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Adaptive synchronization of Lorenz systems using a reduced number of control signals and parameters without knowing bounds on system parameters and trajectories

EFREDY DELGADO-AGUILERA, MANUEL A. DUARTE-MERMOUD AND NORELYS AGUILA-CAMACHO*
Electrical Engineering Department and Advanced Mining Technology Center, University of Chile, Av. Tupper 2007, Santiago 8370451, Chile

*Corresponding author. Email: naguila@ing.uchile.cl

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This article analyzes the synchronization of two Lorenz systems with unknown but matched parameters. The proposed control strategies use a reduced number of control signals and adjustable parameters, without any assumption on the boundedness of the master system state trajectories. The stability and convergence of the synchronization errors are analytically demonstrated. Some simulations studies are presented, in order to verify the effectiveness of the proposed control strategy and comparing with known results.

Keywords: adaptive synchronization; Lorenz system; chaotic systems.

1. Introduction

The goal of the synchronization of two dynamical systems evolving separately, one called master and the other called slave, is that those systems will be sharing a common trajectory from a certain time onward. This synchronization can be performed under the hypothesis that system parameters are known (non-adaptive synchronization, or simply synchronization) or, if those parameters are unknown (adaptive synchronization) (Freeman, 2000). The synchronization of chaotic systems has been widely studied due to its theoretical challenges and its applications in important areas such as secure communications, chemical reactions and modelling brain activities, among others (Chen *et al.*, 2004).

Many control techniques have been used to synchronize systems with known parameters, such as sliding mode control (Zhang, 2014), impulsive control (Li & Fu, 2012) and state feedback (Oancea *et al.*, 2009), among others.

When the system parameters are unknown, adaptive control techniques must be used in order to achieve the synchronization (see for instance Ravoori *et al.*, 2009; Sorrentino *et al.*, 2010). For the case of Lorenz systems, there is a wide number of works using some kind of adaptive control to achieve synchronization. It can be mentioned the works by Liao (1998), Xu *et al.* (2009), Zhang (2010), Yi & Sun (2010), Zhan & Dengi (2014), among others. However, those works use the maximum possible number of control signals to achieve synchronization, and in some cases they assume that the trajectories of the systems are bounded.

There are some studies where a reduced number of control signals are used to achieve the synchronization (Wang *et al.*, 2006, 2008; Xiao *et al.*, 2009), however in those works the authors assume that the orbits of the chaotic systems are bounded, in order to conclude that the systems state variables are bounded, or they assume some known upper bound on the system parameters uncertainty, and thus concluding about the stability and convergence of the errors.

In this article we study the adaptive synchronization of two Lorenz systems with unknown but matched parameters, using a reduced number of control signals and adjustable parameters, and also without assuming boundedness on the master system trajectories or known upper bounds for the parameters, which is a solution not reported in literature. The adaptive synchronization is achieved using a direct approach, that is, the controller parameters are directly adjusted, without identifying plant parameters. Basics of the direct approach can be found in Narendra & Annaswamy (2005), as well as some other approaches not used in this work, for instance the indirect and the combined approach.

Firstly, we analyze the three possible cases where two control signals and one adjustable parameter are used. Next we analyze a case where only one control signal and one adjustable parameter are employed. No assumptions on the system states boundedness are made in any case. No upper bounds for system parameters are used neither. The stability of the controlled systems and the convergence of the synchronization errors are proved in all cases, using the Lyapunov direct method and the Barbalat Lemma, respectively.

The article is organized as follows. Section 2 presents the statement of the adaptive synchronization problem, and the proposed solutions. The theoretical stability analysis of the controlled system and the convergence of the errors are also presented for each case. Section 3 presents the simulation results for all the solutions proposed in Section 2. An analysis of the influence of the adaptive gain and initial conditions of the adjustable parameter, together with a comparison with other control strategies already reported in literature is also performed. Finally, Section 4 presents the main conclusions of the work.

2. Problem statement and solutions

Let us consider the synchronization of two Lorenz systems (Lorenz, 1963) formulated in the state space, one called master system and the other called slave system, described by following equations

$$\text{Master}\begin{cases}\dot{x}_m(t) = \sigma\left(y_m(t) - x_m(t)\right)\\ \dot{y}_m(t) = \gamma x_m(t) - x_m(t)\,z_m(t) - y_m(t)\\ \dot{z}_m(t) = x_m(t)\,y_m(t) - \beta z_m(t)\end{cases} \tag{2.1}$$

and

$$\text{Slave}\begin{cases}\dot{x}_s(t) = \sigma\left(y_s(t) - x_s(t)\right) + U_1(t)\\ \dot{y}_s(t) = \gamma x_s(t) - x_s(t)\,z_s(t) - y_s(t) + U_2(t)\\ \dot{z}_s(t) = x_s(t)\,y_s(t) - \beta z_s(t) + U_3(t),\end{cases} \tag{2.2}$$

where $X_m(t) = \begin{bmatrix} x_m(t) & y_m(t) & z_m(t)\end{bmatrix}^T \in \mathbb{R}^3$ and $X_s(t) = \begin{bmatrix} x_s(t) & y_s(t) & z_s(t)\end{bmatrix}^T \in \mathbb{R}^3$ are the states of the master and slave systems, respectively. $U(t) = [U_1(t) \quad U_2(t) \quad U_3(t)]^T \in \mathbb{R}^3$ is the control signal applied to the slave system, designed to achieve the synchronization of both systems. The goal is to find $U(t)$ such that

$$\lim_{t\to\infty} \|X_m(t) - \alpha X_s(t)\| = 0, \tag{2.3}$$

i.e. to synchronize both systems except for a scaling factor $\alpha \in \mathbb{R}$ (generalized projective synchronization (Wang & Guan, 2006)), which in this study is a scalar and constant factor.

It is well known that Lorenz systems (Lorenz, 1963) exhibit chaotic behavior for the following parameter values:

$$\sigma = 10 \quad \gamma = 28 \quad \beta = 8/3.$$

Let us define the synchronization error as $e(t) = \left[x_m - x_s \;\; y_m - y_s \;\; z_m - z_s\right]^T = [e_1(t) \;\; e_2(t) \;\; e_3(t)]^T = X_m(t) - \alpha X_s(t) \in \mathbb{R}^3$, where $\alpha \in \mathbb{R}^+$ is the scale factor. Then using (2.1) and (2.2), the resulting equations describing the evolution of the synchronization error are

$$\begin{aligned} \dot{e}_1(t) &= -\sigma e_1(t) + \sigma e_2(t) - \alpha U_1(t) \\ \dot{e}_2(t) &= \gamma e_1(t) - e_2(t) - x_m(t) z_m(t) + \alpha x_s(t) z_s(t) - \alpha U_2(t) \\ \dot{e}_3(t) &= -\beta e_3(t) + x_m(t) y_m(t) - \alpha x_s(t) y_s(t) - \alpha U_3(t). \end{aligned} \tag{2.4}$$

When $\alpha = 1$, equation (2.4) turns out to be

$$\begin{aligned} \dot{e}_1(t) &= -\sigma e_1(t) + \sigma e_2(t) - U_1(t) \\ \dot{e}_2(t) &= \gamma e_1(t) - e_2(t) - x_s(t) e_3(t) - z_m(t) e_1(t) - U_2(t) \\ \dot{e}_3(t) &= -\beta e_3(t) + x_m(t) e_2(t) + y_s(t) e_1(t) - U_3(t). \end{aligned} \tag{2.5}$$

The question to be answered is how to synchronize both systems (2.1) and (2.2) to achieve and maintain a common regime as t goes to infinity. Moreover, it is desired to accomplish this task without the knowledge of the parameters σ, γ, β, seeking for solutions involving a reduced number of control and states signals, as well as with a reduced number of adjustable parameters and without any assumption on the boundedness of the system parameters and trajectories.

In this study we will distinguish four different cases. We will analyze first the three cases of adaptive synchronization using two control signals and one adjustable parameter, and later it is analyzed a case using one control signal and one adjustable parameter.

Although the control strategies proposed in this article are designed for Lorenz systems, we would like to mention that the methodology used to find the control signals—using the Lyapunov direct method and the Barbalat Lemma—could be used to propose similar control strategies for other kind of chaotic systems (Rössler, Chen, Chua, etc.) using a reduced number of control signals and parameters.

2.1. *Adaptive synchronization using control signals $U_2(t)$ and $U_3(t)$ and one adjustable parameter $\theta(t)$*

Here we present a solution to the adaptive synchronization of Lorenz systems, using two control signals ($U_2(t)$ and $U_3(t)$), together with one adjustable parameter ($\theta(t)$). The parameters σ, γ, β are assumed to be unknown, but we assume that $\sigma, \beta > 0$.

LEMMA 2.1 Let us consider the master system (2.1) and the slave system (2.2) with unknown parameters σ, γ, β and $\sigma, \beta > 0$. Let us assume that all the states of the master and slave systems are accessible.

Using the following control signals in (2.2)

$$\begin{aligned} U_1(t) &= 0 \\ U_2(t) &= \frac{1}{\alpha}\left(\theta(t)\, e_1(t) - x_m(t)\, z_m(t) + \alpha x_s(t)\, z_s(t)\right) \\ U_3(t) &= \frac{1}{\alpha}\left(x_m(t)\, y_m(t) - \alpha x_s(t)\, y_s(t)\right), \end{aligned} \tag{2.6}$$

together with an adaptive law for the adjustable parameter $\theta(t) \in \mathbb{R}$ given by

$$\dot{\theta}(t) = \delta e_1(t)\, e_2(t), \tag{2.7}$$

then the synchronization of the master and slave systems is achieved. In (2.7), $\delta \in \mathbb{R}^+$ corresponds to the adaptive gain that can be used to handle the convergence speed.

Proof. Let us define the parameter error as $\phi(t) = \gamma - \theta(t) \in \mathbb{R}$. Then, using the control signals (2.6) in (2.4), the evolution of the synchronization errors and the parameter error can be expressed as

$$\begin{aligned} \dot{e}_1(t) &= -\sigma e_1(t) + \sigma e_2(t) \\ \dot{e}_2(t) &= \phi(t)\, e_1(t) - e_2(t) \\ \dot{e}_3(t) &= -\beta e_3(t) \\ \dot{\phi}(t) &= -\dot{\theta}(t) = -\delta e_1(t)\, e_2(t). \end{aligned} \tag{2.8}$$

In order to prove the stability of the controlled system, let us use the Lyapunov direct method (Narendra & Annaswamy, 2005). We propose the following Lyapunov candidate function, which is positive definite and decrescent

$$V = \frac{1}{2\sigma} e_1^2(t) + \frac{1}{2} e_2^2(t) + \frac{1}{2} e_3^2(t) + \frac{1}{2\delta}\phi^2(t). \tag{2.9}$$

Thus, the first derivative of (2.9) along the system trajectories (2.8) results in

$$\dot{V} = -\frac{1}{2}\left(e_1(t) - e_2(t)\right)^2 - \frac{1}{2} e_1^2(t) - \frac{1}{2} e_2^2(t) - \beta e_3^2(t). \tag{2.10}$$

As can be seen from (2.10), the first derivative of the Lyapunov function is negative semidefinite, then it can be concluded that the origin of system (2.8) is uniformly stable, and therefore $e_1(t), e_2(t)$, $e_3(t), \phi(t) \in \mathcal{L}^\infty$.

To analyze the convergence of the synchronization errors to zero, let us integrate equation (2.10) between 0 and ∞. In that way we obtain

$$V(\infty) - V(0) = -\frac{1}{2}\int_0^\infty \left(e_1(t) - e_2(t)\right)^2 dt - \frac{1}{2}\int_0^\infty e_1^2(t)\, dt - \frac{1}{2}\int_0^\infty e_2^2(t)\, dt - \beta \int_0^\infty e_3^2(t)\, dt. \tag{2.11}$$

From expression (2.11), using the fact that $e_1(t), e_2(t)$, $e_3(t), \phi(t) \in \mathcal{L}^\infty$, and therefore $V(0), V(\infty) \in \mathcal{L}^\infty$, it can be concluded that $e_1(t), e_2(t), e_3(t) \in \mathcal{L}^2$.

Since $e_1(t), e_2(t), e_3(t), \phi \in \mathcal{L}^\infty$, from (2.8) it can be concluded also that $\dot{e}_1(t), \dot{e}_2(t), \dot{e}_3(t) \in \mathcal{L}^\infty$. Then, using the Corollary of the Barbalat Lemma (Narendra & Annaswamy, 2005), it follows that

$$\lim_{t\to\infty} e_1(t) = \lim_{t\to\infty} e_2(t) = \lim_{t\to\infty} e_3(t) = 0, \tag{2.12}$$

that is, the synchronization is achieved, and this concludes the proof. □

REMARK 2.1 The reason for assuming $\sigma, \beta > 0$ in Lemma 2.1 and in the consequent lemmas, is because these parameters are used as part of the Lyapunov functions or they appear as part of the first time derivative of the Lyapunov functions. Since the Lyapunov function and its first time derivative have to satisfy some sign definite properties to guarantee stability, parameters σ, β have to be positive. We do not need to know the values of σ, β, but we do need to assure that they are positive.

2.2. *Adaptive synchronization using control signals $U_1(t)$ and $U_3(t)$ and one adjustable parameter $\theta(t)$*

Let us present now a solution for the adaptive synchronization using control signals $U_1(t)$ and $U_3(t)$ and one adjustable parameter $\theta(t)$. The proposed solution is valid when the scaling factor $\alpha = 1$, and it is assumed that $\sigma, \beta > 0$.

LEMMA 2.2 Let us consider the master system (2.1) and the slave system (2.2) with unknown parameters σ, γ, β and $\sigma, \beta > 0, \alpha = 1$. Let us assume that all the states of the master and slave systems are accessible. Using the following control signals in (2.2)

$$\begin{aligned} &U_1(t) = -z_m(t)\, e_2(t) + \theta(t)\, e_2(t) \\ &U_2(t) = 0 \\ &U_3(t) = x_m(t)\, e_2(t) + y_s(t)\, e_1(t) - x_s(t)\, e_2(t), \end{aligned} \tag{2.13}$$

together with the following adaptive law for the adjustable parameter $\theta(t) \in \mathbb{R}$

$$\dot{\theta}(t) = \delta e_1(t)\, e_2(t), \tag{2.14}$$

then the synchronization of the master and slave systems is achieved. In (2.14), $\delta \in \mathbb{R}^+$ corresponds to the adaptive gain that can be used to handle the convergence speed.

Proof. Let us define the parameter error as $\phi(t) = \sigma + \gamma - \theta(t)$. The proof for this case can be made following the same procedure than in the previous case, using the Lyapunov direct method with a Lyapunov function candidate given by

$$V = \frac{1}{2}e_1^2(t) + \frac{1}{2}e_2^2(t) + \frac{1}{2}e_3^2(t) + \frac{1}{2\delta}\phi^2(t). \tag{2.15}$$

This allows concluding that $e_1(t), e_2(t), e_3(t), \phi(t) \in \mathcal{L}^\infty$. Then using the corollary of the Barbalat Lemma it can be concluded also that

$$\lim_{t\to\infty} e_1(t) = \lim_{t\to\infty} e_2(t) = \lim_{t\to\infty} e_3(t) = 0, \tag{2.16}$$

and this concludes the proof. □

2.3. *Adaptive synchronization using control signals $U_1(t)$ and $U_2(t)$ and one adjustable parameter $\theta(t)$*

Another solution for the adaptive synchronization using control signals $U_1(t)$ and $U_2(t)$ and one adjustable parameter $\theta(t)$ is analyzed in this section. The proposed solution is valid only for the case $\alpha = 1$, and it is assumed that $\sigma, \beta > 0$.

LEMMA 2.3 Let us consider the master system (2.1) and the slave system (2.2) with unknown parameters σ, γ, β and $\sigma, \beta > 0, \alpha = 1$. Let us assume that all the states of the master and slave systems are accessible. Using the following control signals in (2.2)

$$\begin{aligned} U_1(t) &= y_s(t)\, e_3(t) + \theta(t)\, e_2(t) \\ U_2(t) &= -x_s(t)\, e_3(t) - z_m(t)\, e_1(t) + x_m(t)\, e_3(t) \\ U_3(t) &= 0, \end{aligned} \tag{2.17}$$

together with the following adaptive law for the adjustable parameter $\theta(t) \in \mathbb{R}$

$$\dot{\theta}(t) = \delta e_1(t)\, e_2(t), \tag{2.18}$$

then the synchronization of the master and slave systems is achieved. In (2.18), $\delta \in \mathbb{R}^+$ corresponds to the adaptive gain that can be used to handle the convergence speed.

Proof. Let us define the parameter error as $\phi(t) = \sigma + \gamma - \theta(t)$. The proof of this Lemma can be made following the same procedure used in the two previous cases, using the Lyapunov direct method with a Lyapunov function candidate given by

$$V = \frac{1}{2}e_1^2(t) + \frac{1}{2}e_2^2(t) + \frac{1}{2}e_3^2(t) + \frac{1}{2\delta}\phi^2(t). \tag{2.19}$$

This analysis allows proving that $e_1(t), e_2(t), e_3(t), \phi(t) \in \mathcal{L}^\infty$, and then using the corollary of the Barbalat Lemma it follows that

$$\lim_{t\to\infty} e_1(t) = \lim_{t\to\infty} e_2(t) = \lim_{t\to\infty} e_3(t) = 0. \tag{2.20}$$

□

2.4. *Adaptive synchronization using only control signal $U_1(t)$ and one adjustable parameter $\theta(t)$*

Finally, we present a solution for the adaptive synchronization using only one control signal, $U_1(t)$ in this case, and one adjustable parameter $\theta(t)$. The proposed solution is valid for the case where the scaling factor $\alpha = 1$, and it is assumed that $\sigma, \beta > 0$.

LEMMA 2.4 Let us consider the master system (2.1) and the slave system (2.2) with unknown parameters σ, γ, β and $\sigma, \beta > 0, \alpha = 1$. Let us assume that all the states of the master and slave systems are accessible. Using the following control signals in (2.2)

$$\begin{aligned} U_1(t) &= \theta(t)\, e_2(t) - y_m(t)\, z_s(t) + y_s(t)\, z_m(t) \\ U_2(t) &= 0 \\ U_3(t) &= 0, \end{aligned} \tag{2.21}$$

together with the following adaptive law for the adjustable parameter $\theta(t) \in \mathbb{R}$

$$\dot{\theta}(t) = \delta e_1(t)\, e_2(t), \tag{2.22}$$

then the synchronization of the master and slave systems is achieved. In (2.22), $\delta \in \mathbb{R}^+$ corresponds to the adaptive gain that can be used to handle the convergence speed.

Proof. Let us define the parameter error as $\phi(t) = \sigma + \gamma - \theta(t)$. As it was made in the three previous cases, the proof here can be made using the Lyapunov direct method with a Lyapunov function candidate given by

$$V = \frac{1}{2}e_1^2(t) + \frac{1}{2}e_2^2(t) + \frac{1}{2}e_3^2(t) + \frac{1}{2\delta}\phi^2(t). \tag{2.23}$$

From this analysis it can be concluded that $e_1(t), e_2(t), e_3(t), \phi(t) \in \mathcal{L}^\infty$ and using the corollary of the Barbalat Lemma it can be concluded that

$$\lim_{t\to\infty} e_1(t) = \lim_{t\to\infty} e_2(t) = \lim_{t\to\infty} e_3(t) = 0. \tag{2.24}$$

□

REMARK 2.2 Adaptive synchronization can also be achieved using only control signal $U_2(t)$ and the methodology presented in this article. However, this solution was not included in this document, because it needs the assumption of boundedness of the master state trajectories, like in the cases already available in literature, so there is no contribution in this case.

Solution using only control signal $U_3(t)$ could not be found using the methodology proposed in this article, that is why this case is not presented here either.

REMARK 2.3 The control strategies reported in literature to achieve synchronization of Lorenz systems can be divided in two groups. One group corresponds to non-adaptive strategies and the other group corresponds to adaptive strategies.

The non-adaptive strategies cannot be implemented in the problem we address in this article, since they need the knowledge of the parameters σ, β, γ to make the control, and in this problem σ, β, γ are considered unknown.

Regarding the adaptive strategies that are already available in literature, some of them do not need the knowledge of the parameters σ, β, γ, so they could be implemented under these circumstances. However, many of these control strategies use three control signals and more than one adjustable parameter (instead of a reduced number of control signals and one adjustable parameter as we do), and/or they assume that the orbits of the master system (x_m, y_m, z_m) are always bounded (we do not assume this), and/or some upper bounds on the system parameters σ, γ, β are known (we do not assume this either).

TABLE 1 *Summary of the results presented in Section 2*

Section	Control signals	Adjustable parameter	Scale factor	Parameter assumptions	Required master signals	Stability and convergence
2.1	U_2, U_3 Eq. (2.6)	θ Eq. (2.7)	$\alpha > 0$	$\sigma, \beta > 0$	x_m, y_m, z_m	u.a.s. $\lim_{t\to\infty} e_1, e_2, e_3 = 0$
2.2	U_1, U_3 Eq. (2.13)	θ Eq. (2.14)	$\alpha = 1$	$\sigma, \beta > 0$	x_m, y_m, z_m	u.a.s. $\lim_{t\to\infty} e_1, e_2, e_3 = 0$
2.3	U_1, U_2 Eq. (2.17)	θ Eq. (2.18)	$\alpha = 1$	$\sigma, \beta > 0$	x_m, y_m, z_m	u.a.s. $\lim_{t\to\infty} e_1, e_2, e_3 = 0$
2.4	U_1 Eq. (2.21)	θ Eq. (2.22)	$\alpha = 1$	$\sigma, \beta > 0$	x_m, y_m, z_m	u.a.s. $\lim_{t\to\infty} e_1, e_2, e_3 = 0$

REMARK 2.4 Regarding the information needed in this problem to achieve the synchronization, we would like to mention the following. The problem addressed in this article deals with two existing and independent systems (master system (2.1) and slave system (2.2)) which have the same structure and matched parameters. The structure of both systems is considered to be known in the problem. The parameter values, however, are considered unknown for both systems. This information is not available for the master system and neither for the slave system. It is only assumed that parameters σ, β are greater than zero. There is no need to know the parameter values of master and slave systems in order to achieve the synchronization. It can be seen that none of the control signals (2.6), (2.13), (2.17) and (2.21) use the values of parameters σ, β, γ. The adaptive laws (2.7), (2.14), (2.18) and (2.22) do not use these values either. The adaptive control strategies used here to solve the synchronization problem follow a direct approach. This means that no estimation of the unknown system parameters is attempted, but only direct adjustment of the controller parameters is employed. If an indirect approach would it be chosen to address this problem, then the estimation of the parameter values σ, γ, β would it be needed.

Although synchronization of two Lorenz systems with unknown but matched parameters could seem restrictive, real world applications can lead to this kind of problem. In Zhang (2015), for instance, a lag synchronization of two Lorenz systems with matched parameters is proposed, with applications to communication.

Nevertheless, research is currently undergoing on how to achieve adaptive synchronization using a reduced number of control signals and parameters, when the structure of master and slave systems is the same, but whose parameters can be different and unknown.

As a summary, we present in Table 1 the main strategies.

3. Numerical results and simulations

From the approaches presented in Section 2, it can be concluded that adaptive synchronization of Lorenz systems can be reached, by handling either one or two control signals, with a single adjustable parameter and without assumptions on the boundedness of the master system trajectories or upper bounds on the system parameters.

3.1. *Behavior of synchronization control strategies proposed in this article*

This section presents some representative simulations, to illustrate the effectiveness of the proposed synchronization schemes of Section 2. For these simulations, the fourth order Runge Kutta method for solving differential equations in Matlab/Simulink was used. The parameter values for the master system (2.1) and slave system (2.2), which are assumed to be unknown, are $\sigma = 10$, $\gamma = 28$ and $\beta = 8/3$, values at which the Lorenz system exhibits a chaotic behavior. The initial conditions for master and slave systems were chosen as $[-8, -5, 6]^T$ and $[10, -10, -10]^T$, respectively. For this study we took a scale factor $\alpha = 1$ as well as $\alpha = 2$, in the case where it is possible. In all simulations, a unity adaptive gain δ was chosen for simplicity, although the analysis is also valid for any $\delta > 0$. The initial condition for the estimated parameter $\theta(t)$ was chosen as $\theta(0) = 0$.

Figure 1 shows the evolution of the synchronization errors for the case of using control signals U_2 and U_3, one adjustable parameter θ and two cases for the scale factor, $\alpha = 1$ and $\alpha = 2$, as described in Section 2.1.

As can be seen from Fig. 1, the three components of the synchronization error converge to zero no matter the scale factor α used, as it was expected from the theoretical proof made in Section 2.1. It can be seen that the component e_3 is strictly decreasing, since the closed loop equation for this error is of the form $\dot{e}_3 = -\beta e_3$. In the case of e_2 and e_3, there are some initial oscillations, due to the adaptation process, but they rapidly converge to zero. In the case using $\alpha = 2$, it can be seen that the convergence speed is quite similar to the case with $\alpha = 1$, but the magnitude of the initial oscillations is higher. Figure 2 shows the state trajectories of both, master and slave systems, for the case of using $\alpha = 2$. As can be seen, the synchronization is effectively achieved, and the use of the scale factor different from 1 can be observed.

Figure 3 shows the evolution of the synchronization error for the case of using control signals U_1 and U_3, one adjustable parameter θ and a scale factor $\alpha = 1$, as described in Section 2.2.

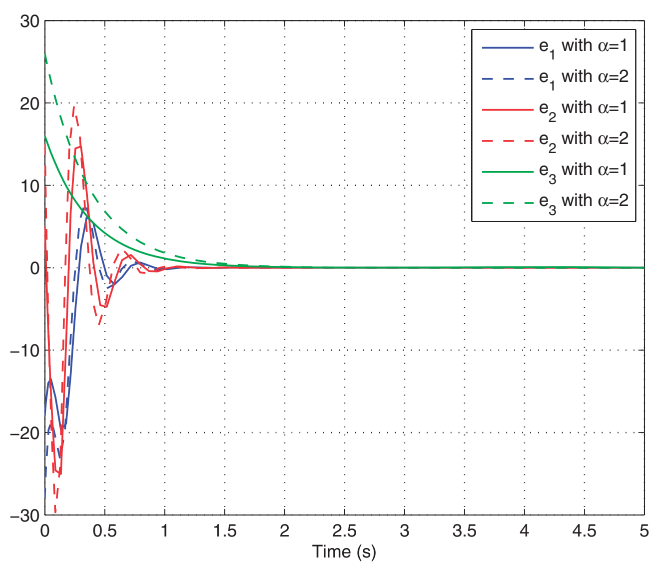

FIG. 1. Evolution of the synchronization error when using two control signals (U_2 and U_3), one adjustable parameter (θ) and a scale factor $\alpha = 1$ and $\alpha = 2$.

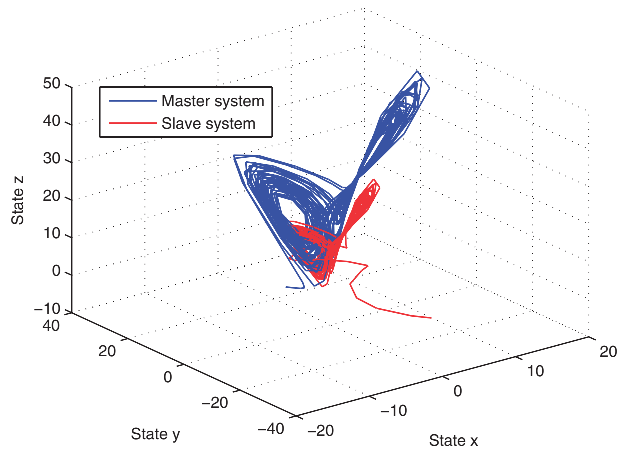

FIG. 2. State trajectories of the master system and the slave system, for the case using two control signals (U_2 and U_3), one adjustable parameter (θ) and a scale factor $\alpha = 2$.

As can be observed, the three components of the synchronization error converge to zero, that is, the synchronization is achieved, as proved theoretically in Section 2.2.

However, in this case the transient response of the three components of the synchronization error presents some oscillations, due to the adaptation process. In the previous case (using U_2 and U_3), the component e_3 was not affected by the adaptation process, since the closed loop equation for this error has the form $\dot{e}_3 = -\beta e_3$. In this case, however, it does not have that form, and the adjustable parameter θ affects directly e_1, and indirectly e_2 and consequently e_3.

Regarding the magnitudes of the error in the transient response, they are higher than in the case of using U_2 and U_3 with the same scale factor $\alpha = 1$.

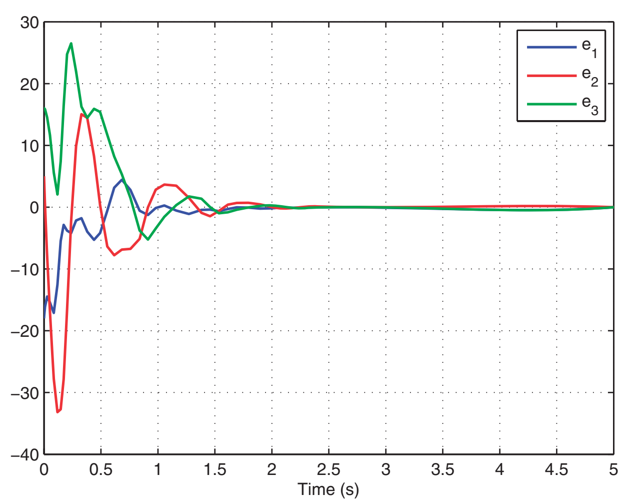

FIG. 3. Evolution of the synchronization error when using two control signals (U_1 and U_3), one adjustable parameter (θ) and a scale factor $\alpha = 1$.

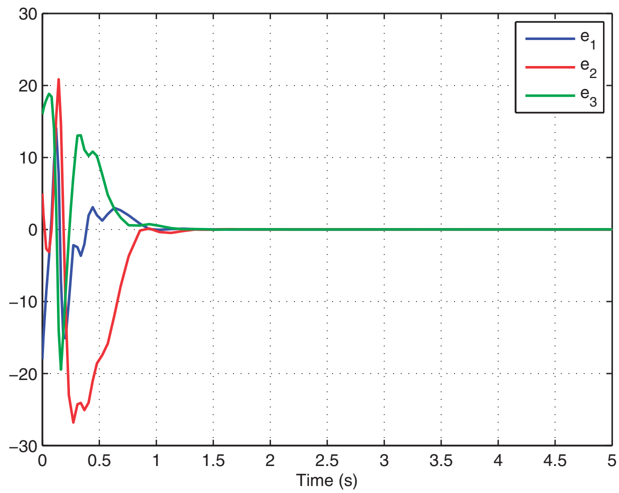

FIG. 4. Evolution of the synchronization error when using two control signals (U_1 and U_2), one adjustable parameter (θ) and a scale factor $\alpha = 1$.

Figure 4 shows the evolution of the synchronization errors for the case of using control signals U_1 and U_2, one adjustable parameter θ and a scale factor $\alpha = 1$, as indicated in Section 2.3.

It can be observed from Fig. 4 that the three components of the synchronization error converge to zero, as it was expected from the theoretical analysis in Section 2.3.

In this case, the transient response of the three components of the synchronization error present oscillations, due to the adaptation process. The same explanation given for the case of using U_1 and U_3 is valid here. The initial magnitudes of the synchronization errors are higher than in the case of using U_2 and U_3, and similar to the case of using U_1 and U_3, although the convergence speed of the errors is a little bit faster than in those two cases.

Figure 5 shows the evolution of the synchronization error for the case of using only the control signal U_1, one adjustable parameter θ and a scale factor $\alpha = 1$, as described in Section 2.4. From Fig. 5 it is observed that the synchronization is achieved as in the three previous cases, as was expected from the results in Section 2.4.

However, in this case, the transient response of the errors is more oscillatory than in the three previous cases using two control signals, although the convergence speed is similar. Certainly, the main advantage in this case is the possibility to achieve synchronization using only one control signal.

3.2. *Analysis of the effect of adaptive gain δ and initial condition $\theta(0)$*

It is important to point out that in the simulations already presented, the adaptive gain δ was chosen as $\delta = 1$ and the initial condition for the estimated parameter was chosen as $\theta(0) = 0$ in all the cases. However, these are design parameters that we can handle in the adaptive scheme, so it is important to analyze their influence in the convergence time and the transient responses of the synchronization error.

To that extent, many other simulations were performed with adaptive gains different from unity and initial conditions for the estimated parameter different from zero.

To illustrate the influence of the adaptive gain, we present Fig. 6, where the synchronization is achieved using control signals U_2 and U_3, with the same initial conditions used in the previous

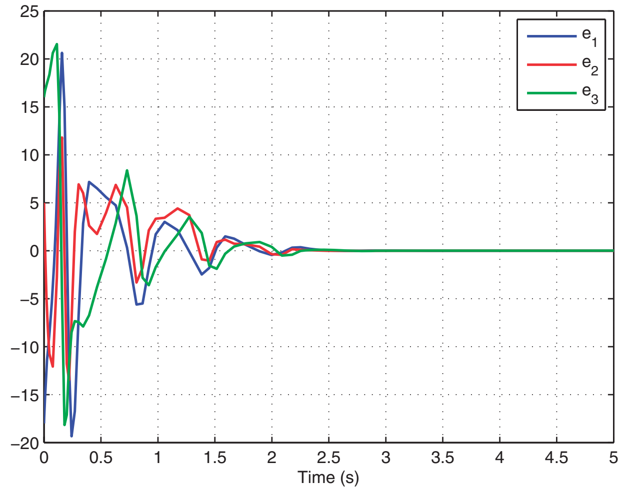

FIG. 5. Evolution of the synchronization error when using one control signal (U_1), one adjustable parameter (θ) and a scale factor $\alpha = 1$.

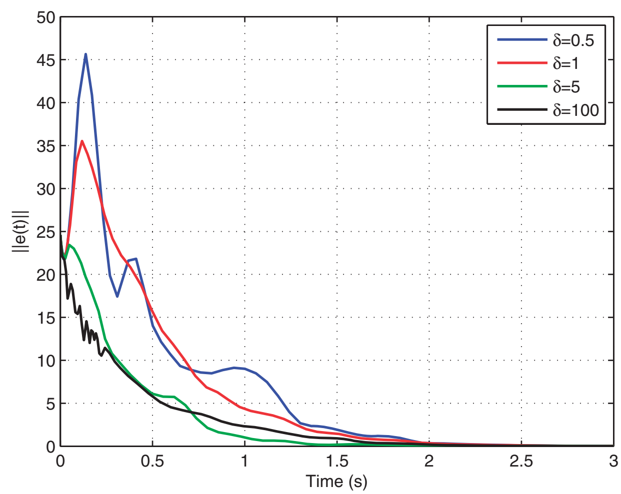

FIG. 6. Evolution of the norm of the synchronization error when using control signals U_2 and U_3, one adjustable parameter (θ), a scale factor $\alpha = 1$ and different values for the adaptive gain δ.

simulations, but using different adaptive gains in each case. For the sake of space, the norm of the synchronization error has been plotted in this case, instead of the three components.

As can be seen from Fig. 6, a faster convergence of the errors can be achieved using adaptive gains greater than 1. The magnitude of the oscillations in the transient responses decreases with the increasing of δ as well, however the frequency of the oscillations can be higher for larger values of the adaptive gain (see for example the case $\delta = 100$). Therefore, this design parameter must be wisely used inside the synchronization scheme.

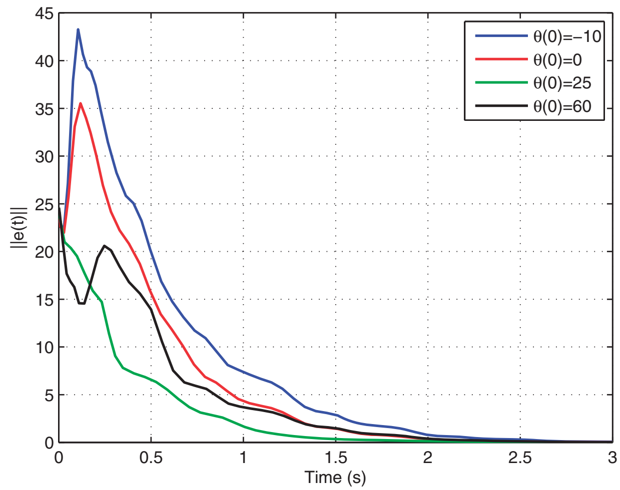

FIG. 7. Evolution of the norm of the synchronization error when using control signals U_2 and U_3, one adjustable parameter (θ), a scale factor $\alpha = 1$ and different values for the initial condition $\theta\,(0)$.

Let us now analyze the influence of the initial conditions of the estimated parameter $\theta\,(0)$. Figure 7 shows the evolution of the norm of the synchronization error, using control signals U_2 and U_3. In this case a constant adaptive gain $\delta = 1$ and different values for the initial condition $\theta\,(0)$ are used. For this problem, the parameter error is defined as $\phi\,(t) = \gamma - \theta\,(t)$, as can be seen in the proof of Lemma 2.1. Thus, the real unknown value of the estimated parameter θ is 28. Although this value is unknown in this problem, to see how the transient response and the convergence time are affected for the initial value $\theta\,(0)$, in the simulations of Fig. 7 we use values for $\theta\,(0)$ that are close and far from the real value.

As can be seen from Fig. 7, the convergence is faster and the transient response is less oscillatory when the initial value $\theta\,(0)$ gets closer to the real value 28. This implies that if we have an idea of the real value of the estimated parameter, we can use this information to adjust the initial condition $\theta\,(0)$, obtaining faster convergence and less oscillatory transient responses. However, we may note that this is not always the case in adaptive schemes, where usually this information is not available.

Finally, we want to point out that although the simulations presented in this subsection correspond to the case using control signals U_2 and U_3, the conclusions were similar for the rest of the cases treated in this article.

3.3. *Comparison with other control strategies proposed in the technical literature*

As it was mentioned before, the work presented in this article has advantage over other tecniques since the system parameters σ, γ, β are assumed to be unknown, a reduced number of control signals and adjustable parameters are used, and no bounds on the system trajectories or system parameters are needed.

Although there is no similar works reported in the literature accomplishing all these conditions, we would like to make some comparison between the results reported in this article, and some other results reported in literature for the same system, no matter the information they need to implement the control. The idea is to analyze the convergence time and the transient response of the synchronization errors for each case.

Thus, two reported control strategies were implemented to establish the comparison. In what follows, we are going to refer to the control strategy proposed in this article using control signals U_2 and U_3 and one adjustable parameter described in Section 2.1 as 'Control strategy 1'. The 'Control strategy 2' corresponds to that reported in Liao (1998), which does not need any knowledge of the system parameters or any bounds, but it uses three control signals U_1, U_2, U_3 and three adjustable parameters. The 'Control strategy 3', on the other hand, corresponds to that reported in Xiao *et al.* (2009), which uses only one control signal U_1 and one adjustable parameter, but it needs to know the value of the system parameter β to find additional controller parameters, and also uses the assumption that the system trajectories are bounded in order to prove the convergence of the errors.

Figure 8 shows the evolution of the synchronization error using these three control strategies. The initial conditions for the master and slave systems are the same as in previous simulations (see Figs 1 and 2) adaptive gains are unitary and the initial conditions for all the adjustable parameters are zero.

As can be observed in Fig. 8, no big differences can be seen about the convergence time of e_1 for the three control strategies. However, for the cases of e_2 and e_3 the Control strategy 3 presents higher convergence times than Control strategy 1 and 2.

Regarding the transient responses, the control strategy proposed in this article presents a better transient response with respect to the other two for the case of e_3, since no oscillations are appreciated. For the cases of e_1 and e_2, the magnitudes of the initial oscillations are pretty similar for the three strategies.

Summarizing, the behavior of the convergence time and transient response of the synchronization error for the control strategy proposed in this article is similar or better than for the other two reported control strategies, even when the one proposed in this article uses less control signals and adjustable

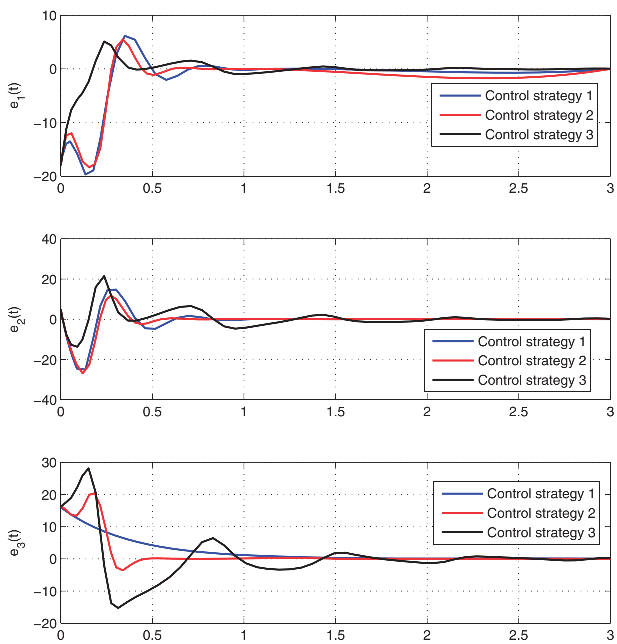

FIG. 8. Evolution of the synchronization error for different control strategies.

parameters than the Control strategy 2 (Liao, 1998) and uses less information about the system than the Control strategy 3 (Xiao *et al.*, 2009).

4. Conclusions

In this article the analysis of the adaptive synchronization of two Lorenz systems has been presented, using a reduced number of control signals and adjustable parameters. The synchronization was studied based on theoretical results and complemented by simulations describing the behavior of the synchronization errors.

The study performed in this article indicates that adaptive synchronization of Lorenz systems can be achieved with a reduced number of parameters and signals, without assumptions of boundedness of the master system trajectories and without knowledge of upper bounds on system parameters or system uncertainties. This can be done by using two control signals and one adjustable parameter, or even using one control signal and one adjustable parameter. The main differences between these solutions appear mainly in the transient response of the synchronization errors, being the convergence speed quite similar for all the cases.

It is important to emphasize the fact that incorporating adaptive gains in the adaptive laws, allows reducing the convergence time of the synchronization errors, and in some cases allows reducing the initial magnitudes of the transient response. However, using large adaptive gains can also lead to more oscillatory transient responses, existing a trade off between the value of this design parameter and the speed and transient quality of the resulting adaptive system.

Comparisons with other control strategies show that the proposed control strategies can lead to shorter convergence times and better transient responses, even when a reduced number of control signal and adjustable parameters are used, and less information about the system is required.

Research is currently under way to analyze the minimal synchronization of fractional order Lorenz systems.

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