A New Class of Graphs That Satisfies the Chen-Chvátal Conjecture

P. Aboulker,[1] M. Matamala,[2,3] P. Rochet,[3,4] and J. Zamora[5]

[1] PROJECT COATI, I3S (CNRS, UNSA) AND INRIA, SOPHIA ANTIPOLIS, FRANCE

[2] DEPTO. INGENIERÍA MATEMÁTICA (DIM), UNIVERSIDAD DE CHILE, CHILE

[3] CENTRO DE MODELAMIENTO MATEMÁTICO (CMM, UMI 2807 CNRS), UNIVERSIDAD DE CHILE, CHILE

[4] LABORATOIRE DE MATHÉMATIQUES JEAN LERAY, UNIVERSITÉ DE NANTES, FRANCE

[5] DEPTO MATEMÁTICAS, UNIVERSIDAD ANDRES BELLO, CHILE

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Abstract: A well-known combinatorial theorem says that a set of n non-collinear points in the plane determines at least n distinct lines. Chen and Chvátal conjectured that this theorem extends to metric spaces, with an appropriated definition of line. In this work, we prove a slightly stronger

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1. INTRODUCTION

A classic result in Euclidean geometry asserts that every noncollinear set of n points in the Euclidean plane determines at least n distinct lines.

Erdős [14] showed that this result is a consequence of the Sylvester-Gallai theorem that asserts that every noncollinear set of n points in the plane determines a line containing precisely two points. Coxeter [12] showed that the Sylvester–Gallai theorem holds in a more basic setting known as *ordered geometry*. Here, the notions of distance and angle are not used and, instead, a ternary relation of *betweenness* is employed. We write $[abc]$ for the statement that b lies between a and c. In this notation, a *line* $\overline{xy}$ is defined (for any two distinct points x and y) as:

$$\overline{xy} = \{x, y\} \cup \{u : [uxy] \text{ or } [xuy] \text{ or } [xyu]\} \tag{1}$$

Betweenness in metric spaces was first studied by Menger [16] and further on by Chvátal [10]. In a metric space (V, d), we define

$$[abc] \Leftrightarrow d(a, b) + d(b, c) = d(a, c).$$

Hence, in any metric space (V, d), we can define the line $\overline{uv}$ induced by two points u and v as in (1). A line of a metric space (V, d) is *universal* if it contains all points of V. With this definition of lines in metric spaces, Chen and Chvátal [7] proposed the following beautiful conjecture.

Conjecture 1.1. *Every metric space on n points, where $n \geq 2$, either has at least n distinct lines or has a universal line.*

The best-known lower bound for the number of lines in metric spaces with no universal line is $\Omega(\sqrt{n})$ [2].

As it is explained in [3], it suffices to prove Conjecture 1.1 for metric spaces with integral distances. This motivates looking at two particular types of metric spaces. First, for a positive integer k, we define a *k-metric space* to be a metric space in which all distances are integral and are at most k. Chvátal [11] proved that every 2-metric space on n points ($n \geq 2$) either has at least n distinct lines or has a universal line. The question is open for $k \geq 3$. Aboulker et al. [2] proved that, for all $k \geq 3$, a k-metric space with no universal line has at least $n/5k$ distinct lines. The conjecture has also been studied in the context of hypergraphs (see [1, 4, 15]) and for metric spaces (see [8]).

A second type of metric space with integral distances arises from graphs. Any finite connected graph induces a metric space on its vertex set, where the distance between two vertices u and v is defined as the length of a shortest path linking u and v. Such metric spaces are called *graph metrics* and are the subject of this article. The best-known lower bound on the number of lines in a graph metric with no universal line is $\Omega(n^{4/7})$ [2]. In [5] and [3] it is proved that Conjecture 1.1 holds for chordal graphs and for distance-hereditary graphs respectively.

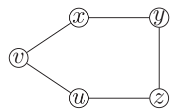

FIGURE 1. A line can be a proper subset of another line: $\overline{xz} = \{x, y, z\} \subseteq \overline{xy} = \{v, x, y, z\}$

The main result of this article is to prove Conjecture 1.1 for all graphs that can be constructed from chordal graphs by repeated substitutions and gluing along vertices. This generalizes chordal and distance hereditary graphs.

2. STATEMENT OF THE MAIN THEOREM

Let $G = (V, E)$ be a graph. We denote by $|G|$ the number of vertices of G. Let a, b, c be three distinct vertices in $V(G)$. The distance $d_G(a, b)$ (or simply $d(a, b)$ when the context is clear) between a and b is the length of a shortest path linking a and b if such a path exists; otherwise, $d_G(a, b) = \infty$. We write $[abc]^G$ (or simply $[abc]$) when $d(a, b) + d(b, c) = d(a, c) < \infty$. Observe that $[abc] \Leftrightarrow [cba]$. We denote by $\overline{ab}^G$ (or simply $\overline{ab}$) the line determined by two distinct vertices a, b, where $\overline{ab}^G = \{a, b\} \cup \{x : [abx] \text{ or } [axb] \text{ or } [xab]\}$. Notice that with this definition, the line defined by two vertices a, b lying in different connected components is $\{a, b\}$. We denote by $\mathcal{L}(G)$ the set of distinct lines in G and by $\ell(G) = |\mathcal{L}(G)|$ the number of distinct lines in G.

The lines on this set can have strange properties. For example, two lines might have more than one common point, and it is even possible for a line to be a proper subset of another line as in Figure 1.

We denote by $N_G(v)$ the set of all neighbors of a vertex v in G. For a set of vertices S, we denote by $N_G(S)$ (or simply $N(S)$) the set of all vertices *outside* S having a neighbor in S . A set S is *dominating* if $S \cup N_G(S) = V(G)$.

A set of vertices M of a graph $G = (V, E)$ is a *module* if for each $a, b \in M$, $u \notin M$, $au \in E$ if and only if $bu \in E$. It is a *nontrivial* module if $|G| > |M| \geq 2$. If M is a dominating set, we call it a *dominating module*. In this situation, $N(M)$ is also a module unless $M = V$. When $\{u, v\}$ is a module, we say that (u, v) is *a pair of twins*. If u and v are adjacent they are called *true* twins; otherwise, they are called *false* twins.

A *bridge* ab is an edge whose deletion increases the number of connected components of the graph. We denote by $br(G)$ the number of bridges of G. If $br(G) = 0$, we say that G is *bridgeless*. If ab is a bridge of a connected graph G, then for every vertex $p \in V(G) \setminus \{a, b\}$, we either have $[pab]$ or $[abp]$. Hence $\overline{ab}^G = V(G)$ and thus Conjecture 1.1 is only interesting for bridgeless graphs.

We are going to define a class of graphs that includes chordal graphs (graphs without induced cycles of length more than three) and distance hereditary graphs (graphs whose induced paths are shortest paths) (see [6] for further references).

Let $\mathcal{C}$ be the class of graphs G such that every induced subgraph of G either is a chordal graph, or has a cut-vertex, or has a nontrivial module. By definition, this class is *hereditary*, that is, if $G \in \mathcal{C}$, then every induced subgraph of G is also in $\mathcal{C}$.

Let $\mathcal{F} = \{C_4, K_{2,3}, W_4, W_4', K_6', K_8'\}$ (see Fig. 2). In this work, we prove the following theorem.

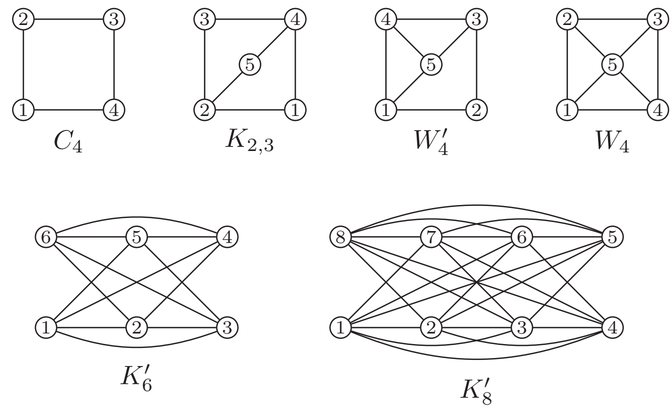

FIGURE 2. Graphs in $\mathcal{F}$.

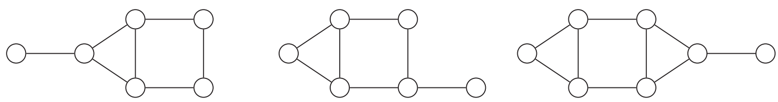

FIGURE 3. The three known minimal counter-examples with a bridge to $\ell(G) + br(G) \geq |G|$.

Theorem 2.1. *For each connected graph $G \in \mathcal{C} \setminus \mathcal{F}$, $\ell(G) + br(G) \geq |G|$.*

As a consequence we have that Chen-Chvatal conjecture holds for $\mathcal{C}$, as it holds for graphs in $\mathcal{F}$ (they all have a universal line). Since all distance-hereditary graphs contain either a pendant edge or a pair of twins, $\mathcal{C}$ is a super class of distance hereditary graphs. It is also clearly a super class of chordal graphs.

The difference between our result and the original conjecture is that we count a universal line as any other line but, since each bridge defines a universal line, we keep track of the number of ways that universal lines are caused by bridges. It is tempting to conjecture that the property $\ell(G) + br(G) \geq |G|$ holds for all but a finite number of graphs. We know this to be false in general, as it was pointed out to us by Yori Zwols, owing to counter-examples that results from replacing a bridge by paths of arbitrary length. The three minimal counterexamples known so far that contain a bridge are shown in Figure 3. It remains unknown, however, whether all counter-examples to $\ell(G) + br(G) \geq |G|$ can be obtained from a finite set of graphs by replacing a bridge by a path. Since the bridge is a pendant edge for these three graphs, we venture to propose the following conjecture.

Conjecture 2.2. *There is a finite set of graphs $\mathcal{F}_0$ such that every connected graph $G \notin \mathcal{F}_0$ either has a pendant edge or satisfies $\ell(G) + br(G) \geq |G|$.*

So far, we know that if such a family $\mathcal{F}_0$ exists, it contains the list of graphs in Figures 2 and 4. An interesting variation of the conjecture can be stated as follows, denoting by $ul(G)$ the number of pairs of vertices in G that induce a universal line.

Conjecture 2.3. *For every connected graph G, $\ell(G) + ul(G) \geq |G|$.*

Although less general (a bridge always induces a universal line but not all universal lines are induced by bridges), this conjecture has the merit of being true for all the known graphs in $\mathcal{F}_0 \setminus \mathcal{F}$ (see Fig. 4). Thus, there is no known counter-example to Conjecture

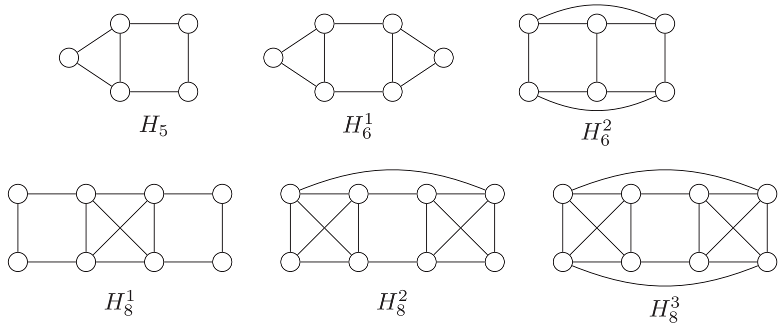

FIGURE 4. Known graphs in $\mathcal{F}_0 \setminus \mathcal{F}$.

2.3 to this day. Moreover, it remains stronger than the original Chen-Chvatal conjecture without ruling out graphs with universal lines as trivial solutions.

3. PRELIMINARIES

In this section, we give some results on the number of lines of graphs in $\mathcal{F}$ or that are constructed from a graph in $\mathcal{F}$ adding a vertex.

Lemma 3.1. $\ell(C_4) = 1$, $\ell(K'_6) = 4$, *and for* $H \in \mathcal{F} \setminus \{C_4, K'_6\}$, $\ell(H) = |H| - 1$.

Proof. Remark that the graphs K'_6 and K'_8 belong to the family of graphs obtained from complete graphs K_2n by removing a perfect matching. In Case 4.1 of the proof of Theorem 2.1 we shall see that $\ell(K'_2n) = \binom{n}{2} + 1$, when $n \geq 3$. For $n = 3, 4$ we have that $\ell(K'_6) = 4$ and $\ell(K'_8) = 7$. The lines of the remaining graphs are computed by brute force (See Fig. 5.) ■

Lemma 3.2. *Let* $G \in \mathcal{C} \setminus \mathcal{F}$ *be a graph.*
1) If G has a pendant vertex v such that $G - v \in \mathcal{F} \setminus \{C_4\}$*, then* $\ell(G) + br(G) = \ell(G) + 1 \geq |G|$.
2) If G contains a nontrivial module M and $G - v \in \mathcal{F} \setminus \{C_4\}$*, for some* $v \in M$*, then* $\ell(G) \geq |G| + 1$.

Proof. The proofs of both statements are easy although tedious. In the first case, if u is the neighbor of v in G, then $\overline{wv}^G$ defines different lines, when w varies over the neighbors of u in $G - v$. These lines are not in $\mathcal{L}(G - v)$ if $G - v \in \mathcal{F} \setminus \{C_4\}$. Since the graphs in $\mathcal{F}$ have no vertex of degree one, we obtain at least two new lines.

In the second case, for each $G' \in \mathcal{F}$ we need to consider all graphs G arising from G' by adding a copy v of a vertex v' in G' so as (v, v') is a pair of twins (true or false) in G. We do this with the help of a computer program [1]. ■

[1] For the sake of completeness, the R code and environment used to check all the cases are available in http://www.math.sciences.univ-nantes.fr/˜rochet/recherche/Code_lines.R http://www.math.sciences.univ-nantes.fr/˜rochet/recherche/Code_lines.R with the environment http://www.math.sciences.univ-nantes.fr/˜rochet/

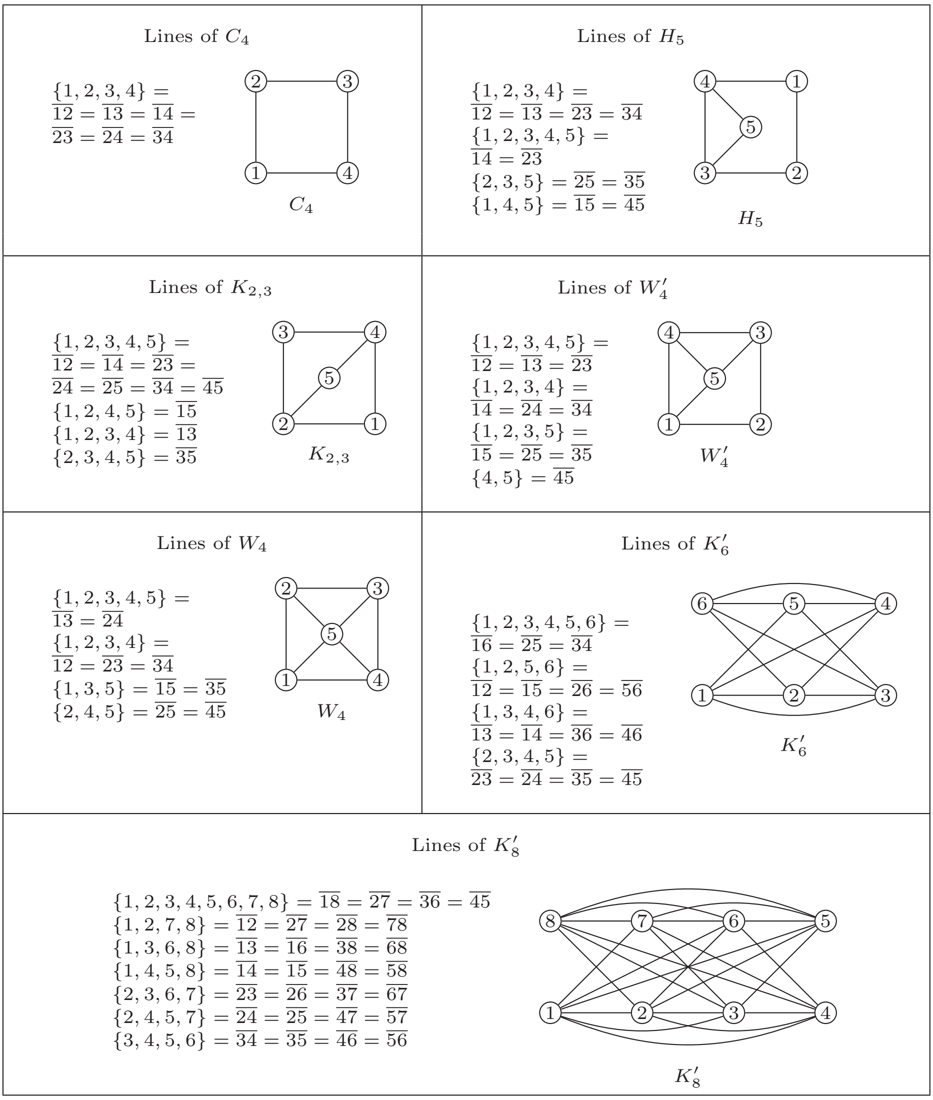

FIGURE 5. H_5 and the graphs in $\mathcal{F}$. For each graph in $\mathcal{F} \cup \{H_5\}$, we have computed all their lines.

4. PROOF OF THEOREM 2.1

We prove Theorem 2.1 by induction on the number of vertices of G. Let $G \in \mathcal{C} \setminus \mathcal{F}$ with $|G| = n$. By the definition of $\mathcal{C}$, the proof splits into four parts: (1) G has a bridge, (2) G has no bridge, and has a cut-vertex, (3) G is 2-connected and chordal, (4) G is 2-connected and has a nontrivial module.

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recherche/env_lines.RData http://www.math.sciences.univ-nantes.fr/˜rochet/ recherche/env_lines.RData to be downloaded separately.

Part 1: G has a bridge.

Let $e = u_1u_2$ be a bridge of G. Let G_1 and G_2 be the connected components of $G - e$ that contain respectively u_1 and u_2. We denote by G/e the graph obtained by *contracting* the edge e of G. That is, the edge e is deleted from G and its ends are identified. Name u the vertex of G/e appeared after the contraction.

We shall prove that $\ell(G) \geq \ell(G/e)$. From the definition of a line, it follows that for each $x, y, z \in V(G) \setminus \{u_1, u_2\}$ we have that $z \in \overline{xy}^G$ if and only if there is a shortest path in G containing $\{z, x, y\}$.

To each path P in G we associate a path P' in G/e as follows. When $V(P) \subseteq V(G/e)$, we set $P' = P$. If e is an edge of P, then $P' = P/e$. If $\{u_1, u_2\} \cap V(P) = \{u_i\}$, then P' is obtained from P by replacing u_i by u. It is easy to see that for P and P' so defined, P is a shortest path in G if and only if P' is a shortest path in G/e. Moreover, $V(P) \setminus \{u_1, u_2\} = V(P') \setminus \{u\}$.

Let $x, y \in V(G) \setminus \{u_1, u_2\}$. Since x and y belong to a shortest path in G if and only if they belong to a shortest path in G/e, for each $z \in V(G) \setminus \{u_1, u_2\}$ we have that $z \in \overline{xy}^G$ if and only if $z \in \overline{xy}^{G/e}$. Thus, $\overline{xy}^G \setminus \{u_1, u_2\} = \overline{xy}^{G/e} \setminus \{u\}$. Similarly, we also have that $\overline{xu_i}^G \setminus \{u_1, u_2\} = \overline{xu}^{G/e} \setminus \{u\}$, for each $i = 1, 2$. Moreover, $u \in \overline{xy}^{G/e}$ if and only if $\{u_1, u_2\} \cap \overline{xy}^G \neq \emptyset$.

By using this information, we can describe lines in G in terms of lines in G/e as follows.

- $\overline{xy}^G = \overline{xy}^{G/e}$, if $u \notin \overline{xy}^{G/e}$.
- $\overline{xy}^G = (\overline{xy}^{G/e} - \{u\}) \cup \{u_i\}$, if $x, y \in V(G_i)$, $[xuy]^{G/e}$ and $i \in \{1, 2\}$.
- $\overline{xy}^G = (\overline{xy}^{G/e} - \{u\}) \cup \{u_1, u_2\}$ in the remaining two situations: First, if $x, y \in V(G_i)$ for $i \in \{1, 2\}$ and $[xyu]^{G/e}$ or $[yxu]^{G/e}$; Second, if $x \in V(G_i)$ and $y \in V(G_{3-i})$ for $i \in \{1, 2\}$.

In fact, the first equality is direct from the above discussion. The second equality comes from the fact that when $[xuy]^{G/e}$ and $x, y \in V(G_i)$ ($i \in \{1, 2\}$), then there is a shortest path P' in G/e between x and y that contains u and its associated path P contains u_i but does not contain u_{3-1}. Hence, $u_i \in \overline{xy}^G$ and no path between x and y in G contains u_{3-i}. Thus, $\overline{xy}^G = (\overline{xy}^{G/e} - \{u\}) \cup \{u_i\}$. The third equality occurs in two situations. In the first situation, when $[xyu]^{G/e}$ or $[yxu]^{G/e}$, there is a shortest path in G/e containing $\{x, y, u\}$ and ending in u. Its associated path in G contains $\{x, y, u_i\}$ and ends in u_i. Hence, by adding the edge e we obtain a shortest path in G that contains $\{x, y, u_1, u_2\}$. Thus, $\overline{xy}^G = (\overline{xy}^{G/e} - \{u\}) \cup \{u_1, u_2\}$. In the second situation, we can assume that $x \in V(G_1)$ and $y \in V(G_2)$. It is clear that a shortest path between x and y in G/e contains u, and that a shortest path between them in G contains $\{u_1, u_2\}$. Therefore, in this situation, we also have $\overline{xy}^G = (\overline{xy}^{G/e} - \{u\}) \cup \{u_1, u_2\}$.

The description we have just obtained shows an injective correspondence between $\mathcal{L}(G/e)$ and $\mathcal{L}(G)$. This implies that $\ell(G) \geq \ell(G/e)$. Moreover, it is clear that $br(G) = br(G/e) + 1$. If $G/e \in \mathcal{C} \setminus \mathcal{F}$, then by induction we have $\ell(G/e) + br(G/e) \geq |G/e| = |G| - 1$ and thus $\ell(G) + br(G) \geq |G|$ and we are done.

So we may assume that $G/e \in \mathcal{F}$. Since graphs in $\mathcal{F}$ are 2-connected, it implies that u_1u_2 is a pendant edge of G. Then the result follows by Lemma 3.2 when $G/e \neq C_4$, and it is easily checkable when $G/e = C_4$.

Part 2: G has no bridge and has a cut-vertex.

Let u be a cut-vertex of G. Let C_1 be a connected component of $G - \{u\}$ and let C_2 be the union of the other connected components of G. Set $G_1 = G[V(C_1) \cup \{u\}]$, and $G_2 = G[V(C_2) \cup \{u\}]$. Observe that, since G is bridgeless, G_1 and G_2 are also bridgeless.

We can argue as in the previous case. For each $i = 1, 2$, any path in G between two vertices x and y in $V(G_i)$ is also a path in G_i. Hence, $\overline{xy}^{G_i} = \overline{xy}^G \cap V(G_i)$. Moreover, if such a path end in u, then for each $z \in V(G_{3-i})$ this path can be extended to a path in G between x and z.

Notice that a shortest path containing $\{u, x, y\}$ and ending in u exists if and only if $[xyu]^G$ or $[yxu]^G$. This happens if and only for some $z \in V(G_{3-i})$, there is a shortest path containing $\{z, x, y\}$ that contains a subpath containing $\{u, x, y\}$ that ends in u.

This shows the following property.

(1) *For $i = 1, 2$ and for all $x, y \in V(G_i)$ we have*:
- if $[xyu]$ or $[yxu]$, then $\overline{xy}^G = \overline{xy}^{G_i} \cup V(C_{3-i})$,
- otherwise $\overline{xy}^G = \overline{xy}^{G_i}$ and in particular $\overline{xy}^G \cap V(C_{3-i}) = \emptyset$.

This implies that, for $i = 1, 2$, a line induced by two vertices in $V(G_i)$ is either disjoint from $V(C_{3-i})$ or contains $V(C_{3-i})$. In particular, it implies that a line induced by two vertices in $V(G_1)$ is distinct from a line induced by two vertices in $V(G_2)$, except in the case where this line is universal.

We next prove the following lower bound for $\ell(G)$.

(2) $\ell(G) \geq \ell(G_1) + \ell(G_2) - 1 + |N_{G_1}(u)||N_{G_2}(u)|$.

For $i = 1, 2$, let $\mathcal{L}_i = \{\overline{ab}^G : a, b \in V(G_i)\}$. By (1) $|\mathcal{L}_i| = \ell(G_i)$, and the only possible line in $\mathcal{L}_1 \cap \mathcal{L}_2$ is the universal line. Hence $|\mathcal{L}_1 \cup \mathcal{L}_2| \geq \ell(G_1) + \ell(G_2) - 1$. Moreover, for all lines l in $\mathcal{L}_1 \cup \mathcal{L}_2$, l contains either $V(C_1)$ or $V(C_2)$.

For $i = 1, 2$, let u_i be a neighbor of u in G_i. We have that $\overline{u_1u_2}^G \cap N(u) = \{u_1, u_2\}$ and since u has at least two neighbors in both G_1 and G_2 (because G is bridgeless), then $\overline{u_1u_2}^G \cap V(G_i) \notin \{\emptyset, V(G_i)\}$ for each $i = 1, 2$ and thus it is distinct from all lines in $\mathcal{L}_1 \cup \mathcal{L}_2$. Moreover, for every u_i, v_i neighbors of u in G_i, for each $i = 1, 2$, if $\{u_1, u_2\} \neq \{v_1, v_2\}$, then $\overline{u_1u_2}^G \neq \overline{v_1v_2}^G$. Therefore, there are at least $|N_{G_1}(u)||N_{G_2}(u)|$ lines in $\mathcal{L}(G) \setminus (\mathcal{L}_1 \cup \mathcal{L}_2)$.

Since $|N_{G_1}(u)||N_{G_2}(u)| \geq 4$ we get $\ell(G) \geq \ell(G_1) + \ell(G_2) + 3$. If $\ell(G_1) + \ell(G_2) \geq |G_1| + |G_2| - 4$, then $\ell(G) \geq |G_1| + |G_2| - 1 = |G|$. If $\ell(G_1) + \ell(G_2) \leq |G_1| + |G_2| - 5$, by the induction hypothesis, we conclude that both G_1 and G_2 are in $\mathcal{F}$. Moreover, from Lemma 3.1 we get that $G_1 = G_2 = C_4$ or $\{G_1, G_2\} = \{C_4, K_6'\}$. We have verified that in the first case we have 11 lines, while in the second, we have 20 lines (see Fig. 6).

Part 3: G is 2-connected and chordal.

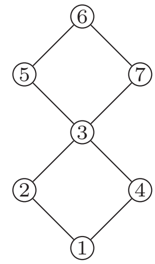
In [5] it was proved that Conjecture 1.1 holds for chordal graphs. The proof of this part is the same as their proof. We first need Lemma 2 of [5]:

Lemma 4.1. *Let G be a chordal graph and let s, x, y in $V(G)$ such that $[sxy]$. If $\overline{sx} = \overline{sy}$, then x is a cut-vertex of G.*

A vertex of a graph is called *simplicial* if its neighbors are pairwise adjacent. By a classic result of Dirac [13], a chordal graph has at least two simplicial vertices. Let s be a simplicial vertex of G. Since s is simplicial for any pair of vertices $x, y \in V(G) \setminus \{s\}$, $[xsy]$ does not hold. Hence, if $\overline{sx} = \overline{sy}$, we must have $[sxy]$ or $[syx]$ and thus, by Lemma 4.1, x or y is a cut vertex, a contradiction.

Lines of $C_4 \oplus C_4$

$\{1,2,3,4,5,6,7\}$		$= \overline{12} = \overline{13} = \overline{14}$	$= \overline{16} = \overline{23} = \overline{34}$
		$= \overline{35} = \overline{36} = \overline{37}$	$= \overline{56} = \overline{67}$
$\{1,2,3,4,5,6\}$	$= \overline{15}$	$\{1,2,3,4,6,7\}$	$= \overline{17}$
$\{1,2,3,4\}$	$= \overline{24}$	$\{1,2,3,5,6\}$	$= \overline{25}$
$\{1,2,3,5,6,7\}$	$= \overline{26}$	$\{1,2,3,6,7\}$	$= \overline{27}$
$\{1,3,4,5,6\}$	$= \overline{45}$	$\{1,3,4,5,6,7\}$	$= \overline{46}$
$\{1,3,4,6,7\}$	$= \overline{47}$	$\{3,5,6,7\}$	$= \overline{57}$



Lines of $C_4 \oplus K_6'$

$\{1,2,3,4,5,6,7,8,9\}$	$= \overline{34} = \overline{37}$	$= \overline{38}$	
	$= \overline{39} = \overline{48}$	$= \overline{78} = \overline{89}$	
$\{2,3,4,5,7,8,9\}$	$= \overline{23} = \overline{24}$	$= \overline{35} = \overline{45}$	
$\{1,2,5,6\}$	$= \overline{12} = \overline{15}$	$= \overline{26} = \overline{56}$	
$\{1,3,4,6,7,8,9\}$	$= \overline{13} = \overline{14}$	$= \overline{36} = \overline{46}$	
$\{1,2,3,4,5,6\}$	$= \overline{16} = \overline{25}$		
$\{1,2,3,4,5,6,7,8\}$	$= \overline{47}$		
$\{1,2,3,4,5,6,8,9\}$	$= \overline{49}$		
$\{1,3,4,7,8\}$	$= \overline{17}$	$\{1,3,4,7,8,9\}$	$= \overline{18}$
$\{1,3,4,8,9\}$	$= \overline{19}$	$\{2,3,4,7,8\}$	$= \overline{27}$
$\{2,3,4,7,8,9\}$	$= \overline{28}$	$\{2,3,4,8,9\}$	$= \overline{29}$
$\{3,4,5,7,8\}$	$= \overline{57}$	$\{3,4,5,7,8,9\}$	$= \overline{58}$
$\{3,4,5,8,9\}$	$= \overline{59}$	$\{3,4,6,7,8\}$	$= \overline{67}$
$\{3,4,6,7,8,9\}$	$= \overline{68}$	$\{3,4,6,8,9\}$	$= \overline{69}$
$\{3,7,8,9\}$	$= \overline{79}$		

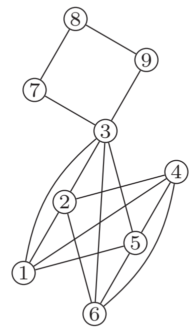


FIGURE 6. Lines of the graphs $C_4 \oplus C_4$ and $C_4 \oplus K_6'$ obtained by gluing, respectively, two copies of the graph C_4 and a copy of C_4 and a copy of K_6'.

Hence, the set $\{\overline{su} : u \in V(G) \setminus \{s\}\}$ has $n-1$ distinct lines. Observe that all these lines contain s. Now, since G is 2-connected, s has at least two neighbors a, b and $s \notin \overline{ab}$. Thus, the graph G has at least n different lines.

Part 4: G is 2-connected, nonchordal and has a nontrivial module

We first consider the case when G has a nontrivial nondominating module.

Let $M = \{v_1, \ldots v_s\}$ be a nontrivial nondominating module of G with neighborhood $N(M)$ of minimal size. Set $G' := G - \{v_1\}$. If $G' \in \mathcal{F} \setminus \{C_4\}$, we are done by Lemma 3.2. If $G' = C_4$, then G is either $K_{2,3}$ or W_4', a contradiction with $G \notin \mathcal{F}$. So we may assume that $G' \notin \mathcal{F}$ and from the induction hypothesis we get $\ell(G') + br(G') \geq |G'| = |G| - 1$.

Set $\mathcal{L}' = \{\overline{xy}^G : x, y \in V(G')\}$. Since G' is an isometric subgraph of G (i.e. for all $x, y \in V(G')$, $d_{G'}(x,y) = d_G(x,y)$), we have, for all $a, b \in V(G')$, $\overline{ab}^G = \overline{ab}^{G'}$ or $\overline{ab}^{G'} \cup \{v_1\}$. Hence

$$|\mathcal{L}'| = \ell(G') \geq |G| - 1 - br(G'). \tag{2}$$

Moreover, each line in $\mathcal{L}'$ that contains v_1 must contain at least one other vertex of M. In effect, if $v_1 \in \overline{ab} \in \mathcal{L}'$ and $a, b \notin M$, then $\overline{ab}$ contains M. Hence, each line in $\mathcal{L}'$ that contains v_1 must contain at least one other vertex of M.

Let $t \in G - (M \cup N(M))$. It is clear that v_1 is the unique vertex in M that belongs to the line $\overline{v_1t}^G$. Hence, $\overline{v_1t}^G \notin \mathcal{L}'$ and thus, if $br(G') = 0$, we are done by (2).

So we may assume that G' has at least one bridge. Let ab be a bridge of G', and let G_a, G_b be the connected components of $G' - ab$ that contain respectively a and b. We are going to prove that one of G_a, G_b consists of one vertex of degree exactly 2 and that this vertex is in $N(M)$ (and so $|M| = 2$). Since a vertex in G_a has at most one common neighbor with a vertex in G_b and $|N(M)| \geq 2$, because G is 2-connected and thus vertices in $M - \{v_1\}$ have at least two common neighbors in G', $M - \{v_1\}$ cannot intersect both G_a and G_b. So we may assume without loss of generality that $M - \{v_1\} \subseteq V(G_a)$. Since ab is not a bridge of G, v_1 must have neighbors in both G_a and G_b. Hence the only neighbor of v_1 in G_b is b, $v_2 = a$ and $M = \{v_1, v_2\}$. Moreover, since G has no cut-vertex, $G_b = \{b\}$. Finally, by minimality of $N(M)$, v_2b is the unique bridge of G' (If G' has another bridge, then there exists a vertex $b' \neq b$ such that v_2b' is a bridge of G' and $N_G(b') = \{v_1, v_2\}$; hence $\{b, b'\}$ is a nontrivial nondominating module of G and $|N(\{b, b'\})| < |N(\{v_1, v_2\})|$, a contradiction).

Consider now the line $\overline{v_1v_2}$. We claim that $\overline{v_1v_2} \notin \mathcal{L}' \cup \{\overline{v_1t}\}$ that gives the result by (2). If v_1v_2 is an edge of G, then $\overline{v_1v_2} = \{v_1, v_2\}$ and the result holds. Hence we may assume that v_1v_2 is not an edge and thus $\overline{v_1v_2} = M \cup N(M)$. So $\overline{v_1v_2} \neq \overline{v_1t}$ and we may assume for contradiction that $\overline{v_1v_2} \in \mathcal{L}'$ that implies there exists $x, y \in N(M) \cup M - \{v_1\}$ such that $\overline{xy} = M \cup N(M)$. If $\{x, y\} \cap \{b, v_2\} \neq \emptyset$, then $\overline{xy}$ must contain some vertices of $V(G) - (M \cup N(M))$, so we may assume that $\{x, y\} \subseteq N(M) \setminus \{b\}$, but then $b \notin \overline{xy}$.

We now consider the case where all the nontrivial modules of G are dominating.

In this case, G has diameter 2. It was proven in [11] that for every graph G of diameter 2, G either has an universal line or it has at least $|G|$ distinct lines. Since what we want to prove is stronger, we cannot use this result. We will need the following lemma that was already proved in [9].

Lemma 4.2. *Let G be a graph of diameter two and let x, a, b be three vertices of G such that $\overline{xa} = \overline{xb}$. Then either (a, b) is a pair of false twins and $d(x, a) = d(x, b) = 1$, or $d(x, a) \neq d(x, b)$.*

Proof. Assume that $d(x, a) = d(x, b)$. If $d(x, a) = 2$, then $a \notin \overline{xb}$, a contradiction, so $d(x, a) = 1$. If a and b are adjacent, then again $a \notin \overline{xb}$, so a and b are not adjacent. Assume now that there exists a vertex c adjacent to a but not to b, i.e. $d(c, a) = 1$ and $d(c, b) = 2$. If $d(c, a) = 1$, then $c \in \overline{xb}$ and $c \notin \overline{xa}$, and if $d(c, x) = 2$, then $c \notin \overline{xb}$ and $c \in \overline{xa}$, a contradiction in both cases. So (a, b) is a pair of false twins. ■

Notice that for any nontrivial dominating module M, the set $N(M)$ is a module as well and $M \cup N(M) = V(G)$. Moreover, for each $u \in M$ and each $v \in N(M)$ the line $\overline{uv}$ is given by

$$\overline{uv} = (M - N(u)) \cup (N(M) - N(v)).$$

We assume first that G does not contain pairs of false twins. Let M be a module of G. For $u, u' \in M$ and $v, v' \in N(M)$ with $\{u, v\} \neq \{u', v'\}$ we have that $\overline{uv} \neq \overline{u'v'}$. Hence, $\ell(G) \geq |M||N(M)|$. Since $|M| \geq 2$, then the equality $|M||N(M)| = |M| + |N(M)| + (|M| - 1)(|N(M)| - 1) - 1$ implies that $\ell(G) \geq |G|$ when $N(M)$ is not a singleton.

If $N(M) = \{x\}$, then all the lines $\overline{xv}$ are distinct, when v varies over M. This gives us $|G| - 1$ distinct lines, all containing x. Since M has no pair of false twins, it contains at least one edge ab, and $\overline{ab}$ is a new line since $x \notin \overline{ab}$.

Hence, we can assume that every nontrivial module M is a dominating set and the graph G contains pairs of false twins.

Let $(u_1, v_1), (u_2, v_2), \ldots, (u_t, v_t)$ be the pairs of false twins of G and set $T = \{u_1, v_1, \ldots, u_t, v_t\}$. Since $\{u_i, v_i\}$ is a nontrivial module for $i = 1, \ldots, t$, it must be a dominating module and thus $N(u_i) = N(v_i) = V(G) \setminus \{u_i, v_i\}$. This implies that all vertices in $\{u_1, v_1, \ldots, u_t, v_t\}$ are pairwise distinct, i.e. $|T| = 2t$, and that T induces a complete graph minus a perfect matching.

Set $U = \{u_1, \ldots, u_t\}$ and $\mathcal{L}_U = \{\overline{u_i u_j} : 1 \leq i \neq j \leq t\} \cup \{\overline{u_1 v_1}\}$. For $1 \leq i \neq j \leq t$, we have $\overline{u_i u_j} = \{u_i, v_i, u_j, v_j\}$ and $\overline{u_1 v_1} = V(G)$. So $\binom{|\mathcal{L}_U|=}{t2+1}$.

Set $R = V(G) \setminus T$. We split the rest of the proof into three cases.

Case 4.1: R is empty.
In this case $|G| = 2t$. If $t \in \{2, 3, 4\}$, then $G \in \{C_4, K'_6, K'_8\}$, which is a contradiction. If $t \geq 5$, then $\binom{\ell(G)\geq|\mathcal{L}_U|\geq}{t2\geq 2t}$ and we are done.

Case 4.2: R is a clique.
Set $|R| = k \geq 1$. Set $\mathcal{L}_R = \{\overline{xy} : x, y \in R\}$. Notice that each pair of vertices $x, y \in R$ are true twins, resulting in $\overline{xy} = \{x, y\}$, which means that each pair of vertices of R determines a different line. Now, for each $x \in R$, set $\mathcal{L}_{xU} = \{\overline{xu_i} : i = 1, \ldots, t\}$. Observe that $\overline{xu_i} = \{x, u_i, v_i\}$. It follows that lines in $\cup_{x\in R}\mathcal{L}_{xU}$ are all pairwise distinct and disjoint of $\mathcal{L}_U$. Moreover, lines in $\cup_{x\in R}\mathcal{L}_{xU}$ are not universal (except when $|R| = 1$ and $T = \{u_1, v_1\}$, but then the graph is not 2-connected). Hence, all lines in $\mathcal{L}_U$, $\mathcal{L}_R$, and $\cup_{x\in R}\mathcal{L}_{xU}$ are pairwise distinct. So if $|R|$ and t are greater than 2 we have that:

$$\ell(G) \geq \binom{t}{2} + \binom{|R|}{2} + t|R| + 1 \geq 2t + |R| = |G| \tag{3}$$

If $|R| = 1$ and $t \geq 2$, $\ell(G) \geq \binom{t}{2} + t + 1$.If $t \geq 3$ this quantity is greater than $|G|$. If $t = 2$ then $G = W_4$ that is a contradiction because $W_4 \in \mathcal{F}$. If $|R| = 1$ and $t = 1$, then G is not 2-connected. Hence, R is not a clique.

Case 4.3: R is nonempty and is not a clique.
There exists $x, y \in R$ such that xy is not an edge. Since (x, y) is not a pair of false twin, we may assume that there exists $z \in R \setminus \{x, y\}$ such that z is adjacent to y but not to x. Set $\mathcal{L}_x = \{\overline{xa} : a \in U \text{ or } a \in R \setminus \{x\}\}$.

Suppose $\mathcal{L}_x$ contains an universal line $\overline{xa}$. If $a \in R$, then $d(x, a) = 2$ and all the other vertices in R are at distance 1, but this would imply that (x, a) is a pair of false twins. Hence, $a = u_i$ for some $i \in \{1, 2, \ldots, t\}$. Notice that $\overline{xu_i} = \{v_i\} \cup (R \setminus N(x))$. If it is universal, then $i = 1$, $t = 1$ and $N(x) = T$. But then, $R \setminus \{x\}$ is a nontrivial nondominating module that is a contradiction. Hence, we can assume that $\mathcal{L}_x$ does not contain an universal line.

Recall that all lines in $\mathcal{L}_U$ are subsets of T except for the universal line. Since no line in $\mathcal{L}_x$ is universal, then $\mathcal{L}_x \cap \mathcal{L}_U = \emptyset$. Moreover, since $x \notin \overline{yz}$, $\overline{yz} \notin \mathcal{L}_x$ and since $\overline{yz} \cap T = \emptyset$, $\overline{yz} \notin \mathcal{L}_U$. Hence, we have that $\binom{\ell(G)\geq}{t2+2+|\mathcal{L}_x|}$ if $t \geq 2$ or $\ell(G) \geq 2 + |\mathcal{L}_x|$ if $t = 1$.

In both cases if $|\mathcal{L}_x| \geq |R| + t - 1$ then $\ell(G) \geq 2t + |R| = |G|$. Thus it is enough to prove that for all $a, b \in U \cup R \setminus \{x\}$, we have $\overline{xa} \neq \overline{xb}$.

Let $a, b \in U \cup R \setminus \{x\}$ and let us prove that $\overline{xa} \neq \overline{xb}$. Since $a, b \in U \cup R \setminus \{x\}$, (a, b) is not a pair of false twins, and thus, by Lemma 4.2, we may assume that $d(x, a) \neq d(x, b)$. Without loss of generality, $d(x, a) = 1$ and $d(x, b) = 2$ that implies in particular that $b \in R \setminus \{x\}$. If $a \in R \setminus \{x\}$, then $T \subseteq \overline{xb}$ and $T \cap \overline{xa} = \emptyset$. So we may assume that $a \in U$. One of the vertices y, z is distinct from b, say $y \neq b$. We have $[xay]$, so $y \in \overline{xa}$. But $d(x, y) = d(x, b) = 2$ that implies that $y \notin \overline{xb}$.

Thus, $\ell(G) \geq |R| + 2 = |G|$ that proves the Theorem.

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