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Minimal heteroclinics for a class of fourth order O.D.E. systems

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ABSTRACT

We prove the existence of minimal heteroclinic orbits for a class of fourth order O.D.E. systems with variational structure. In our general set-up, the set of equilibria of these systems is a union of manifolds, and the heteroclinic orbits connect two disjoint components of this set.

1. Introduction and main results

Given a smooth nonnegative function $W : \mathbb{R}^m \times \mathbb{R}^m \to [0,\infty)$ $(m \geq 1)$, we define for every $(u,v) := (u_1,\ldots,u_m,v_1,\ldots,v_m) \in \mathbb{R}^m \times \mathbb{R}^m$, the vector $W_u(u,v) := (\frac{\partial W}{\partial u_1}(u,v),\ldots,\frac{\partial W}{\partial u_m}(u,v)) \in \mathbb{R}^m$, and the matrices $W_{uv}(u,v) := \big(\frac{\partial^2 W}{\partial u_j \partial v_i}(u,v)\big)_{1\leq i,j\leq m}$, $W_{vv}(u,v) := \big(\frac{\partial^2 W}{\partial v_j \partial v_i}(u,v)\big)_{1\leq i,j\leq m}$, where i (respectively j) stands for the row (resp. the column). Next, we consider the system:

$$\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} + W_u(u,u') - W_{uv}(u,u')u' - W_{vv}(u,u')u'' = 0,\ u : \mathbb{R} \to \mathbb{R}^m, \tag{1}$$

which is the Euler–Lagrange equation of the energy functional:

$$J_{\mathbb{R}}(u) = \int_{\mathbb{R}} \Big(\frac{1}{2}|u''|^2 + W(u,u')\Big),\ u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m). \tag{2}$$

In the scalar case $(m = 1)$, setting $W(u,v) = \frac{1}{4}(u^2-1)^2 + \frac{\beta}{2}v^2$, where $\beta > 0$,[1] we obtain the Extended Fisher–Kolmogorov equation

$$\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} - \beta u'' + u^3 - u = 0,\ u : \mathbb{R} \to \mathbb{R}, \tag{3}$$

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[1] In the case where $\beta < 0$, the corresponding equation is known as the Swift–Hohenberg equation.

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which was proposed in 1988 by Dee and van Saarloos [5] as a higher-order model equation for bistable systems. Eq. (3) has been extensively studied by different methods: topological shooting methods, Hamiltonian methods, variational methods, and methods based on the maximum principle (cf. [3,13], and the references therein, in particular [9–11], and [12]). In recent years, it has become evident that the structure of solutions of (3) is considerably richer than the structure of solutions of the Allen–Cahn O.D.E.:

$$u'' = u^3 - u,\ u : \mathbb{R} \to \mathbb{R}, \tag{4}$$

or equivalently $u'' = W'(u)$, with $W(u) = \frac{1}{4}(u^2-1)^2$. Depending on the value of β, we mention below some properties of the heteroclinic orbits[2] of (3), connecting at $\pm\infty$ the two equilibria ± 1, in the sense that

$$\lim_{x\to\pm\infty}(u(x), u'(x), u''(x), u'''(x)) = (\pm 1, 0, 0, 0) \text{ in the phase-space.} \tag{5}$$

When $\beta \geq \sqrt{8}$,[3] the structure of bounded solutions of (3) exactly mirrors that of (4). In particular, (3) has (up to translations) a unique heteroclinic orbit connecting -1 to 1, which is monotone. However, as soon as β passes the critical value $\sqrt{8}$ from above, an infinity of heteroclinics appears immediately, and these orbits are no longer monotone. Actually, they oscillate around the equilibria ± 1, and may jump from -1 to 1 and back a number of times. Also note that as β decreases from $\sqrt{8}$, these orbits continue to exist up to $\beta = 0$, and even somewhat beyond.

Another major difference between the second order model (3) and (4), lies in the existence of *pulses* for $\beta < \sqrt{8}$, i.e. nontrivial solutions $u : \mathbb{R} \to \mathbb{R}$ of (3) such that

$$\lim_{|x|\to\infty}(u(x), u'(x), u''(x), u'''(x)) = (1, 0, 0, 0) \text{ or } (-1, 0, 0, 0). \tag{6}$$

This situation which is excluded for the scalar equation (4), may occur if we consider the system $u'' = \nabla W(u)$ with a multiple well potential $W : \mathbb{R}^2 \to [0, \infty)$ (cf. [1, Remark 2.6] and [15, Section 2]).

A more general version of the canonical equation (3) is given by

$$\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} - g(u)u'' - \frac{g'(u)}{2}(u')^2 + f'(u) = 0,\ u : \mathbb{R} \to \mathbb{R},\ W(u, v) = \frac{g(u)}{2}v^2 + f(u), \tag{7}$$

where $f : \mathbb{R} \to \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$, are smooth functions (cf. [2,4]). For instance in [2], a double well potential $f \geq 0$ is considered, and g is allowed to take negative values to an extent that is balanced by f. Provided that $\inf g$ is bigger than a negative constant depending on the nondegeneracy of the minima of f, the variational method can be applied to construct heteroclinics of (7).

The scope of this paper is to establish the existence of minimal heteroclinics for system (1) in a general set-up, similar to that considered in [1] for the Hamiltonian system $u'' = \nabla W(u)$. In particular, we allow the function W to vanish on submanifolds, and we are interested in connecting two disjoint subsets of minima of W.

We assume that $W \in C^2(\mathbb{R}^m \times \mathbb{R}^m; [0, \infty))$ is a nonnegative function such that

H$_1$: The set $A := \{u \in \mathbb{R}^m : W(u, 0) = 0\}$ is partitioned into two nonempty disjoint compact subsets A^- and A^+.

H$_2$: There exists an open set $\Omega \subset \mathbb{R}^m$ such that $A^- \subset \Omega$, $A^+ \cap \overline{\Omega} = \emptyset$, and $W(u, v) > 0$ holds for every $u \in \partial\Omega$, and for every $v \in \mathbb{R}^m$.

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[2] The existence of heteroclinic solutions of (3) via variational arguments was investigated for the first time by L. A. Peletier, W. C. Troy and R. C. A. M. VanderVorst [14], and W. D. Kalies, R. C. A. M. VanderVorst [8].

[3] The linearization of (3) at ± 1 reads $\frac{\mathrm{d}^4 v}{\mathrm{d}x^4} - \beta v'' + 2v = 0$. The four roots of the associated characteristic equation $\lambda^4 - \beta\lambda^2 + 2 = 0$ are all real if and only if $\beta \geq \sqrt{8}$.

$\mathbf{H}_3$: $\liminf_{|u|\to\infty} W(u,v) > 0$, uniformly in $v \in \mathbb{R}^m$.

In $\mathbf{H}_1$, we define the sets A^- and A^+ that we are going to connect. On the other hand, Hypothesis $\mathbf{H}_2$ ensures that the energy required to connect a neighborhood of A^- to a neighborhood of A^+ cannot become arbitrarily small. As a consequence an orbit with finite energy may travel from A^- to A^+ and back, only a finite number of times (cf. Lemma 2.4). Also note that W is allowed to vanish if $u \notin \partial\Omega$, and $v \neq 0$. Finally, Hypothesis $\mathbf{H}_3$ is assumed to derive the boundedness of finite energy orbits (cf. Lemma 2.2).

Some typical examples of functions satisfying $\mathbf{H}_i$, $i = 1, 2, 3$, are given by $W(u,v) = F(u)$, $W(u,v) = F(u) + \frac{\beta}{2}|v|^2$ (vector analog of (3)), $W(u,v) = F(u) + \frac{G(u)}{2}|v|^2$ (vector analog of (7)), where $F : \mathbb{R}^m \to [0,\infty)$ is a multiple well potential such that $\liminf_{|u|\to\infty} F(u) > 0$, $G : \mathbb{R}^m \to [0,\infty)$, and $\beta > 0$. In particular, our results apply to the system

$$\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} + \nabla F(u) = 0,\ u : \mathbb{R} \to \mathbb{R}^m, \tag{8}$$

to the vector Extended Fisher–Kolmogorov equation

$$\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} - \beta u'' + \nabla F(u) = 0,\ u : \mathbb{R} \to \mathbb{R}^m,\ \beta > 0, \tag{9}$$

or to

$$\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} - G(u)u'' + \frac{\nabla G(u)}{2}|u'|^2 + \nabla F(u) - (\nabla G(u)\cdot u')u' = 0,\ u : \mathbb{R} \to \mathbb{R}^m. \tag{10}$$

Let $q \in \big(0, \frac{d(A^-,A^+)}{2}\big)$, be such that

$$\{u \in \mathbb{R}^m : d(u, A^-) \leq q\} \subset \Omega,\ \text{and}\ \{u \in \mathbb{R}^m : d(u, A^+) \leq q\} \cap \overline{\Omega} = \emptyset.$$

We define the class $\mathcal{A}$ by:

$$\mathcal{A} = \left\{ u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m) : \begin{array}{l} d(u(x), A^-) \leq q,\ \text{for } x \leq x_u^-, \\ d(u(x), A^+) \leq q,\ \text{for } x \geq x_u^+, \end{array} \text{for some } x_u^- < x_u^+ \right\},$$

where d stands for the Euclidean distance. Note that no limitation is imposed on the numbers $x_u^- < x_u^+$ that may largely depend on u. Our main theorem establishes the existence of a connecting minimizer in the class $\mathcal{A}$:

Theorem 1.1. *Assume W satisfies $\mathbf{H}_i$, $i = 1, 2, 3$. Then $J_{\mathbb{R}}(u)$ admits a minimizer $\bar{u} \in \mathcal{A}$:*

$$J_{\mathbb{R}}(\bar{u}) = \min_{u\in\mathcal{A}} J_{\mathbb{R}}(u) < +\infty.$$

Moreover it results that[4]

(i) $\bar{u} \in C^4(\mathbb{R};\mathbb{R}^m)$ *solves* (1)
(ii) $\lim_{x\to\pm\infty} d(\bar{u}(x), A^\pm) = 0$,
(iii) $\lim_{x\to\pm\infty} (\bar{u}'(x), \bar{u}''(x), \bar{u}'''(x)) = (0, 0, 0)$,
(iv) $H := \frac{1}{2}|\bar{u}''|^2 - W(\bar{u}, \bar{u}') + W_v(\bar{u}, \bar{u}') \cdot \bar{u}' - \bar{u}''' \cdot \bar{u}' \equiv 0$, $\forall x \in \mathbb{R}$.

An immediate consequence of Theorem 1.1 is

[4] The existence of a minimizer $\bar{u}$ satisfying (ii) is ensured provided that W is continuous (cf. the proof in Section 2). On the other hand, the C^1 smoothness of W and W_u is required to establish properties (i), (iii) and (iv).

Corollary 1.2. *Assume that $A = \{a_1, \ldots, a_N\}$ for some $N \geq 2$, and given $a^- \in A$, set $A^- = \{a^-\}$ and $A^+ = A \setminus \{a^-\}$. Then under the assumptions of Theorem 1.1, there exists $a^+ \in A^+$ such that the minimizer $\bar{u}$ satisfies $\lim_{x\to\pm\infty} \bar{u}(x) = a^\pm$.*

By construction, the minimizer $\bar{u}$ of Theorem 1.1 is a minimal solution of (1), in the sense that

$$J_{\operatorname{supp}\phi}(\bar{u}) \leq J_{\operatorname{supp}\phi}(\bar{u} + \phi)$$

for all $\phi \in C_0^\infty(\mathbb{R}; \mathbb{R}^m)$. This notion of minimality is standard for many problems in which the energy of a localized solution is actually infinite due to non compactness of the domain. The *Hamiltonian* H introduced in property (iv) of Theorem 1.1, is a constant function for every solution of (1). In the case of system $u'' = \nabla W(u)$, we have $H = \frac{1}{2}|u'|^2 - W(u)$, and every heteroclinic orbit satisfies the equipartition relation $H = 0 \Leftrightarrow \frac{1}{2}|u'|^2 = W(u)$. We also point out that in the general set-up of Theorem 1.1, the minimizer $\bar{u}$ is a heteroclinic orbit only in a weak sense, since $\bar{u}$ approaches the sets $A^\pm$ at $\pm\infty$, but the limits of $\bar{u}$ at $\pm\infty$ may not exist. In Section 3, we will study the asymptotic convergence of $\bar{u}$, and establish an exponential estimate under a convexity assumption on W in a neighborhood of the smooth orientable surfaces $A^\pm$. From this estimate, it follows that the limits of $\bar{u}$ exist at $\pm\infty$. As a consequence, in many standard situations, the orbit of $\bar{u}$ actually connects two points $a^\pm \in A^\pm$.

The next Section contains the proof of Theorem 1.1. In contrast with [1], we avoid utilizing comparison arguments, since this method applied to higher order problems requires a lot of calculation. Indeed, to modify $W^{2,2}$ Sobolev maps, we also have to ensure the continuity of the first derivatives. Two ideas in Lemma 2.4 are crucial in the proof of Theorem 1.1. Firstly, the fact that a finite energy orbit may travel from A^- to A^+ and back, only a finite number of times in view of $\mathbf{H}_2$. Secondly, an inductive argument to consider appropriate translations of the minimizing sequence, and fix the loss of compactness issue due to the translation invariance of (1).

2. Proof of Theorem 1.1

We first establish the following Lemmas:

Lemma 2.1. *There exists $u_0 \in \mathcal{A}$ satisfying*

$$J_{\mathbb{R}}(u_0) < +\infty. \tag{11}$$

Proof. Indeed, let $a^\pm \in A^\pm$ be such that $d(a^-, a^+) = d(A^-, A^+)$. We define

$$u_0(x) = \begin{cases} a^-, & \text{for } x \leq 0, \\ a^- + (2x^2 - x^4)(a^+ - a^-), & \text{for } 0 \leq x \leq 1, \\ a^+, & \text{for } x \geq 1, \end{cases}$$

which clearly belongs to $\mathcal{A}$ and satisfies (11). □

From (11) it follows that

$$\inf_{u\in\mathcal{A}} J_{\mathbb{R}}(u) = \inf_{u\in\mathcal{A}_b} J_{\mathbb{R}}(u) < +\infty,$$

where

$$\mathcal{A}_b = \{u \in \mathcal{A} : \ J_{\mathbb{R}}(u) \leq J_{\mathbb{R}}(u_0)\}.$$

Lemma 2.2. *The maps $u \in \mathcal{A}_b$ and their first derivatives are uniformly bounded. In addition, the derivatives u' of the maps $u \in \mathcal{A}_b$ are equicontinuous.*

Proof. We first notice that the first derivative of a map $u \in \mathcal{A}_b$ is Hölder continuous, since by the Cauchy–Schwarz inequality we have

$$|u'(y) - u'(x)| \leq \left(\int_x^y |u''|^2\right)^{1/2} \sqrt{y-x} \leq \sqrt{2J_{\mathbb{R}}(u_0)}\sqrt{y-x},\ \forall x < y. \tag{12}$$

This proves that the derivatives u' of the maps $u \in \mathcal{A}_b$ are equicontinuous.

Next, we establish the uniform boundedness of the maps $u \in \mathcal{A}_b$. Let $R > 0$ be large enough and such that $d(u, A^- \cup A^+) \leq q$ implies that $|u| < R$. According to Hypothesis $\mathbf{H}_3$, we can find a constant $w_R > 0$ such that $W(u, v) \geq w_R$, for every $u \in \mathbb{R}^m$ such that $|u| \geq R$, and for every $v \in \mathbb{R}^m$. It follows that for every map $u \in \mathcal{A}_b$ we have $w_R \mathcal{L}^1(\{x \in \mathbb{R} : |u(x)| \geq R\}) \leq \int_{\mathbb{R}} W(u) \leq J_{\mathbb{R}}(u_0)$, where $\mathcal{L}^1$ denotes the one dimensional Lebesgue measure. As a consequence, if u takes a value $u(x_2) = L\nu$ with $L > R$ and ν a unit vector, we can find an interval $x_1 < x_2$ such that $|u(x_1)| = R$ and $|u(x)| \geq R$, $\forall t \in [x_1, x_2]$. Then, we have $L - R \leq \int_{x_1}^{x_2} u'(x)\cdot\nu \, \mathrm{d}x$, and this implies the existence of $y_1 \in [x_1, x_2]$ such that $u'(y_1)\cdot\nu \geq \frac{(L-R)}{x_2 - x_1} \geq \frac{(L-R)w_R}{J_{\mathbb{R}}(u_0)}$. Similarly, we can find $x_3 > x_2$ such that $|u(x_3)| = R$ and $|u(x)| \geq R$, $\forall t \in [x_2, x_3]$. As previously, there exists $y_3 \in [x_2, x_3]$ such that $u'(y_3) \cdot \nu \leq -\frac{(L-R)w_R}{J_{\mathbb{R}}(u_0)}$, and by construction $y_3 - y_1 \leq \frac{J_{\mathbb{R}}(u_0)}{w_R}$. Finally in view of (12) we obtain

$$\frac{2(L-R)w_R}{J_{\mathbb{R}}(u_0)} \leq |u'(y_3) - u'(y_1)| \leq \sqrt{2J_{\mathbb{R}}(u_0)}\sqrt{y_3 - y_1} \leq J_{\mathbb{R}}(u_0)\sqrt{\frac{2}{w_R}},$$

and deduce that $L \leq M := R + \frac{1}{\sqrt{2}} \frac{(J_{\mathbb{R}}(u_0))^2}{w_R^{3/2}}$, which proves the uniform bound for $u \in \mathcal{A}_b$.

Now, suppose that $u'(x_0) = \Lambda\nu$ with $\Lambda > \sqrt{2J_{\mathbb{R}}(u_0)}$ and ν a unit vector. Utilizing again (12) we have $u'(x) \cdot \nu \geq \Lambda - \sqrt{2J_{\mathbb{R}}(u_0)}$ for $x \in [x_0 - 1, x_0 + 1]$. In particular since $\|u\|_{L^\infty} \leq M$, we conclude that $2M \geq \int_{x_0-1}^{x_0+1} (u'(x) \cdot \nu)\mathrm{d}x \geq 2(\Lambda - \sqrt{2J_{\mathbb{R}}(u_0)})$ which implies that $\Lambda \leq M + \sqrt{2J_{\mathbb{R}}(u_0)}$. This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let $u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^m)$ be such that $J_{\mathbb{R}}(u) \leq J_{\mathbb{R}}(u_0)$, and u as well as u' are bounded, and uniformly continuous. Then,*

$$\lim_{x\to\pm\infty} d(u(x), A^- \cup A^+) = 0, \ \textit{and} \ \lim_{x\to\pm\infty} u'(x) = 0. \tag{13}$$

Proof. We first assume by contradiction that $\lim_{x\to\pm\infty} u'(x) = 0$ does not hold. Without loss of generality, we consider a sequence $\{x_k\}$ such that $\lim_{k\to\infty} x_k = +\infty$, and $\lim_{k\to\infty} u'(x_k) = \lambda\nu$, with $\lambda \neq 0$, and ν a unit vector. Let k_0 be large enough, such that $u'(x_k) \cdot \nu \geq \frac{3\lambda}{4}$ for every $k \geq k_0$, and let $I_k = [a_k, b_k]$ be the largest interval containing x_k and such that $u' \cdot \nu \geq \lambda/2$ holds on I_k. Since $\|u\|_{L^\infty} \leq M$, it is clear that $\mathcal{L}^1(I_k) \leq \frac{4M}{\lambda}$, where $\mathcal{L}^1$ denotes the one dimensional Lebesgue measure. Moreover, we have $u'(a_k)\cdot\nu = \lambda/2$. Applying (12) in the interval $[a_k, x_k]$, it follows that $\frac{\lambda^3}{4^3 M} \leq \int_{I_k} |u''|^2$. Since, by passing to a subsequence if necessary, we can assume that the intervals I_k are disjoint, this contradicts $J_{\mathbb{R}}(u) \leq J_{\mathbb{R}}(u_0)$.

Next, we assume by contradiction that $\lim_{x\to\pm\infty} d(u(x), A^- \cup A^+) = 0$ does not hold. Without loss of generality, we consider a sequence $\{x_k\}$ such that $\lim_{k\to\infty} x_k = +\infty$, $\lim_{k\to\infty} u(x_k) = l \notin A^- \cup A^+$, and $\lim_{k\to\infty} u'(x_k) = 0$. Since u as well as u' are bounded and uniformly continuous, the function $x \to W(u(x), u'(x))$ is also uniformly continuous. In view of $\mathbf{H}_1$, there exists $\delta > 0$ independent of k such that $W(u(x), u'(x)) \geq W(l, 0)/2 > 0$, for every $x \in [x_k - \delta, x_k + \delta]$, and $k \geq k_0$ large enough. In particular we have $J_{[x_k-\delta, x_k+\delta]}(u) \geq \delta W(l, 0)$, for $k = k_0, k_0 + 1, \ldots$. Since, by passing to a subsequence if necessary, we can assume that the intervals $[x_k - \delta, x_k + \delta]$ are disjoint, we reach again a contradiction. □

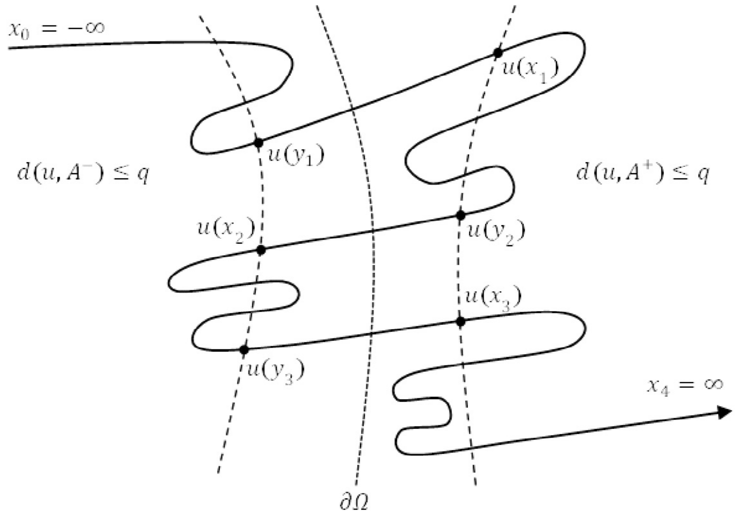

Fig. 1. The sequence $-\infty = x_0 < y_1 < x_1 < y_2 < x_2 < \cdots < x_{2N} = \infty$, $(N = 2)$.

Lemma 2.4. *There exists* $\bar{u} \in \mathcal{A}_b$ *satisfying* $J_{\mathbb{R}}(\bar{u}) = \min_{u \in \mathcal{A}_b} J_{\mathbb{R}}(u) < +\infty$, *and property (ii) of Theorem 1.1.*

Proof. We consider a sequence $u_k \in \mathcal{A}_b$ such that $\lim_{k\to\infty} J(u_k) = \inf_{u \in \mathcal{A}_b} J_{\mathbb{R}}(u)$. For every k we define the sequence

$$-\infty < x_1(k) < x_2(k) < \cdots < x_{2N_k-1}(k) < x_{2N_k}(k) = \infty$$

by induction:

- $x_1(k) = \sup\{t \in \mathbb{R} : d(u_k(s), A^+) \geq q, \forall s \leq t\} < \infty$,
- $x_{2i}(k) = \sup\{t \in \mathbb{R} : d(u_k(s), A^-) \geq q, \forall s \in [x_{2i-1}(k), t]\} \leq \infty$,
- $x_{2i+1}(k) = \sup\{t \in \mathbb{R} : d(u_k(s), A^+) \geq q, \forall s \in [x_{2i}(k), t]\} < \infty$, if $x_{2i}(k) < \infty$,

where $i = 1, \ldots$. In addition, we set

- $y_{2i-1}(k) = \sup\{t \leq x_{2i-1}(k) : d(u_k(t), A^-) \leq q\}$,
- $y_{2i}(k) = \sup\{t \leq x_{2i}(k) : d(u_k(t), A^+) \leq q\}$, if $x_{2i}(k) < \infty$ (see Fig. 1).

By Lemma 2.2, we have the uniform bounds $M := \sup_k \|u_k\|_{L^\infty} < \infty$, and $\Lambda := \sup_k \|u_k'\|_{L^\infty} < \infty$. Let $\delta > 0$ be such that

- $B_\delta(z) \cap \{u \in \mathbb{R}^m : d(u, A^-) \leq q\} = \emptyset$, and $B_\delta(z) \cap \{u \in \mathbb{R}^m : d(u, A^+) \leq q\} = \emptyset$, $\forall z \in \partial\Omega \cap B_M$,
- $W(u, v) > 0$ holds on $\{(u, v) \in B_M \times B_\Lambda : d(u, \partial\Omega \cap B_M) \leq \delta\}$ (cf. Hypothesis $\mathbf{H}_2$),

where $B_R(z) \subset \mathbb{R}^m$ denotes the closed ball of radius R centered at $z \in \mathbb{R}^m$, and B_R the closed ball of radius R centered at the origin.

Next, we notice that in every interval $[y_j(k), x_j(k)]$ $(j = 1, \ldots, 2N_k - 1)$, there exists $z_j(k) \in [y_j(k), x_j(k)]$ such that $u_k(z_j(k)) \in \partial\Omega$. Let $I_j(k) = [a_j(k), b_j(k)]$ be the largest interval containing $z_j(k)$, and such that $|u_k(x) - u_k(z_j(k))| \leq \delta$, $\forall x \in I_j(k)$. Since $|u_k(a_j(k)) - u_k(z_j(k))| = \delta$, and $|u_k(b_j(k)) - u_k(z_j(k))| = \delta$, it is clear that

$$2\delta \leq \int_{a_j(k)}^{b_j(k)} |u_k'| \leq \Lambda(b_j(k) - a_j(k)),$$

and

$$\int_{a_j(k)}^{b_j(k)} W(u_k, u_k') \geq w_\delta (b_j(k) - a_j(k)) \geq w_\delta \frac{2\delta}{\Lambda},$$

where $w_\delta := \inf\{W(u, v) : d(u, \partial\Omega \cap B_M) \leq \delta, |u| \leq M, |v| \leq \Lambda\} > 0$. Since the intervals $[a_j(k), b_j(k)] \subset [y_j(k), x_j(k)]$ are disjoint for every $j = 1, \ldots, 2N_k - 1$, it follows that

$$(2N_k - 1) w_\delta \frac{2\delta}{\Lambda} \leq \int_{\mathbb{R}} W(u_k, u_k') \leq J_{\mathbb{R}}(u_0),$$

and thus the integers N_k are uniformly bounded. By passing to a subsequence, we may assume that N_k is a constant integer $N \geq 1$.

Our next claim is that up to subsequence, there exist an integer i_0 ($1 \leq i_0 \leq N$) and an integer j_0 ($i_0 \leq j_0 \leq N$) such that

- (a) the sequence $x_{2j_0-1}(k) - x_{2i_0-1}(k)$ is bounded,
- (b) $\lim_{k\to\infty}(x_{2i_0-1}(k) - x_{2i_0-2}(k)) = \infty$,
- (c) $\lim_{k\to\infty}(x_{2j_0}(k) - x_{2j_0-1}(k)) = \infty$,

where for convenience we have set $x_0(k) := -\infty$.

Indeed, we are going to prove by induction on $N \geq 1$, that given $2N + 1$ sequences $-\infty \leq x_0(k) < x_1(k) < \cdots < x_{2N}(k) \leq \infty$, such that $\lim_{k\to\infty}(x_1(k) - x_0(k)) = \infty$, and $\lim_{k\to\infty}(x_{2N}(k) - x_{2N-1}(k)) = \infty$, then up to subsequence the properties (a), (b), and (c) above hold, for two fixed indices $1 \leq i_0 \leq j_0 \leq N$. When $N = 1$, the assumption holds by taking $i_0 = j_0 = 1$. Assume now that $N > 1$, and let $l \geq 1$ be the largest integer such that the sequence $x_l(k) - x_1(k)$ is bounded. Note that $l < 2N$. If l is odd, we are done, since the sequence $x_{l+1}(k) - x_l(k)$ is unbounded, and thus we can extract a subsequence $\{n_k\}$ such that $\lim_{k\to\infty}(x_{l+1}(n_k) - x_l(n_k)) = \infty$. Otherwise $l = 2m$ (with $1 \leq m < N$), and the sequence $x_{2m+1}(k) - x_{2m}(k)$ is unbounded. We extract a subsequence $\{n_k\}$ such that $\lim_{k\to\infty}(x_{2m+1}(n_k) - x_{2m}(n_k)) = \infty$. Then, we apply the inductive statement with $N' = N - m$, to the $2N' + 1$ sequences $x_{2m}(n_k) < x_{2m+1}(n_k) < \cdots < x_{2N}(n_k)$.

At this stage, we consider appropriate translations of the sequence $\{u_k\}$, by setting $\bar{u}_k(x) = u_k(x - x_{2i_0-1}(k))$. It is obvious that $\{\bar{u}_k\}$ is still a minimizing sequence. In view of Lemma 2.2 we obtain by the theorem of Ascoli via a diagonal argument that $\lim_{k\to\infty}\bar{u}_k = \bar{u}$ in C^1_{loc} (up to subsequence). On the other hand, since $\int_{\mathbb{R}} |\bar{u}_k''|^2 \leq 2J_{\mathbb{R}}(u_0)$ we deduce that $\bar{u}_k'' \rightharpoonup \bar{v}$ weakly in $L^2(\mathbb{R}; \mathbb{R}^m)$ (up to subsequence). One can check that actually $\bar{u} \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^m)$, and $\bar{u}'' = \bar{v}$. Finally, we have by lower semicontinuity

$$\int_{\mathbb{R}} |\bar{u}''|^2 \leq \liminf_{k\to\infty} \int_{\mathbb{R}} |\bar{u}_k''|^2, \tag{14}$$

and by Fatou's Lemma

$$\int_{\mathbb{R}} W(\bar{u}, \bar{u}') \leq \liminf_{k\to\infty} \int_{\mathbb{R}} W(\bar{u}_k, \bar{u}_k'). \tag{15}$$

It follows from (14) and (15) that $J_{\mathbb{R}}(\bar{u}) \leq \inf_{u\in\mathcal{A}_b} J_{\mathbb{R}}(u) < +\infty$. To complete the proof it remains to show that $\bar{u} \in \mathcal{A}$. Indeed, in the interval $[x_{2i_0-2}(k), x_{2i_0-1}(k)]$ we have $d(u_k(x), A^+) \geq q$, thus since $\lim_{k\to\infty}(x_{2i_0-1}(k) - x_{2i_0-2}(k)) = \infty$, we deduce that $d(\bar{u}(x), A^+) \geq q$, for $x \leq 0$. Similarly, in the interval $[x_{2j_0-1}(k), x_{2j_0}(k)]$ we have $d(u_k(x), A^-) \geq q$, thus since $\lim_{k\to\infty}(x_{2j_0}(k) - x_{2j_0-1}(k)) = \infty$, while the sequence $x_{2j_0-1}(k) - x_{2i_0-1}(k)$ is bounded, it follows that $d(\bar{u}(x), A^-) \geq q$, in a neighborhood of $+\infty$. To conclude, Lemma 2.3 applied to $\bar{u}$, implies that $\lim_{x\to\pm\infty} d(\bar{u}(x), A^\pm) = 0$, and thus $\bar{u} \in \mathcal{A}$. □

Now, we complete the proof of Theorem 1.1. By definition of the class $\mathcal{A}$, for every $\phi \in C_0^\infty(\mathbb{R}; \mathbb{R}^m)$, we have $\bar{u} + \phi \in \mathcal{A}$. Thus, the minimizer $\bar{u}$ satisfies the Euler–Lagrange equation:

$$\int_{\mathbb{R}} \Big(\bar{u}'' \cdot \phi'' + W_u(\bar{u}, \bar{u}') \cdot \phi + W_v(\bar{u}, \bar{u}') \cdot \phi'\Big) = 0, \ \forall \phi \in C_0^\infty(\mathbb{R}; \mathbb{R}^m). \tag{16}$$

This is the weak formulation of (1). Since $W \in C^2(\mathbb{R}^m \times \mathbb{R}^m; \mathbb{R})$, it follows that $\bar{u} \in C^4(\mathbb{R}; \mathbb{R}^m)$, and $\bar{u}$ is a classical solution of system (1).

Next we establish property (iii). The limit $\lim_{x\to\pm\infty} \bar{u}'(x) = 0$, is a consequence of Lemma 2.3. To see that $\lim_{x\to\pm\infty} \bar{u}''(x) = 0$, we recall the interpolation inequality

$$\int_a^{a+h} |v'|^2 \leq C\Big(\int_a^{a+h} |v|^2 + \int_a^{a+h} |v''|^2\Big), \ \forall v \in W^{2,2}([a, a+h]; \mathbb{R}^m), \tag{17}$$

that holds for a constant C independent of $a \in \mathbb{R}$, and $h \in [1, \infty)$. In view of (1), it is clear that

$$\int_a^{a+1} \Big|\frac{\mathrm{d}^4 \bar{u}}{\mathrm{d}x^4}\Big|^2 \leq C_1 + C_2 \int_a^{a+1} |\bar{u}''|^2 \leq C_1 + 2C_2 J_{\mathbb{R}}(\bar{u}) \leq C_3, \tag{18}$$

where C_i are constants independent of a. Moreover, applying (17) to $v = \bar{u}''$, we can find another constant C_4 independent of a, such that

$$\int_a^{a+1} |\bar{u}'''|^2 \leq C_4, \tag{19}$$

and as a consequence $\bar{u}''$ is uniformly continuous (see the proof of Lemma 2.2). Since $\int_{\mathbb{R}} |\bar{u}''|^2 < \infty$ it follows that $\lim_{x\to\pm\infty} \bar{u}''(x) = 0$. Finally, $\mathbf{H}_1$ and (1) imply that $\lim_{x\to\pm\infty} \frac{\mathrm{d}^4\bar{u}}{\mathrm{d}x^4}(x) = 0$. thus, we also have $\lim_{x\to\pm\infty} \bar{u}'''(x) = 0$ (cf. [7, §3.4 p. 37]).

To prove property (iv), consider an arbitrary solution u of (1). By integrating the inner product of (1) by u', one can see that the Hamiltonian $H := \frac{1}{2}|u''|^2 - W(u, u') + W_v(u, u') \cdot u' - u''' \cdot u'$ is constant along solutions. In the case of the minimizer $\bar{u}$, the Hamiltonian is zero by properties (ii) and (iii), and by Hypothesis $\mathbf{H}_1$.

3. Asymptotic convergence of the minimizer $\bar{u}$

A natural question arises in the case where the set $A^{\pm}$ defined in $\mathbf{H}_1$ are manifolds or union of manifolds: does the minimizer $\bar{u}$ converge to a point of A^+ (respectively A^-) at $\pm\infty$? Before answering this question, we are going to establish by a variational method the following exponential estimate:

Proposition 3.1. *Assume that $A^- \subset \mathbb{R}^m$ is a C^2 compact orientable surface with unit normal $\mathbf{n}$, and that W satisfies*

$$\frac{\mathrm{d}^2 W}{\mathrm{d}s^2}(a + s\mathbf{n}, s\nu)\Big|_{s=0} > 0, \ \forall a \in A^-, \ \forall \nu \in \mathbb{R}^m \text{ such that } |\nu| = 1. \tag{20}$$

Then, the minimizer $\bar{u}$ constructed in Theorem 1.1 satisfies $d(\bar{u}(x), A^-) \leq Ke^{kx}$, and $|\bar{u}'(x)| \leq Ke^{kx}$, $\forall x \leq 0$, for some constants $K, k > 0$.

Proof. In view of (20), there exists $\lambda > 0$ small enough, such that $\mathcal{U} := \{u \in \mathbb{R}^m : d(u, A^-) < \lambda\}$ is a tubular neighborhood of A^- (cf [6]), and moreover

$$m(d^2(u, A^-) + |v|^2) \leq W(u, v) \leq M(d^2(u, A^-) + |v|^2), \ \forall u \in \mathcal{U}, \forall v \in \mathbb{R}^m : |v| < \lambda, \tag{21}$$

for some constants $0 < m < M$. Let x_0 be such that $d(\bar{u}(x), A^-) < \lambda/8$, and $|\bar{u}'(x)| < \lambda/4$, $\forall x \leq x_0$. For fixed $x \leq x_0$, we set $\phi(x) := \bar{u}(x) - \frac{1}{2}\bar{u}'(x)$. One can see that $\phi(x) \in \mathcal{U}$, since actually $d(\phi(x), A^-) \leq d(\bar{u}(x), A^-) + \frac{1}{2}|\bar{u}'(x)| < \lambda/4$. We also introduce the point $a(x) \in A^-$ such that $d(\phi(x), a(x)) = d(\phi(x), A^-)$. Next we define the comparison map

$$z(t) = \begin{cases} \bar{u}(x) + \big((t-x) + \dfrac{(t-x)^2}{2}\big)\bar{u}'(x) & \text{for } x-1 \leq t \leq x, \\ \phi(x) + (2(t-x+1)^2 - (t-x+1)^4)(a(x) - \phi(x)) & \text{for } x-2 \leq t \leq x-1, \\ a(x) & \text{for } t \leq x-2. \end{cases} \tag{22}$$

An easy computation shows that

(a) $z \in W^{2,2}_{\text{loc}}((-\infty, x]; \mathbb{R}^m)$,
(b) $z(x) = \bar{u}(x)$, and $z'(x) = \bar{u}'(x)$,
(c) $d(z(t), A^-) \leq d(\bar{u}(x), A^-) + \frac{1}{2}|\bar{u}'(x)| < \lambda/4$, $\forall t \leq x$,
(d) $|z'(t)| \leq 4d(\bar{u}(x), A^-) + 2|\bar{u}'(x)| < \lambda$, $\forall t \leq x$,
(e) $J_{(-\infty,x]}(z) \leq C(d^2(\bar{u}(x), A^-) + |\bar{u}'(x)|^2)$, for a constant $C > 0$ independent of x.

At this stage, we set $\theta(x) := \int_{-\infty}^{x}(d^2(\bar{u}(t), A^-)+|\bar{u}'(t)|^2)\mathrm{d}t$, and it is clear that $\theta'(x) = d^2(\bar{u}(x), A^-)+|\bar{u}'(x)|^2$. The variational characterization of $\bar{u}$, (21), and (e) above, imply that

$$m\theta(x) \leq \int_{-\infty}^{x} W(\bar{u}(t), \bar{u}'(t))\mathrm{d}t \leq J_{(-\infty,x]}(\bar{u}) \leq J_{(-\infty,x]}(z) \leq C\theta'(x). \tag{23}$$

After an integration of inequality (23), we obtain that $\theta(x) \leq \theta(x_0)e^{c(x-x_0)}$, for every $x \leq x_0$, and for some constant $c > 0$. Since the functions $x \to d^2(\bar{u}(x), A^-)$, and $x \to |\bar{u}'(x)|^2$ are Lipschitz continuous (cf. Theorem 1.1), the statement of Proposition 3.1 follows from

Lemma 3.2. *Let $f : (-\infty, x_0] \to [0, \infty)$ be a function such that*

- $|f(x) - f(y)| \leq M|x - y|$, $\forall x, y \leq x_0$,
- $\int_{-\infty}^{x} f(t)\mathrm{d}t \leq Ce^{cx}$, $\forall x \leq x_0$,

where M, c and C are positive constants. Then, $f(x) \leq 2\sqrt{MC}e^{\frac{c}{2}x}$, $\forall x \leq x_0$.

Proof. Let $x \leq x_0$ be fixed and let $\lambda := f(x)$. For $t \in [x - \frac{\lambda}{2M}, x]$, we have $f(t) \geq f(x) - M|t - x| \geq \frac{\lambda}{2}$. Thus,

$$\frac{\lambda^2}{4M} \leq \int_{x-\frac{\lambda}{2M}}^{x} f(t)\mathrm{d}t \leq Ce^{cx} \Rightarrow \lambda = f(x) \leq 2\sqrt{MC}e^{\frac{c}{2}x}. \quad \square \quad \square$$

Corollary 3.3. *Under the assumptions of Proposition 3.1, there exists $l \in A^-$ such that $\bar{u}(x) \to l$, as $x \to -\infty$.*

Proof. The exponential estimates provided by Proposition 3.1 imply that $x \to |\bar{u}('x)|$ is integrable in a neighborhood of $-\infty$. As a consequence, it is easy to see that the limit of $\bar{u}$ at $-\infty$ exists and belongs to A^-. $\square$

Proposition 3.4. *Assume that $A^- = \{a^-\}$, and that W satisfies*

$$m|u|^2 \leq W(u, v) \text{ in a neighborhood of } (a^-, 0), \text{ for a constant } m > 0, \tag{24}$$

Then, the minimizer $\bar{u}$ constructed in Theorem 1.1 satisfies $|\bar{u}(x)-a^-| \leq Ke^{kx}$, and $|\bar{u}'(x)| \leq Ke^{kx}$, $\forall x \leq 0$, for some constants $K, k > 0$.

Proof. We proceed as in the proof of Proposition 3.1. For $\lambda > 0$ small enough, we have

$$m|u|^2 \leq W(u, v) \leq M(|u - a^-|^2 + |v|^2),\ \forall u \in \mathbb{R}^m : |u - a^-| < \lambda, \forall v \in \mathbb{R}^m : |v| < \lambda. \tag{25}$$

Let x_0 be such that $|\bar{u}(x) - a^-| < \lambda/8$, and $|\bar{u}'(x)| < \lambda/4$, $\forall x \leq x_0$. For fixed $x \leq x_0$, we set again $\phi(x) := \bar{u}(x) - \frac{1}{2}\bar{u}'(x)$. Next we consider the comparison map (22), where $a(x)$ is now replaced by a^-. This map z still satisfies properties (a)–(e) in Proposition 3.1. Setting $\theta(x) := \int_{-\infty}^{x}(|\bar{u}(t) - a^-|^2 + |\bar{u}'(t)|^2)\mathrm{d}t$, we obtain

$$m\int_{-\infty}^{x} |\bar{u}(t) - a^-|^2\mathrm{d}t + \frac{1}{2}\int_{-\infty}^{x} |\bar{u}''(t)|^2\mathrm{d}t \leq J_{(-\infty,x]}(\bar{u}) \leq J_{(-\infty,x]}(z) \leq C\theta'(x). \tag{26}$$

Finally, we utilize the interpolation inequality (17), to bound $\int_{-\infty}^{x} |\bar{u}'(t)|^2 \mathrm{d}t$ by a constant multiplied by the left hand-side of (26). Hence, $\theta(x) \leq c\theta'(x)$, $\forall x \leq x_0$, and for a constant $c > 0$. Then we conclude as in Proposition 3.1. □

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