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ORIGINAL PAPER

On semicoercive sweeping process with velocity constraint

Samir Adly[1] · **Ba Khiet Le**[2]

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Abstract In this note, by solving a variational inequality at each iteration, we study the existence of solutions for a class of sweeping processes with velocity in the moving set, originally introduced in a recent paper (Adly et al. in Math Program Ser B 148(1):5–47, 2014). Our aim is to improve Adly et al. (2014, Theorem 5.1) to allow possibly unbounded moving sets. The theoretical result is supported by some examples in nonregular electrical circuits.

Keywords Sweeping processes · Unbounded moving sets · Semicoercive variational inequalities

1 Introduction

Sweeping processes were introduced and thoroughly studied in the seventies by Jean-Jacques Moreau in his serial seminars at the University of Montpellier [9,10] to model quasi-static evolution in elastoplasticity, contact dynamics, friction dynamics, granular material, which was the original motivation. Nowadays, abundant applications of sweeping processes can be also found in nonsmooth mechanics, convex optimization, mathematical economics, dynamic networks, switched electrical circuits, modeling of crowd motion ... (see, e.g., [1,2,4,7]). The simplest sweeping process has the form of a differential inclusion defined as follows

✉ Samir Adly
samir.adly@unilim.fr

1 XLIM UMR-CNRS 7252, Université de Limoges, 87060 Limoges, France

2 Centro de Modelamiento Matemático (CMM), Universidad de Chile, Santiago, Chile

$$\begin{cases} \dot{u}(t) \in -N_{C(t)}(u(t)) \text{ a.e. } t \in [0, T], \\ u(0) = u_0 \in C(0), \end{cases} \tag{1}$$

where the right-hand side is the normal cone of a moving closed convex set $C(t)$ at the state $u(t)$ in a Hilbert space H. Implicitly the constraint $u(t) \in C(t)$ is satisfied for almost every $t \in [0, T]$. The following interpretation [9,10] justifies the name "sweeping process" and arises for the way how the point $u(t)$ is "*sweept*" by the set $C(t)$: as long as the point $u(t)$ lies in the interior of $C(t)$ (assumed to be nonempty), the normal cone $N_{C(t)}(u(t))$ is reduced to zero, so $u(t)$ does not move. When the point is "*caught up*" by the boundary of $C(t)$ it moves, subject to an inward normal direction, as if pushed by this boundary. There exist in the literature many variants of the sweeping process. Recently in [2], the authors proposed a new variant with velocity in the moving set and gave various examples in electrical circuits which can be recast into the following form

$$(\mathcal{S}) \begin{cases} A_1\dot{u}(t) + A_0 u(t) - f(t) \in -N_{C(t)}(\dot{u}(t)) \text{ a.e. } t \in [0, T], \\ u(0) = u_0, \end{cases}$$

where $A_1, A_0 : H \to H$ are two bounded symmetric linear semi-definite operators and $f : [0, T] \to H$ is a continuous mapping. Let us mention that the system $(\mathcal{S})$ is equivalent to the following evolution variational inequality which consists to find $u : [0, T] \longrightarrow H$, with $u(0) = u_0 \in H$ such that $\dot{u}(t) \in C(t)$ a.e. $t \in [0, T]$ and

$$a_0(u(t), v - \dot{u}(t)) + a_1(\dot{u}(t), v - \dot{u}(t)) \geq \langle f(t), v - \dot{u}(t)\rangle, \quad \text{for all } v \in C(t). \tag{2}$$

Here $a_0(\cdot,\cdot)$ and $a_1(\cdot,\cdot)$ are real bilinear, bounded and symmetric forms associated to the operators A_1 and A_0 respectively, $f \in W^{1,2}([0, T], H)$.
The evolution variational inequalities of type (2) are widely used in applied mathematics, unilateral mechanics and various fields of sciences and engineering such as for instance traffic networks, energy market, transportation, elastoplasticity etc …(see e.g. [5]). In nonsmooth mechanics, the moving set is usually expressed in the form of inequality as follows

$$C(t) := \{x \in H : g_i(t, x) \leq 0, i = 1, 2, \ldots, m\} \tag{3}$$

for some regular convex functions $g_i : [0, T] \times \mathbb{R}^n \to \mathbb{R}$, $(i = 1, 2, \ldots, m)$. Traditionally the above inequalities correspond to the so-called unilateral constraint (that is, one-sided constraint). An interesting case also in applications is given when the set $C(t)$ is polyhedral, i.e.

$$C(t) = \{x \in \mathbb{R}^n : \langle a_i(t), x\rangle \leq b_i(t),\ i = 1, 2, \ldots, m\}, \tag{4}$$

with $a_i(\cdot)$ and $b_i(\cdot)$ are some given functions. In [2], the moving set is supposed to move in an continuous way with bounded initial position $C(0)$. Consequently, it is easy to see that $C(\cdot)$ is bounded on $[0, T]$. The boundedness of the moving set limits the applications for the new variant. However, without additional assumption, the unboundedness of $C(\cdot)$ can make the differential inclusion ($\mathcal{S}$) having no absolutely continuous solutions, even in one-dimensional space, as showed in the beginning of [2, Section 5]. In this note, we relax the boundedness condition by a constraint on the operator A_1 and $C(0)$, which covers the case considered in [2] and allows the possibly unbounded moving set.

The note is organized as follows. In Sect. 2, we recall some standard notations used in the sequel. The main result about existence of solutions for ($\mathcal{S}$) is provided in Sect. 3. We end the paper in Sect. 4 with some conclusions and perspectives.

2 Notations and preliminaries

We begin with some notations used throughout the paper. Let H be a separable Hilbert space. Denote by $\langle\cdot,\cdot\rangle$, $\|\cdot\|$ the scalar product and the corresponding norm in H. Denote by $\mathbb{B}$ the unit ball and $\mathbb{B}_r := r\mathbb{B},\ \mathbb{B}_r(x) := x + r\mathbb{B}$. The *distance* from a point s to a set C is denoted by $d(s, C)$ or $d_C(s)$ and

$$d(s, C) := \inf_{x\in C} \|s - x\|.$$

The *projection* of s onto C is the set of all points in C that are nearest to s, denoted by

$$P_C(s) := \{x \in C : \|s - x\| = d(s, C)\}.$$

The *Hausdorff distance* between two sets C_1 and C_2 is defined by

$$d_H(C_1, C_2) := \max\left\{\sup_{x_1\in C_1} d(x_1, C_2), \sup_{x_2\in C_2} d(x_2, C_1)\right\}.$$

We define the *indicator function* $i_S(\cdot)$, the *characteristic function* $\mathbf{1}_S(\cdot)$ and the *support function* $\sigma(S,\cdot)$ of a given set S as follows

$$i_S(x) := \begin{cases} 0 & \text{if } x \in S, \\ +\infty & \text{if } x \notin S, \end{cases} \quad \mathbf{1}_S(x) := \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases} \quad \text{and } \sigma(S, x) := \sup_{\xi\in S} \langle x, \xi\rangle.$$

Let $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous function. The *Legendre* $-$ *Fenchel conjugate* of φ is defined as

$$\varphi^*(x^*) := \sup_{x\in H}\{\langle x^*, x\rangle - \varphi(x)\}.$$

For $\varphi(x)$ finite, one has

$$x^* \in \partial\varphi(x) \Leftrightarrow \varphi(x) + \varphi^*(x^*) = \langle x^*, x\rangle \tag{5}$$

where $\partial\varphi(x)$ denotes the *subdifferential* of φ at x,

$$\partial\varphi(x) := \{x^* \in H : \langle x^*, y - x\rangle \leq \varphi(y) - \varphi(x),\ \forall y \in H\}. \tag{6}$$

In particular, $\partial\varphi$ is surjective if and only if φ^* is surjective. The *normal cone* of a closed convex set S is defined as follows

$$N_S(x) := \partial i_S(x) = \{x^* \in H : \langle x^*, y - x\rangle \leq 0,\ \forall y \in S\}.$$

It is easy to see that

$$x^* \in N_S(x) \Leftrightarrow \sigma(S, x^*) = \langle x^*, x\rangle \ \text{ and } \ x \in S. \tag{7}$$

Let $D : H \to H$ be a linear bounded operator. It is said to be *coercive* if there exists $c > 0$ such that

$$\langle Dx, x\rangle \geq c\|x\|^2, \quad \text{for all } x \in H.$$

It is said to be *semi-coercive* if there exists $c > 0$ such that

$$\langle Dx, x\rangle \geq c\|Qx\|^2 \ \text{ for all } x \in H,$$

where $Q = I - P_{\ker(D+D^T)}$, I is the identity operator and $P_{\ker(D+D^T)}$ is the orthogonal projection onto $\ker(D + D^T)$. It is said to be *semi-positive definite* if

$$\langle Dx, x\rangle \geq 0 \ \text{ for all } x \in H.$$

It is known that [6] if the linear bounded operator D is monotone and $\mathrm{Im}\,(D + D^T)$ is closed, then D is semi-coercive. In particular a semi-positive definite matrix in $\mathbb{R}^{n\times n}$ is semi-coercive.

3 Main results

In this section, we study the existence of solutions for $(\mathcal{S})$ by using a discretization technique. It is an improvement of [2, Theorem 5.1] to relax the boundedness of the moving set by using only, for example, the boundedness of the projection of $C(0)$ onto $\ker(A_1)$. Let us assume the following two assumptions.

Assumption 1 The set-valued mapping $C : [0, T] \rightrightarrows H$ has nonempty closed convex values and is continuous in the sense that there is some continuous function $v : [0, T] \to \mathbb{R}$ such that

$$d_H(C(s), C(t)) \leq |v(s) - v(t)|, \quad \forall s, t \in [0, T]. \tag{8}$$

Assumption 2 Let $A_1, A_0 : H \to H$ be two bounded symmetric linear semi-definite operators and $f : [0, T] \to H$ be a continuous mapping. Assume that there exist $\alpha > 0$ and $\beta > 0$ such that

$$\langle A_1 x, x\rangle \geq \alpha \|x\|^2 - \beta, \quad \forall\, x \in C(0). \tag{9}$$

Remark 1 (i) Since $C(\cdot)$ moves in a continuous way, it is equivalent to consider the inequality in (9) for all $x \in C(t),\ t \in [0, T]$. Obviously, the condition (9) holds if A_1 is coercive.

(ii) If $C(0) \subset \gamma\mathbb{B}$, then the condition (9) is satisfied with $\alpha = 1$ and $\beta = \gamma^2$. Thus Theorem 1 covers the case considered in [2, Theorem 5.1].

(iii) If the projection of $C(0)$ onto $\ker(A_1)$ is bounded by some $\gamma > 0$ and A_1 is semi-coercive then (9) also holds. Indeed, there exists $\alpha > 0$ such that for all $x \in C(0)$ one has $\langle A_1 x, x\rangle \geq \alpha\|x - P_{\ker(A_1)}x\|^2 \geq \alpha\|x\|^2 - \alpha\gamma^2$, where $P_{\ker(A_1)}x$ is the projection of x onto $\ker(A_1)$. Consequently, in finite dimensional spaces, we only need the boundedness of the projection of $C(0)$ onto $\ker(A_1)$.

The following lemmas will be useful.

Lemma 1 *Let Assumptions 1 and 2 hold. Then for each fixed $t \in [0, T]$, the set-valued mapping $x \mapsto N_{C(t)}(x) + A_1 x$ is surjective.*

Proof Note that the set-valued mapping $N_{C(t)} + A_1 = \partial(i_{C(t)} + \varphi_{A_1}) = \partial\varphi$, where $\varphi_{A_1}(x) := \frac{1}{2}\langle A_1 x, x\rangle$ and $\varphi := i_{C(t)} + \varphi_{A_1}$ is a proper convex function from H to $\mathbb{R} \cup \{+\infty\}$. Thus in order to prove the surjection of $N_{C(t)} + A_1$, it is enough to show that $\mathrm{dom}(\varphi^*) = H$. Let $x^* \in H$, by using Assumption 1 and Remark 1(i), one has

$$\begin{aligned}\varphi^*(x^*) &= \sup_{x\in C(t)} \{\langle x^*, x\rangle - \varphi_{A_1}(x)\}\\ &\leq \sup_{x\in C(t)} \left\{\langle x^*, x\rangle - \frac{\alpha}{2}x^2 + \frac{\beta}{2}\right\}\\ &\leq \frac{\|x^*\|^2}{2\alpha} + \frac{\beta}{2} < +\infty,\end{aligned}$$

which implies that $x^* \in \mathrm{dom}(\varphi^*)$. □

Lemma 2 *There exists a constant $c > 0$ which depends only on the initial data such that for all $t \in [0, T]$, we can find some $y_t \in C(t)$ satisfying that $\|y_t\| \leq c$.*

Proof Fix some $y_0 \in C(0)$. Let $y_t := \mathrm{proj}_{C(t)}(y_0)$. From Assumption 1, one has

$$\|y_t - y_0\| = \mathrm{d}(y_0; C(t)) \leq \mathrm{d}_H(C(0), C(t)) \leq |v(t) - v(0)| \leq |v(0)| + \max_{t\in[0,T]} |v(t)|.$$

By choosing $c := |v(0)| + \max_{t\in[0,T]} |v(t)| + \|y_0\|$, the conclusion follows. □

Now, we are ready to state the main result.

Theorem 1 *Let Assumptions 1 and 2 hold. Then for any initial condition, problem $(\mathcal{S})$ has at least one Lipschitz solution.*

Proof Let be given some positive integer n, define $h_n := T/n$ and $t_i^n := ih$ for $0 \leq i \leq n$. For $0 \leq i \leq n-1$ and given u_i^n, we want to find v_{i+1}^n, u_{i+1}^n such that

$$\begin{cases} A_1 v_{i+1}^n + A_0 u_i^n \in -N_{C(t_{i+1}^n)}\left(v_{i+1}^n\right) + f_i^n, \\ u_{i+1}^n = u_i^n + h_n v_{i+1}^n, \end{cases} \tag{10}$$

where $f_i^n = f(t_i^n)$. It is easy to see that the first line of (10) can be rewritten as follows

$$f_i^n - A_0 u_i^n \in \left(N_{C(t_{i+1}^n)} + A_1\right)\left(v_{i+1}^n\right). \tag{11}$$

The mapping $A_1 + N_{C(t_{i+1}^n)}$ is surjective thanks to Lemma 1. Thus, we can choose $v_{i+1}^n \in (N_{C(t_{i+1}^n)} + A_1)^{-1}(f_i^n - A_0 u_i^n)$. We have the following algorithm to construct the sequences $(u_i^n)_{i=0}^n$, $(v_i^n)_{i=0}^n$, $(f_i^n)_{i=0}^n$ which are well-defined.

Modified catching-up algorithm:

- $u_0^n = u_0$, $f_0^n = f(0)$.

  For $0 \leq i \leq n-1$:
- Find v_{i+1}^n by solving the following variational inequality

$$v_{i+1}^n \in \left(N_{C(t_{i+1}^n)} + A_1\right)^{-1}\left(f_i^n - A_0 u_i^n\right) \text{ and set } u_{i+1}^n = u_i^n + h_n v_{i+1}^n, f_{i+1}^n = f\left(t_{i+1}^n\right).$$

Next we prove that the sequences $(u_i^n)_{i=0}^n$, $(v_i^n)_{i=0}^n$ are uniformly bounded. Indeed, from (11) and the convexity of $C(t_{i+1}^n)$, one has

$$\langle A_1 v_{i+1}^n - \left(f_i^n - A_0 u_i^n\right), y - v_{i+1}^n \rangle \geq 0, \quad \forall y \in C\left(t_{i+1}^n\right). \tag{12}$$

Since we can choose some $y \in C(t_{i+1}^n)$ which is bounded by some constant depending only on initial data, the inequalities (12) and (9) imply the existence of some constant $a > 0$ such that

$$\|v_{i+1}^n\| \leq a\left(\|u_i^n\| + 1\right). \tag{13}$$

Hence,

$$\|u_{i+1}^n\| + 1 \leq \|u_i^n\| + 1 + h_n\|v_{i+1}^n\| \leq (1 + h_n a)\left(\|u_i^n\| + 1\right). \tag{14}$$

By induction, one has

$$\|u_i^n\| + 1 \leq (1 + h_n a)^n (\|u_0\| + 1) \leq e^{aT}(\|u_0\| + 1). \tag{15}$$

It follows that

$$\|u_i^n\| \leq e^{aT}(\|u_0\| + 1) - 1 \text{ and } \|v_i^n\| \leq ae^{aT}(\|u_0\| + 1) \ \forall n, i. \tag{16}$$

Consequently, the sequences $(u_i^n)_{i=0}^n$, $(v_i^n)_{i=0}^n$, (f_i^n) are uniformly bounded by some real number $M > 0$, where

$$M := \max\left\{e^{aT}(\|u_0\|+1)-1, ae^{aT}(\|u_0\|+1), \sup_{t\in[0,T]}\|f(t)\|\right\}. \tag{17}$$

We construct the sequences of functions $(u_n(\cdot))_n, (f_n(\cdot))_n : [0,T] \to H$ and $(\theta_n(\cdot))_n, (\eta_n(\cdot))_n : [0,T] \to [0,T]$ as follows: $u_n(0) = u_0$, $f_n(0) = f(0)$ and on $(t_i^n, t_{i+1}^n]$ for $0 \le i \le n-1$, we set

$$u_n(t) := u_i^n + \frac{u_{i+1}^n - u_i^n}{h_n}\left(t - t_i^n\right), \quad f_n(t) := f_i^n, \quad \theta_n(t) := t_i^n, \quad \eta_n(t) := t_{i+1}^n.$$

Then, for all $t \in (t_i^n, t_{i+1}^n)$

$$\dot{u}_n(t) = \frac{u_{i+1}^n - u_i^n}{h_n} = v_{i+1}^n \in C\left(t_{i+1}^n\right),$$

and

$$\max\left\{\sup_{t\in[0,T]}|\theta_n(t)-t|, \sup_{t\in[0,T]}|\eta_n(t)-t|\right\} \le h_n \to 0 \text{ as } n \to +\infty. \tag{18}$$

It is easy to see that the sequence of functions $(u_n(\cdot))_n$ is bounded in norm and variation. By using [8, Theorem 0.2.1], there exist some bounded variation function $u : [0,T] \to H$ and a subsequence, still denoted by $(u_n(\cdot))_n$ such that

- $u_n(t)$ converges weakly to $u(t)$ for all $t \in [0,T]$;
- $\dot{u}_n(\cdot)$ converges weakly to some $\xi(\cdot)$ in $L^2(0,T;H)$.

Clearly, $u(0) = u_0$ and $u(\cdot)$ is M-Lipschitz continuous since for all $s, t \in [0,T]$, one has

$$\|u(t) - u(s)\| \le \liminf_{n\to+\infty}\|u_n(t) - u_n(s)\| \le M|t-s|.$$

□

Consequently, $u(\cdot)$ is differentiable almost every $t \in [0,T]$. Fix $t \in [0,T]$ and let $z := u(t) - u_0 - \int_0^t \xi(s)ds$. We have

$$\begin{aligned}\left\langle z, u(t) - u_0 - \int_0^t \xi(s)ds\right\rangle &= \lim_{n\to+\infty}\left\langle z, u_n(t) - u_0 - \int_0^t \xi(s)ds\right\rangle \\ &= \lim_{n\to+\infty}\left\langle z, \int_0^t (\dot{u}_n(s) - \xi(s))ds\right\rangle \\ &= \lim_{n\to+\infty}\int_0^T \langle \mathbf{1}_{[0,t]}(s)z, \dot{u}_n(s) - \xi(s)\rangle ds = 0.\end{aligned}$$

It implies that $u(t) = u_0 + \int_0^t \xi(s)ds$ for all $t \in [0, T]$ and hence $\dot{u}(t) = \xi(t)$ for almost every $t \in [0, T]$. Next, let us prove that $\dot{u}(t) \in C(t)$ for almost every $t \in [0, T]$. Indeed, given any t such that $\dot{u}(t)$ exists, we have

$$\dot{u}_n(t) \in C(\eta_n(t)) \subset C(t) + |v(t) - v(\eta_n(t))|\mathbb{B} \subset C(t) + \omega(h_n)\mathbb{B}, \tag{19}$$

where $\omega(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ denotes the modulus of continuity of $v(\cdot)$, defined by

$$\omega(h) := \sup_{|x-y|\leq h} |v(x) - v(y)|.$$

Let

$$D := \{w \in L^2(0, T; H) : w(t) \in C(t) \;\; a.e.\; t \in [0, T]\}.$$

Then D is a closed, convex subset of $L^2(0, T; H)$. Note that $\dot{u}_n(\cdot)$ converges weakly to $\dot{u}(\cdot)$ in $L^2(0, T; H)$. In addition, from (19), it is easy to see that

$$\dot{u}_n \in D + \varepsilon_n \mathbb{B}_{L^2},$$

where $\varepsilon_n := \omega(h_n)\sqrt{T} \to 0$ as $n \to +\infty$ and $\mathbb{B}_{L^2}$ denotes the closed unit ball in $L^2(0, T; H)$. Using [7, Lemma 2], one deduces that $\dot{u} \in D$, which means that $\dot{u}(t) \in C(t)$ for almost every $t \in [0, T]$. Finally, we show that

$$A_1\dot{u}(t) + A_0 u(t) - f(t) \in -N_{C(t)}(\dot{u}(t)) \text{ a.e. } t \in [0, T]. \tag{20}$$

Indeed, from (10), one obtains for almost every $t \in [0, T]$ that

$$z_n(t) := -A_1\dot{u}_n(t) - A_0 u_n(\theta_n(t)) + f_n(t) \in N_{C(\eta_n(t))}(\dot{u}_n(t)), \tag{21}$$

which is equivalent to

$$\sigma(C(\eta_n(t)); z_n(t)) + \langle A_1\dot{u}_n(t) + A_0 u_n(\theta_n(t)) - f_n(t), \dot{u}_n(t)\rangle \leq 0.$$

Thus,

$$\int_0^T \sigma(C(\eta_n(t)); z_n(t))dt + \int_0^T \langle A_1\dot{u}_n(t) + A_0 u_n(\theta_n(t)) - f_n(t), \dot{u}_n(t)\rangle dt \leq 0. \tag{22}$$

It is easy to see that the function $x \mapsto \int_0^T \langle A_1 x(t), x(t)\rangle dt$ is weakly lower semicontinuous on $L^2(0, T; H)$ since it is convex and continuous on $L^2(0, T; H)$. Note that $\dot{u}_n$ converges weakly to $\dot{u}$ in $L^2(0, T; H)$. Therefore

$$\int_0^T \langle A_1\dot{u}(t), \dot{u}(t)\rangle dt \leq \liminf_{n\to+\infty} \int_0^T \langle A_1\dot{u}_n(t), \dot{u}_n(t)\rangle dt. \tag{23}$$

On the other hand,

$$\begin{aligned}\int_0^T \langle A_0 u(t), \dot{u}(t)\rangle dt &= \frac{1}{2}\langle A_0 u(T), u(T)\rangle - \frac{1}{2}\langle A_0 u(0), u(0)\rangle \\ &\le \liminf_{n\to+\infty}\left(\frac{1}{2}\langle A_0 u_n(T), u_n(T)\rangle - \frac{1}{2}\langle A_0 u_n(0), u_n(0)\rangle\right) \\ &\le \liminf_{n\to+\infty}\int_0^T \langle A_0 u_n(t), \dot{u}_n(t)\rangle dt,\end{aligned}$$

and

$$\left|\liminf_{n\to+\infty}\int_0^T \langle A_0 u_n(t) - A_0 u_n(\theta(t)), \dot{u}_n(t)\rangle dt\right| \le \liminf_{n\to+\infty} M^2\|A_0\| h_n T = 0.$$

Hence,

$$\int_0^T \langle A_0 u(t), \dot{u}(t)\rangle dt \le \liminf_{n\to+\infty}\int_0^T \langle A_0 u_n(\theta(t)), \dot{u}_n(t)\rangle dt. \tag{24}$$

One also has

$$\int_0^T \langle f(t), \dot{u}(t)\rangle dt = \lim_{n\to+\infty}\int_0^T \langle f_n(t), \dot{u}_n(t)\rangle dt, \tag{25}$$

since f_n converges strongly to f and $\dot{u}_n$ converges weakly to $\dot{u}$ in $L^2(0, T; H)$. From (23), (24) and (25), one obtains that

$$\begin{aligned}&\int_0^T \langle A_1\dot{u}(t) + A_0 u(t) - f(t), \dot{u}(t)\rangle dt \\ &\quad\le \liminf_{n\to+\infty}\int_0^T \langle A_1\dot{u}_n(t) + A_0 u_n(\theta_n(t)) - f_n(t), \dot{u}_n(t)\rangle dt.\end{aligned} \tag{26}$$

Let us recall that the convex mapping $x \mapsto \int_0^T \sigma(C(t), x(t))dt$ is weakly lower semicontinuous on $L^2(0, T; H)$ ([11], see also [2]) and $z_n = -A_1\dot{u}_n - A_0 u_n \circ \theta_n + f_n$ converges weakly to $z := -A_1\dot{u} - A_0 u + f$ in $L^2(0, T; H)$. Consequently, one has

$$\int_0^T \sigma(C(t), z(t))dt \le \liminf_{n\to+\infty}\int_0^T \sigma(C(t), z_n(t))dt. \tag{27}$$

Since $C(t) \subset C(\eta(t)) + |v(\eta(t)) - v(t)|\mathbb{B}$ and $\|z_n(t)\| \le (\|A_1\| + \|A_0\|)M$ for all $t \in [0, T]$, we deduce that

$$\begin{aligned}\liminf_{n\to+\infty}\int_0^T \sigma(C(t), z_n(t))dt &\le \liminf_{n\to+\infty}\int_0^T \sigma(C(\eta(t)), z_n(t))dt \\ &\quad+(\|A_1\| + \|A_0\|)M \liminf_{n\to+\infty}\int_0^T |v(\eta(t)) - v(t)|dt \\ &= \liminf_{n\to+\infty}\int_0^T \sigma(C(\eta(t)), z_n(t))dt.\end{aligned} \tag{28}$$

From (27) and (28), we obtain that

$$\int_0^T \sigma(C(t), z(t))dt \le \liminf_{n\to+\infty} \int_0^T \sigma(C(\eta(t)), z_n(t))dt. \tag{29}$$

From (22), (26) and (29), one has

$$\int_0^T \sigma(C(t), z(t))dt + \int_0^T \langle A_1\dot{u}(t) + A_0u(t) - f_n(t), \dot{u}(t)\rangle dt \le 0, \tag{30}$$

which implies that

$$\sigma(C(t), z(t)) + \langle A_1\dot{u}(t) + A_0u(t) - f(t), \dot{u}(t)\rangle = 0 \;\text{ a.e. } t \in [0, T],$$

or equivalently,

$$A_1\dot{u}(t) + A_0u(t) - f(t) \in -N_{C(t)}(\dot{u}(t)) \text{ a.e. } t \in [0, T],$$

which shows that u is a solution of $(\mathcal{S})$. The proof of Theorem 1 is thereby completed. □

Remark 2 The construction of the existence of a solution is based on solving at each iteration the variational inequality (11). From a numerical point of view, there exist tremendous algorithms for solving such problems depending on the structure of the constraint set $C(t)$. If for each t, $C(t)$ is a closed convex cone, then complementarity problems algorithms can be used to solve (11). Numerical optimization algorithms, like interior points methods or SQP, can be used in the case $C(t)$ is of the form of inequalities constraints (3).

Example 1 Let us consider $H = \mathbb{R}^2$ with $T = 1$, some continuous function $f : [0, 1] \to \mathbb{R}^2$ and

$$A_1 = A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C(t) = [t, +\infty) \times [0, 1].$$

Clearly, all the assumptions of Theorem 1 are satisfied. Then for any initial condition, there exists at least one Lipschitz continuous solution for problem $(\mathcal{S}_1)$. However, if we consider $C(t) = [t, +\infty) \times [0, +\infty)$ with $f(t) = (0 \;\; 1)^T$ for all $t \in [0, 1]$, then for any initial condition, there are no solutions for problem $(\mathcal{S}_1)$. It is easy to see in this case, that condition (9) is not satisfied.

4 Application in nonregular electrical circuits

Let us consider the following complementarity system with a velocity constraint:

$$\begin{cases} g\big(t, x(t), \dot{x}(t), \lambda(t), u(t)\big) = 0 \\ y(t) = h\big(t, x(t), \dot{x}(t), \lambda(t), u(t)\big) \\ 0 \le \lambda(t) \perp y(t) \ge 0, \end{cases}$$

where $g : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, $h : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ are two given functions and $\lambda(t) \perp y(t)$ means that $\lambda(t)^T y(t) = \sum_{i=1}^m \lambda_i(t) y_i(t) = 0$.

A particular interesting case in applications is given when g and h are linear, and is known in the literature as Linear Complementarity Systems, i.e.

$$(LCS) \begin{cases} A_1 \dot{x}(t) + A_0 x(t) = B\lambda(t) - Eu(t) \\ y(t) = C_0 x(t) + C_1 \dot{x}(t) + D\lambda(t) + Gu(t) + F(t) \\ 0 \leq \lambda(t) \perp y(t) \geq 0, \end{cases}$$

where $A_1, A_0, B, E, C_0, C_1, D, G$ are given matrices with suitable dimensions and $F : [0, T] \to \mathbb{R}^m$ is a given function. If we suppose that $C_0 = 0$ and $D = 0$, then $y(t) = C_1 \dot{x}(t) + Gu(t) + F(t)$.

Let us assume that there exist a symmetric and positive definite matrix $P = P^T > 0$ such that

$$P^2 B = C_1^T. \tag{31}$$

The assumption (31) is linked with the well-known Kalman–Yakubovich–Popov Lemma (see e.g. [3] and references therein). By setting $z = Px$, we show that the system (S) is equivalent to

$$PA_1 P^{-1} \dot{z}(t) + PA_0 P^{-1} z(t) + PEu(t) \in -N_{C(t)}(\dot{z}(t)), \tag{32}$$

where

$$C(t) = \left\{ Px : C_1 x + Gu(t) + F(t) \in \mathbb{R}^n_+ \right\}.$$

It is easy to see that problem (32) is of the form ($\mathcal{S}$) (see [3] for more details).
Let us mention that many problems in electrical and mechanical engineering can be modeled by linear (or more generally nonlinear) complementarity systems of the (LCS).

Example 2 Let us consider the following electrical circuit depicted in Fig. 1 which involves a load resistance $R > 0$, capacitors $C_1,\ C_2 > 0$, two diodes D_1, D_2 and two current sources $c_1(\cdot), c_2(\cdot)$.
Using Kirchhoff's laws, we have

$$\begin{cases} V_R + V_{C_1} + V_{C_2} = -V_{D_1} \\ V_{C_1} - V_{C_2} = -V_{D_2}. \end{cases}$$

Let us assume the Ampere-Volt characteristics of the two diodes D_1 and D_2 are given by:

$$V_{D_1} \in N_{\mathbb{R}_+}(i_1) \text{ and } V_{D_2} \in -N_{[a,b]}(i_2),$$

where V_{D_k} and i_k are respectively the voltage and the current across the diode D_k $(k = 1, 2)$ and $a, b \in \mathbb{R}$.

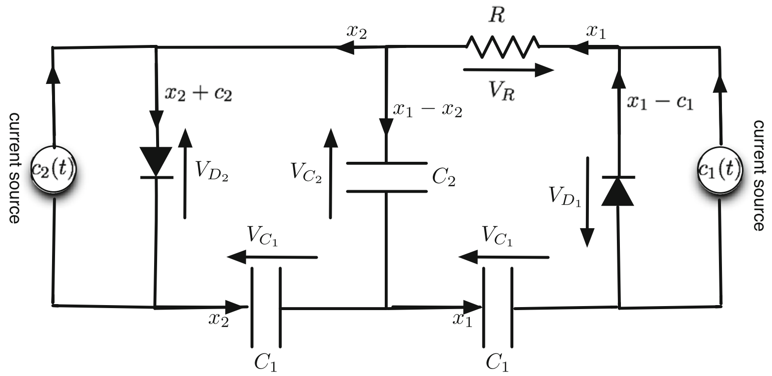

Fig. 1 A RLCD electrical circuit

Therefore the dynamics of this circuit is given by

$$\overbrace{\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}}^{A_1}\overbrace{\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}}^{\dot{q}} + \overbrace{\begin{pmatrix} \frac{1}{C_1}+\frac{1}{C_2} & -\frac{1}{C_2} \\ -\frac{1}{C_2} & \frac{1}{C_1}+\frac{1}{C_2} \end{pmatrix}}^{A_0}\overbrace{\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}}^{q} \in -N_{C(t)}(\dot{q}(t)), \tag{33}$$

with $C(t) = [c_1(t), +\infty[\times[-c_2(t)+a, -c_2(t)+b]$ and $\dot{q}_i(t) = x_i(t),\ i = 1, 2$. We note that the moving set $C(t)$ is unbounded in $\mathbb{R}^2$ and that the matrix A_1 is symmetric and semi-coercive while the matrix A_0 is symmetric and positive definite. It is easy to check that $\ker(A_1) \cap C(t)$ is bounded if the function $t \mapsto c_2(t)$ is bounded in $\mathbb{R}$. Then, all assumptions of Theorem 1 are satisfied. Hence, problem (33) has at least one solution. We can take D_2 as an ideal diode like D_1, however in this case we can add a resistor $R_2 > 0$ which will force $\ker(A_1) = \{0\}$ like in the example of [2].

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