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FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
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DELAUNAY SOLUTIONS TO THE CAHN-HILLIARD EQUATION

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ÁLVARO ANDRÉS HERNÁNDEZ URIBE

PROFESOR GUÍA:
MICHAŁ KOWALCZYK

MIEMBROS DE LA COMISIÓN:
JUAN DÁVILA BONCZOS
DUVAN HENAO MANRIQUE
CLAUDIO MUÑOZ CERÓN

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POR: Álvaro Andrés Hernández Uribe
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PROF. GUÍA: Michał Kowalczyk

DELAUNAY SOLUTIONS TO THE CAHN-HILLIARD EQUATION

En esta tesis doctoral se construyen soluciones rotacionalmente simétricas de la ecuación de Cahn-Hilliard en $\mathbb{R}^d$ y se estudian sus propiedades de estabilidad.

En el Capítulo 1 se presenta la ecuación de Cahn-Hilliard y se explica su origen e interpretación física. Además se repasan varios resultados conocidos, se presenta la notación y se exponen los dos resultados más importantes de esta tesis: el primero establece la existencia de soluciones rotacionalmente simétricas cuyos conjuntos de nivel se aproximan a los unduloides de Delaunay. El segundo resultado afirma que las propiedades de estabilidad de los unduloides de Delanay heredan propiedades de estabilidad de las soluciones encontradas, en el sentido que son no degeneradas y tienen 6 campos de Jacobi con crecimiento moderado.

En el Capítulo 2 se presentan en detalle los principales ingredientes que se necesitan para probar los Teoremas 1.1 y 1.2, a saber las coordenadas de Fermi cerca de una superficie de curvatura media constante, los unduloides de Delaunay y su operador de Jacobi. También se muestra la primera aproximación de la solución anunciada en el Teorema 1.1.

En el Capítulo 3 se demuestra el Teorema 1.1. Usamos una versión refinada del método de reducción del Lyapunov-Schmidt que simplifica varios aspectos técnicos de construcciones de problemas similares. Los resultados de este capítulo fueron obtenidos en colaboración con mi Profesor Guía, Dr. Michał Kowalczyk y fueron publicados en la revista *Discrete and Continous Dynamical Systems* bajo el título *Rotationally Symmetric Solutions to the Cahn-Hillard Equation* [31].

Una demostración del Teorema 1.2 se da el Capítulo 4. La clave es relacionar el núcleo del operador linearizado alrededor de nuestra solución con los campos de Jacobi que provienen de invariancias geométricas. Esta relación se puede realizar debido a que es posible separar las variables una vez que se ha aplicado la transformada de Fourier-Laplace. Los resultados de este capítulo también fueron obtenidos con mi profesor Guía y han sido aceptados para su publicación en la revista *Indiana University Mathematics Journal* bajo el título *Nondegeneracy and the Jacobi Fields of Rotationally Symmetric Solutions to the Cahn-Hillard Equation* [32].

DELAUNAY SOLUTIONS TO THE CAHN-HILLIARD EQUATION

In this PhD thesis rotationally symmetric solutions to the Cahn-Hilliard equation are constructed. Also we study its stability properties.

In Chapter 1 we present the Cahn-Hilliard equation in $\mathbb{R}^d$ and explain its origin and physical interpretation. We also review several known results, introduce some basic notation and present the two main results of this thesis. The first one states the existence of radially symmetric solutions to the Cahn-Hilliard equation which nodal sets approaches to Delaunay unduloids, and the second one claims that stability properties of the Delaunay unduloids inherit stability properties of the solutions we found in the sense that our solutions are non degenerated and have 6 Jacobi fields with temperate growth.

Chapter 2 is devoted to present in detail the main ingredients we need to prove Theorem 1.1 and Theorem 1.2, namely Fermi coordinates near a constant mean curvature (CMC), the Delaunay unduloids and its Jacobi operator. We also present the construction of the first approximation of the solutions announced in Theorem 1.1.

In Chapter 3 we prove Theorem 1.1. We use a refined version of the Lyapunov-Schmidt reduction method which simplifies very technical aspects of previous constructions for similar problems. The results of this chapter were obtained in collaboration with my thesis advisor Dr. Michał Kowalczyk and published in *Discrete and Continuous Dynamical Systems* [31].

A proof of Theorem 1.2 is given in Chapter 4. The key is to relate the kernel of the linearized operator about our solution with the Jacobi fields that comes from the geometric invariances. This relation can be performed since we are able to separate the variables once the Laplace-Fourier transform is applied. The results of this chapter were obtained in collaboration with my thesis advisor Dr. Michał Kowalczyk and admitted for publication in *Indiana University Mathematics Journal* [32].

Dedicado a mi esposa Andrea y a mis padres Iris y Gabriel.

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Tabla de Contenido

Chapter 1

Introduction

The evolutionary Cahn-Hilliard equation

$$\begin{cases} u_t = -\Delta(\varepsilon^2 \Delta u - F'(u)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial}{\partial \nu}(\varepsilon^2 \Delta u - F'(u)) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where F is a *double-well potential*, is a model introduced in 1958 (see [10]) that describes the process of phase separation of two components of a binary alloy. Here $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a bounded domain that represents the region where the isolation of the components takes place, and ν, as usual, denotes the outer normal on $\partial\Omega$. The function u represents the concentration of one of the components and ε is the range of intermolecular forces. The double-well potential $F(u)$ corresponds to the free energy density at low temperatures, and in what follows we will take

$$F(u) = \frac{1}{4}\left(1 - u^2\right)^2, \quad f(u) := -F'(u) = u(1 - u^2).$$

Note that the constant functions $u = \pm 1$ are stable solutions of (1.1).

Equation (1.1) can be derived from the gradient flow of the Helmholtz free energy functional

$$E_\varepsilon(u) = \int_\Omega \left(F(u(x)) + \frac{1}{2}\varepsilon^2 |\nabla u(x)|^2 \right) dx, \tag{1.2}$$

for $u \in H^{-1}(\Omega)$, subject to the average concentration to be constant, i.e.

$$\frac{1}{|\Omega|} \int_\Omega u \, dx = m, \tag{1.3}$$

where $m \in [-1, 1]$ (see [27, 25, 26] for details). In this case the constant functions $u \equiv \pm 1$ are minimizers of this functional subject to the constraint $m = \pm 1$.

Stationary solutions of (1.1) satisfy the Euler-Lagrange equation

$$\begin{cases} \varepsilon^2 \Delta u + f(u) = \delta_\varepsilon & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \dfrac{1}{|\Omega|} \displaystyle\int_\Omega u\, dx = m, \end{cases} \tag{1.4}$$

where $\delta_\varepsilon \in \mathbb{R}$ is the Lagrange multiplier associated to the constraint (1.3).

Standard methods of Calculus of Variation can be applied in order to find minimizers (or more generally critical points) of (1.2), nevertheless understanding the behavior of these minimizers has been the goal of studies for recent years. Using Γ-convergence approach Modica [51] showed that global minimizers u_ε of (1.2) under the constraint (1.3) Γ-converge to the function $1 - 2\chi_{A_0}$ as $\varepsilon \to 0$, where χ_{A_0} is the characteristic function of an open set $A_0 \subset \Omega$. Moreover $\partial A_0 \cap \Omega$ is locally a surface of constant mean curvature (CMC surface for short). In [8, 9] Caffarelli and Córdoba shown that the nodal sets of the minimizers u_ε, that is $\{x : u_\varepsilon(x) = \lambda\}$, with $\lambda \in (-1, 1)$, converge uniformly over compacts to $\partial A_0 \cap \Omega$. As a matter of fact, the set A_0 minimizes of the following isoperimetric problem: minimize the perimeter functional $\mathrm{Per}_\Omega(A)$ (i.e. the Hausdorff measure $\mathcal{H}^n(\partial A)$) among the sets $A \subset \Omega$ whose volume is fixed. A generalisation of these results was given by Sternberg [59]. Furthermore Hutchinson and Tonegawa [34] studied limits of general critical points (1.2) and showed that their limits are locally minimal or CMC surfaces. On the other hand Kohn and Sternberg [39] proved that if a set $A \subset \Omega$ is an isolated minimizer of the perimeter functional subject to the constant volume constraint then there exists a sequence of minimizers u_ε of (1.2) which Γ-converges to A, this allowed to Chen and Kowalczyk [12] to obtain solutions of (1.4) at least in dimension 2. Some of these results were generalized in the following sense: let (M, g) be a compact Riemannian manifold with or without boundary. The corresponding problem is to study the behavior of the critical points of the functional

$$E_\varepsilon(u) = \int_M \left(F(u(x)) + \frac{1}{2}\varepsilon^2 |\nabla u(x)|_g^2 \right) d_g v, \tag{1.5}$$

subject to constraint

$$\frac{1}{|M|} \int_M u\, d_g v = m. \tag{1.6}$$

Under suitable non degeneracy assumptions (that we will explain in more detail in a while) Pacard and Ritoré in [55] found critical points of (1.5)-(1.6) that converges uniformly over compacts to the function $1 - 2\chi_N$ where $N \subset M$ is a CMC submanifold.

The counterpart of this theory for the time dependent problem (1.1), in particular the the dynamics of its transition layers solutions, has been the interest of several authors, for instance Alikakos, Bates and Chen [6] proved that as $\varepsilon \to 0$ the time evolution of interfaces is governed by the Helle-Shaw problem, where of course CMC surfaces are stationary points of the flow. More detailed description of the Cahn-Hilliard flow and key spectral tools can be found for instance in [3], [5], [4], [2], [11] and the references therein.

In the case $d = 1$ Grinfeld and Novick-Cohen [28, 29] completely determined the solutions of (1.4). On the other hand several examples of stationary solutions for the singular perturbation problem in a bounded domain in higher dimensions have been constructed. We refer to [64, 66, 63, 65, 62, 7]. In [64] Wei and Winter constructed boundary spike solutions, that is, if $m \in (\sqrt{3}/3, 1)$ and $p_0 \in \partial\Omega$ is a non degenerated critical point of the mean curvature, then there exists a solution u_ε of (1.4) such that $u_\varepsilon \to m$ as $\varepsilon \to 0$ for $x \in \bar{\Omega} \setminus \{p_0\}$. Moreover, u_ε has only one local minimum $p_\varepsilon \in \partial\Omega$, and $p_\varepsilon \to p_0$ as $\varepsilon \to 0$, and $u_\varepsilon(p_\varepsilon) \to \beta$ as $\varepsilon \to 0$, where β is some number $\beta \in (0, m)$. The same authors also proved existence of multi-boundary spike solutions, see [66]. In [63] Wei and Winter were able to find interior spike solutions under some geometric assumptions on the domain Ω, and in [65] the authors constructed multi-interior-spike solutions.

Another result related to the equation (1.4) was also obtained by Wei and Winter in [62], they constructed bubble solutions, namely, u_ε is a bubble solution if there exists a point $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, r)$, the ball of radius r centered at x_0, is contained in Ω and $u_\varepsilon \to 1 - 2\chi_{B(x_0,r)}$.

It is worth to pointing out that the so called Lyapunov-Schmidt reduction is the fundamental method used in most of these works, which is also the main tool we apply in our study. We also mention that this method can also be applied to the Allen-Cahn equation which is closely related to the Cahn-Hilliard equation as we proceed to explain briefly. The Allen-Cahn equation

$$\begin{cases} \varepsilon^2 \Delta u + f(u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ is related to the Cahn-Hilliard equation (1.4) in the sense that no mass conservation constraint (1.3) is assumed, this yields $\delta = 0$.

Scaling $x \mapsto x/\varepsilon$ in (1.4) (respectively in (1.7)) and letting $\varepsilon \to 0$ leads in a natural way to the following way to the following problem

$$\Delta u + f(u) = \delta \quad \text{in } \mathbb{R}^d. \tag{1.8}$$

Assuming $\delta = 0$ and $d = 1$ there exists an obvious solution of (1.8), namely the unique odd and monotonically increasing heteroclinic solution H of the ODE

$$\begin{gathered} H'' + f(H) = 0, \quad \text{in } \mathbb{R}, \\ H(\pm\infty) = \pm 1. \end{gathered} \tag{1.9}$$

We also notice that if $\mathtt{a} \in \mathbb{R}^d$ is a unit vector and $b \in \mathbb{R}$ then the function

$$u(x) = H(\mathtt{a} \cdot x + b), \quad x \in \mathbb{R}^d$$

is also a solution of (1.8) with $\delta = 0$. On the other hand, when $\delta \neq 0$ there exist radially symmetric solutions to (1.8) (see [56]). Note that in both cases the level sets of the solutions are CMC surfaces, in the former case they correspond to $(d-1)$-dimensional hyperplanes in

$\mathbb{R}^d$ which mean curvature is 0 and in the latter case they correspond to the $(d-1)$-dimensional spheres in $\mathbb{R}^d$ of radius R_0 which mean curvature is $(d-1)/R_0$.We also remark that radially symmetric solutions in $\mathbb{R}^{d-1}$ can be lifted trivially to $\mathbb{R}^d$ giving solutions whose nodal sets are cylinders, which again are CMC surfaces in this case the mean curvature is $(d-2)/R_0$. This kind of property can also be obtained for a larger class of surfaces, namely in [22] the authors using the Lyapunov-Schmidt reduction method found an entire solution to the Allen-Cahn equation in $\mathbb{R}^d$, whose nodal sets resembles a large dilation of a given complete embedded, non degenerate, total finite curvature and minimal surface M. The surface M is also assumed to have m unbounded components or *ends*, each of them away of a large ball $B(0, R)$ resembles either a plane or a catenoid. The existence of this type of surfaces is guaranteed according to the results of [58].

The aim of this work is to prove existence of entire solutions to

$$\Delta u + u(1-u^2) = \delta \quad \text{in } \mathbb{R}^d, \tag{1.10}$$

whose nodal sets are not cylinders or spheres, but rather more general surfaces. From now on we will assume that $d \geq 3$ and $\delta > 0$ is a small parameter.

Note that dilating of the independent variable by a (large) factor $\varepsilon^{-1} > 0$

$$\mathtt{x} \longmapsto \varepsilon^{-1}\mathtt{x},$$

we obtain the equivalent equation:

$$\varepsilon \Delta u + \frac{1}{\varepsilon} u(1-u^2) = \ell_\varepsilon \quad \text{in } \mathbb{R}^d, \tag{1.11}$$

where we have denoted $\delta/\varepsilon = \ell_\varepsilon$. Clearly, if u_ε is a solution of (1.11) then $v(\mathtt{x}) = u_\varepsilon(\varepsilon \mathtt{x})$ is a solution of (1.10). On the other hand, if v is a solution of (1.10) then $u_\varepsilon(\mathtt{x}) = v(\mathtt{x}/\varepsilon)$ is a solution of (1.11). In particular this means that while phase transition of the solutions of (1.10) are of order 1, for the solutions of (1.11) they are of order ε. Thus the latter are more "concentrated".

In the sequel we will focus on solving (1.11). From what we have said above about the singular perturbation problem it is clear that level sets of these solutions should converge, as ε tends to 0, to a CMC surfaces in $\mathbb{R}^d$. In fact we expect (on the basis of formal calculations in Section 2.2) that the Lagrange multiplier

$$\ell_\varepsilon = -\frac{1}{2} H_\Sigma \int_{\mathbb{R}} H'(s)^2 \, ds + \mathcal{O}(\varepsilon),$$

where Σ is the surface of the phase transition and H_Σ is its mean curvature. We introduce a family of embedded CMC surfaces good candidates to be the limiting surfaces. Let us consider the case $d = 3$ first. The Delaunay *unduloids* [23, 24, 45] are a one parameter family D_τ, $\tau \in (0, 1)$ of embedded, periodic CMC surfaces of revolution. When the real parameter τ tends to 1^- the surfaces D_τ approach to the straight cylinder while when $\tau \to 0^+$ they become an array of identical spheres arranged along the x_3 axis.

It turns out that Delaunay surfaces can be constructed in any dimension $d > 3$ that will be denoted by D_τ, $\tau \in (0, \tau_*)$, we note that the parameter τ_* is given by:

$$\tau_* = \frac{(d-2)^{(d-2)/(d-1)}}{d-1}.$$

Again, in the limit $\tau \to \tau_*^-$ the surfaces D_τ approach the straight cylinder with x_d as symmetry axis, and when $\tau \to 0^+$, the surfaces D_τ resemble a array of spheres arranged along the x_d axis. It is convenient to "normalize" the Delaunay surface and suppose that the mean curvature of D_τ is 1 for all $\tau \in (0, \tau_*)$. We will also denote by N_τ the vector field normal to D_τ. Let us notice that the surface D_τ divides the space into two disjoint components $\Omega_\tau^\pm$, such that $\mathbb{R}^d \setminus D_\tau = \Omega_\tau^+ \cup \Omega_\tau^-$, where N_τ points towards Ω_τ^+ By changing the orientation of D_τ if necessary we can chose N_τ in such a way that Ω_τ^+ contains the x_d axis.

Our first result is

Theorem 1.1 For all $\tau \in (0, \tau_*)$ when $d = 3$ and with a possible exception of a finite set of τ when $d > 3$, there exists a sufficiently small $\varepsilon_\tau > 0$ such that for all $\varepsilon \in (0, \varepsilon_\tau)$ the problem

$$\varepsilon \Delta u + \frac{1}{\varepsilon}(u - u^3) = \ell_\varepsilon \quad \text{in } \mathbb{R}^d \tag{1.12}$$

has a solution $u_{\tau,\varepsilon}$ which is one-periodic along the x_d-axis and rotationally symmetric with respect to rotations about the same axis. Also it holds that $\ell_\varepsilon = 1 + \mathcal{O}(1)$ as $\varepsilon \to 0$ and $u_{\tau,\varepsilon}$ satisfies

$$u_{\tau,\varepsilon} \to 1 \text{ as } \varepsilon \to 0 \text{ in } \Omega_\tau^+$$
$$u_{\tau,\varepsilon} \to -1 \text{ as } \varepsilon \to 0 \text{ in } \Omega_\tau^-$$

uniformly over compacts.

Remark 1.1. We took $f(u) = u - u^3$, which is the standard nonlinearity for the Cahn-Hilliard equation. Theorem 1.1 holds for more general nonlinearities of bistable, balanced type, namely $f \in C^3$ such that $f(u) = -F'(u)$ where F is a double well, even potential with non degenerate wells at ± 1. Rather straightforward modifications required in the proof of the more general setting are easy to get.

Remark 1.2. In the statement of the Theorem 1.1 we assume that $\tau \neq 0, \tau_*$. In fact solutions for these extreme values of the Delaunay parameter are known: when $\tau = 0$ they are simply the radially symmetric solutions in $\mathbb{R}^d$ and when $\tau = \tau_*$ they are radial symmetric solutions in $\mathbb{R}^{d-1}$ lifted to $\mathbb{R}^d$. However our construction does not cover the boundary values of τ. On the one hand it has to do with the difficulty of finding an approximate solution which will give uniformly small error when $\tau \to 0$ and on the other hand with the extra degeneracy of the linearized operator when $\tau \to \tau_*$. The latter case could be possibly dealt with within our construction but we have not purse this since this would not give any new result.

Remark 1.3. Solutions we construct here are rotationally symmetric, periodic with period T_τ, the same period of D_τ, and also symmetric with respect to the hyperplane $x_d = T_\tau/2$. This could be used to show existence of solutions to the Cahn-Hilliard equation obeying these symmetries using for instance variational methods in the spirit similar to [14] or [18]. However, without further analysis it is not immediately clear how to make sure that the zero level set of such a solution would be a Delaunay surface for ε sufficiently small. This is important if one wants to use them as a basis of a connected sum construction of solutions of the Cahn-Hilliard equation whose zero level sets is a non degenerate, non compact constant mean surface with k Delaunay ends.

Before we present our second result, we recall that the Jacobi operator on an embedded surface Σ is defined as the linearization of the mean curvature with respect to normal perturbations and its formula is

$$\mathcal{J}_\Sigma \phi = \Delta_\Sigma \phi + |A_\Sigma|^2 \phi,$$

here Δ_Σ is the Laplace-Beltrami operator on Σ and A_Σ its second fundamental form. The elements of the kernel of $\mathcal{J}_\Sigma$, also called *Jacobi fields*, correspond to variations of the surface which preserve the mean curvature to the second order. In [45] the authors give a complete structure of the kernel of $\mathcal{J}_\Sigma$ when $\Sigma = D_\tau$ (this structure is reviewed in more detail in Section 2.4). In particular they realize that *all* elements of the kernel of $\mathcal{J}_{D_\tau}$ with temperate grow in the direction of the axis of rotation is a finite dimensional family that come from global deformations which arise from explicit geometric motions, such as translations, rotations and variance of the Delaunay parameter τ, that are called *geometric Jacobi Fields*. It is also remarkable that non of these elements are in $L^2(D_\tau)$, but they are periodic (hence bounded) or at most linearly growing in the along direction of D_τ. The main tool used to completely describe the structure of the kernel of $\mathcal{J}_{D_\tau}$ is the possibility of separate the variables, which leads to the study of the normal modes of an ODE.

The existence result by Pacard and Ritoré in [55] described earlier relies on the assumption that the Jacobi operator associated to the minimal $(n-1)$-dimensional submanifold N of a compact closed n-dimensional manifold M which in this case reads $J_N = \Delta_N + |A_N|^2 + \mathrm{Ric}_g(\nu_N, \nu_N)$ (here Δ_N is the Laplace-Beltrami operator on N, $|A_N|^2$ is the norm of the second fundamental form, Ric_g is the Ricci tensor on M and ν_N is the normal vector to N) is nondegenerate in $L^2(N)$, that is there are no nontrivial solution $\phi \in L^2(N)$ of $J_N \phi = 0$, or equivalently it is injective in $L^2(N)$. In particular it follows that N divides M into two disjoint components $M^\pm(N)$, similarly as D_τ divides $\mathbb{R}^3$.

The assumption of non degeneracy is key in other works, for instance in [19] del Pino et. al. study the structure space of entire solutions to the Allen-Cahn equation

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^2.$$

which, at infinity are asymptotic, modulo the action of some rigid motion, to $2k$ copies of $\pm H$ (here H is the heteroclinic solution H (see (1.9)) whose nodal sets are, away from a compact set, asymptotic at infinity to $2k$-oriented half affine lines. They show, under some non degeneracy assumption, that this space is a smooth manifold whose dimension is equal to $2k$. The non degeneracy they need is that the linearized operator of the Allen-Cahn equation about one of the solutions described above, that is $\Delta + f'(u)$, defined on the space of functions that decays or grow exponentially along its nodal set and decays exponentially in the transversal direction of its nodal set, is said to be non degenerated if it is injective, or equivalently there are no nontrivial solutions of $\Delta\phi + f'(u)\phi = 0$ in $L^2(\mathbb{R}^2)$.

Our existence result in Theorem 1.1 is also based on the non degeneracy of the Delaunay surfaces, which in this case means that their Jacobi operator does not have kernel in $L^2(\mathbb{R}^2)$, and moreover it uses the fact the the Jacobi fields of these surfaces can be classified.

The second result of this Thesis is related to this type of nondegeneracy, but now of the linerized operator that comes from the equation (1.12) and the solution we have found in Theorem 1.1. Assuming $d = 3$ and according to Theorem 1.1, the solution $u_{\tau,\varepsilon}$ (what we now name simply by $w_\tau = w_\tau(\mathtt{x})$, $\mathtt{x} = (x_1, x_2, z)$) is rotationally symmetric about the z axis and z-periodic. This means that if we consider a vector $\mathtt{h} = \sum_{i=1}^{3} h_i \mathtt{e}_i$, where $\mathtt{e}_i$ are the canonical vectors and a rotation $\mathcal{R}_\vartheta(\mathtt{x}) = \mathcal{R}_{\theta_1,\theta_2}(\mathtt{x})$, where θ_i is the angle of rotation about the x_i axis, and a number η such that $|\eta|$ is small, then the function

$$\Phi_{\mathtt{h},\vartheta,\eta}(w_\tau) := (w_{\tau+\eta} \circ \mathcal{R}_\vartheta)(\mathtt{x} + \mathtt{h})$$

is also a solution. It follows that it is natural to study the linearized operator about w_τ:

$$L_{w_\tau}\phi := \varepsilon\Delta\phi + \frac{1}{\varepsilon} f'(w_\tau)\phi.$$

Keeping this in mind we say that ϕ is a *Jacobi field* of L_{w_τ} if it is a smooth null element.

The 5 symmetries described above by $\Phi_{\mathtt{h},\vartheta,\eta}(w_\tau)$ and $\partial_\tau w_\tau$ determine 6 null elements of L_{w_τ}. As we will see in Lemma 4.2.1 the behavior of these null elements is governed by Jacobi fields of $\mathcal{J}_{D_\tau}$ and the first approximation of w_τ in a sort of "separated variable" version. In particular we obtain that the former are actually Jacobi fields of L_{w_τ}, that we refer as *geometric* Jacobi Fields. Thus, in analogy with the description of the kernel of $\mathcal{J}_{D_\tau}$, and the nondegeneracy property required in similar operators, it is natural to ask whether *all* Jacobi fields of L_{w_τ} are the geometric Jacobi fields. The answer is provided by our second main result:

Theorem 1.2 For all $\tau \in (0, 1)$ and all small ε the operator L_{w_τ} is nondegenerate in the sense that $H^2(\mathbb{R}^3) \cap \ker L_{w_\tau} = \emptyset$. Moreover, there exists $a > 0$ such that the linear subspace of $H^2_{loc}(\mathbb{R}^3)$ solutions of

$$L_{w_\tau}\phi = 0,$$

with temperate growth in the direction of the axis of rotation of w_τ i.e. such that

$$\|\phi \cosh^{-a} z\|_{L^2(\mathbb{R}^3)} < \infty, \tag{1.13}$$

has dimension 6 and coincides with the linear subspace of the geometric Jacobi fields.

Note however that Theorem 1.2 does not exclude the possibility of existence of a solution of $L_{w_\tau}\phi = 0$ such that ϕ satisfies (1.13) with some large value of a.

Theorem 1.2 provides a classification of the Jacobi fields of the family w_τ of rotationally symmetric solutions of (1.11) which is key for problems that can be considered for future work: the construction of more complicated solutions build upon more complicated CMC surfaces in $\mathbb{R}^3$. It is clear that this study depends on the invertibility theory of L_{w_τ} and on the precise knowledge of the Jacobi fields we have described, (see [36], [37] ,[46] , [45], [43], [35], [34] for examples of such constructions in the geometry of CMC surfaces) and the already mentioned work [19] (see also [41] in the context of the Allen-Cahn equation on the plane).

To explain this let us recall that a non compact, Alexandrov embedded, complete CMC surface with finite topology outside of a compact set consists of finitely many half Delaunay surfaces ([50], [49], [52]) called Delaunay ends. In addition if the number of ends of such surface is k and this surface is non degenerate then set of nearby CMC surfaces is an analytic manifold of dimension $3k$. This was proven by Kusner, Mazzeo and Pollack in [42] and the argument of their paper is in many ways inspired by the similar result for the singular Yamabe problem [48]. One of the problems is to decide whether a given CMC surface is non degenerate and this is rather difficult problem except for the Delaunay surface for which separation of variables and ODE methods can be used to prove non degeneracy (see also [40]). Pushing these arguments further one can also classify Jacobi fields with temperate growth [47] and show that all of them came from the natural invariances of the family of Delaunay surfaces. Starting from non degenerate Delaunay surface with k ends one can built more complicated examples by gluing to it either an extra end or another non degenerate surface and thus obtain CMC surfaces with arbitrary many ends. In some cases these new surfaces are also nondegenerate, see for instance [47], [46], [36], [35].

Theorem 1.2 is the precise analog of the result proven in [47] but in the case of the Cahn-Hilliard equation. Given what we said about the linear properties of the Delaunay surface its assertion is expected, which does not mean that the proof is equally obvious. Certainly what needs to be done is to connect the stability properties of the Delaunay surface D_τ and the corresponding solution w_τ of (1.12) and this can be achieved by expressing w_τ in the Fermi coordinates of D_τ (Section 2.1). While w_τ is localized near D_τ this kind of expression is only valid in a neighbourhood of the surface and this is what complicates the situation (see Section 4.1). In order to deal with this in this paper we replace the operator L_{w_τ} with another operator $\mathbb{L}_{w_\tau}$ (Section 4.2), which locally agrees with the original one but which is easier to analyze.

In the rest of this Introduction we explain explain briefly the tools, the schemes and differences with related works of the proof of Theorem 1.1 and Theorem 1.2. Let us begin with Theorem 1.1. One of the main tools we use is a variant of the Lyapunov-Schmidt reduction method. Let us first discuss the differences between our approach and the older implementations which can be found in [55] and [15], [16]. To set ideas we recall the standard Lyapunov-Schmidt reduction method in its abstract version (see [13]). Given Banach spaces X, Y and a linear operator $A\colon X \to Z$ and a continuous, nonlinear operator $N\colon X \to Z$, we are to solve the problem:

$$Ax - N(x) = 0. \tag{1.14}$$

Let

$$\mathcal{N}(A) = Y \subset X, \qquad \mathcal{R}(A) = W \subset Z,$$

and let π_Y, π_W be the projections on the corresponding subspaces. There exists a bounded linear operator $K\colon W \to \mathcal{R}(I - \pi_Y)$ (the right inverse of A) such that $AK = I$ on W and $KA = I - \pi_Y$, and moreover the equation (1.14) is equivalent to the equation

$$\begin{aligned} x = y + z, \qquad y \in Y, \quad z \in \mathcal{R}(I - \pi_Y)\\ z - K\pi_W N(y+z) = 0,\\ (I - \pi_W)N(y+z) = 0. \end{aligned} \tag{1.15}$$

In applications the Lyapunov-Schmidt method consists of reducing (1.14) to (1.15), solving the first equation for z with y given (which usually can be done by a fixed point argument) and replacing this solution in the second equation to obtain *the reduced problem*

$$(I-\pi_W)N(y+z(y))=0.$$

In practice several complications may arise and we will illustrate this considering a related to our problem, which was treated by Pacard and Ritoré [55], and in many aspects it is similar to the problem we consider here. Let M be a compact, closed manifold of dimension n and $N\subset M$ a minimal $n-1$ dimensional sub manifold which divides M into two disjoint components. Consider the problem

$$\varepsilon^2\Delta_M u+u(1-u^2)=0 \qquad \text{on } M. \tag{1.16}$$

We say that N is non degenerate if the Jacobi operator of N

$$J_N=\Delta_N+|A_N|^2+\mathrm{Ric}_g\left(\nu_N,\nu_N\right)$$

has empty kernel. As we already said earlier the result they proved is: given a non degenerate, minimal sub manifold N of M for each sufficiently small ε there exists a solution u_ε of (1.16) such that the zero level set of u_ε approaches N as $\varepsilon\to 0$. Moreover, u_ε converges to ± 1 uniformly over compacts of the two disjoint components of $M\setminus N$. Let us explain now their implementation of the Lyapunov-Schmidt reduction. It is expected that for $x\in M$ near N we should have $u_\varepsilon(x)=H(\varepsilon^{-1}\mathrm{dist}\,(x,N))+\varphi$, where $\mathrm{dist}\,(\cdot,N)$ is the signed distance function on M, H is the unique odd, monotonically increasing solution of $-H''=H(1-H^2)$ in $\mathbb{R}$ and φ is a small perturbation. The problem to solve for φ amounts to inverting the linearized operator around $H(\varepsilon^{-1}\mathrm{dist}\,(x,N))$ which has form

$$L=\Delta_M+f'(H(\varepsilon^{-1}\mathrm{dist}\,(\cdot,N))).$$

It is known that the norm of L^{-1} is large due to local translational invariance of the problem. Thus we need to perturb N as well. To describe this perturbation we consider a manifold N_h to be a normal graph over N described by a smooth and small function $h\colon N\to\mathbb{R}$. Furthermore we let $t_h(x)=\mathrm{dist}\,(x,N_h)$ to be the signed distance from N_h. Then we look for a solution of the form

$$u=H\left(\frac{t_h}{\varepsilon}\right)+\varphi.$$

Now both h and φ are unknowns. The problem to solve for φ is

$$L_h\varphi=\mathcal{F}(h,\varphi),$$

where L_h is the linearized operator around $H\left(\frac{t_h}{\varepsilon}\right)$. The Lyapunov-Schmidt reduction strategy amounts to projection of the above equation onto the function $H'\left(\frac{t_h}{\varepsilon}\right)$ and its complement, denote this last projection by π_h. This leads to a problem for φ

$$\pi_h L_h\varphi=\pi_h\mathcal{F}(h,\varphi),$$

which we solve first for a given h, and the problem for h

$$J_{N_h}h=\mathcal{G}(h), \tag{1.17}$$

which we solve next (J_{N_h} is the Jacobi operator of N_h). Let us discuss (1.17). We notice that the expression of J_{N_h} in local coordinates will depend in general on h and its derivatives up to order 3, while the Jacobi operator is itself only a second order operator. This loss of regularity was dealt with in [55] using a regularisation procedure. In a series of papers [15], [16], [17] del Pino, Kowalczyk and Wei introduced a slightly different approach to circumvent this problem. It amounts to considering perturbation in the normal direction of the *fixed* manifold N so that $u = H\left(\frac{t+h}{\varepsilon}\right) + \cdots$, where now t is the signed distance from N and h is a smooth, unknown function defined on N. Equation (1.17) takes form

$$J_N h = \mathcal{G}(h),$$

and the problem of the loss of regularity is thus avoided. The problem is now reduced to finding a fixed point of $J_N \circ \mathcal{G}(h)$, using for example Banach fixed point theorem. To do this we need to know that $\mathcal{G}$ is at least Lipschitz in h. In both implementations of the Lyapunov-Schmidt reduction described above this is rather complicated technical point since $\mathcal{G}$ depends in a non explicit, non local and non linear way on h. This is mainly due to the fact that the linearized operator L_h still depends on h through the potential $f'(H(\frac{t+h}{\varepsilon}))$. Thus difficulty is to some extend circumvented in [53] where the presentation of the Lyapunov-Schmidt reduction is state of the art. In this approach modifying the nonlinear problem by composing it (twice) with a carefully chosen diffeomorphism (and its inverse) both the loss of regularity and the nonlocal dependence on the perturbation h are avoided, in fact h appears only algebraically in the problem.

We propose still another modification to the method. The idea is simple: instead of working with an approximation of the form $u = H\left(\frac{t+h}{\varepsilon}\right) + \cdots$ with h unknown we will improve the initial approximation to $w(t, y) = H\left(\frac{t}{\varepsilon}\right) + \cdots$, t being the signed distance to N and $y \in N$ in such a way that we do not need to "move" N anymore. In other words h will be determined with some sufficient precision before setting up the Lyapunov-Schmidt reduction, which with this modification will look like the abstract setting described at the beginning. This way we avoid both the loss of regularity and technical difficulties due to complicated character of the nonlinear function $\mathcal{G}(h)$. This is described in detail in Chapter 2.

Regarding Theorem 1.2, we start setting the asymptotic behavior of the geometric Jacobi fields of L_{w_τ}, in particular, we realize that near the surface D_τ and asymptotically as $\varepsilon \to 0$ they are proportional to $H'(\varepsilon^{-1}\text{dist}(\cdot, D_\tau))$ a Jacobi field of $\mathcal{J}_{D_\tau}$. Setting the variable

$$\mathtt{t} = \frac{1}{\varepsilon}\text{dist}(\cdot, D_\tau)$$

we notice that, again near D_τ, the operator L_{w_τ} resembles

$$\mathcal{L} = \varepsilon^{-1}(\partial_{\mathtt{t}}^2 + f'(H)) + \varepsilon \mathcal{J}_{D_\tau},$$

whose kernel is fairly easy to determine using separation of variables, indeed taking $u = H'(\mathtt{t})\psi$, we have that

$$\mathcal{L}(H'\psi) = \varepsilon H' \mathcal{J}_{D_\tau} \psi,$$

thus Jacobi fields of $\mathcal{J}_{D_\tau}$ determine the Jacobi fields of $\mathcal{L}$ and as we can see they have, at least near D_τ, the same structure of the geometric Jacobi Fields. One of the main difficulties we have to face is that above formulas are valid near D_τ and written in local coordinates, but our problem is established in global coordinates. Our research is inspired in the work of del Pino et. al. [19] where the authors look for null elements of the linearization of the

Allen-Cahn equation about a non degenerated solution in the plane. They were able to apply Fourier transform and the problem is reduced to the study of the Allen-Cahn equation in $\mathbb{R}$ whose structure it is well known. In our context we have that the operator is z periodic, thus in order to separate the variables it is more appropriated to apply the Fourier-Laplace transform, which lead us to the study of a complex parameter operator.

Chapter 2

Preliminaries

In this chapter we introduce definitions and recall in detail some results needed for our work. In specific we first exhibit the Fermi coordinates near a CMC surface, then we introduce the first approximation for the solution to equation (1.12). Thereafter we review the construction of Delaunay unduloids and finally recall the properties of the Jacobi operator on a Delaunay surface.

2.1. Fermi coordinates near any CMC surface

Let Σ in $\mathbb{R}^d$ be a CMC surface with mean curvature H_Σ and denote by N its unit outer normal. We will assume that there exists a tubular neighborhood $\mathcal{N}_\delta$ of Σ of width 2δ in which we can introduce the Fermi system of coordinates $(y,t) \in \Sigma \times (-\delta,\delta)$ by setting

$$\begin{aligned} Y\colon \mathcal{N}_\delta &\longrightarrow \Sigma \times (-\delta,\delta), \\ y + tN(y) = \mathtt{x} &\longmapsto (y,t). \end{aligned}$$

This map is in fact a diffeomorphism from $\mathcal{N}_\delta$ to $\Sigma \times (-\delta,\delta)$ whenever δ is taken sufficiently small. In the sequel we will use the inverse of this map

$$\begin{aligned} Y^{-1}\colon \Sigma \times (-\delta,\delta) &\longrightarrow \mathcal{N}_\delta, \\ (y,t) &\longmapsto \mathtt{x}. \end{aligned}$$

which allows to define the pullback Y^*w to $\Sigma \times (-\delta,\delta)$ of any real function $w\colon \mathcal{N}_\delta \to \mathbb{R}$, namely

$$Y^*w(y,t) = w \circ Y^{-1}(y,t).$$

For technical reasons we will chose later the size of the tubular neighborhood δ depending on ε but for now on we just take δ small.

We will now derive formulas expressing the Laplace operator Δ in $\mathbb{R}^d$ in terms of the Fermi coordinates $(y,t) \in \Sigma \times (-\delta,\delta)$. We define for each $t \in (-\delta,\delta)$ the normal surface

$$\Sigma_t = \{\mathtt{x} \in \mathcal{N}_\delta \mid \operatorname{dist}(\Sigma,\mathtt{x}) = t\}.$$

In other words Σ_t is the surface obtained from Σ by translation in the direction of the normal by t. Then the well known formula gives

$$\Delta = \Delta_{\Sigma_t} + \partial_t^2 - H_{\Sigma_t}\partial_t, \tag{2.1}$$

where Δ_{Σ_t} denotes the Laplace-Beltrami operator on Σ_t and H_{Σ_t} is the mean curvature of Σ_t. We need to expand these operators in terms of the variable t. By g and g_t, respectively, we will denote the metric on Σ, Σ_t (induced from $\mathbb{R}^d$). Let us fix a point on Σ and some local parametrisation $X(u)$, $u \in \mathcal{U} \subset \mathbb{R}^{d-1}$ of Σ in a neighbourhood of this point (X could be the isothermal coordinates but any parametrisation will do). In terms of these local coordinates we get the following relation

$$g_{t,ij} = g_{ij} + ta_{ij} + t^2 b_{ij},$$

where

$$\begin{aligned} g_{ij} &= (\partial_{u_j} X \cdot \partial_{u_i} X), \quad a_{ij} = (\partial_{u_j} X \cdot \partial_{u_i} N) + (\partial_{u_i} X \cdot \partial_{u_j} N), \\ b_{ij} &= (\partial_{u_i} N \cdot \partial_{u_j} N). \end{aligned}$$

Then, for the matrix $g_t^{-1} = (g_t^{ij})_{i,j=1,\dots,d-1}$ we get, provided that $|t|$ is sufficiently small

$$g_t^{-1} = g^{-1} + tM_1 + t^2 M_2,$$

where

$$M_1 = M_1(u) \quad \text{and} \quad M_2 = M_2(u,t)$$

are smooth matrix functions. The expression for the Laplace-Beltrami operator on Σ in local coordinates is

$$\begin{aligned} \Delta_\Sigma &= \frac{1}{\sqrt{\det(g)}} \partial_{u_j} \left(\sqrt{\det(g)} g^{ij} \partial_{u_i} \right) \\ &= g^{ij} \partial_{u_i u_j} + \frac{1}{\sqrt{\det(g)}} \partial_{u_j} \left(\sqrt{\det(g)} g^{ij} \right) \partial_{u_i} \\ &= g^{ij} \partial_{u_i u_j} - g^{k\ell} \Gamma^i_{k\ell} \partial_{u_i}, \end{aligned}$$

where $\Gamma^i_{k\ell}$ are the Christoffel symbols. A similar formula holds for Σ_t. Using this we can write

$$\Delta_{\Sigma_t} = \Delta_\Sigma + c_{ij} \partial_{u_i u_j} + d_i \partial_{u_i},$$

where

$$\begin{aligned} c_{ij} &= g_t^{ij} - g^{ij}, \\ d_i &= g_t^{k\ell} \left(\Gamma^i_{t,k\ell} - \Gamma^i_{k\ell} \right) + \Gamma^i_{k\ell} \left(g_t^{k\ell} - g^{k\ell} \right). \end{aligned} \tag{2.2}$$

Expressions in local coordinates for c_{ij}, d_i can be further derived using the above expansions, however their exact form is not crucial here. The point is that, formally, these functions are small in terms of $|t|$

$$|c_{ij}(u,t)| + |d_i(u,t)| \le C|t|. \tag{2.3}$$

With a choice of local coordinates on Σ the constant in the above estimate does not depend on the point on Σ.

Next, we will expand the mean curvature H_{Σ_t}. To this end by $\Bbbk_j$, $j = 1, \ldots, d-1$ we will denote the principal curvatures of Σ. Then we have

$$\begin{aligned} H_{\Sigma_t} &= \sum_{j=1}^{d-1} \frac{\Bbbk_j}{1 - t\Bbbk_j} \\ &= \sum_{j=1}^{d-1} \Bbbk_j + t \sum_{j=1}^{d-1} \Bbbk_j^2 + \mathbb{Q}_{\Sigma_t} \\ &= H_\Sigma + t|A_\Sigma|^2 + \mathbb{Q}_{\Sigma_t}, \end{aligned}$$

where

$$\mathbb{Q}_{\Sigma_t}(y,t) = t^2 \sum_{j=1}^{d-1} \Bbbk_j^3 + t^3 \sum_{j=1}^{d-1} \Bbbk_j^4 + \ldots$$

and $|A_\Sigma|$ is the norm of the second fundamental form on Σ. Summarizing all this using (2.1) we can express the Laplace operator in Fermi coordinates as follows

$$\Delta = \partial_{tt} + \Delta_\Sigma - \left(H_\Sigma + t|A_\Sigma|^2 + \mathbb{Q}_{\Sigma_t}\right) \partial_t + \mathbb{A}_{\Sigma_t}, \tag{2.4}$$

where $\mathbb{A}_{\Sigma_t}$ is a differential operator whose expression in local coordinates are given in (2.2) and satisfy (2.3).

Next we introduce stretched Fermi coordinates

$$\mathtt{t} = \frac{t}{\varepsilon}, \quad \mathtt{y} = y.$$

As before we have a diffeomorphism Y_ε and its inverse $Y_\varepsilon^{-1} \colon \Sigma \times \left(-\frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon}\right) \to \mathcal{N}_\delta$, and for any function $w \colon \mathcal{N}_\delta \to \mathbb{R}$ we define its pullback by Y_ε by

$$Y_\varepsilon^* w(\mathtt{y}, \mathtt{t}) = w \circ Y_\varepsilon^{-1}(\mathtt{y}, \mathtt{t}).$$

Taking into account formula (2.4) we get

$$\Delta = \varepsilon^{-2}\partial_{\mathtt{tt}} - \varepsilon^{-1}(H_\Sigma + \varepsilon\mathtt{t}|A_\Sigma|^2 + \mathbb{Q}_\varepsilon)\partial_{\mathtt{t}} + \Delta_\Sigma + \mathbb{A}_\varepsilon, \tag{2.5}$$

where

$$\mathbb{Q}_\varepsilon(\mathtt{y}, \mathtt{t}) = \mathbb{Q}_{\Sigma_z}(\mathtt{y}, \varepsilon\mathtt{t}), \qquad \mathbb{A}_\varepsilon(\mathtt{y}, \mathtt{t}) = \mathbb{A}_{\Sigma_z}(\mathtt{y}, \varepsilon\mathtt{t}).$$

We now define *shifted* Fermi coordinates. To do this we let $h \colon \Sigma \to \mathbb{R}$ be a given smooth function such that the map

$$\mathtt{x} \longmapsto (y,t), \qquad \text{where } \mathtt{x} = y + (t + h(y))\, N(y),$$

is a diffeomorphism from $\mathcal{N}_\delta$ into $\Sigma \times (-\delta, \delta)$. We denote this map by Y_h and by Y_h^{-1} we denote its inverse, finally by $Y_h^* w$ we will denote the pullback of $w \colon \mathcal{N}_\delta \to \mathbb{R}$ by Y_h

$$Y_h^* w(y,t) = w \circ Y_h^{-1}(y,t).$$

Now we compute the expressions for the Laplacian in shifted Fermi coordinates. To derive it, we denote $t\mathbb{B}_{\Sigma,t} = \Delta_{\Sigma_t} - \Delta_\Sigma$. The operator $\mathbb{B}_{\Sigma,t}$ is a second order differential operator. From this it is easy to obtain a formula for the Laplacian in the shifted Fermi coordinates

$$\Delta = \Delta_\Sigma + \partial_t^2 - (H_\Sigma + \Delta_\Sigma h + (t+h)|A_\Sigma|^2)\partial_t + (t+h)\mathbb{B}_{\Sigma,t+h} + (t+h)^2\mathbb{Q}_{\Sigma,t+h}. \tag{2.6}$$

Finally we combine both stretched and shifted Fermi coordinates: by $Y_{\varepsilon,h}$ we denote a diffeomorphism and its inverse

$$Y_{\varepsilon,h}^{-1}\colon \Sigma \times \left(-\frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon}\right) \longrightarrow \mathcal{N}_\delta,$$

and for any function $w\colon \mathcal{N}_\delta \to \mathbb{R}$ we define its pullback by $Y_{\varepsilon,h}$ by

$$Y_{\varepsilon,h}^* w(\mathtt{y},\mathtt{t}) = w \circ Y_{\varepsilon,h}^{-1}(\mathtt{y},\mathtt{t}).$$

Taking onto account formula (2.6) we get

$$\Delta = \Delta_\Sigma + \varepsilon^{-2}\partial_{\mathtt{t}}^2 - \varepsilon^{-1}(H_\Sigma + \Delta_\Sigma h + (\varepsilon\mathtt{t}+h)|A_\Sigma|^2)\partial_{\mathtt{t}} \\ + (\varepsilon\mathtt{t}+h)\mathbb{B}_{\Sigma,\varepsilon\mathtt{t}+h} + (\varepsilon\mathtt{t}+h)^2\mathbb{Q}_{\Sigma,\varepsilon\mathtt{t}+h}. \tag{2.7}$$

2.2. Formal approximation of the solution concentrating on any CMC surface Σ

Before continuing with our work, let us remark that all the formulas described in the previous Section are valid in the tubular neighbourhood $\mathcal{N}_\delta$. In Chapter 3 we study equation 1.12 in the whole space, so we have to split the space into two components. We proceed as follows: given *any* CMC surface Σ we introduce, by formal means, an approximate solution w that depends on the stretched and shifted Fermi coordinates $(\mathtt{y},\mathtt{t})$, that is, is defined only near Σ. The properties we need in order to solve our problem comes from when we choose a specific CMC surfaces: in Chapter 3 we impose $\Sigma = D_\tau$ the $d-1$ dimensional Delaunay unduloid The problem we have to deal with is that w is not enough, since w is defined only near the corresponding CMC surface, nevertheless w can be modified to improve the approximation.

We introduce the first approximate solution in the following way

$$Y_{\varepsilon,h}^* w(\mathtt{y},\mathtt{t}) = U(\mathtt{t}) + \varepsilon^2\psi_0(\mathtt{y},\mathtt{t}), \tag{2.8}$$

for some functions U and ψ_0 which we will determine. Moreover, we will assume that the function h in (2.7) has the form $h = \varepsilon^2 h_0$, where h_0 is a constant to be chosen.

The expression for the Laplacian in local coordinates indicates the form that U and ψ_0 should take, in that sense we have that the error $N_\varepsilon(w) - \ell_\varepsilon := \varepsilon\Delta w + \frac{1}{\varepsilon}w(1-w^2) - \ell_\varepsilon$, can be written as

$$
\begin{aligned}
N_\varepsilon(w) - \ell_\varepsilon = \varepsilon\big\{ & \varepsilon^{-2}\partial_{\mathtt{t}}^2 U - \varepsilon^{-1} H_\Sigma \partial_{\mathtt{t}} U + \varepsilon^{-2} f(U) - \varepsilon^{-1}\ell_\varepsilon \\
& + \partial_{\mathtt{t}}^2\psi_0 - \varepsilon H_\Sigma \partial_{\mathtt{t}}\psi_0 + f'(U)\psi_0 - (\mathtt{t} + \varepsilon h_0)|A_\Sigma|^2 \partial_{\mathtt{t}} U \\
& + \varepsilon^2 \Delta_\Sigma \psi_0 - \varepsilon^2(\mathtt{t} + \varepsilon h_0)|A_\Sigma|^2 \partial_{\mathtt{t}}\psi_0 \\
& + \left[(\varepsilon\mathtt{t} + h)\mathbb{B}_{\Sigma,\varepsilon\mathtt{t}+h} + (\varepsilon\mathtt{t}+h)^2 \mathbb{Q}_{\Sigma,\varepsilon\mathtt{t}+h}\right](U + \varepsilon^2\psi_0)\big\} \\
& + \frac{1}{\varepsilon} f(w) - \frac{1}{\varepsilon} f(U) - \varepsilon f'(U)\psi_0.
\end{aligned}
\tag{2.9}
$$

In order to get as small as possible this approximation we have to get rid of the first two lines of the expression above, since they show the lower powers in ε. To write things compactly let S_0 and L_0 be the operators that these lines define, that is

$$
\begin{aligned}
S_0(w) &:= \partial_{\mathtt{t}}^2 w - \varepsilon H_\Sigma \partial_{\mathtt{t}} w + f(w), \\
L_0 w &:= \partial_{\mathtt{t}}^2 w - \varepsilon H_\Sigma \partial_{\mathtt{t}} w + f'(U) w.
\end{aligned}
$$

With this notation and the ansatz (2.8) we can write the problem in the form

$$
S_0(U) + \varepsilon^2 L_0 \psi_0 + Q_0(U + \varepsilon^2\psi_0) = \varepsilon \ell_\varepsilon.
$$

where $Q(U + \varepsilon^2\psi_0)$ represents the rest of the terms in (2.9), namely

$$
\begin{aligned}
Q(U + \varepsilon^2\psi_0) = & -\varepsilon^2(\mathtt{t} + \varepsilon h_0)|A_\Sigma|^2 \partial_{\mathtt{t}} U \\
& + \varepsilon^4 \Delta_\Sigma \psi_0 - \varepsilon^4(\mathtt{t} + \varepsilon h_0)|A_\Sigma|^2 \partial_{\mathtt{t}}\psi_0 \\
& + \varepsilon^2 \left[(\varepsilon\mathtt{t} + h)\mathbb{B}_{\Sigma,\varepsilon\mathtt{t}+h} + (\varepsilon\mathtt{t}+h)^2 \mathbb{Q}_{\Sigma,\varepsilon\mathtt{t}+h}\right](U + \varepsilon^2\psi_0) \\
& + f(w) - f(U) - f'(U)\varepsilon^2\psi_0,
\end{aligned}
$$

and we also have to determine the Lagrange multiplier ℓ_ε. Thus we set U as the the function that yields the following lemma.

Lemma 2.2.1 There exists a monotonically increasing solution of the equation

$$
\begin{aligned}
S_0(U) = U'' - \varepsilon H_\Sigma U' + f(U) &= \varepsilon \ell_\varepsilon, \quad \text{in } \mathbb{R}, \\
f(U(\pm\infty)) &= \varepsilon \ell_\varepsilon.
\end{aligned}
\tag{2.10}
$$

This function is easy to find by perturbing the heteroclinic solution H introduced in (1.9). Note that since f is odd symmetric we have $U(\pm\infty) = \pm 1 + \sigma_\varepsilon^\pm$, where $\sigma_\varepsilon^\pm$ is the small constant such that

$$
f(\pm 1 + \sigma_\varepsilon^\pm) = \varepsilon \ell_\varepsilon.
$$

Also, we have

$$
\ell_\varepsilon = \ell_0 + \mathcal{O}(\varepsilon), \quad \ell_0 = -\frac{1}{2} H_\Sigma \int_{\mathbb{R}} H'(s)^2\, ds,
$$

and at main order

$$
U(\mathtt{t}) = H(\mathtt{t}) + \mathcal{O}(\varepsilon),
$$

We first have to set function space where it is possible to find a solution of this equation and the later ones, thus we introduce following weighted uniform norm space

$$C^{\infty}_{\mu}(\mathbb{R}) = \left\{ v \in C^{\infty}(\mathbb{R}) \,\middle|\, \|v\|_{C^{\infty}_{\mu}(\mathbb{R})} := \sup_{\mathtt{t}\in\mathbb{R}} |v(\mathtt{t})|(\cosh \mathtt{t})^{\mu} < \infty, \text{ for } \mu \in \mathbb{R} \right\}$$

PROOF OF LEMMA 2.2.1. We consider the ansatz $U = H + \varepsilon\phi$, where H is the heteroclinic solution of (1.9), and ϕ the corresponding perturbation, we find that ϕ must satisfy

$$H'' + \varepsilon\phi'' - \varepsilon H_{\Sigma} H' - \varepsilon^2 H_{\Sigma}\phi' + f(H + \varepsilon\phi) = \varepsilon\ell_{\varepsilon} \quad \text{in } \mathbb{R}. \tag{2.11}$$

Taking into account that we can write $f(H+\varepsilon\phi) = f(H) + \varepsilon f'(H)\phi + [f(H+\varepsilon\phi) - f(H) - \varepsilon f'(H)\phi]$, we have that (2.11) is equivalent to

$$L\phi := \phi'' + f'(H)\phi = \ell_{\varepsilon} + H_{\Sigma}(H' + \varepsilon\phi') + \mathcal{O}(\varepsilon\phi^2) \quad \text{in } \mathbb{R}. \tag{2.12}$$

We already know that H' is an element of the kernel of the second order linear operator L, the other one can be found with the ansatz vH', plugging this ansatz into the corresponding equation we find that $v = \int \frac{1}{(H')^2}$, and it easy to see that vH' is exponentially increasing in fact $vH' = \mathcal{O}(e^{\pm \mathtt{t}/\sqrt{2}})$ as $\mathtt{t} \to \pm\infty$. In order to find a bounded particular solution of $L\phi = g$ we first notice that the Wronskian $W(H', vH') = 1$, therefore using variation of parameters, a particular solution has the form

$$y_p(\mathtt{t}) = -H'(\mathtt{t}) \int_0^{\mathtt{t}} v(s)H'(s)g(s)\,ds + v(\mathtt{t})H'(\mathtt{t}) \int_{-\infty}^{\mathtt{t}} H'(s)g(s)\,ds.$$

This solution is bounded if the following orthogonality condition is satisfied:

$$\int_{\mathbb{R}} H'(t)g(t)\,dt = 0.$$

Thus (2.12) can be written as the integral equation

$$\begin{aligned}\phi(\mathtt{t}) = -H'(\mathtt{t}) &\int_0^{\mathtt{t}} v(s)H'(s)[\ell_{\varepsilon} + H_{\Sigma}(H'(s) + \varepsilon\phi'(s)) + \mathcal{O}(\varepsilon\phi^2(s))]\ ds \\ &+ v(\mathtt{t})H'(\mathtt{t}) \int_{-\infty}^{\mathtt{t}} H'(s)[\ell_{\varepsilon} + H_{\Sigma}(H'(s) + \varepsilon\phi'(s)) + \mathcal{O}(\varepsilon\phi^2(s))]\ ds.\end{aligned} \tag{2.13}$$

Taking derivative one has:

$$\begin{aligned}\phi'(\mathtt{t}) = -H''(\mathtt{t}) &\int_0^{\mathtt{t}} v(s)H'(s)[\ell_{\varepsilon} + H_{\Sigma}(H'(s) + \varepsilon\phi'(s)) + \mathcal{O}(\varepsilon\phi^2(s))]\ ds \\ + [H''(\mathtt{t})v(\mathtt{t}) + H'(\mathtt{t})v'(\mathtt{t})] &\int_{-\infty}^{\mathtt{t}} H'(s)[\ell_{\varepsilon} + \varepsilon H_{\Sigma}(H'(s) + \varepsilon\phi'(s)) + \mathcal{O}(\varepsilon\phi^2(s))]\ ds.\end{aligned} \tag{2.14}$$

Since the linear operator L has as H' as the only bounded element of its kernel it follows that the the right hand side of this equation has to be orthogonal to H' and this condition reads

$$\int_{\mathbb{R}} \ell_{\varepsilon} H'(s)\ ds + \int_{\mathbb{R}} H_{\Sigma}(H'(s) + \varepsilon\phi'(s))H'(s)\ ds + \int_{\mathbb{R}} \mathcal{O}(\varepsilon\phi^2(s))H'(s)\ ds = 0$$

or equivalently

$$\ell_\varepsilon = -\frac{1}{\int_{\mathbb{R}} H'(s)\ ds} H_\Sigma \int_{\mathbb{R}} H'(s)^2\ ds + \mathcal{O}(\varepsilon) \int_{\mathbb{R}} \phi^2(s) H'(s)\ ds + \mathcal{O}(\varepsilon) H_\Sigma \int_{\mathbb{R}} \phi'(s) H'(s)\ ds$$
$$= -\frac{1}{2} H_\Sigma \int_{\mathbb{R}} H'(s)^2\ ds + \mathcal{O}(\varepsilon),$$

as we anticipated. Therefore we can write the problem as

$$\begin{pmatrix} \phi \\ \phi' \end{pmatrix} = \mathcal{M} \begin{pmatrix} \phi \\ \phi' \end{pmatrix} = \begin{pmatrix} \mathcal{F}(\varepsilon\phi^2, \varepsilon\phi') \\ \mathcal{G}(\varepsilon\phi^2, \varepsilon\phi') \end{pmatrix}$$

where $\mathcal{F}$ and $\mathcal{G}$ are the right-hand-side operators from (2.13) and (2.14) respectively.

The presence of the ε factor on the non linearity of the above operator allows an implementation of the Banach fixed point theorem which can be done if we consider $\mathcal{B} \subset C^\infty_\mu(\mathbb{R}) \times C^\infty_\mu(\mathbb{R})$ (equipped with the norm $\|(u,v)\|_{C^\infty_\mu(\mathbb{R}) \times C^\infty_\mu(\mathbb{R})} = \text{máx}\{\|u\|_{C^\infty_\mu(\mathbb{R})}, \|v\|_{C^\infty_\mu(\mathbb{R})}\}$) of functions that satisfy $\|(\phi,\phi')\|_{C^\infty_\mu(\mathbb{R}) \times C^\infty_\mu(\mathbb{R})} \le K$, where K is a large constant to be chosen, in fact we have that

$$\|(\mathcal{F},\mathcal{G})\|_{C^\infty_\mu(\mathbb{R}) \times C^\infty_\mu(\mathbb{R})} = \mathcal{O}(\varepsilon) \|(\phi,\phi')\|_{C^\infty_\mu(\mathbb{R}) \times C^\infty_\mu(\mathbb{R})}$$

thus we get that $\mathcal{M} : \mathcal{B} \to \mathcal{B}$ has a fixed point.

□

We continue with the description of the approximate solution: ψ_0 should be chosen as the bounded solution of

$$L_0\psi_0 = \partial^2_{\mathtt{t}}\psi_0 - \varepsilon H_\Sigma \partial_{\mathtt{t}}\psi_0 + (1 - 3U^2)\psi_0 = (\mathtt{t} + \varepsilon h_0)|A_\Sigma|^2 \partial_{\mathtt{t}} U, \tag{2.15}$$

thus we have the following

Lemma 2.2.2 There exists a negative constant η such that for all $\mu \in (\eta + \varepsilon H_\Sigma, -\eta]$ the following holds. Given $g_0 = g_0(\mathtt{t}) \in C^\infty_\mu(\mathbb{R})$ satisfying the following orthogonal condition

$$\int_{\mathbb{R}} g_0(\mathtt{t}) \partial_{\mathtt{t}} U(\mathtt{t}) e^{-\varepsilon H_\Sigma \mathtt{t}}\, d\mathtt{t} = 0, \tag{2.16}$$

the problem

$$L_0 v(\mathtt{t}) = g_0(\mathtt{t}) \tag{2.17}$$

has a unique bounded solution. In addition v satisfies the estimate

$$\|v\|_{C^\infty_\mu(\mathbb{R})} \le C \|g_0\|_{C^\infty_\mu(\mathbb{R})}.$$

Thus, the solution of this equation defines a linear and bounded operator $\mathcal{G}$ in $C^\infty_\mu(\mathbb{R})$, namely $v = \mathcal{G}(g_0)$.

PROOF. The fundamental set of the homogeneous equation $L_0 v = 0$ can be found by the variation of parameters formula, indeed they correspond to the functions

$$V(\mathtt{t}) = \partial_{\mathtt{t}} U(\mathtt{t}) = \mathcal{O}((\cosh \mathtt{t})^{\eta^{\pm}}), \quad W(\mathtt{t}) = \mathcal{O}((\cosh \mathtt{t})^{\nu^{\pm}}), \qquad \text{as } \mathtt{t} \to \pm\infty.$$

where

$$\eta^{\pm} = \frac{1}{2}\varepsilon H_{\Sigma} - \frac{1}{2}\sqrt{-4\iota(\pm\infty) + \varepsilon^2 H_{\Sigma}^2}, \quad \mu^{\pm} = \frac{1}{2}\varepsilon H_{\Sigma} + \frac{1}{2}\sqrt{-4\iota(\pm\infty) + \varepsilon^2 H_{\Sigma}^2},$$

here $\iota(\pm\infty) = 1 - 3(\pm 1 + \sigma_{\varepsilon}^{\pm})^2$.

Let us notice that for $\sigma_{\varepsilon}^{\pm}$ small, one has $\eta^{\pm}, -\nu^{\pm} < 0$ for all $\varepsilon > 0$, thus V is exponentially decaying in $\mathtt{t}$. We can assume that the Wronskian at the origin is equal to 1, therefore, if g_0 satisfies (2.16) then we can write $\varphi = \mathcal{G}(g_0)$ where

$$\mathcal{G}(v)(\mathtt{t}) = -V(\mathtt{t}) \int_0^{\mathtt{t}} W(s) e^{-\varepsilon H_{\Sigma} s} v(s)\, ds + W(\mathtt{t}) \int_{-\infty}^{\mathtt{t}} V(s) e^{-\varepsilon H_{\Sigma} s} v(s)\, ds.$$

For the estimate of the lemma, let us notice that the orthogonality condition (2.16) guarantees that the function $\mathcal{G}(g_0)$ is exponentially decaying whenever g_0 is exponentially decaying. To be more precise let us assume for instance that if $\mu \in (\eta + \varepsilon H_{\Sigma}, -\eta]$, where $\eta = \text{máx}\{\eta^-, \eta^+\} < 0$, and

$$\|g_0\|_{C_{\mu}^{\infty}(\mathbb{R})} \leq C,$$

then we have

$$\|v\|_{C_{\mu}^{\infty}(\mathbb{R})} \leq C,$$

as well. □

Now we apply this lemma to the problem (2.15) with $g(\mathtt{y}, \mathtt{t}) = (\mathtt{t} + \varepsilon h_0)|A_{\Sigma}|^2 \partial_{\mathtt{t}} U(\mathtt{t})$ as the right hand side, that is $\psi_0 = \mathcal{G}((\mathtt{t} + h_0)\partial_{\mathtt{t}} U)$, note that this function depends also of $\mathtt{y}$. In order to get decaying at $\pm\infty$ (in $\mathtt{t}$) we must satisfy the orthogonal condition in (2.17), thus

$$\int_{\mathbb{R}} (\mathtt{t} + \varepsilon h_0)|A_{\Sigma}|^2 (\partial_{\mathtt{t}} U(\mathtt{t}))^2 e^{-\varepsilon H_{\Sigma} \mathtt{t}}\, d\mathtt{t} = 0,$$

this expression tells us how we must pick h_0, namely

$$h_0 = \frac{-\displaystyle\int_{\mathbb{R}} \mathtt{t}\, (V(\mathtt{t}))^2\, e^{-\varepsilon H_{\Sigma} \mathtt{t}}\, d\mathtt{t}}{\varepsilon \displaystyle\int_{\mathbb{R}} (V(\mathtt{t}))^2\, e^{-\varepsilon H_{\Sigma} \mathtt{t}}\, d\mathtt{t}} = \mathcal{O}(1),$$

which is a bounded quantity.

With this choice we define:

$$\psi_0(\mathtt{y}, \mathtt{t}) = \mathcal{G}\left((\mathtt{t} + \varepsilon h_0) V(\mathtt{t})\right) |A_{\Sigma}|^2.$$

Note that we have:

$$\|(|\mathtt{t}| + \varepsilon|h_0|) V(\mathtt{t})\|_{C_{\mu}^{\infty}(\mathbb{R})} \leq C, \quad 0 < \mu < -\eta,$$

and as a consequence

$$\|\psi_0(\mathtt{y},\mathtt{t})\|_{C^\infty_\mu} \le C, \quad 0 < \mu < -\eta.$$

Sometimes it is convenient to derive a more refined estimate taking into account the fact that the leading order term on the right hand side is $\mathtt{t}V(\mathtt{t}) = \mathcal{O}\left(|\mathtt{t}|(\cosh \mathtt{t})^{\eta^\pm})\right)$ as $|\mathtt{t}| \to \pm\infty$. Thus, we consider (2.17) with right hand side $g(\mathtt{y},\mathtt{t})$ satisfying

$$\|g(\mathtt{y},\mathtt{t})(1+|\mathtt{t}|)^\beta\|_{C^\infty_\mu(\mathbb{R})} \le C,$$

with $\mu \in (\eta + \varepsilon H_\Sigma, -\eta]$, and $\beta \in \mathbb{R}$. By a simple argument we have as well:

$$\|\mathcal{G}(g)(\mathtt{y},\mathtt{t})(1+|\mathtt{t}|)^\beta\|_{C^\infty_\mu(\mathbb{R})} \le C.$$

2.3. The Delaunay unduloids

The Delunay unduloids D_τ, $\tau \in (0,1)$ are CMC embedded surfaces of revolution in $\mathbb{R}^3$ [23, 24]. In this section we review how to obtain the equations that describe them. Consider the parametrization obtained by rotate a curve $(\phi(z), 0, z)$, $z \in \mathbb{R}$, around the z-axis, namely

$$X(z,\theta) = (\phi(z)\cos\theta, \phi(z)\sin\theta, z),$$

easy computations gives the following formulas

$$A_{D_\tau} = \frac{\phi_{zz}}{\sqrt{1+\phi_z^2}}\, dz^2 - \frac{\phi}{\sqrt{1+\phi_z^2}}\, d\theta^2,$$
$$H_{D_\tau} = \phi_{zz}(1+\phi_z^2)^{-3/2} - \frac{1}{\phi}(1+\phi_z^2)^{-1/2}$$

for the second fundamental form and the mean curvature, respectively. In order to have constant mean curvature surfaces, say -1 (or 1 depending on the chosen orientation) we impose

$$\phi_{zz} - \frac{1}{\phi}(1+\phi_z^2) + (1+\phi_z^2)^{3/2} = 0. \tag{2.18}$$

One can easily find two special solutions to (2.18), namely

$$\phi_1 = 1 \text{ and } \phi_0 = \sqrt{4-(z-2)^2}, \text{ for } |z-2| \le 2.$$

and they arise the cylinder of radius 1 and the sphere of radius 2 centered at $(0,0,2)$ respectively.

Let us notice, by differentiating, that ϕ solves (2.18) if

$$\mathcal{H}(\phi,\phi_z) = \phi^2 - \frac{2\phi}{\sqrt{1+\phi_z^2}}$$

is constant. In particular this quantity should be negative in order to get embedded surfaces (see [24]). The existence of Delaunay unduloids are guaranteed according to the following

Proposición 2.1 ([45], Proposition 1) *For all* $\varepsilon \in (0,1)$ *the problem*

$$\begin{cases} \phi_{zz} - \frac{1}{\phi}(1+\phi_z^2) + (1+\phi_z^2)^{3/2} = 0, \\ \phi(0) = \varepsilon, \\ \phi'(0) = 0, \end{cases} \tag{2.19}$$

has a solution ϕ_ε *which is periodic and*

$$\varepsilon \leq \phi_\varepsilon(z) \leq 2 - \varepsilon,$$

and $\lim_{\varepsilon\to 0^+} \phi_\varepsilon(z) = \phi_0(z) = \sqrt{4-(z-2)^2}$, $\lim_{\varepsilon\to 1^-} \phi_\varepsilon(z) = \phi_1(z) = 1$ *the solutions described above.*

Note also that in particular we have $\mathcal{H}(\phi_\varepsilon, (\phi_\varepsilon)_x) = \varepsilon^2 - 2\varepsilon < 0$, that allows to introduce the parameter $\tau = \sqrt{2\varepsilon - \varepsilon^2} \in (0,1)$ and renaming by ϕ_τ the corresponding function and the limit surfaces are the described in the introduction: as $\tau \to 0^+$ we obtain the array of spheres of radius 2 and as $\tau \to 1^-$ we get the cylinder of radius 1.

We would like to have an isothermal parametrization, for that reason we change the independent and dependent variable: $z = k_\tau(s)$, $\tau\sigma_\tau(s) = \phi_\tau(k_\tau(s))$ The new parametrization reads

$$X_\tau(s,\theta) = (\tau e^{\sigma_\tau(s)}\cos\theta, \tau e^{\sigma_\tau(s)}\sin\theta, k_\tau(s)). \tag{2.20}$$

And σ_τ and k_τ must satisfy the following equations (see Proposition 2 in [45])

$$\begin{cases} \left(\dfrac{\partial \sigma_\tau}{\partial s}\right)^2 + \tau^2\cosh^2\sigma_\tau = 1, \\ \tau^2\cosh\sigma_\tau(0) = 1, \end{cases} \qquad \begin{cases} \dfrac{\partial k_\tau}{\partial s} = \dfrac{\tau^2}{2}(1+e^{2\sigma_\tau}), \\ k_\tau(0) = 0. \end{cases}$$

Under this notation the main curvatures are

$$\kappa_1 = e^{-\sigma_\tau}\sinh\sigma_\tau, \quad \kappa_2 = e^{-\sigma_\tau}\cosh\sigma_\tau$$

and from here we obtain

$$|A_\tau|^2 = \kappa_1^2 + \kappa_2^2 = e^{-2\sigma_\tau}\cosh 2\sigma_\tau.$$

If we denote by $2s_\tau$ the period of σ_τ we get that D_τ is $2T_\tau = \kappa_\tau(2s_\tau)/2$ periodic, thus

$$D_\tau = D_\tau + 2T_\tau \mathsf{e}_3.$$

The exposed above is the standard way to introduce the Delaunay surfaces, nevertheless there are other two. The first one corresponds to a variational approach where the aim is to find extrema for the area functional of a surface of revolution subject to the constraint of the corresponding volume is constant. In [24] the author shows that the Euler-Lagrange equation associated to this problems is (2.19). The second (see [24]) one is the problem of the roulette of a conic that rolls along a straight line where its focus is the generator. Depending on the conic we get different curves and their corresponding surfaces: for the parabola we obtain the catenary and the catenoid, for the ellipse we obtain the undulary and Delaunay unduloid and for the hyperbola we obtain the nodary and its surface is the Delaunay nodoid.

It is also known that the Delaunay surfaces can be constructed in any dimension $d \geq 3$, see [33]. By D_τ, $\tau \in (0, \tau_*)$ we will denote the family of embedded Delaunay surfaces in $\mathbb{R}^d$ with axis of symmetry $\mathsf{e}_d\mathbb{R}$. We note that the parameter τ_* satisfies

$$\tau_* = \frac{(d-2)^{(d-2)/(d-1)}}{d-1},$$

and by T_τ we keep denoting its period, thus $D_\tau = D_\tau + 2T_\tau \mathsf{e}_d$.

2.4. Jacobi operator on Delaunay surface

Let us recall that normal graph surface of Σ is

$$\Sigma_\phi = \{y + \phi(y)N(y)\},$$

where $\phi(y) \in C^2(\Sigma)$ is small. The mean curvature of Σ_ϕ, denoted by H_ϕ expressed in terms of ϕ can be written by

$$\mathcal{N}(\phi) := 2(H_\phi - 1) = \Delta_\Sigma + |A_\Sigma|^2\phi + Q(\phi, D\phi, D^2\phi),$$

where Δ_Σ denotes the Laplace-Beltrami operator on Σ, A_Σ corresponds to the second fundamental form and Q is a quadratic nonlinearity that can be found explicitly (see for instance in [38]). In order to obtain a CMC from Σ_ϕ necessarily $H_\phi = 1$, or equivalently $\mathcal{N}(\phi) = 0$, moreover the linearized operator of $\mathcal{N}$ about $\phi = 0$ yields the Jacobi operator on Σ

$$\mathcal{J}_\Sigma\phi = \frac{\partial}{\partial s}\left(\mathcal{N}(1 + s\phi)\right)\Big|_{s=0} = \Delta_\Sigma\phi + |A_\Sigma|^2\phi,$$

and in particular its null elements rise the so called Jacobi fields that corresponds to variations of Σ which preserve the mean curvature to the second order.

A surface Σ is called *nondegenerate* if the Jacobi operator of this surface is injective. We will describe briefly the invertibility theory of this operator in the case where Σ is a Delaunay surface D_τ. The corresponding Jacobi operator will be denoted by $\mathcal{J}_{D_\tau} = \Delta_{D_\tau} + |A_{D_\tau}|^2$.

For our study it is important to understand the kernel of $\mathcal{J}_{D_\tau}$ or the so called Jacobi fields. These elements are of the three types:

(1) *Jacobi fields arising from infinitesimal translations.* Given $\mathsf{e} \in \mathbb{R}^3$, $|\mathsf{e}| = 1$ the constant Killing field associated to translations

$$\mathtt{x} \longmapsto \mathsf{e}$$

induces the following Jacobi fields

$$\Phi_\tau^{T,\mathsf{e}} = \mathsf{e} \cdot N_\tau,$$

where N_τ is the unit normal vector to D_τ. The coordinate vectors $\mathtt{e}_j$, $j = 1, 2, 3$ generate three linearly independent Jacobi fields $\Phi_\tau^{T,\mathtt{e}_j}$ corresponding to translations of D_τ in the respective directions of the coordinate axis.

It is important to notice that the Jacobi fields $\Phi_\tau^{T,\mathtt{e}_j}$ are bounded and periodic in s.

(2) *Jacobi fields arising from infinitesimal rotations.* Let $\mathtt{e} \in \mathbb{R}^3$, $|\mathtt{e}| = 1$ be such that $\mathtt{e} \cdot \mathtt{e}_3 = 0$. The Killing vector field corresponding to the rotation about the vector $\mathtt{e}$ is:

$$\mathtt{x} \longmapsto (\mathtt{x} \cdot \mathtt{e})\mathtt{e}_3 - (\mathtt{x} \cdot \mathtt{e}_3)\mathtt{e}.$$

We define the Jacobi field associated to this vector field by:

$$\Phi_\tau^{R,\mathtt{e}} = [(\mathtt{x} \cdot \mathtt{e})\mathtt{e}_3 - (\mathtt{x} \cdot \mathtt{e}_3)\mathtt{e}] \cdot N_\tau.$$

There are clearly two linearly independent Jacobi fields associated to the rotations. They are:

$$\Phi_\tau^{R,\mathtt{e}_1} \quad \text{and} \quad \Phi_\tau^{R,\mathtt{e}_2},$$

and they correspond to rotations about the coordinate axis. Note that in isothermal coordinates functions $\Phi_\tau^{R,\mathtt{e}_j}$, $j = 1, 2$ grow linearly as functions of s.

(3) *Jacobi field associated with the variation of the Delaunay parameter.* We define:

$$\Phi_\tau^D = -\partial_\tau X_\tau \cdot N_\tau.$$

This Jacobi field is somewhat harder to write explicitly, however it can be shown that the function $\Phi_\tau^D(s)$ is linearly growing (see [44] for the explicit formula)

In summary, the Jacobi operator $\mathcal{J}_{D_\tau}$ has at least 6 explicit Jacobi fields, three of them are bounded, and the other 3 are linearly growing. By a result of Mazzeo and Pacard in [45] we know that these corresponds to *all* Jacobi fields with at most linear growth. In fact the key property to show this is that in isothermal coordinates $\mathcal{J}_{D_\tau}$ separates variables into a sequence of second order ODEs

$$J_{\tau,j}\varphi = 0, \quad J_{\tau,j} = \partial_s^2 + \tau^2 \cosh(2\sigma_\tau) - j^2, \quad |j| = 0, 1, \ldots.$$

Proposition 2.4.1 ([45]) The homogeneous problem $J_{\tau,j}\varphi = 0$ has the following solutions:

1. one periodic and one linearly growing solution when $j = 0$ or $|j| = 1$;
2. two solutions $\varphi_{\tau,j}^\pm(s)$ which satisfy:

$$\varphi_{\tau,j}^\pm(s + s_\tau) = e^{\pm\zeta_{\tau,j} s_\tau} \varphi_{\tau,j}^\pm(s),$$

with

$$\gamma_{\tau,j} = \operatorname{Re} \zeta_{\tau,j} > 0,$$

when $|j| > 2$. The numbers $\zeta_{\tau,j}$ are the indicial roots of the operators $L_{\tau,j}$ and correspond to the behavior of the solutions of the homogeneous problem at $\pm\infty$.

The numbers $\zeta_{\tau,j}$ correspond to the indicial roots of the operators $J_{\tau,j}$ and determine the behavior of the solutions of the homogeneous problem at $\pm\infty$. In fact $e^{\pm\zeta_{\tau,j}s}\varphi^{\pm}_{\tau,j}(s)$ are periodic, in consequence

$$|\varphi^{+}_{\tau,j}(s)| \leq Ce^{-\zeta_{\tau,j}s}, \quad |\varphi^{-}_{\tau,j}(s)| \leq Ce^{\zeta_{\tau,j}s}, \quad \forall s \in \mathbb{R}$$

in fact $\varphi^{+}_{\tau_j}(-s) = \varphi^{-}_{\tau_j}(s)$. In light of points 1. and 2. of the proposition it is natural to define $\zeta_{\tau,0} = \zeta_{\tau,\pm 1} = 0$.

These basic facts can be generalized for the Jacobi operator of Delaunay surfaces in $\mathbb{R}^d$, $d > 3$, namely, there exist $2d$ Jacobi fields with at most linear growth: $\Phi^{T,\mathsf{e}_j}_{\tau}(s,\Theta)$, $j = 1,\ldots,d-1$, $\Phi^{T,\mathsf{e}_j}_{\tau}(s)$, $\Phi^{R,\mathsf{e}_j}_{\tau}(s,\Theta)$, $j = 1,\ldots,d-1$, $\Phi^{D}_{\tau}(s,\Theta)$, where $(s,\Theta) \in \mathbb{R}\times S^{d-2}$ are the corresponding parameters of the isothermal parametrisation.

Chapter 3

Proof of the Theorem 1.1

3.1. The Lyapunov-Schmidt reduction

While our formal considerations in Chapter 2 are valid for any embedded CMC surface Σ in $\mathbb{R}^d$, in what follows we will focus on the special case when $\Sigma = D_\tau$, i.e. it is a Delaunay unduloid. Since we are interested in functions which are periodic in the direction of the x_d axis with the minimal period equal to that of D_τ we will introduce the manifold $\mathring{D}_\tau$ which is obtained by identifying the set $D_\tau \cap \{x_d = 0\}$ with the set $D_\tau \cap \{x_d = 2T_\tau\}$. Let us observe that set $\mathring{D}_\tau$ is homeomorphic to the $d-1$ dimensional torus $\mathbb{T}^{d-1}$. We first notice that the function w defined in (2.8) is a good approximation of our problem, nevertheless it is only defined in the tubular neighborhood $\mathcal{N}_\delta$. Since $\lim_{\mathtt{t}\to\pm\infty} U(\mathtt{t}) = \pm 1 + \sigma_\varepsilon^\pm$, it is natural to define

$$\mathbb{H}(\mathtt{x}) = \begin{cases} 1 + \sigma_\varepsilon^+ & \text{if } \mathtt{x} \in \mathring{D}_\tau^+, \\ -1 + \sigma_\varepsilon^- & \text{if } \mathtt{x} \in \mathring{D}_\tau^-, \end{cases}$$

where $\mathring{D}_\tau^+$, $\mathring{D}_\tau^-$ denote, respectively, the exterior and the interior of $\mathring{D}_\tau$. Let us notice that the function w approaches $\mathbb{H}$ exponentially. Indeed, we have

$$|Y_{\varepsilon,h}^* w(\mathtt{y},\mathtt{t}) - Y_{\varepsilon,h}^*\mathbb{H}(\mathtt{y},\mathtt{t})| \le C_\mu e^{-\mu|\mathtt{t}|} \qquad \mathtt{y} \in \mathring{D}_\tau, \mathtt{t} \in \left(-\frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon}\right),$$

for any $\mu \in (0, |\eta|)$. Now we "glue" w and $\mathbb{H}$: take χ a cut-off function such that $\chi(s) = 1$, if $|s| \le 1/2$, and $\chi(s) = 0$ if $|s| \ge 1$. Next, we define the cut-off function χ^* supported in $\mathcal{N}_\delta$ by:

$$Y_{\varepsilon,h}^* \chi^*(\mathtt{t}) = \chi\left(\frac{\varepsilon \mathtt{t}}{\delta}\right).$$

We define the global approximate solution w^* by

$$w^*(\mathtt{x}) = w(\mathtt{x})\chi^*(\mathtt{x}) + \mathbb{H}(\mathtt{x})\left[1 - \chi^*(\mathtt{x})\right].$$

Now we look for a solution of the equation (1.12) in the form

$$u = w^* + \varphi.$$

where φ is a small (in a way to be specified) function. Thus our problem can be stated: find $\varphi\colon \mathbb{R}^d \to \mathbb{R}$, which is one-periodic with period $2T_\tau$ in the x_d direction, such that

$$N_\varepsilon(w^* + \varphi) = \ell_\varepsilon, \quad \text{in } \mathbb{R}^{d-1} \times S^1_{2T_\tau}, \tag{3.1}$$

where $S^1_{2T_\tau}$ is the circle of radius $2T_\tau$ and we recall

$$N_\varepsilon(u) = \varepsilon \Delta u + \frac{1}{\varepsilon} u(1-u^2).$$

Remark 3.1. Let us recall that we want our solution to be rotationally symmetric. That is, if by $\mathcal{R}_\Theta$ we denote the rotation of $\mathbb{R}^d$ about the x_d axis by angle Θ then we should have

$$(w^* + \varphi)(\mathtt{x}) = (w^* + \varphi)(\mathcal{R}_\Theta \mathtt{x}).$$

Since we already have (by definition)

$$w^*(\mathtt{x}) = w^*(\mathcal{R}_\Theta \mathtt{x}),$$

then as a result we will have $\varphi(\mathtt{x}) = \varphi(\mathcal{R}_\Theta \mathtt{x})$ as well, as it can be seen easily from the proceeding construction.

Since the function φ appearing in (3.1) is expected to be small it is natural to expand the nonlinear operator N_ε and study the following equation

$$L_\varepsilon \varphi = -N_\varepsilon(w^*) - Q_\varepsilon(\varphi) + \ell_\varepsilon,$$

where

$$\begin{aligned} L_\varepsilon(\varphi) &:= DN_\varepsilon(w^*)\varphi = \varepsilon \Delta \varphi + \frac{1}{\varepsilon} f'(w^*)\varphi, \\ Q_\varepsilon(\varphi) &:= N_\varepsilon(w^* + \varphi) - N_\varepsilon(w^*) - L_\varepsilon \varphi. \end{aligned}$$

The strategy, based on the Lyapunov-Schmidt reduction is clear. Indeed, using the fact that, in a certain way, L_ε separates variables and due to the d dimensional bounded and periodic kernel of the Jacobi operator $\mathcal{J}_{\mathring{D}_\tau}$ given by $\{\Phi_\tau^{T,\mathtt{e}_j}\}$ $j = 1, \ldots, d$ which is associated to translations of D_τ in the directions of the coordinate axis $\mathtt{e}_j$, the linear operator L_ε should have a d dimensional kernel spanned, roughly speaking, by the functions $Z_{\tau,\varepsilon}^{T,\mathtt{e}_j}$ where

$$\begin{aligned} Y^*_{\varepsilon,h} Z_{\tau,\varepsilon}^{T,\mathtt{e}_j}(\mathtt{y},\mathtt{t}) &= \mathtt{V}(\mathtt{y},\mathtt{t}) \Phi_\tau^{T,\mathtt{e}_j}(\mathtt{y}), \quad j = 1, \ldots, d, \\ \mathtt{V}(\mathtt{y},\mathtt{t}) &= \partial_\mathtt{t} w = \partial_\mathtt{t} U(\mathtt{t}) + \varepsilon^2 \partial_\mathtt{t} \psi_0(\mathtt{y},\mathtt{t}). \end{aligned}$$

Notice also that in general any function Z such that

$$Y^*_{\varepsilon,h} Z(\mathtt{y},\mathtt{t}) = \mathtt{V}(\mathtt{y},\mathtt{t}) \Phi(\mathtt{y}), \tag{3.2}$$

is also "almost" in the kernel of L_ε, in the sense that $Y^*_{\varepsilon,h}(L_\varepsilon Z) = o(1)$.

At this point we introduce the following general set

$$\mathcal{X} = \left\{ \varphi(\mathtt{y},\mathtt{t}) \in L^2(\mathring{D}_\tau \times \mathbb{R}) \,\middle|\, \int_\mathbb{R} \mathtt{V}(\mathtt{y},\mathtt{t}) \varphi(\mathtt{y},\mathtt{t})\, d\mathtt{t} = 0 \right\},$$

Notice that any function $Z \in \mathcal{X}^\perp$, the orthogonal complement of $\mathcal{X}$, has the form

$$\mathcal{X}^\perp = \left\{ Z(\mathtt{y},\mathtt{t}) \,\middle|\, Y^*_{\varepsilon,h} Z(\mathtt{y},\mathtt{t}) = \mathtt{V}(\mathtt{y},\mathtt{t})\Phi(\mathtt{y}) \right\}$$

for some function Φ that we can even compute explicitly

$$\Phi(\mathtt{y}) = \frac{\displaystyle\int_{\mathbb{R}} \mathtt{V}(\mathtt{y},\mathtt{t}) Z(\mathtt{y},\mathtt{t}) \; d\mathtt{t}}{\displaystyle\int_{\mathbb{R}} \mathtt{V}^2(\mathtt{y},\mathtt{t})(\mathtt{t}) \; d\mathtt{t}}.$$

By Π we denote the orthogonal projection on $\mathcal{X}$. We set $\varphi = \varphi^{\parallel} + \varphi^{\perp}$, where

$$Y^*_{\varepsilon,h}\varphi^{\parallel} \in \mathcal{X}, \quad Y^*_{\varepsilon,h}\varphi^{\perp} = \mathtt{V}\Phi \in \mathcal{X}^\perp.$$

Finally we split our problem into two equations:

$$\Pi \circ Y^*_{\varepsilon,h}[N_\varepsilon(w^* + \varphi^{\parallel} + \varphi^{\perp}) - \ell_\varepsilon] = 0, \tag{3.3}$$

$$(\mathrm{Id}\, - \Pi) \circ Y^*_{\varepsilon,h}[N_\varepsilon(w^* + \varphi^{\parallel} + \varphi^{\perp}) - \ell_\varepsilon] = 0. \tag{3.4}$$

When solving (3.3) we use the fact that the associated linear operator is coercive on $\mathcal{X} \cap H^1(\mathring{D}_\tau \times \mathbb{R})$. To solve (3.4) we will make use of the theory of solvability of the Jacobi operator $\mathcal{J}_{\mathring{D}_\tau}$. It is important to note that that this is not direct, basically due two facts: first, and somewhat technical step which we have omitted in this informal discussion, is to "transfer" the original problem from the space of functions defined on $\mathbb{R}^{d-1} \times S^1_{2T_\tau}$ to a space of functions defined on $\mathring{D}_\tau \times \mathbb{R}$. We will explain these details in Section 3.3. And second, and more delicate, the operator $\mathcal{J}_{\mathring{D}_\tau}$ has non trivial bounded elements in its kernel, roughly speaking and suggested by (3.2), the span of the finitely dimensional set

$$\mathcal{Z} = \left\{ Z^{T,\mathtt{e}_j}_{\tau,\varepsilon}(\mathtt{y},\mathtt{t}) \in \mathcal{X}^\perp \,\middle|\, Z^{T,\mathtt{e}_j}_{\tau,\varepsilon}(\mathtt{y},\mathtt{t}) = \mathtt{V}(\mathtt{y},\mathtt{t})\Phi^{T,\mathtt{e}_j}_{\tau}(\mathtt{y}) \right\}. \tag{3.5}$$

In order to remedy this issue we use the Lyapunov-Schmidt reduction. Indeed, instead of solving (3.3)–(3.4) we study the problem

$$\Pi \left[N_\varepsilon(w^* + \varphi^{\parallel} + \varphi^{\perp}) - \ell_\varepsilon + \chi^* \sum_{j=1}^{d} \mathtt{c}_j Z^{T,\mathtt{e}_j}_{\tau,\varepsilon} \right] = 0, \tag{3.6}$$

$$(\mathrm{Id}\, - \Pi) \left[N_\varepsilon(w^* + \varphi^{\parallel} + \varphi^{\perp}) - \ell_\varepsilon + \chi^* \sum_{j=1}^{d} \mathtt{c}_j Z^{T,\mathtt{e}_j}_{\tau,\varepsilon} \right] = 0. \tag{3.7}$$

The remaining step amounts to showing that the constants $\mathtt{c}_j = 0$ are actually all zero. To prove this we will use balancing formulae.

In Section 3.2 and 3.3 we develop the linear theory associated to the problem (3.6)–(3.7), and in Section 3.4 we provide the existence results of those equations and we also conclude the proof showing that actually $\mathtt{c}_j = 0 \; \forall j = 1, \ldots, d$.

3.2. Linear theory for the model problem

In this section we will develop the necessary theory to deal with the operator L_ε. To this end we will consider the operator

$$\mathbb{L}_\varepsilon = \varepsilon\Delta_{\mathring{D}_\tau} - (H_{\mathring{D}_\tau} + \varepsilon(\mathtt{t}+\varepsilon h_0)\chi(\varepsilon\mathtt{t}/\delta)|A_{\mathring{D}_\tau}|^2)\partial_\mathtt{t} + \varepsilon^{-1}\partial_\mathtt{t}^2 + \varepsilon^{-1}f'(w), \tag{3.8}$$

where $\chi(s)$ is a cutoff function supported in $(-1,1)$ and equal to 1 in $(-1/2,1/2)$. Note that this operator is defined for functions $\phi\colon \mathring{D}_\tau \times \mathbb{R} \to \mathbb{R}$ (and not just functions defined on $\mathring{D}_\tau \times (-\frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon})$). It is clear that $Y^*_{\varepsilon,h}L_\varepsilon \approx \mathbb{L}_\varepsilon$. Although the function $w = U + \varepsilon^2\psi_0$ depends on both variables $(\mathtt{y},\mathtt{t})$ in some sense the operator $\mathbb{L}_\varepsilon$ separates variables. To see this, with $\partial_\mathtt{t} w = \partial_\mathtt{t}(U+\varepsilon^2\psi_0) = \mathtt{V}$, we consider functions of the form

$$Z(\mathtt{y},\mathtt{t}) = \mathtt{V}(\mathtt{y},\mathtt{t})\Phi(\mathtt{y}) \in \mathcal{X}^\perp.$$

Observe that, by construction of $w = U + \varepsilon^2\psi_0$, combining equations (2.10) and (2.15) multiplied by ε^2 we get

$$\begin{aligned}\varepsilon^{-1}\partial_\mathtt{t}^2 w - (H_{\mathring{D}_\tau} + \varepsilon(\mathtt{t}+\varepsilon h_0)|A_{\mathring{D}_\tau}|^2)\partial_\mathtt{t} w + \varepsilon^{-1}f(w)&\\ = \ell_\varepsilon - \varepsilon^3(\mathtt{t}+\varepsilon h_0)|A_{\mathring{D}_\tau}|^2)\partial_\mathtt{t}\psi_0 + \varepsilon^{-1}Q(\varepsilon\psi_0),&\end{aligned}$$

where $Q(v) = f(U+v) - f(U) - f'(U)v$. Differentiating this equation in $\mathtt{t}$ we get for $\mathtt{V} = \partial_\mathtt{t} w$:

$$\begin{aligned}\varepsilon^{-1}\partial_\mathtt{t}^2\mathtt{V} - (H_{\mathring{D}_\tau} + \varepsilon(\mathtt{t}+\varepsilon h_0)|A_{\mathring{D}_\tau}|^2)\partial_\mathtt{t}\mathtt{V} - \varepsilon|A_{\mathring{D}_\tau}|^2\mathtt{V} + \varepsilon^{-1}f'(w)\mathtt{V}&\\ = -\varepsilon^3|A_{\mathring{D}_\tau}|^2\partial_\mathtt{t}\psi_0 - \varepsilon^3(\mathtt{t}+\varepsilon h_0)|A_{\mathring{D}_\tau}|^2\partial_\mathtt{t}^2\psi_0 - \varepsilon^{-1}\partial_\mathtt{t} Q(\varepsilon^2\psi_0).&\end{aligned} \tag{3.9}$$

From this, using the definition of $\mathbb{L}_\varepsilon$ in (3.8) we get:

$$\mathbb{L}_\varepsilon(\varphi) = \mathbb{L}_\varepsilon(\mathtt{V}\Phi) = \varepsilon\mathtt{V}\mathcal{J}_{\mathring{D}_\tau}\Phi + B_\varepsilon(\Phi) \tag{3.10}$$

where $\mathcal{J}_{\mathring{D}_\tau}$ is the Jacobi operator on $\mathring{D}_\tau$ and

$$\begin{aligned}B_\varepsilon(\Phi) = -\varepsilon(\mathtt{t}+\varepsilon h_0)(1-\chi^*)|A_{\mathring{D}_\tau}|^2\Phi\partial_\mathtt{t}\mathtt{V} + 2\varepsilon\nabla_{\mathring{D}_\tau}\mathtt{V}\cdot\nabla_{\mathring{D}_\tau}\Phi&\\ + \varepsilon\Phi\Delta_{\mathring{D}_\tau}\mathtt{V} - \varepsilon^3\left[(\mathtt{t}+\varepsilon h_0)|A_{\mathring{D}_\tau}|^2\partial_\mathtt{t}^2\psi_0 + |A_{\mathring{D}_\tau}|^2\partial_\mathtt{t}\psi_0\right]\Phi&\\ + \varepsilon^{-1}\left[f'(w)\partial_\mathtt{t} w - f'(U)\partial_\mathtt{t} w - \varepsilon^2 f''(U)\partial_\mathtt{t} U\psi_0\right]\Phi.&\end{aligned}$$

We note that

$$B_\varepsilon(\Phi) = \mathcal{O}(\varepsilon^3)\|\Phi\|_{\mathcal{C}^1(\mathring{D}_\tau)}. \tag{3.11}$$

Identity (3.9) and its consequence (3.10) is the key calculation which allows to use the usual Lyapunov-Schmidt reduction scheme, as we explained in the introduction. Indeed, if we had taken as the approximate solution only the function U then differentiating the equation (2.10) for $U' = \partial_\mathtt{t} U$ we would have gotten

$$\varepsilon^{-1}\partial_\mathtt{t}^2 U' - H_{\mathring{D}_\tau}\partial_\mathtt{t} U' + \varepsilon^{-1}f'(U)U' = 0.$$

This equation, unlike (3.9), does not carry any information about the geometry of $\mathring{D}_\tau$ besides its mean curvature which is constant. Following the method of [55] or [15], [16], [17], [53] we would have to perturb the surface $\mathring{D}_\tau$ additionally introducing new unknown functions in our problem. With the approach presented here this is no longer necessary and the Lyapunov-Schmidt procedure in this version is in this sense simpler. Recalling that the linearization of the mean curvature operator is the Jacobi operator which depends on the second fundamental form, we see that the operator $\mathbb{L}_\varepsilon$ is naturally compatible with the geometric context of our problem. To put it differently: the operator $\mathbb{L}_\varepsilon$ is, up to negligible terms, the correct linearization of the Cahn-Hilliard operator near a solution whose zero level set is the constant curvature surface $\mathring{D}_\tau$.

To develop invertibility theory for $\mathbb{L}_\varepsilon$ we will we employ two basic facts. First, we observe that on the subspace of $\mathcal{X}^\perp$

$$\mathcal{Y} := \left\{ Z(\mathtt{y},\mathtt{t}) = \mathtt{V}(\mathtt{y},\mathtt{t})\Phi(\mathtt{y}) \in \mathcal{X}^\perp \,\middle|\, \int_{\mathring{D}_\tau} \Phi(\mathtt{y})\Phi_\tau^{T,\mathtt{e}_j}(\mathtt{y})\, d\mathtt{y} = 0, \quad j = 1,\dots,d \right\}$$

we have that the bilinear form defined by

$$a(\varphi_1,\varphi_2) = \langle \mathbb{L}_\varepsilon \varphi_1, \varphi_2 \rangle_{L^2(\mathring{D}_\tau\times\mathbb{R})} = \int_{\mathring{D}_\tau\times\mathbb{R}} (\mathbb{L}_\varepsilon\varphi_1)\varphi_2\, d\mathtt{y}d\mathtt{t}$$

satisfies the coercive condition

$$a(Z,Z) \geq C\varepsilon \|Z\|^2_{L^2(\mathring{D}_\tau\times\mathbb{R})} \quad \forall Z \in \mathcal{Y}\cap H^1(\mathring{D}_\tau\times\mathbb{R}).$$

This follows basically from the invertibility of the Jacobi operator $\mathcal{J}_{\mathring{D}_\tau}$ on orthogonal functions to its kernel.

Second, when we consider $\varphi\in\mathcal{X}$ and $g\in L^2(\mathring{D}_\tau\times\mathbb{R})$ such that φ is a bounded solution of the problem

$$\mathtt{L}_\varepsilon\varphi = g,$$

where $\mathtt{L}_\varepsilon = \varepsilon\Delta_{\mathring{D}_\tau} + \varepsilon^{-1}\partial_{\mathtt{t}}^2 + \varepsilon^{-1}f'(H)$, then we have

$$\|\varphi\|_{L^2(\mathring{D}_\tau\times\mathbb{R})} \leq C\varepsilon\|g\|_{L^2(\mathring{D}_\tau\times\mathbb{R})}. \tag{3.12}$$

To prove this estimate we use Cauchy-Schwarz inequality and the well known fact that from

$$\int_{\mathbb{R}} |v'|^2 - f'(H)v^2 \geq C\|v\|^2_{L^2(\mathbb{R})} \quad \text{if} \quad \int_{\mathbb{R}} v(t)H'(t)\,dt = 0.$$

It follows that the bilinear form

$$\mathtt{B}_\varepsilon(\varphi) := \langle \mathtt{L}_\varepsilon\varphi,\varphi\rangle_{L^2(\mathring{D}_\tau\times\mathbb{R})},$$

is coercive on $\mathcal{X}\cap H^1(\mathring{D}_\tau\times\mathbb{R})$. In fact

$$\mathtt{B}_\varepsilon(\varphi) \geq C\varepsilon^{-1}\|\varphi\|^2_{L^2(\mathring{D}_\tau\times\mathbb{R})} \quad \forall\varphi\in\mathcal{X}\cap H^1(\mathring{D}_\tau\times\mathbb{R}).$$

In the same way as (3.12) is shown one can prove

$$\varepsilon^2\|\nabla_{\mathring{D}_\tau}\varphi\|_{L^2(\mathring{D}_\tau\times\mathbb{R})} + \|\partial_{\mathtt{t}}\varphi\|_{L^2(\mathring{D}_\tau\times\mathbb{R})} + \|\varphi\|_{L^2(\mathring{D}_\tau\times\mathbb{R})} \leq C\|g\|_{L^2(\mathring{D}_\tau\times\mathbb{R})}. \tag{3.13}$$

We refer the reader to [15] or [17] where results similar to estimates (3.12) and (3.13) were proven.

At the same time we can use (3.13) and the coercivity of the bilinear form $\mathtt{B}_\varepsilon(\varphi)$ to solve the equation

$$\Pi_{\mathcal{X}}\mathbb{L}_\varepsilon\varphi = g. \tag{3.14}$$

To do this we write $\mathbb{L}_\varepsilon = \mathtt{L}_\varepsilon + (\mathbb{L}_\varepsilon - \mathtt{L}_\varepsilon)$ and use a perturbation argument. The solution will still satisfy estimate (3.13). The perturbation argument is as follows: for $g \in \mathcal{X}$ we solve

$$\mathtt{L}_\varepsilon\varphi = g + cH',$$

where $c = \frac{\int gH'\,dt}{\int (H')^2\,dt}$. Then we define a map

$$G_{\mathcal{X}}(g) = \varphi - \mathtt{V}\frac{\int \mathtt{V}\varphi\,dt}{\int \mathtt{V}^2\,dt}.$$

Note that $G_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X}$ and it is well defined since bounded elements of the kernel of $\mathtt{L}_\varepsilon$ are precisely given by the span of H' (see also Lemma 3.2.2). Next we check

$$\|\Pi_{\mathcal{X}}\mathbb{L}_\varepsilon G_{\mathcal{X}}(g) - g\|_{L^2(\mathring{D}_\tau\times\mathbb{R})} \le o(1)\|g\|_{L^2(\mathring{D}_\tau\times\mathbb{R})}.$$

Indeed, since $\mathtt{V} = H' + \mathcal{O}(\varepsilon)$ by the construction of $w = H + \mathcal{O}(\varepsilon)$ in Section 2.2, we have

$$|c| \le C\varepsilon\|\phi\|_{L^2(\mathring{D}_\tau\times\mathbb{R})} \le C\varepsilon\|g\|_{L^2(\mathring{D}_\tau\times\mathbb{R})}.$$

Moreover, from

$$\mathbb{L}_\varepsilon - \mathtt{L}_\varepsilon = -(H_{\mathring{D}_\tau} + \varepsilon(\mathtt{t} + \varepsilon h_0)\chi(\varepsilon\mathtt{t}/\delta)|A_{\mathring{D}_\tau}|^2)\partial_{\mathtt{t}} + \varepsilon^{-1}[f'(w) - f'(H)]$$

we find

$$\|(\mathbb{L}_\varepsilon - \mathtt{L}_\varepsilon)\phi\|_{L^2(\mathring{D}_\tau\times\mathbb{R})} \le C(\|\partial_{\mathtt{t}}\phi\|_{L^2(\mathring{D}_\tau\times\mathbb{R})} + \|\phi\|_{L^2(\mathring{D}_\tau\times\mathbb{R})}) \le C\varepsilon\|g\|_{L^2(\mathring{D}_\tau\times\mathbb{R})}$$

by estimate (3.13). Therefore $\Pi_{\mathcal{X}}\mathbb{L}_\varepsilon G_{\mathcal{X}}$ is invertible as a map from $\mathcal{X}$ to itself and we can define

$$(\Pi_{\mathcal{X}}\mathbb{L}_\varepsilon)^{-1} = G_{\mathcal{X}}(\Pi_{\mathcal{X}}\mathbb{L}_\varepsilon G_{\mathcal{X}})^{-1}. \tag{3.15}$$

Moreover it is rather straightforward to show that a solution to (3.14) will satisfy estimates (3.12) and (3.13).

We will use these observations to solve the following model equation

$$\mathbb{L}_\varepsilon\varphi = g(\mathtt{y}, \mathtt{t}),$$

where we will assume initially that $g \in L^2(\mathring{D}_\tau \times \mathbb{R})$. We look for a solution in the form $\varphi = \varphi^{\parallel} + \varphi^{\perp}$, where

$$\varphi^{\parallel} \in \mathcal{X}, \quad \varphi^{\perp} = \mathtt{V}\Phi \in \mathcal{X}^{\perp}. \tag{3.16}$$

We write

$$\mathbb{L}_\varepsilon\varphi = \Pi_{\mathcal{X}}\mathbb{L}_\varepsilon(\varphi^{\parallel} + \varphi^{\perp}) + \Pi_{\mathcal{X}^\perp}\mathbb{L}_\varepsilon(\varphi^{\parallel} + \varphi^{\perp}),$$

and then we need to solve

$$\Pi_{\mathcal{X}}\mathbb{L}_\varepsilon\varphi^{\|} = \Pi_{\mathcal{X}} g - \Pi_{\mathcal{X}}\mathbb{L}_\varepsilon\varphi^{\perp}$$
$$\Pi_{\mathcal{X}^\perp}\mathbb{L}_\varepsilon\varphi^{\perp} = \Pi_{\mathcal{X}^\perp} g - \Pi_{\mathcal{X}^\perp}\mathbb{L}_\varepsilon\varphi^{\|}.$$

The idea is that terms $\Pi_{\mathcal{X}}\mathbb{L}_\varepsilon\varphi^{\perp}$ and $\Pi_{\mathcal{X}^\perp}\mathbb{L}_\varepsilon\varphi^{\|}$ are of smaller order because $\mathbb{L}_\varepsilon \mathtt{V} = o(1)$ so that the coupling between the two equations is rather weak. Another important point is that

$$\mathbb{L}_\varepsilon\varphi^{\perp} = \varepsilon \mathtt{V}\mathcal{J}_{\mathring{D}_\tau}\Phi + B_\varepsilon(\Phi),$$

where $B_\varepsilon(\Phi)$ is small (see (3.11)). We decompose accordingly

$$g = g^{\|} + g^{\perp}, \quad g^{\perp} = \mathtt{V}\Xi \tag{3.17}$$

and $B_\varepsilon(\Phi) = B_\varepsilon^{\|}(\Phi) + B_\varepsilon^{\perp}(\Phi)$, $B_\varepsilon^{\perp}(\Phi) = \Upsilon_\varepsilon(\Phi)\mathtt{V}$ and look for a solution of the system

$$\begin{aligned} \Pi_{\mathcal{X}}\mathbb{L}_\varepsilon\varphi^{\|} &= g^{\|} - B_\varepsilon^{\|}(\Phi), \\ \varepsilon\mathcal{J}_{\mathring{D}_\tau}\Phi + \Upsilon_\varepsilon(\Phi) &= \Xi - \frac{1}{\int_{\mathbb{R}} \mathtt{V}^2\, d\mathtt{t}}\int_{\mathbb{R}} (\mathbb{L}_\varepsilon\varphi^{\|})\mathtt{V}\, d\mathtt{t} + \sum_{j=1}^{d} c_j \Phi_\tau^{T,\mathtt{e}_j}. \end{aligned} \tag{3.18}$$

Note that in the second equation we have introduced Lagrange multipliers c_j to be determined, and we anticipated in the previous section. Let us notice that the main terms to study in the above system are

$$\begin{aligned} \Pi_{\mathcal{X}}\mathbb{L}_\varepsilon\varphi^{\|} &= g^{\|}, \\ \varepsilon\mathcal{J}_{D_\tau}\Phi &= \Xi + \sum_{j=1}^{d} c_j \Phi_\tau^{T,\mathtt{e}_j}, \end{aligned} \tag{3.19}$$

which is a decoupled system that we write as $\mathfrak{L}_\varepsilon(\varphi^{\|}, \Phi, c_j) = (g^{\|}, \Xi)$. As we saw above, the first equation in this system can be solved according to (3.15), meanwhile, for the second equation we have that the bilinear form associated

$$a_{\mathring{D}_\tau}(\Phi) = \langle \varepsilon\mathcal{J}_{\mathring{D}_\tau}\Phi, \Phi\rangle_{L^2(\mathring{D}_\tau)} = \int_{\mathring{D}_\tau} \varepsilon(\mathcal{J}_{\mathring{D}_\tau}\Phi(\mathtt{y}))\Phi(\mathtt{y})\; d\mathtt{y}$$

is coercive in $Y \cap H^1(\mathring{D}_\tau)$, where

$$Y = \left\{ \Phi(\mathtt{y}) \in L^2(\mathring{D}_\tau) \,\middle|\, \int_{\mathring{D}_\tau} \Phi(\mathtt{y})\Phi_\tau^{T,\mathtt{e}_j}(\mathtt{y})\; d\mathtt{y} = 0 \right\}.$$

In fact, we have the estimate

$$a_{\mathring{D}_\tau}(\Phi) = \langle \varepsilon\mathcal{J}_{\mathring{D}_\tau}\Phi, \Phi\rangle_{L^2(\mathring{D}_\tau)} \geq \varepsilon\|\Phi\|^2_{L^2(\mathring{D}_\tau)} \quad \forall \Phi \in Y \cap H^1(\mathring{D}_\tau).$$

therefore, taking into account that the bounded elements of the kernel of $\mathcal{J}_{\mathring{D}_\tau}$ are the span of the orthogonal Jacobi fields $\Phi_\tau^{T,\mathtt{e}_j}$, we obtain that for all $\Xi \in L^2(\mathring{D}_\tau)$ there exists $\Phi \in L^2(\mathring{D}_\tau)$ such that the second equation in (3.19) is solved whenever its right hand side is in Y, which leads us to the following condition on the coefficients c_j

$$c_j = -\frac{\int_{\mathring{D}_\tau} \Xi(\mathtt{y})\Phi_\tau^{T,\mathtt{e}_j}(\mathtt{y})\; d\mathtt{y}}{\int_{\mathring{D}_\tau} (\Phi_\tau^{T,\mathtt{e}_j})^2(\mathtt{y})\; d\mathtt{y}}. \tag{3.20}$$

Also, it holds the estimate

$$\|\Phi\|_{L^2(\mathring{D}_\tau)} \leq \varepsilon^{-1}\|\Xi\|_{L^2(\mathring{D}_\tau)}. \tag{3.21}$$

In summary we have:

Lemma 3.2.1 Let $\varphi, g \in L^2(\mathring{D}_\tau \times \mathbb{R})$ and consider their respective decomposition onto $\mathcal{X}$ and $\mathcal{X}^\perp$ as in (3.16) and (3.17) respectively, and assume that $\Xi \in Y$. The system (3.19), or equivalently $\mathfrak{L}_\varepsilon(\varphi^\parallel, \Phi, c_j) = (g^\parallel, \Xi)$ has a solution that defines an operator

$$\begin{aligned} \mathcal{G}\colon \mathcal{X} \times L^2(\mathring{D}_\tau) &\to \mathcal{X} \times L^2(\mathring{D}_\tau) \times \mathbb{R}^d, \\ (g^\parallel, \Xi) &\mapsto (\varphi^\parallel, \Phi, c_j), \end{aligned}$$

that satisfy estimates (3.12) and (3.13) and (3.21). In addition, coefficients c_j are given by (3.20)

Finally we can write system (3.18) in a fixed point scheme of the operator $\mathcal{G}$.

An alternative approach is to use a perturbation as the given recently. We introduce the notation $\tilde{\mathfrak{L}}_\varepsilon(\varphi^\parallel, \Phi, c_j) = (g^\parallel, \Xi)$ for the system (3.18). Let us notice that

$$\|\tilde{\mathfrak{L}}_\varepsilon\mathcal{G}(g^\parallel, \Xi) - (g^\parallel, \Xi)\|_{\mathcal{X}\times L^2(\mathring{D}_\tau)} = o(1)\|(g^\parallel, \Xi)\|_{\mathcal{X}\times L^2(\mathring{D}_\tau)},$$

which follows basically from the fact that $\tilde{\mathfrak{L}}_\varepsilon - \mathfrak{L}_\varepsilon$ only contains negligible terms, and in consequence the operator

$$\tilde{\mathfrak{L}}_\varepsilon\mathcal{G}\colon \mathcal{X} \times L^2(\mathring{D}_\tau) \to \mathcal{X} \times L^2(\mathring{D}_\tau)$$

is invertible, therefore we can define

$$\mathfrak{A}\colon \mathcal{X} \times L^2(\mathring{D}_\tau) \to \mathcal{X} \times L^2(\mathring{D}_\tau) \times \mathbb{R}^d$$

as $\mathfrak{A}(g^\parallel, \Xi) = \mathcal{G}((\tilde{\mathfrak{L}}_\varepsilon\mathcal{G})^{-1})(g^\parallel, \Xi)$. From this it easy to see that $\tilde{\mathfrak{L}}_\varepsilon\mathfrak{A}(g^\parallel, \Xi) = (g^\parallel, \Xi)$, that is $\mathfrak{A}$ is a right inverse of $\tilde{\mathfrak{L}}_\varepsilon$. Finally if we write $\tilde{\mathfrak{L}}_\varepsilon = \mathfrak{L}_\varepsilon + (\tilde{\mathfrak{L}}_\varepsilon - \mathfrak{L}_\varepsilon)$ we see that $\mathfrak{A}(g^\parallel, \Xi) = (\varphi^\parallel, \Phi, c_j)$ still satisfy estimates (3.12), (3.13) and (3.21).

Given that we can solve (3.18) our purpose is to find suitable estimates for the solution of the problem

$$\mathbb{L}_\varepsilon\varphi = g(\mathtt{y}, \mathtt{t})$$

on $\mathcal{X}$ assuming that

$$g(\mathtt{y}, \mathtt{t}) = \mathcal{O}(e^{-\mu|\mathtt{t}|}), \quad |\mathtt{t}| \to \infty.$$

In particular we would like to know that $\varphi(\mathtt{t}, \mathtt{y}) = \mathcal{O}(e^{-\mu|\mathtt{t}|})$ as well. This is straightforward by comparison principle once we know for example that φ is bounded. Thus the main issue is to obtain L^∞ control for $\varphi^\parallel$. We will go a little further now and show how to control *a priori* certain weighted Hölder norms of $\varphi^\parallel$ and $\varphi^\perp$.

To this end we consider $\mathring{D}_\tau \times \mathbb{R}$ equipped with the product metric and the associated Levi-Civita connection and define the following weighted Hölder norms

$$\begin{aligned} \|u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} &= \sup_{\mathtt{t}\in\mathbb{R}}(\cosh \mathtt{t})^\mu \|u\|_{\mathcal{C}^{0,\alpha}(\mathring{D}_\tau\times(\mathtt{t}-1,\mathtt{t}+1))}, \\ \|u\|_{\mathcal{C}^{1,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} &= \|u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \|\nabla_{\mathring{D}_\tau\times\mathbb{R}}u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}, \\ \|u\|_{\mathcal{C}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} &= \|u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \|\nabla_{\mathring{D}_\tau\times\mathbb{R}}u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \|\nabla^2_{\mathring{D}_\tau\times\mathbb{R}}u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}, \end{aligned} \tag{3.22}$$

where $\nabla_{\mathring{D}_\tau\times\mathbb{R}}$ and $\nabla^2_{\mathring{D}_\tau\times\mathbb{R}}$ correspond to the gradient and Hessian operators on $\mathring{D}_\tau\times\mathbb{R}$. By $D_{\mathring{D}_\tau}$ and $D^2_{\mathring{D}_\tau}$ we denote the gradient and the Hessian operator on $\mathring{D}_\tau$ respectively. In isothermal coordinates they read

$$\begin{aligned} D_{\mathring{D}_\tau} &= \frac{1}{\tau^2 e^{2\sigma_\tau(s)}}(\partial_s,\partial_\Theta),\\ D^2_{\mathring{D}_\tau} &= \frac{1}{\tau^4 e^{4\sigma_\tau(s)}}\begin{pmatrix}\partial_s & 0\\ 0 & \partial_\Theta\end{pmatrix}. \end{aligned}$$

These formulas can be obtained easily, since the metric on $\mathring{D}_\tau$ is $g_\tau=\tau^2 e^{2\sigma_\tau(s)}(ds^2+d\Theta^2)$.In consequence we obtain the following formulas

$$\begin{aligned} \|\nabla_{\mathring{D}_\tau\times\mathbb{R}}u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} &= \|\partial_{\mathtt{t}}u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}+\|D_{\mathring{D}_\tau}u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})},\\ \|\nabla^2_{\mathring{D}_\tau\times\mathbb{R}}u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} &= \|\partial^2_{\mathtt{t}}u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}+\|\partial_{\mathtt{t}}D_{\mathring{D}_\tau}u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}+\|D^2_{\mathring{D}_\tau}u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}. \end{aligned}$$

In order to simplify several arguments below and avoid keeping track of negative powers of ε appearing on the right hand side of various estimates we will rescale the $\mathtt{y}$ variable. Thus we introduce

$$\tilde{\mathtt{y}}=\frac{\mathtt{y}}{\varepsilon},\quad \tilde{\mathtt{t}}=\mathtt{t}.$$

We also denote $\mathring{D}_{\tau,\varepsilon}=\frac{1}{\varepsilon}\mathring{D}_\tau$, and we consider the manifold $\mathring{D}_{\tau,\varepsilon}\times\mathbb{R}$ again equipped with the product metric and the associated Levi-Civita connection. Finally let us define the following weighted Hölder norms on $\mathring{D}_{\tau,\varepsilon}\times\mathbb{R}$, where as usual $0<\alpha<1$

$$\begin{aligned} \|u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})} &= \sup_{\tilde{\mathtt{t}}\in\mathbb{R}}(\cosh\tilde{\mathtt{t}})^\mu\|u\|_{\mathcal{C}^{0,\alpha}(\mathring{D}_{\tau,\varepsilon}\times(\tilde{\mathtt{t}}-1,\tilde{\mathtt{t}}+1))},\\ \|u\|_{\mathcal{C}^{1,\alpha}_\mu(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})} &= \|u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})}+\|\nabla_{\mathring{D}_{\tau,\varepsilon}\times\mathbb{R}}u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})},\\ \|u\|_{\mathcal{C}^{2,\alpha}_\mu(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})} &= \|u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})}+\|\nabla_{\mathring{D}_{\tau,\varepsilon}\times\mathbb{R}}u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})}+\|\nabla^2_{\mathring{D}_{\tau,\varepsilon}\times\mathbb{R}}u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})}. \end{aligned}\tag{3.23}$$

We note that if for a given a real function u defined on $\mathring{D}_\tau\times\mathbb{R}$ we set $\tilde{u}(\tilde{\mathtt{y}},\tilde{\mathtt{t}})=u(\varepsilon\tilde{\mathtt{y}},\tilde{\mathtt{t}})$ then we have

$$\|\tilde{u}\|_{\mathcal{C}^{\ell,\alpha}_\mu(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})}=\sum_{0\le k+m\le\ell}\varepsilon^m\|\partial^k_{\mathtt{t}}D^m_{\mathring{D}_\tau}u\|_{\mathcal{C}^0_\mu(\mathring{D}_\tau\times\mathbb{R})}+\sum_{0\le k+m\le\ell}\varepsilon^{m+\alpha}[\partial^k_{\mathtt{t}}D^m_{\mathring{D}_\tau}u]_{\alpha,\mu,\mathring{D}_\tau\times\mathbb{R}},\tag{3.24}$$

where $\|\cdot\|_{\mathcal{C}^0_\mu(\mathring{D}_\tau\times\mathbb{R})}$ and $[\cdot]_{\alpha,\mu,\mathring{D}_\tau\times\mathbb{R}}$ correspond to the weighted Hölder norm and weighted Hölder seminorm respectively, that is

$$\begin{aligned} \|u\|_{\mathcal{C}^0_\mu(\mathring{D}_\tau\times\mathbb{R})} &= \sup_{\mathtt{t}\in\mathbb{R}}(\cosh\mathtt{t})^\mu\|u\|_{C^0(\mathring{D}_\tau\times(\mathtt{t}-1,\mathtt{t}+1))},\\ [u]_{\alpha,\mu,\mathring{D}_\tau\times\mathbb{R}} &= \sup_{\mathtt{t}\in\mathbb{R}}(\cosh\mathtt{t})^\mu[u]_{\alpha,\mathring{D}_\tau\times(\mathtt{t}-1,\mathtt{t}+1)}. \end{aligned}$$

Consequently by $\mathcal{E}^{\ell,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})$ we denote the space of functions on $\mathring{D}_\tau\times\mathbb{R}$ whose norm

$$\|u\|_{\mathcal{E}^{\ell,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}:=\sum_{0\le k+m\le\ell}\varepsilon^m\|\partial^k_{\mathtt{t}}D^m_{\mathring{D}_\tau}u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}$$

is bounded. In consequence we have the following result which is based on similar results from [15],[22], [17].

Proposition 3.2.1 For all $\varepsilon > 0$ sufficiently small the following holds. Given $\mu \in [0, -\eta)$ and g satisfying

$$\|g\|_{\mathcal{E}^{0,\alpha}_\mu(\mathring{D}_\tau \times \mathbb{R})} < \infty,$$
$$\int_{\mathbb{R}} g(\mathtt{y}, \mathtt{t}) \mathtt{V}(\mathtt{y}, \mathtt{t}) \, d\mathtt{t} = 0$$

there exists a positive constant C such that any solution of

$$\Pi_{\mathcal{X}} \mathbb{L}_\varepsilon u = g \tag{3.25}$$

satisfies

$$\|u\|_{\mathcal{E}^{\ell,\alpha}_\mu(\mathring{D}_\tau \times \mathbb{R})} \leq C\varepsilon^{1-\alpha} \|g\|_{\mathcal{E}^{0,\alpha}_\mu(\mathring{D}_\tau \times \mathbb{R})},$$

for $\ell = 0, 1, 2$.

Proof. We split the proof in several steps.

Step 1. *Rescaling the variables.* Given functions u and g on $\mathring{D}_\tau \times \mathbb{R}$ we introduce

$$\tilde{u}(\tilde{\mathtt{y}}, \tilde{\mathtt{t}}) = u(\varepsilon\tilde{\mathtt{y}}, \tilde{\mathtt{t}}), \quad \tilde{w}(\tilde{\mathtt{y}}, \tilde{\mathtt{t}}) = w(\varepsilon\tilde{\mathtt{t}}, \tilde{\mathtt{t}}), \quad \tilde{g}(\tilde{\mathtt{y}}, \tilde{\mathtt{t}}) = \varepsilon g(\varepsilon\tilde{\mathtt{y}}, \tilde{\mathtt{t}}),$$

and set

$$\tilde{\mathbb{L}}_\varepsilon \tilde{u} = \Delta_{\mathring{D}_{\tau,\varepsilon}} \tilde{u} + \partial^2_{\tilde{\mathtt{t}}} \tilde{u} + f'(\tilde{w})\tilde{u} + \varepsilon\tilde{q}\partial_{\tilde{\mathtt{t}}}\tilde{u},$$

where

$$\tilde{q} = 1 + \varepsilon(\tilde{\mathtt{t}} + \varepsilon h_0)\chi(\varepsilon\tilde{\mathtt{t}}/\delta)|A_{\mathring{D}_\tau}(\varepsilon\tilde{\mathtt{y}})|^2$$

is a bounded function. The linear problem (3.25) is equivalent to

$$\Pi_{\mathcal{X}} \tilde{\mathbb{L}}_\varepsilon \tilde{u} = \tilde{g} \quad \text{in } \mathring{D}_{\tau,\varepsilon} \times \mathbb{R}, \tag{3.26}$$

where we assume

$$\int_{\mathbb{R}} \tilde{g}(\tilde{\mathtt{y}}, \tilde{\mathtt{t}}) \mathtt{V}(\varepsilon\tilde{\mathtt{y}}, \tilde{\mathtt{t}}) \, d\tilde{\mathtt{t}} = 0.$$

Step 2. *Description of the kernel of the associated operator.* We consider a problem of the form

$$\Delta_y \phi + \partial_t^2 \phi + f'(H(t))\phi = 0 \quad \text{in } \mathbb{R}^{d-1} \times \mathbb{R}. \tag{3.27}$$

The following result is known (c.f. [22], Lemma 5.1)

Lemma 3.2.2 Let ϕ be a bounded solution of (3.27). Then $\phi = cH'(\mathtt{t})$, with some constant c.

Step 3. *A priori estimate of the associated operator.*

Consider now equation (3.26) and assume that $\tilde{g} \in \mathcal{C}^{0,\alpha}_\mu(\mathring{D}_{\tau,\varepsilon} \times \mathbb{R})$, with $\mu \in (0, |\eta|)$

Lemma 3.2.3 There exists a constant $C > 0$ such that for all sufficiently small ε any bounded solution of (3.26) with

$$\int_{\mathbb{R}} \tilde{u}(\tilde{\mathtt{y}}, \tilde{\mathtt{t}}) \mathtt{V}(\varepsilon \tilde{\mathtt{y}}, \tilde{\mathtt{t}})\, d\tilde{\mathtt{t}} = 0.$$

satisfies

$$\|\tilde{u}\|_{\mathcal{C}^{2,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})} \leq C\|\tilde{g}\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})}.$$

A proof of this lemma, is given bellow.

Step 4. *Returning to the original variables.* Note that with the definitions of the norms (3.22)–(3.23) we have

$$\|u\|_{\mathcal{E}^{0,\alpha}_{\mu}(\mathring{D}_{\tau}\times\mathbb{R})} = \|u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau}\times\mathbb{R})},$$

while with the notation of (3.24) we have

$$C^{-1}\|\tilde{u}\|_{\mathcal{C}^{\ell,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})} \leq \|u\|_{\mathcal{E}^{\ell,\alpha}_{\mu}(\mathring{D}_{\tau}\times\mathbb{R})} \leq C\varepsilon^{-\alpha}\|\tilde{u}\|_{\mathcal{C}^{\ell,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})}.$$

And from this the desired result. □

Proof of Lemma 3.2.3. The proof of this lemma relies on Lemma 3.2.2, a contradiction argument and the comparison principle. It follows the same lines as those of similar results in [22] (see also [20]). By local elliptic estimates we first prove by contradiction the following estimate

$$\|\tilde{u}\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})} \leq C\|\tilde{g}\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})} \tag{3.28}$$

In fact, if (3.28) were not valid, we would be able to find $0 < \varepsilon_n \to 0$ and $\tilde{u}_n$ such that

$$\|\tilde{u}_n\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})} = 1 \quad \forall n, \tag{3.29}$$

$$\int_{\mathbb{R}} \tilde{u}_n(\tilde{\mathtt{y}}, \tilde{\mathtt{t}}) \mathtt{V}(\varepsilon \tilde{\mathtt{y}}, \tilde{\mathtt{t}})\, d\tilde{\mathtt{t}} = 0.$$

and $\tilde{g}_n \in \mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})$ such that $\varepsilon_n \to 0$ as $n \to \infty$, $\|\tilde{g}_n\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})} \to 0$ as $n \to \infty$. That satisfy

$$\Pi_{\mathcal{X}}\tilde{\mathbb{L}}_{\varepsilon}\tilde{u}_n = \tilde{g}_n \quad \text{in } \mathring{D}_{\tau,\varepsilon}\times\mathbb{R}$$

Since $\|\tilde{\varphi}_n\| = 1$ it is possible to find $(\tilde{\mathtt{y}}_n, \tilde{\mathtt{t}}_n) \in \mathring{D}_{\tau,\varepsilon}\times\mathbb{R}$ such that

$$|\tilde{\varphi}_n(\tilde{\mathtt{y}}_n, \tilde{\mathtt{t}}_n)| > \frac{1}{2}.$$

Since $\mathring{D}_{\tau,\varepsilon}$ is periodic we may assume that $\varepsilon_n\tilde{\mathtt{y}}_n \to \tilde{\mathtt{y}}_0 \in \mathring{D}_{\tau,\varepsilon}$. In terms of isothermal parametrizations we have

$$X_{\tau,\varepsilon}(\tilde{s}, \tilde{\Theta}) = \frac{1}{\varepsilon}X_{\tau,\varepsilon}(\varepsilon\tilde{s}, \varepsilon\tilde{\Theta}), \quad (\tilde{s}, \tilde{\Theta}) = \frac{1}{\varepsilon}(s, \Theta), \quad (s, \Theta) \in \mathbb{R}\times S^{d-2},$$

thus $\tilde{\mathtt{y}}_0 = X_{\tau,\varepsilon}(s_0, \Theta_0)$, and $\varepsilon_n\tilde{\mathtt{y}}_0 = X_{\tau,\varepsilon}(s_n, \Theta_n)$. For some (s_0, Θ_0) and (s_n, Θ_n) we next center at (s_0, Θ_0), that is we define

$$(s, \Theta) = \frac{1}{\varepsilon}(s_n, \Theta_n) + (\mathtt{s}, \mathtt{\Theta}),$$

Now we analyze two cases, namely $\tilde{\mathtt{t}}_n$ bounded or unbounded.

Case $\tilde{\mathtt{t}}_n$ *bounded.* Let us define the bounded function

$$\bar{u}_n(\mathtt{s},\Theta,\tilde{\mathtt{t}}) = \tilde{u}_n((1/\varepsilon)(s_n,\Theta_n)+(\mathtt{s},\Theta),\tilde{\mathtt{t}}).$$

Here and in what follows we write the functions in terms of the parameters of $X_{\tau,\varepsilon}$, instead of the variable $\mathtt{y}$. Let's assume for a while that $d=3$. In this case we have a explicit expression for the Laplace-Beltrami operator. It is straightforward to check that $\bar{\varphi}_n$ solves

$$\tau^2 e^{2\sigma_\tau(s_n+\varepsilon_n \mathtt{s})}(\partial_{\mathtt{s}}^2+\partial_\theta^2)\bar{u}_n(\mathtt{s},\Theta,\tilde{\mathtt{t}}) + \partial_{\tilde{\mathtt{t}}}^2\bar{u}_n(\mathtt{s},\Theta,\tilde{\mathtt{t}}) + f'(\tilde{w})\bar{u}_n(\mathtt{s},\theta,\tilde{\mathtt{t}}) \\ + \varepsilon_n\tilde{q}(\mathtt{s},\Theta,\tilde{\mathtt{t}}) = \bar{g}_n(\mathtt{s},\Theta,\tilde{\mathtt{t}}),$$

where $\bar{g}_n(\mathtt{s},\Theta,\tilde{\mathtt{t}}) = \tilde{g}_n((s_n,\theta_n)+\varepsilon(\mathtt{s},\Theta),\tilde{\mathtt{t}})$. Using elliptic estimates we can obtain bounds for the gradient, thus by Arzela-Ascoli's Theorem a subsequence of $\bar{u}_n$ converges uniformly over compacts to a function $\bar{u}$ that satisfies

$$\Delta_{\mathring{D}_{\tau,\varepsilon}}\bar{u} + \partial_{\tilde{\mathtt{t}}}^2\bar{u} + f'(\tilde{w})\bar{u} = 0. \tag{3.30}$$

On the other hand

$$0 = \int_{\mathbb{R}} \bar{u}_n(\mathtt{s},\Theta,\tilde{\mathtt{t}})\mathtt{V}(\varepsilon\tilde{\mathtt{y}},\varepsilon\tilde{\mathtt{t}})\, d\tilde{\mathtt{t}} \to \int_{\mathbb{R}} \bar{u}(\mathtt{s},\Theta,\tilde{\mathtt{t}})\mathtt{V}(\varepsilon\tilde{\mathtt{y}},\tilde{\mathtt{t}})\, d\tilde{\mathtt{t}} \quad \text{as } n\to\infty$$

this implies necessarily $\bar{u}\equiv 0$ but this is impossible, because of the first condition in (3.29). For the general case for d, we use the fact that the Laplace-Beltrami operator can be written in terms of bounded functions, namely

$$\Delta_{\mathring{D}_\tau} = a_{i,j}(\mathtt{y})\partial_{i,j} + b_i(\mathtt{y})\partial_i, \quad \mathtt{y} = X_\tau(s,\Theta), \quad (s,\Theta)\in\mathbb{R}\times S^{d-2},$$

then (3.30) becomes

$$a_{ij}\partial_{ij}\bar{u}_n(\mathtt{s},\Theta,\tilde{\mathtt{t}}) + b_i\partial_i\bar{u}_n(\mathtt{s},\Theta,\tilde{\mathtt{t}}) + \partial_{\tilde{\mathtt{t}}}^2\bar{u}_n(\mathtt{s},\Theta,\tilde{\mathtt{t}}) + f'(\tilde{w})\bar{\varphi}_n(\mathtt{s},\Theta,\tilde{\mathtt{t}}) = 0.$$

And the same arguments can be applied.

Case $\mathtt{t}_n$ *unbounded.* The situation is quite similar and in this case we define

$$\bar{u}_n(\mathtt{s},\Theta,\tilde{\mathtt{t}}) = e^{\mu(\tilde{\mathtt{t}}_n+\tilde{\mathtt{t}})}\tilde{\varphi}_n((1/\varepsilon)(s_n,\Theta_n)+(\mathtt{s},\Theta),\tilde{\mathtt{t}}_n+\tilde{\mathtt{t}}),$$
$$\bar{g}_n(\mathtt{s},\Theta,\tilde{\mathtt{t}}) = e^{\mu(\tilde{\mathtt{t}}_n+\tilde{\mathtt{t}})}\tilde{g}_n((s_n,\Theta_n)+\varepsilon(\mathtt{s},\Theta),\tilde{\mathtt{t}}_n+\tilde{\mathtt{t}})$$

And $\bar{\varphi}_n$ again is uniformly bounded and $g_n\to 0$ in $L^\infty_{\text{loc}}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})$, and in in this case we have

$$a_{ij}(s_n+\varepsilon_n\mathtt{s})\partial_{ij}\bar{u}_n(\mathtt{s},\Theta,\tilde{\mathtt{t}}) + \varepsilon_n b_i(s_n+\varepsilon_n\mathtt{s})\partial_i\bar{u}_n(\mathtt{s},\Theta,\tilde{\mathtt{t}}) + \partial_{\tilde{\mathtt{t}}}^2\bar{u}_n(\mathtt{s},\Theta,\tilde{\mathtt{t}}) \\ - 2\mu_\varepsilon\partial_{\tilde{\mathtt{t}}}\bar{u}_n(\mathtt{s},\Theta,\tilde{\mathtt{t}}) + f'(\tilde{w}(\tilde{\mathtt{y}},\tilde{\mathtt{t}}+\tilde{\mathtt{t}}_n)) + \mu^2\bar{u}_n(\mathtt{s},\Theta,\tilde{\mathtt{t}}) = \bar{g}_n$$

And at the limit we obtain

$$a^*_{i,j}\partial_{i,j}\bar{u} + \partial_{\tilde{\mathtt{t}}}^2\varphi - 2\mu_\varepsilon\partial_{\tilde{\mathtt{t}}}\bar{u} - (\mu_+^2-\mu^2)\bar{u} = 0 \quad \text{in } \mathring{D}_{\tau,\varepsilon}\times\mathbb{R}.$$

where $a_{i,j}^*$ is a positive definite constant matrix and $\varphi \neq 0$. Since ${\mu_\varepsilon^+}^2 - \mu_\varepsilon^2 > 0$ and by the maximum principle this implies that $\varphi \equiv 0$, achieving a contradiction. Using (3.28) we can compare with functions in the following form

$$W_\nu(\mathtt{y},\mathtt{t}) = (e^{-\mu\mathtt{t}} + \nu e^{\mu\mathtt{t}})k(\mathtt{y})$$

here $\nu > 0$ chosen such that $\mu^2 + 4\nu^2 \leq 2$ and where $k(\mathtt{y})$ is a proper function of the operator $\Delta_{\mathring{D}_\tau}$, this can of functions can be obtained by spectral theory and Fourier decomposition, for the case $d = 2$ this theory is developed in [45, 44]. Thus we obtain

$$(\Delta_{\mathring{D}_{\tau,\varepsilon}} + \partial_{\mathtt{t}}^2 - 2)W_\nu = -(2 - \mu^2 - \nu^2)W_\nu$$

and for $|\mathtt{t}| \gg 1$ we obtain

$$\Delta_{\mathring{D}_{\tau,\varepsilon}}\tilde{u} + \partial_{\mathtt{t}}^2 + f'(\tilde{w})\tilde{u} \leq -\left(\frac{2-\mu^2}{4}\right)e^{-\mu|\mathtt{t}|} \quad \text{for } |\mathtt{t}| \gg 1$$

We next define the function

$$\bar{u} = \frac{\tilde{u}}{\|(\cosh\mathtt{t})^\mu \tilde{g}\|_{L^\infty(\mathring{D}_\tau\times\mathbb{R})}}$$

hence $\|(\cosh\mathtt{t})^\mu \tilde{\mathbb{L}}_\varepsilon \bar{\varphi}\|_{L^\infty(\mathring{D}_\tau\times\mathbb{R})} \leq 1$, and

$$\Pi_{\mathcal{X}}\tilde{\mathbb{L}}_\varepsilon W_\nu \leq Ce^{-\mu\mathtt{t}} \leq \Pi_{\mathcal{X}}\tilde{\mathbb{L}}_\varepsilon \bar{u} \quad \mathtt{t} \gg 1$$

Using the maximum principle we can compare $\bar{u}$ with W_ν in the range above, obtaining $\bar{u} \leq W_\nu$, thus

$$\tilde{u} \leq W_\nu \|(\cosh\mathtt{t})^\mu \tilde{g}\|_{L^\infty(\mathring{D}_\tau\times\mathbb{R})}$$

letting $\nu \to 0$ we obtain

$$\tilde{\varphi}(\mathtt{t},\mathtt{y}) \leq e^{-\mu\mathtt{t}}\|(\cosh\mathtt{t})^\mu \tilde{g}\|_{L^\infty(\mathring{D}_\tau\times\mathbb{R})} \quad \text{for } \mathtt{t} \gg 1$$

□

Proposition 3.2.1 shows that we can control the size of $\mathcal{E}_\mu^{\ell,\alpha}(\mathring{D}_\tau \times \mathbb{R})$-norm of the solution of the first equation in (3.18) using the fact that $U \to H$ as $\varepsilon \to 0$ uniformly over compacts in $\mathring{D}_\tau \times \mathbb{R}$. Meanwhile for the second equation one can use elliptic theory to get Hölder estimates. Thus at the end we get

Proposition 3.2.2 For all $\varepsilon > 0$ small enough and $\mu \in [0, -\eta)$ and $\ell = 0, 1, 2$, there exits a positive constant C such that if $\|g\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} < \infty$ then we have that every solution of both equations in (3.18) satisfies

$$\|\varphi^\parallel\|_{\mathcal{E}_\mu^{\ell,\alpha}(\mathring{D}_\tau\times\mathbb{R})} \leq C\varepsilon^{1-\alpha}(\|g^\parallel\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^2\|g^\perp\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})}), \tag{3.31}$$

$$\|\varphi^\perp\|_{\mathcal{C}_\mu^{\ell,\alpha}(\mathring{D}_\tau\times\mathbb{R})} \leq C\varepsilon^{-1}\|g^\perp\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})}, \tag{3.32}$$

3.3. The linear problem in the whole space.

Now we will use the theory outlined in the previous section to solve the following problem:

$$\varepsilon\Delta\varphi + \frac{1}{\varepsilon}f'(w^*)\varphi = g(\mathtt{x}) \quad \text{in } \mathbb{R}^{d-1}\times S_{2T_\tau}.$$

From what we have said above it is in general not possible to find a solution with a reasonably bounded norm unless the right hand side satisfies some extra conditions, or equivalently, we need to introduce some Lagrange multipliers. Thus, we will solve

$$\varepsilon\Delta\varphi + \frac{1}{\varepsilon}f'(w^*)\varphi = g(\mathtt{x}) + \chi^*\sum_{j=1}^{d}\mathtt{c}_j Z^{T,\mathtt{e}_j}_{\tau,\varepsilon} \quad \text{in } \mathbb{R}^{d-1}\times S_{2T_\tau}, \tag{3.33}$$

where

$$\begin{aligned} Y^*_{\varepsilon,h}\chi^*(\mathtt{t}) &= \chi(\varepsilon\mathtt{t}/\delta),\\ Y^*_{\varepsilon,h}Z^{T,\mathtt{e}_j}_{\tau,\varepsilon} &= \mathtt{V}(\mathtt{y},\mathtt{t})\Phi^{T,\mathtt{e}_j}_{\tau}(\mathtt{y}) \quad j=1,\dots,d, \end{aligned}$$

that is, $Z^{T,\mathtt{e}_j}_{\tau,\varepsilon}$ are the elements of the base of $\mathcal{Z}$ (c.f. (3.5)). The idea is to solve (3.33) by *gluing* a solution defined near $\mathring{D}_\tau$ and another one defined away from $\mathring{D}_\tau$. To describe this construction rigorously we need some preparation.

We introduce the function $q(\mathtt{x})$ as follows:

$$q(\mathtt{x}) = \begin{cases} f'(1+\sigma^+_\varepsilon), & \text{dist}\,(\mathtt{x},\mathring{D}_\tau) > \delta/2,\\ f'(-1+\sigma^-_\varepsilon), & \text{dist}\,(\mathtt{x},\mathring{D}_\tau) < -\delta/2, \end{cases}$$

and otherwise $q(\mathtt{x})$ is a smooth function such that

$$\text{mín}\{f'(1+\sigma^+_\varepsilon), f'(-1+\sigma^-_\varepsilon)\} < q(\mathtt{x}) \le \text{máx}\{f'(1+\sigma^+_\varepsilon), f'(-1+\sigma^-_\varepsilon)\}.$$

Above, $\text{dist}\,(\mathtt{x},\mathring{D}_\tau)$ is the signed distance function chosen in such a way that the Fermi coordinate $z(\mathtt{x}) = \text{dist}\,(\mathtt{x},\mathring{D}_\tau)$ for $\mathtt{x}\in\mathcal{N}_\delta$. Note that $q(\mathtt{x}) = -2+\mathcal{O}(\varepsilon)$.

Finally, we need another cutoff function $\widetilde{\chi}$ such that $\widetilde{\chi}\chi^* = \chi^*$ (take for instance $Y^*_{\varepsilon,h}\chi^*(\mathtt{t}) = \chi(\varepsilon\mathtt{t}/2\delta)$ and chose δ smaller so that the Fermi coordinates are defined in $\mathcal{N}_{2\delta}$).

We want to find a solution of (3.33) in the form

$$\varphi = \chi^*\check{\varphi}\circ Y_{\varepsilon,h} + \psi,$$

where the function $\check{\varphi}$ solves:

$$\begin{aligned} \mathbb{L}_\varepsilon\check{\varphi} = Y^*_{\varepsilon,h}\Big(\widetilde{\chi}\{g + \chi^*\sum_{j=1}^{d}\mathtt{c}_j Z^{T,\mathtt{e}_j}_{\tau,\varepsilon} + (\mathbb{L}_\varepsilon - L_\varepsilon)\check{\varphi}\circ Y_{\varepsilon,h}\\ - [\chi^*, L_\varepsilon]\check{\varphi}\circ Y_{\varepsilon,h} + \varepsilon^{-1}[q - f'(w^*)]\psi\}\Big) \quad \text{in } \mathring{D}_\tau\times\mathbb{R}, \end{aligned} \tag{3.34}$$

and the function ψ solves

$$\varepsilon\Delta\psi + \varepsilon^{-1}[(1-\chi^*)f'(w^*) + \chi^* q]\psi = (1-\chi^*)\{g + \chi^*\sum_{j=1}^{d}\mathtt{c}_j Z_{\tau,\varepsilon}^{T,\mathtt{e}_j} - [\chi^*, L_\varepsilon]\breve{\varphi}\circ Y_{\varepsilon,h}\} \quad \text{in } \mathbb{R}^{d-1}\times S_{2T_\tau}, \tag{3.35}$$

where $[\chi^*, L_\varepsilon]\breve{\varphi} = 2\varepsilon\nabla_{\mathbb{R}^{d-1}\times S_{2T_\tau}}\breve{\varphi}\nabla_{\mathbb{R}^{d-1}\times S_{2T_\tau}}\chi^* + \varepsilon\breve{\varphi}\Delta_{\mathbb{R}^{d-1}\times S_{2T_\tau}}\chi^*$. In fact, multiplying (3.34) by χ^*, adding both equations and using the fact that $\widetilde{\chi}\chi^* = \chi^*$ we get the equivalent relation. In the above and in what follows we abuse slightly notation writing for instance $\breve{\varphi}$ as a function defined on $\mathring{D}_\tau\times\mathbb{R}$ and as a function defined on $\mathbb{R}^{d-1}\times S_{2T_\tau}$. It is understood that in the latter case we take $\breve{\varphi}\circ Y_{\varepsilon,h}$. To avoid complicated notions we will omit the composition with $Y_{\varepsilon,h}$ or $Y_{\varepsilon,h}^{-1}$ whenever it does not cause confusion. Thus the commutator $[\chi^*, L_\varepsilon]\breve{\varphi}$ in $\mathbb{R}^{d-1}\times S_{2T_\tau}$ is

$$[\chi^*, L_\varepsilon]\breve{\varphi} = 2\varepsilon\nabla\breve{\varphi}\circ Y_{\varepsilon,h}\nabla\chi^* + \varepsilon\breve{\varphi}\circ Y_{\varepsilon,h}\Delta\chi^*,$$

while in $\mathring{D}_\tau\times\mathbb{R}$ we have to first express L_ε in local coordinate $(\mathtt{y},\mathtt{t})$ (written as $Y_{\varepsilon,h}^* L_\varepsilon$) and calculate $[\chi^*, Y_{\varepsilon,h}^* L_\varepsilon]\breve{\varphi}$. In what follows we will assume that the function g on the right hand side of this equation satisfies the following general assumptions on its asymptotic behaviour

$$\begin{aligned} \|\,(g\chi^*)^{\|}\,\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} &\le C,\\ \|\,(g\chi^*)^{\perp}\,\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} &\le C,\\ \|(1-\chi^*)g\|_{\mathcal{C}^{0,\alpha}(\mathcal{C}_{2T_\tau})} &\le Ce^{-c_0/\varepsilon}, \end{aligned} \tag{3.36}$$

for some positive constants c_0, C, where $\mathcal{C}_{2T}$ is the cylinder $\mathbb{R}^{d-1}\times S_{2T}^1$. In addition we assume that g is rotationally symmetric about the x_d axis, namely if by $\mathcal{R}_\theta$ we denote the rotation of $\mathbb{R}^d$ about the x_d axis by angle θ then

$$g(\mathcal{R}_\theta\mathtt{x}) = g(\mathtt{x}). \tag{3.37}$$

In order to solve this coupled system we need to make sure that all terms on the right hand side that involve $\breve{\varphi}$ and ψ are small in suitable weighted Hölder and Hölder norms respectively. It is at this point that we need to chose the parameter δ in the definition of the tubular neighbourhood $\mathcal{N}_\delta$ small and dependent on ε. Thus we take $\delta(\varepsilon) = \varepsilon^{2/3}$. This means in particular that for $\mathtt{x}\in\mathcal{N}_\delta$ we have $\varepsilon\mathtt{t}(\mathtt{x}) = \mathcal{O}(\varepsilon^{2/3})$. For reasons that will become clear soon we will also chose the Hölder exponent α in the definition of $\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})$ and $\mathcal{C}^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})$ to be in the interval $(0,\frac{1}{10})$. Finally, the parameter μ will be always taken in the interval $(0,-\eta)$. The main result of this section is

Lemma 3.3.1 For each sufficiently small ε there exists a unique solution of (3.33) in the form $\varphi = \chi^*\breve{\varphi}\circ Y_{\varepsilon,h} + \psi$. In addition, assuming that (3.36)–(3.37) hold, then the solution φ satisfy the following estimates

$$\begin{aligned} \|\breve{\varphi}^{\|}\|_{\mathcal{E}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})} &\le C\varepsilon^{1-\alpha}\Big\{\|(\widetilde{\chi}g)^{\|}\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^{-1}\delta(\varepsilon)\|(\widetilde{\chi}\,g)^{\perp}\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})}\\ &\qquad + \varepsilon^{-3-\alpha}\delta(\varepsilon)\|(1-\chi^*)g\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})}\Big\},\\ \|\breve{\varphi}^{\perp}\|_{\mathcal{C}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})} &\le C\varepsilon^{-1}\Big\{\|(\widetilde{\chi}g)^{\perp}\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^{1-\alpha}\|(\widetilde{\chi}\,g)^{\|}\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})}\\ &\qquad + \varepsilon^{-2-\alpha}\|(1-\chi^*)g\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})}\Big\}, \end{aligned} \tag{3.38}$$

and

$$\|\psi\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})} \le C\varepsilon^{-1-\alpha}\|(1-\chi^*)g\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})} + \mathcal{O}(e^{-c\varepsilon^{-1/8}})\|\widetilde{\chi}g\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})}. \quad (3.39)$$

Proof. We first analyze (3.35). Taking into account that $[(1-\chi^*)f'(w^*)+\chi^* q]<0$ and the exponential decay of the functions g we get that if u is a solution of

$$\varepsilon\Delta u - \frac{1}{\varepsilon}[(1-\chi^*)f'(w^*)+\chi^* q]u = g, \quad \text{in } \mathbb{R}^{d-1}\times S_{2T_\tau},$$

then we have an *a priori* estimate:

$$\|u\|_{\mathcal{C}^{\ell,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})} \le C\varepsilon^{1-\ell-\alpha}\|g\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})}.$$

The proof of this fact is straightforward and it is omitted, for similar results see for instance Lemma 4.1 in [21]. From this we get readily an *a priori* estimate for (3.35)

$$\|\psi\|_{\mathcal{C}^{\ell,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})} \le C\varepsilon^{1-\ell-\alpha}\|(1-\chi^*)g\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})} + e^{-c\varepsilon^{-1/4}}\left(\sum_{j=1}^{d}|\mathsf{c}_j| + \|\check{\varphi}\|_{\mathcal{C}^{1,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})}\right). \quad (3.40)$$

From the theory developed in the previous section we can also obtain an *a priori* estimate for (3.34). If we write

$$\mathfrak{g} = \widetilde{\chi}\left\{g + \chi^*\sum_{j=1}^{d}\mathsf{c}_j Z^{T,\mathsf{e}_j}_{\tau,\varepsilon} + (\mathbb{L}_\varepsilon - L_\varepsilon)\check{\varphi} - [\chi^*, L_\varepsilon]\check{\varphi} + \varepsilon^{-1}[q-f'(w^*)]\psi\right\},$$

then we have using (3.31)

$$\|\check{\varphi}^{\parallel}\|_{\mathcal{E}^{\ell,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})} \le C\varepsilon^{1-\alpha}(\|\mathfrak{g}^{\parallel}\|_{\mathcal{E}^{0,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^2\|\mathfrak{g}^{\perp}\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})}), \quad (3.41)$$

and using (3.32)

$$\|\check{\varphi}^{\perp}\|_{\mathcal{C}^{\ell,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})} \le C\varepsilon^{-1}\|\mathfrak{g}^{\perp}\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})}. \quad (3.42)$$

We note that the weighted norms we use for $\check{\phi}^{\parallel}$ and $\check{\phi}^{\perp}$ are scaled differently with ε. This slight nuisance is a result of our choice of the original scaling of the Cahn-Hilliard equation. We observe as well that with our definitions $\|\cdot\|_{\mathcal{E}^{0,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})} = \|\cdot\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})}$.

We will now estimate $\|\mathfrak{g}^{\parallel}\|_{\mathcal{E}^{0,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})}$. To do this we observe that, with $\check{\phi} = \check{\phi}^{\parallel} + \check{\phi}^{\perp}$, we have

$$\begin{aligned}
\|\big(\widetilde{\chi}(\mathbb{L}_\varepsilon - L_\varepsilon)\check{\varphi}\big)^{\parallel}\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})} &\le C\delta(\varepsilon)(\varepsilon^{-1}\|\check{\varphi}^{\parallel}\|_{\mathcal{E}^{2,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon\|\check{\varphi}^{\perp}\|_{\mathcal{C}^{2,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})}),\\
\|\big(\widetilde{\chi}[\chi^*, L_\varepsilon]\check{\varphi}\big)^{\parallel}\|_{\mathcal{E}^{0,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})} &\le C\delta(\varepsilon)^{-1}(\|\check{\varphi}^{\parallel}\|_{\mathcal{E}^{1,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})} + \mathcal{O}(e^{-c\varepsilon^{-1/3}})\|\check{\varphi}^{\perp}\|_{\mathcal{C}^{1,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})}),\\
\|\big(\widetilde{\chi}\varepsilon^{-1}[q-f'(w^*)]\psi\big)^{\parallel}\|_{\mathcal{E}^{0,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})} &\le C\varepsilon^{-1}\|\psi\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})}.
\end{aligned}$$

Next we estimate the orthogonal complement of these functions

$$
\begin{aligned}
\big\|\big(\widetilde{\chi}(\mathbb{L}_\varepsilon - L_\varepsilon)\check{\varphi}\big)^\perp\big\|_{\mathcal{E}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} &\le C(\|\check{\varphi}^\parallel\|_{\mathcal{E}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^2\|\check{\varphi}^\perp\|_{\mathcal{C}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}),\\
\|\big(\widetilde{\chi}[\chi^*, L_\varepsilon]\check{\varphi}\big)^\perp\|_{\mathcal{E}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} &\le \mathcal{O}(e^{-c\varepsilon^{-1/3}})\big(\|\check{\varphi}^\parallel\|_{\mathcal{E}^{1,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \|\check{\varphi}^\perp\|_{\mathcal{C}^{1,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}\big),\\
\big\|\big(\widetilde{\chi}\varepsilon^{-1}[q - f'(w^*)]\psi\big)^\perp\big\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} &\le C\varepsilon^{-1}\|\psi\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})}.
\end{aligned}
$$

We can estimate the parameters c_j by projection of $\mathfrak{g}$ onto $Z^{T,\mathsf{e}_j}_{\tau,\varepsilon}$. Using the above estimate we get

$$
\begin{aligned}
|\mathsf{c}_j| \le{}& C\{\|(\widetilde{\chi}\ g)^\perp\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \big\|\big(\widetilde{\chi}(\mathbb{L}_\varepsilon - L_\varepsilon)\check{\varphi}\big)^\perp\big\|_{\mathcal{E}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}\\
&+ \|\big(\widetilde{\chi}[\chi^*, L_\varepsilon]\check{\varphi}\big)^\perp\|_{\mathcal{E}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \big\|\big(\widetilde{\chi}\varepsilon^{-1}[q - f'(w^*)]\psi\big)^\perp\big\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}\}\\
\le{}& C\Big\{\|(\widetilde{\chi}\ g)^\perp\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \|\check{\varphi}^\parallel\|_{\mathcal{E}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^2\|\check{\varphi}^\perp\|_{\mathcal{C}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}\\
&+ \mathcal{O}(e^{-c\varepsilon^{-1/3}})\big(\|\check{\varphi}^\parallel\|_{\mathcal{E}^{1,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \|\check{\varphi}^\perp\|_{\mathcal{C}^{1,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}\big)\\
&+ \varepsilon^{-1}\|\psi\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})}\Big\}\\
\le{}& C\Big\{\|(\widetilde{\chi}\ g)^\perp\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \|\check{\varphi}^\parallel\|_{\mathcal{E}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^2\|\check{\varphi}^\perp\|_{\mathcal{C}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}\\
&+ \varepsilon^{-1}\|\psi\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})}\Big\}.
\end{aligned}
$$

Now we use estimates (3.41)–(3.42). After rearranging terms suitably and using

$$
\varepsilon^{1-\alpha}\delta^{-1}(\varepsilon) = o(1)
$$

to absorb $\check{\varphi}^\parallel$ in the first inequality below we get

$$
\begin{aligned}
\|\check{\varphi}^\parallel\|_{\mathcal{E}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} \le{}& C\varepsilon^{1-\alpha}\{\|(\widetilde{\chi}\ g)^\parallel\|_{\mathcal{E}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^2\|(\widetilde{\chi}\ g)^\perp\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}\\
&+ \varepsilon^{-1}\|\psi\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})} + \delta(\varepsilon)\|\check{\varphi}^\perp\|_{\mathcal{C}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}\},\\
\|\check{\varphi}^\perp\|_{\mathcal{C}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} \le{}& C\varepsilon^{-1}\{\|(\widetilde{\chi}\ g)^\perp\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \|\check{\varphi}^\parallel\|_{\mathcal{E}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}\\
&+ \varepsilon^{-1}\|\psi\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})}\}.
\end{aligned}
\tag{3.43}
$$

From (3.40) we get as well for $\ell = 0, 1, 2$

$$
\begin{aligned}
\|\psi\|_{\mathcal{C}^{\ell,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})} \le{}& C\varepsilon^{1-\ell-\alpha}\{\|(1-\chi^*)g\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^3)}\\
&+ e^{-c\varepsilon^{-1/8}}(\|\widetilde{\chi}g\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \|\check{\varphi}^\parallel\|_{\mathcal{E}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}\\
&+ \|\check{\varphi}^\perp\|_{\mathcal{C}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \|\psi\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})})\}.
\end{aligned}
$$

Using the fact that $\delta(\varepsilon)\varepsilon^{-\alpha} = o(1)$ to absorb term $\delta(\varepsilon)\|\check{\varphi}^\perp\|_{\mathcal{C}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}$ appearing on the right hand side of the first inequality in (3.43) and combining these estimates we get (3.38) and (3.39). □

Using these *a priori* estimates we can solve the system (3.34)–(3.35) by a standard fixed point argument. To do this we replace the functions $\check{\varphi}^\parallel, \check{\varphi}^\perp, \psi$ on the right hand side of the system by known functions $\check{\Phi}^{\parallel}, \check{\Phi}^{\perp}, \Psi$ which satisfy estimates of the same type as (3.38)–(3.39) but with constants bigger that those appearing in (3.38)–(3.39). Then we have a map

$$(\check{\Phi}^{\parallel}, \check{\Phi}^{\perp}, \Psi) \longmapsto (\check{\varphi}^{\parallel}, \check{\varphi}^{\perp}, \psi),$$

from a certain ball in the space $\mathcal{E}^{2,\alpha}_{\mu}(\mathring{D}_\tau \times \mathbb{R}) \times \mathcal{C}^{2,\alpha}_{\mu}(\mathring{D}_\tau \times \mathbb{R}) \times \mathcal{C}^{2,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})$ into itself. This and the Lipschitz character of this map being evident from the way we have derived *a priori* estimates allows for an application of the Banach fixed point theorem.

3.4. Proof of Theorem 1.1

Now we can finish solving the nonlinear problem

$$L_\varepsilon \varphi = \ell_\varepsilon - N_\varepsilon(w^*) - Q_\varepsilon(\varphi).$$

As we saw above we need to modify this equation by introducing Lagrange multipliers. Thus we will consider

$$L_\varepsilon \varphi = \ell_\varepsilon - N_\varepsilon(w^*) - Q_\varepsilon(\varphi) + \chi^* \sum_{j=1}^{d} \mathsf{c}_j Z^{T,\mathsf{e}_j}_{\tau,\varepsilon}. \tag{3.44}$$

To solve this problem we use a fixed point argument and the linear theory in Lemma 3.3.1 above. The first task is to calculate the size of the error of the approximation $\ell_\varepsilon - N_\varepsilon(w^*)$. This is straightforward using the definition of w^* and formula (2.7). We recall here that $h = \varepsilon^2 h_0$, where h_0 is a constant and consequently this last formula simplifies significantly. We can write:

$$\ell_\varepsilon - N_\varepsilon(w^*) = \chi^*[\ell_\varepsilon - N_\varepsilon(w)] + [N(w^*) - \chi^* N(w) - (1-\chi^*)N(\mathbb{H})] \equiv A_1 + A_2,$$

since $\ell_\varepsilon = N_\varepsilon(\mathbb{H})$ in $\operatorname{supp}(1-\chi^*)$. Using exponential decay of $w - (\pm 1 + \sigma^{\pm}_\varepsilon)$ when $\mathtt{t} \to \pm\infty$ we get easily:

$$\|Y^*_{\varepsilon,h} \widetilde{\chi} A_2\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_\tau \times \mathbb{R})} \le C_0 e^{-c_\mu \varepsilon^{-1/3}},$$
$$\|(1-\chi^*) A_2\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})} \le C_0 e^{-\theta \varepsilon^{-1}}.$$

To estimate A_1 some standard calculations which we will omit are needed (c.f Section 2.2). As a result we get

$$\|[Y^*_{\varepsilon,h} \widetilde{\chi} A_1]^{\parallel}\|_{\mathcal{E}^{0,\alpha}_{\mu}(\mathring{D}_\tau \times \mathbb{R})} \le C_0 \varepsilon^3,$$
$$\|[Y^*_{\varepsilon,h} \widetilde{\chi} A_1]^{\perp}\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_\tau \times \mathbb{R})} \le C_0 \varepsilon^3,$$
$$\|(1-\chi^*) A_1\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})} \le C_0 e^{-\theta \varepsilon^{-1}},$$

where C_0, c_μ and θ are positive constants. Now we use the linear theory developed in the previous section to solve the nonlinear problem (3.44). Thus we write $\varphi = \chi^* \check{\varphi} \circ Y_{\varepsilon,h} + \psi$, and further decompose $\check{\varphi} = \check{\varphi}^{\parallel} + \check{\varphi}^{\perp}$ where $\check{\varphi}^{\parallel} \in \mathcal{X} \cap \mathcal{C}^{2,\alpha}_{\mu}(\mathring{D}_\tau \times \mathbb{R})$, $\varphi^{\perp} \in \mathcal{Y} \cap \mathcal{C}^{2,\alpha}_{\mu}(\mathring{D}_\tau \times \mathbb{R})$ and $\psi \in \mathcal{C}^{2,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})$. To set up a fixed point scheme we fix functions $\widetilde{\varphi}^{\parallel}$, $\widetilde{\varphi}^{\perp}$ and $\widetilde{\psi}$ in these sets such that

$$\begin{aligned} \|\widetilde{\varphi}^{\parallel}\|_{\mathcal{E}^{2,\alpha}_{\mu}(\mathring{D}_\tau \times \mathbb{R})} &\le K \varepsilon^{4-\alpha}, \\ \|\widetilde{\varphi}^{\perp}\|_{\mathcal{C}^{2,\alpha}_{\mu}(\mathring{D}_\tau \times \mathbb{R})} &\le K \varepsilon^{2}, \\ \|\widetilde{\psi}\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})} &\le K e^{-\bar{\theta} \varepsilon^{-1}}, \end{aligned} \tag{3.45}$$

where K is a large constant to be chosen and $\bar{\theta} \in (\theta/2, \theta)$ is a constant. Let us denote the right hand side of (3.44) by $\mathfrak{g}$. It is evident that under the assumptions (3.45), and with a suitable choice of the constants $\alpha > 0$ and $\mu \in (0, |\eta|)$ we can solve the problem (3.44) for functions $(\breve{\varphi}^{\|}, \breve{\varphi}^{\perp}, \psi)$ which again satisfy (3.45). Thus we have a non-linear map

$$(\widetilde{\varphi}^{\|}, \widetilde{\varphi}^{\perp}, \widetilde{\psi}) \longmapsto (\breve{\varphi}^{\|}, \breve{\varphi}^{\perp}, \psi), \tag{3.46}$$

of this set into itself. To show that this map is a contraction is straightforward, using the quadratic nature of the nonlinear function $Q(\varphi)$. At the end we have a solution of the problem:

$$\varepsilon \Delta u + \frac{1}{\varepsilon} f(u) = \ell_\varepsilon + \sum_{j=1}^{d} \chi^* \mathsf{c}_j Z_{\tau,\varepsilon}^{T,\mathsf{e}_j}, \quad \text{in } \mathbb{R}^{d-1} \times S_{2T_\tau}, \tag{3.47}$$

where $Z_{\tau,\varepsilon}^{T,\mathsf{e}_j}$ is the (approximate) element of the kernel of the linear operator $\mathbb{L}_\varepsilon$ associated with translation in the direction of the x_j axis. To show that in fact

$$\mathsf{c}_j = 0, \quad j = 1, \ldots, d,$$

we need:

Lemma 3.4.1 (Balancing formula) Let $X = \sum a_j \partial_{x_j}$ be the infinitesimal generator of translations or rotations in $\mathbb{R}^d$. For any $\mathcal{C}^2(\mathbb{R}^d)$ function it holds:

$$\operatorname{div} \left[\left(\frac{\varepsilon}{2} |\nabla u|^2 - \frac{1}{\varepsilon} F(u) \right) X(u) - \varepsilon X(u) \nabla u \right] = - \left[\varepsilon \Delta u + \frac{1}{\varepsilon} F'(u) \right] X(u).$$

Proof. Simple computations yields

$$\begin{aligned} -\Delta u (X \cdot \nabla u) &= -\operatorname{div} ((X \cdot \nabla u) \nabla u) + \nabla (X \cdot \nabla) \cdot \nabla u \\ \operatorname{div} \left(\frac{1}{2} |\nabla u|^2 X \right) &= \frac{1}{2} \nabla \left(|\nabla u|^2 \right) \cdot X \\ &= ((\nabla u \cdot \nabla) \nabla u) \cdot X \\ &= \nabla (X \cdot \nabla u) \cdot \nabla u \\ \operatorname{div}(F(u) X) &= F'(u) \nabla u \cdot X. \end{aligned}$$

□

We will take $X_j = \partial_{x_j}$ for some $1 \le j \le d$ and integrate the balancing formula over the cylinder $\mathcal{C}_R = B_R \times S_{2T_\tau}$. Using (3.47) and Gauss theorem we get:

$$\underbrace{\int_{\partial \mathcal{C}_R} \left(\frac{\varepsilon}{2} |\nabla u|^2 - \frac{1}{\varepsilon} F(u) + \ell_\varepsilon u \right) n_j \, dS}_{I_R^1} - \underbrace{\int_{\partial \mathcal{C}_R} \partial_{x_j} u \partial_n u \, dS}_{I_R^2} = - \underbrace{\int_{\mathcal{C}_R} \left(\sum_{j'=1}^{d} \chi^* \mathsf{c}_{j'} Z_{\tau,\varepsilon}^{T,\mathsf{e}_{j'}} \right) \partial_{x_j} u}_{I_R^3} .$$

We first analyze $d = 3$, and distinguish the cases $j = 3$ and $j = 1$ or 2.

Case $j = 3$. The first integral I_R^1 can be decomposed by the sum of the integrals over the top, the bottom and the cylindrical surface. The integrals over the top and the bottom cancel out, because u is periodic, on the other hand on the cylinder surface, say $M_R = \partial B_R \times S_{2T_\tau}$, we have that the outer normal is $\hat{r} = (x_1, x_2, 0)/r$ where $r = \sqrt{x_1^2 + x_2^2}$, hence $n_i := \hat{r} \cdot \mathtt{e}_i = x_i/r$, $n_1 = \sin\theta$, $n_2 = \cos\theta$, and $\nu_\varepsilon = -\tau \cosh\sigma_\tau(s)\hat{r} + \sigma_\tau'(s)\mathtt{e}_3$. correspond to the normal vectors. But the integration on M_R contains the term $\mathtt{e}_3 \cdot \hat{r} = 0$, hence this integral also vanishes. The integral I_R^2 admits the same decomposition and again integration on the top an the bottom cancel out. Let us show that

$$\lim_{R\to\infty} \int_{M_R} \partial_{x_3} u \frac{\partial u}{\partial \hat{r}} \, dS = 0$$

We note that we may write

$$u = H(\mathtt{t}) + \sigma_\varepsilon + \varepsilon^2 \psi_0(\mathtt{y}, \mathtt{t}) \tag{3.48}$$

near $\mathring{D}_\tau$, thus $\nabla u = H'(\mathtt{t})\nabla \mathtt{t} + \varepsilon^2 \nabla \psi_0$, where $\nabla t = \nu_\tau$. From the estimates

$$|\psi_0| + |\nabla \psi_0| \leq C(\cosh \mathtt{t})^\mu$$

we obtain

$$\nabla u = H'(\mathtt{t})\nu_\tau + \mathcal{O}\left(\varepsilon^2 (\cosh \mathtt{t})^{-\mu}\right). \tag{3.49}$$

Using the fact that $Z_{\varepsilon,\tau}^{T,\mathtt{e}_j} = \nu_\tau \cdot \mathtt{e}_j$, we obtain $\nabla u \cdot \mathtt{e}_j = H'(\mathtt{t})\nu_\tau \cdot \mathtt{e}_j + \mathcal{O}\left(\varepsilon^2 (\cosh \mathtt{t})^{-\mu}\right)$, hence

$$\partial_{x_3} u \nabla u \cdot \hat{r} = H^2(\mathtt{t})\tau\sigma_\tau' \cosh\sigma_\tau \mathcal{O}\left((\cosh \mathtt{t})^{-\mu}\right) + \mathcal{O}\left(\varepsilon^2 (\cosh \mathtt{t})^{-\mu}\right),$$

therefore

$$\int_{M_R} \partial_{x_3} u \nabla u \cdot \hat{r} \, dA = \mathcal{O}\left(\frac{1}{R^2}\right) \int_{RS^1} \int_0^{s_\tau} \tau\sigma_\tau' \cosh\sigma + \mathcal{O}\left(\frac{1}{R^2}\right) = \mathcal{O}\left(\frac{1}{R}\right).$$

Case $j = 1, 2$. Similarly as before we have the top and the bottom integrals in I_R cancel out, equivalently for I_R^2. Let us show that

$$\lim_{R\to\infty} \int_{M_R} \left(\frac{\varepsilon}{2}|\nabla u|^2 - \frac{1}{\varepsilon}F(u) + \ell_\varepsilon u\right) n_2 + \partial_{x_2} \nabla u \cdot \hat{r} \, dS = 0.$$

One of the integral involved is

$$\int_{M_R} \partial_{x_2} u \cdot \hat{r} \, dS = \mathcal{O}\left(\frac{1}{R^2}\right) \int_{RS^1} \int_0^{s_\tau} (\tau \cosh\sigma_\tau)^2 + \mathcal{O}\left(\frac{1}{R^2}\right) = \mathcal{O}\left(\frac{1}{R}\right).$$

Using (3.49) we obtain

$$|\nabla u(\mathtt{x})|^2 = |H'(\mathtt{t})|^2 + \mathcal{O}\left(\varepsilon^2 (\cosh \mathtt{t})^{-\mu}\right),$$

therefore we have

$$\begin{aligned}\int_{M_R} \frac{\varepsilon}{2} |\nabla u|^2 n_2 \, dS &= \int_0^{s_\tau} \int_0^{2\pi} \varepsilon (H'(\mathtt{t}(z, R)))^2 \sin\theta \, d\theta \, dz + \mathcal{O}\left(1/R^2\right) \\ &= \int_0^{2\pi} \varepsilon \sin\theta \, d\theta R \int_0^{s_\tau} (H'(\mathtt{t}(z, R))^2 + \mathcal{O}(1/R) = \mathcal{O}(1/R),\end{aligned}$$

because the first integral equals zero and the second one is bounded, thus

$$\lim_{R\to\infty}\int_{M_R}\frac{\varepsilon}{2}|\nabla u|^2 n_2\ dS=0.$$

With similar arguments we obtain

$$\lim_{R\to\infty}\int_{M_R}\ell_\varepsilon u n_2\ dS=\lim_{R\to\infty}\int_{M_R}\frac{1}{\varepsilon}F(u)n_2\ dS=0,$$

in fact, using the approximate estimate (3.48) for u we get

$$\begin{aligned}\int_{M_R}\ell_\varepsilon u n_2\ dS&=\int_0^{s_\tau}\int_0^{2\pi}\varepsilon H(\mathtt{t}(z,R))\sin\theta\ d\theta\ dz+\mathcal{O}\left(1/R^2\right)\\&=\int_0^{2\pi}\ell_\varepsilon\sin\theta\ d\theta R\int_0^{s_\tau}H(\mathtt{t}(z,R)+\mathcal{O}\left(1/R\right)=\mathcal{O}\left(1/R\right).\end{aligned}$$

Finally, we multiply (3.47) by each element of the approximate kernel an integrate over $\mathbb{R}^2\times S_{2T_\tau}$, using the computations we have just found, we obtain that for each j

$$\lim_{R\to\infty}I_R^3=\mathtt{c}_j\int_{\mathring{D}_\tau\times\mathbb{R}}|Z_{\tau,\varepsilon}^{T,\mathtt{e}_j}|^2+o(1)\sum_{j'=1}^{d}\mathtt{c}_{j'},$$

from which we get immediately $\mathtt{c}_j=0$, $j=1,2,3$. This conclude the case $d=3$. For the general case we argue in a similar way, in fact, using (2.20) we obtain that $\nu_\tau=\Lambda_\tau^1\Theta+\Lambda_\tau^2\mathtt{e}_d$, for some bounded functions Λ_τ^i, therefore we distinguish two cases $j=d$ and $j=1,\dots,d-1$ which are the equivalent cases as before.

Chapter 4

Proof of the Theorem 1.2

4.1. Further properties of the radially symmetric solution and the linearised operator for the periodic Delaunay solution

For sake of simplicity from now one we will assume that the dimension is $d = 3$, and rename by w_τ the rotationally symmetric solution we have found in Theorem 1.1. In the present section we will provide extra properties of w_τ that are needed for the proof of Theorem 1.2. From what we have done in the previous chapter, we can describe in more detail the local behavior of w_τ. To begin with, we express w_τ near D_τ in the local stretched Fermi coordinates $(\mathtt{y}, \mathtt{t})$: since w_τ is z periodic it suffices to restrict this last variable to one period of D_τ. Thus by $\mathring{D}_\tau$ we denote one period piece of D_τ with the top and bottom identified. We recall weighted Hölder norms on $\mathring{D}_\tau \times \mathbb{R}$ defined in (3.22)

$$\|u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} = \sup_{\mathtt{t}\in\mathbb{R}} (\cosh \mathtt{t})^\mu \|u\|_{\mathcal{C}^{0,\alpha}(\mathring{D}_\tau\times(\mathtt{t}-1,\mathtt{t}+1))},$$
$$\|u\|_{\mathcal{C}^{1,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} = \|u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \|\nabla_{\mathring{D}_\tau\times\mathbb{R}} u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})},$$
$$\|u\|_{\mathcal{C}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} = \|u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \|\nabla_{\mathring{D}_\tau\times\mathbb{R}} u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \|\nabla^2_{\mathring{D}_\tau\times\mathbb{R}} u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})},$$

where $\nabla_{D_\tau\times\mathbb{R}} = \nabla_{\mathring{D}_\tau} + \partial_{\mathtt{t}}$. With these norms and the results of the previous Chapter there exists $\mu_0 > 0$ and $\alpha_0 > 0$ such that for $0 < \mu < \mu_0$ and $0 < \alpha < \alpha_0$ it holds

$$Y_\varepsilon^* w_\tau(\mathtt{y}, \mathtt{t}) = U(\mathtt{t}) + \mathcal{O}_{\mathcal{C}^{2,\alpha}_\mu(D_\tau\times\mathbb{R})}(\varepsilon^{2-\alpha}). \tag{4.1}$$

Above the symbol $\mathcal{O}_{\mathcal{C}^{2,\alpha}_\mu(D_\tau\times\mathbb{R})}(\varepsilon^{2-\alpha})$ denotes functions whose $\mathcal{C}^{2,\alpha}_\mu(D_\tau\times\mathbb{R})$ norm is bounded by a constant times $\varepsilon^{2-\alpha}$. This formula is valid in a tubular neighbourhood $\mathcal{N}_\delta = \mathcal{N}_\delta(\varepsilon)$ of $\mathring{D}_\tau$, where $\delta(\varepsilon) = \mathcal{O}(\varepsilon^{2/3})$. In local variables this means $|\mathtt{t}| \leq C\varepsilon^{-1/3}$. Outside of this neighbourhood we have

$$w_\tau = \pm 1 + \sigma_\varepsilon^\pm + \psi, \qquad \text{where } \|\psi\|_{\mathcal{C}^2(\mathbb{R}^3)} \leq Ce^{-c/\varepsilon^{1/3}}. \tag{4.2}$$

In fact more is true: we claim that w_τ converges exponentially to constants $\pm 1 + \sigma_\varepsilon^\pm$ away from D_τ. More precisely, if D_τ is given as the surface of revolution of the curve $x_1 = \rho_\tau(z)$

then

$$|w_\tau(r,z)+1-\sigma_\varepsilon^-|\le C\exp\left(-\frac{c}{\varepsilon}(r-\rho_\tau(z))\right),\quad r>\rho_\tau(z),\quad r=\sqrt{x_1^2+x_2^2}, \tag{4.3}$$

with similar estimate when $r<\rho_\tau(z)$. To prove this we note that (4.3) valid in a tubular neighbourhood of D_τ by (4.1) and the fact that $\mathtt{t}$ and $\frac{(r-\rho_\tau(z))}{\varepsilon}$ are comparable in this neighbourhood. Far form D_τ we use the fact that $w_\tau=\pm1+\sigma_\varepsilon^\pm+\psi$, where ψ is an exponentially small in ε function (see (4.2)) and a comparison argument. These estimates can be made more precise as far as the rate of exponential decay but we will not need such a precision here.

One property that we will need in the sequel is

Proposition 4.1.1 w_τ is differentiable with respect to τ.

Proof. This is consequence of the version of the fixed point Banach theorem (see [30] Theorem 3.2). Since w_τ can be written as the fixed solution of the system (3.46), where all terms are differentiable functions of τ. For instance, one of these terms is the ansatz $U(\mathtt{t})$ which in Fermi coordinate we have on D_τ

$$\partial_\tau\mathtt{t}=\varepsilon^{-1}\partial_\tau X_\tau\cdot N_\tau=-\varepsilon^{-1}\Phi_\tau^D,$$

where Φ_τ^D is the Jacobi field on D_τ associated with the change of the Delaunay parameter. □

4.2. The linearized operator near D_τ

Our main objective in the next section will be to study the linearized operator of the Delaunay solution

$$L_{w_\tau}=\varepsilon\Delta+\frac{1}{\varepsilon}f'(w_\tau),$$

as an operator defined for functions on $\mathbb{R}^3$ and here we will introduce some basic observations and notations needed later.

Using (2.5) we find expression of L_{w_τ} in stretched Fermi coordinates in $\mathcal{N}_\delta$

$$Y_\varepsilon^*L_{w_\tau}u=\varepsilon^{-1}\left[\partial_{\mathtt{tt}}u-\varepsilon(H_{D_\tau}+\varepsilon\mathtt{t}|A_{D_\tau}|^2+\mathbb{Q}_\varepsilon)\partial_{\mathtt{t}}u+f'(w_\tau)u\right]+\varepsilon\Delta_{D_\tau}u+\varepsilon\mathbb{A}_\varepsilon u,$$

where with some abuse of notation we write u and w_τ instead of Y_ε^*u and $Y_\varepsilon^*w_\tau$ (we will consistently abuse notation this way whenever it is unambiguous). One technical problem we will have to face is the fact that while the operator L_{w_τ} is defined in $\mathbb{R}^3$ its expression in local coordinates $Y_\varepsilon^*L_{w_\tau}$ makes sense only in $\mathcal{N}_\delta$ and not as we would like in $D_\tau\times\mathbb{R}$. There are possibly many ways to extend $Y_\varepsilon^*L_{w_\tau}$ and we will chose one of them for the rest of the paper. Let $\chi(s)$ be a smooth nonnegative cut-off function equal to 1 for $|s|\le1$ and equal to 0 for $|s|>2$. We set

$$\chi_{\varepsilon/\delta}(\mathtt{t})=\chi\left(\frac{\varepsilon\mathtt{t}}{\delta}\right). \tag{4.4}$$

We need to extend the function $Y_\varepsilon^* w_\tau$ in such a way that it is defined outside of $\mathcal{N}_\delta$. To this end we set

$$\mathtt{w}_\tau = \chi_{\varepsilon/\delta}(\mathtt{t}) Y_\varepsilon^* w_\tau + \left(1 - \chi_{\varepsilon/\delta}(\mathtt{t})\right) U(\mathtt{t}).$$

Next we define the extension of the operator $Y_\varepsilon^* L_{w_\tau}$ by

$$\mathbb{L}_{w_\tau} u = \varepsilon^{-1} \left[\partial_{\mathtt{tt}} u - \varepsilon \left(H_{D_\tau} + \varepsilon \mathtt{t} \chi_{\varepsilon/\delta}(\mathtt{t}) |A_{D_\tau}|^2 + \chi_{\varepsilon/\delta}(\mathtt{t}) \mathbb{Q}_\varepsilon\right) \partial_{\mathtt{t}} u + f'(\mathtt{w}_\tau) u\right] + \varepsilon \Delta_{D_\tau} u + \varepsilon \chi_{\varepsilon/\delta}(\mathtt{t}) \mathbb{A}_\varepsilon u. \quad (4.5)$$

As we will see $\mathbb{L}_{w_\tau}$ resembles the operator

$$\mathcal{L} u = \frac{1}{\varepsilon} \left[\partial_{\mathtt{tt}} u + f'(H) u\right] + \varepsilon \left[\Delta_{D_\tau} u + |A_{D_\tau}|^2 u\right]$$

whose kernel is fairly easy to determine by separation of variables. Indeed, taking $u = H'(\mathtt{t}) \psi(\mathtt{y})$ we get

$$\mathcal{L}(H' \psi) = \varepsilon H' \left[\Delta_{D_\tau} \psi + |A_{D_\tau}|^2 \psi\right] \quad (4.6)$$

and therefore the Jacobi fields of D_τ determine the Jacobi fields of $\mathcal{L}$. Let us explain in what sense L_{w_τ} and $\mathcal{L}$ are similar. To do this we will use the operator $\mathbb{L}_{w_\tau}$ (our theory of the operator L_{w_τ} is based on exploiting this link). First we need a function which will play a role of $H'(\mathtt{t})$. Since our proof is based on a perturbation argument there is no unique way to define such a function but a natural candidate seems to be $\partial_{\mathtt{t}} Y_\tau^* w_\tau$. An important observation to make is that $\partial_{\mathtt{t}}$ and Δ do not commute so we do not have $\mathbb{L}_{w_\tau} \partial_{\mathtt{t}} w_\tau = 0$ and as we will see below the commutator $[\Delta, \partial_{\mathtt{t}}]$ gives rise to the term $|A_{D_\tau}|^2$ in $\mathcal{L}$. Since $\partial_{\mathtt{t}} Y_\tau^* w_\tau$ is defined only in $\mathcal{N}_\delta$ we define the extension of this function to $D_\tau \times \mathbb{R}$ by

$$\mathtt{W} = \chi_{\varepsilon/\delta}(\mathtt{t}) \partial_{\mathtt{t}} Y_\varepsilon^* w_\tau + \left(1 - \chi_{\varepsilon/\delta}(\mathtt{t})\right) U'(\mathtt{t}), \quad (4.7)$$

where U is the solution of (2.10). Note that $\mathtt{W}$ depends on $\mathtt{t} \in \mathbb{R}$ and $\mathtt{y} \in D_\tau$ but using (4.1) we get

$$\mathtt{W}(\mathtt{y}, \mathtt{t}) = U'(\mathtt{t}) + \mathcal{O}_{C^{1,\alpha}_\mu(D_\tau \times \mathbb{R})}(\varepsilon^{2-\alpha})$$

globally on $D_\tau \times \mathbb{R}$, which means that the dependence on $\mathtt{y}$ is mild. Next we calculate

$$\mathbb{L}_{w_\tau} \mathtt{W} = \chi_{\varepsilon/\delta} \mathbb{L}_{w_\tau} \partial_{\mathtt{t}} Y_\varepsilon^* w_\tau + \left(1 - \chi_{\varepsilon/\delta}\right) \mathbb{L}_{w_\tau} U' + \left[\mathbb{L}_{w_\tau}, \chi_{\varepsilon/\delta}\right] \partial_{\mathtt{t}} Y_\varepsilon^* w_\tau + \left[\mathbb{L}_{w_\tau}, 1 - \chi_{\varepsilon/\delta}\right] U'. \quad (4.8)$$

The first term above is the most complicated. For brevity let us denote $v_\tau = \partial_{\mathtt{t}} Y_\varepsilon^* w_\tau$. With this notation differentiating the equation satisfied by w_τ in $\mathcal{N}_\delta$ with respect to $\mathtt{t}$ we have

$$\mathbb{L}_{w_\tau} v_\tau - \varepsilon |A_{D_\tau}|^2 v_\tau = -\varepsilon \left[\mathbb{Q}_\varepsilon, \partial_{\mathtt{t}}\right] v_\tau + \varepsilon \left[\mathbb{A}_\varepsilon, \partial_{\mathtt{t}}\right] Y_\varepsilon^* w_\tau,$$

where $[A, B] = AB - BA$. By definition of $\mathbb{Q}_\varepsilon$ we see that

$$\left[\mathbb{Q}_\varepsilon, \partial_{\mathtt{t}}\right] v_\tau = -\varepsilon^2 \left(2\mathtt{t} \sum_{j=1}^{2} \mathbb{k}_j^3 + 3\varepsilon \mathtt{t}^2 \sum_{j=1}^{2} \mathbb{k}_j^4 + \dots\right) v_\tau = \mathcal{O}_{C^{1,\alpha}_\mu(D_\tau \times \mathbb{R})}(\varepsilon^2).$$

The differential operator $\mathbb{A}_\varepsilon$ contains derivatives in $\mathtt{y} \in D_\tau$ only while $Y_\varepsilon^* w_\tau$ is, up to order $\mathcal{O}_{C^{2,\alpha}_\mu(D_\tau \times \mathbb{R})}(\varepsilon^{2-\alpha})$, a function of $\mathtt{t}$. This gives

$$\varepsilon \left[\mathbb{A}_\varepsilon, \partial_{\mathtt{t}}\right] Y_\varepsilon^* w_\tau = \varepsilon \mathbb{A}_\varepsilon \partial_{\mathtt{t}} Y_\varepsilon^* w_\tau - \varepsilon \partial_t \mathbb{A}(\mathtt{y}, \varepsilon \mathtt{t}) Y_\varepsilon^* w_\tau = \mathcal{O}_{C^{0,\alpha}_\mu(D_\tau \times \mathbb{R})}(\varepsilon^{3-\alpha}).$$

It follows that in $\mathcal{N}_\delta$ we get

$$\mathbb{L}_{w_\tau} v_\tau - \varepsilon |A_{D_\tau}|^2 v_\tau = \mathcal{O}_{C^{0,\alpha}_\mu(D_\tau\times\mathbb{R})}(\varepsilon^{3-\alpha})$$

Considering other terms in (4.8) from the fact that $\chi_{\varepsilon/\delta} = \chi_{\varepsilon/\delta}(\mathtt{t})$ and (4.1) we get

$$\left[\mathbb{L}_{w_\tau}, \chi_{\varepsilon/\delta}\right] \partial_{\mathtt{t}} Y^*_\tau w_\tau = \mathcal{O}_{C^{0,\alpha}_\mu(D_\tau\times\mathbb{R})}(\varepsilon^{3-\alpha}).$$

Similar estimates hold for terms involving $U'(\mathtt{t})$. In summary we get

$$\mathbb{L}_{w_\tau}\mathtt{W} - \varepsilon |A_{D_\tau}|^2 \mathtt{W} = \mathcal{O}_{C^{0,\alpha}_\mu(D_\tau\times\mathbb{R})}(\varepsilon^{3-\alpha}). \tag{4.9}$$

Now let $\psi \in C^{2,\alpha}(D_\tau)$ be fixed. Using (4.9) we get

$$\begin{aligned}
\mathbb{L}_{w_\tau}(\psi\mathtt{W}) &= \psi\mathbb{L}_{w_\tau}\mathtt{W} + \varepsilon\mathtt{W}\left(\Delta_{D_\tau} + \chi_{\varepsilon/\delta}\mathbb{A}_\varepsilon\right)\psi + \varepsilon\left[\Delta_{D_\tau} + \chi_{\varepsilon/\delta}\mathbb{A}_\varepsilon, \mathtt{W}\right]\psi \\
&= \psi\left(\mathbb{L}_{w_\tau}\mathtt{W} - \varepsilon|A_{D_\tau}|^2\mathtt{W}\right) + \varepsilon\mathtt{W}\left(\Delta_{D_\tau} + |A_{D_\tau}|^2 + \chi_{\varepsilon/\delta}\mathbb{A}_\varepsilon\right)\psi + \varepsilon\left[\Delta_{D_\tau} + \chi_{\varepsilon/\delta}\mathbb{A}_\varepsilon, \mathtt{W}\right]\psi \\
&= \varepsilon\mathtt{W}\left(\mathcal{J}_{D_\tau} + \chi_{\varepsilon/\delta}\mathbb{A}_\varepsilon\right)\psi + \varepsilon\left[\Delta_{D_\tau} + \chi_{\varepsilon/\delta}\mathbb{A}_\varepsilon, \mathtt{W}\right]\psi + \mathcal{O}_{C^{0,\alpha}_\mu(D_\tau\times\mathbb{R})}(\varepsilon^{3-\alpha})\psi.
\end{aligned} \tag{4.10}$$

For future reference we note that

$$\left\|\left[\Delta_{D_\tau} + \chi_{\varepsilon/\delta}\mathbb{A}_\varepsilon, \mathtt{W}\right]\psi\right\|_{C^{0,\alpha}_\mu(D_\tau\times\mathbb{R})} \le C\varepsilon^{2-\alpha}\|\psi\|_{C^{1,\alpha}(D_\tau)}.$$

Observe that formula (4.10) is quite similar to (4.6) and in particular it is clear that if $\mathbb{L}_{w_\tau}(\psi\mathtt{W}) \approx 0$ then ψ should be a Jacobi field on D_τ, and as a consequence we should get an approximate Jacobi field of w_τ. Indeed we can easily describe explicit Jacobi fields of the two ended Delaunay solution w_τ which are approximately of the form $\psi\mathtt{W}$. Let $\mathtt{h} = \sum_{i=1}^3 h_i\mathtt{e}_i$ be a vector, $\mathcal{R}_\vartheta(\mathtt{x}) = \mathcal{R}_{\vartheta_1,\vartheta_2}(\mathtt{x})$ be a rotation in $\mathbb{R}^3$, where ϑ_i is the angle of the rotation about the x_i axis $i = 1, 2$, and η be a number such that $|\eta|$ is small. Then the function

$$\Phi_{\mathtt{h},\vartheta,\eta}(w_\tau) = (w_{\tau+\eta}\circ\mathcal{R}_\vartheta)(\mathtt{x}+\mathtt{h}),$$

is also a solution of the Cahn-Hilliard equation (1.12). In particular, taking derivatives of $\Phi_{\mathtt{h},\vartheta,\eta}(w_\tau)$ with respect to the parameters we get

$$\begin{aligned}
L_{w_\tau}\partial_{h_i}\Phi_{\mathtt{h},\vartheta,\eta}(w_\tau)\,|_{\mathtt{h},\vartheta,\eta=0} &= 0, \quad i = 1,2,3, \\
L_{w_\tau}\partial_{\vartheta_i}\Phi_{\mathtt{h},\vartheta,\eta}(w_\tau)\,|_{\mathtt{h},\vartheta,\eta=0} &= 0, \quad i = 1,2, \\
L_{w_\tau}\partial_{\eta}\Phi_{\mathtt{h},\vartheta,\eta}(w_\tau)\,|_{\mathtt{h},\vartheta,\eta=0} &= 0,
\end{aligned}$$

and hence the 6 dimensional linear space

$$\mathcal{I}_{w_\tau} = \operatorname{span}\{\partial_{h_i}\Phi_{\mathtt{h},\vartheta,\eta}(w_\tau)\,|_{\mathtt{h},\vartheta,\eta=0}, \quad \partial_{\vartheta_i}\Phi_{\mathtt{h},\vartheta,\eta}(w_\tau)\,|_{\mathtt{h},\vartheta,\eta=0}, \quad \partial_{\eta}\Phi_{\mathtt{h},\vartheta,\eta}(w_\tau)\,|_{\mathtt{h},\vartheta,\eta=0}\}. \tag{4.11}$$

These are the geometric Jacobi fields of L_{w_τ} introduced already in the introduction. For future use we state the following lemma

Lemma 4.2.1 With the above notations the following formulas hold in a tubular neighbourhood $\mathcal{N}_\delta(\varepsilon)$, $\delta(\varepsilon) = \mathcal{O}(\varepsilon^{\frac{2}{3}})$

$$\begin{aligned}
Y^*_\varepsilon\partial_{h_i}\Phi_{\mathtt{h},\vartheta,\eta}(w_\tau)\,|_{\mathtt{h},\vartheta,\eta=0} &= \varepsilon^{-1}\Phi^{T,\mathtt{e}_i}_\tau\mathtt{W} + \mathcal{O}_{C^{1,\alpha}_\mu(D_\tau\times\mathbb{R})}(1), \\
Y^*_\varepsilon\partial_{\vartheta_i}\Phi_{\mathtt{h},\vartheta,\eta}(w_\tau)\,|_{\mathtt{h},\vartheta,\eta=0} &= \varepsilon^{-1}\Phi^{R,\mathtt{e}_i}_\tau\mathtt{W} + \mathcal{O}_{C^{1,\alpha}_\mu(D_\tau\times\mathbb{R})}(1), \\
Y^*_\varepsilon\partial_{\eta}\Phi_{\mathtt{h},\vartheta,\eta}(w_\tau)\,|_{\mathtt{h},\vartheta,\eta=0} &= \varepsilon^{-1}\Phi^{D}_\tau\mathtt{W} + \mathcal{O}_{C^{1,\alpha}_\mu(D_\tau\times\mathbb{R})}(1).
\end{aligned} \tag{4.12}$$

Proof. We recall that by (4.1) in $\mathcal{N}_{\delta(\varepsilon)}$ we have

$$Y_\varepsilon^* w_\tau(\mathtt{y},\mathtt{t}) = U(\mathtt{t}) + \mathcal{O}_{\mathcal{C}^{2,\alpha}_\mu(D_\tau\times\mathbb{R})}(\varepsilon^{2-\alpha}). \tag{4.13}$$

In $\mathcal{N}_{\delta(\varepsilon)}$ we can write explicitly using the isothermal coordinates on D_τ

$$\mathtt{x} = X_\tau(s,\theta) + \varepsilon\mathtt{t}N_\tau(s,\theta). \tag{4.14}$$

Now, fix a unit vector $\mathtt{e}\in\mathbb{R}^3$ and denote $\mathtt{x}_h = \mathtt{x} + h\mathtt{e}$. Taking derivative in h of (4.14) and evaluating at $h=0$ we get

$$\mathtt{e} = \varepsilon\partial_{\mathtt{e}}\mathtt{t}N_\tau + \partial_{\mathtt{e}}s[\partial_s X_\tau + \varepsilon\mathtt{t}\partial_s N_\tau] + \partial_{\mathtt{e}}\theta[\partial_\theta X_\tau + \varepsilon\mathtt{t}\partial_\theta N_\tau].$$

Taking the scalar product with N_τ, $\partial_s X_\tau$ and $\partial_\theta X_\tau$ we find expression for $\partial_{\mathtt{e}}\mathtt{t}$, $\partial_{\mathtt{e}}s$, $\partial_{\mathtt{e}}\theta$. Note in particular that $\partial_{\mathtt{e}_i}\mathtt{t} = \varepsilon^{-1}\mathtt{e}_i\cdot N_\tau = \varepsilon^{-1}\Phi_\tau^{T,\mathtt{e}_i}$. Then, taking derivatives $\partial_{\mathtt{e}_i}$ of (4.13) we get the first formula in (4.12). We follow a similar argument to show the two remaining identities. □

This means that the elements of $\mathcal{I}_{w_\tau}$ are at most linearly functions in the along direction of D_τ and exponentially decaying in the transversal direction of D_τ, thus they correspond to Jacobi fields of L_{w_τ} with temperate grow in the direction of of the axis of rotation of w_τ. It is reasonable to conjecture that all Jacobi fields which are at most linearly growing are of this form. Proving this fact is a very important element in our analysis since the invertibility theory of L_{w_τ} depends on the precise knowledge of this type of Jacobi fields. In order to establish the precise meaning of a at most linearly growing Jacobi field we need to set up several weighted Sobolev spaces. First, let $\mathrm{dist}\,(\mathtt{x}, D_\tau)$ denote the signed distance function, where we chose the orientation of D_τ in such a way that the sign of $\mathrm{dist}\,(\mathtt{x}, D_\tau)$ agrees with that of $\rho_\tau(z) - r$. We have globally

$$|\mathrm{dist}\,(\mathtt{x}, D_\tau)| \le |r - \rho_\tau(z)|,$$

and the two quantities are comparable near D_τ. Recall that above we have denoted $\mathtt{t} = \frac{1}{\varepsilon}\mathrm{dist}\,(\mathtt{x}, D_\tau)$ as long as $|\mathrm{dist}\,(\mathtt{x}, D_\tau)| \le \delta$.

We will define the weighted Sobolev norms we will use in the sequel. First, let us consider Sobolev spaces $L^2(D_\tau\times\mathbb{R})$ and $H^\ell(D_\tau\times\mathbb{R})$. Since functions in these spaces can be expressed in terms of the isothermal coordinates (s,θ) and the Fermi coordinate $\mathtt{t}$ we define

$$\begin{aligned} L^2_{a,\gamma}(D_\tau\times\mathbb{R}) &= \cosh^{-a}(s)\cosh^{-\gamma}(\mathtt{t})L^2(D_\tau\times\mathbb{R}) \\ H^\ell_{a,\gamma}(D_\tau\times\mathbb{R}) &= \cosh^{-a}(s)\cosh^{-\gamma}(\mathtt{t})H^\ell(D_\tau\times\mathbb{R}). \end{aligned}$$

Second, let us consider the subspace $H^\ell(\mathbb{R}^3)_+$ (respectively $H^\ell(\mathbb{R}^3)_-$) of $H^\ell(\mathbb{R}^3)$ which consists of functions supported in the set $\{z \ge -1\}$ (respectively in $\{z \le 1\}$). We define weighted Sobolev norms in these subspaces as follows

$$\begin{aligned} \|u\|_{H^\ell_{a,\gamma}(\mathbb{R}^3)_+} &= \sum_{|\alpha|=0}^{\ell} \|e^{az}e^{\gamma\left(\frac{r-\rho_\tau(z)}{\varepsilon}\right)}D^\alpha u\|_{L^2(\mathbb{R}^3)_+}, \\ \|u\|_{H^\ell_{a,\gamma}(\mathbb{R}^3)_-} &= \sum_{|\alpha|=0}^{\ell} \|e^{-az}e^{\gamma\left(\frac{r-\rho_\tau(z)}{\varepsilon}\right)}D^\alpha u\|_{L^2(\mathbb{R}^3)_-}, \end{aligned} \tag{4.15}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi index and derivatives are taken with respect to (x_1, x_2, z). We agree

$$\|u\|_{L^2_{a,\gamma}(\mathbb{R}^3)_+} = \|u\|_{H^0_{a,\gamma}(\mathbb{R}^3)_+},$$
$$\|u\|_{L^2_{a,\gamma}(\mathbb{R}^3)_-} = \|u\|_{H^0_{a,\gamma}(\mathbb{R}^3)_-}.$$

Note that γ measures the rate of decay or growth of the functions in the transversal direction to D_τ and a measures the rate of decay or growth along the axis of D_τ in the positive (respectively negative) direction. Next, we define

$$L^2_{a,\gamma}(\mathbb{R}^3) = L^2_{a,\gamma}(\mathbb{R}^3)_+ \oplus L^2_{a,\gamma}(\mathbb{R}^3)_-,$$
$$H^\ell_{a,\gamma}(\mathbb{R}^3) = H^\ell_{a,\gamma}(\mathbb{R}^3)_+ \oplus H^\ell_{a,\gamma}(\mathbb{R}^3)_-.$$

With these definitions when $\gamma > 0$, $a > 0$ our spaces consist of exponentially decaying functions, in the opposite case they are exponentially increasing. Combinations of signs for γ and a are of course allowed.

Finally, we define the weighted Sobolev spaces

$$\bar{L}_{a,\gamma}(\mathbb{R}^3) := L^2_{a,\gamma}(\mathbb{R}^3) \cap L^2_{a,-\gamma}(\mathbb{R}^3), \quad \bar{H}^\ell_{a,\gamma}(\mathbb{R}^3) := H^\ell_{a,\gamma}(\mathbb{R}^3) \cap H^s_{a,-\gamma}(\mathbb{R}^3).$$

Note that $u \in \bar{L}_{a,\gamma}(\mathbb{R}^3)$ decays away from D_τ as $\cosh^{-\gamma}\left(\frac{r-\rho_\tau(z)}{\varepsilon}\right)$ if $\gamma > 0$, and decays (for $a > 0$) or grows (for $a < 0$) along D_τ at the rate $\cosh^{-a} z$.

Theorem 4.1 For all $\tau \in (0,1)$, and $\varepsilon > 0$ sufficiently small there exits $\delta_\tau > 0$ and a finite set $\mathcal{S}_0$ such that for all a, γ satisfying $a^2 + \gamma^2 < \delta_\tau$ and $a \notin \mathcal{S}_0$, the operator L_{w_τ} it is injective in the space $\bar{L}_{a,\gamma}(\mathbb{R}^3)$ for some a, γ. Moreover, the Jacobi fields of L_{w_τ} that belong to $\bar{L}^2_{a,\gamma}$ correspond to $\mathcal{I}_{w_\tau}$.

It is clear that Theorem 1.2 is a direct consequence of Theorem 4.1, thus we focus on proving the last one.

The norms $H^\ell_{a,\gamma}(D_\tau \times I_{\delta/\varepsilon})$ (where $I_{\delta/\varepsilon} = (-\delta/\varepsilon, \delta/\varepsilon)$) and $H^\ell_{a,\gamma}(\mathbb{R}^3 \cap \{|\mathrm{dist}\,(\mathtt{x}, D_\tau)| < \delta\})$ are equivalent in the following sense

$$\begin{aligned} \|\phi\|_{L^2_{a,\gamma}(\mathbb{R}^3 \cap \{|\mathrm{dist}\,(\mathtt{x},D_\tau)|<\delta\})} &\le C\varepsilon^{1/2}\|Y^*_\varepsilon\phi\|_{L^2_{a^*,\gamma^*}(D_\tau\times I_{\delta/\varepsilon})}, \\ \|\phi\|_{L^2_{a,\gamma}(\mathbb{R}^3 \cap \{|\mathrm{dist}\,(\mathtt{x},D_\tau)|<\delta\})} &\ge C\varepsilon^{1/2}\|Y^*_\varepsilon\phi\|_{L^2_{a_*,\gamma_*}(D_\tau\times I_{\delta/\varepsilon})}, \end{aligned} \tag{4.16}$$

where in general constants a, γ, a^*, γ^* and a_*, γ_* are different. In addition, relating the norms of gradients and second derivatives we expect to loose powers of ε. For instance

$$\begin{aligned} \|\nabla\phi\|_{L^2_{a,\gamma}(\mathbb{R}^3 \cap \{|\mathrm{dist}\,(\mathtt{x},D_\tau)|<\delta\})} &\le C\varepsilon^{-1/2}\|\nabla Y^*_\varepsilon\phi\|_{L^2_{a^*,\gamma^*}(D_\tau\times I_{\delta/\varepsilon})}, \\ \|\nabla\phi\|_{L^2_{a,\gamma}(\mathbb{R}^3 \cap \{|\mathrm{dist}\,(\mathtt{x},D_\tau)|<\delta\})} &\ge C\varepsilon^{1/2}\|\nabla Y^*_\varepsilon\phi\|_{L^2_{a_*,\gamma_*}(D_\tau\times I_{\delta/\varepsilon})}. \end{aligned} \tag{4.17}$$

Similar estimates hold for the second derivatives. We will use these estimates later on.

4.3. The Fourier-Laplace transform of L_{w_τ}

We will consider the linear operator L_{w_τ} acting on the space $L^2_{a,\gamma}(\mathbb{R}^3)$ with dense domain $D(L_{w_\tau}) = H^2_{a,\gamma}(\mathbb{R}^3)$ defined by

$$L_{w_\tau} : H^2_{a,\gamma}(\mathbb{R}^3) \longmapsto L^2_{a,\gamma}(\mathbb{R}^3),$$
$$u \longmapsto L_{w_\tau} u.$$

The important property of the operator L_{w_τ} is the fact that it is periodic in z. This will allow us to define the Fourier-Laplace transform of L_{w_τ} (this idea was originated by Taubes [61], [60] and developed in the form that we adopt here in [48] and [45]).

To begin we define the Fourier-Laplace transform for functions on $\mathbb{R}$ by

$$\hat{h}(\sigma,\zeta) = \mathcal{F}(h) = \sum_{-\infty<k<\infty} e^{-i(k+\sigma)\zeta} h(\sigma+k), \quad \sigma \in [0,1), \quad \zeta = \mu + i\nu. \tag{4.18}$$

Observe that with this definition we have

$$\hat{h}(\sigma+1,\zeta) = \sum_{-\infty<k<\infty} e^{-i(k+1+\sigma)\zeta} h(\sigma+k+1) = \hat{h}(\sigma,\zeta). \tag{4.19}$$

Note that the definition we adopt here is slightly different from the one in [48] the two differ by a factor $e^{-i\sigma\zeta}$ and this factor turns $\hat{h}$ into a periodic function.

The Fourier-Lapalace transform can be inverted and the inverse is given by an explicit formula. To state it let $s \in \mathbb{R}$ be given and denote the fractional part of s by $s \bmod 1$. With this notation we have

$$h(s) = \mathcal{F}^{-1}(\hat{h})(s) = \frac{1}{2\pi}\int_{\mu=0}^{2\pi} e^{is\zeta}\hat{h}(s \bmod 1,\zeta)\, d\mu, \tag{4.20}$$

where we integrate along the line $\operatorname{Im}\zeta = \nu$, $\zeta = \mu + i\nu$ (see [60]). The Fourier-Laplace transform is well defined in the Schwartz class $\mathcal{S}$ and, by Cauchy's theorem, the value of the integral in the inversion formula does not depend on ν, since the segment along which we integrate can be vertically shifted. However, for our purpose it is convenient to consider the class of functions which are allowed to grow exponentially at $+\infty$ (or at $-\infty$). Suppose for instance that h is a continuous function, supported in $[-1,\infty)$ and such that $|e^{as}h(s)| < \infty$. Then the series in (4.18) is well defined as long as $\operatorname{Im}\zeta = \nu < a$. Likewise, we can define the transform on a subspace

$$H^\ell_a(\mathbb{R})_+ = e^{-as}H^\ell(\mathbb{R}),$$

of the Sobolev space $H^\ell(\mathbb{R})$ consisting of functions supported in $[-1,\infty)$, where a is the rate of exponential decay or growth. In a similar way we define the subspace $H^\ell_a(\mathbb{R})_-$ of $H^\ell(\mathbb{R})$ consisting of exponentially decaying or growing functions supported in $(-\infty,1]$. As long as $\operatorname{Im}\zeta = \nu \le a$ the Fourier-Laplace transform of $h \in H^\ell_a(\mathbb{R})_+$ is well defined. Moreover the function h can be recovered from $\hat{h}(\cdot,\zeta)$ if the path of integration in the formula (4.20) is taken in the lower half plane $\mathbb{H}^-_a = \{\operatorname{Im}(\zeta) = \nu \le a\}$. The situation is similar when instead we consider the Fourier-Laplace transform in the space of functions $H^\ell_a(\mathbb{R})_-$, except that now the transform is defined in the upper half plane $\mathbb{H}^+_a = \{\operatorname{Im}\zeta = \nu \ge a\}$.

We observe that from Plancherel's formula

$$\int_{\mu=0}^{2\pi}\int_0^1 |\hat{h}(\sigma,\zeta)|^2\,d\sigma\,d\mu = \int_{\mathbb{R}} |e^{\nu s}h(s)|^2\,ds, \qquad \zeta=\mu+i\nu,$$

it follows that L^2 norms of the Fourier-Laplace transforms are equal to the exponentially weighted L^2 norm of functions. This property is crucial for our purpose.

Note that if $u(\sigma,\zeta)$ is an $L^2([0,1])$ function which is analytic as a function of ζ with values in $L^2([0,1])$ in the lower half plane $\mathbb{H}_a^-$ then, by Cauchy's theorem, the path of integration in the inversion formula (4.20) can be shifted down to any path $\zeta=\mu+i\nu$, $\nu<a$. If in addition $u(\cdot,\zeta)$ is bounded by $e^{-\nu}$ along such paths then the inverse transform $\mathcal{F}^{-1}u(s)$ is supported in $[-1,\infty)$. This explains the reason we have paid so much attention to functions defined on a half-line. On the other hand Fourier-Laplace transforms of functions in $H_a^\ell(\mathbb{R})_+$ have the property described above.

The Fourier-Laplace transform plays a similar role as the Fourier transform in the theory of linear PDEs with constant coefficients when the differential operator at hand is periodic with respect to the independent variable. To fix attention on a concrete example let us suppose that $A(s)\colon L_a^2(\mathbb{R})_+\to L_a^2(\mathbb{R})_+$, $s\in\mathbb{R}$ is a family of densely defined, linear operators. Then it is natural to define

$$(\hat{A}\hat{h})(\sigma,\zeta)=\widehat{A(s)h}(\sigma,\zeta).$$

Now, let us suppose that A is periodic with period 1, i.e. $A(s)=A(s+1)$. We have

$$(\hat{A}\hat{h})(\sigma,\zeta)=\sum_{-\infty<k<\infty} e^{-i(k+\sigma)\zeta}A(\sigma+k)h(\sigma+k)=e^{-i\zeta\sigma}A(\sigma)e^{i\zeta\sigma}\hat{h},$$

hence explicitly

$$\hat{A}(\sigma,\zeta)=e^{-i\zeta\sigma}A(\sigma)e^{i\zeta\sigma}.$$

With our definition of the Fourier-Laplace transform we have $\hat{h}(\sigma)=\hat{h}(\sigma+1)$ and also $\hat{A}(\sigma,\zeta)=\hat{A}(\sigma+1,\zeta)$. It follows that the operator $\hat{A}(\sigma,\zeta)$ is naturally defined on functions in the space of L^2 functions defined on S^1. Through the identification $u(\sigma)=\tilde{u}(e^{2\pi i\sigma})$ we consider this as a space of periodic functions on $[0,1]$ and denote it by $L^2_{per}([0,1])$.

Often one has to deal with operators that are periodic with period $T>0$ that is not necessarily equal to 1. It is elementary to modify our definitions of the Fourier-Laplace transform of a function and a linear operator in this case. For a given function h and $T>0$ our objective is to define the Fourier-Lapalace transform of h which is periodic of period T. We set $h_T(x)=h(Tx)$ and let naturally $\hat{h}(\xi,\zeta)=\hat{h}_T(\xi/T,\zeta)$ so that

$$\hat{h}(\xi,\zeta)=\sum_{-\infty<k<\infty} e^{-i(\xi+Tk)\zeta/T}h(\xi+Tk), \qquad h(x)=\frac{1}{2\pi}\int_{\mu=0}^{2\pi} e^{ix\zeta/T}\hat{h}(x \bmod T,\zeta/T)\,d\mu,$$

the Plancherel's formula is

$$\int_{\mu=0}^{2\pi}\int_0^T |\hat{h}(\sigma,\zeta)|^2\,d\sigma\,d\mu = \int_{\mathbb{R}} |e^{\nu s/T}h(s)|^2\,ds, \qquad \zeta=\mu+i\nu,$$

and the Fourier-Laplace transform of a T periodic operator A is

$$\hat{A}(\xi,\zeta) = e^{-i\xi\zeta/T} A(\xi) e^{i\xi\zeta/T}.$$

The operator $\hat{A}(\xi,\zeta)$ acts on a space of functions $L^2_{per}([0,T])$. Note that from the Plancherel's formula we see that if $h \in L^2_a(\mathbb{R})_+$ and its Fourier-Laplace transform is T periodic then it is natural to take $\zeta = \mu + i\nu = \mu + iTa$, $\mu \in (0,2\pi)$ as the path of integration.

In many applications, and this will be in particular the case in our context, the family of operators $\hat{A}(\sigma,\zeta)$ is Fredholm and depends holomorphically on the variable ζ. If this is the case one can use the analytic Fredholm theorem to conclude that either $\hat{A}(\sigma,\zeta)$ is nowhere invertible or it is invertible in the set of all admissible ζ except possibly a discrete set. If the latter happens then in order to solve the equation

$$A(x)h = g,$$

we can pass to the Fourier-Laplace transform

$$\hat{A}(\xi,\zeta)\hat{h}(\xi,\zeta) = \hat{g}(\xi,\zeta) \Longrightarrow h(x) = \frac{1}{2\pi}\int_{\mu=0}^{2\pi} e^{ix\zeta/T}(\hat{A}^{-1}\hat{g})(x \bmod T, \zeta/T)\, d\mu, \tag{4.21}$$

where in the last integral the path of integration should avoid the poles of $\hat{A}^{-1}\hat{g}(x \bmod T,\zeta)$. If between two such paths there is no pole of $\hat{A}^{-1}\hat{g}(x \bmod T,\zeta)$ then the path of integration can be shifted from one of the paths to the other horizontally without changing the value of the integral. This follows by Cauchy's theorem, since the integrals over the vertical segments cancel out due to (4.19). This means for instance that we can get the inverse of $A(x)$ in a space of functions $L^2_a(\mathbb{R})_+$ whenever $\hat{A}^{-1}\hat{g}(\xi,\zeta)$ is analytic in some neighbourhood of the segment $\zeta = \mu + iTa$, $\mu \in [0,2\pi]$. Alternatively, this means that $\hat{A}^{-1}\hat{g}(\xi,\zeta)$ is well defined in the space $L^2_{per}([0,T])$, for $\zeta = \mu + iT\nu$, $|\nu - a| < \kappa$ with some $\kappa > 0$. It may however happen that $\hat{A}^{-1}\hat{g}(\xi,\zeta)$ is analytic along two paths $\zeta_j = \mu + iT\nu_j$, $j = 1,2$, $\mu \in [0,2\pi]$ and $\nu_1 < \nu_2$, but it has a pole at some $\zeta^* = \mu^* + iT\nu^*$, with $\nu_1 < \nu^* < \nu_2$, $\mu^* \in (0,2\pi)$. In this case formula (4.21) would give two solutions h_1 and h_2 (by integrating over the paths $\zeta = \mu + iT\nu_j$, $j = 1,2$) which would differ by an element of the kernel of $A(x)$. This corresponds to the residue of $\hat{A}^{-1}\hat{g}(\xi,\zeta)$, $\zeta^* = \mu^* + iT\nu^*$.

4.4. Mapping properties of L_{w_τ} in weighted Sobolev spaces

Going back to our context, we see that since L_{w_τ} is T_τ periodic in the z variable, and so is it induces a family of operators on $L^2_{\gamma,per}(\mathbb{R}^2 \times [0,T_\tau])$, which is densely defined and holomorphic, as a function of ζ, in a neighbourhood of the segment $[0,2\pi]$. Here and below $H^\ell_{\gamma,per}(\mathbb{R}^2 \times [0,T_\tau])$ is a subspace of $H^\ell(\mathbb{R}^2 \times [0,T_\tau])$ which consists of functions that are periodic in z and whose grow (decay) away from D_τ is controlled by $e^{-\gamma\left(\frac{r-\rho_T(z)}{\varepsilon}\right)}$, cf. (4.15). In detail

$$H^\ell_{\gamma,\mathrm{per}}(\mathbb{R}^2 \times [0,T_\tau]) = \{u \in H^\ell(\mathbb{R}^2 \times [0,T_\tau]) | u \text{ is } T_\tau\text{-peridic in } z, \|u\|_{H^\ell_{\gamma,\mathrm{per}}(\mathbb{R}^2\times[0,T_\tau])} < \infty\},$$

where

$$\|u\|_{H^{\ell}_{\gamma,\mathrm{per}}(\mathbb{R}^2\times[0,T_\tau])} = \sum_{|\alpha|\le\ell} \left\| e^{\gamma\left(\frac{r-\rho_\tau(\xi)}{\varepsilon}\right)} D^\alpha u \right\|_{L^2(\mathbb{R}^2\times[0,T_\tau])}.$$

Later on we will also consider the space of functions $H^{\ell}_{\gamma,per}(\mathring{D}_\tau\times\mathbb{R})$ consisting of functions defined on $\mathring{D}_\tau$ (here by $\mathring{D}_\tau$ we denote a one period portion of D_τ with the top and the bottom identified) and whose decay away from $\mathring{D}_\tau$ is controlled by $e^{-\gamma\mathtt{t}}$, *i.e.*

$$\|u\|_{H^{\ell}_{\gamma,\mathrm{per}}(\mathring{D}_\tau\times\mathbb{R})} = \sum_{|\alpha|\le\ell} \left\| e^{\gamma\mathtt{t}} D^\alpha_{\mathring{D}_\tau\times\mathbb{R}} Y^*_\varepsilon u(\mathtt{y},\mathtt{t}) \right\|_{L^2(\mathring{D}_\tau\times\mathbb{R})}$$

We also agree that

$$L^2_{\gamma,\mathrm{per}}(\mathbb{R}\times[0,T_\tau]) = H^0_{\gamma,\mathrm{per}}(\mathbb{R}\times[0,T_\tau]),$$
$$L^2_{\gamma,\mathrm{per}}(\mathring{D}_\tau\times\mathbb{R}) = H^0_{\gamma,\mathrm{per}}(\mathring{D}_\tau\times\mathbb{R}).$$

These two norms are related locally, near $\mathring{D}_\tau$, by formulas analogous to (4.16)–(4.17).

If we restrict L_{w_τ} to the subspace of $L^2_{a,\gamma}(\mathbb{R}^3)_+$ of $L^2_{a,\gamma}(\mathbb{R}^3)$ functions that are supported in the set $z>-1$, and consider it as acting on Fourier-Laplace transforms of such functions, then we can obtain a parametrix for the operator L_{w_τ} via the Fourier-Laplace inversion formula (4.21). As we pointed out earlier the advantage in working with the family $\hat{L}_{w_\tau}(\zeta)$, is the fact that we can use the theory developed in [43] and [54].

Using the Fourier-Laplace transform we can consider the family of operators $\hat{L}_{w_\tau}(\zeta)$ instead of L_{w_τ}. We will write the operator $\hat{L}_{w_\tau}(\zeta)$ in terms of variables (x_1,x_2,ξ) (here $\xi\in[0,T_\tau]$)

$$\hat{L}_{w_\tau}(\zeta) = \varepsilon[\Delta + T_\tau^{-2}(\partial_{\xi\xi} + 2i\zeta\partial_\xi - \zeta^2)] + \frac{1}{\varepsilon}f'(w_\tau) \qquad (\text{here}\quad \Delta = \partial^2_{x_1} + \partial^2_{x_2}).$$

This operator is defined for functions in $H^2_{\gamma,per}(\mathbb{R}^2\times[0,T_\tau])$ and induces a densely defined operator on $L^2_{\gamma,per}(\mathbb{R}^2\times[0,T_\tau])$. In order that the inversion formula for the Fourier-Laplace transform made sense we need to know the Fredholm property at least for $\zeta=\mu+i\nu$, where $\mu\in[0,2\pi]$ and $|\nu|$ is small, or in other words when ζ is in a neighbourhood of the segment $[0,2\pi]$. In order to prove that this operator is Fredholm we use the following

Lemma 4.4.1 Let $A_R=\{(\mathtt{x},\xi)\mid r-\rho_\tau(\xi)\in(-R,R), r=|\mathtt{x}|=\sqrt{x_1^2+x_2^2}, \xi\in[0,T_\tau]\}$ and let $M>0$ be such that $f'(w_\tau)<-\frac{\sqrt{2}}{2}$ in $\mathbb{R}^2\times[0,T_\tau]\setminus A_{\varepsilon M}$. There exists $\delta_\tau>0$ such that for all $\zeta=\mu+ia$, $\mu\in[0,2\pi]$, and γ such that $a^2+\gamma^2<\delta_\tau$, and all sufficiently small ε, it holds

$$\varepsilon\|\nabla\phi\|^2_{L^2_{\gamma,per}(\mathbb{R}^2\times[0,T_\tau])} + \varepsilon^{-1}\|\phi\|^2_{L^2_{\gamma,per}(\mathbb{R}^2\times[0,T_\tau])} \le C\|\hat{L}_{w_\tau}(\zeta)\phi\|^2_{L^2_{\gamma,per}(\mathbb{R}^2\times[0,T_\tau])} + C\varepsilon^{-1}\|\phi\|^2_{L^2(A_{\varepsilon M})}.$$

for any function $\phi\in H^2_{\gamma,per}(\mathbb{R}^2\times[0,T_\tau])$. The constant C above depends on ζ, M and γ.

Proof. This type of estimate is well known and it can be found for instance in [1]. We will outline the proof here (following the proof of a similar result in [19]). We agree that Γ is one of the functions

$$\Gamma = e^{\gamma\left(\frac{r-\rho_\tau(\xi)}{\varepsilon}\right)}, \qquad \Gamma = \cosh^\gamma\left(\frac{r-\rho_\tau(\xi)}{\varepsilon}\right).$$

We take a cutoff function $\chi_{\varepsilon M}$ which is supported in the complement of the set $A_{\varepsilon M/2}$ and is identically equal to 1 in the complement of the set $A_{\varepsilon M}$. Let us denote

$$\phi_\zeta = e^{i\zeta\xi/T_\tau}\phi,$$

so that

$$\hat{L}_{w_\tau}\phi = e^{-i\zeta\xi/T_\tau}[\varepsilon\Delta + \frac{1}{\varepsilon}f'(w_\tau)]\phi_\zeta = g.$$

Multiply the left hand side of the last equation by $\bar{\phi}\Gamma\chi^2_{\varepsilon M}$ and integrate by parts. This gives

$$\int_{\mathbb{R}^2\times[0,T_\tau]}\hat{L}_{w_\tau}(\zeta)\phi\bar{\phi}\Gamma\chi^2_{\varepsilon M} = -\varepsilon\int_{\mathbb{R}^2\times[0,T_\tau]}[|\nabla\phi_\zeta|^2+\zeta^2|\phi_\zeta|^2]\Gamma\chi^2_{\varepsilon M}$$
$$+\frac{1}{\varepsilon}\int_{\mathbb{R}^2\times[0,T_\tau]}f'(w_\tau)|\phi_\zeta|^2\Gamma\chi^2_{\varepsilon M} - \varepsilon\int_{\mathbb{R}^2\times[0,T_\tau]}\nabla\phi_\zeta\bar{\phi}_\zeta\nabla(\Gamma\chi^2_{\varepsilon M}) \quad (4.22)$$

Young's inequality gives for example

$$\varepsilon|\nabla\phi_\zeta\cdot\nabla\Gamma\bar{\phi}_\zeta| \le \varepsilon\kappa|\nabla\phi_\zeta|^2\Gamma + \frac{\varepsilon}{4\kappa}|\phi_\zeta|^2\frac{|\nabla\Gamma|^2}{\Gamma} \le \varepsilon\kappa|\nabla\phi_\zeta|^2\Gamma + \frac{C\gamma}{4\varepsilon\kappa}|\phi_\zeta|^2\Gamma.$$

Combining similar manipulations and adjusting the constants in the Young's inequality and the exponent γ suitably we find

$$\varepsilon\int_{\mathbb{R}^2\times[0,T_\tau]}|\nabla\phi_\zeta|^2\Gamma\chi^2_{\varepsilon M} + \frac{1}{\varepsilon}\int_{\mathbb{R}^2\times[0,T_\tau]}|\phi_\zeta|^2\Gamma\chi^2_{\varepsilon M} \le C\int_{\mathbb{R}^2\times[0,T_\tau]}|g|^2\Gamma\chi^2_{\varepsilon M}$$
$$+ C\varepsilon\int_{\mathbb{R}^2\times[0,T_\tau]}|\phi_\zeta|^2|\nabla\chi_{\varepsilon M}|^2. \quad (4.23)$$

As $|\nabla\chi_{\varepsilon M}| = \mathcal{O}(\varepsilon^{-1})$ and

$$|\phi_\zeta| = e^{\xi\mathrm{Im}\zeta/T_\tau}|\phi|, \quad |\nabla\phi| = |\nabla(e^{-i\zeta\xi/T_\tau}\phi_\zeta)| \le e^{|\xi\mathrm{Im}\zeta|/T_\tau}|\nabla\phi_\zeta| + \frac{|\zeta|}{T_\tau}e^{|\xi\mathrm{Im}\zeta|/T_\tau}|\phi_\zeta|,$$

the Lemma follows from this. □

Remark 4.1. Estimate (4.23) is of separate interest and it and its variants will be used for instance when we analyse the operator L_{w_τ} below. In particular we will need such a variant in the proof Lemma 4.4.4 (to follow). To explain this let us suppose that the weight function Γ depends on z as well, say $\Gamma = (\cosh z)^a e^{\gamma\left(\frac{r-\rho_\tau(z)}{\varepsilon}\right)}$ and consider the problem

$$L_{w_\tau}\phi = g,$$

where $\phi, g \in L^2_{a,\gamma}(\mathbb{R}^3)$. Choosing the cutoff function $\chi_{\varepsilon M}$ as above (understood now as a function on $\mathbb{R}^3$) and multiplying by $\phi\Gamma\chi^2_{\varepsilon M}$ we see that the term we need to control is of the form

$$\varepsilon|\nabla\phi\cdot\nabla\Gamma\phi| \le \varepsilon\kappa|\nabla\phi_\zeta|^2\Gamma + \frac{C\gamma}{4\varepsilon\kappa}|\phi_\zeta|^2\Gamma,$$

where the last inequality follows since we still have

$$\frac{|\nabla\Gamma|}{\Gamma} \leq C\varepsilon^{-1}.$$

As a consequence we get an estimate of the same type as (4.23) but with integrals taken over the whole space $\mathbb{R}^3$.

Lemma 4.4.2 The operator $\hat{L}_{w_\tau}(\zeta)$ acting on $H^2_{\gamma,per}(\mathbb{R}^2 \times [0, T_\tau])$ is Fredholm.

Proof. We need to show that $\hat{L}_{w_\tau}(\zeta)$ has finite dimensional kernel, closed range and that codimension of the range is also finite. To see that the $\dim \operatorname{Ker} \hat{L}_{w_\tau}(\zeta)$ is finite we argue by contradiction. Using notation of Lemma 4.4.1 let

$$\mathcal{B}_1 = \left\{ \phi \in H^2_{\gamma,per}(\mathbb{R}^2 \times [0, T_\tau]) \mid \hat{L}_{w_\tau}(\zeta)\phi = 0, \quad \|\phi\|_{L^2(A_{\varepsilon M})} = 1 \right\}.$$

By Lemma 4.4.1 we know that set $\mathcal{B}_1$ is bounded in $H^1_{\gamma,per}(\mathbb{R}^2 \times [0, T_\tau])$ and then by Sobolev embedding it is compact in $L^2(A_{\varepsilon M})$ and thus it must be finite dimensional. To show that $\hat{L}_{w_\tau}(\zeta)$ has finite range we argue similarly (see for instance [54] for a detailed proof). To show that the codimension of the range is finite we use the fact that $\dim \operatorname{Ker} \hat{L}_{w_\tau}(\zeta) = \operatorname{codim} \operatorname{Range}(\hat{L}_{w_\tau}(\bar{\zeta}))$, by duality (the dual of $L^2_{\gamma,per}(\mathbb{R}^2 \times [0, T_\tau])$ being $L^2_{-\gamma,per}(\mathbb{R}^2 \times [0, T_\tau])$). □

We will use this in proving

Proposition 4.4.1 There exists $\delta_\tau > 0$ and a finite set $\mathcal{S}_0$, such that for all a, γ with $a^2 + \gamma^2 < \delta_\tau$, $a \notin \mathcal{S}_0$, for all sufficiently small ε and for all $g \in L^2_{a,\gamma}(\mathbb{R}^3)$ there exists a solution of the problem

$$L_{w_\tau}\phi = g, \tag{4.24}$$

where $\phi \in H^2_{-|a|,\gamma}(\mathbb{R}^3)$.

Note that even if the right hand side of (4.24) is decaying as $z \to \pm\infty$ (i.e. $a > 0$) we get a solution which in general may be increasing as $z \to \pm\infty$ at the exponential rate proportional to $e^{|a||z|}$.

Proof of Proposition 4.4.1. The idea of the proof is to show that $\hat{L}_{w_\tau}(\zeta)$ is an isomorphism for ζ in some neighbourhood of $[0, 2\pi]$, except possibly a finite set of points, and then use the parametrix formula to solve (4.24). Since $\hat{L}_{w_\tau}(\zeta)$ is a Fredholm family of holomorphic operators in an open set $\mathcal{U} \subset \mathbb{C}$ with $[0, 2\pi] \subset \mathcal{U}$ it is either non invertible everywhere in $\mathcal{U}$ or it is invertible except a discrete subset of $\mathcal{U}$ [57]. In particular, if we consider $\zeta \in [0, 2\pi]$ (note that the operator $\hat{L}_{w_\tau}(\zeta)$ is self adjoint for $\zeta \in \mathbb{R}$) and are able to show that it is injective there except possibly a discrete set of points then we will conclude that it is invertible in $[0, 2\pi]$ except the discrete set and then the same will be true at least in a neighbourhood $\mathcal{U}$ of this segment.

To carry out this plan we consider $\hat{L}_{w_\tau}$ taken with respect to variable z. This operator is defined on the space of functions in $H^2_{per}(\mathbb{R}^2 \times [0, T_\tau])$ which consists of functions which are periodic with period T_τ. Recall that we have

$$\hat{L}_{w_\tau}(\zeta) = e^{-i\zeta\xi/T_\tau}\{\varepsilon\Delta + \varepsilon^{-1} f'(w_\tau)\}e^{i\zeta\xi/T_\tau}.$$

We want to express $\hat{L}_{w_\tau}$ in terms of the stretched Fermi co-ordinates in $\mathcal{N}_\delta$. Let $\mathring{D}_\tau$ be the one period piece of D_τ (i.e. $0 \le z \le T_\tau$) with the top and the bottom identified. The natural domain for the expression of the Fourier-Laplace transforms of functions in $L^2_{per}(\mathbb{R}^2 \times [0, T_\tau])$ in the stretched Fermi coordinates is $\mathring{D}_\tau \times [-\delta/\varepsilon, \delta/\varepsilon]$. For example from the definition of the shifted Fermi coordinates we see that

$$(Y_\varepsilon^* \xi)(\mathtt{y}, \mathtt{t}) = [\mathtt{y} + \varepsilon \mathtt{t} N_\tau(\mathtt{y})] \cdot \mathtt{e}_3.$$

It is convenient to extend this function from $\mathring{D}_\tau \times [-\delta/\varepsilon, \delta/\varepsilon]$ to $\mathring{D}_\tau \times \mathbb{R}$. We will use for this purpose the cutoff function $\chi_{\delta/\varepsilon}$ defined in (4.4) and set

$$\xi^* = \chi_{\varepsilon/\delta}(Y_\varepsilon^* \xi) + (1 - \chi_{\varepsilon/\delta})\mathtt{y} \cdot \mathtt{e}_3$$

for the extension of $(Y_\varepsilon^* \xi) = \xi^*$, understanding that this is a function of $(\mathtt{y}, \mathtt{t})$.

We use the operator $\mathbb{L}_{w_\tau}$ (see (4.5)) to define also a natural extension of $Y_\varepsilon^* \widehat{L}_{w_\tau}$ to $\mathring{D}_\tau \times \mathbb{R}$

$$\tilde{\mathbb{L}}_{w_\tau}(\zeta) = e^{-i\zeta\xi^*/T_\tau} \mathbb{L}_{w_\tau} e^{i\zeta\xi^*/T_\tau}.$$

The operator $\tilde{\mathbb{L}}_{w_\tau}(\zeta)$ is "almost" the Fourier-Laplace transform of $\mathbb{L}_{w_\tau}$. Note that $\tilde{\mathbb{L}}_{w_\tau}(0) = \mathbb{L}_{w_\tau}$. The strategy of the proof is to show first that the operator $\tilde{\mathbb{L}}_{w_\tau}(\zeta)$ is injective and then conclude from this that $\hat{L}_{w_\tau}(\zeta)$ is injective.

We will study the kernel of $\tilde{\mathbb{L}}_{w_\tau}(\zeta)$ in the space of functions $L^2_{\gamma,per}(\mathring{D}_\tau \times \mathbb{R})$. Let us suppose that for some γ, $|\gamma| < \delta_\tau$, and $\zeta \in [0, 2\pi]$ there exists a function $\phi_0 \in H^2_{\gamma,per}(\mathring{D}_\tau \times \mathbb{R})$

$$\tilde{\mathbb{L}}_{w_\tau}(\zeta)\phi_0 = \mathbb{L}_{w_\tau}(e^{i\zeta\xi^*/T_\tau}\phi_0) = \mathbb{L}_{w_\tau}\phi_{0\zeta} = 0,$$

where we have denoted

$$\phi_{0\zeta} = e^{i\zeta\xi^*/T_\tau}\phi_0.$$

We can normalize $\|\phi_{0\zeta}\|_{L^2_\gamma(\mathring{D}_\tau \times \mathbb{R})} = 1$ and then by elliptic estimates for any $M > 0$ in the set $\mathring{D}_\tau \times (-M, M)$ the function $\phi_{0\zeta}$ is bounded (we bound the real and imaginary parts of $\phi_{0\zeta}$ separately). Take M large so that $f'(w_\tau) < -2 + \eta$ with some small $\eta > 0$. Take δ_τ in the statement of the Proposition small so that $\gamma \in (-\sqrt{2-\eta}, \sqrt{2-\eta})$. Using the comparison principle for the operator $\tilde{\mathbb{L}}_{w_\tau}$ it is then easy to show that in fact

$$|\phi_{0\zeta}(\mathtt{y}, \mathtt{t})| = |\phi_0| \le C e^{-\sqrt{2-\eta}|\mathtt{t}|} \tag{4.25}$$

and therefore $\phi_0 \in H^2(\mathring{D}_\tau \times \mathbb{R})$.

For complex valued functions $\phi_1, \phi_2 \in L^2(\mathring{D}_\tau \times \mathbb{R})$ we define Hermitian inner product

$$\langle \phi_1, \phi_2 \rangle = \int_{\mathring{D}_\tau \times \mathbb{R}} \phi_1 \bar{\phi}_2 dV_{\mathring{D}_\tau} d\mathtt{t}.$$

Above $\nabla_{\mathring{D}_\tau}$ and $dV_{\mathring{D}_\tau}$ are respectively the gradient and the volume element on $\mathring{D}_\tau$. We introduce an orthogonal decomposition in $L^2(\mathring{D}_\tau \times \mathbb{R})$ as follows. Let $\mathtt{W}$ be the function defined in (4.7) (we recall that it is an extension of $\partial_{\mathtt{t}} Y_\varepsilon^* w_\tau$). Given a function $\phi \in L^2(\mathring{D}_\tau \times \mathbb{R})$ we denote $\phi_\zeta = e^{i\zeta\xi^*/T_\tau}\phi$ and decompose

$$\phi_\zeta = \phi_\zeta^{\parallel} + \psi_\zeta \mathtt{W},$$

where

$$\phi_\zeta^{\parallel} \in \mathcal{X}_\gamma := \left\{ \phi \in L^2_{\gamma,per}(\mathring{D}_\tau \times \mathbb{R}) \,\middle|\, \int_{\mathbb{R}} \phi \mathtt{W}\, d\mathtt{t} = \int_{\mathbb{R}} \bar{\phi} \mathtt{W}\, d\mathtt{t} = 0 \right\}, \qquad \psi_\zeta = \frac{\int_{\mathbb{R}} \phi_\zeta \mathtt{W}\, d\mathtt{t}}{\int_{\mathbb{R}} \mathtt{V}^2\, d\mathtt{t}}.$$

In particular for $\phi_0 \in \operatorname{Ker} \tilde{\mathbb{L}}_{w_\tau}(\zeta)$ we have

$$\mathbb{L}_{w_\tau}\phi_{0\zeta} = \mathbb{L}_{w_\tau}\phi_{0\zeta}^{\parallel} + \mathbb{L}_{w_\tau}(\psi_{0\zeta}\mathtt{W}) = 0 \tag{4.26}$$

and

$$\left\langle -\mathbb{L}_{w_\tau}\phi_{0\zeta}^{\parallel}, \phi_{0\zeta}^{\parallel} \right\rangle = -\left\langle \mathbb{L}_{w_\tau}\psi_{0\zeta}\mathtt{W}, \phi_{0\zeta}^{\parallel} \right\rangle. \tag{4.27}$$

We will use this identity to estimate $\phi_{0\zeta}^{\parallel}$ in terms of suitable norm of $\psi_{0\zeta}$. To do so we need

Lemma 4.4.3 It holds

$$\left| \langle -\mathbb{L}_{w_\tau}\phi_\zeta^{\parallel}, \phi_\zeta^{\parallel} \rangle \right| \geq \frac{C}{\varepsilon} \left(\|\partial_{\mathtt{t}}\phi_\zeta^{\parallel}\|^2_{L^2(\mathring{D}_\tau\times\mathbb{R})} + \|\phi_\zeta^{\parallel}\|^2_{L^2(\mathring{D}_\tau\times\mathbb{R})} \right) + C\varepsilon \|\nabla_{\mathring{D}_\tau}\phi_\zeta^{\parallel}\|^2_{L^2(\mathring{D}_\tau\times\mathbb{R})}.$$

Proof. We recall the well known fact: with $H(x) = \tanh\left(\frac{x}{\sqrt{2}}\right)$ the bilinear form

$$\int_{\mathbb{R}} |\psi'|^2 - f'(H)\psi^2 \tag{4.28}$$

is positive definite on the space of functions $L^2(\mathbb{R})$ orthogonal to $H'(x)$. Consider a quadratic form

$$\mathcal{B}(\phi,\phi) = \frac{1}{\varepsilon} \int_{\mathring{D}_\tau\times\mathbb{R}} |\partial_{\mathtt{t}}\phi|^2 + \varepsilon^2 |\nabla_{\mathring{D}_\tau}\phi|^2 - f'(H)|\phi|^2$$

for $\phi \in \mathcal{X}_\gamma$. Write

$$\phi = \phi_1 + \phi_2 H', \qquad \text{where} \qquad (\phi_1, H') = 0, \qquad \phi_2 = \frac{(\phi, H')}{(H', H')}$$

and where we have denoted

$$(\phi, \psi) = \int_{\mathbb{R}} \phi\bar{\psi}\, d\mathtt{t}.$$

We have

$$0 = (\phi, \mathtt{W}) = (\phi, H') + (\phi, \mathtt{W} - H') = \phi_2\,(H', H') + (\phi, \mathtt{W} - H')$$

and also

$$\begin{aligned} 0 = \nabla_{\mathring{D}_\tau}(\phi, \mathtt{W}) &= \left(\nabla_{\mathring{D}_\tau}\phi, \mathtt{W}\right) + \left(\phi, \nabla_{\mathring{D}_\tau}\mathtt{W}\right) \\ &= \nabla_{\mathring{D}_\tau}\phi_2\,(H', H')^2 + \left(\nabla_{\mathring{D}_\tau}\phi, \mathtt{W} - H'\right) + \left(\phi, \nabla_{\mathring{D}_\tau}\mathtt{W}\right). \end{aligned}$$

Since

$$\mathtt{W} - H' = U' - H' + \mathcal{O}_{\mathcal{C}^{1,\alpha}_{\mu}\mathring{D}_\tau\times\mathbb{R}}(\varepsilon^{2-\alpha}) = \mathcal{O}_{\mathcal{C}^{1,\alpha}_{\mu}\mathring{D}_\tau\times\mathbb{R}}(\varepsilon), \qquad \nabla_{\mathring{D}_\tau}\mathtt{W} = \mathcal{O}_{\mathcal{C}^{1,\alpha}_{\mu}\mathring{D}_\tau\times\mathbb{R}}(\varepsilon^{2-\alpha}),$$

we get

$$\|\phi_2 H'\|_{H^1(\mathring{D}_\tau\times\mathbb{R})} \leq C\varepsilon\|\phi\|_{H^1(\mathring{D}_\tau\times\mathbb{R})}. \tag{4.29}$$

By (4.28)

$$\mathcal{B}(\phi_1,\phi_1) \geq \frac{C}{\varepsilon}\left(\|\phi_1\|^2_{L^2(\mathring{D}_\tau\times\mathbb{R})} + \|\partial_{\mathtt{t}}\phi_1\|^2_{L^2(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^2\|\nabla_{\mathring{D}_\tau}\phi_1\|^2_{L^2(\mathring{D}_\tau\times\mathbb{R})}\right),$$

hence from (4.29)

$$\mathcal{B}(\phi,\phi) \geq \frac{C}{\varepsilon}\left(\|\phi\|^2_{L^2(\mathring{D}_\tau\times\mathbb{R})} + \|\partial_{\mathtt{t}}\phi\|^2_{L^2(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^2\|\nabla_{\mathring{D}_\tau}\phi\|^2_{L^2(\mathring{D}_\tau\times\mathbb{R})}\right),$$

for any $\phi \in \mathcal{X}_\gamma$.

We get

$$\begin{aligned}
\langle -\mathbb{L}_{w_\tau}\phi^{\parallel}_\zeta, \phi^{\parallel}_\zeta\rangle &= \mathcal{B}(\phi^{\parallel}_\zeta,\phi^{\parallel}_\zeta) - \frac{1}{\varepsilon}\int_{\mathring{D}_\tau\times\mathbb{R}}[f'(\mathtt{w}_\tau) - f'(H)]|\phi_\zeta|^2\, dV_{\mathring{D}_\tau}d\mathtt{t} \\
&\quad - \frac{\varepsilon}{2}\int_{\mathring{D}_\tau\times\mathbb{R}}\partial_{\mathtt{t}}\left(\mathtt{t}\chi_{\varepsilon/\delta}\right)|A_{\mathring{D}_\tau}|^2|\phi_\zeta|^2\, dV_{\mathring{D}_\tau}d\mathtt{t} \\
&\quad + \langle\chi_{\varepsilon/\delta}\mathbb{Q}_\varepsilon\partial_{\mathtt{t}}\phi_\zeta,\phi_\zeta\rangle + \varepsilon\langle\chi_{\varepsilon/\delta}\mathbb{A}_\varepsilon\phi_\zeta,\phi_\zeta\rangle \\
&= \mathcal{B}(\phi^{\parallel}_\zeta,\phi^{\parallel}_\zeta) + \left(\mathcal{O}(1) + \mathcal{O}(\varepsilon/\delta) + \mathcal{O}(\delta^2)\right)\left(\|\phi^{\parallel}_\zeta\|^2_{L^2(\mathring{D}_\tau\times\mathbb{R})} + \|\partial_{\mathtt{t}}\phi^{\parallel}_\zeta\|^2_{L^2(\mathring{D}_\tau\times\mathbb{R})}\right) \\
&\quad + \mathcal{O}(\varepsilon\delta)\|\nabla_{\mathring{D}_\tau}\phi^{\parallel}_\zeta\|^2_{L^2(\mathring{D}_\tau\times\mathbb{R})}.
\end{aligned}$$

Since δ can be taken as small as we wish the assertion of the Lemma follows. □

Now, we need to control the mixed term in (4.27)

$$\begin{aligned}
\langle -\mathbb{L}_{w_\tau}(\psi_{0\zeta}\mathtt{W}), \phi^{\parallel}_{0\zeta}\rangle &= -\varepsilon\left\langle \mathtt{W}\left(\mathcal{J}_{D_\tau} + \chi_{\varepsilon/\delta}\mathbb{A}_\varepsilon\right)\psi_{0\zeta}, \phi^{\parallel}_{0\zeta}\right\rangle - \varepsilon\left\langle\left[\Delta_{D_\tau} + \chi_{\varepsilon/\delta}\mathbb{A}_\varepsilon, \mathtt{W}\right]\psi_{0\zeta}, \phi^{\parallel}_{0\zeta}\right\rangle \\
&\quad + \left\langle \mathcal{O}_{C^{0,\alpha}_\mu(D_\tau\times\mathbb{R})}(\varepsilon^{3-\alpha})\psi_{0\zeta}, \phi^{\parallel}_{0\zeta}\right\rangle \\
&= \left\langle \mathcal{O}_{C^{0,\alpha}_\mu(D_\tau\times\mathbb{R})}(\varepsilon^{2})\nabla_{\mathring{D}_\tau}\psi_{0\zeta}, \nabla_{\mathring{D}_\tau}\phi^{\parallel}_{0\zeta}\right\rangle + \left\langle \mathcal{O}_{C^{0,\alpha}_\mu(D_\tau\times\mathbb{R})}(\varepsilon^{3-\alpha})\psi_{0\zeta}, \phi^{\parallel}_{0\zeta}\right\rangle,
\end{aligned}$$

where the last equality follows because the coefficients of the operator $\mathbb{A}_\varepsilon$ are bounded by $\varepsilon\mathtt{t}$ and $\mathtt{W}$ is exponentially decaying in $\mathtt{t}$. By the Cauchy-Schwarz inequality for any some $\eta > 0$ small we get

$$\left|\left\langle -\mathbb{L}_{w_\tau}(\psi_{0\zeta}\mathtt{W}), \phi^{\parallel}_{0\zeta}\right\rangle\right| \leq \eta\left(\varepsilon^{-1}\|\phi^{\parallel}_{0\zeta}\|^2_{H^1(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon\|\phi^{\parallel}_{0\zeta}\|^2_{H^1(\mathring{D}_\tau\times\mathbb{R})}\right) + C_\eta\varepsilon^3\|\psi_{0\zeta}\|^2_{H^1(\mathring{D}_\tau)}.$$

It follows from (4.27) and Lemma 4.4.3

$$\frac{1}{\varepsilon}\left(\|\partial_{\mathtt{t}}\phi^{\parallel}_{0\zeta}\|^2_{L^2(\mathring{D}_\tau\times\mathbb{R})} + \|\phi^{\parallel}_{0\zeta}\|^2_{L^2(\mathring{D}_\tau\times\mathbb{R})}\right) + \varepsilon\|\nabla_{\mathring{D}_\tau}\phi^{\parallel}_{0\zeta}\|^2_{L^2(\mathring{D}_\tau\times\mathbb{R})} \leq C\varepsilon^3\|\psi_{0\zeta}\|^2_{H^1(\mathring{D}_\tau)}.$$

Now consider the orthogonal complement of $\mathcal{X}_\gamma$. From (4.10) we obtain

$$\mathbb{L}_{w_\tau}(\psi_{0\zeta}\mathtt{W}) = \varepsilon\mathtt{W}\left(\mathcal{J}_{\mathring{D}_\tau} + \chi_{\varepsilon/\delta}\mathbb{A}_\varepsilon\right)\psi_{0\zeta} + \varepsilon\left[\Delta_{D_\tau} + \chi_{\varepsilon/\delta}\mathbb{A}_\varepsilon, \mathtt{W}\right]\psi_{0\zeta} + \mathcal{O}_{C^{0,\alpha}_\mu(D_\tau\times\mathbb{R})}(\varepsilon^{3-\alpha})\psi_{0\zeta},$$

Using this and projecting (4.26) onto W and integrating over $\mathbb{R}$ we get

$$\mathcal{J}_{\mathring{D}_\tau}\psi_{0\zeta} = T(\psi_{0\zeta}, \phi_{0\zeta}), \tag{4.30}$$

where

$$\begin{aligned}\|T(\psi_{0\zeta}, \phi_{0\zeta})\|_{L^2(\mathring{D}_\tau)} &\le C\delta\|\psi_{0\zeta}\|_{H^2(\mathring{D}_\tau)} + C\left(\varepsilon^{-1}\|\phi^{\|}_{0\zeta}\|_{L^2(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^{2-\alpha}\|\phi^{\|}_{0\zeta}\|_{H^1(\mathring{D}_\tau\times\mathbb{R})}\right)\\ &\le C\left(\delta\|\psi_{0\zeta}\|_{H^2(\mathring{D}_\tau)} + \varepsilon\|\psi_{0\zeta}\|_{H^1(\mathring{D}_\tau)}\right).\end{aligned} \tag{4.31}$$

We claim that from this it follows that for any $\zeta\in(0,1)$ there exists $\varepsilon_\zeta > 0$ such that for any $\varepsilon\in(0,\varepsilon_\zeta)$ we have $\psi_{0\zeta} = 0$ and hence $\phi_0 = 0$. To show this claim we note that by definition

$$\begin{aligned}\psi_{0\zeta} &= \frac{(\phi_{0\zeta}, \mathtt{W})}{(\mathtt{W}, \mathtt{W})}\\ &= \frac{\left(e^{i\zeta\xi^*/T_\tau}\phi_0, \mathtt{W}\right)}{(\mathtt{W}, \mathtt{W})}\\ &= \frac{\left(e^{i\zeta(\mathtt{y}_3+\varepsilon\chi_{\varepsilon/\delta}\mathtt{t}N_\tau\cdot\mathtt{e}_3)/T_\tau}\phi_0, \mathtt{W}\right)}{(\mathtt{W}, \mathtt{W})}\\ &= \frac{e^{i\zeta\mathtt{y}_3/T_\tau}\left(e^{i\zeta\varepsilon\chi_{\varepsilon/\delta}\mathtt{t}N_\tau\cdot\mathtt{e}_3)/T_\tau}\phi_0, \mathtt{W}\right)}{(\mathtt{W}, \mathtt{V})}\\ &= e^{i\zeta\mathtt{y}_3/T_\tau}\tilde{\psi}_{0\zeta},\end{aligned}$$

where $\tilde{\psi}_{0\zeta}$ is periodic in $\mathtt{y}_3$ with period T_τ. We see that $\psi_{0\zeta}$ satisfies

$$\psi_{0\zeta}(\mathtt{y}_1, \mathtt{y}_2, \mathtt{y}_3 + T_\tau) = e^{i\zeta}\psi_{0\zeta}(\mathtt{y}_1, \mathtt{y}_2, \mathtt{y}_3), \qquad \mathtt{y} = (\mathtt{y}_1, \mathtt{y}_2, \mathtt{y}_3)\in D_\tau,$$

with similar relation for $\partial_{\mathtt{y}_3}\psi_{0\zeta}$. By Proposition 4.2 in [45] we know that the operator $J_{\mathring{D}_\tau}$ is invertible in the space of functions satisfying these conditions as long as $\zeta\in(0,2\pi)$ with an inverse whose norm depends on τ. The claim now follows from (4.30) and (4.31).

In particular we conclude that the operator $\tilde{\mathbb{L}}_{w_\tau}(\zeta)$ is injective for $\zeta\in(0,2\pi)$ and by the same argument for $\zeta\in(-2\pi,0)$ (note that $\tilde{\mathbb{L}}^*_{w_\tau}(\zeta) = \tilde{\mathbb{L}}_{w_\tau}(-\zeta)$). A version of Lemma 4.4.1 for $\tilde{\mathbb{L}}_{w_\tau}(\zeta)$ shows that this operator is Fredholm, depends analytically on ζ and, as a consequence, it is invertible in a neighbourhood of $[0,2\pi]$ except for a discrete set.

Now let us suppose that for some $\zeta\in(0,2\pi)$ there exists a function $\phi_0\in H^2_{\gamma,per}(\mathbb{R}^2\times[0,T_\tau])$, with some γ, $|\gamma|$ small, such that $\hat{L}_{w_\tau}(\zeta)\phi_0 = 0$. Since ϕ_0 is bounded locally near $\mathring{D}_\tau$ we can use comparison principle to show that ϕ_0 is decaying away from $\mathring{D}_\tau$ at least like $e^{-\sqrt{2-\eta}\frac{|r-\rho(\xi)|}{\varepsilon}}$ (the argument is similar to the one leading to (4.25)). Using Lemma 4.4.1 we get

$$\|\phi_0\|_{H^2_{\pm\gamma,per}(\mathbb{R}^2\times[0,T_\tau])} \le C\varepsilon^{-1}\|\phi_0\|_{L^2_{per}(\mathbb{R}^2\times[0,T_\tau])}. \tag{4.32}$$

We normalize $\|\phi_0\|_{H^1_{per}(\mathbb{R}^2\times[0,T_\tau])} = 1$ and set $\tilde{\phi}_0 = \chi_{\varepsilon/\delta}Y^*_\varepsilon\phi_0$. With this notation (see (4.16))

$$\|\tilde{\phi}_0\|_{L^2(\mathring{D}_\tau\times\mathbb{R})} \sim \varepsilon^{-1/2}\|\phi_0\|_{L^2(\{\mathrm{dist}\,(x,\mathring{D}_\tau)<\delta\})},$$

since $d\mathtt{t}dV_{\mathring{D}_\tau} \sim \varepsilon^{-1}dx$. Similarly, we have

$$\|\tilde{\phi}_0\|_{H^1(\mathring{D}_\tau\times\mathbb{R})} \sim \varepsilon^{1/2}\|\phi_0\|_{H^1(\{\mathrm{dist}\,(x,\mathring{D}_\tau)<\delta\})}.$$

Next, we observe that since ϕ_0 is decaying exponentially away from $\mathring{D}_\tau$ we have by (4.32)

$$\begin{aligned}\|\phi_0\|^2_{H^1_{per}(\mathbb{R}^2\times[0,T_\tau])} &= \|\phi_0\|^2_{H^1(\{\mathrm{dist}\,(x,\mathring{D}_\tau)<\delta\})} + \|\phi_0\|^2_{H^1(\{\mathrm{dist}\,(x,\mathring{D}_\tau)>\delta\})}\\ &\le \|\phi_0\|^2_{H^1(\{\mathrm{dist}\,(x,\mathring{D}_\tau)<\delta\})} + \mathcal{O}(e^{-c\delta/\varepsilon})\|\phi\|^2_{H^1_{\gamma,per}(\mathbb{R}^2\times[0,T_\tau])\cap H^1_{-\gamma,per}(\mathbb{R}^2\times[0,T_\tau])}\\ &\le \|\phi_0\|^2_{H^1(\{\mathrm{dist}\,(x,\mathring{D}_\tau)<\delta\})} + \mathcal{O}(\varepsilon^{-1}e^{-c\delta/\varepsilon})\|\phi_0\|^2_{H^1_{per}(\mathbb{R}^2\times[0,T_\tau])},\end{aligned}$$

hence

$$\|\phi_0\|^2_{H^1(\{\mathrm{dist}\,(x,\mathring{D}_\tau)<\delta\})} \ge \frac{1}{2}\|\phi_0\|^2_{H^1_{per}(\mathbb{R}^2\times[0,T_\tau])} = \frac{1}{2}. \tag{4.33}$$

Given all this we claim that we can find a nontrivial function $\tilde{\phi} = \tilde{\phi}_0 + \tilde{\phi}_1$, $\psi \in L^2(\mathring{D}_\tau\times\mathbb{R})$, such that $\tilde{\mathbb{L}}_{w_\tau}(\zeta)\tilde{\phi} = 0$, by solving

$$\begin{aligned}\tilde{\mathbb{L}}_{w_\tau}(\zeta)\tilde{\phi}_1 = &-\left[\chi_{\varepsilon/\delta}, \tilde{\mathbb{L}}_{w_\tau}(\zeta)\right]\tilde{\phi}_0\\ &- e^{-i\zeta\xi^*/T_\tau}\chi_{\varepsilon/\delta}(1-\chi_{\varepsilon/\delta})\left[\left(\varepsilon\mathtt{t}|A_{D_\tau}|^2 + \mathbb{Q}_\varepsilon\right)\partial_\mathtt{t} + \varepsilon\mathbb{A}_\varepsilon\right]e^{i\zeta\xi^*/T_\tau}\tilde{\phi}_0\\ := &\, R_{\varepsilon/\delta}(\mathtt{y},\mathtt{t}).\end{aligned}$$

In fact, since $R_{\varepsilon/\delta}$ is supported in the set $\delta/2\varepsilon \le |\mathtt{t}| \le \delta/\varepsilon$ therefore

$$\|R_{\varepsilon/\delta}\|_{L^2(\mathring{D}_\tau\times\mathbb{R})} \le Ce^{-c\delta/\varepsilon}\|\phi_0\|_{H^1_{\gamma,per}(\mathbb{R}^2\times[0,T_\tau])\cap H^1_{-\gamma,per}(\mathbb{R}^2\times[0,T_\tau])}.$$

Next we decompose $\tilde{\phi}_1 = \tilde{\phi}_1^{\|} + \tilde{\psi}_1\mathtt{W}$ and use (with only slight modifications) the argument that we have used to show that $\tilde{\mathbb{L}}_{w_\tau}(\zeta)$ is injective to get

$$\|\tilde{\phi}_1\|_{H^1(\mathring{D}_\tau\times\mathbb{R})} \le C\varepsilon^{-1}\|R_{\varepsilon/\delta}\|_{L^2(\mathring{D}_\tau\times\mathbb{R})} \le Ce^{-c\delta/\varepsilon}\|\phi_0\|_{L^2_{per}(\mathbb{R}^2\times[0,T_\tau])}.$$

From (4.33) it now follows

$$\begin{aligned}\|\tilde{\phi}\|_{H^1(\mathring{D}_\tau\times\mathbb{R})} &\ge \|\tilde{\phi}_0\|_{H^1(\mathring{D}_\tau\times\mathbb{R})} - \|\tilde{\phi}_1\|_{H^1(\mathring{D}_\tau\times\mathbb{R})}\\ &\ge C\varepsilon^{1/2}\|\phi_0\|^2_{H^1(\{\mathrm{dist}\,(x,\mathring{D}_\tau)<\delta\})} + \mathcal{O}(e^{-c\delta/\varepsilon})\|\phi_0\|_{L^2_{per}(\mathbb{R}^2\times[0,T_\tau])} > 0,\end{aligned}$$

for ε sufficiently small. This contradicts the fact that $\tilde{\mathbb{L}}_{w_\tau}(\zeta)$ is injective. Taking this into account we see that $\hat{L}_{w_\tau}(\zeta)$ is invertible at least for $\zeta \in (0, 2\pi)$, and thus by the Fredholm alternative is invertible for all ζ such that $|\mathrm{Im}\,\zeta| < \delta_\tau$, expect possibly a finite set where $\hat{L}^{-1}_{w_\tau}(\zeta)$ has poles. We claim that the required properties of L_{w_τ} follow now by taking the inverse Fourier-Laplace transform at any a for which $\hat{L}^{-1}_{w_\tau,\gamma}(\zeta)$ is well defined for $\zeta = \mu + iT_\tau a$, $\mu \in [0, 2\pi]$. Indeed, given $g \in L^2_{a,\gamma}(\mathbb{R}^3)$ with $a^2 + \gamma^2 < \delta_\tau$ and cutoff functions $\chi^\pm(z)$ such that $\chi^+(z) + \chi^-(z) = 1$ and $\mathrm{supp}\,\chi^+ = (-1, \infty)$ we can solve

$$L_{w_\tau}\phi^\pm = \chi^\pm g.$$

To do this we let $\hat{g}^\pm$ to be the Fourier-Laplace transforms of $g^\pm$. Then we solve

$$\hat{L}_{w_\tau}(\zeta)\hat{\phi}^\pm = \hat{g}^\pm \Longrightarrow \hat{\phi}^\pm = \hat{L}_{w_\tau}(\zeta)^{-1}\hat{g}^\pm,$$

and by taking the inverse of the Fourier-Laplace transform $\mathcal{F}$ we determine

$$\phi^{\pm} = \mathcal{F}^{-1}\big(\hat{L}_{w_\tau}(\zeta)^{-1}\hat{g}^{\pm}\big),$$

and define

$$\phi = G_{w_\tau}(g) := \phi^- + \phi^+.$$

This ends the proof. $\square$

Remark 4.2. We will describe a useful consequence of local elliptic estimates. Let us suppose that we know *a priori* $\phi, g \in L^2_{a,\gamma}(\mathbb{R}^3)$ where

$$\Delta\phi = g.$$

The goal is to obtain weighted Sobolev estimates for the derivatives of ϕ. First, consider a cube $Q_r(x_0)$ centered at $x_0 \in \mathbb{R}^3$ and with its sides equal to r. Standard elliptic estimates show

$$r\|D^2\phi\|_{L^2(Q_r(x_0))} + \|\nabla\phi\|_{L^2(Q_r(x_0))} \le C\|g\|_{L^2(Q_{2r}(x_0))} + Cr^{-1}\|\phi\|_{L^2(Q_{2r}(x_0))}.$$

If $r = \varepsilon$ then we get from this

$$\varepsilon\|D^2\phi\|_{L^2_{a,\gamma}(Q_\varepsilon(x_0))} + \|\nabla\phi\|_{L^2_{a,\gamma}(Q_\varepsilon(x_0))} \le C\|g\|_{L^2_{a,\gamma}(Q_{2\varepsilon}(x_0))} + C\varepsilon^{-1}\|\phi\|_{L^2_{a,\gamma}(Q_{2\varepsilon}(x_0))},$$

since the exponential weights are comparable on the sets with diameters proportional to ε. Arranging now a countable collection of cubes $\{Q_\varepsilon(x_j)\}_{j\in\mathbb{N}}$ in such a way that for each x_j the number of cubes $Q_{2\varepsilon}(x_{j'})$, $j' \neq j$, whose intersection with $Q_\varepsilon(x_j)$ is nonempty is finite and bounded independently on j, while at the same time $\mathbb{R}^3 = \cup_{j\in\mathbb{N}}Q_\varepsilon(x_j)$, we see that above local estimates can be summed up to yield

$$\varepsilon\|D^2\phi\|_{L^2_{a,\gamma}(\mathbb{R}^3)} + \|\nabla\phi\|_{L^2_{a,\gamma}(\mathbb{R}^3)} \le C\|g\|_{L^2_{a,\gamma}(\mathbb{R}^3)} + C\varepsilon^{-1}\|\phi\|_{L^2_{a,\gamma}(\mathbb{R}^3)}.$$

Lemma 4.4.4 Let $\phi \in L^2_{a,\gamma'}(\mathbb{R}^3)$ be a solution of $L_{w_\tau}\phi = g$ with $g \in L^2_{a,\gamma}(\mathbb{R}^3)$ where $\gamma > 0$, $\gamma' < \gamma$ and $a^2 + \gamma^2 < \delta_\tau$, $a^2 + \gamma'^2 < \delta_\tau$. Then $\phi \in L^2_{a,\gamma}(\mathbb{R}^3)$. An analogous statement holds when we assume that $\gamma < 0$ and $\gamma < \gamma'$.

Proof. We follow the proof of a similar result in [19]. Let χ be a cutoff function supported in the set $\varepsilon M < r - \rho_\tau(z)$, $r^2 = x_1^2 + x_2^2$, and such that $\chi \equiv 1$ in the set $r - \rho_\tau(z) > 2\varepsilon M$ where M is chosen so that $f'(w_\tau) < -1$ for $r - \rho_\tau(z) > \varepsilon M$. We calculate

$$L_{w_\tau}(\chi\phi) = \chi g + [\varepsilon\Delta, \chi]\phi \equiv g_1, \quad [\varepsilon\Delta, \chi]\phi = \varepsilon\Delta(\chi\phi) - \varepsilon\chi\Delta\phi.$$

We have $\varepsilon\nabla\chi = \mathcal{O}(1)$ and $\varepsilon\Delta\chi = \mathcal{O}(\varepsilon^{-1})$. Moreover, by local elliptic estimates applied to the equation

$$\Delta\phi = \varepsilon^{-1}g + \varepsilon^{-2}f'(w_\tau)\phi$$

we can show that (see Remark 4.2)

$$\|\varepsilon\nabla\chi\cdot\nabla\phi\|_{L^2_{a,\gamma}(\mathbb{R}^3)} \le C\varepsilon^{-1}\|g\|_{L^2_{a,\gamma}(\mathbb{R}^3)} + C\varepsilon^{-2}\|\phi\|_{L^2_{a,\gamma'}(\mathbb{R}^3)}.$$

We find from this

$$\|\varepsilon\nabla\chi\cdot\nabla\phi\|_{L^2_{a,\gamma}(\mathbb{R}^3)} + \|\varepsilon\phi\Delta\chi\|_{L^2_{a,\gamma}(\mathbb{R}^3)} \le C\varepsilon^{-1}\|g\|_{L^2_{a,\gamma}(\mathbb{R}^3)}\| + C\varepsilon^{-2}\|\phi\|_{L^2_{a,\gamma'}(\mathbb{R}^3)}.$$

Above, we use the fact that the weighed norms $\|\cdot\|_{L^2_{a,\gamma}(\mathbb{R}^3)}$ and $\|\cdot\|_{L^2_{a,\gamma'}(\mathbb{R}^3)}$ are comparable in the set $r-\rho_\tau(z)\in[\varepsilon M, 2\varepsilon M]$. From this we obtain

$$\|g_1\|_{L^2_{a,\gamma}(\mathbb{R}^3)} \le C\varepsilon^{-1}\|g\|_{L^2_{a,\gamma}(\mathbb{R}^3)} + C\varepsilon^{-2}\|\phi\|_{L^2_{a,\gamma'}(\mathbb{R}^3)}.$$

Now we solve the problem

$$\begin{aligned} L_{w_\tau}\phi_{1,R} &= g_1 \qquad \text{in } \Omega_{\varepsilon M,R},\\ \phi_{1,R} &= 0 \qquad \text{on } \Omega_{\varepsilon M,R}, \end{aligned}$$

in a bounded set $\Omega_{\varepsilon M,R}=\{\varepsilon M< r-\rho_\tau(z)<R, |z|<R\}$. Using similar argument as the one leading to (4.23) in the proof of Lemma 4.4.1 we get

$$\begin{aligned} \|\phi_{1,R}\|_{H^1_{a,\gamma}(\Omega_{\varepsilon M,R})} &\le C\varepsilon^{-1/2}\|g_1\|_{L^2_{a,\gamma}(\mathbb{R}^3)}\\ &\le C\varepsilon^{-3/2}\|g\|_{L^2_{a,\gamma}(\mathbb{R}^3)} + C\varepsilon^{-5/2}\|\phi\|_{L^2_{a,\gamma'}(\mathbb{R}^3)}. \end{aligned}$$

Note the that in the first of the above inequalities only the right hand side of the equation appears, which is due to the fact that we assumed homogeneous Dirichlet boundary conditions on $\phi_{1,R}$ and we do not need to introduce the cut off function $\chi_{\varepsilon M}$ in proving a version of Lemma 4.4.1 needed here. Letting $R\to\infty$ we get a solution ϕ_1 of the equation $L_{w_\tau}\phi_1=g_1$ but now in the set $\varepsilon M<r-\rho_\tau(z)$, such that

$$\|\phi_1\|_{H^1_{a,\gamma}(\mathbb{R}^3)} \le C\varepsilon^{-3/2}\|g\|_{L^2_{a,\gamma}(\mathbb{R}^3)} + C\varepsilon^{-5/2}\|\phi\|_{L^2_{a,\gamma'}(\mathbb{R}^3)}.$$

We also have $L_{w_\tau}(\chi\phi-\phi_1)=0$ and $\phi_1=\chi\phi=0$ along the surface $\varepsilon M=r-\rho_\tau(z)$. Then, an estimate similar to (4.23), shows that actually $\chi\phi=\phi_1$. Similar argument applied in the set $-\varepsilon M>r-\rho_\tau(z)$ ends the proof. □

4.5. The deficiency space and the kernel of L_{w_τ}

Let us summarize our results so far. Let $g\in L^2_{a,\gamma}(\mathbb{R}^3)$ with $a^2+\gamma^2<\delta_\tau$, and cutoff functions $\chi^\pm(z)$ such that $\chi^+(z)+\chi^-(z)=1$ and $\operatorname{supp}\chi^+=[-1,\infty)$ be given.

(i) As in Proposition 4.4.1 we can solve

$$L_{w_\tau}\phi^\pm=\chi^\pm g$$

where $\phi^\pm\in H^2_{-|a|,\gamma}(\mathbb{R}^3)_\pm$, (except for a finite set of a).

(ii) If we have $L_{w_\tau}\phi=g$, $\phi\in L^2_{a,\gamma'}(\mathbb{R}^3)$, $g\in L^2_{a,\gamma}(\mathbb{R}^3)$ with $\gamma>0$ and $\gamma'<\gamma$ then $\phi\in H^2_{a,\gamma}(\mathbb{R}^3)$. In particular if g is decaying exponentially away from the surface D_τ, so that we have $g\in L^2_{a,\gamma}(\mathbb{R}^3)\cap L^2_{a,-\gamma}(\mathbb{R}^3)$ then $\phi\in H^2_{a,\gamma}(\mathbb{R}^3)\cap H^2_{a,-\gamma}(\mathbb{R}^3)$. This means that the decay rate of the solution away from the nodal set improves together with the rate of decay of the right hand side.

(iii) When the right hand side decays both along the nodal set and in the direction transversal to it, for example $g\in L^2_{a,\gamma}(\mathbb{R}^3)\cap L^2_{a,-\gamma}(\mathbb{R}^3)$, with $a>0$, then we can use the parametrix to solve the equation $L_{w_\tau}\phi^+=\chi^+g$ and determine a solution ϕ^+ such that

$\chi^+\phi^+ \in L^2_{a,\gamma}(\mathbb{R}^3)_+ \cap L^2_{a,-\gamma}(\mathbb{R}^3)_+$. At the same time we can find another solution ϕ_1^+, such that $\chi^+\phi_1^+ \in L^2_{-a,\gamma}(\mathbb{R}^3)_+ \cap L^2_{-a,-\gamma}(\mathbb{R}^3)_+$ and we get the following decomposition

$$\phi^+ = \sum_{j=1}^{k} Z_j^+ + \phi_1^+,$$

where Z_j^+ are in the kernel of the operator L_{w_τ}. Then we have

$$\phi^+ = \chi^+\phi^+ + \chi^-\phi^+ = \chi^+\phi^+ + \chi^-\phi_1^+ + \sum_{j=1}^{k} Z_j^+\chi^-,$$

where $(\chi^+\phi^+ + \chi^-\phi_1^+) \in H^2_{a,\gamma}(\mathbb{R}^3)$. Of course we can argue similarly for the equation $L_{w_\tau}\phi^- = \chi^- g$ and thus at the end we get the following formula

$$\phi = \phi_0 + \sum_{j=1}^{k} \chi^- Z_j^+ + \sum_{j=1}^{k} \chi^+ Z_j^-,$$

where $\phi_0 \in H^2_{a,\gamma}(\mathbb{R}^3)$. This is the so called *linear decomposition formula*. It says that any solution to $L_{w_\tau}\phi = g$ can be decomposed into an exponentially decaying part and and a linear combination of $2k$ functions which are related to the residues of $\hat{L}^{-1}_{w_\tau,\gamma}(\zeta)$ at its poles. We say that these functions belong to the *deficiency space*. Clearly the elements of the kernel of L_{w_τ} (which is k dimensional) belong to the deficiency space and thus removing them from it we obtain a space on which L_{w_τ} is an isomorphism (see Lemma (4.5.1) below).

Before stating precisely the next Lemma we introduce weighted Sobolev spaces

$$\bar{L}_{a,\gamma}(\mathbb{R}^3) := L^2_{a,\gamma}(\mathbb{R}^3) \cap L^2_{a,-\gamma}(\mathbb{R}^3), \quad \bar{H}^\ell_{a,\gamma}(\mathbb{R}^3) := H^\ell_{a,\gamma}(\mathbb{R}^3) \cap H^\ell_{a,-\gamma}(\mathbb{R}^3).$$

Note that $\phi \in \bar{L}_{a,\gamma}(\mathbb{R}^3)$ decays away from D_τ as $\cosh^{-|\gamma|}\left(\frac{r-\rho_\tau(z)}{\varepsilon}\right)$ if $\gamma > 0$, and decays (for $a > 0$) or grows (for $a < 0$) along D_τ at the rate $\cosh^{-a} z$. Based on observations (i)–(iii) we have

Lemma 4.5.1 Let $\gamma > 0$, $a > 0$, with $a^2 + \gamma^2 < \delta_\tau$ and let us define the deficiency space

$$\mathcal{D}_{w_\tau} = \operatorname{span}\left\{\chi^+ Z_j, \chi^- Z_j, j = 1, \dots k, Z_j \in \operatorname{Ker} L_{w_\tau}\right\}.$$

We further decompose $\mathcal{D}_{w_\tau} = \mathcal{K}_{w_\tau} \oplus \mathcal{E}_{w_\tau}$, where $\mathcal{K}_{w_\tau} = \operatorname{Ker} L_{w_\tau}$. Then the operator

$$\begin{aligned} L_{w_\tau} \colon \bar{H}^2_{a,\gamma}(\mathbb{R}^3) \oplus \mathcal{E}_{w_\tau} &\longrightarrow \bar{L}^2_{a,\gamma}(\mathbb{R}^3) \\ \phi &\longmapsto L_{w_\tau}\phi. \end{aligned}$$

is an isomorphism.

Note that $\dim \mathcal{E}_{w_\tau} = k = \dim \mathcal{K}_{w_\tau}$ and that we know already that $k \geq 6 = \dim \mathcal{I}_{w_\tau}$ where the linear subspace $\mathcal{I}_{w_\tau}$ was defined in (4.11). We will show next that indeed $k = 6$.

Proposition 4.5.1 We have $\mathcal{K}_{w_\tau} = \mathcal{I}_{w_\tau}$.

PROOF OF PROPOSITION 4.5.1. The idea of the proof is to relate the kernel of the operator L_{w_τ} with the space of the Jacobi fields of the operator $\mathcal{J}_{D_\tau}$ that is explicitly known and in particular its dimension is 6. Let us consider a $\phi \in \mathcal{K}_{w_\tau}$. *A priori* it may happen that ϕ is exponentially increasing in the z variable but we know already (see the argument leading to (4.32) an also Lemma 4.4.4 and Remark 4.1) that it must be decaying at least like $\cosh^{-\gamma}(\frac{r-\rho_\tau(z)}{\varepsilon})$ with some $\gamma > 0$. In particular all integrations with respect to the transversal direction to D_τ that will appear below are justified.

Next, we note that formula (4.10) suggests that near the surface D_τ the elements of $\mathcal{K}_{w_\tau}$ should be proportional, asymptotically as $\varepsilon \to 0$, to $\mathtt{W}$ times a function on D_τ. To make this rigorous we first prove the following

Lemma 4.5.2 Let $\phi \in \mathcal{K}_{w_\tau}$ be such that

$$\int_{\mathbb{R}} (Y_\varepsilon^* \phi)(\mathtt{y},\mathtt{t}) \mathtt{W}(\mathtt{y},\mathtt{t}) \chi_{\varepsilon/\delta}(\mathtt{t})\, d\mathtt{t} = 0 \quad \forall \mathtt{y} \in D_\tau. \tag{4.34}$$

Then we have $\phi \equiv 0$.

PROOF OF LEMMA 4.5.2. As we have pointed out it is not hard to show that ϕ decays exponentially like $\cosh^{-\gamma}\left(\frac{r-\rho_\tau(z)}{\varepsilon}\right)$ and so we can compute

$$\int_{\mathbb{R}^2} \phi^2(x,z)\, dx = h(z), \qquad x = (x_1, x_2).$$

Direct calculation shows

$$\frac{\varepsilon}{2}\frac{d^2 h}{dz^2} = \int_{\mathbb{R}^2} \left[\varepsilon |\nabla_x \phi|^2 - \frac{1}{\varepsilon} f'(w_\tau)\phi^2\right] dx + \varepsilon \int_{\mathbb{R}^2} |\partial_z \phi|^2\, dx. \tag{4.35}$$

We claim that the orthogonality condition (4.34) implies

$$\int_{\mathbb{R}^2} \left[\varepsilon |\nabla_x \phi|^2 - \frac{1}{\varepsilon} f'(w_\tau)\phi^2\right] dx \geq \frac{\kappa}{\varepsilon} \int_{\mathbb{R}^2} \phi^2(x,z)\, dx = \frac{\kappa}{\varepsilon} h, \tag{4.36}$$

with some constant $\kappa > 0$. To prove this claim we need

Lemma 4.5.3 There exists a constant $\kappa > 0$ such that for any sufficiently large R and any $v \in H^1((-R,R))$ it holds

$$\int_{-R}^{R} |v'|^2 - f'(H)v^2 \geq \kappa \int_{-R}^{R} v^2 \quad \text{whenever} \quad \int_{-R}^{R} vH'\chi_R = 0,$$

where χ_R is a smooth cutoff function supported in $(-R,R)$ such that $\chi_R(x) = 1$ in $(-R/2, R/2)$.

A proof of this Lemma (using for instance (4.28) as a point of departure) is omitted.

Changing to Fermi coordinates we have in $\mathcal{N}_\delta$

$$\varepsilon|\nabla_x\phi|^2 = \frac{1}{\varepsilon}|\partial_{\mathtt{t}} Y_\varepsilon^*\phi|^2 + \mathcal{O}(\varepsilon)|\partial_s Y_\varepsilon^*\phi|^2 + \mathcal{O}(\varepsilon)|\partial_\theta Y_\varepsilon^*\phi|^2.$$

Next, for a fixed z we consider a diffeomorphism $(x_1, x_2) \mapsto (\theta, \mathtt{t})$ defined by

$$x_j = (X_\tau(s,\theta) + \varepsilon \mathtt{t} N_\tau(s,\theta)) \cdot \mathtt{e}_j$$

where $s = s(\theta, \mathtt{t}; z)$ is determined from

$$z = (X_\tau(s,\theta) + \varepsilon \mathtt{t} N_\tau(s,\theta)) \cdot \mathtt{e}_3.$$

The Jacobian matrix of this map can be calculated explicitly but for our purpose it is enough to note that

$$dx_1 dx_2 = \varepsilon\mu_0(\theta)\, d\theta d\mathtt{t} + \varepsilon^2 \mathtt{t}\mu_1(\theta,\mathtt{t})\, d\theta d\mathtt{t},$$

where μ_0, μ_1 are positive densities and

$$|\mu_1(\theta,\mathtt{t})| \leq C.$$

From this we find

$$\begin{aligned}\int_{\mathbb{R}^2} \left[\varepsilon|\nabla_x\phi|^2 - \frac{1}{\varepsilon}f'(w_\tau)\phi^2\right] dx \geq & \int_0^{2\pi} \left\{\int_{|\mathtt{t}|\leq\delta/\varepsilon} \left[|\partial_{\mathtt{t}} Y_\varepsilon^*\phi|^2 - f'\left(H(\mathtt{t})\right)|Y_\varepsilon^*\phi|^2\right] d\mathtt{t}\right\} \mu_0 d\theta \\ & + \int_{\mathbb{R}^2\setminus\mathcal{N}_\delta} \left[\varepsilon|\nabla_x\phi|^2 + \frac{1}{\varepsilon}\phi^2\right] dx \\ & - K(\delta+\varepsilon)\int_{\mathbb{R}^2} \left[\varepsilon|\nabla_x\phi|^2 + \frac{1}{\varepsilon}\phi^2\right] dx\end{aligned}$$

The potential $f'(w_\tau)$ in the first line on the left can be replaced by $f'(H)$ on the right of this line since $Y_\varepsilon^* w_\tau = H + \mathcal{O}(\varepsilon)$. The term in the second line above appears because $f'(w_\tau) < -2 + \eta$ in the complement of $\mathcal{N}_\delta$. Finally, all the other terms are of smaller size and can be controlled by the integral in the third line times $K(\delta+\varepsilon)$, where K is a constant. Using Lemma 4.5.3 and going back to the original variables we get

$$\int_0^{2\pi} \left\{\int_{|\mathtt{t}|\leq\delta/\varepsilon} \left[|\partial_{\mathtt{t}} Y_\varepsilon^*\phi|^2 - f'\left(H(\mathtt{t})\right)|Y_\varepsilon^*\phi|^2\right] d\mathtt{t}\right\} \mu_0 d\theta \geq \frac{C}{\varepsilon}\int_{\mathcal{N}_\delta} \phi^2\, dx.$$

It follows

$$\int_{\mathbb{R}^2} \left[\varepsilon|\nabla_x\phi|^2 - \frac{1}{\varepsilon}f'(w_\tau)\phi^2\right] dx \geq \frac{C}{\varepsilon}\int_{\mathbb{R}^2} \phi^2\, dx - K(\varepsilon+\delta)\int_{\mathbb{R}^2} \left[\varepsilon|\nabla_x\phi|^2 + \frac{1}{\varepsilon}\phi^2\right] dx,$$

hence

$$[1 + K(\varepsilon+\delta)]\int_{\mathbb{R}^2} \left[\varepsilon|\nabla_x\phi|^2 - \frac{1}{\varepsilon}f'(w_\tau)\phi^2\right] dx \geq \frac{C}{\varepsilon}\int_{\mathbb{R}^2} \phi^2\, dx - \frac{K(\varepsilon+\delta)}{\varepsilon}\int_{\mathbb{R}^2} [1 + f'(w_\tau)]\phi^2\, dx$$

which gives (4.36) provided that ε and δ are small enough. From (4.36) and (4.35) we find

$$\frac{\varepsilon}{2}\frac{d^2 h}{dz^2} - \frac{\kappa}{\varepsilon}h > 0.$$

By Lemma 4.5.1 we know *a priori* that ϕ, hence h, is growing in z at $\pm\infty$ at some exponential rate which is independent on ε. Applying the comparison principle we see that h, and hence ϕ, is actually decaying as $z \to \pm\infty$, at some exponential rate proportional to ε^{-1}. Using again orthogonality condition (4.34) we calculate

$$\langle -L_{w_\tau}\phi, \phi\rangle = \int_{\mathbb{R}^3} \left[\varepsilon|\nabla\phi|^2 - \frac{1}{\varepsilon} f'(w_\tau)\phi^2\right] dxdz \geq c\varepsilon^{-1}\|\phi\|^2_{L^2(\mathbb{R}^3)},$$

hence $\phi \equiv 0$ as claimed. This ends the proof of the Lemma. □

We continue with the proof of the Proposition. For a given $\phi \in \mathcal{K}_{w_\tau}$ we define

$$\varphi = (Y^*_\varepsilon\phi)\chi_{\delta/\varepsilon}.$$

The function φ is a cutoff of $Y^*_\varepsilon\phi$ and is supported in $\mathcal{N}_\delta$. Since $\phi \in \bar{H}^2_{a,\gamma}(\mathbb{R}^3)$ with some $a \in \mathbb{R}$, and $\gamma > 0$ both small (ϕ decays or grows in z like $\cosh^{-a} z$, and it decays like $\cosh^{-\gamma}\left(\frac{r-\rho_\tau(z)}{\varepsilon}\right)$ away from D_τ) we have that $\varphi \in \bar{H}^2_{a_*,\gamma_*}(D_\tau \times \mathbb{R})$ with some $a_* \in \mathbb{R}$ and $\gamma_* > 0$ both small. We also have

$$\|\varphi\|_{\bar{L}^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})} \leq C\varepsilon^{-1/2}\|\phi\|_{\bar{L}^2_{a,\gamma}(\mathbb{R}^3)},$$

with similar estimates for other Sobolev norms. Note that since ϕ decays like $\cosh^{-\gamma}\left(\frac{r-\rho_\tau(z)}{\varepsilon}\right)$ away from D_τ then φ decays at least like $\cosh^{-\bar{\gamma}}\mathtt{t}$ with some $\bar{\gamma} > 0$. Above estimate holds then for any $\gamma_* < \bar{\gamma}$ and we will consider only γ_* restricted this way.

In what follows we will argue by contradiction and we will assume that $\dim\mathcal{K}_{w_\tau} > 6$. Since we know explicitly six linearly independent elements in $\mathcal{K}_{w_\tau}$, which are the geometric Jacobi fields spanning the subspace $\mathcal{I}_{w_\tau}$ defined in (4.11) we can find a function $\phi \in \mathcal{K}_{w_\tau}$ such that $\phi \notin \mathcal{I}_{w_\tau}$ and in particular we can assume

$$\int_{D_\tau\times\mathbb{R}} \chi_{\delta/\varepsilon}(Y^*_\varepsilon\phi)(Y^*_\varepsilon\Phi_\bullet)\cosh^{a_*}(s)\, dV_{D_\tau}d\mathtt{t} = 0, \qquad \forall\Phi_\bullet \in \mathcal{I}_{w_\tau}. \tag{4.37}$$

We decompose

$$\varphi = \psi\mathtt{W} + \varphi^{\parallel}, \qquad \int_{\mathbb{R}} \chi_{\varepsilon/\delta}\varphi^{\parallel}(\mathtt{y},\mathtt{t})\mathtt{W}(\mathtt{y},\mathtt{t})\, d\mathtt{t} = 0.$$

From Lemma 4.5.2 we know that $\psi \neq 0$ and therefore we can assume $\|\psi\|_{L^2_{a^*}(D_\tau)} = 1$ (indeed we expect $\|\varphi^{\parallel}\|_{L^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})} = o(1)$). We compute

$$\mathbb{L}_{w_\tau}\varphi = (Y^*_\varepsilon L_{w_\tau})\varphi + [\mathbb{L}_{w_\tau} - (Y^*_\varepsilon L_{w_\tau})]\varphi \equiv g,$$

where, more explicitly,

$$(Y^*_\varepsilon L_{w_\tau})\varphi = \varepsilon^{-1}[(Y^*_\varepsilon\phi)\partial_{\mathtt{tt}}\chi_{\varepsilon/\delta} + 2\partial_{\mathtt{t}}(Y^*_\varepsilon\phi)\partial_{\mathtt{t}}\chi_{\varepsilon/\delta}] - (H_{D_\tau} + \varepsilon\mathtt{t}|A_{D_\tau}|^2 + \mathbb{Q}_\varepsilon)(Y^*_\varepsilon\phi)\partial_{\mathtt{t}}\chi_{\varepsilon/\delta}$$

$$[\mathbb{L}_{w_\tau} - (Y^*_\varepsilon L_{w_\tau})]\varphi = (1-\chi_{\varepsilon/\delta})(\varepsilon\mathtt{t}|A_{D_\tau}|^2 + \mathbb{Q}_\varepsilon)\partial_{\mathtt{t}}\varphi - \varepsilon(1-\chi_{\varepsilon/\delta})\mathbb{A}_\varepsilon\varphi.$$

It is not hard to see that

$$\|\chi_{\varepsilon/\delta}g\|_{\bar{L}^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})} \leq \mathcal{O}(e^{-c\delta/\varepsilon})\|\varphi\|_{\bar{H}^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})},$$

since $\gamma_* < \bar{\gamma}$. Using this we can calculate

$$\int_{\mathbb{R}} \chi_{\varepsilon/\delta} \mathbb{L}_{w_\tau} \varphi \mathtt{W}\, d\mathtt{t} = \int_{\mathbb{R}} \chi_{\varepsilon/\delta} g \varphi \mathtt{W}\, d\mathtt{t}$$

which gives

$$\mathcal{J}_{D_\tau} \psi = T(\varphi^{\parallel}, \psi),$$

where T is a linear operator satisfying

$$\begin{aligned} \|T(\varphi^{\parallel}, \psi)\|_{L^2_{a_*}(D_\tau)} \le C\varepsilon^{-1} \|\varphi^{\parallel}\|_{\bar{L}^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})} + C\varepsilon \|\varphi^{\parallel}\|_{\bar{H}^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})} \\ + C\varepsilon^{1-\alpha}\|\psi\|_{H^1_{a_*}(D_\tau)} + C\delta \|\psi\|_{H^2_{a_*}(D_\tau)}, \quad (4.38) \end{aligned}$$

with some $\alpha \in (0,1)$. Next we will estimate $\varphi^{\parallel}$. Since this argument is similar to that of Proposition 4.4.1 we will outline the main points omitting some tedious but straightforward calculations. Let $K > 0$ be a large constant and $\chi^{\pm}\colon \mathbb{R} \to \mathbb{R}_+$ be smooth cutoff functions such that $\chi^+ + \chi^- \equiv 1$, $\chi^+(s) = 1$ when $s > 1$ and $\chi^+(s) = 0$ when $s < -K$ and additionally $K|\chi^{\pm}_s| + K^2|\chi^{\pm}_{ss}| \le C$.

We define $\varphi^{\parallel,\pm} = \chi^{\pm}\varphi^{\parallel}$. Taking the Fourier-Laplace transform (with respect to s) we get

$$\begin{aligned} (\mathbb{L}_{w_\tau}\varphi^{\parallel,\pm})^\wedge &= (\chi^{\pm}\mathbb{L}_{w_\tau}\varphi^{\parallel})^\wedge + \left(\left[\mathbb{L}_{w_\tau}, \chi^{\pm}\right]\varphi^{\parallel}\right)^\wedge \\ &= (\chi^{\pm} g)^\wedge - \left(\chi^{\pm}\mathbb{L}_{w_\tau}(\psi\mathtt{W})\right)^\wedge + \left(\left[\mathbb{L}_{w_\tau}, \chi^{\pm}\right]\varphi^{\parallel}\right)^\wedge . \end{aligned}$$

We can project

$$\begin{aligned} \int_{[0,T_\tau]\times[0,2\pi]\times\mathbb{R}} \hat{\varphi}^{\parallel,\pm}(\mathbb{L}_{w_\tau}\varphi^{\parallel,\pm})^\wedge = \int_{[0,T_\tau]\times[0,2\pi]\times\mathbb{R}} \hat{\varphi}^{\parallel,\pm}(\chi^{\pm} g)^\wedge \\ - \int_{[0,T_\tau]\times[0,2\pi]\times\mathbb{R}} \hat{\varphi}^{\parallel,\pm}\left(\chi^{\pm}\mathbb{L}_{w_\tau}(\psi\mathtt{W})\right)^\wedge \quad (4.39) \\ + \int_{[0,T_\tau]\times[0,2\pi]\times\mathbb{R}} \hat{\varphi}^{\parallel,\pm}\left(\left[\mathbb{L}_{w_\tau}, \chi^{\pm}\right]\varphi^{\parallel}\right)^\wedge . \end{aligned}$$

Since we have

$$\int_{\mathbb{R}} \chi_{\varepsilon/\delta}\hat{\varphi}^{\parallel,\pm}\mathtt{W}\, d\mathtt{t} = 0,$$

therefore the bilinear form on the left hand side in (4.39) is positive definite and by an argument similar to the one in Proposition 4.4.1 we get

$$\left| \int_{[0,T_\tau]\times[0,2\pi]\times\mathbb{R}} \hat{\varphi}^{\parallel,\pm}(\mathbb{L}_{w_\tau}\varphi^{\parallel,\pm})^\wedge \right| \ge \frac{C}{\varepsilon}\|\hat{\varphi}^{\parallel,\pm}\|^2_{L^2([0,T_\tau]\times[0,2\pi]\times\mathbb{R})} \ge \frac{C}{\varepsilon}\|\varphi^{\parallel,\pm}\|^2_{L^2_{a_*}(D_\tau\times\mathbb{R})_\pm},$$

where the last inequality follows from Plancherel's identity. Using Cauchy-Schwarz inequality and Plancherel identity again on the right hand side of (4.39) we find

$$\begin{aligned} \varepsilon^{-1}\|\varphi^{\parallel,\pm}\|_{L^2_{a_*}(D_\tau\times\mathbb{R})_\pm} \le C\Big(\|\chi^{\pm} g\|_{L^2_{a_*}(D_\tau\times\mathbb{R})_\pm} + \|\chi^{\pm}\mathbb{L}_{w_\tau}(\psi\mathtt{W})\|_{L^2_{a_*}(D_\tau\times\mathbb{R})_\pm} \\ + \left\|\left[\mathbb{L}_{w_\tau}, \chi^{\pm}\right]\varphi^{\parallel}\right\|_{L^2_{a_*}(D_\tau\times\mathbb{R})_\pm}\Big). \end{aligned}$$

Using an argument similar to the one indicated in Remark 4.1 and Remark 4.2 we can show from this

$$\varepsilon^{-1}\|\varphi^{\|,\pm}\|_{L^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})_\pm}+\varepsilon\|\nabla\varphi^{\|,\pm}\|_{L^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})_\pm}+\varepsilon\|D^2\varphi^{\|,\pm}\|_{L^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})_\pm}\le CR, \tag{4.40}$$

where

$$R\equiv\left(\|\chi^\pm g\|_{L^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})_\pm}+\|\chi^\pm\mathbb{L}_{w_\tau}(\psi\mathtt{W})\|_{L^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})_\pm}+\left\|\left[\mathbb{L}_{w_\tau},\chi^\pm\right]\varphi^{\|}\right\|_{L^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})_\pm}\right).$$

We have

$$\begin{aligned}\|\chi^\pm g\|_{L^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})_\pm}&\le\mathcal{O}(e^{-c\delta/\varepsilon})\left(\|\varphi^{\|}\|_{H^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})}+\|\psi\|_{H^2_{a_*}(D_\tau)}\right),\\ \|\chi^\pm\mathbb{L}_{w_\tau}(\psi\mathtt{W})\|_{L^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})_\pm}&\le C\varepsilon\|\psi\|_{H^2_{a_*}(D_\tau)},\\ \left\|\left[\mathbb{L}_{w_\tau},\chi^\pm\right]\varphi^{\|}\right\|_{L^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})_\pm}&\le\frac{C\varepsilon}{K}\|\varphi^{\|}\|_{H^1_{a_*,\gamma_*}(D_\tau\times\mathbb{R})}.\end{aligned}$$

Combining these inequalities we get from (4.40)

$$\varepsilon^{-1}\|\varphi^{\|}\|_{L^2_{a_*}(D_\tau\times\mathbb{R})}+\varepsilon\|\nabla\varphi^{\|}\|_{L^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})}+\varepsilon\|D^2\varphi^{\|}\|_{L^2_{a_*,\gamma_*}(D_\tau\times\mathbb{R})}\le C\varepsilon\|\psi\|_{H^2_{a_*}(D_\tau)}. \tag{4.41}$$

This and estimate (4.38) imply

$$\|T(\varphi^{\|},\psi)\|_{L^2_{a_*}(D_\tau)}\le C(\varepsilon^{1-\alpha}+\varepsilon)\|\psi\|_{H^2_{a_*}(D_\tau)}+C\delta\|\psi\|_{H^2_{a_*}(D_\tau)}.$$

Decomposing $\psi=\psi^++\psi^-$, where $\psi^\pm=\chi^\pm\psi$ we can use the Fourier-Laplace transform to show that

$$\psi=\psi_0+\psi_1$$

where ψ_0 is a linear combination of the the geometric Jacobi fields and

$$\|\psi_1\|_{H^2_{a_*}(D_\tau)}\le C\|T(\varphi^{\|},\psi)\|_{L^2_{a_*}(D_\tau)}\le C(\varepsilon^{1-\alpha}+\varepsilon)\|\psi\|_{H^2_{a_*}(D_\tau)}+C\delta\|\psi\|_{H^2_{a_*}(D_\tau)}. \tag{4.42}$$

When $a_*>0$ then $\psi_0\equiv 0$ and (4.42) implies that $\psi\equiv 0$. In case $a_*<0$ from (4.37), (4.41) and Lemma 4.2.1 we see that ψ satisfies

$$\int_{D_\tau}\psi\Phi^\bullet_\tau\cosh^{a_*}s\,dV_{D_\tau}=0\Longrightarrow\int_{D_\tau}\psi_0\Phi^\bullet_\tau\cosh^{a_*}s\,dV_{D_\tau}=-\int_{D_\tau}\psi_1\Phi^\bullet_\tau\cosh^{a_*}s\,dV_{D_\tau}$$

for each geometric Jacobi field $\Phi^\bullet_\tau$ of J_{D_τ}. It follows that

$$\|\psi_0\|_{L^2_{a_*}(D_\tau)}\le C\|\psi_1\|_{L^2_{a_*}(D_\tau)}$$

which, together with (4.42), implies $\psi_1\equiv 0$ hence $\psi\equiv 0$. In both cases this is a contradiction. The proof of the proposition is complete. □

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