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# STRUCTURAL IMPLICATIONS OF A CLASS OF FLEXIBLE FUNCTIONAL FORMS FOR PROFIT FUNCTIONS*

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In 1973, Diewert proposed the use of various Flexible Functional Forms (FFF) for profit functions. Since then, the use of FFF specifications for profit functions in empirical production analysis has become increasingly popular (Woodland [1977]; Kohli [1978]; Cowing [1978]; Sidhu and Baanante [1981], etc.). A number of alternative FFF specifications are available which may seem equally plausible. In fact, the choice among FFF for empirical applications is typically a purely arbitrary decision. The central problem considered in this paper is whether some FFF impose more or less *a priori* restrictions on the structure of production. The purpose of this note is to show that indeed an important class of FFF, when used to represent profit functions, impose quite undesirable restrictions on the production technology. These restrictions include quasi-homotheticity and certain additional separability structures of the underlying production technology. A paper by Blackorby, Primont and Russell [1977] shed some doubt on the flexibility of FFF when certain separability conditions are imposed. It proved that the flexibility of these forms rest indeed on very feeble grounds, being extremely sensitive to weak separability restrictions. These forms do not provide second order local approximations to an arbitrary weakly separable function. What we demonstrate here is that an important family of FFF does impose serious structural rigidities on the underlying production structure even if weak separability is not imposed.

We first present a simple taxonomy of flexible functional forms which allows us to classify them into two major families according to certain key differences. Next, we show that one of these families imposes quasi-homotheticity and certain separability conditions on the underlying production technology. In section 3, we provide some general comments concerning the implications of these results as a potential basis for discriminating among FFF in empirical analysis. We end this note with a summary of the major conclusions.

## 1. TWO CLASSES OF FFF REPRESENTATIONS FOR PROFIT FUNCTIONS

Consider a profit function, $\pi = \pi(q)$ where $q$ is a vector of $M$ output and $N$ input prices. Most FFF of the above function can be represented by (Blackorby, Primont and Russell [1978])

$$\phi(\pi) = a_0 + \alpha\tilde{d} + 1/2\, \tilde{d}'B\tilde{d} \tag{1}$$

where $\phi(\cdot)$ is any arbitrary, monotonic increasing function, $\tilde{d} = [Z(q)]$ is a vector

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of $M+N$ monotonic increasing functions of the $M$ output prices and $N$ input prices, i.e., $Z(q)=[Z^1(q_1), Z^2(q_2),\ldots, Z^{M+N}(q_{M+N})]$, $a_0$ is a parameter, $\alpha$ is a vector of $M+N$ parameters, and $B=[b_{ij}]$ is an $(M+N)\times(M+N)$ symmetric matrix of parameters.

An alternative equivalent way of representing equation (1) is the following:

$$\phi(\pi) = g[Z_1(q_1), Z_2(q_2),\ldots, Z_{N+M}(q_{N+M})] \tag{2}$$

where $g(\cdot)$ is a quadratic function. Since $\phi$ is a monotonic increasing function, it is invertible and, therefore, we can write:

$$\pi = \psi[g(Z_1, Z_2,\ldots, Z_N)] \tag{3}$$

and hence

$$\pi_{ij} = \frac{\partial Z_i}{\partial q_i}\frac{\partial^2 \pi}{\partial Z_i \partial Z_j}\frac{\partial Z_j}{\partial q_j} \tag{4}$$

where $\pi_{ij} \equiv \dfrac{\partial^2 \pi}{\partial q_i \partial q_j}$. Since the expression

$$\frac{\partial^2 \pi}{\partial Z_i \partial Z_j} = \psi' \frac{\partial^2 g}{\partial Z_i \partial Z_j} + \psi'' \frac{\partial g}{\partial Z_j}\frac{\partial g}{\partial Z_i} \tag{5}$$

we obtain that the ratio of the second derivative of $\pi$ with respect to prices is

$$\frac{\pi_{ij}}{\pi_{ik}} = \frac{\left[\psi' \dfrac{\partial^2 g}{\partial Z_i \partial Z_j} + \psi'' \dfrac{\partial g}{\partial Z_j}\dfrac{\partial g}{\partial Z_i}\right]\dfrac{\partial Z_i}{\partial q_i}\dfrac{\partial Z_j}{\partial q_j}}{\left[\psi' \dfrac{\partial^2 g}{\partial Z_i \partial Z_k} + \psi'' \dfrac{\partial g}{\partial Z_i}\dfrac{\partial g}{\partial Z_k}\right]\dfrac{\partial Z_i}{\partial q_i}\dfrac{\partial Z_k}{\partial q_k}}. \tag{6}$$

Note that if the transformation $\phi(\cdot)$ is linear, then $\psi''=0$ and thus (6) collapses to

$$\frac{\pi_{ij}}{\pi_{ik}} = \frac{\partial Z_j(q_j)}{\partial q_j}\left[\frac{\partial Z_k(q_k)}{\partial q_k}\right]^{-1} . B_{jk} \qquad \forall i, j, k=1,\ldots N+M \tag{7}$$

where $B_{jk}$ is a constant because $g(\cdot)$ is quadratic and, hence, is independent of all prices except $q_j$ and $q_k$.

Thus, we can classify FFF according to whether they are characterized by a linear (LFFF) or non-linear (NLFFF) transformation on the left-hand-side variable, $\pi$. Widely used functional forms of the class LFFF are the Generalized Leontief and the Normalized Quadratic while the Translog form is the best known function corresponding to the NLFFF group. In the Generalized Leontief form $\phi(\pi)=\pi$ and $Z(q_i)=q_i^{1/2}$ and in the Translog case $\phi(\pi)=\ln \pi$ and $Z(q_i)=\ln q_i$.

## 2. LFFF SPECIFICATIONS FOR MULTIOUTPUT TECHNOLOGIES[1]

Consider the following identity:

[1] The results obtained in this and other sections are shown for long-run LFFF profit functions (i.e., when all factors are variable). However, the results can be trivially generalized to the case when some inputs are fixed.

(8) $$x_s(p_1, p_2, \ldots, p_M, w) = \mu_s[y_1(p, w), y_2(p, w), \ldots\ y_M(p, w), w]$$

where $x_s = -\dfrac{\partial \pi(p, w)}{\partial w_s}$ is the Marshallian factor demand derived from the profit function $\pi(p, w)$, $\mu_s \equiv \dfrac{\partial c(w, y)}{\partial w_s}$ is the Hicksian demand function derived from the cost function $c(w, y)$, $p$ is a vector of $M$ output prices, $w$ is a vector of $N$ input prices and $y$ is a vector of $M$ output quantities. From (8) we obtain

(9) $$\frac{\partial x_s}{\partial p_i} = \frac{\partial \mu_s}{\partial p_i} = \sum_{k=1}^{M} \frac{\partial \mu_s}{\partial y_k} \frac{\partial y_k}{\partial p_i}.$$

Using Hotelling's lemma in our definition of LFFF specifications one can verify that:

(10) $$\frac{\partial x_s / \partial p_i}{\partial x_m / \partial p_i} = B_{sm} \frac{z'(w_s)}{z'(w_m)} \qquad \forall\ s, m, i$$

where $z'(w_s) \equiv \dfrac{\partial z(w_s)}{\partial w_s}$ and $z'(w_m) = \dfrac{\partial z(w_m)}{\partial w_m}$.
Thus, a LFFF implies that (10) is independent of all output prices and of all factor prices except $w_s$ and $w_m$. Using (9) and (10) it follows that

(11) $$\frac{\partial x_s / \partial p_i}{\partial x_m / \partial p_i} = \frac{\sum_{k=1}^{M} \frac{\partial \mu_s}{\partial y_k} \frac{\partial y_k}{\partial p_i}}{\sum_{k=1}^{M} \frac{\partial \mu_m}{\partial y_k} \frac{\partial y_k}{\partial p_i}} \qquad \forall\ s, m, i.$$

That is, the right-hand side of (11) is also independent of all output prices $p$ and of all factor prices except $w_m$ and $w_s$. This should hold for all levels of factor and output prices. Therefore, the right side of (11) is independent of all outputs since output levels are not, in general, independendent of output prices. Independence of output levels is necessary and sufficient for independence of (11) of $p$ because the Hicksian factor demands $\mu_s[y_1(p, w), y_2(p, w), \ldots, y_m(p, w), w]$ and, hence, their derivatives are dependent on output prices via the output quantities only. Obviously, neither the numerator nor the denominator of the right side of (11) can be reasonably assumed to be independent of all output prices. Hence, only if the total effect of output prices on $\mu_s$ is identical to the corresponding effect on $\mu_m$ can (11) be independent of all output prices. In this case the total output price effects on $\mu_s$ and $\mu_m$ would cancel out in (11). This can only occur if the marginal effect of output levels on each input is identical up to a scale factor which is independent of the vector of outputs. That is, integrable Hicksian factor demands should have a structure of the form

(12) $$\mu_s = \psi(y_1, y_2, \ldots, y_M) \cdot H_s(w) + B_s(w).$$

If this structure prevails, then

$$(13)\qquad \frac{\partial\mu_s/\partial p_i}{\partial\mu_m/\partial p_i} = \frac{H_s(w)\sum_{k=1}^{m}\frac{\partial\psi}{\partial y_k}\frac{\partial y_k}{\partial p_i}}{H_m(w)\sum_{k=1}^{m}\frac{\partial\psi}{\partial y_k}\frac{\partial y_k}{\partial p_i}} = \frac{H_s(w)}{H_m(w)} \qquad \forall\ s, m, i$$

which is independent of all output prices.

The structures of (12) and (13) are not as restricted as (10) because (13) is dependent on all factor prices and not only on $w_m$ and $w_s$ as (10) suggests. Therefore, it is necessary to specialize the functions $H_s(\cdot)$ and $H_m(\cdot)$ to be dependent only on $w_s$ and $w_m$ respectively rather than on the full vector of input prices, $w$. Hence, the Hicksian factor demand functions consistent with the restrictions of a LFFF multioutput profit function (i.e., consistent with (10)) are of the form:

$$(14)\qquad \mu_s = \varepsilon(y_1, y_2, \ldots, y_M)\cdot N(w)H_s(w_s) + B_s(w) \quad s = 1, \ldots, N.$$

It can be easily seen that the associated cost function is quasi-homothetic, i.e.,

$$(15)\qquad c = \varepsilon(y_1, y_2, \ldots, y_M)\cdot H[\sum_{l=1}^{N} g^l(w_l)] + B(w).$$

The structure of the cost function in (15) is well known (see, for example, Blackorby, Boyce and Russell [1978]). It corresponds to a Gorman–Polar form (Gorman, [1953]) which meets the sufficient conditions for aggregation across firms when firms are cost minimizers. It implies that the underlying production function is such that the expansion paths are straight lines that are not necessarily borned in the origin. That is, the isoquants are parallel and the marginal rates of substitution among input pairs are constants independent of the output scale. This implies that all factor demand elasticities with respect to output tend to unity as output rises. The $B(w)$ function in (15) reflects the existence of a minimum set of committed input levels which are used even if no production takes place and even when all factors are variable inputs. The composition of these committed quantities of inputs is dependent on relative factor prices as reflected by the fact that $B(w)$ is not necessarily a linear function of $w$. If $B(w)$ vanishes, then the production function is homothetic to the origin. This structure has been widely used in empirical analysis in the context of consumer demand (Blackorby, Boyce and Russell [1978]) because it allows for aggregation across households. However, in production analysis, this structure is rather inadequate because it appears unrealistic to assume that variable inputs will be used at zero output levels.[2] Moreover, the conveniency associated with the fact that the struc-

[2] Notice that the function $B(w)$ is dependent on variable input prices. That is, the zero output cost of production is a variable cost, i.e., it cannot be interpreted as a fixed cost.

ture in (15) is consistent with aggregation is unnecessary in the context of production when profit maximization (and competitive behavior) is assumed. Under profit maximization with all factors variable and price taking behavior, the aggregation conditions are automatically satisfied.[3] There is no need to impose any structural restrictions on the production technology (except, of course, the conditions for profit maximization) in order to obtain consistent aggregation.

Furthermore, the fact that the function $H(\cdot)$ in (15) is additively separable implies that the slope of the expansion path of each input pair is independent of factor prices other than those corresponding to the input pair. Note from (15) that the Hicksian cross price demand effects are independent of output levels, i.e., $\frac{\partial(\partial\mu_i/\partial w_j)}{\partial y}=0$ for all $i\neq j$.[4]

A third implication of (15) is that such a structure imposes the existence of an aggregate output index, $\varepsilon(y_1, y_2,\dots, y_M)$. That is, outputs are weakly separable from all inputs, and thus, the marginal rates of output substitution are independent of factor quantities. At a general equilibrium level this implies that output prices are independent of factor prices. This suggests that LFFF for profit functions would not be appropriate specifications for complete econometric models.

For the case of a single output technology,[5] (11) simply implies that the ratio $\frac{\partial\mu_s}{\partial y}\Big/\frac{\partial\mu_m}{\partial y}$ is independent of output as well as of all other input prices except $w_m$ and $w_s$. Using Shephard's lemma, it is clear that this signifies that the marginal cost function can be written as $\frac{\partial c(w, y)}{\partial y}=G(y, h(w))$ where $h(w)$ is strongly separable. Integrating this expression with respect to $y$ and using the linear homogeneity condition of $c$ in $w$ yields a quasi-homothetic function.[6]

[3] See Bliss [1975].

[4] The independence of the marginal rate of substitution among input pairs of all other factor prices and outputs is shown as follows. The Hicksian factor demand associated with (15) is:

$$\mu_s = \varepsilon(y)H'g_s^s(w_s) + B_s(w)$$
$$\mu_s - B_s(w) = \varepsilon(y)\, H'\cdot g_s^s(w_s) \qquad \forall s$$
$$\frac{\mu_s - B_s(w)}{\mu_m - B_m(w)} = \frac{g_s^s(w_s)}{g_m^m(w_m)} = L(w_s, w_m) \qquad \forall\ s, m.$$

Hence, $\mu_s - B_s(w) = L(w_s, w_m)[\mu_m - B_m(w)]$ and the slope of the expansion path or marginal rate of substitution is $\frac{\partial\mu_s}{\partial\mu_m}=L(w_s, w_m)$, which is not only independent of outputs, but also of all factor prices except $w_s$ and $w_m$.

[5] The papers by Woodland [1977] and Kohli [1978] are examples of empirical studies which have assumed a single output LFFF for a profit function.

[6] As an example consider the Generalized Leontief (GL) single output profit function which is a highly used (Woodland [1977], Kohli [1978], etc.) LFFF developed by Diewert [1973]. The GL Function is

$$\pi = b_{00}p + 2\sum_{i=1}^{N} b_{0i}\, p^{1/2}w_i^{1/2} + \sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}w_i^{1/2}w_j^{1/2}.$$

By using the definition $\pi(p, w)\equiv\max_y\,[py - c(w, y)]$, it can be shown that the following cost

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It is interesting to analyze the type of revenue function underlying an LFFF profit function. Use the definition

$$\pi(p, w) = \max_x [R(x, p) - wx] \tag{16}$$

where $R(x, p) \equiv \max_y [py - wx \colon T(x, y) = 0]$ is the revenue function and $T(\cdot)$ is a transformation function. From (16) using Hotelling's lemma, it is clear that the following identity holds:

$$y_s(p_1, p_2, \ldots, p_M, w) = \eta_s(p_1, p_2, \ldots, p_M, x(p, w)), \quad \text{for} \quad s = 1, \ldots, M \tag{17}$$

where the LHS are the profit maximizing output levels and the RHS are the revenue maximizing outputs evaluated at the profit maximizing input vector $x$. Differentiating (17) with respect to a factor price $w_i$, we obtain

$$\frac{\partial y_s}{\partial w_i} = \frac{\partial \eta_s}{\partial w_i} = \sum_{k=1}^{N} \frac{\partial \eta_s}{\partial x_k} \frac{\partial x_k}{\partial w_i}. \tag{18}$$

Using Hotelling's lemma and assuming that $\pi(\cdot)$ is a LFFF then

$$\frac{\partial y_s / \partial w_i}{\partial y_1 / \partial w_i} = \frac{z'(p_s)}{z'(p_1)} B_{s1} \qquad \forall\ s, 1, i$$

$$= \frac{\sum_{k=1}^{N} \frac{\partial \eta_s}{\partial x_k} \frac{\partial x_k}{\partial w_i}}{\sum_{k=1}^{N} \frac{\partial \eta_1}{\partial x_k} \frac{\partial x_k}{\partial w_i}}. \tag{19}$$

It is clear that (19) is independent of all factor prices and output prices except $p_s$ and $p_1$. Therefore, using similar arguments as those used in the context of the cost function, we obtain that the revenue maximizing outputs have the following structure

$$\eta_s = \beta(x_1, x_2, \ldots, x_N)\tilde{A}(p)A_s(p_s) + B_s(p) \quad \forall S = 1, \ldots, M \tag{20}$$

and, hence, the revenue function can be written as:

$$R(x, p) = \beta(x_1, x_2, \ldots, x_N)A[\sum_{i=1}^{m} h^i(p_i)] + L(p) \tag{21}$$

where $A[\cdot]$ and $L(\cdot)$ are non-decreasing, linear homogeneous and convex functions. Note that $\tilde{A}(p)$ in (20) is the function $A'[\cdot]$ in (21).

(*Continued*)

function belonging to the class of quasi-homothetic functions yields the GL profit function:

$$c = (b_{00} - y)^{-1}[\sum_i - b_{0i} w_i^{1/2}]^2 - \sum_i \sum_j b_{ij} w_i^{1/2} w_j^{1/2}$$

where $y > b_{00}$.

That is, the GL profit function implies a quasi-homothetic cost which satisfies all the separability restrictions discussed in the text.

The structure of the revenue function in (21) suggests that an aggregate input index $\beta(x_1, x_2, \ldots, x_N)$ exists and hence that inputs are separable from outputs. Thus, using (15) and (21) it is clear that a LFFF imposes both separability of outputs from each input and at the same time separability of inputs from each output. Aggregate output and input indexes exist. The structure of the revenue function in (21) also reveals that there exists a form of quasi-homotheticity in the output space as well. The function $L(p)$ reflects combinations of outputs which are independent of input levels. Expansion paths in the output space emanate from a point in the output surface $L(p)$ and are linear. That is, the shapes of the output transformation curves are identical for any level of revenue above $L(p)$.

Perhaps the most startling fact shown by expressions (15) and (21) is that the functions $H(\cdot)$ and $A(\cdot)$ are not flexible and, therefore, the flexibility of the LFFF rests entirely on the functions $B(w)$ and $L(p)$. Since these latter functions do not reflect interactions between outputs and inputs, one should conclude that LFFF are quite rigid in representing these vital interactions.

## 3. ISSUES ON DISCRIMINATION AMONG FLEXIBLE FORMS

The previous analysis has shown that some FFF effectively impose stronger *a priori* restrictions on the underlying structure of production than others. This may be seen as one possible basis for discriminating among alternative forms. Indeed the quasi-homotheticity and separability restrictions imposed by LFFF are quite undesirable and, since NLFFF do not impose them, one could conclude that the latter forms should be preferred to the former. However, there exist a number of further issues to consider to be able to discriminate among FFF. A particularly important additional consideration is concerned with the global properties of FFF. Most FFF do not globally satisfy certain desirable regularity conditions, i.e., monotonicity and convexity. Previous studies have shown that, in general, it is very difficult to provide much insight into the global behavior of FFF when more than two commodities are considered. Caves and Christensen [1980], for example, have shown that even for two commodities the Generalized Leontief's and Translog's regular regions vary substantially according to the magnitude of the elasticity of substitution. For more than two commodities, the comparison of their respective regular regions becomes substantially less conclusive. The size of the regular regions for each form is, in general, highly sensitive to the particular mixture of substitution elasticities.

It is interesting to note, however, that one highly used LFFF, the Normalized Quadratic, is capable of satisfying one of the regularity conditions (convexity) globally. That is, if the Normalized Quadratic is locally convex, then it is also globally convex. This represents an important counterbalancing advantage of the Normalized Quadratic form. The monotonicity condition is not necessarily globally satisfied by the Normalized Quadratic when such condition is locally met. Thus, the trade-off between the capacity of modelling more complex technologies and of satisfying regularity conditions globally is presented when comparing the Normalized Quadratic and NLFFF. The Normalized Quadratic imposes more prior restrictions on the structure of technology than NLFFF but it is able to at least satisfy the convexity condition globally. It is essentially the same trade-off faced in choosing between non-flexible forms (i.e., Cobb–Douglas and CES) and flexible forms. The trend has been clearly to prefer those forms capable of representing more complex technologies even if their global properties are hard to verify.

## 4. CONCLUSIONS

We have shown that flexible functional form specifications for profit functions which are linear in profits impose important *a priori* restrictions on the structure of the production technology. These undesirable properties of this class of flexible forms are not, however, shared by forms which are non-linear in profits. This suggests that, contrary to what is usually assumed, not all flexible forms are equally general and able to model equally complex production structures. We have proved that both single and multioutput LFFF for profit functions imply the following restrictions for the production technology:

1. Quasi-homotheticity or linear expansion paths which imply that the marginal rate of input substitution is independent of output levels. An undesirable feature of quasi-homotheticity is that it implies that all input demand elasticities with respect to output tend to one as output increases.
2. Certain additive separability restrictions which signify that the marginal rate of substitution among any input pair is not only independent of output levels but also of *all* factor prices except those of the input pair.
3. In the multioutput case, by using LFFF for a profit function, one also imposes separability between inputs and outputs. The implication of this is that the marginal rates of output transformation are independent of factor intensities or factor prices.
4. The quasi-homotheticity and separability restrictions also extend to the underlying revenue function. In particular, the expansion paths in the output space are also linear or, equivalently, the shapes of the output transformation curves are invariant to input levels.

The fact that these restrictions on the production technology are implicitly imposed when one uses a LFFF profit function had not been previously recognized. It is important that applied researchers be aware of these restrictions at the moment of choosing among FFF for profit function specifications.

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