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Strong convergence of the symmetrized Milstein scheme for some CEV-like SDEs

Mireille Bossy*[1] and Héctor Olivero Quinteros†[2]

[1]TOSCA Laboratory, INRIA Sophia Antipolis – Méditerranée, France
[2]Departamento de Ingeniería Matemática, Universidad de Chile, Chile

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Abstract

In this article we study the rate of convergence of a symmetrized version of the Milstein scheme applied to the solution of the SDE

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma|X_s|^\alpha dW_s,\ x_0 > 0,\ \sigma > 0,\ \alpha \in [\tfrac{1}{2}, 1).$$

Under suitable hypotheses we prove a strong rate of convergence of order one, recovering the classical result of Milstein for SDEs with smooth coefficients. Some numerical experiments complement our theoretical analysis.

1 Introduction and main result

The Milstein scheme was introduced by Milstein in [10] for one dimensional Stochastic Differential Equations (SDEs) having smooth diffusion coefficient. Introducing an appropriated correction term, this scheme has better convergence rate for the strong error than the classical Euler-Maruyama scheme. Typically, when the drift and diffusion coefficient of one dimensional SDE are twice continuously differentiable with bounded derivatives, the Milstein scheme is of order one for strong error (see eg. Talay [12]) instead of one-half for the Euler-Maruyama scheme. This well-know fact produces remarks on blogs and internet forums that sometimes recommend to use the Milstein scheme for constant elasticity of variance (CEV) models in finance, or it extension with stochastic volatility as SABR model, (see e.g Delbaen and Shirakawa [7] and Lions and Musiela [9] for a discussion on the (weak) existence of such models); CEV are popular stochastic volatility models of the form

$$dX_t = \mu X_t dt + \sigma X_t^\gamma dW_t$$

with $0 < \gamma < 1$. But the interesting fact in this story is that the rate of convergence of the Milstein scheme, for such family of processes with $0 < \gamma < 1$ is not yet well studied, to the best of our knowledge.

In this article we establish a rate of convergence result for a symmetrized version of the Milstein scheme applied to the solution of the SDE of the form

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma|X_s|^\alpha dW_s, \tag{1.1}$$

where $x_0 > 0$, $\sigma > 0$ and $\frac{1}{2} \le \alpha < 1$. Of course Equation (1.1) does not satisfies the hypothesis to apply the classical result of Milstein [10]. In particular, the diffusion coefficient is only Hölder continuous whereas the classical hypothesis is to have a $\mathcal{C}^2$ coefficient.

*email: mireille.bossy@inria.fr
†email: holivero@dim.uchile.cl

The main picture of our convergence rate result is that Milstein scheme stay of order one in the case of Equation (1.1), but some attention must be paid to the values of $b(0)$, α and σ.

Other strategies for the discretization of solution to (1.1) have been proposed and we refer to the recent review on this topic proposed in Chassagneux et all [6] . In numerical experiments, we compare the symmetrized Milstein scheme with the Implicit Scheme recently proposed by Alfonsi [1][2], with the Modified Euler Scheme proposed by Chassagneux et al [6], and with the simple symmetrized Euler scheme.

In the whole paper, we work under the following basis-hypothesis:

Hypothesis 1.1. *The power parameter α in the diffusion coefficient belongs to $[\frac{1}{2}, 1)$. The drift coefficient b is Lipschitz with constant $K > 0$, and is such that $b(0) > 0$.*

Hypothesis 1.1 is a classical assumption to ensure a unique strong solution valued in $\mathbb{R}^+$. We assume it in all the forthcoming results of the paper, without recall it explicitly. To state the convergence result (see Theorem 1.6), another Hypothesis 1.5 will be added and discussed, that in particular constrains the values α, $b(0)$ and σ.

1.1 The symmetrized Milstein scheme

To complete our task we follow the ideas of Berkaoui, and Diop in [3] who analyze the rate of convergence of the strong error for the symmetrized Euler scheme applied to Equation (1.1). Although, whereas they utilize an argument of change of time, we consider first a weighted error for which we prove a convergence result, and then we utilize this result to prove the convergence of the actual error.

We consider $x_0 > 0$, $T > 0$, and $N \in \mathbb{N}$. We define the constant step size $\Delta t = T/N$ and $t_k = k\Delta t$. Over this discretization of the interval $[0, T]$ we define the Symmetrized Milstein Scheme (SMS) $(\overline{X}_{t_k}, k = 0, \ldots, N)$ by

$$\overline{X}_{t_k} = \begin{cases} x_0, \text{ for } k = 0, \\ \Big| \overline{X}_{t_{k-1}} + b(\overline{X}_{t_{k-1}})\Delta t + \sigma \overline{X}^{\alpha}_{t_{k-1}}(W_{t_k} - W_{t_{k-1}}) \\ \qquad + \dfrac{\alpha\sigma^2}{2}\overline{X}^{2\alpha-1}_{t_{k-1}}\left[(W_{t_k} - W_{t_{k-1}})^2 - \Delta t\right] \Big|, \text{ for } k = 1, \ldots, N. \end{cases}$$

In the following, we use the time continuous version of the SMS, $(\overline{X}_t, 0 \le t \le T)$ satisfying

$$\begin{aligned} \overline{X}_t = \Big| \overline{X}_{\eta(t)} + b(\overline{X}_{\eta(t)})(t - \eta(t)) + \sigma \overline{X}^{\alpha}_{\eta(t)}(W_t - W_{\eta(t)}) \\ + \frac{\alpha\sigma^2}{2}\overline{X}^{2\alpha-1}_{\eta(t)}\left[(W_t - W_{\eta(t)})^2 - (t - \eta(t))\right] \Big|, \end{aligned} \tag{1.2}$$

where $\eta(t) = \sup_{k\in\{1,\ldots,N\}}\{t_k : t_k \le t\}$. We also introduce the increment process $(\overline{Z}_t, 0 \le t \le T)$ defined by

$$\begin{aligned} \overline{Z}_t = \overline{X}_{\eta(t)} + b(\overline{X}_{\eta(t)})(t - \eta(t)) + \sigma \overline{X}^{\alpha}_{\eta(t)}(W_t - W_{\eta(t)}) \\ + \frac{\alpha\sigma^2}{2}\overline{X}^{2\alpha-1}_{\eta(t)}\left[(W_t - W_{\eta(t)})^2 - (t - \eta(t))\right], \end{aligned} \tag{1.3}$$

so that $\overline{X}_t = |\overline{Z}_t|$. Thanks to Tanaka's Formula, the semi-martingale decomposition of $\overline{X}_t$ is given by

$$\begin{aligned} \overline{X}_t = x_0 + \int_0^t \operatorname{sgn}(\overline{Z}_s) b(\overline{X}_{\eta(s)})ds + \frac{1}{2}L^0_t(\overline{X}) \\ + \int_0^t \operatorname{sgn}(\overline{Z}_s)\left[\sigma\overline{X}^{\alpha}_{\eta(s)} + \alpha\sigma^2\overline{X}^{2\alpha-1}_{\eta(s)}(W_s - W_{\eta(s)})\right] dW_s. \end{aligned} \tag{1.4}$$

Moment upper bound estimations for X and $\overline{X}$

We summarize some facts about $(X_t, 0 \le t \le T)$, the proofs of which can be found in Bossy and Diop [5].

Lemma 1.2. *For any $x_0 \geq 0$, for any $q \geq 1$, there exists a positive constant C depending on q, but also on the parameters $b(0)$, K, σ, α and T such that*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} X_t^{2q}\Big] \leq C(1+x_0^{2q}). \tag{1.5}$$

When $\frac{1}{2} < \alpha < 1$, for any $q > 0$,

$$\sup_{0\leq t\leq T} \mathbb{E}\left[X_t^{-q}\right] \leq C(1+x_0^{-q}). \tag{1.6}$$

When $\alpha = \frac{1}{2}$, for any q such that $1 < q < 2b(0)/\sigma^2 - 1$,

$$\sup_{0\leq t\leq T} \mathbb{E}\left[X_t^{-q}\right] \leq C x_0^{-q}. \tag{1.7}$$

Lemma 1.3. *Let $(X_t, 0 \leq t \leq T)$ be the solution of (1.1) with $\frac{1}{2} < \alpha < 1$. For all $\mu \geq 0$, there exist a positive constant $C(T,\mu)$, increasing in μ and T, depending also on b, σ, α and x_0 such that*

$$\mathbb{E}\exp\left(\mu\int_0^T \frac{ds}{X_s^{2(1-\alpha)}}\right) \leq C(T,\mu). \tag{1.8}$$

When $\alpha = \frac{1}{2}$, the inequality (1.8) holds if $b(0) > \sigma^2/2$ and $\mu \leq \sigma^2/8(2b(0)/\sigma^2 - 1)^2$.

Notice that the condition $b(0) > \sigma^2/2$ is also imposed by the Feller test in the case $\alpha = \frac{1}{2}$ for the strict positivity of X, that allows to rewrite Equation (1.1) as

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma\sqrt{X_s}dW_s.$$

Using the semimartingale representation (1.4), we prove the following Lemma regarding the existence of moments of any order for $\overline{X}_t$.

Lemma 1.4. *For any $x_0 \geq 0$, for any $q \geq 1$, there exists a positive constant C depending on q, but also on the parameters $b(0)$, K, σ, α and T such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \overline{X}_t^{2q}\Big] \leq C(1+x_0^{2q}).$$

The proof of this lemma is based on the Lipschitz property of b and classical combination of Itô formula and Young Inequality. For the sake of completeness, we give a short proof in the Appendix.

1.2 Strong rate of convergence

The main result of this works is the strong convergence at rate one of the SMS $\overline{X}$ to the exact process X. The convergence holds in L^{2p} for $p = 1$ or $p \geq 3/2$. To state it, we add to Hypothesis 1.1 the following:

Hypothesis 1.5.

(i) For $\alpha > \frac{1}{2}$ we assume $b(0) > 2\alpha(1-\alpha)^2\sigma^2$. Whereas for $\alpha = \frac{1}{2}$ we assume $b(0) > 3\,(3p+1)\,\sigma^2/2$.

(ii) The drift coefficient b is of class $\mathcal{C}^2(\mathbb{R})$, and b'' has polynomial growth.

About Hypothesis 1.5-(i), notice that for $\alpha > \frac{1}{2}$, Assumption (i) becomes easier to fulfill as α increase, and does not depend on p. On the other hand, for $\alpha = \frac{1}{2}$, Assumption (i) depends on p in a unpleasant manner. However, as we will see later in Section 4 (see Table 1), this kind of dependence in p is expected, and similar conditions are asked in the literature for other approximation schemes in order to obtain similar rate of convergence results.

Also, notice that Hypothesis 1.5-(i) is a sufficient condition: in the numerical experiments we still observe a rate of convergence of order one for parameters that do not satisfy it, but we also observe that for parameters such that $b(0) \ll \sigma^2$, although the convergence occurs, it does in a sublinear fashion.

On the other hand, Hypothesis 1.5-(ii) is the classical requirement for the strong convergence of the Milstein scheme. As we will see later in the proof of the main theorem, with the help of the Itô formula, this hypothesis let us conclude that

$$\mathbb{E}\left[|X_s - \overline{X}_s|^{2p-1}\big(b(X_{\eta(s)}) - b(X_s)\big)\right] \leq C\left(\sup_{u\leq s}\mathbb{E}\left[|X_u - \overline{X}_u|^{2p}\right] + \Delta t^{2p}\right)$$

instead of

$$\mathbb{E}\left[|X_s - \overline{X}_s|^{2p-1}\big(b(X_{\eta(s)}) - b(X_s)\big)\right] \leq C\left(\sup_{u\leq s}\mathbb{E}\left[|X_u - \overline{X}_u|^{2p}\right] + \Delta t^{p}\right),$$

which is the classical bound obtained for the Euler-Maruyama Scheme under a Lipschitz condition for a drift b.

To lighten the notation, we consider for $\alpha \in (\frac{1}{2}, 1)$

$$\begin{cases} b_\sigma(\alpha) := b(0) - 2(1-\alpha)^2\alpha\sigma^2, \\ K(\alpha) := K + \dfrac{\alpha\sigma^2}{2}(2\alpha - 1)[2(1-\alpha)]^{-\frac{2(1-\alpha)}{2\alpha-1}}, \end{cases} \tag{1.9}$$

we extend this definitions to $\alpha = \frac{1}{2}$ taking limits. So, $b_\sigma(1/2) = \lim_{\alpha\to\frac{1}{2}} b_\sigma(\alpha) = b(0) - \sigma^2/4$, and $K(1/2) = \lim_{\alpha\to\frac{1}{2}} K(\alpha) = K$. Notice that $\lim_{\alpha\to 1} K(\alpha) = K + \sigma^2/2$, and since $K(\alpha)$ is continuous on $(\frac{1}{2}, 1)$, we have that $K(\alpha)$ is bounded. This is especially important in the definition of $\Delta_{\max}(\alpha)$ bellow, because tell us that $\alpha \mapsto \Delta_{\max}(\alpha)$ is strictly positive and bounded on $[\frac{1}{2}, 1)$.

We now state our main theorem.

Theorem 1.6. *Assume Hypotheses 1.1 and 1.5. Define a maximum step size $\Delta_{\max}(\alpha)$ as*

$$\Delta_{\max}(\alpha) = \frac{x_0}{(1-\sqrt{\alpha})b_\sigma(\alpha)} \wedge \begin{cases} \dfrac{1}{4\alpha K(\alpha)}, & \textit{for } \alpha \in (\frac{1}{2}, 1) \\ \dfrac{1}{4K} \wedge x_0, & \textit{for } \alpha = \frac{1}{2}. \end{cases} \tag{1.10}$$

Let $(X_t, 0 \leq t \leq T)$ be the process defined on (1.1) and $(\overline{X}_t, 0 \leq t \leq T)$ the symmetrized Milstein scheme given in (1.2). Then for $p = 1$ or $p \geq 3/2$, there exists a constant C depending on p, T, $b(0)$, α, σ, K, and x_0, but not on Δt, such that for all $\Delta t \leq \Delta_{\max}(\alpha)$,

$$\sup_{0\leq t\leq T}\mathbb{E}\left[|X_t - \overline{X}_t|^{2p}\right] \leq C\Delta t^{2p}. \tag{1.11}$$

The rest of the paper goes as follow. In Section 2 we state some preliminary results on the scheme which will be building blocks in the proof of Theorem 1.6. Section 3 is devoted to the proof of the convergence rate. The main idea is first to introduce a weight process in the $L^p(\Omega)$-error, and get the rate of convergence for this weighted error process, and after use this intermediate bound to control the $L^p(\Omega)$-error. In section 4 we display some numerical experiments to show the effectiveness of the theoretical rate of convergence of the scheme, but also to test Hypotheses 1.5-(i) on a set of parameters. In Section 5 we present the proof of the preliminary results on the scheme. Finally we have included in a small appendix a couple of proofs to make this article self contained.

2 Some preliminary results for $\overline{X}$

This short section is devoted to state some results about the behavior of $\overline{X}$, their proofs are postponed to Section 5. All these results hold under Hypothesis 1.5-(i) which is in fact stronger than what we need here. So, we present the next lemmas with their minimal hypotheses (still assuming Hypothesis 1.1).

Lemma 2.1 (Local error). *For any $x_0 \geq 0$, for any $p \geq 1$, there exists a positive constant C, depending on p, T, but also on the parameters of the model $b(0)$, K, σ, α, such that*

$$\sup_{0\leq t\leq T} \mathbb{E}\left[|\overline{X}_t - \overline{X}_{\eta(t)}|^{2p}\right] \leq C\Delta t^p.$$

By construction the scheme $\overline{X}$ is nonnegative, but a key point of the convergence proof resides in the analysis of the behavior of $\overline{X}$ or $\overline{Z}$ visiting the point 0. The next Lemma shows that although $\overline{Z}_t$ is not always positive, the probability of $\overline{Z}_t$ being negative is actually very small under suitable hypotheses.

Lemma 2.2. *For any $\alpha \in [\frac{1}{2}, 1)$, if $b(0) > 2\alpha(1-\alpha)^2\sigma^2$, and $\Delta t \leq 1/(2K(\alpha))$, then there exist positive constants C and γ, depending on the parameters of the model, but not on Δt, such that*

$$\sup_{0\leq t\leq T} \mathbb{P}\left(\overline{Z}_t \leq 0\right) \leq C \exp\left(-\frac{\gamma}{\Delta t}\right).$$

To prove Lemma 2.2, it is necessary to establish before the following one, which although technical, gives some intuition about the difference between the SMS and the Symmetrized Euler scheme presented in [3].

Lemma 2.3. *Fix $\rho \in (0, 1]$, and set $\bar{x}(\alpha) = b_\sigma(\alpha)/K(\alpha)$. For any $\alpha \in [\frac{1}{2}, 1)$ if $b(0) > 2\alpha(1-\alpha)^2\sigma^2$, then for all $t \in [0, T]$,*

$$\mathbb{P}\left[\overline{Z}_t \leq (1-\rho)b_\sigma(\alpha)\Delta t,\ \overline{X}_{\eta(s)} < \rho\bar{x}(\alpha)\right] = 0.$$

Roughly speaking, from this lemma we see that when $\overline{Z}_{\eta(t)} > 0$, $\overline{Z}_t$ becomes negative only when

$$|\overline{Z}_t - \overline{Z}_{\eta(t)}| > \rho\bar{x}(\alpha),$$

but observe that only the left-hand side of this inequality depends on Δt, and its expectation decreases to zero proportionally to $\sqrt{\Delta t}$, according to Lemma 2.1.

Now imposing Δt small enough, we prove an explicit bound for the local time moment of $\overline{X}$.

Lemma 2.4. *For any $\alpha \in [\frac{1}{2}, 1)$, if $b(0) > 2\alpha(1-\alpha)^2\sigma^2$ and $\Delta t \leq 1/(2K(\alpha)) \wedge x_0/[(1-\sqrt{\alpha})b_\sigma(\alpha)]$ then there exist a positive $\gamma > 0$ depending on α, $b(0)$, K, and σ but not in Δt such that*

$$\mathbb{E}\left(L_T^0(\overline{X})^2\right) \leq C\frac{1}{\sqrt{\Delta t}} \exp\left(\frac{-\gamma}{2\Delta t}\right).$$

We end this section with another key preliminary result, which is the convergence rate of order 1 for the *corrected* local error. Although the classical local error is of order 1/2, as stated in Lemma 2.1, the local error seen by the diffusion coefficient function, corrected with the Milstein term stays of order 1.

Lemma 2.5 (Corrected local error process). *Let us fix $p \geq 1$, and $\alpha \in [\frac{1}{2}, 1)$. For $\alpha > \frac{1}{2}$, assume $b(0) > 2\alpha(1-\alpha)^2\sigma^2$, whereas for $\alpha = \frac{1}{2}$, assume $b(0) > 3(2p+1)\sigma^2/2$. Then, there exists $C > 0$, depending on the parameters of the model but not in Δt, such that for all $\Delta t \leq \Delta_{\max}(\alpha)$, the Corrected Local Error satisfies*

$$\sup_{0\leq t\leq T} \mathbb{E}\left[\left|\sigma\overline{X}_t^\alpha - \sigma\overline{X}_{\eta(t)}^\alpha - \alpha\sigma^2\overline{X}_{\eta(t)}^{2\alpha-1}(W_t - W_{\eta(t)})\right|^{2p}\right] \leq C\Delta t^{2p}.$$

3 Proof of the Main Theorem

In what follows we denote

$$\mathcal{E}_t := \overline{X}_t - X_t$$

and

$$\Sigma_t := \text{sgn}(\overline{Z}_t)\left[\sigma\overline{X}_{\eta(t)}^\alpha + \alpha\sigma^2\overline{X}_{\eta(t)}^{2\alpha-1}(W_t - W_{\eta(t)})\right] - \sigma X_t^\alpha$$

so that

$$d\mathcal{E}_t = \left(\text{sgn}(\overline{Z}_t)b(\overline{X}_{\eta(t)}) - b(X_t)\right)dt + \frac{1}{2}dL_t^0(\overline{X}) + \Sigma_t dW_t.$$

Also, to make the notation lighter, we will denote the Corrected Local Error by

$$D_t(\overline{X}) := \sigma\overline{X}_t^\alpha - \sigma\overline{X}_{\eta(t)}^\alpha - \alpha\sigma^2\overline{X}_{\eta(t)}^{2\alpha-1}(W_t - W_{\eta(t)}).$$

3.1 The Weighted Error

Before to prove the main theorem, we establish in the auxiliary Lemma 3.2 the convergence of a weighted error. For $p \geq 1$, and $\delta > 0$, let us consider

$$\beta_t = 2p\|b'\|_\infty + \left(1 + \frac{1}{\delta^2}\right) p(3p-1) + \frac{4\alpha^2(1+\delta^2)p(3p-1)\sigma^2}{X_t^{2(1-\alpha)}}, \tag{3.1}$$

and the Weight Process $(\Gamma_t, 0 \leq t \leq T)$ defined by

$$\Gamma_t = \exp\left(-\int_0^t \beta_s ds\right). \tag{3.2}$$

The Weight Process is adapted, almost surely positive, and bounded by 1. Its paths are non increasing and hence have bounded variation, and also satisfies

$$d\Gamma_t = -\beta_t \Gamma_t dt.$$

Observe that the definition of β_t and Γ_t depend on δ. We omit this dependency for two reasons. First, to keep the notation as simple as possible. And second, because later in the paper we will fix δ to a particular value.

Remark 3.1.

(i) From Lemma 1.2, the process β has polynomial moments of any order for $\alpha > \frac{1}{2}$, and when $\alpha = \frac{1}{2}$, there exist C such that $\mathbb{E}(\beta_t^q) < C$, for all $1 < q < 2b(0)/\sigma^2 - 1$. Since for $\alpha = \frac{1}{2}$, Hypothesis 1.5-(i) is equivalent to $9p + 2 < 2b(0)/\sigma^2 - 1$, it follows that the process β has moments at least up to order $9p + 2$.

(ii) Due to Lemma 1.3, there exist a constant $C(T)$ such that $\mathbb{E}\left(\Gamma_T^{-q}\right) < C(T)$ for all $q > 0$ when $\alpha > \frac{1}{2}$, whereas for $\alpha = \frac{1}{2}$, the q-th negative moment of the weight process is finite, as soon as

$$(1+\delta^2)p(3p-1)\sigma^2\, q \leq \frac{\sigma^2}{8}\left(\frac{2b(0)}{\sigma^2} - 1\right)^2.$$

Notice that, thanks to Hypothesis 1.5-(i), a sufficient condition such that this last inequality holds is

$$(1+\delta^2)8p(3p-1)\, q \leq (9p+2)^2.$$

Lemma 3.2 (Weighted Error)**.** *Under the hypothesis of Theorem 1.6, for $p \geq 1$ and $\alpha \in [\frac{1}{2}, 1)$, there exists a constant C not depending on Δt such that for all $\Delta t \leq \Delta_{\max}(\alpha)$*

$$\sup_{0 \leq t \leq T} \mathbb{E}\left(\Gamma_t^{3/2}\mathcal{E}_t^{3p}\right) \leq C\Delta t^{3p}. \tag{3.3}$$

Proof. We decompose the proof in 3 main steps.

Step 1. Let us prove that

$$\begin{aligned}\mathbb{E}\left(\Gamma_t^{3/2}\mathcal{E}_t^{3p}\right) \leq 3p\int_0^t \mathbb{E}\left(\Gamma_s^{3/2}\mathcal{E}_s^{3p-1}\left[b(X_{\eta(s)}) - b(X_s)\right]\right) ds \\ + 3p\|b'\|_\infty \int_0^t \sup_{0\leq u\leq s} \mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right) ds + C\Delta t^{3p}.\end{aligned} \tag{3.4}$$

By the integration by parts formula,

$$\begin{aligned}\mathbb{E}\left(\Gamma_t^{3/2}\mathcal{E}_t^{3p}\right) = \ & 3p\mathbb{E}\left(\int_0^t \Gamma_s^{3/2}\mathcal{E}_s^{3p-1}\left\{\mathrm{sgn}(\overline{Z}_s)b(\overline{X}_{\eta(s)}) - b(X_s)\right\} ds\right) \\ & + \frac{3}{2}p(3p-1)\mathbb{E}\left(\int_0^t \Gamma_s^{3/2}\mathcal{E}_s^{3p-2}\Sigma_s^2 ds\right) \\ & + \frac{3p}{2}\mathbb{E}\left(\int_0^t \Gamma_s^{3/2}\mathcal{E}_s^{3p-1} dL_s^0(\overline{X})\right) - \mathbb{E}\left(\int_0^t \frac{3}{2}\beta_s\Gamma_s^{3/2}\mathcal{E}_s^{3p} ds\right).\end{aligned}$$

Thanks to Lemma 2.4 and the control in the moments of the exact process in Lemma 1.2 we have

$$\begin{aligned}\mathbb{E}\left(\int_0^t \Gamma_s^{3/2}\mathcal{E}_s^{3p-1} dL_s^0(\overline{X})\right) &\leq \mathbb{E}\left(\int_0^t |X_s^{3p-1}| dL_s^0(\overline{X})\right)\\ &\leq \sqrt{\mathbb{E}\left(\sup_{0\leq s\leq T} X_s^{6p-2}\right)\mathbb{E}\left(L_T^0(\overline{X})^2\right)}\\ &\leq C\Delta t^{3p}.\end{aligned}$$

On the other hand, with $\operatorname{sgn}(x) = 1 - 2\mathbb{1}_{\{x<0\}}$, and since for any $\delta > 0$, $x, y \in \mathbb{R}$

$$(x+y)^2 \leq (1+\delta^2)x^2 + (1+1/\delta^2)y^2,$$

calling $\Delta W_s = W_s - W_{\eta(s)}$, we get for all $0 \leq s \leq t$

$$\Sigma_s^2 \leq (1+\delta^2)\left[\sigma X_s^\alpha - \sigma \overline{X}_s^\alpha\right]^2 + \left(1+\frac{1}{\delta^2}\right)\left[\sigma\overline{X}_s^\alpha - \sigma\overline{X}_{\eta(s)}^\alpha - \alpha\sigma^2\overline{X}_{\eta(s)}^{2\alpha-1}\Delta W_s\right]^2 + R_s^\Sigma \mathbb{1}_{\{\overline{Z}_s<0\}},$$

where we put aside all the terms multiplied by $\mathbb{1}_{\{\overline{Z}_s<0\}}$ in

$$\begin{aligned}R_s^\Sigma :=& 4\left[\sigma\overline{X}_{\eta(s)}^\alpha + \alpha\sigma^2\overline{X}_{\eta(s)}^{2\alpha-1}\Delta W_s\right]\\ &\times\left\{\left[\sigma\overline{X}_{\eta(s)}^\alpha + \alpha\sigma^2\overline{X}_{\eta(s)}^{2\alpha-1}\Delta W_s\right] + \left[\sigma X_s^\alpha - \sigma\overline{X}_s^\alpha\right] + \left[\sigma\overline{X}_s^\alpha - \sigma\overline{X}_{\eta(s)}^\alpha - \alpha\sigma^2\overline{X}_{\eta(s)}^{2\alpha-1}\Delta W_s\right]\right\}.\end{aligned}$$

So, from the previous computations, the Lipschitz property of b, and Young's Inequality, we conclude

$$\begin{aligned}\mathbb{E}\left(\Gamma_t^{3/2}\mathcal{E}_t^{3p}\right) \leq& 3p\int_0^t \mathbb{E}\left(\Gamma_s^{3/2}\mathcal{E}_s^{3p-1}\left[b(X_{\eta(s)}) - b(X_s)\right]\right)ds\\ &+3p\|b'\|_\infty \int_0^t \mathbb{E}\left(\Gamma_{\eta(s)}^{3/2}\mathcal{E}_{\eta(s)}^{3p}\right)ds + 3p\|b'\|_\infty\int_0^t \mathbb{E}\left(\Gamma_s^{3/2}\mathcal{E}_s^{3p}\right)ds\\ &+\frac{3}{2}(1+\delta^2)p(3p-1)\mathbb{E}\left(\int_0^t \Gamma_s^{3/2}\mathcal{E}_s^{3p-2}\left[\sigma X_s^\alpha - \sigma\overline{X}_s^\alpha\right]^2 ds\right)\\ &+\frac{3}{2}\left(1+\frac{1}{\delta^2}\right)p(3p-1)\mathbb{E}\left(\int_0^t \Gamma_s^{3/2}\mathcal{E}_s^{3p}ds\right)\\ &+\frac{3}{2}\left(1+\frac{1}{\delta^2}\right)p(3p-1)\int_0^t \mathbb{E}\left(D_s(\overline{X})^{3p}\right)ds\\ &-\mathbb{E}\left(\int_0^t \frac{3}{2}\beta_s\Gamma_s^{3/2}\mathcal{E}_s^{3p}ds\right) + \int_0^t \mathbb{E}\left(R_s\mathbb{1}_{\{\overline{Z}_s<0\}}\right)ds + C\Delta t^{3p}.\end{aligned}$$

where $R_s = 3p(3p-1)\mathcal{E}_s^{3p-2}R_s^\Sigma + 6p\mathcal{E}_s^{4p-1}b(\overline{X}_{\eta(s)})$, and from Lemma 2.2 we have

$$\mathbb{E}\left(R_s\mathbb{1}_{\{\overline{Z}_s<0\}}\right) \leq C\Delta t^{3p}.$$

Since $\Delta t \leq \Delta_{\max}(\alpha)$ and Hypothesis 1.5-(i) holds, we can apply Lemma 2.5 so $\mathbb{E}\left(D_s(\overline{X})^{3p}\right) \leq C\Delta t^{3p}$.

Introducing these estimations in the previous computations, we have

$$\begin{aligned}\mathbb{E}\left(\Gamma_t^{3/2}\mathcal{E}_t^{3p}\right) \leq & 3p\int_0^t \mathbb{E}\left(\Gamma_s^{3/2}\mathcal{E}_s^{3p-1}\left[b(X_{\eta(s)})-b(X_s)\right]\right)ds \\ & + 3p\|b'\|_\infty \int_0^t \mathbb{E}\left(\Gamma_{\eta(s)}^{3/2}\mathcal{E}_{\eta(s)}^{3p}\right)ds \\ & + \frac{3}{2}\left[2p\|b'\|_\infty + \left(1+\frac{1}{\delta^2}\right)p(3p-1)\right]\mathbb{E}\left(\int_0^t \Gamma_s^{3/2}\mathcal{E}_s^{3p}ds\right) \\ & + \frac{3}{2}(1+\delta^2)p(3p-1)\mathbb{E}\left(\int_0^t \Gamma_s^{3/2}\mathcal{E}_s^{3p-2}\left[\sigma X_s^\alpha - \sigma \overline{X}_s^\alpha\right]^2 ds\right) \\ & - \mathbb{E}\left(\int_0^t \frac{3}{2}\beta_s \Gamma_s^{3/2}\mathcal{E}_s^{3p}ds\right) + C\Delta t^{3p}.\end{aligned}$$

Now we use the particular form of the weight process. Since for all $\frac{1}{2} \leq \alpha \leq 1$,

$$\forall x \geq 0,\ y \geq 0,\ |x^\alpha - y^\alpha|(x^{1-\alpha}+y^{1-\alpha}) \leq 2\alpha|x-y|, \tag{3.5}$$

we have

$$\mathbb{E}\left(\int_0^t \Gamma_s^{3/2}\mathcal{E}_s^{3p-2}\left[\sigma X_s^\alpha - \sigma\overline{X}_s^\alpha\right]^2 ds\right) \leq \mathbb{E}\left(\int_0^t \Gamma_s^{3/2}\mathcal{E}_s^{3p}\frac{4\alpha^2\sigma^2}{X_s^{2(1-\alpha)}}ds\right),$$

and then, from the definition of β in (3.1), we conclude

$$\begin{aligned}\mathbb{E}\left(\Gamma_t^{3/2}\mathcal{E}_t^{3p}\right) \leq 3p & \int_0^t \mathbb{E}\left(\Gamma_s^{3/2}\mathcal{E}_s^{3p-1}\left[b(X_{\eta(s)})-b(X_s)\right]\right)ds \\ & + 3p\|b'\|_\infty \int_0^t \mathbb{E}\left(\Gamma_{\eta(s)}^{3/2}\mathcal{E}_{\eta(s)}^{3p}\right)ds + C\Delta t^{3p}.\end{aligned}$$

from where (3.4) follows.

Step 2. Let us prove that for any $s \leq t$

$$\mathbb{E}\left(\Gamma_s^{3/2}\mathcal{E}_s^{3p-1}[b(X_{\eta(s)})-b(X_s)]\right) \leq C\left[\sup_{u\leq s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right] + C\Delta t^{3p}. \tag{3.6}$$

As we will see soon, we prove (3.6) with the help of the Itô's formula applied to b, and here is where we need b of class $\mathcal{C}^2$ required in Hypotheses 1.5-(ii).

Again, by integration by parts

$$\begin{aligned}\mathbb{E}\left(\Gamma_s^{3/2}\mathcal{E}_s^{3p-1}[b(X_{\eta(s)})-b(X_s)]\right) = & -\mathbb{E}\int_{\eta(s)}^s \frac{3}{2}\Gamma_u^{\frac{1}{2}}\mathcal{E}_u^{3p-1}[b(X_{\eta(s)})-b(X_u)]\beta_u\Gamma_u du \\ & + \mathbb{E}\int_{\eta(s)}^s \Gamma_u^{3/2}d\left(\mathcal{E}_u^{3p-1}[b(X_{\eta(s)})-b(X_u)]\right).\end{aligned} \tag{3.7}$$

Applying Hölder's Inequality to the first term in the right-hand side we have

$$\mathbb{E}\int_{\eta(s)}^s \Gamma_u^{3/2}\mathcal{E}_u^{3p-1}[b(X_{\eta(s)})-b(X_u)]\beta_u du \leq \int_{\eta(s)}^s \mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)^{1-1/3p}\mathbb{E}\left([b(X_{\eta(s)})-b(X_u)]^{3p}\beta_u^{3p}\right)^{1/3p}du.$$

Recalling the remark 3.1, we have that $\mathbb{E}\left(\beta_u^{6p}\right)$ is finite, so applying Lemma 2.1,

$$\mathbb{E}\left([b(X_{\eta(s)})-b(X_u)]^{3p}\beta_u^{3p}\right) \leq \sqrt{\mathbb{E}\left([b(X_{\eta(s)})-b(X_u)]^{6p}\right)\mathbb{E}\left(\beta_u^{6p}\right)} \leq C\Delta t^{3p/2}.$$

Then,

$$\mathbb{E}\int_{\eta(s)}^{s}\Gamma_u^{3/2}\mathcal{E}_u^{3p-1}[b(X_{\eta(s)})-b(X_u)]\beta_u du \le C\left[\sup_{u\le s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]^{1-1/3p}\Delta t^{3/2}.$$

Applying the Itô's Formula to the second term in the right-hand side of (3.7), and taking expectation we get

$$\begin{aligned}
&\mathbb{E}\int_{\eta(s)}^{s}\Gamma_u^{3/2}d\Big(\mathcal{E}_u^{3p-1}[b(X_{\eta(s)})-b(X_u)]\Big)\\
&\qquad=-\sigma\mathbb{E}\int_{\eta(s)}^{s}\Gamma_u^{3/2}\mathcal{E}_u^{3p-1}\left[b'(X_u)b(X_u)+\frac{\sigma^2}{2}b''(X_u)X_u^{2\alpha}\right]du\\
&\qquad\quad+(3p-1)\mathbb{E}\int_{\eta(s)}^{s}\Gamma_u^{3/2}\mathcal{E}_u^{3p-2}[b(X_{\eta(s)})-b(X_u)]\left\{\operatorname{sgn}(\overline{Z}_u)b(\overline{X}_{\eta(s)})-b(X_u)\right\}du\\
&\qquad\quad+\frac{\sigma^2}{2}(3p-1)(3p-2)\mathbb{E}\int_{\eta(s)}^{s}\Gamma_u^{3/2}\mathcal{E}_u^{3p-3}[b(X_{\eta(s)})-b(X_u)]\Sigma_s^2du\\
&\qquad\quad-(3p-1)\sigma^2\mathbb{E}\int_{\eta(s)}^{s}\Gamma_u^{3/2}\mathcal{E}_u^{3p-2}b'(X_u)X_u^{\alpha}\Sigma_s du\\
&\qquad\quad+\frac{(3p-1)}{2}\mathbb{E}\int_{\eta(s)}^{s}\Gamma_u^{3/2}\mathcal{E}_u^{3p-2}[b(X_{\eta(s)})-b(X_u)]dL_u^0(\overline{X})\\
&\qquad=:I_1+I_2+I_3+I_4+I_5.
\end{aligned}$$

By the finiteness of the moment of X, the linear growth of b, and the polynomial growth of b'', applying Holder's inequality, we have

$$\begin{aligned}
I_1&\le C\int_{\eta(s)}^{s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)^{1-1/3p}\mathbb{E}\left(\left[b'(X_u)b(X_u)+\frac{\sigma^2}{2}b''(X_u)X_u^{2\alpha}\right]^{3p}\right)^{1/3p}du\\
&\le C\left[\sup_{u\le s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]^{1-1/3p}\Delta t.
\end{aligned}$$

For the bound of I_2, since b is Lipschitz, and $\operatorname{sgn}(x)=1-2\mathbb{1}_{\{x<0\}}$, we have

$$\begin{aligned}
I_2\le&C\int_{\eta(s)}^{s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p-2}|X_{\eta(s)}-X_u||\overline{X}_{\eta(s)}-\overline{X}_u|\right)du\\
&+C\int_{\eta(s)}^{s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p-1}|X_{\eta(s)}-X_u|\right)du+\int_{\eta(s)}^{s}\mathbb{E}\left(R_u^{(2)}\mathbb{1}_{\{\overline{Z}_u<0\}}\right)du\\
\le&C\left[\sup_{u\le s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]^{1-2/3p}\Delta t^2+C\left[\sup_{u\le s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]^{1-1/3p}\Delta t^{3/2}+C\Delta t^{3p}
\end{aligned}$$

Where again, all the terms multiplied by $\mathbb{1}_{\{\overline{Z}_u<0\}}$ are putted in the rest $R_u^{(2)}$, and the expectation of the product is bounded with Lemma 2.2.

In a similar way for the bound of I_3,

$$\begin{aligned}
I_3\le&C\int_{\eta(s)}^{s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p-3}|X_{\eta(s)}-X_u|D_s(\overline{X})^2\right)du\\
&+C\int_{\eta(s)}^{s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p-3}|X_{\eta(s)}-X_u|\left[\sigma\overline{X}_u-\sigma X_u^{\alpha}\right]^2\right)du\\
&+\int_{\eta(s)}^{s}\mathbb{E}\left(R_u^{(3)}\mathbb{1}_{\{\overline{Z}_u<0\}}\right)du.
\end{aligned}$$

For the first term in the right-hand side we have

$$\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p-3}|X_{\eta(s)}-X_u|D_s(\overline{X})^2\right)\le\left[\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]^{1-1/p}\left[\mathbb{E}\left(|X_{\eta(s)}-X_u|^{3p}\right)\right]^{1/3p}\left[\mathbb{E}\left(D_s(\overline{X})^{3p}\right)\right]^{2/3p}$$
$$\le\left[\sup_{u\le s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]^{1-1/p}\Delta t^{5/2},$$

due to the bound for the increments of the exact process and Lemma 2.5. For the second term, applying (3.5), and noting that under Hypothesis 1.5, the exact process has negative moments up to of order $9p+2$

$$\begin{aligned}
&\mathbb{E}\Big(\Gamma_u^{3/2}\mathcal{E}_u^{3p-3}|X_{\eta(s)}-X_u|\left[\sigma\overline{X}_u-\sigma X_u^\alpha\right]^2\Big)\\
&\quad\le C\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p-1}|X_{\eta(s)}-X_u|\frac{1}{X_u^{2(1-\alpha)}}\right)\\
&\quad\le C\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)^{1-1/3p}\mathbb{E}\left(|X_{\eta(s)}-X_u|^{6p}\right)^{1/6p}\mathbb{E}\left(\frac{1}{X_u^{12(1-\alpha)p}}\right)^{1/6p}\\
&\quad\le\left[\sup_{u\le s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]^{1-1/3p}\Delta t^{\frac{1}{2}}.
\end{aligned}$$

We control the third term in the right-hand side in the bound for I_3 using again Lemma 2.2, so

$$I_3\le C\left[\sup_{u\le s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]^{1-1/p}\Delta t^{7/2}+\left[\sup_{u\le s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]^{1-1/3p}\Delta t^{3/2}+C\Delta t^{3p}.$$

Now we bound I_4.

$$\begin{aligned}
I_4\le{}& C\int_{\eta(s)}^s\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p-2}D_u(\overline{X})b'(X_u)X_u^\alpha\right)du\\
&+C\int_{\eta(s)}^s\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p-2}\left[\sigma\overline{X}_u-\sigma X_u^\alpha\right]b'(X_u)X_u^\alpha\right)du\\
&+\int_{\eta(s)}^s\mathbb{E}\left(R_u^{(4)}\mathbb{1}_{\{\overline{Z}_u<0\}}\right)du.
\end{aligned}$$

We control the first term in the right-hand side using Hölder's inequality, Lemma 2.5 and the control in the moments of the exact process for all $0\le u\le s$

$$\begin{aligned}
\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p-2}D_u(\overline{X})b'(X_u)X_u^\alpha\right)\le{}&\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)^{1-2/3p}\mathbb{E}\left(D_s(\overline{X})^{3p}\right)^{1/3p}\\
&\times\mathbb{E}\left(b'(X_u)^{3p}X_u^{4p\alpha}\right)^{1/3p}\\
\le{}& C\left[\sup_{u\le s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]^{1-2/3p}\Delta t.
\end{aligned}$$

For the second term in the right-hand side of the bound for I_4, we use one more time (3.5), and the existence of negative moments of the exact process X, and then

$$\begin{aligned}
\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p-2}\left[\sigma\overline{X}_u-\sigma X_u^\alpha\right]b'(X_u)X_u^\alpha\right)\le{}& C\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p-1}\frac{1}{X_u^{(1-\alpha)}}b'(X_u)X_u^\alpha\right)\\
\le{}& C\left[\sup_{u\le s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]^{1-1/3p}.
\end{aligned}$$

To control the third term in the right-hand side of the bound for I_4 we use Lemma 2.2 just as before. So

$$I_4 \leq C\left[\sup_{u\leq s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]^{1-2/3p}\Delta t^2 + C\left[\sup_{u\leq s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]^{1-1/3p}\Delta t + C\Delta t^{3p}.$$

Finally,

$$\begin{aligned}
I_5 &= \frac{(3p-1)}{2}\mathbb{E}\int_{\eta(s)}^{s}\Gamma_u^{3/2}X_u^{4p-2}[b(X_{\eta(s)})-b(X_u)]dL_u^0(\overline{X})\\
&\leq C\mathbb{E}\left(\sup_{u\leq s}\left[1+X_u^{4p-1}\right]\left[L_T^0(\overline{X})\right]\right)\\
&\leq C\mathbb{E}\left(\sup_{u\leq s}\left[1+X_u^{4p-1}\right]^2\right)^{\frac{1}{2}}\mathbb{E}\left(\left[L_T^0(\overline{X})\right]^2\right)^{\frac{1}{2}}\\
&\leq C\Delta t^{3p}
\end{aligned}$$

last inequality comes from Lemmas 1.2 and 2.4.

Putting all the last calculations in (3.7) we find

$$\begin{aligned}
\mathbb{E}\left(\Gamma_s^{3/2}\mathcal{E}_s^{3p-1}[b(X_{\eta(s)})-b(X_s)]\right) \leq{}& C\left[\sup_{u\leq s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]^{1-1/3p}\Delta t^{3/2} + C\left[\sup_{u\leq s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]^{1-1/3p}\Delta t\\
&+ C\left[\sup_{u\leq s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]^{1-2/3p}\Delta t^2 + C\left[\sup_{u\leq s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]^{1-1/p}\Delta t^{7/2}\\
&+ C\Delta t^{3p}.
\end{aligned}$$

Applying Young's Inequality in all terms in the right, we get the desired inequality (3.6).

Notice that in this Step 2, all the arguments and computation do not use the definition of Γ, only the fact that the process is bounded. In particular (3.6) is also true when Γ is a constant equals to one.

Step 3. Let us conclude. Introducing the previous computations in (3.4) we get

$$\mathbb{E}\left(\Gamma_t^{3/2}\mathcal{E}_t^{3p}\right) \leq C\int_0^t\left[\sup_{u\leq s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]ds + C\Delta t^{3p}.$$

and then, since the right-hand side is increasing, it follows

$$\sup_{s\leq t}\mathbb{E}\left(\Gamma_s^{3/2}\mathcal{E}_s^{3p}\right) \leq C\int_0^t\left[\sup_{u\leq s}\mathbb{E}\left(\Gamma_u^{3/2}\mathcal{E}_u^{3p}\right)\right]ds + C\Delta t^{3p},$$

from where we conclude the result thanks to Gronwall's Inequality. □

3.2 Proof of the Theorem

Proof of Theorem 1.6. By the Itô's formula we have

$$\begin{aligned}
\mathbb{E}\left(\mathcal{E}_t^{2p}\right) ={}& 2p\mathbb{E}\int_0^t\mathcal{E}_s^{2p-1}\left\{\text{sgn}(\overline{Z}_s)b(\overline{X}_{\eta(s)})-b(X_s)\right\}ds\\
&+ 2p\mathbb{E}\int_0^t\mathcal{E}_s^{2p-1}dL_s^0(\overline{X}) + p(2p-1)\mathbb{E}\int_0^t\mathcal{E}_s^{2p-2}\Sigma_s^2 ds.
\end{aligned}$$

As we have seen before, $\mathbb{E}\int_0^t \mathcal{E}_s^{2p-1} dL_s^0(\overline{X}) \leq C\Delta t^{2p}$, and $\operatorname{sgn}(x) = 1 - 2\mathbb{1}_{\{x<0\}}$, so

$$\begin{aligned}\mathbb{E}\left(\mathcal{E}_t^{2p}\right) \leq 2p\mathbb{E}\int_0^t \mathcal{E}_s^{2p-1}\left[b(\overline{X}_{\eta(s)}) - b(X_{\eta(s)})\right] ds \\ + 2p\mathbb{E}\int_0^t \mathcal{E}_s^{2p-1}\left[b(X_{\eta(s)}) - b(X_s)\right] ds \\ + 8p(2p-1)\mathbb{E}\int_0^t \mathcal{E}_s^{2p-2}\left[\sigma\overline{X}_s^{\alpha} - \sigma X_s^{\alpha}\right]^2 ds \\ + 8p(2p-1)\mathbb{E}\int_0^t \mathcal{E}_s^{2p-2} D_s(\overline{X})^2 ds + \mathbb{E}\int_0^t R_s \mathbb{1}_{\{\overline{Z}_s<0\}} ds + C\Delta t^{2p},\end{aligned} \tag{3.8}$$

where $R_s = 4p(2p-1)\mathcal{E}_s^{2p-2}\left[\sigma\overline{X}_{\eta(s)}^{\alpha} + \alpha\sigma^2\overline{X}_{\eta(s)}^{2\alpha-1}\left(W_s - W_{\eta(s)}\right)\right] + 8p\mathcal{E}_s^{2p-1} b(\overline{X}_{\eta(s)})$. If we use the Lipschitz property of b, and Young's inequality in the first term in the right of (3.8), Lemma 2.5 in the fourth one, and Lemma 2.2 in the fifth one, we have

$$\begin{aligned}\mathbb{E}\left(\mathcal{E}_t^{2p}\right) \leq C\int_0^t \sup_{u\leq s}\mathbb{E}\left(\mathcal{E}_u^{2p}\right) ds + 2p\int_0^t \mathbb{E}\left(\mathcal{E}_s^{2p-1}\left[b(X_{\eta(s)}) - b(X_s)\right]\right) ds \\ + 8p(2p-1)\int_0^t \mathbb{E}\left(\mathcal{E}_s^{2p-2}\left\{\sigma\overline{X}_s^{\alpha} - \sigma X_s^{\alpha}\right\}^2\right) ds + C\Delta t^{2p}.\end{aligned} \tag{3.9}$$

Using the inequality (3.6) for $p \geq 3/2$ and the weight Γ constant equal to one, we immediately get that

$$\mathbb{E}\left(\mathcal{E}_s^{2p-1}[b(X_{\eta(s)}) - b(X_s)]\right) \leq C\left[\sup_{u\leq s}\mathbb{E}\left(\mathcal{E}_u^{2p}\right)\right] + C\Delta t^{2p}.$$

This last inequality is also true for $p = 1$, the proof of this fact reproduces in a simpler case the arguments in the step 2 of the proof of Lemma 3.2.

On the other hand, using again (3.5), we have

$$\mathbb{E}\left(\mathcal{E}_s^{2p-2}\left\{\sigma\overline{X}_s^{\alpha} - \sigma X_s^{\alpha}\right\}^2\right) \leq C\mathbb{E}\left(\mathcal{E}_s^{2p} X_s^{-2(1-\alpha)}\right) = C\mathbb{E}\left(\Gamma_s\mathcal{E}_s^{2p} X_s^{-2(1-\alpha)}\Gamma_s^{-1}\right),$$

and applying Holder's inequality,

$$\mathbb{E}\left(\Gamma_s\mathcal{E}_s^{2p}\frac{1}{X_s^{2(1-\alpha)}}\Gamma_s^{-1}\right) \leq \left[\mathbb{E}\left(\Gamma_s^{3/2}\mathcal{E}_s^{3p}\right)\right]^{\frac{2}{3}} \times \left[\mathbb{E}\left(\frac{1}{X_s^{6(1-\alpha)}}\Gamma_s^{-3}\right)\right]^{\frac{1}{3}}.$$

The first term in the right-hand side is the weight error controlled by Lemma 3.2. To control the second one, let us recall Remark 3.1. For $\alpha > \frac{1}{2}$, the exact process and the weight process Γ have negative moments of any order, therefore the second term in the last inequality is bounded by a constant. On the other hand, when $\alpha = \frac{1}{2}$, thanks to Hypothesis 1.5-(i) the $(9p+2)$-th negative moment of the exact process is finite, and since for δ^2 small enough, for example for $\delta^2 = 1/8$, we have

$$(1+\delta^2)8p(3p-1)\,3\frac{9p+2}{9p-1} < (9p+2)^2,$$

according with the second point of Remark 3.1, the $3(9p+2)/(9p-1)$-th negative moment of the weight process Γ is also finite. Therefore, when $\alpha = \frac{1}{2}$,

$$\begin{aligned}\mathbb{E}\left(\frac{1}{X_s^{6(1-\alpha)}}\Gamma_s^{-3}\right) &= \mathbb{E}\left(\frac{1}{X_s^3}\Gamma_s^{-3}\right) \\ &\leq \left[\mathbb{E}\left(\frac{1}{X_s^{9p+2}}\right)\right]^{\frac{3}{9p+2}}\left[\mathbb{E}\left(\Gamma_s^{-3\frac{9p+2}{9p-1}}\right)\right]^{\frac{9p-1}{9p+2}} \leq C,\end{aligned}$$

and then in any case

$$\mathbb{E}\left(\Gamma_s \mathcal{E}_s^{2p} \frac{1}{X_s^{2(1-\alpha)}} \Gamma_s^{-1}\right) \leq C\Delta t^{2p}.$$

Introducing all the last computations in (3.9) we get

$$\mathbb{E}\left(\mathcal{E}_t^{2p}\right) \leq C \int_0^t \sup_{u \leq s} \mathbb{E}\left(\mathcal{E}_u^{2p}\right) ds + C\Delta t^{2p}.$$

Since the right-hand side is increasing, it follows we conclude the proof thanks to Gronwall's Inequality. □

Remark 3.3. *The choice of δ is arbitrary, but has a impact in the size of the constant C appearing in the right-hand side of convergence inequality in* (1.11)*, and in the constrain for the parameters $b(0)$ and σ^2 appearing in Hypothesis 1.5-(i). If δ is small, we get a better constrain (less restrictive) but a bigger constant, whereas if δ is big we get a better constant, but a worse constrain. Since our objective was to prove the result for a set of parameters as big as possible, we have chosen δ small.*

Remark 3.4. *Let us mention an example of extension of our convergence result, based on simple transformation method.*

Let us consider the $3/2$-model. That is, the solution of

$$r_t = r_0 + \int_0^t c_1 r_s(c_2 - r_s)ds + \int_0^t c_3 r_s^{3/2} dW_s.$$

If we apply the Itô's Formula to the function $f(x) = x^{-1}$, and we define $v_t = f(r_t)$, we have

$$v_t = v_0 + \int_0^t c_1 + c_3^2 - c_1 c_2 v_s ds + \int_0^t c_3 v_s^{\frac{1}{2}} dB_s,$$

where $B_s = -W_s$ is a Brownian motion. We can approximate v with the SMS, and then define $\bar{r}_t := 1/\bar{v}_t$. Then we can prove the order one strong convergence of $\bar{r}_t$ to r_t from our previous results.

Transformation methods can be used in a more exhaustive manner, in the context of CEV-like SDEs and we refer to [6] for approximation results and examples, using this approach.

4 Numerical Experiments and conclusion

In this section we show the result of numerical simulations performed to study the behavior of the SMS. We compute the error of the scheme as a function of the step size Δt for different values of the parameters α and σ, and we compare the performance of the SMS with other schemes proposed in the literature. For $\alpha > \frac{1}{2}$ we compare the SMS with the Symmetrized Euler Scheme (SES) introduced in [3]. Whereas for $\alpha = \frac{1}{2}$, in addition to compare with the aforementioned scheme, we will also compare with the Modified Euler Scheme (MES) proposed in [6], and with the Alfonsi Implicit Scheme (AIS) proposed in [1]. In order to include this last scheme we consider for all simulations a linear drift

$$b(x) = 10 - 10x.$$

A priori, the AIS can be apply also for $\alpha \in (\frac{1}{2}, 1)$, but is relevant to observe that only when $\alpha = \frac{1}{2}$, the AIS is an explicit scheme, whereas in any other case is not. This implies that in order to compute the AIS for $\alpha > \frac{1}{2}$ at each time step it is necessary to solve numerically a non linear equation. This extra step in the implementation of the scheme brings questions about the impact of the error of this subroutine in the error of the scheme, and about the computing performance of the scheme. Since this questions are beyond the scope of the present work, we include the AIS in the comparison only in the CIR case. In this context, the AIS can be use only if $\sigma^2 > 4b(0)$, for other values of the parameters the AIS is not defined.

About the theoretical convergence estimations of the different schemes

When $\alpha = \frac{1}{2}$, according to Theorem 2 in Alfonsi [2], the AIS converges in the $L^p(\Omega)$-norm, for $p \geq 1$, at rate Δt when $(1 \vee 3p/4)\sigma^2 < b(0)$. On the other hand, according to Theorem 2.2 in [3], the rate of convergence of the SES is $\sqrt{\Delta t}$ under suitable conditions for $b(0)$, σ^2 and K. Finally, the rate of convergence in the $L^1(\Omega)$-norm of the MES depend on the parameters, being Δt if σ^2 is big enough compared with $b(0)$, and Δt^ρ with $\rho < 1$ in other case. We present a summary of this conditions in Table 1.

When $\alpha > \frac{1}{2}$, the AIS (see Section 3 of [2]), the SES (see Theorem 2.2 in [3]), and the MES (see Proposition 4.1 in [6]) converge as soon as $b(0) > 0$, at rate Δt to the exact solution, meanwhile our result need a little more stronger restriction over the parameters in this case. See Table 2.

Scheme	Norm	Theoretical rate	Convergence's Condition
SMS	$L^{2p}, p \geq 1$	1	$b(0) > 3(3p+1)\sigma^2/2$
SES[3]	$L^{2p}, p \geq 1$	1/2	$b(0) > \left[\sqrt{8\mathcal{K}(p)/\sigma^2} + 1\right]\sigma^2/2$, $\mathcal{K}(p) = K(16p-1) \vee 4\sigma^2(8p-1)^2$
AIS [2]	$L^p, p \in [1, \frac{4b(0)}{3\sigma^2})$	1	$b(0) > (1 \vee \frac{3}{4}p)\sigma^2$
MES [6]	L^1	1	$b(0) > \frac{5\sigma^2}{2}$
MES [6]	L^1	1/2	$b(0) > \frac{3\sigma^2}{2}$
MES [6]	L^1	$\left(\frac{1}{6}, \frac{1}{2} - \frac{\sigma^2}{2b(0)+\sigma^2}\right)$	$b(0) > \sigma^2$

Table 1: Summary of the condition over the parameters for the different schemes for the CIR process.

Scheme	Norm	Theoretical rate	Convergence's Condition
SMS	$L^{2p}, p \geq 1$	1	$b(0) > 2\alpha(1-\alpha)^2\sigma^2$
SES[3]	$L^{2p}, p \geq 1$	1/2	$b(0) > 0$
AIS [2]	$L^p, p \geq 1$	1	$b(0) > 0$
AIS [2]	L^1	1	$b(0) > 0$
MES [6]	L^1	1	$b(0) > 0$

Table 2: Summary of the condition over the parameters for the different schemes when $\alpha > \frac{1}{2}$.

In our simulations we consider a time horizon $T = 1$, and to measure the error of every scheme we estimate its $L^1(\Omega)$-norm. A priori our result bounds the $L^2(\Omega)$-norm of the error, but we prefer to work in the simulations with $L^1(\Omega)$-norm to compare our condition in the parameters in the "worst case scenario".

Let $\mathbb{E}|\mathcal{E}_T^{\text{SMS}}|$, $\mathbb{E}|\mathcal{E}_T^{\text{SES}}|$, $\mathbb{E}|\mathcal{E}_T^{\text{MES}}|$, and $\mathbb{E}|\mathcal{E}_T^{\text{AIS}}|$ be the $L^1(\Omega)$-norm of the error for the SMS, SES, MES and AIS respectively, to estimate these quantities, we consider as a reference solution the SMS for $\Delta t = \Delta_{\max}(\alpha)/2^{10}$. Then for each

$$\Delta t \in \left\{ \frac{\Delta_{\max}(\alpha)}{2^n}, n = 1, \dots 9 \right\},$$

we estimate $\mathbb{E}|\mathcal{E}_T^{\cdot}|$ by computing 5×10^4 trajectories of the corresponding scheme, and comparing them with the reference solution. The results of these simulations appears in Figures 1 and 2.

In Figure 1, we shows the result for the CIR case. From Table 1 we observe that we can distinguish five cases for the parameters. The first case is $b(0) > 6\sigma^2$, in which the SMS, the MES, and the AIS have a theoretical rate of convergence equal to Δt, whereas the SES has a theoretical rate of convergence equal to $\sqrt{\Delta t}$. In Figure 1a we observe these expected rates of convergence. Notice how the SMS and the MES have a very similar performance.

The second case is $b(0) \in (5\sigma^2/2, 6\sigma^2)$, now only the AIS and the MES have a theoretical rate of convergence equal to Δt. However, how we can see in Figure 1b the SMS still shows a linear behavior in this case. Recall that the condition over the parameters is a sufficient condition and we believe that could be improved. In Figure 1c, we illustrate the third case, that is $b(0) \in (3\sigma^2/2, 5\sigma^2/2)$. In this case the theoretical rate of the MES is $\sqrt{\Delta t}$, but we observe a linear behavior for the MES and the SMS. The fourth case is $b(0) \in (\sigma^2, 3\sigma^2/2)$, which we display in Figure 1d. For this values of the parameters only the AIS has a theoretical rate of convergence equal to Δt, but we observe that all the schemes seems to reach their optimal convergence rates. Finally, the fifth case is $b(0) < \sigma^2$. In this case all the schemes have a sublinear behavior. We display the result of the simulation for this case in Figure 1e. Since the MES performs considerable worst than the other schemes we show in Figure 1f the same results without the MES, so we can appreciate better the differences between the other schemes.

In Figure 2, we present the results of the simulations for $\alpha = 0.6$, and $\alpha = 0.7$.

In this cases, it can be observed in numerical experiments that the MES scheme needs smaller Δt to achieve its theoretical order one convergence rate, unless one tunes the projection operator in the manner of Remark 5.1 in [6]. For this reason, we present only some comparisons between the symmetrized Euler scheme and its Milstein correction version.

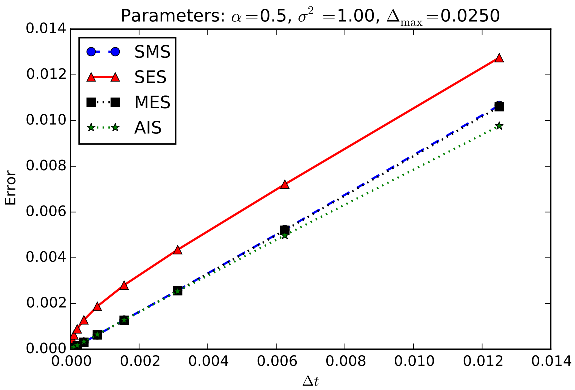
We have observed in the numerical experiments three cases for the parameters. The first one is when $b(0) > 2\alpha(1-\alpha)^2\sigma^2$. In this case Theorem 1.6 holds and we observe the order one convergence (see Figures 1a and 1b). The second case is when the parameters do not satisfy $b(0) > 2\alpha(1-\alpha)^2\sigma^2$, and then we can not apply Theorem 1.6, but in the numerical simulations we still observe the order one convergence (see Figures 1c and 1d). Finally the third case, is when $\sigma \gg b(0)$, and then we do not observe a linear convergence anymore (see Figures 1e and 1f). The second and third case show us that some restriction has to be impose on the parameters to observe the one order convergence of the error, but our restriction, although sufficient, it seems to be too strong, specially for α close to one.

Conclusion

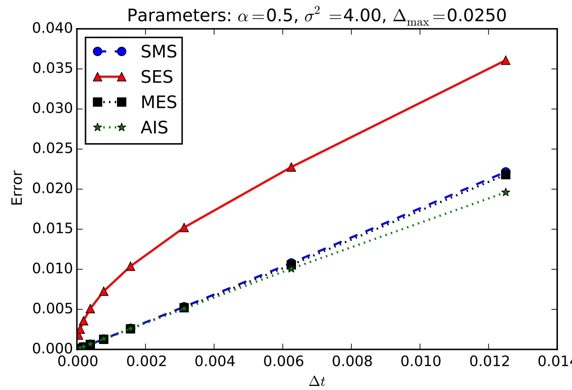
In this paper we have recovered the classical rate of convergence of the Milstein scheme in a context of non smooth diffusion coefficient, although we had to impose some restrictions over the parameters of the equation (1.1) to ensure the right order of convergence. To be precise, if the quotient $b(0)/\sigma^2$ is big enough we will observe the optimal convergence rate. This phenomena was already noted through numerical simulations by Alfonsi in [1].

In the numerical simulations we have observed that, despite the fact it is necessary to impose some restriction over the parameters of the equation (1.1) to obtain the order one convergence, Hypothesis 1.5 seems to be not optimal, specially for $\alpha = \frac{1}{2}$.

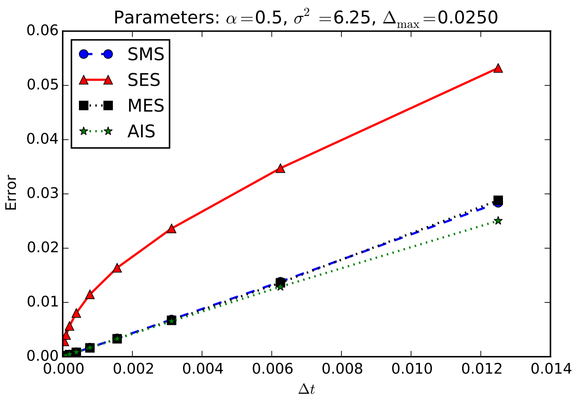
We end this paper doing a small comment on two recent articles addressing similar problematics. In [2], the author propose a general implicit scheme for equations with constant diffusion coefficient, and then, applying Lamperti's transformation, apply this implicit scheme to the CIR model obtaining in fact an explicit scheme. In this case we can compare the AIS with our scheme, and observe that the AIS can be apply to a wider set of parameters, whereas our scheme is explicit and can be applied to a wider class of drifts coefficients. On the other hand, in [6], the authors also make use of Lamperti's transformation to remove the lack of regularity in the diffusion coefficient and obtain a very nice result with several applications. Although, again our result has the more restrictive hypothesis in terms of



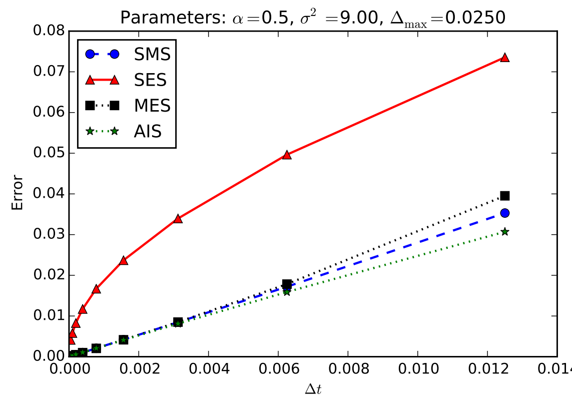
(a) Parameters in case 1: $b(0) > 6\sigma^2$.



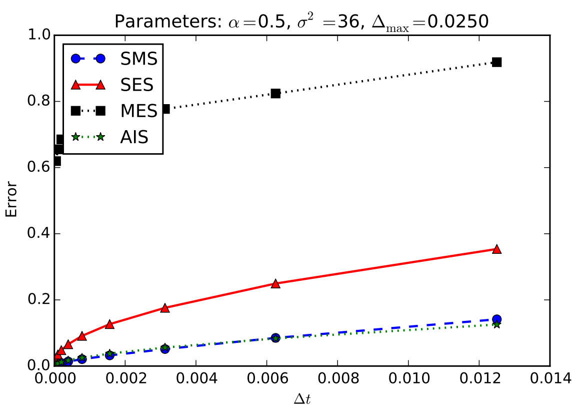
(b) Parameters in case 2: $b(0) \in (5\sigma^2/2, 6\sigma^2)$.



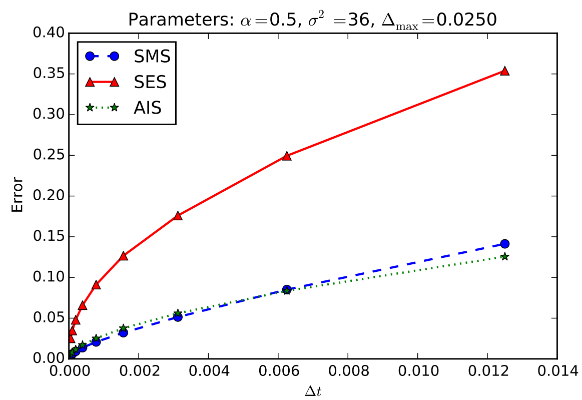
(c) Parameters in case 3: $b(0) \in (3\sigma^2/2, 5\sigma^2/2)$.



(d) Parameters in case 4: $b(0) \in (\sigma^2, 3\sigma^2/2)$.



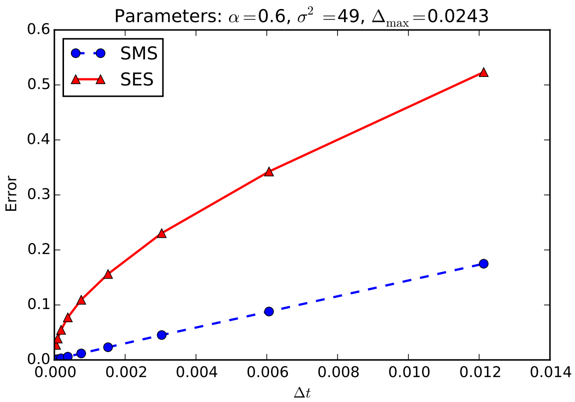
(e) Parameters in case 5: $b(0) < \sigma^2$.



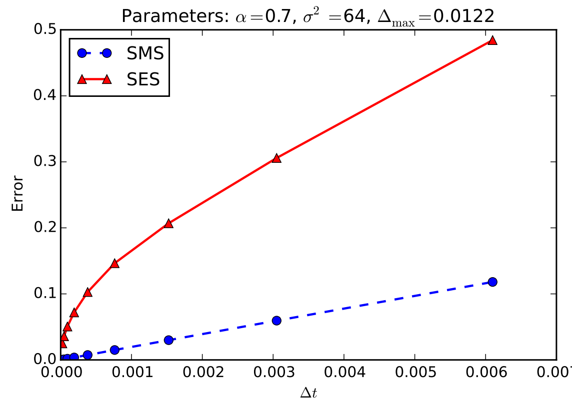
(f) Same plot of Figure 1e without the MES to observer the behavior of the other schemes.

Figure 1: Step size Δt versus the estimated $L^1(\Omega)$-strong error for the CIR Process.

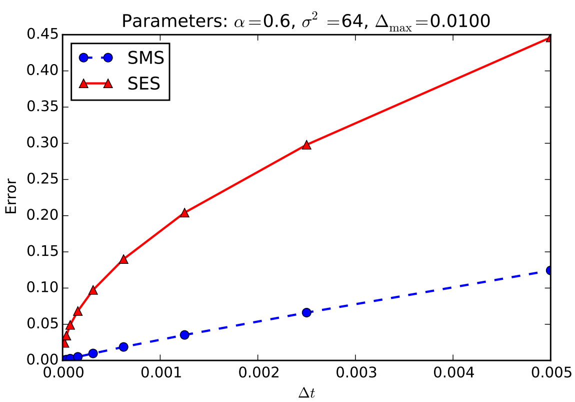
parameters, it can be applied to a more general class of drifts functions. In conclusion, we think that our result, and the recent ones in the literature, are complementary and useful in different contexts.



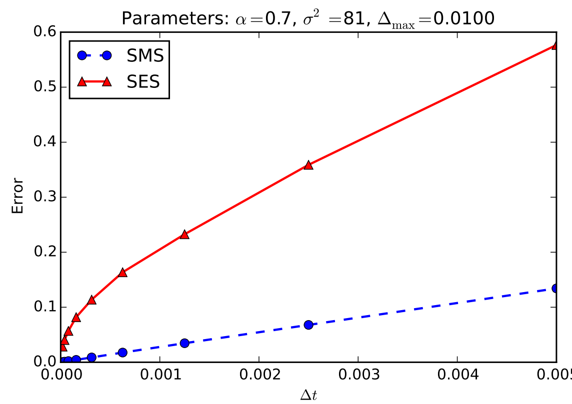
(a) Parameters in case 1: $\alpha = 0.60$.



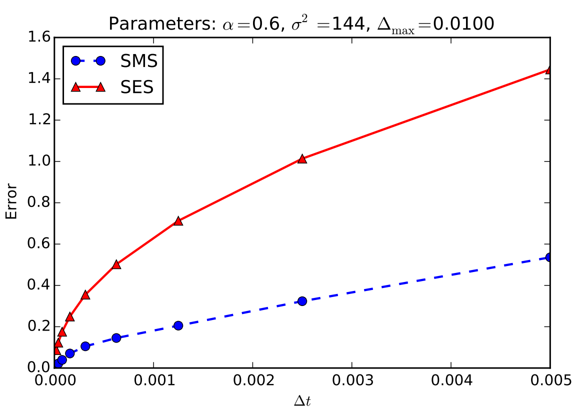
(b) Parameters in case 1: $\alpha = 0.70$.



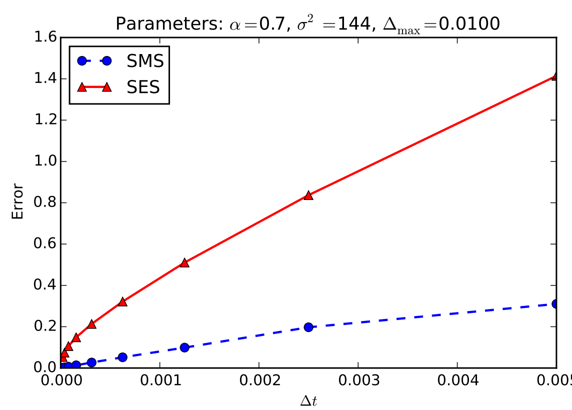
(c) Parameters in case 2: $\alpha = 0.60$.



(d) Parameters in case 2: $\alpha = 0.70$.



(e) Parameters in case 3: $\alpha = 0.60$



(f) Parameters in case 3: $\alpha = 0.70$

Figure 2: Step size Δt versus the estimated $L^1(\Omega)$-error for $\alpha > \frac{1}{2}$ and different values for σ^2.

5 Proofs for preliminary lemmas

5.1 On the Local Error of the SMS

Proof of Lemma 2.1. From the definition of $\overline{X}$, and the algebraic inequality for positive real numbers $(a_1 + \ldots + a_n)^p \leq n^p(a_1^p + \ldots + a_n^p)$ we have

$$\left|\overline{X}_t - \overline{X}_{\eta(t)}\right|^{2p} \leq 3^{2p}\left(b(\overline{X}_{\eta(t)})^{2p}(t-\eta(t))^{2p} + \sigma^{2p}\overline{X}_{\eta(t)}^{2\alpha p}(W_t - W_{\eta(t)})^{2p} + \frac{\alpha^{2p}\sigma^{4p}}{2^{2p}}\overline{X}_{\eta(t)}^{(2\alpha-1)2p}\left[(W_t - W_{\eta(t)})^2 - (t-\eta(t))\right]^{2p}\right).$$

Thanks to the linear growth of b, Lemma 1.4 and the properties of the Brownian Motion it is quite easy to conclude the existence of a constant C such that

$$\mathbb{E}\left[|\overline{X}_t - \overline{X}_{\eta(t)}|^{2p}\right] \leq C\Delta t^p,$$

from where the result follows. □

5.2 On the Probability of SMS being close to zero

From $b_\sigma(\alpha)$ and $K(\alpha)$ defined in (1.9), let us recall the notation

$$\bar{x}(\alpha) := \frac{b_\sigma(\alpha)}{K(\alpha)}$$

introduced in Lemma 2.3. As $b_\sigma(\alpha) > 0$ under Hypothesis 1.5-(i), $\bar{x}(\alpha)$ is bounded away from 0. In particular,

$$\lim_{\alpha\to\frac{1}{2}}\bar{x}(\alpha) = \frac{(b(0) - \sigma^2/4)}{K}, \quad \text{whereas} \quad \lim_{\alpha\to 1}\bar{x}(\alpha) = \frac{b(0)}{K + \sigma^2/2}.$$

Proof of Lemma 2.3. Denoting $\Delta W_s = (W_s - W_{\eta(s)})$, and $\Delta s = s - \eta(s)$, we have for all $s \in [0,T]$,

$$\overline{Z}_s = \frac{\alpha\sigma^2}{2}\overline{X}_{\eta(s)}^{2\alpha-1}\Delta W_s^2 + \sigma\overline{X}_{\eta(s)}^{\alpha}\Delta W_s + \overline{X}_{\eta(s)} + \left(b(\overline{X}_{\eta(s)}) - \frac{\alpha\sigma^2}{2}\overline{X}_{\eta(s)}^{2\alpha-1}\right)\Delta s.$$

From the Lipschitz property of b and the following bound for any $x > 0$

$$x^{2\alpha-1} \leq 4(1-\alpha)^2 + (2\alpha-1)[2(1-\alpha)]^{-\frac{2(1-\alpha)}{2\alpha-1}}x, \tag{5.1}$$

we have

$$\overline{Z}_s \geq \frac{\alpha\sigma^2}{2}\overline{X}_{\eta(s)}^{2\alpha-1}\Delta W_s^2 + \sigma\overline{X}_{\eta(s)}^{\alpha}\Delta W_s + \overline{X}_{\eta(s)} + \left(b_\sigma(\alpha) - K(\alpha)\overline{X}_{\eta(s)}\right)\Delta s. \tag{5.2}$$

So,

$$\mathbb{P}\Big[\overline{Z}_s \leq (1-\rho)b_\sigma(\alpha)\Delta s,\ \overline{X}_{\eta(s)} < \rho\bar{x}(\alpha)\Big] \leq \mathbb{P}\Big[\frac{\alpha\sigma^2}{2}\overline{X}_{\eta(s)}^{2\alpha-1}\Delta W_s^2 + \sigma\overline{X}_{\eta(s)}^{\alpha}\Delta W_s + \overline{X}_{\eta(s)} + \left(\rho b_\sigma(\alpha) - K(\alpha)\overline{X}_{\eta(s)}\right)\Delta s \leq 0,\ \overline{X}_{\eta(t)} < \rho\bar{x}(\alpha)\Big].$$

From the independence of ΔW_s with respect to $\mathcal{F}_{\eta(s)}$, if we denote by N a standard Gaussian variable, we have

$$\mathbb{P}\Big[\overline{Z}_s \leq (1-\rho)b_\sigma(\alpha)\Delta s,\ \overline{X}_{\eta(s)} < \rho\bar{x}(\alpha)\Big] \leq \mathbb{E}\Big[\mathbb{P}\Big(\frac{\alpha\sigma^2}{2}x^{2\alpha-1}\Delta s N^2 + \sigma\sqrt{\Delta s}x^\alpha N + x + [\rho b_\sigma(\alpha) - K(\alpha)x]\,\Delta s \leq 0\Big)\Big|_{x=\overline{X}_{\eta(s)}}\mathbb{1}_{\{\overline{X}_{\eta(s)} < \rho\bar{x}(\alpha)\}}\Big].$$

Notice that in the right-hand side we have a quadratic polynomial of a standard Gaussian random variable. Let us compute its discriminant:

$$\begin{aligned}\Delta(x,\alpha) &= \sigma^2 x^{2\alpha}\Delta s - 2\alpha\sigma^2 x^{2\alpha-1}\Delta s\left(x + \left(\rho b_\sigma(\alpha) - K(\alpha)x\right)\Delta s\right)\\ &= -(2\alpha-1)\sigma^2 x^{2\alpha}\Delta s - 2\alpha\sigma^2 x^{2\alpha-1}\Delta s^2\left(\rho b_\sigma(\alpha) - K(\alpha)x\right).\end{aligned}$$

Since $b_\sigma(\alpha) > 0$, we have $\Delta(x,\alpha) < 0$ for all $\alpha \in [\frac{1}{2}, 1)$, and $x \leq \rho\bar{x}(\alpha)$. So, for all $\Delta s \leq \Delta t$ we have

$$\mathbb{P}\left[\overline{Z}_s \leq (1-\rho)b_\sigma(\alpha)\Delta s,\ \overline{X}_{\eta(s)} < \rho\bar{x}(\alpha)\right] = 0,$$

taking $\Delta s = \Delta t$ we conclude on the Lemma. □

Proof of Lemma 2.2. We keep the notation from the last proof. From Lemma 2.3,

$$\mathbb{P}\left[\overline{Z}_s \leq (1-\rho)b_\sigma(\alpha)\Delta s,\ \overline{X}_{\eta(s)} < \rho\bar{x}(\alpha)\right] = 0.$$

Then

$$\mathbb{P}\left[\overline{Z}_s \leq (1-\rho)b_\sigma(\alpha)\Delta s\right] = \mathbb{P}\left[\overline{Z}_s \leq (1-\rho)b_\sigma(\alpha)\Delta s,\ \overline{X}_{\eta(s)} \geq \rho\bar{x}(\alpha)\right].$$

On the other hand, from 5.2 we have

$$\overline{Z}_s \geq \sigma\overline{X}^{\alpha}_{\eta(s)}\Delta W_s + \left(1 - K(\alpha)\Delta s\right)\overline{X}_{\eta(s)} + b_\sigma(\alpha)\Delta s.$$

Then

$$\begin{aligned}&\mathbb{P}\left[\overline{Z}_s \leq (1-\rho)b_\sigma(\alpha)\Delta s,\ \overline{X}_{\eta(s)} \geq \rho\bar{x}(\alpha)\right]\\ &\qquad\qquad \leq \mathbb{P}\left(\sigma\overline{X}^{\alpha}_{\eta(s)}\Delta W_s \leq -(1-K(\alpha)\Delta s)\overline{X}_{\eta(s)},\ \overline{X}_{\eta(s)} \geq \rho\bar{x}(\alpha)\right).\end{aligned}$$

Since ΔW_s is independent to $\mathcal{F}_{\eta(s)}$, and Normally distributed, and $\Delta t \leq 1/(2K(\alpha))$ we can apply the exponential bound for Gaussian tails and get

$$\begin{aligned}&\mathbb{P}[\overline{Z}_s \leq (1-\rho)b_\sigma(\alpha)\Delta s, \overline{X}_{\eta(s)} \geq \rho\bar{x}(\alpha)]\\ &\qquad \leq \mathbb{E}\left[\exp\left(\frac{-1}{2\sigma^2\Delta s\overline{X}^{2\alpha}_{\eta(s)}}(1-K(\alpha)\Delta s)^2\overline{X}^2_{\eta(s)}\right)\mathbb{1}_{\left\{\overline{X}_{\eta(s)} \geq \rho\bar{x}(\alpha)\right\}}\right].\\ &\qquad \leq \exp\left(\frac{-(1-K(\alpha)\Delta s)^2\rho\bar{x}(\alpha)^{2(1-\alpha)}}{2\sigma^2\Delta s}\right).\end{aligned}$$

Finally

$$\mathbb{P}\left[\overline{Z}_s \leq (1-\rho)b_\sigma(\alpha)\Delta s\right] \leq \exp\left(\frac{-(1-K(\alpha)\Delta s)^2\rho\bar{x}(\alpha)^{2(1-\alpha)}}{2\sigma^2\Delta s}\right),$$

and since $\Delta s \leq \Delta t$ it follows

$$\mathbb{P}\left[\overline{Z}_s \leq (1-\rho)b_\sigma(\alpha)\Delta t\right] \leq \exp\left(\frac{-(1-K(\alpha)\Delta t)^2\rho\bar{x}(\alpha)^{2(1-\alpha)}}{2\sigma^2\Delta t}\right).$$

We conclude on Lemma 2.2 by taking $\rho = 1$ and $\gamma = \bar{x}(\alpha)^{2(1-\alpha)}/(8\sigma^2)$. □

5.3 On the Local Time of the SMS at Zero

The Stopping Times $(\Theta_\alpha, \frac{1}{2} \leq \alpha < 1)$

In what follows, we consider

$$\Theta_\alpha = \inf\left\{s > 0 : \overline{X}_s < (1-\sqrt{\alpha})b_\sigma(\alpha)\Delta t\right\}. \tag{5.3}$$

Lemma 5.1. *Assume $b(0) > 2\alpha(1-\alpha)^2\sigma^2$, and $\Delta t \leq 1/(2K(\alpha)) \wedge x_0/[(1-\sqrt{\alpha})b_\sigma(\alpha)]$. Then there exists a positive constant γ depending on α, $b(0)$, K and σ but not on Δt such that*

$$\mathbb{P}(\Theta_\alpha \leq T) \leq \frac{T}{\Delta t}\exp\left(\frac{-\gamma}{\Delta t}\right). \tag{5.4}$$

Proof. To start notice that the condition $\Delta t < x_0/[(1-\sqrt{\alpha})b_\sigma(\alpha)]$ ensures that the stopping time Θ_α is almost surelluy strictly positive.

To enlighten the notation along the proof, let us call $l_\sigma(\alpha) = (1-\sqrt{\alpha})b_\sigma(\alpha)$, and $i_k = \inf_{t_k < s \leq t_{k+1}} \overline{Z}_s$. We split the proof in three steps.

Step 1. Let us prove that for a suitable function $\psi : \mathbb{R} \to \mathbb{R}$ and a set $A_k \in \mathcal{F}_{t_k}$:

$$\mathbb{P}(\Theta_\alpha \leq T) \leq \sum_{k=0}^{N-1} \mathbb{E}\left(\psi(\overline{X}_{t_k})\mathbb{1}_{A_k}\right) \tag{5.5}$$

Indeed,

$$\mathbb{P}(\Theta_\alpha \leq T) \leq \sum_{k=0}^{N-1} \mathbb{P}\left(i_k \leq l_\sigma(\alpha)\Delta t,\ \ \overline{X}_{t_k} > l_\sigma(\alpha)\Delta t\right).$$

But, for each $k = 0, \dots, N-1$

$$\begin{aligned}\mathbb{P}\left(i_k \leq l_\sigma(\alpha)\Delta t,\ \ \overline{X}_{t_k} > l_\sigma(\alpha)\Delta t\right) &= \mathbb{P}\left(i_k \leq l_\sigma(\alpha)\Delta t,\ \ \overline{X}_{t_k} > l_\sigma(\alpha)\Delta t,\ \ \overline{X}_{t_k} < \bar{x}(\alpha)\sqrt{\alpha}\right) \\ &\quad + \mathbb{P}\left(i_k \leq l_\sigma(\alpha)\Delta t,\ \ \overline{X}_{t_k} > l_\sigma(\alpha)\Delta t,\ \ \overline{X}_{t_k} \geq \bar{x}(\alpha)\sqrt{\alpha}\right).\end{aligned}$$

Since $l_\sigma(\alpha)\Delta t = (1-\sqrt{\alpha})b_\sigma(\alpha)\Delta t \leq \bar{x}(\alpha)\sqrt{\alpha}$, we have

$$\begin{aligned}&\mathbb{P}\Big(i_k \leq l_\sigma(\alpha)\Delta t,\ \ \overline{X}_{t_k} > l_\sigma(\alpha)\Delta t,\ \overline{X}_{t_k} < \bar{x}(\alpha)\sqrt{\alpha}\Big) \\ &\qquad \leq \mathbb{P}\left(i_k \leq l_\sigma(\alpha)\Delta t,\ \ \overline{X}_{t_k} < \bar{x}(\alpha)\sqrt{\alpha}\right) \\ &\qquad \leq \sum_{s \in \mathbb{Q}\cap(t_k, t_{k+1}]} \mathbb{P}\left(\overline{Z}_s \leq l_\sigma(\alpha)\Delta t,\ \ \overline{X}_{t_k} < \bar{x}(\alpha)\sqrt{\alpha}\right) = 0,\end{aligned}$$

thanks to Lemma 2.3. On the other hand, we have

$$\begin{aligned}&\mathbb{P}\Big(i_k \leq l_\sigma(\alpha)\Delta t,\ \overline{X}_{t_k} > l_\sigma(\alpha)\Delta t,\ \overline{X}_{t_k} \geq \bar{x}(\alpha)\sqrt{\alpha}\Big) \\ &\qquad = \mathbb{P}\left(i_k \leq l_\sigma(\alpha)\Delta t,\ \ \overline{X}_{t_k} \geq \bar{x}(\alpha)\sqrt{\alpha}\right) \\ &\qquad \leq \mathbb{P}\Big(\inf_{t_k < s \leq t_{k+1}} \frac{\overline{X}^{1-\alpha}_{\eta(s)}}{\sigma} + \frac{\left(b_\sigma(\alpha) - K(\alpha)\overline{X}_{\eta(s)}\right)\Delta s}{\sigma \overline{X}^{\alpha}_{\eta(s)}} + \Delta W_s \leq \frac{l_\sigma(\alpha)\Delta t}{\sigma \overline{X}^{\alpha}_{\eta(s)}},\ \ \overline{X}_{t_k} \geq \bar{x}(\alpha)\sqrt{\alpha}\Big) \\ &\qquad = \mathbb{E}\Big(\psi(\overline{X}_{t_k})\mathbb{1}_{\left\{\overline{X}_{t_k} > \bar{x}(\alpha)\sqrt{\alpha}\right\}}\Big),\end{aligned}$$

where the inequality comes from (5.2), and the last equality holds thanks to the Markov Property of the Brownian motion, for

$$\psi(x) = \mathbb{P}\Big(\inf_{0 < u \leq \Delta t} \frac{x^{1-\alpha}}{\sigma} + \frac{b_\sigma(\alpha) - K(\alpha)x}{\sigma x^\alpha}u + B_u \leq \frac{l_\sigma(\alpha)\Delta t}{\sigma x^\alpha}\Big),$$

where (B_t) denotes a Brownian Motion independent of (W_t). Summarizing

$$\mathbb{P}\left(i_k \leq l_\sigma(\alpha)\Delta t,\ \ \overline{X}_{t_k} > l_\sigma(\alpha)\Delta t\right) \leq \mathbb{E}\Big(\psi(\overline{X}_{t_k})\mathbb{1}_{\{\overline{X}_{t_k} > \bar{x}(\alpha)\sqrt{\alpha}\}}\Big),$$

and we have (5.5) for $A_k = \left\{\overline{X}_{t_k} > \bar{x}(\alpha)\sqrt{\alpha}\right\}$.

Step 2. Let us prove for all $x \geq \bar{x}(\alpha)\sqrt{\alpha}$:

$$\psi(x) \leq \exp\left(-\frac{(1-K(\alpha)\Delta t)^2(\bar{x}(\alpha)\sqrt{\alpha})^{2(1-\alpha)}}{2\sigma^2\Delta t}\right). \tag{5.6}$$

If $(B_t^\mu, 0 \leq t \leq T)$ is a Brownian motion with drift μ, starting at y_0, then for all $y \leq y_0$, we have (see [4]):

$$\begin{aligned}\mathbb{P}\left(\inf_{0<s\leq t} B_s^\mu \leq y\right) = \frac{1}{2}\operatorname{erfc}\left(\frac{y_0-y}{\sqrt{2t}} + \frac{\mu\sqrt{t}}{\sqrt{2}}\right)\\ + \frac{1}{2}\exp\left(-2\mu(y_0-y)\right)\operatorname{erfc}\left(\frac{y_0-y}{\sqrt{2t}} - \frac{\mu\sqrt{t}}{\sqrt{2}}\right),\end{aligned}$$

where for $z \in \mathbb{R}$, $\operatorname{erfc} z = \sqrt{2/\pi}\int_{\sqrt{2}z}^{\infty}\exp\left(-u^2/2\right)du$. So,

$$\begin{aligned}\psi(x) &= \frac{1}{2}\operatorname{erfc}\left(\frac{[(1-K(\alpha)\Delta t)x + \sqrt{\alpha}b_\sigma(\alpha)\Delta t]}{\sqrt{2\Delta t}\sigma x^\alpha}\right)\\ &\quad + \frac{1}{2}\exp\left(-\frac{2[b_\sigma(\alpha) - K(\alpha)x][x-(1-\sqrt{\alpha})b_\sigma(\alpha)\Delta t]}{\sigma^2 x^{2\alpha}}\right)\\ &\quad \times \operatorname{erfc}\left(\frac{1}{\sqrt{2\Delta t}\sigma x^\alpha}\left[(1+K(\alpha)\Delta t)x - (2-\sqrt{\alpha})b_\sigma(\alpha)\Delta t\right]\right)\\ &=: A(x) + B(x).\end{aligned}$$

Since $\Delta t \leq 1/(2K(\alpha))$, and $\operatorname{erfc}(z) \leq \exp(-z^2)$ for all $z > 0$ we have

$$\begin{aligned}A(x) &\leq \frac{1}{2}\exp\left(-\frac{[(1-K(\alpha)\Delta t)x + \sqrt{\alpha}b_\sigma(\alpha)\Delta t]^2}{2\sigma^2\Delta t x^{2\alpha}}\right)\\ &\leq \frac{1}{2}\exp\left(-\frac{(1-K(\alpha)\Delta t)^2 x^{2(1-\alpha)}}{2\sigma^2\Delta t}\right).\end{aligned}$$

On the other hand, for $x \geq \bar{x}(\alpha)\sqrt{\alpha}$, and $\Delta t \leq 1/(2K(\alpha))$, it follows

$$x > (2-\sqrt{\alpha})b_\sigma(\alpha)\Delta t/(1+K\Delta t),$$

so the argument of the function erfc in B is positive, and then

$$\begin{aligned}B(x) &\leq \frac{1}{2}\exp\left(-\frac{2[b_\sigma(\alpha)-K(\alpha)x][x-(1-\sqrt{\alpha})b_\sigma(\alpha)\Delta t]}{\sigma^2 x^{2\alpha}}\right)\\ &\quad \times \exp\left(-\frac{[(1+K(\alpha)\Delta t)x - (2-\sqrt{\alpha})b_\sigma(\alpha)\Delta t]^2}{2\Delta t\sigma^2 x^{2\alpha}}\right)\\ &= \frac{1}{2}\exp\left(-\frac{[(1-K(\alpha)\Delta t)x + \sqrt{\alpha}b_\sigma(\alpha)\Delta t]^2}{2\Delta t\sigma^2 x^{2\alpha}}\right)\\ &\leq \frac{1}{2}\exp\left(-\frac{(1-K(\alpha)\Delta t)^2 x^{2(1-\alpha)}}{2\sigma^2\Delta t}\right).\end{aligned}$$

So

$$\psi(x) = A(x) + B(x) \leq \exp\left(-\frac{(1-K(\alpha)\Delta t)^2 x^{2(1-\alpha)}}{2\sigma^2\Delta t}\right),$$

and since the right-hand side is decreasing on x, we have (5.6).

Step 3. Let us conclude. Putting together (5.5) and (5.6) we have

$$\begin{aligned}\mathbb{P}(\Theta_\alpha \le T) &\le \sum_{k=0}^{N-1} \mathbb{E}\left(\psi(\overline{X}_{t_k})\mathbb{1}_{\{\overline{X}_{t_k} > \bar{x}(\alpha)\sqrt{\alpha}\}}\right) \\ &\le \sum_{k=0}^{N-1} \exp\left(-\frac{(1-K(\alpha)\Delta t)^2(\bar{x}(\alpha)\sqrt{\alpha})^{2(1-\alpha)}}{2\sigma^2\Delta t}\right)\mathbb{P}\left(\overline{X}_{t_k} > \bar{x}(\alpha)\sqrt{\alpha}\right) \\ &\le \frac{C}{\Delta t}\exp\left(-\frac{\gamma}{\Delta t}\right)\end{aligned}$$

with $\gamma = (\bar{x}(\alpha)\sqrt{\alpha})^{2(1-\alpha)}/(8\sigma^2)$. □

Proof of Lemma 2.4. From (1.4), standard arguments show that $\mathbb{E}\left(L_T^0(\overline{X})^4\right) \le C(T)$. On the other hand, thanks to Corollary VI.1.9 on Revuz and Yor [11, p. 212], we have almost surely

$$L^0_{T\wedge\Theta_\alpha}(\overline{X}) = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^{T\wedge\Theta_\alpha}\mathbb{1}_{[0,\varepsilon)}(\overline{X}_s)d\langle\overline{X}\rangle_s = 0,$$

because for $\varepsilon < (1-\sqrt{\alpha})b_\sigma(\alpha)\Delta t$, and $s \le T\wedge\Theta_\alpha$, $\mathbb{1}_{[0,\varepsilon)}(\overline{X}_s) = 0$. a.s. Now, since

$$L^0_T(\overline{X}) = L^0_T(\overline{X})\mathbb{1}_{\{\Theta_\alpha < T\}} + L^0_{T\wedge\Theta_\alpha}(\overline{X})\mathbb{1}_{\{T\le\Theta_\alpha\}} = L^0_T(\overline{X})\mathbb{1}_{\{\Theta_\alpha<T\}},$$

we can conclude

$$\begin{aligned}\mathbb{E}\left(L_T^0(\overline{X})^2\right) =&\mathbb{E}\left(L_T^0(\overline{X})^2\mathbb{1}_{\{\Theta_\alpha<T\}}\right) \\ &\le \sqrt{\mathbb{E}\left(L_T^0(\overline{X})^4\right)\mathbb{P}(\Theta_\alpha<T)} \le C\sqrt{\frac{1}{\Delta t}\exp\left(-\frac{\gamma}{\Delta t}\right)}.\end{aligned}$$

□

5.4 On the negative moments of the stopped increment process $(\overline{Z}_{t\wedge\Theta_\alpha})$

To prove Lemma 2.5 we need to control the negative moments of the increment process $(\overline{Z}_t, 0 \le t \le T)$. We can not do it directly because there is positive probability of $\overline{Z}_t$ hitting 0 for some $t \in [0,T]$. So we will instead, study the negative moments of the stopped process $\{\overline{Z}_{t\wedge\Theta_\alpha}\}_{0\le t\le T}$.

Existence of Negative Moments. Case $\alpha = \frac{1}{2}$

The proof of the existence of Negative Moments of $\overline{Z}_{t\wedge\Theta_\alpha}$ has two parts. First we study the quotient $\overline{X}_{\eta(s)}/\overline{Z}_s$, and then we proof the main result of the section.

Lemma 5.2. *For* $\alpha = \frac{1}{2}$, *and* $\Delta t \le 1/(4K)\wedge x_0$ *we have*

$$\sup_{0\le s\le T}\mathbb{P}\left(\overline{Z}_s \le \frac{\overline{X}_{\eta(s)}}{2}\right) \le C\Delta t^{\frac{15b_\sigma(1/2)}{8\sigma^2}}. \tag{5.7}$$

To prove this lemma, we need the following auxiliary result, the proof of which is postponed in Appendix A as a straightforward adaptation of the Lemma 3.6 in [5].

Lemma 5.3. *Assume Hypothesis 1.1 holds, and* $b(0) > \sigma^2/4$. *Assume also that* $\Delta t \le 1/(4K)\wedge x_0$. *Then, for any* $\gamma \ge 1$ *there exists a constant* C *depending on the parameters* $b(0)$, K, σ, x_0, T, *and also on* γ, *such that*

$$\sup_{k=0,\ldots,N}\mathbb{E}\exp\left(-\frac{\overline{X}_{t_k}}{\gamma\sigma^2\Delta t}\right) \le C\left(\frac{\Delta t}{x_0}\right)^{\frac{2b_\sigma(1/2)}{\sigma^2}\left(1-\frac{1}{2\gamma}\right)}.$$

Proof of Lemma 5.2. We star by proving

$$\sup_{0\le s\le T}\mathbb{P}\left(\overline{Z}_s\le\frac{\overline{X}_{\eta(s)}}{2}\right)\le\sup_{k=0,\ldots,N}\mathbb{E}\exp\left(-\frac{\overline{X}_{t_k}}{\gamma\sigma^2\Delta t}\right). \tag{5.8}$$

Indeed, if we call $\Delta s=s-\eta(s)$, and $\Delta W_s=(W_s-W_{\eta(s)})$, then

$$\begin{aligned}\mathbb{P}\left(\overline{Z}_s\le\frac{\overline{X}_{\eta(s)}}{2}\right)&\le\mathbb{P}\left(\sigma\sqrt{\overline{X}_{\eta(s)}}\Delta W_s+b_\sigma(1/2)\Delta s+(1-K\Delta s)\overline{X}_{\eta(s)}\le\frac{\overline{X}_{\eta(s)}}{2}\right)\\&\le\mathbb{E}\left(\mathbb{P}\left(\frac{\Delta W_s}{\sqrt{\Delta s}}\le\frac{b_\sigma(1/2)\Delta s+(\frac{1}{2}-K\Delta s)\overline{X}_{\eta(s)}}{\sigma\sqrt{\tau\overline{X}_{\eta(s)}}}\,\middle|\,\mathcal{F}_{\eta(s)}\right)\right)\\&\le\mathbb{E}\left(\exp\left(-\frac{(b_\sigma(1/2)\Delta s+(\frac{1}{2}-K\Delta s)\overline{X}_{\eta(s)})^2}{2\sigma^2\Delta s\overline{X}_{\eta(s)}}\right)\right)\\&\le\mathbb{E}\left(\exp\left(-\frac{(1-2K\Delta t)^2\overline{X}_{\eta(s)}}{8\sigma^2\Delta t}\right)\right).\end{aligned}$$

From here, the bound (5.8) follows easily, and then we conclude using Lemma 5.3. □

Lemma 5.4. *Let $\Theta_{\frac{1}{2}}$ be the stopping time defined in (5.3), and $q\ge1$. If $\Delta t\le\Delta_{\max}(1/2)$, and*

$$b(0)>\frac{3\sigma^2(q+1)}{2}. \tag{5.9}$$

Then there exists a constant C depending on $b(0)$, σ, α, T and q but not on Δt, such that

$$\forall t\in[0,T],\quad\mathbb{E}\left(\frac{1}{\overline{Z}^q_{t\wedge\Theta_{\frac{1}{2}}}}\right)\le C\left(1+\frac{1}{x_0^q}\right).$$

Proof. Let us call $\Delta W_s:=(W_s-W_{\eta(s)})$, and $\Delta s:=(s-\eta(s))$. By Ito's formula

$$\begin{aligned}\mathbb{E}\left(\frac{1}{\overline{Z}^q_{t\wedge\Theta_{\frac{1}{2}}}}\right)=&\frac{1}{x_0^q}-q\mathbb{E}\left(\int_0^{t\wedge\Theta_{\frac{1}{2}}}\frac{b(\overline{X}_{\eta(s)})}{\overline{Z}_s^{q+1}}ds\right)\\&+\frac{q(q+1)}{2}\mathbb{E}\left(\int_0^{t\wedge\Theta_{\frac{1}{2}}}\frac{1}{\overline{Z}_s^{q+2}}\left\{\sigma\sqrt{\overline{X}_{\eta(s)}}+\frac{\sigma^2}{2}\Delta W_s\right\}^2ds\right).\end{aligned} \tag{5.10}$$

But,

$$\left\{\sigma\sqrt{\overline{X}_{\eta(s)}}+\frac{\sigma^2}{2}\Delta W_s\right\}^2\le\sigma^2\overline{X}_{\eta(s)}+\sigma^2\overline{Z}_s\ \ \mathbb{P}-\text{a.s.} \tag{5.11}$$

Indeed,

$$\begin{aligned}\left\{\sigma\sqrt{\overline{X}_{\eta(s)}}+\frac{\sigma^2}{2}\Delta W_s\right\}^2=&\sigma^2\overline{X}_{\eta(s)}+\sigma^2\left(\sigma\sqrt{\overline{X}_{\eta(s)}}\Delta W_s+\frac{\sigma^2}{4}\Delta W_s\right)\\=&\sigma^2\overline{X}_{\eta(s)}+\sigma^2\overline{Z}_s-\sigma^2\left(\overline{X}_{\eta(s)}+b(\overline{X}_{\eta(s)})\Delta s-\frac{\sigma^2}{4}\Delta s\right).\end{aligned}$$

But, thanks to the Lipschitz property of b,

$$\begin{aligned}\overline{X}_{\eta(s)} + b(\overline{X}_{\eta(s)})\Delta s - \frac{\sigma^2}{4}\Delta s &\geq \overline{X}_{\eta(s)} + \left(b(0) - K\overline{X}_{\eta(s)}\right)\Delta s - \frac{\sigma^2}{4}\Delta s \\ &= b_\sigma(1/2)\Delta s + (1 - K\Delta s)\overline{X}_{\eta(s)} \geq 0,\end{aligned}$$

since $\Delta s \leq \Delta t \leq 1/(2K)$, and $b_\sigma(1/2) > 0$. So we have (5.11).

Introducing (5.11) in (5.10), and using $b(x) \geq b(0) - Kx$, we have

$$\begin{aligned}\mathbb{E}\left(\frac{1}{\overline{Z}^q_{t\wedge\Theta_{\frac{1}{2}}}}\right) \leq & \frac{1}{x_0^q} - q\mathbb{E}\left(\int_0^{t\wedge\Theta_{\frac{1}{2}}} \frac{b(0)}{\overline{Z}_s^{q+1}}ds\right) + qK\mathbb{E}\left(\int_0^{t\wedge\Theta_{\frac{1}{2}}} \frac{\overline{X}_{\eta(s)}}{\overline{Z}_s^{q+1}}ds\right) \\ & + \frac{q(q+1)}{2}\sigma^2\mathbb{E}\left(\int_0^{t\wedge\Theta_{\frac{1}{2}}} \frac{1}{\overline{Z}_s^{q+2}}\left\{\overline{X}_{\eta(s)} + \overline{Z}_s\right\}ds\right).\end{aligned} \tag{5.12}$$

Since,

$$\left(\frac{\overline{X}_{\eta(s)}}{\overline{Z}_s}\right) \leq \left(\frac{\overline{X}_{\eta(s)}}{\overline{Z}_s}\right)\mathbb{1}_{\{\overline{Z}_s \leq \overline{X}_{\eta(s)}/2\}} + 2,$$

and applying Hölder's Inequality for some $\varepsilon > 0$, we have

$$\begin{aligned}\mathbb{E}\left(\frac{1}{\overline{Z}^q_{t\wedge\Theta_{\frac{1}{2}}}}\right) \leq & \frac{1}{x_0^q} - q\mathbb{E}\left(\int_0^{t\wedge\Theta_{\frac{1}{2}}} \frac{b(0)}{\overline{Z}_s^{q+1}}ds\right) + 2qK\mathbb{E}\left(\int_0^{t\wedge\Theta_{\frac{1}{2}}} \frac{1}{\overline{Z}_s^{q}}ds\right) \\ & + \frac{3q(q+1)}{2}\sigma^2\mathbb{E}\left(\int_0^{t\wedge\Theta_{\frac{1}{2}}} \frac{1}{\overline{Z}_s^{q+1}}ds\right) \\ & + \frac{C}{\Delta t^{q+2}}\int_0^T \mathbb{E}\left(\overline{X}_{\eta(s)}^{1/\varepsilon}\right)^\varepsilon \mathbb{P}\left(\overline{Z}_s \leq \overline{X}_{\eta(s)}/2\right)^{1-\varepsilon}ds.\end{aligned}$$

Since $b(0) > 3\sigma^2(q+1)/2$, we have $15b_\sigma(1/2)/8\sigma^2 > 2q+2$, so choosing $\varepsilon = q/(2q+2)$, and applying Lemma 5.2 we have

$$\mathbb{P}\left(\overline{Z}_s \leq \overline{X}_{\eta(s)}/2\right)^{1-\varepsilon} \leq C\Delta t^{q+2},$$

and then

$$\begin{aligned}\mathbb{E}\left(\frac{1}{\overline{Z}^q_{t\wedge\Theta_{\frac{1}{2}}}}\right) \leq & \frac{1}{x_0^q} + 2qK\mathbb{E}\left(\int_0^{t\wedge\Theta_{\frac{1}{2}}} \frac{1}{\overline{Z}_s^{q}}ds\right) \\ & + q\left(\frac{3(q+1)}{2}\sigma^2 - b(0)\right)\mathbb{E}\left(\int_0^{t\wedge\Theta_{\frac{1}{2}}} \frac{1}{\overline{Z}_s^{q+1}}ds\right) + C.\end{aligned}$$

Since from the Hypotheses the third term in the right-hand side is negative, we can conclude thanks to Gronwall's Lemma. □

Existence of Negative Moments. Case $\alpha > \frac{1}{2}$

Lemma 5.5. *For $\alpha \in (\frac{1}{2}, 1)$, if $b(0) > 2\alpha(1-\alpha)^2\sigma^2$ and $\Delta t \leq 1/(4\alpha K(\alpha))$, there exists $\gamma > 0$ such that*

$$\sup_{0\leq s\leq T}\mathbb{P}\left(\overline{Z}_s \leq \left(1 - \frac{1}{2\alpha}\right)\overline{X}_{\eta(s)}\right) \leq \exp\left(-\frac{\gamma}{\Delta t}\right). \tag{5.13}$$

Proof. Let us call $\Delta W_s := (W_s - W_{\eta(s)})$, $\Delta s := (s - \eta(s))$, and

$$q(\overline{X}_{\eta(s)}, \Delta W_s) = \frac{\alpha\sigma^2}{2}\overline{X}_{\eta(s)}^{2\alpha-1}\Delta W_s^2 + \sigma\overline{X}_{\eta(s)}^{\alpha}\Delta W_s + \frac{\overline{X}_{\eta(s)}}{2\alpha} + \left(b_\sigma(\alpha) - K(\alpha)\overline{X}_{\eta(s)}\right)\Delta s.$$

Notice that for fix $x \in \mathbb{R}$, $q(x,\cdot)$ is a quadratic polynomial. Using (5.2), we have

$$\begin{aligned}\mathbb{P}\Big(\overline{Z}_s \leq \left(1 - \frac{1}{2\alpha}\right)\overline{X}_{\eta(s)}\Big) &\leq \mathbb{P}\Big(q(\overline{X}_{\eta(s)}, \Delta W_s) \leq 0, \overline{X}_{\eta(s)} \leq \bar{x}(\alpha)\Big) \\ &\quad + \mathbb{P}\Big(q(\overline{X}_{\eta(s)}, \Delta W_s) \leq 0, \overline{X}_{\eta(s)} \geq \bar{x}(\alpha)\Big),\end{aligned}$$

where recall, $\bar{x}(\alpha) = b_\sigma(\alpha)/K(\alpha)$. But

$$\mathbb{P}\Big[q(\overline{X}_{\eta(s)}, \Delta W_s) \leq 0, \overline{X}_{\eta(s)} \leq \bar{x}(\alpha)\Big] = \mathbb{E}\left[\mathbb{P}\left(q(x, \sqrt{\Delta s}N) \leq 0\right)\Big|_{x=\overline{X}_{\eta(s)}} \mathbb{1}_{\{\overline{X}_{\eta(s)} \leq \bar{x}(\alpha)\}}\right],$$

where N stands for a normal for a standard Gaussian random variable. As in the Lemma 2.3, we have a quadratic polynomial in N, its discriminant is

$$\begin{aligned}\Delta &= \sigma^2 x^{2\alpha}\Delta s - 2\alpha\sigma^2 x^{2\alpha-1}\Delta s\left[\frac{x}{2\alpha} + \left(b_\sigma(\alpha) - K(\alpha)x\right)\Delta s\right] \\ &= -2\alpha\sigma^2 x^{2\alpha-1}\Delta s^2\left(b_\sigma(\alpha) - K(\alpha)x\right),\end{aligned}$$

so if $x \leq \bar{x}(\alpha)$, $\Delta < 0$ and the quadratic form in N has not real roots, and in particular is non negative almost surely. Then

$$\mathbb{P}\Big(q(\overline{X}_{\eta(s)}, \Delta W_s) \leq 0, \overline{X}_{\eta(s)} \leq \bar{x}(\alpha)\Big) = 0.$$

On the other hand,

$$\begin{aligned}&\mathbb{P}\Big(q(\overline{X}_{\eta(s)}, \Delta W_s) \leq 0, \overline{X}_{\eta(s)} \geq \bar{x}(\alpha)\Big) \\ &\qquad \leq \mathbb{E}\left[\mathbb{P}\left(N \leq -\frac{b_\sigma(\alpha)\Delta s + \left(\frac{1}{2}\alpha - K(\alpha)\Delta s\right)x}{\sigma x^\alpha\sqrt{\Delta s}}\right)\Bigg|_{x=\overline{X}_{\eta(s)}} \mathbb{1}_{\{\overline{X}_{\eta(s)} \geq \bar{x}(\alpha)\}}\right],\end{aligned}$$

and since $\Delta t \leq 1/(4\alpha K(\alpha))$ we can conclude with the same argument of Lemma 2.2. □

Lemma 5.6. *Let Θ_α be the stopping time defined in (5.3). Let us assume for $\alpha \in (\frac{1}{2}, 1)$, $b(0) > 2\alpha(1-\alpha)^2$, and $\Delta t \leq \Delta_{\max}(\alpha)$, then for all $q \geq 1$, there exists a constant C depending on $b(0)$, σ, α, T and p but not on Δt, such that*

$$\forall t \in [0,T], \quad \mathbb{E}\left(\frac{1}{\overline{Z}_{t\wedge\Theta_\alpha}^q}\right) \leq C\left(1 + \frac{1}{x_0^q}\right).$$

Proof. Let us call $\Delta W_s := W_s - W_{\eta(s)}$. By Ito's formula and the Lipschitz property of b,

$$\begin{aligned}\mathbb{E}\left(\frac{1}{\overline{Z}_{t\wedge\Theta_\alpha}^q}\right) &\leq \frac{1}{x_0^q} - q\mathbb{E}\left(\int_0^{t\wedge\Theta_\alpha}\frac{b(0)}{\overline{Z}_s^{q+1}}ds\right) + qK\mathbb{E}\left(\int_0^{t\wedge\Theta_\alpha}\frac{\overline{X}_{\eta(s)}}{\overline{Z}_s^{q+1}}ds\right) \\ &\quad + \frac{q(q+1)}{2}\mathbb{E}\left(\int_0^{t\wedge\Theta_\alpha}\frac{1}{\overline{Z}_s^{q+2}}\left\{\sigma\overline{X}_{\eta(s)}^\alpha + \alpha\sigma^2\overline{X}_{\eta(s)}^{2\alpha-1}\Delta W_s\right\}^2 ds\right). \quad (5.14)\end{aligned}$$

Following the same ideas to prove (5.11) we can prove for all $s \in [0,t]$, that almost surely

$$\left\{\sigma\overline{X}_{\eta(s)}^\alpha + \alpha\sigma^2\overline{X}_{\eta(s)}^{2\alpha-1}\Delta W_s\right\}^2 \leq \sigma^2\overline{X}_{\eta(s)}^{2\alpha} + 2\alpha\sigma^2\overline{X}_{\eta(s)}^{2\alpha-1}\overline{Z}_s.$$

Introducing this bound in the previous inequality, we have

$$\mathbb{E}\left(\frac{1}{\overline{Z}^q_{t\wedge\Theta_\alpha}}\right) \leq \frac{1}{x_0^q} - q\mathbb{E}\left(\int_0^{t\wedge\Theta_\alpha} \frac{b(0)}{\overline{Z}_s^{q+1}}ds\right) + qK\mathbb{E}\left(\int_0^{t\wedge\Theta_\alpha} \frac{\overline{X}_{\eta(s)}}{\overline{Z}_s^{q+1}}ds\right) + \frac{q(q+1)}{2}\sigma^2\mathbb{E}\left(\int_0^{t\wedge\Theta_\alpha} \frac{1}{\overline{Z}_s^{q+2}}\left\{\overline{X}^{2\alpha}_{\eta(s)} + 2\alpha\overline{X}^{2\alpha-1}_{\eta(s)}\overline{Z}_s\right\}ds\right). \tag{5.15}$$

For $r \in \{1, 2\alpha-1, 2\alpha\}$, we have

$$\left(\frac{\overline{X}_{\eta(s)}}{\overline{Z}_s}\right)^r \leq \left(\frac{\overline{X}_{\eta(s)}}{\overline{Z}_s}\right)^r \mathbb{1}_{\{\overline{Z}_s \leq \overline{X}_{\eta(s)}(1-\frac{1}{2}\alpha)\}} + \left(\frac{2\alpha}{2\alpha-1}\right)^r.$$

So,

$$\begin{aligned}\mathbb{E}\left(\frac{1}{\overline{Z}^q_{t\wedge\Theta_\alpha}}\right) \leq{}& \frac{1}{x_0^q} - q\mathbb{E}\left(\int_0^{t\wedge\Theta_\alpha} \frac{b(0)}{\overline{Z}_s^{q+1}}ds\right) + \frac{2\alpha}{2\alpha-1}qK\mathbb{E}\left(\int_0^{t\wedge\Theta_\alpha} \frac{1}{\overline{Z}_s^{q}}ds\right)\\ &+ \frac{q(q+1)}{2}\sigma^2\frac{(2\alpha)^{2\alpha+1}}{(2\alpha-1)^{2\alpha}}\mathbb{E}\left(\int_0^{t\wedge\Theta_\alpha} \frac{1}{\overline{Z}_s^{q+2(1-\alpha)}}ds\right)\\ &+ C\mathbb{E}\left(\int_0^{t\wedge\Theta_\alpha}\left\{\frac{\overline{X}_{\eta(s)}}{\overline{Z}_s^{q+1}} + \frac{\overline{X}^{2\alpha}_{\eta(s)}}{\overline{Z}_s^{q+2}} + \frac{\overline{X}^{2\alpha-1}_{\eta(s)}}{\overline{Z}_s^{q+1}}\right\}\mathbb{1}_{\{\overline{Z}_s \leq \overline{X}_{\eta(s)}(1-\frac{1}{2}\alpha)\}}ds\right).\end{aligned}$$

The last term in the previous inequality is bounded because of the definition of Θ_α and the Lemma 5.5. Indeed,

$$\begin{aligned}&\mathbb{E}\Big(\int_0^{t\wedge\Theta_\alpha}\left\{\frac{\overline{X}_{\eta(s)}}{\overline{Z}_s^{q+1}} + \frac{\overline{X}^{2\alpha}_{\eta(s)}}{\overline{Z}_s^{q+2}} + \frac{\overline{X}^{2\alpha-1}_{\eta(s)}}{\overline{Z}_s^{q+1}}\right\}\mathbb{1}_{\{\overline{Z}_s \leq \overline{X}_{\eta(s)}(1-\frac{1}{2}\alpha)\}}ds\Big)\\ &\quad\leq \frac{C}{\Delta t^{q+2}}\int_0^T\sqrt{\mathbb{E}\left[\left(\overline{X}_{\eta(s)} + \overline{X}^{2\alpha}_{\eta(s)} + \overline{X}^{2\alpha-1}_{\eta(s)}\right)^2\right]\mathbb{P}\left(\overline{Z}_s \leq \overline{X}_{\eta(s)}\left(1-\frac{1}{2\alpha}\right)\right)}ds\\ &\quad\leq \frac{C}{\Delta t^{q+2}}\exp\left(\frac{\gamma}{\Delta t}\right) \leq C.\end{aligned}$$

So, (5.15) becomes

$$\mathbb{E}\left(\frac{1}{\overline{Z}^q_{t\wedge\Theta_\alpha}}\right) \leq \frac{1}{x_0^q} - q\mathbb{E}\left(\int_0^{t\wedge\Theta_\alpha} \frac{b(0)}{\overline{Z}_s^{q+1}}ds\right) + \frac{2\alpha}{2\alpha-1}qK\mathbb{E}\left(\int_0^{t\wedge\Theta_\alpha} \frac{1}{\overline{Z}_s^{q}}ds\right) + \frac{q(q+1)}{2}\sigma^2\frac{(2\alpha)^{2\alpha+1}}{(2\alpha-1)^{2\alpha}}\mathbb{E}\left(\int_0^{t\wedge\Theta_\alpha} \frac{1}{\overline{Z}_s^{q+2(1-\alpha)}}ds\right) + C. \tag{5.16}$$

But, for any $A_1, A_2 > 0$, the function

$$\frac{A_1}{z^{q+2(1-\alpha)}} - \frac{A_2}{z^{q+1}},$$

is bounded, and (5.16) becomes

$$\mathbb{E}\left(\frac{1}{\overline{Z}^q_{t\wedge\Theta_\alpha}}\right) \leq \frac{1}{x_0^q} + 2qK\mathbb{E}\left(\int_0^{t\wedge\Theta_\alpha} \frac{1}{\overline{Z}_s^{q}}ds\right) + C,$$

from where we can conclude applying Gronwall's Lemma.

□

5.5 On the corrected local error process

Proof of Lemma 2.5. Let us recall the notation in the proof of the main Theorem

$$D_s(\overline{X}) := \sigma\overline{X}_s^{\alpha} - \sigma\overline{X}_{\eta(s)}^{\alpha} - \alpha\sigma^2\overline{X}_{\eta(s)}^{2\alpha-1}(W_s - W_{\eta(s)}), \tag{5.17}$$

and also introduce the notations

$$S_{u\wedge\Theta_\alpha}(\overline{X}) = \left\{\sigma\overline{X}_{\eta(s\wedge\Theta_\alpha)}^{\alpha} + \alpha\sigma^2\overline{X}_{\eta(s\wedge\Theta_\alpha)}^{2\alpha-1}(W_{u\wedge\Theta_\alpha} - W_{\eta(s\wedge\Theta_\alpha)})\right\},$$

and $\Delta W_s = (W_s - W_{\eta(s)})$.

Using Lemma 5.1, and the finiteness of the moments of D, is easy to prove

$$\mathbb{E}\left(D_s(\overline{X})^{2p}\right) \leq C\mathbb{E}\left(D_{s\wedge\Theta_\alpha}(\overline{X})^{2p}\mathbb{1}_{\{\Theta_\alpha\geq\eta(s)\}}\right) + C\Delta t^{2p}.$$

Then, to prove the Lemma we only have to prove

$$\mathbb{E}\left(D_{s\wedge\Theta_\alpha}(\overline{X})^{2p}\mathbb{1}_{\{\Theta_\alpha\geq\eta(s)\}}\right) \leq C\Delta t^{2p}. \tag{5.18}$$

Notice that $\overline{X}_{s\wedge\Theta_\alpha} = \overline{Z}_{s\wedge\Theta_\alpha}$, so

$$D_{s\wedge\Theta_\alpha}(\overline{X})\mathbb{1}_{\{\Theta_\alpha\geq\eta(s)\}} = \left[\sigma\overline{Z}_{s\wedge\Theta_\alpha}^{\alpha} - \sigma\overline{X}_{\eta(s\wedge\Theta_\alpha)}^{\alpha} - \alpha\sigma^2\overline{X}_{\eta(s\wedge\Theta_\alpha)}^{2\alpha-1}\Delta W_{s\wedge\Theta_\alpha}\right]\mathbb{1}_{\{\Theta_\alpha\geq\eta(s)\}},$$

then applying Itô's Formula to the function $\sigma|x|^{\alpha}$ which is $\mathcal{C}^2$ for $x \geq C\Delta t$, we have

$$\begin{aligned} D_{s\wedge\Theta_\alpha}(\overline{X})\mathbb{1}_{\{\Theta_\alpha\geq\eta(s)\}} = \Bigg\{ &\int_{\eta(s\wedge\Theta_\alpha)}^{s\wedge\Theta_\alpha}\left(\frac{\alpha\sigma}{\overline{Z}_{u\wedge\Theta_\alpha}^{1-\alpha}} - \frac{\alpha\sigma}{\overline{X}_{\eta(s\wedge\Theta_\alpha)}^{1-\alpha}}\right)\sigma\overline{X}_{\eta(s\wedge\Theta_\alpha)}^{\alpha}dW_u \\ &+ \int_{\eta(s\wedge\Theta_\alpha)}^{s\wedge\Theta_\alpha}\frac{\alpha^2\sigma^3\overline{X}_{\eta(s\wedge\Theta_\alpha)}^{2\alpha-1}}{\overline{Z}_{u\wedge\Theta_\alpha}^{1-\alpha}}\Delta W_{u\wedge\Theta_\alpha}dW_u \\ &+ \int_{\eta(s\wedge\Theta_\alpha)}^{s\wedge\Theta_\alpha}\frac{\alpha\sigma}{\overline{Z}_{u\wedge\Theta_\alpha}^{1-\alpha}}b(\overline{X}_{\eta(s\wedge\Theta_\alpha)})du \\ &- \int_{\eta(s\wedge\Theta_\alpha)}^{s\wedge\Theta_\alpha}\frac{1}{2}\frac{\alpha(1-\alpha)\sigma}{\overline{Z}_{u\wedge\Theta_\alpha}^{2-\alpha}}S_{u\wedge\Theta_\alpha}(\overline{X})^2du\Bigg\}\mathbb{1}_{\{\Theta_\alpha\geq\eta(s)\}} \\ =:\ & I_1 + I_2 + I_3 - I_4. \end{aligned} \tag{5.19}$$

Notice that on the event $\{\eta(s) \leq \Theta_\alpha\}$ we have $\eta(s) = \eta(s\wedge\Theta_\alpha)$, and then

$$\mathbb{E}\left[I_1^{2p}\right] = \mathbb{E}\left[\int_{\eta(s)}^{s\wedge\Theta_\alpha}\mathbb{1}_{\{\Theta_\alpha\geq\eta(s)\}}\left(\frac{\alpha\sigma}{\overline{Z}_{u\wedge\Theta_\alpha}^{1-\alpha}} - \frac{\alpha\sigma}{\overline{X}_{\eta(s\wedge\Theta_\alpha)}^{1-\alpha}}\right)\sigma\overline{X}_{\eta(s\wedge\Theta_\alpha)}^{\alpha}dW_u\right]^{2p}.$$

By the Burkholder-Davis-Gundy inequality (see [8, p. 166]) there exists a constant C_p depending only on p such that

$$\begin{aligned} &\mathbb{E}\left[\int_{\eta(s)}^{s\wedge\Theta_\alpha}\mathbb{1}_{\{\Theta_\alpha\geq\eta(s)\}}\left(\frac{\alpha\sigma}{\overline{Z}_{u\wedge\Theta_\alpha}^{1-\alpha}} - \frac{\alpha\sigma}{\overline{X}_{\eta(s\wedge\Theta_\alpha)}^{1-\alpha}}\right)\sigma\overline{X}_{\eta(s\wedge\Theta_\alpha)}^{\alpha}dW_u\right]^{2p} \\ &\qquad \leq (\alpha\sigma^2)^{2p}C_p\mathbb{E}\left[\int_{\eta(s)}^{s\wedge\Theta_\alpha}\left(\frac{\overline{X}_{\eta(s\wedge\Theta_\alpha)}^{1-\alpha} - \overline{Z}_{u\wedge\Theta_\alpha}^{1-\alpha}}{\overline{Z}_{u\wedge\Theta_\alpha}^{1-\alpha}\overline{X}_{\eta(s\wedge\Theta_\alpha)}^{1-\alpha}}\right)^2\overline{X}_{\eta(s\wedge\Theta_\alpha)}^{2\alpha}\mathbb{1}_{\{\Theta_\alpha\geq\eta(s)\}}du\right]^{p}, \end{aligned}$$

Noting that the integrand in the right-hand side is positive, and we have

$$\mathbb{E}\left[I_1^{2p}\right] = (\alpha\sigma^2)^{2p} C_p \mathbb{E}\left[\int_{\eta(s)}^{s} \left(\frac{\overline{X}^{1-\alpha}_{\eta(s\wedge\Theta_\alpha)} - \overline{Z}^{1-\alpha}_{u\wedge\Theta_\alpha}}{\overline{Z}^{1-\alpha}_{u\wedge\Theta_\alpha}}\right)^2 \overline{X}^{4\alpha-2}_{\eta(s\wedge\Theta_\alpha)} \mathbb{1}_{\{\Theta_\alpha \geq \eta(s)\}} du\right]^p$$

$$\leq C_p \mathbb{E}\left[\int_{\eta(s)}^{s} \left(\frac{\left(\overline{X}^{1-\alpha}_{\eta(s\wedge\Theta_\alpha)} - \overline{Z}^{1-\alpha}_{u\wedge\Theta_\alpha}\right)}{\overline{Z}^{1-\alpha}_{u\wedge\Theta_\alpha}} \frac{\left(\overline{X}^{\alpha}_{\eta(s\wedge\Theta_\alpha)} + \overline{Z}^{\alpha}_{u\wedge\Theta_\alpha}\right)}{\overline{Z}^{\alpha}_{u\wedge\Theta_\alpha}}\right)^2 \overline{X}^{4\alpha-2}_{\eta(s\wedge\Theta_\alpha)} \mathbb{1}_{\{\Theta_\alpha \geq \eta(s)\}} du\right]^p.$$

but for $x, y \geq 0$, and $\beta \in [0, \frac{1}{2})$ it holds $|x^\beta - y^\beta|(x^{1-\beta} + y^{1-\beta}) \leq 2|x - y|$, so

$$\mathbb{E}\left[I_1^{2p}\right] \leq C\mathbb{E}\left[\int_{\eta(s)}^{s} \left(\overline{X}_{\eta(s\wedge\Theta_\alpha)} - \overline{Z}_{u\wedge\Theta_\alpha}\right)^2 \mathbb{1}_{\{\Theta_\alpha \geq \eta(s)\}} \frac{\overline{X}^{4\alpha-2}_{\eta(s\wedge\Theta_\alpha)}}{\overline{Z}^{2}_{u\wedge\Theta_\alpha}} du\right]^p$$

$$\leq C\Delta t^{p-1} \int_{\eta(s)}^{s} \mathbb{E}\left[\left(\overline{X}_{\eta(s\wedge\Theta_\alpha)} - \overline{Z}_{u\wedge\Theta_\alpha}\right)^{2p} \mathbb{1}_{\{\Theta_\alpha \geq \eta(s)\}} \frac{\overline{X}^{2p(2\alpha-1)}_{\eta(s\wedge\Theta_\alpha)}}{\overline{Z}^{2p}_{u\wedge\Theta_\alpha}}\right] du.$$

Let $a > 1$. Thanks to Hölder's inequality we have

$$\mathbb{E}\left[\left(\overline{X}_{\eta(s\wedge\Theta_\alpha)} - \overline{Z}_{u\wedge\Theta_\alpha}\right)^{2p} \mathbb{1}_{\{\Theta_\alpha \geq \eta(s)\}} \frac{\overline{X}^{2p(2\alpha-1)}_{\eta(s\wedge\Theta_\alpha)}}{\overline{Z}^{2p}_{u\wedge\Theta_\alpha}}\right]$$

$$\leq \left(\mathbb{E}\left[\left[\overline{X}_{\eta(s\wedge\Theta_\alpha)} - \overline{Z}_{u\wedge\Theta_\alpha}\right]^{\frac{2ap}{(a-1)}} \mathbb{1}_{\{\Theta_\alpha \geq \eta(s)\}}\right]\right)^{1-1/a} \left(\mathbb{E}\left[\frac{\overline{X}^{2ap(2\alpha-1)}_{\eta(s\wedge\Theta_\alpha)}}{\overline{Z}^{2ap}_{u\wedge\Theta_\alpha}}\right]\right)^{1/a}.$$

We can use Lemma 2.1 to bound the Local Error of the scheme, and the Lemma 5.1 to bound the probability of Θ_α being smaller than s or $\eta(s)$, and prove

$$\mathbb{E}\left[\left(\overline{X}_{\eta(s\wedge\Theta_\alpha)} - \overline{Z}_{u\wedge\Theta_\alpha}\right)^{\frac{2ap}{(a-1)}} \mathbb{1}_{\{\Theta_\alpha \geq \eta(s)\}}\right] \leq C\Delta t^{\frac{ap}{(a-1)}},$$

On the other hand, when $\alpha > \frac{1}{2}$, we have control of any negative moment of $\overline{Z}_{u\wedge\Theta_\alpha}$, so

$$\mathbb{E}\left[\frac{\overline{X}^{4ap(2\alpha-1)}_{\eta(s\wedge\Theta_\alpha)}}{\overline{Z}^{2ap}_{u\wedge\Theta_\alpha}}\right] \leq \sqrt{\mathbb{E}\left[\overline{X}^{2ap(2\alpha-1)}_{\eta(s\wedge\Theta_\alpha)}\right] \mathbb{E}\left[\frac{1}{\overline{Z}^{4ap}_{u\wedge\Theta_\alpha}}\right]} \leq C,$$

whereas when $\alpha = \frac{1}{2}$, we choose $a > 1$, such that $2b(0)/\sigma^2 > 3(2ap + 1)$, so we have control of the $2ap$-th negative moment of $\overline{Z}_{u\wedge\Theta_\alpha}$. And then

$$\mathbb{E}\left[\frac{\overline{X}^{4ap(2\alpha-1)}_{\eta(s\wedge\Theta_\alpha)}}{\overline{Z}^{2ap}_{u\wedge\Theta_\alpha}}\right] = \mathbb{E}\left[\frac{1}{\overline{Z}^{2ap}_{u\wedge\Theta_\alpha}}\right] \leq C.$$

So, in any case we have

$$\mathbb{E}\left[\left(\overline{X}_{\eta(s\wedge\Theta_\alpha)} - \overline{Z}_{u\wedge\Theta_\alpha}\right)^{2p} \mathbb{1}_{\{\Theta_\alpha \geq \eta(s)\}} \frac{\overline{X}^{2p(2\alpha-1)}_{\eta(s\wedge\Theta_\alpha)}}{\overline{Z}^{2p}_{u\wedge\Theta_\alpha}}\right] \leq C\Delta t^p.$$

And then we can conclude $\mathbb{E}\left[I_1^{2p}\right] \leq C\Delta t^{2p}$.

Using the same arguments for $\mathbb{E}\left(I_2^{2p}\right)$, we have

$$
\begin{aligned}
\mathbb{E}\left(I_2^{2p}\right) &\leq C_p \mathbb{E}\left(\int_{\eta(s)}^{s\wedge\Theta_\alpha} \mathbb{1}_{\{\Theta_\alpha \geq \eta(s)\}} \frac{\alpha^2\sigma^6 \overline{X}_{\eta(s)}^{2(2\alpha-1)}}{\overline{Z}_u^{2(1-\alpha)}} \Delta W_u^2 du\right)^p \\
&\leq C\Delta t^{p-1} \int_{\eta(s)}^{s} \mathbb{E}\left(\mathbb{1}_{\{\Theta_\alpha \geq \eta(s)\}} \frac{\overline{X}_{\eta(s)}^{2(2\alpha-1)p}}{\overline{Z}_{u\wedge\Theta_\alpha}^{2(1-\alpha)p}} \Delta W_{u\wedge\Theta_{\frac{1}{2}}}^{2p}\right) du \\
&\leq C\Delta t^{p-1} \int_{\eta(s)}^{s} \sqrt{\mathbb{E}\left(\frac{\overline{X}_{\eta(s)}^{4(2\alpha-1)p}}{\overline{Z}_{u\wedge\Theta_\alpha}^{4(1-\alpha)p}}\right)} \sqrt{\mathbb{E}\left(\mathbb{1}_{\{\Theta_\alpha \geq \eta(s)\}} \Delta W_{u\wedge\Theta_{\frac{1}{2}}}^{4p}\right)} du \\
&\leq C\Delta t^{2p}.
\end{aligned}
$$

To bound $\mathbb{E}\left(I_3^{2p}\right)$ we proceed as follows

$$
\begin{aligned}
\mathbb{E}\left(I_3^{2p}\right) &= \mathbb{E}\left(\int_{\eta(s)}^{s\wedge\Theta_\alpha} \mathbb{1}_{\{\Theta_\alpha \geq \eta(s)\}} \frac{\alpha\sigma}{\overline{Z}_{u\wedge\Theta_\alpha}^{1-\alpha}} b(\overline{X}_{\eta(s\wedge\Theta_\alpha)}) du\right)^{2p} \\
&\leq (\alpha\sigma)^{2p} \Delta t^{2p-1} \int_{\eta(s)}^{s} \mathbb{E}\left(\frac{1}{\overline{Z}_{u\wedge\Theta_\alpha}^{2(1-\alpha)p}} b\left(\overline{X}_{\eta(s\wedge\Theta_\alpha)}\right)^{2p}\right) du \\
&\leq (\alpha\sigma)^{2p} \Delta t^{2p-1} \int_{\eta(s)}^{s} \mathbb{E}\left(\frac{1}{\overline{Z}_{u\wedge\Theta_\alpha}^{2p}}\right)^{1-\alpha} \mathbb{E}\left(b\left(\overline{X}_{\eta(s\wedge\Theta_\alpha)}\right)^{\frac{2p}{\alpha}}\right)^{\alpha} du \\
&\leq C\Delta t^{2p}.
\end{aligned}
$$

Finally for $\mathbb{E}\left(I_4^{2p}\right)$ we consider first $\alpha > \frac{1}{2}$, in this case we have control of any negative moment of $\overline{Z}_{u\wedge\Theta_\alpha}$ so proceeding as before

$$
\begin{aligned}
\mathbb{E}\left(I_4^{2p}\right) &= \mathbb{E}\left(\int_{\eta(s)}^{s\wedge\Theta_\alpha} \mathbb{1}_{\{\Theta_\alpha \geq \eta(s)\}} \frac{1}{2} \frac{\alpha(1-\alpha)\sigma}{\overline{Z}_{u\wedge\Theta_\alpha}^{2-\alpha}} S_{u\wedge\Theta_\alpha}(\overline{X})^2 du\right)^{2p} \\
&\leq C\Delta t^{2p-1} \int_{\eta(s)}^{s} \mathbb{E}\left(\frac{1}{\overline{Z}_{u\wedge\Theta_\alpha}^{2p(2-\alpha)}} S_{u\wedge\Theta_\alpha}(\overline{X})^{4p}\right) du \\
&\leq C\Delta t^{2p}.
\end{aligned}
$$

The case $\alpha = \frac{1}{2}$ is a little more delicate. Let us recall that in the proof of (5.11) we find the following equality

$$
\left\{\sigma \overline{X}_{\eta(s\wedge\Theta_{\frac{1}{2}})}^{\alpha} + \frac{\sigma^2}{2} \Delta W_{u\wedge\Theta_{\frac{1}{2}}}\right\}^2 = \sigma^2 \overline{Z}_{u\wedge\Theta_{\frac{1}{2}}} - \sigma^2 \left(b(\overline{X}_{\eta(s\wedge\Theta_{\frac{1}{2}})}) - \frac{\sigma^2}{4}\right)(u\wedge\Theta_{\frac{1}{2}} - \eta(s\wedge\Theta_{\frac{1}{2}})),
$$

so, we have from the definition of $\Theta_{\frac{1}{2}}$

$$
\begin{aligned}
\left\{\sigma \overline{X}_{\eta(s\wedge\Theta_{\frac{1}{2}})}^{\alpha} + \frac{\sigma^2}{2} \Delta W_{u\wedge\Theta_{\frac{1}{2}}}\right\}^{4p} &\leq C\left(\overline{Z}_{u\wedge\Theta_{\frac{1}{2}}}^{2p} + \left(b(\overline{X}_{\eta(s\wedge\Theta_{\frac{1}{2}})}) - \frac{\sigma^2}{4}\right)^{2p} \Delta t^{2p}\right) \\
&\leq C\left(1 + \left(b(\overline{X}_{\eta(s\wedge\Theta_{\frac{1}{2}})}) - \frac{\sigma^2}{4}\right)^{2p}\right) \overline{Z}_{u\wedge\Theta_{\frac{1}{2}}}^{2p},
\end{aligned}
$$

and then

$$\mathbb{E}\left(I_4^{2p}\right) \leq C\Delta t^{2p-1}\int_{\eta(s)}^{s}\mathbb{E}\left(\frac{1}{\overline{Z}^{3p}_{u\wedge\Theta_\alpha}}\left\{\sigma\overline{X}^{\alpha}_{\eta(s\wedge\Theta_{\frac{1}{2}})}+\frac{\sigma^2}{2}\Delta W_{u\wedge\Theta_{\frac{1}{2}}}\right\}^{4p}\right)du$$
$$\leq C\Delta t^{2p-1}\int_{\eta(s)}^{s}\mathbb{E}\left(\frac{1}{\overline{Z}^{p}_{u\wedge\Theta_\alpha}}\left\{1+\left(b(\overline{X}_{\eta(s\wedge\Theta_{\frac{1}{2}})})-\frac{\sigma^2}{4}\right)^{2p}\right\}\right)du$$
$$\leq C\Delta t^{2p-1}\int_{\eta(s)}^{s}\sqrt{\mathbb{E}\left(\frac{1}{\overline{Z}^{2p}_{u\wedge\Theta_\alpha}}\right)}\sqrt{\mathbb{E}\left(1+\left(b(\overline{X}_{\eta(s\wedge\Theta_{\frac{1}{2}})})-\frac{\sigma^2}{4}\right)^{4p}\right)}du$$
$$\leq C\Delta t^{2p}.$$

So, for every $\alpha\in[\frac{1}{2},1)$

$$\mathbb{E}\left(I_4^{2p}\right)\leq C\Delta t^{2p},$$

from where we conclude on the Lemma. □

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A Appendix

Proof of Lemma 1.4. Let us recall the notations $\Delta s = s-\eta(s)$, and $\Delta W_s = W_s - W_{\eta(s)}$. Let us define $\tau_m = \inf\{t\geq 0 : \overline{X}_t \geq m\}$. Then by Itô's Formula, Young's inequality and the Lipschitz property of b, we have

$$\begin{aligned}\mathbb{E}\left(\overline{X}^{2p}_{t\wedge\tau_m}\right) &\leq x_0^{2p} + C\mathbb{E}\int_0^{t\wedge\tau_m}\overline{X}^{2p}_s + C + \overline{X}^{2p}_{\eta(s)}ds\\ &+ C\mathbb{E}\int_0^{t\wedge\tau_m}\left[\sigma\overline{X}^{\alpha}_{\eta(s)}+\alpha\sigma^2\overline{X}^{2\alpha-1}_{\eta(s)}\Delta W_s\right]^{2p}ds.\end{aligned} \tag{A.1}$$

From the definition of $\overline{X}$, a straightforward computation shows that for all $s\in[0,t]$ almost surely

$$\overline{X}^{2p}_s \leq C\left(1+\overline{X}^{2p}_{\eta(s)}+\Delta W_s^{\frac{2p}{1-\alpha}}+\left[\Delta W_s^2-\Delta s\right]^{\frac{2p}{2(1-\alpha)}}\right).$$

Putting this in (A.1), we have

$$\begin{aligned}\mathbb{E}\left(\overline{X}^{2p}_{t\wedge\tau_m}\right) \leq& x_0^{2p} + C\mathbb{E}\int_0^{t\wedge\tau_m} 1+\overline{X}^{2p}_{\eta(s)}+\left[\sigma\overline{X}^{\alpha}_{\eta(s)}+\alpha\sigma^2\overline{X}^{2\alpha-1}_{\eta(s)}\Delta W_s\right]^{2p}ds\\ &+C\mathbb{E}\int_0^{t\wedge\tau_m}\Delta W_s^{\frac{2p}{1-\alpha}}+\left[\Delta W_s^2-(s-\eta(s))\right]^{\frac{2p}{2(1-\alpha)}}ds\\ &\leq x_0^{2p}+C\mathbb{E}\int_0^{t\wedge\tau_m}1+\overline{X}^{2p}_{\eta(s)}+\overline{X}^{2p\alpha}_{\eta(s)}+\overline{X}^{2p(2\alpha-1)}_{\eta(s)}\Delta W_s^{2p}ds\\ &+C\int_0^T\mathbb{E}\left(\Delta W_s^{\frac{2p}{1-\alpha}}\right)+\mathbb{E}\left(\left[\Delta W_s^2-\Delta s\right]^{\frac{2p}{2(1-\alpha)}}\right)ds.\end{aligned}$$

Since $\alpha\in[\frac{1}{2},1)$ we have $\overline{X}^{2p\alpha}_{\eta(s)}\leq 1+\overline{X}^{2p}_{\eta(s)}$, and then, using Young's Inequality and the finiteness of the moments of Gaussian random variables, we conclude

$$\mathbb{E}\left(\overline{X}^{2p}_{t\wedge\tau_m}\right)\leq Cx_0^{2p}+C\mathbb{E}\int_0^{t\wedge\tau_m}\overline{X}^{2p}_{\eta(s)}ds\leq Cx_0^{2p}+C\int_0^t\sup_{u\leq s}\mathbb{E}\left(\overline{X}^{2p}_{u\wedge\tau_m}\right)ds.$$

Since the right-hand side is increasing, we can take supremum in the left-hand side and from here, applying Gronwall's inequality, and taking $m \to \infty$ we get

$$\sup_{t \le T} \mathbb{E}\left(\overline{X}_t^{2p}\right) \le C x_0^{2p}$$

since

$$\sup_{s \le t} \lim_{m \to \infty} \mathbb{E}\left(\overline{X}_{s \wedge \tau_m}^{2p}\right) \le \lim_{m \to \infty} \sup_{s \le t} \mathbb{E}\left(\overline{X}_{s \wedge \tau_m}^{2p}\right).$$

From here, following a standard argument using Burkholder-Davis-Gundy inequality we can conclude on Lemma 1.4. □

Proof of lemma 5.3. First, from the definition of $\overline{X}_{t_k}$ we have

$$\overline{X}_{t_k} \ge \overline{X}_{t_{k-1}} + (b_\sigma(1/2) - K\overline{X}_{t_{k-1}})\Delta t + \sigma\sqrt{\overline{X}_{t_{k-1}}}\left(W_{t_k} - W_{t_{k-1}}\right),$$

then

$$\mathbb{E}\exp\left(-\mu_0 \overline{X}_{t_k}\right) \le \mathbb{E}\exp\Big(-\mu_0\Big[\overline{X}_{t_{k-1}} + (b_\sigma(1/2) - K\overline{X}_{t_{k-1}})\Delta t + \sigma\sqrt{\overline{X}_{t_{k-1}}}\left(W_{t_k} - W_{t_{k-1}}\right)\Big]\Big),$$

where $\mu_0 = 1/\gamma\sigma^2\Delta t$. From here, just as in Lemma 3.6 in [5], we conclude

$$\mathbb{E}\exp\left(-\mu_0\overline{X}_{t_k}\right) \le \exp\left(-\mu_0 b_\sigma(1/2)\Delta t\right)\mathbb{E}\exp\left(-\mu_0\overline{X}_{t_{k-1}}\left[1 - K\Delta t - \frac{\sigma^2\Delta t}{2}\mu_0\right]\right). \tag{A.2}$$

Then if we introduce the same sequence $(\mu_j)_j \ge 0$ of Lemma 3.6 in [5], given by

$$\mu_j = \begin{cases} \frac{1}{\gamma\sigma^2\Delta t}, & j = 0, \\ \mu_{j-1}\left[1 - K\Delta t - \frac{\sigma^2\Delta t}{2}\mu_{j-1}\right], & j \ge 1. \end{cases}$$

We can repeat the proof in [5] and find out that if $\Delta t \le 1/(2K)$ then, the sequence $(\mu_j)_j \ge 0$ is nonnegative, decreasing and satisfies the following bound

$$\mu_j \ge \mu_1\left(\frac{1}{1 + \frac{\sigma^2}{2}\Delta t(j-1)\mu_0}\right) - K\left(\frac{\Delta t(j-1)\mu_0}{1 + \frac{\sigma^2}{2}\Delta t(j-1)\mu_0}\right), \ \forall j \ge 1.$$

On the other hand making the same calculations to obtain (A.2) we can get for any $j \in \{0, \ldots, k-1\}$,

$$\mathbb{E}\exp\left(-\mu_j\overline{X}_{t_{k-j}}\right) \le \exp\left(-\mu_j b_\sigma(1/2)(\tfrac{1}{2})\Delta t\right)\mathbb{E}\exp\left(-\mu_j\overline{X}_{t_{k-j-1}}\left[1 - K\Delta t - \frac{\sigma^2\Delta t}{2}\mu_{j+1}\right]\right),$$

from where, by an induction argument we have

$$\mathbb{E}\left(-\mu_0\overline{X}_{t_k}\right) \le \exp\left(-b_\sigma(1/2)\sum_{j=0}^{k-1}\mu_j\Delta t\right)\exp\left(x_0\mu_k\right).$$

From here, and the bound for the sequence $(\mu_j)_j \ge 0$, we have

$$\mathbb{E}\left(-\mu_0\overline{X}_{t_k}\right) \le C\left(\frac{\Delta t}{x_0}\right)^{\frac{2b_\sigma(1/2)}{\sigma^2}\left(1 - \frac{1}{2\gamma}\right)}.$$

From where we see immediately

$$\sup_{k=0,\ldots,N}\mathbb{E}\exp\left(-\frac{\overline{X}_{t_k}}{\gamma\sigma^2\Delta t}\right) \le C\left(\frac{\Delta t}{x_0}\right)^{\frac{2b_\sigma(1/2)}{\sigma^2}\left(1 - \frac{1}{2\gamma}\right)}. \quad \square$$

References

[1] A. Alfonsi. On the discretization schemes for the cir (and bessel squared) processes. *Monte Carlo Methods and Applications*, 11:355–384, December 2005. 2, 13, 15

[2] A. Alfonsi. Strong order one convergence of a drift implicit euler scheme: Application to cir process. *Statistic and Probability Letters*, (83):602–607, 2013. 2, 14, 15

[3] A. Berkaoui, M. Bossy, and A. Diop. Euler sheme for SDEs with non-Lipschitz diffusion coefficient : strong convergence. *ESAIM Probability and Statistics*, 12:1–11, 2008. 2, 5, 13, 14

[4] A.N. Borodin and P. Salminen. *Handbook of Brownian Motion: Facts and Formulae*. Operator Theory, Advances and Applications. Birkhäuser, 2002. 21

[5] M. Bossy and A. Diop. Weak convergence analysis of the symmetrized Euler scheme for one dimensional SDEs with diffusion coefficient $|x|^a$, a in $[\frac{1}{2}, 1)$. *Preprint arXiv:1330617*. 2, 22, 31

[6] J.-F. Chassagneux, A. Jacquier, and I. Mihaylov. An explicit euler scheme with strong rate of convergence for financial sdes with non-lipschitz coefficients. *Preprint arXiv:1405.3561*. 2, 13, 14, 15

[7] F. Delbaen and H. Shirakawa. A note on option pricing for the constant elasticity of variance model. *Asia-Pacific Financial Markets*, 9(2):85–99, 2002. 1

[8] I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics. Springer, 2nd edition, 1991. 27

[9] P.-L. Lions and M. Musiela. Correlations and bounds for stochastic volatility models. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, 24(1):1 – 16, 2007. 1

[10] G.N. Milstein. Approximate integration of stochastic differential equations. *Theory of Probability & Its Applications*, 19:557–562, 1974. 1

[11] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. Grundlehren der mathematischen Wissenschaften. Springer-Verlag Berlin Heidelberg, 1st edition, 1991. 22

[12] D. Talay. Probabilistic numerical methods for partial differential equations: elements of analysis. In D. Talay and L. Tubaro, editors, *Probabilistic Models for Nonlinear Partial Differential Equations*, volume 1627 of *Lecture Notes in Mathematics*, pages 148–196. Springer-Verlag, CIME Lectures, 1996. 1

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