergence theorem

$$\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\nabla'\omega_k\to\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}\nabla'\omega,$$

in $L^1_{loc}(S)$. Similarly

$$\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\omega_k\to\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}\omega,$$

in $L^1_{loc}(S)$. If $p\geq 2$

$$\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}|\nabla'\omega_k|\leq c\left(|\omega_k|^{p-1}+|\nabla'\omega_k|^{p-1}\right),$$

and we conclude again by Vitali's convergence theorem that the previous convergences hold. Since

$$\begin{aligned}&\int_{S_k}\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\langle\nabla'\omega_k.\nabla'\zeta\rangle dS\\&\quad=(p-1)\beta_{S_k}(\beta_{S_k}-\beta_0)\int_{S_k}\left(\beta_{S_k}^2\omega_k^2+|\nabla'\omega_k|^2\right)^{\frac{p-2}{2}}\omega_k\zeta dS\end{aligned}$$

we conclude that ω is a weak solution of (1.1) with $\beta=\beta_S$.

2.3. Approximations from outside

Proof of Theorem C. Since $\overline{S}^c$ has a non-empty interior, the existence of a sequence $\{\omega'_k\}$ corresponding to solutions of (1.1) in S'_k with $\beta=\beta_{S'_k}$ is the consequence of [11]. The fact that $\{\beta_{S'_k}\}$ is increasing follows from Proposition 2.2. We denote by $\hat{\beta}:=\hat{\beta}_S$ its limit, and it is smaller or equal to β_S. Estimates (2.4) are valid with S'_k, ω'_k and $\beta_{S'_k}$ instead of S, ω and β. If we extend ω'_k by 0 in $S'^{\,c}_k$ these estimates are valid with S^{N-1} instead of S'_k. Then up to a subsequence the exists $\omega\in W^{1,p}(S^{N-1})$ and a subsequence still denoted by $\{k\}$ such that $\omega'_k\rightharpoonup\omega$ weakly in $W^{1,p}(S^{N-1})$, strongly in $L^p(S^{N-1})$ and a.e. in S^{N-1}. Furthermore, as in the proof of Theorem A, for any compact set $K\subset S$, $\nabla'\omega'_k\to\nabla'\omega'$ in $L^p(K)$. This is sufficient to assert that ω is a weak solution of

$$-div'\left(\left(\hat{\beta}^2\omega'^{\,2}+|\nabla'\omega'|^2\right)^{\frac{p-2}{2}}\nabla'\omega'\right)=(p-1)\hat{\beta}(\hat{\beta}-\beta_0)\left(\hat{\beta}^2\omega^2+|\nabla'\omega'|^2\right)^{\frac{p-2}{2}}\omega'\quad\text{in } S.$$

Moreover $\omega'\lfloor_{S'_k}$ belongs to $W_0^{1,p}(S'_k)$ for all k. Since $\omega'_k = 0$ in S_k^c and converges a.e. to ω, this last function vanishes a.e. in $\cup_k S_k^c = (\cap_k S_k)^c = \overline{S}^c$. Therefore ω vanishes a.e. in $\overline{S}^c$ and since it is quasi continuous, it vanishes, $(1-p)$-quasi everywhere in $\overline{S}^c$. From Netrusov's theorem (see [1, Th 10.1.1]-(iii)) there exists a sequence $\{\eta_n\} \subset C_0^\infty(S)$ which converges to ω in $W^{1,p}(S)$, thus $\omega \in W_0^{1,p}(S)$. □

3. Uniqueness

3.1. Uniqueness of exponent β

Proof of Theorem D. If S is Lipschitz, C_S is also Lipschitz. We fix $z \in S \approx S^{N-1} \cap \partial C_S$ and we apply [8, Th 2] in $G_z = C_S \cap B_{\frac{1}{2}}(z)$ to two separable p-harmonic functions $u(r,\sigma) = r^{-\beta}\omega(\sigma)$ and $u'(r,\sigma) = r^{-\beta'}\omega'(\sigma)$. There exist $\gamma \in (0, \frac{1}{2})$, $c_{10} > 0$ and $\alpha \in (0,1)$ such that

$$\left| \ln \frac{u(y_1)}{u'(y_1)} - \ln \frac{u(y_2)}{u'(y_2)} \right| \leq c_{10} \left| y_1 - y_2 \right|^\alpha \quad \forall\, y_1, y_2 \in C_S \cap B_\gamma(z). \tag{3.24}$$

Assume $|y_1| = |y_2| = 1$, then

$$\left| \ln \frac{\omega(y_1)}{\omega'(y_1)} - \ln \frac{\omega(y_2)}{\omega'(y_2)} \right| \leq c_{10} \left| y_1 - y_2 \right|^\alpha \quad \forall\, y_1, y_2 \in S \cap B_\gamma(z). \tag{3.25}$$

We denote by $\ell(x,y)$ the geodesic distance on S^{N-1} and by $\ell(x,K)$ the geodesic distance from a point $x \in S^{N-1}$ to a subset K. Since the set $S_\gamma = \{\sigma \in S : \ell(\sigma, \partial S) \leq \frac{\gamma}{2}\}$ can be covered by a finite number of balls with center on ∂S, we infer that

$$\left| \ln \frac{\omega(y_1)}{\omega'(y_1)} - \ln \frac{\omega(y_2)}{\omega'(y_2)} \right| \leq c_{11} \qquad \forall\, y_1, y_2 \in S_\gamma. \tag{3.26}$$

In $S \setminus \overline{S}_{\frac{\gamma}{2}}$ we can use Harnack inequality to obtain

$$-c_{12} \leq \ln \frac{\omega(y_1)}{\omega(y_2)} \leq c_{12} \qquad \forall\, y_1, y_2 \in S \setminus \overline{S}_{\frac{\gamma}{2}} \text{ s.t. } \ell(y_1, y_2) \leq \tfrac{\gamma}{2}. \tag{3.27}$$

Hence there exists a constant $c_{13} > 0$ such that (3.27) holds for any $y_1, y_2 \in S \setminus \overline{S}_{\frac{\gamma}{2}}$, with c_{12} replaced by c_{13}. Furthermore ω' satisfies the same inequality in $S \setminus \overline{S}_{\frac{\gamma}{2}}$. Combining the two inequalities we obtain

$$-2c_{13} \leq \ln \frac{\omega(y_1)}{\omega(y_2)} - \ln \frac{\omega'(y_1)}{\omega'(y_2)} \leq 2c_{13} \qquad \forall\, y_1, y_2 \in S \setminus \overline{S}_{\frac{\gamma}{2}}. \tag{3.28}$$

Combining this estimate with (3.25) we derive that it holds for all $y_1, y_2 \in S$. This implies

$$e^{-2c_{13}}\frac{\omega(y_2)}{\omega'(y_2)} \leq \frac{\omega(y_1)}{\omega'(y_1)} \leq e^{2c_{13}}\frac{\omega(y_2)}{\omega'(y_2)} \qquad \forall\, y_1, y_2 \in S. \tag{3.29}$$

Assume now that there exist two exponents $\beta > \beta' > 0$ such that $r^{-\beta}\omega(.)$ and $r^{-\beta'}\omega'(.)$ are p-harmonic and positive in the cone C_S and vanishes on ∂C_S. Put $\theta = \frac{\beta}{\beta'}$, $\eta = \omega'^{\,\theta}$ and

$$\mathcal{T}(\eta) = -div'\left(\left(\beta^2\eta^2 + |\nabla'\eta|^2\right)^{\frac{p-2}{2}}\nabla'\eta\right) - (p-1)\beta(\beta-\beta_0)\left(\beta^2\eta^2 + |\nabla'\eta|^2\right)^{\frac{p-2}{2}}\eta,$$

then

$$\mathcal{T}(\eta) = -\theta^{p-2}\left(\beta'^{\,2}\omega'^{\,2} + |\nabla'\omega'|^2\right)^{\frac{p-2}{2}}\left((\beta-\beta')\omega'^{\,2} + (p-1)\theta(\theta-1)\,|\nabla'\omega'|^2\right) \leq 0.$$

Up to multiplying ω' by λ, we can assume that $\eta \leq \omega$ and that the graphs of η and ω are tangent in $\overline{S}$. Since $\omega' \leq c\omega$, $\eta = o(\omega)$ near ∂S. Hence there exists $\sigma_0 \in S$ such that $\omega(\sigma_0) = \eta(\sigma_0)$ and the coincidence set of η and ω is a compact subset of S. We put $w = \omega - \eta$, since $\nabla\omega(\sigma_0) = \nabla\eta(\sigma_0)$ we proceed as in [12, Th 4.1] (see also [3] in the flat case) and derive that w satisfies, in a system of local coordinates $(\sigma_1, ..., \sigma_{N-1})$ near σ_0,

$$\mathcal{L}w := -\sum_{\ell,j}\frac{\partial}{\partial\sigma_\ell}\left(A_{j,\ell}\frac{\partial w}{\partial\sigma_j}\right) + \sum_j C_j\frac{\partial w}{\partial\sigma_\ell} + Cw \geq 0,$$

where the matrix $(A_{j,\ell})$ is smooth, symmetric and positive near σ_0 and the C_j and C are bounded. Hence w is locally zero. By a standard argument of connectedness, this implies that the zero set of w must be empty, contradiction. Hence $\beta = \beta'$.

3.2. Uniqueness of eigenfunction

The proof is based upon a delicate adaptation of the characterisation of the p-Martin boundary obtained in [8], but we first give a proof in the convex case.

3.2.1. The convex case

Theorem 3.1. *Assume S is a convex spherical subdomain. Then two positive solutions of (1.1) are proportional.*

Proof. We recall that a domain S is (geodesically) convex if a minimal geodesic joining two points of S is contained in S. If $S \subset S^{N-1}$ is convex, the cone C_S is convex too. Since S is convex, it is Lipschitz and by Theorem D, $\beta_S = \hat{\beta}_S := \beta$. Let ω and ω' be two positive solutions of (1.1) satisfying $\sup_S \omega = \sup_S \omega' = 1$. We denote by $u_\omega(x) = |x|^{-\beta}\omega(.)$ and $u_{\omega'}(x) = |x|^{-\beta}\omega'(.)$ the corresponding separable p-harmonic functions defined in C_S. If $0 < a < b$, we set $C_S^{a,b} = C_S \cap (B_b \setminus \overline{B}_a)$. Then for $0 < \epsilon < 1$ we denote by u_ϵ the unique function which satisfies

$$\begin{aligned} -\Delta_p u_\epsilon &= 0 && \text{in } C_S^{\epsilon,1} \\ u_\epsilon &= \epsilon^{-\beta}\omega && \text{in } C_S \cap \partial B_\epsilon \\ u_\epsilon &= 0 && \text{in } (C_S \cap \partial B_1) \cup \left(\partial C_S \cap (\overline{B}_1 \setminus B_\epsilon)\right). \end{aligned} \tag{3.30}$$

Then

$$(u_\omega - 1)_+ \leq u_\epsilon \leq u_\omega \qquad \text{in } C_S^{\epsilon,1}. \tag{3.31}$$

Furthermore $\epsilon \mapsto u_\epsilon$ is increasing. When $\epsilon \downarrow 0$, $u_\epsilon \downarrow u_0$ where u_0 is positive and p-harmonic in $C_S^{1,0}$, vanishes on $\partial C_S^{1,0} \setminus \{0\}$ and satisfies (3.30) with $\epsilon = 0$. In particular

$$\lim_{r\to 0} r^\beta u_0(r,\sigma) = \omega(\sigma) \qquad \text{locally uniformly in } S. \tag{3.32}$$

We construct the same approximation u'_ϵ in $C_S^{\epsilon,1}$ with ω' instead of ω. Mutadis mutandis (3.31) holds and $u'_\epsilon \downarrow u'_0$ which is positive and p-harmonic in C_S^1, satisfies

$$(u_{\omega'} - 1)_+ \leq u'_0 \leq u_{\omega'} \qquad \text{in } C_S^{1,0},$$

and thus

$$\lim_{r\to 0} r^\beta u'_0(r,\sigma) = \omega'(\sigma) \qquad \text{locally uniformly in } S. \tag{3.33}$$

However, by [8, Th 4] u_0 and u'_0 are proportional. Combined with (3.32), (3.33) it implies the claim.

3.2.2. Proof of Theorem E

In what follows we borrow most of our construction from [8] that we adapt to the case of an infinite cone a make explicit for the sake of completeness. The next *nondegeneracy property* of positive p-harmonic functions is proved in [8, Lemma 4.28].

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and $1 < p < \infty$. Then there exist constants $\rho > 0$, $c_{14}, c_{15} > 0$ depending respectively on Ω (for ρ), and p, N and the Lipschitz norm M of $\partial\Omega$ (for c_{14} and c_{15}) with the property that for any $w \in \partial\Omega$ and any positive p-harmonic function u in Ω, continuous in $\overline{\Omega} \cap \overline{B}_{2\rho}(w)$ and vanishing on $\partial\Omega \cap B_\rho(w)$, one can find $\xi \in S^{N-1}$, independent of u, such that*

$$c_{14}^{-1} \frac{u(y)}{\operatorname{dist}(y,\partial\Omega)} \leq \langle \nabla u(y), \xi \rangle \leq |\nabla u(y)| \leq c_{14} \frac{u(y)}{\operatorname{dist}(y,\partial\Omega)}, \tag{3.34}$$

for all $y \in C_S \cap \overline{B}_{\frac{\rho|w|}{c_{15}}}(w)$.

If Ω is replaced by a cone C_S, the nondegeneracy property still holds uniformly on $\partial C_S \setminus \{0\}$.

Corollary 3.3. *Let $1 < p < \infty$, $S \subset S^{N-1}$ is a Lipschitz domain and C_S the cone generated by S.*

(i) Then there exist constants $\rho < \frac{1}{2}$, $c_{14}, c_{15} > 0$ depending respectively on S (for ρ), and p, N and the Lipschitz norm M of ∂S and diam(S) (for c_{14} and c_{15}) with the property that for any $w \in \partial C_S$ and any positive p-harmonic function u in C_S, continuous in $\overline{C}_S \cap \overline{B}_{2\rho|w|}(w)$ and vanishing on $\partial C_S \cap \overline{B}_{\rho|w|}(w)$ continuous, one can find $\xi \in S^{N-1}$, independent of u, such that

$$c_{14}^{-1} \frac{u(y)}{\text{dist}\,(y, \partial C_S)} \leq \langle \nabla u(y), \xi \rangle \leq |\nabla u(y)| \leq c_{14} \frac{u(y)}{\text{dist}\,(y, \partial C_S)}, \tag{3.35}$$

for all $y \in B_{\frac{\rho}{c_{15}}}(w) \cap C_S$.

(ii) Then there exist positive constants κ and c_{16}, c_{17} depending on S (for κ), and p, N and the Lipschitz norm M of ∂S and diam(S) (for c_{16}, c_{17}) such that for any $a > 0$ and any positive p-harmonic function u in C_S^a vanishing on $\partial C_S \cap B_a^c$, there holds

$$c_{16}^{-1} \frac{u(y)}{\text{dist}\,(y, \partial C_S)} \leq |\nabla u(y)| \leq c_{16} \frac{u(y)}{\text{dist}\,(y, \partial C_S)}$$
$$\forall y \in C_S^{c_{17}a} \quad s.t. \;\; \text{dist}\,(y, \partial C_S) \leq \kappa\, |y|\,. \tag{3.36}$$

Let $\omega, \omega' \in W_0^{1,p}(S) \cap C(\overline{S})$ be positive solutions (1.1). Since $\frac{\omega}{\omega'}$ is bounded from above and from below in S by positive constants, we can assume, as in the proof of Theorem D, that $\omega \geq \omega'$ in S and that the graphs of ω and ω' are tangent. Hence, if $\omega \neq \omega'$, then $\omega > \omega'$ in S and there exists a sequence $\{\sigma_n\}$ converging to $\sigma_0 \in \partial S$ as $n \to \infty$ such that

$$\lim_{n \to \infty} \frac{\omega'(\sigma_n)}{\omega(\sigma_n)} = 1.$$

We define $\delta_1 = \sup\{\delta > 0 : \delta\omega < \omega'\}$. For $t \in (\delta_1, 1)$, we set

$$\phi_t = \sup\{\omega', t\omega\} \quad \text{and} \quad \psi_t = \inf\left\{\frac{t}{\delta_1}\omega', \omega\right\} \tag{3.37}$$

We also set

$$v_{\phi_t}(r, \sigma) = r^{-\beta}\phi_t(\sigma) \quad \text{and} \quad v_{\psi_t}(r, \sigma) = r^{-\beta}\psi_t(\sigma) \qquad \forall\, (r, \sigma) \in (0, \infty) \times S. \tag{3.38}$$

Lemma 3.4. *The functions ϕ_t and ψ_t are respectively a subsolution and a supersolution of (1.1) in $W_0^{1,p}(S)$, v_{ϕ_t} and v_{ψ_t} are respectively a subsolution and a supersolution of $-\Delta_p$ in C_S, and there exists $\eta \in W_0^{1,p}(S)$ solution of (1.1) such that*

$$\omega' \leq \phi_t \leq \eta \leq \psi_t \leq \omega \qquad \forall\, t \in (\delta_1, 1). \tag{3.39}$$

If S_t is the subset of $\eta \in W_0^{1,p}(S)$ solutions of (1.1) and satisfying (3.39), then $\omega_t = \sup\{\eta : \eta \in S_t\}$ belongs to S_t. It is increasing with respect to t with uniform limits ω' when $t \downarrow \delta_1$ and ω when $t \uparrow 1$. Finally, if $\theta_t = \frac{t-\delta_1}{1-\delta_1}$, there holds

$$\phi_t \leq \theta_t \omega + (1-\theta_t)\omega' \leq \psi_t. \tag{3.40}$$

Proof. Clearly ϕ_t and ψ_t are respectively a subsolution and a supersolution of the operator $\mathcal{T}$, they belong to $W_0^{1,p}(S) \cap L^\infty(S)$ and they satisfy $\omega' \leq \phi_t \leq \psi_t \leq \omega$. Furthermore, by Dini convergence theorem

$$\lim_{t\uparrow 1} \phi_t = \omega = \lim_{t\uparrow 1} \psi_t \quad \text{and} \quad \lim_{t\downarrow \delta_1} \phi_t = \omega' = \lim_{t\downarrow \delta_1} \psi_t,$$

uniformly in $\overline{S}$. Moreover, in spherical coordinates,

$$-\Delta_p u(r,\sigma) = -\left(\left(u_r^2 + r^{-2}\left|\nabla' u\right|^2\right)^{\frac{p-2}{2}} u_r\right)_r - \frac{N-1}{r}\left(u_r^2 + r^{-2}\left|\nabla' u\right|^2\right)^{\frac{p-2}{2}} u_r$$
$$- \frac{1}{r^2} div'\left(\left(u_r^2 + r^{-2}\left|\nabla' u\right|^2\right)^{\frac{p-2}{2}} \nabla' u\right).$$

Hence, if $u(r,\sigma) = r^{-\beta}\eta(\sigma)$,

$$-\Delta_p u(r,\sigma) = \beta^{p-2} r^{-(p-1)(\beta+1)-1}\mathcal{T}(\eta).$$

Thus v_{ϕ_t} is a subsolution $-\Delta_p$ in C_S and v_{ψ_t} is a supersolution. Since the operator $\mathcal{T}$ is a Leray-Lions operator, it follows by [2] that there exists $\eta \in W_0^{1,p}(S) \cap L^\infty(S)$ satisfying $\mathcal{T}(\eta) = 0$ and $\phi_t \leq \eta \leq \psi_t$ in S. We denote by S_t the set of $\eta \in W_0^{1,p}(S) \cap L^\infty(S)$ satisfying $\mathcal{T}(\eta) = 0$ and $\phi_t \leq \eta \leq \psi_t$ in S. Then there exists a sequence $\{\eta_n\} \subset S_t$ and $\omega_t \in W_0^{1,p}(S) \cap L^\infty(S)$ such that $\eta_n(\sigma) \uparrow \omega_t(\sigma)$ for all $\sigma \in \Sigma$, where Σ is a countable dense subset of S. By Lemma 2.1 $\{\eta_n\}$ is bounded in $L^p(S)$, hence in $C^\gamma(S)$ for some $\gamma \in (0,1)$. By the estimates of the proof of Theorem B-Step 2, $\{\eta_n\}$ is bounded in $W_0^{1,p}(S)$. By standard regularity theory, we can also assume that $\eta_n \to \omega_t$ in the $C^1_{loc}(S)$-topology. Hence ω_t is a weak solution of (1.1), it belongs to $W_0^{1,p}(S) \cap L^\infty(S)$ and satisfies $\phi_t \leq \omega_t \leq \psi_t$. Therefore it is the maximal element of S_t. The monotonity of ω_t is a consequence of the monotonicity of ϕ_t and ψ_t and the last statement (3.40) is a straightforward computation. □

Next we recall the deformation of p-harmonic functions already used in [8]. If $\tau \in (0,1)$ and $0 < a < b$, we denote by $v_{\tau,a,b}$ the p-harmonic function defined in $C_S^{a,b}$ satisfying

$$v_{\tau,a,b}(x) = \begin{cases} a^{-\beta}(\tau\omega + (1-\tau)\omega')(\frac{x}{|x|}) & \text{if } x \in C_S \cap \partial B_a \\ 0 & \text{if } x \in C_S \cap \partial B_b \\ 0 & \text{if } x \in \partial C_S \cap \left(\overline{B}_b \setminus B_a\right). \end{cases} \tag{3.41}$$

Lemma 3.5. *The mapping* $(\tau, b) \mapsto v_{\tau,a,b}$ *is continuous and increasing. If* $v_{\tau,a} = \lim_{b\to\infty} v_{\tau,a,b}$*, then it is a positive p-harmonic function in* $C_S^{a,\infty}$ *vanishing on* $\partial S \cap B_a^c$*, and there holds*

$$u_{\omega'}(x) \leq v_{\phi_{\tau^*}}(x) \leq v_{\tau,a}(x) \leq v_{\psi_{\tau^*}}(x) \leq u_\omega(x) \quad \forall\, x \in C_S^{a,\infty}, \tag{3.42}$$

where $\tau^* = (1-\delta_1)\tau + \delta_1$ *and as a consequence*

$$\lim_{\tau\uparrow 1} \sup_{|x|\geq a} |x|^\beta \left(u_\omega(x) - v_{\tau,a}(x)\right) = 0 \quad \text{and} \quad \lim_{\tau\downarrow 0} \sup_{|x|\geq a} |x|^\beta \left(v_{\tau,a}(x) - u_{\omega'}(x)\right) = 0 \tag{3.43}$$

Furthermore

$$0 \leq \frac{v_{\tau',a} - v_{\tau,a}}{\tau' - \tau} \leq \left(\frac{1}{\delta_1} - 1\right) v_{\tau',a} \quad \forall\, 0 \leq \tau < \tau' \leq 1. \tag{3.44}$$

Proof. The uniqueness and the (strict) monotonicity of $(\tau, b) \mapsto v_{\tau,a,b}$ follow from the monotonicity of $\tau \mapsto \tau\omega + (1-\tau)\omega'$ and the strong maximum principle. The continuity is a consequence of uniqueness and regularity theory for p-harmonic functions. It follows from (3.40) with $t = \tau^*$ and the fact that $v_{\phi_{\tau^*}}$ and $v_{\psi_{\tau^*}}$ are respectively a subsolution and a supersolution of $-\Delta_p$, that we have

$$u_{\omega'}(x) \leq v_{\phi_{\tau^*}}(x) \leq v_{\tau,a,b}(x) \leq v_{\psi_{\tau^*}}(x) \leq u_\omega(x) \quad \forall\, x \in C_S^{a,b},$$

which yields (3.42). Similarly, we have on $\partial C_S^{a,b}$

$$0 \leq \frac{v_{\tau',a,b} - v_{\tau,a,b}}{\tau' - \tau} = u_\omega - u_{\omega'} \leq (\delta_1^{-1} - 1)u_{\omega'} \leq (\delta_1^{-1} - 1)v_{\tau,a,b}, \tag{3.45}$$

equivalently

$$0 \leq v_{\tau',a,b} \leq \left(1 + (\tau' - \tau)(\delta_1^{-1} - 1)\right) v_{\tau,a,b}. \tag{3.46}$$

By the maximum principle (3.45) holds in $C_S^{a,b}$. This implies (3.44). □

As a consequence of (3.44), $\partial_\tau v_{\tau,a}$ exists for almost all $\tau \in (0,1)$ in $W_0^{1,p}(C_S^{a,b})$ for all $b > a$ and it is a solution of

$$\begin{aligned} \mathbb{L}w &= \nabla.\left((p-2)\left|\nabla v_{\tau,a}\right|^{p-4} \langle \nabla v_{\tau,a}.\nabla Z\rangle \nabla v_{\tau,a}\right) \\ &= \sum_{i,j} \frac{\partial}{\partial x_j}\left(b_{i,j}(x)\frac{\partial w}{\partial x_i}\right) = 0 \end{aligned} \tag{3.47}$$

where

$$b_{i,j}(x) = |\nabla v_{\tau,a}|^{p-4}\left((p-2)\frac{\partial v_{\tau,a}}{\partial x_j}\frac{\partial v_{\tau,a}}{\partial x_i} + \delta_{ij}|\nabla v_{\tau,a}|^2\right).$$

$\mathbb{L}$ satisfies the following ellipticity condition

$$\min\{1,p-1\}|\nabla v_{\tau,a}|^2|\xi|^2 \leq \sum_{i,j} b_{i,j}(x)\xi_i\xi_j \leq \max\{1,p-1\}|\nabla v_{\tau,a}|^2|\xi|^2 \quad \forall \xi \in \mathbb{R}^N. \tag{3.48}$$

It is important to notice that $\mathbb{L}v_{\tau,a} = (p-1)\Delta_p v_{\tau,a} = 0$. The estimate (3.48) combined with (3.36) and the decay of $v_{\tau,a}$ and $\partial_\tau v_{\tau,a}$ implies that they satisfy Harnack inequality and boundary Harnack inequality in C_S^a. There exists a constant $\hat{c} > c_{17} > 1$ (see 3.36) such that

$$\frac{1}{\hat{c}}\frac{\partial_\tau v_{\tau,a}(x_a)}{v_{\tau,a}(x_a)} \leq \frac{\partial_\tau v_{\tau,a}(x)}{v_{\tau,a}(x)} \leq \hat{c}\frac{\partial_\tau v_{\tau,a}(x_a)}{v_{\tau,a}(x_a)} \qquad \forall x \in C_S^{\hat{c}a}, \tag{3.49}$$

where $x_a = (\hat{c}a, \sigma_0)$ for some $\sigma_0 \in S$ fixed. We set

$$M(t) = \sup_{x\in C_S^t}\frac{\partial_\tau v_{\tau,a}(x)}{v_{\tau,a}(x)} \quad \text{and} \quad m(t) = \inf_{x\in C_S^t}\frac{\partial_\tau v_{\tau,a}(x)}{v_{\tau,a}(x)} \qquad \forall t > a \tag{3.50}$$

Lemma 3.6. *For $t > \hat{c}a$ there holds*

$$M(\hat{c}t) - m(\hat{c}t) \leq \frac{\hat{c}^2 - 1}{\hat{c}^2 + 1}\left(M(t) - m(t)\right). \tag{3.51}$$

Proof. There holds

$$\partial_\tau v_{\tau,a}(x) - m(t)v_{\tau,a}(x) \geq 0 \quad \text{and} \quad M(t)v_{\tau,a}(x) - \partial_\tau v_{\tau,a}(x) \geq 0 \qquad \forall x \in C_S^t.$$

Estimate (3.49) is valid for any couple of positive solutions (h_1, h_2) of $\mathbb{L}h = 0$ in C_S^a vanishing on $\partial C_S^a \cap B_a^c$, in particular for $(\partial_\tau v_{\tau,a} - m(t)v_{\tau,a}, v_{\tau,a})$ and $(M(t)v_{\tau,a} - \partial_\tau v_{\tau,a}, v_{\tau,a})$. Hence

$$\frac{1}{\hat{c}}\left(\frac{\partial_\tau v_{\tau,a}(x_a)}{v_{\tau,a}(x_a)} - m(t)\right) \leq \frac{\partial_\tau v_{\tau,a}(x)}{v_{\tau,a}(x)} - m(t) \leq \hat{c}\left(\frac{\partial_\tau v_{\tau,a}(x_a)}{v_{\tau,a}(x_a)} - m(t)\right) \quad \forall x \in C_S^t. \tag{3.52}$$

This implies

$$\frac{1}{\hat{c}}\left(\frac{\partial_\tau v_{\tau,a}(x_a)}{v_{\tau,a}(x_a)} - m(t)\right) \leq m(\hat{c}t) - m(t),$$

and

$$\frac{\partial_\tau v_{\tau,a}(x)}{v_{\tau,a}(x)} - m(t) \le \hat{c}^2(m(\hat{c}t) - m(t)) \quad \forall x \in C_S^t.$$

Finally

$$M(\hat{c}t) - m(t) \le \hat{c}^2(m(\hat{c}t) - m(t)). \tag{3.53}$$

Similarly

$$M(t) - m(\hat{c}t) \le \hat{c}^2(M(t) - M(\hat{c}t)). \tag{3.54}$$

Summing the two inequalities we get

$$(M(t) - m(t)) + (M(\hat{c}t) - m(\hat{c}t)) \le \hat{c}^2\left((M(t) - m(t)) - (M(\hat{c}t) - m(\hat{c}t))\right),$$

which yields (3.51).

End of the proof. By the differentiability property of $v_{\tau,a}$ with respect to τ, there exists two countable dense sets $\{(r_\nu\} \subset [a,\infty)$ and $\{\sigma_\mu\} \subset [a,\infty)$ such that $\partial_\tau v_{\tau,a}(r_\nu,\sigma_\mu)$ exists for almost all τ. We put $x_{\nu,\mu} = (r_\nu,\sigma_\mu)$, hence

$$\begin{aligned}\ln\left(\frac{\omega(\sigma_\mu)}{\omega'(\sigma_\mu)}\right) - \ln\left(\frac{\omega(\sigma_{\mu'})}{\omega'(\sigma_{\mu'})}\right) &= \ln\left(\frac{v_{1,a}(x_{\nu,\mu})}{v_{0,a}(x_{\nu,\mu})}\right) - \ln\left(\frac{v_{1,a}(x_{\nu,\mu'})}{v_{0,a}(x_{\nu,\mu'})}\right) \\ &= \int_0^1 \left(\frac{\partial_\tau v_{\tau,a}(x_{\nu,\mu})}{v_{\tau,a}(x_{\nu,\mu})} - \frac{\partial_\tau v_{\tau,a}(x_{\nu,\mu'})}{v_{\tau,a}(x_{\nu,\mu'})}\right) d\tau.\end{aligned} \tag{3.55}$$

Using the continuity of $\frac{\omega}{\omega'}$ and the density of $\{\sigma_m\}$ we derive

$$\left|\ln\left(\frac{\omega(\sigma)}{\omega'(\sigma)}\right) - \ln\left(\frac{\omega(\sigma')}{\omega'(\sigma')}\right)\right|. \le M(r_\nu) - m(r_\nu) \qquad \forall(\sigma,\sigma') \in S \times S. \tag{3.56}$$

We can assume that $r_\nu \ge \hat{c}^{\nu_n} a$ for some sequence $\{\nu_n\}$ tending to infinity with n, hence

$$\left|\ln\left(\frac{\omega(\sigma)}{\omega'(\sigma)}\right) - \ln\left(\frac{\omega(\sigma')}{\omega'(\sigma')}\right)\right| \le \theta^n\left(M(\hat{c}^{\nu_1}) - m(\hat{c}^{\nu_1})\right) \qquad \forall(\sigma,\sigma') \in S \times S \quad \forall n \in \mathbb{N}^*, \tag{3.57}$$

where $\theta = \frac{\hat{c}^2-1}{\hat{c}^2+1} < 1$. Letting $n \to \infty$ implies the claim.

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