Unbounded State-Dependent Sweeping Processes with Perturbations in Uniformly Convex and q-Uniformly Smooth Banach Spaces

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Unbounded State-Dependent Sweeping Processes with Perturbations in Uniformly Convex and q-Uniformly Smooth Banach Spaces.

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Abstract

In this paper, the existence of solutions for a class of first and second order unbounded state-dependent sweeping processes with perturbation in uniformly convex and q-uniformly smooth Banach spaces are analyzed by using a discretization method. The sweeping process is a particular differential inclusion with a normal cone to a moving set and is of a great interest in many concrete applications. The boundedness of the moving set, which plays a crucial role for the existence of solutions in many works in the literature, is not necessary in the present paper. The compactness assumption on the moving set is also improved.

Keywords: Unbounded state-dependent sweeping processes, Differential Inclusions, Normal cones, Variational Analysis, Uniformly Convex and q-Uniformly Smooth Banach Spaces.

AMS Classification: 34A60, 49J52, 49J53.

Contents

1 Introduction

Sweeping processes, initially proposed and thoroughly studied by J. J. Moreau [18, 19, 20, 21, 22] in the seventies, have played an essential role in mechanics and applied mathematics from both

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theoretical and numerical points of view. In [22], Moreau considered the absolutely continuous solution of the differential inclusion in Hilbert spaces of the following form

$$\begin{cases} \dot{u}(t) \in -N_{C(t)}(u(t)) \text{ a.e. } t \in [0,T], \\ u(0) = u_0 \in C(0), \end{cases} \tag{1}$$

where $N_{C(t)}(\cdot)$ denotes the (outward) normal cone to the moving closed and convex set $C(t)$ in the sense of convex analysis. Since then, there have been extensive results with various variants in Hilbert spaces (see, e.g., [1, 2, 13, 16, 17]) and in reflexive Banach spaces recently (see, e.g., [8, 9, 10]). In [8], the authors considered two classes of first order and one class of second order sweeping processes with perturbation in p-uniformly convex and q-uniformly smooth Banach spaces with $p, q > 1$. The first order *state-dependent* without perturbation sweeping processes was also studied in [10] in p-uniformly convex and q-uniformly smooth Banach spaces with $p, q > 2$ and left the perturbed problem as an open problem (see Remark 2 in [10]). On the other hand, it is remarkable that the assumption on the boundedness of the moving sets is essential in the previous works for technical reasons, see [8, 9, 10] for examples.

In this paper, we study the existence of solutions for the first order state-dependent sweeping processes as in [10] but with perturbation and possibly unbounded sweeping sets

$$(\mathcal{S}_1) \qquad \begin{cases} (y^*)'(t) \in -N_{C(t,J^*y^*(t))}(J^*y^*(t)) - F(t, J^*y^*(t)) \quad a.e. \;\; t \in [0,T], \\ y^*(0) = Jy_0, \; y_0 \in C(0, y_0), \end{cases}$$

where $y^* : [0,T] \to X^*$, $C : [0,T] \times X \rightrightarrows X$, $F : [0,T] \times X \rightrightarrows X^*$. Here X is a uniformly convex and q-uniformly smooth Banach space ($q \geq 2$) with its duality mapping J and J^* is the duality mapping of X^*, the dual space of X. The moving set $C : (t,x) \mapsto C(t,x)$ is non-empty closed convex and possibly unbounded. The perturbation part F is an upper semicontinuous set-valued mapping with convex weak* compact values in X^* and satisfies the weak linear growth condition, i.e., the intersection between the perturbation map and the ball with linear growth is non-empty, see (13) for more detail. Another contribution of this paper is the improvement of the compactness assumption compare to the works [8, 9, 10]. Furthermore, when there is no perturbation, i.e. $F \equiv 0$, a kind of Lipschitz assumption on the moving set is proposed which is easier to check than the one used in [8] since it involves only on the usual distance function. Using similar technique, we also consider the second order unbounded state-dependent sweeping processes with perturbation

$$(\mathcal{S}_2) \;\; \begin{cases} (y^*)''(t) \in -N_{C(t,J^*y^*(t))}(J^*(y^*)'(t)) - F(t, y^*(t), (y^*)'(t)) \;\; a.e. \; t \in [0,T], \\ y^*(0) = Jy_0, \; (y^*)'(0) = Ju_0, \; u_0 \in J(C(0,y_0)), \end{cases}$$

where $y^* : [0,T] \to X^*$, $C : [0,T] \times X \rightrightarrows X$, $F : [0,T] \times X^* \times X^* \rightrightarrows X^*$.

The paper is organized as follows. In Section 2, we recall some definitions and useful results on the convexity, smoothness of reflexive Banach spaces and the generalized projection. The main results concerning the existence of solutions of $(\mathcal{S}_1)$ and $(\mathcal{S}_2)$ are proved in Section 3. Some conclusions and perspectives end the paper in Section 4.

2 Notation and Mathematical Backgrounds

Let us first introduce some notations that will be used in the sequel. Let X be a real separable reflexive Banach space with dual space X^*. We denote both norms of X and X^* by $\|\cdot\|$ if there is no confusion and we write $\langle x^*, x\rangle$ instead of $x^*(x)$ for all $x \in X, x^* \in X^*$. The space X is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for all $x, y \in X$ satisfying $\|x\| = \|y\| = 1$. The *modulus of convexity and smoothness* of X are defined respectively as follows

$$\delta_X(\epsilon) = \inf \Big\{1 - \|x+y\|/2 : \|x\| = \|y\| = 1 \text{ and } \|x-y\| = \epsilon\Big\}, \;\; 0 \le \epsilon \le 2,$$

and

$$\rho_X(t) = \inf \Big\{(\|x+y\| + \|x-y\|)/2 - 1 : \|x\| = 1, \|y\| = t\Big\}, \;\; t > 0.$$

The space X is said to be *strictly convex* if $\delta_X(2) = 1$. It is said to be *uniformly convex* if $\delta_X(\epsilon) > 0$ for all $0 < \epsilon \le 2$ and *uniformly smooth* if $\lim_{t\to 0^+} \frac{\rho_X(t)}{t} = 0$. Any uniformly convex space is also strictly convex. Let $p, q > 1$. It is said to be *p-uniformly convex* (*q-uniformly smooth*, resp.) if there exists $a > 0$ such that $\delta_X(\epsilon) \ge a\epsilon^p$ ($\rho_X(t) \le at^q$, resp.). It is known that (see, e.g., [14]) X is smooth if and only if X^* is strictly convex. Furthermore, if X is p-uniformly convex (q-uniformly smooth, resp.), then X^* is p'-uniformly smooth (q'-uniformly convex, resp.) where $p' = \frac{p}{p-1}, q' = \frac{q}{q-1}$ are the conjugate numbers of p, q respectively.

Denote by J the *duality mapping* of X, defined as follows

$$Jx = \{x^* \in X^* : \|x^*\|^2 = \|x\|^2 = \langle x, x^*\rangle\}.$$

Here are some important properties of J (see, e.g., [3, 4, 8] for more properties).

1. Jx is non-empty and $J(\alpha x) = \alpha Jx$, for any $\alpha \in \mathbb{R}, x \in X$.
2. If X^* is uniformly convex, then J is single-valued uniformly continuous on bounded subsets of X and $J^{-1} = J^*$, the duality mapping of X^*.
3. If X is a smooth strictly convex Banach space, then $J^{-1} = J^*, JJ^* = I_{X^*}, J^*J = I_X$.

Let $V : X^* \times X \to \mathbb{R}$ be defined by

$$V(x^*, x) = \|x^*\|^2 - 2\langle x^*, x\rangle + \|x\|^2, \;\; x^* \in X^*, x \in X. \tag{2}$$

It is easy to check the following properties

1. $(\|x^*\| - \|x\|)^2 \le V(x^*, x) \le (\|x^*\| + \|x\|)^2$, for all $x^* \in X^*, x \in X$.
2. $V(x^*, x) = 0$ if and only if $x^* \in J(x)$.

Let us define the distance function $d_S^V : X^* \to \mathbb{R}$ associated with V to the set S as follows

$$d_S^V(x^*) := \inf_{x\in S} V^{1/2}(x^*, x). \tag{3}$$

Note that in Hilbert spaces, the distance function d_S^V becomes the usual distance function d_S. Let us recall now the definition of generalized projection of $x^* \in X^*$ onto $S \subset X$ in Banach spaces [3, 4].

Definition 1. Let $x^* \in X^*$ and $S \subset X$. The generalized projection of x^* on S is defined as follows

$$\pi_S(x^*) = \{\bar{x} \in S : V(x^*, \bar{x}) = \inf_{x \in S} V(x^*, x)\} = \{\bar{x} \in S : V^{1/2}(x^*, \bar{x}) = d_S^V(x^*)\}. \tag{4}$$

Proposition 2. *(see, e.g., [3, 4, 8, 10]) Let X be a reflexive Banach space with dual space X^* and S be a nonempty, closed and convex subset of X. Let $x^* \in X^*$.The following properties hold*
(i) $\pi_S(x^*) \neq \emptyset$.
(ii) If X is also smooth, then $x^* \in N_S(\bar{x})$ if and only if, $\exists a > 0$ so that $\bar{x} \in \pi_S(J\bar{x} + ax^*)$.
(iii) $\pi_S(x^*)$ is singleton for all $x^* \in X^*$ if and only if X is strictly convex.

Proposition 3. *(see, e.g., [11]) Let S be a nonempty, closed and convex subset of a Banach space X and x be a point in S. Then*

$$N_S(x) \cap \mathbb{B}_* = \partial d_S(x),$$

where $\mathbb{B}_*$ denotes the unit ball in X^*.

We recall the following result taken from [4] (see Theorem 7.5).

Theorem 4. *For all $x, y \in X$, one has*

$$8C^2 \delta_X\Big(\frac{\|x-y\|}{4C}\Big) \leq V(Jx, y) \leq 4C^2 \rho_X\Big(\frac{4\|x-y\|}{C}\Big), \tag{5}$$

where $C = \sqrt{(\|x^2\| + \|y^2\|)/2}$.

Corollary 5. *Let $1 < p \leq 2, q \geq 2$ and X be a p-uniformly convex, q-uniformly smooth Banach space. Then there exist some constants a, b depending only on X such that for all $x, y \in X$, one has*

$$a\|x-y\|^2 \leq V(Jx, y) \leq b\|x-y\|^2. \tag{6}$$

Proof. Since X is p-uniformly convex and q-uniformly smooth, there exist c, d depending only X such that

$$\delta_X(\epsilon) \geq c\epsilon^p, \quad \rho_X(t) \leq dt^q \text{ for all } 0 \leq \epsilon \leq 2, t > 0.$$

Using Theorem 4 and noting that $\|x - y\| \leq 2\sqrt{(\|x^2\| + \|y^2\|)/2}$, one has

$$V(Jx, y) \geq 8C^2 c\Big(\frac{\|x-y\|}{4C}\Big)^p \geq 8C^2 c\Big(\frac{\|x-y\|}{4C}\Big)^2 = \frac{c}{2}\|x-y\|^2,$$

and

$$\begin{aligned} V(Jx, y) &\leq 4C^2 \rho_X\Big(\frac{4\|x-y\|}{C}\Big) \leq 4C^2 d\Big(\frac{4\|x-y\|}{C}\Big)^q = 4C^2 d8^q\Big(\frac{\|x-y\|}{2C}\Big)^q \\ &\leq 4C^2 d8^q\Big(\frac{\|x-y\|}{2C}\Big)^2 = d8^q\|x-y\|^2. \end{aligned}$$

The result follows with $a = c/2$ and $b = d8^q$. □ □

Corollary 6. *Let $q \geq 2$ and X be a q-uniformly smooth Banach space. Then there exists some $a_X, b_X > 0$ depending only on X such that for all $x, y \in X$, one has*

$$V^{1/2}(Jx, y) \leq b_X\|x-y\|, \tag{7}$$

and

$$a_X\|x^*-y^*\| \le V_*^{1/2}(J^*y^*,x^*), \quad \text{for all } x^*,y^*\in X^*, \tag{8}$$

where $V_*: X^{**}\times X^*\to\mathbb{R}$ *is defined by*

$$V_*(x^{**},x^*)=\|x^{**}\|^2-\langle x^{**},x^*\rangle+\|x^*\|^2 \ \text{ for all } x^{**}\in X^{**}, x^*\in X^*.$$

Consequently, for any subset $S\subset X$ *and* $x\in X$*, one has*

$$d_S^V(Jx)\le b_X d_S(x). \tag{9}$$

Proof. Using Corollary 5 and noting that X^* is $q'-$ uniform convex, where $1<q'\le 2$ is the conjugate number of q. The last statement (9) can be rewritten as follows

$$d_S^V(Jx)=\inf_{y\in S}V^{1/2}(Jx,y)\le b_X\inf_{y\in S}\|x-y\|=b_Xd_S(x).$$

□ □

Let us recall a discrete version of the Gronwall's inequality.

Lemma 7. *Let* $\alpha>0$ *and* (u_n)*,* (β_n) *be nonnegative real sequences satisfying*

$$u_n\le\alpha+\sum_{k=0}^{n-1}\beta_k u_k, \ \ \forall n=0,1,2,\dots \ \ (\text{with } \beta_{-1}=0). \tag{10}$$

Then for all $n\in\mathbb{N}$*, we have*

$$u_n\le\alpha\exp\Big(\sum_{k=0}^{n-1}\beta_k\Big).$$

Finally, we recall the Kuratowski measure of non-compactness for a bounded set B in X, which is defined as follows

$$\gamma(B):=\inf\Big\{r>0 \ : \ B=\bigcup_{i=1}^{n}B_i \ \text{ for some } n\in\mathbb{N}^* \text{ and } B_i \text{ with } \operatorname{diam}(B_i)\le r\Big\}.$$

One has the following lemma (see, e.g., [15, Proposition 9.1]).

Lemma 8. *Let* B, B_1 *and* B_2 *be bounded sets of* X*,* $\lambda\in\mathbb{R}$*. Then,*

1. $\gamma(B_1)=0\Leftrightarrow B_1$ *is relatively compact.*
2. *If* $B_1\subset B_2$*, then* $\gamma(B_1)\le\gamma(B_2)$*.*
3. γ *is a semi-norm, i.e.,* $\gamma(\lambda B)=|\lambda|\gamma(B)$ *and* $\gamma(B_1+B_2)\le\gamma(B_1)+\gamma(B_2)$*.*
4. $\gamma(x_0+r\mathbb{B})=2r$ *for any* $x_0\in X$ *and* $r>0$*.*

3 First-order Unbounded perturbed State-Dependent Sweeping Processes

In this section, we study the existence of solutions for the unbounded state-dependent sweeping processes with perturbation $(\mathcal{S}_1)$ in a uniformly convex and q-uniformly smooth Banach space $(q \geq 2)$ by using a discretization technique based on Moreau's catching-up algorithm [19, 22]. The possibly unbounded moving set $C(\cdot,\cdot)$ varies in a Lipchitz continuous way while the perturbation F is upper semi-continuous with convex weak* compact values in X^* and satisfies the following weak linear growth condition (13). In detail, let us first make the following assumptions for problem $(\mathcal{S}_1)$.

Assumption 1 The set-valued map $C : [0,T] \times X \rightrightarrows X$ is $(\lambda_1, \lambda_2, \lambda_3)$-Lipschitz continuous where $\lambda_1, \lambda_2, \lambda_3$ are nonnegative real number with $\lambda_3 < a_X$ (a_X is defined in (8)), in the sense that, for all $t_1, t_2 \in [0,T]$, $x_1, x_2 \in X$ and $y_1, y_2 \in X^*$, one has

$$|d^V_{C(t_1,x_1)}(y_1) - d^V_{C(t_2,x_2)}(y_2)| \leq \lambda_1 |t_1 - t_2| + \lambda_2 \|y_1 - y_2\| + \lambda_3 \|Jx_1 - Jx_2\|. \tag{11}$$

Assumption 2 There exists a convex compact set $K \subset X^*$ such that

$$J\big(C(t, M\mathbb{B})\big) \cap M\mathbb{B}^* \subset K, \tag{12}$$

where M is a constant depending only on initial data defined in (19).
Assumption 3 The set-valued mapping $F : [0,T] \times X \rightrightarrows X^*$ is upper semi-continuous with convex, weak* compact values in X^* and satisfies the following weak linear growth condition, i.e., there exists $k_F > 0$ such that for all $t \geq 0$ and $x \in X$, one has

$$F(t,x) \cap k_F(1 + \|x\|)\mathbb{B}^* \neq \emptyset, \tag{13}$$

where $\mathbb{B}^*$ is the unit ball in X^*.

Theorem 9. *Let X be a uniformly convex and q-uniformly smooth Banach space $(q \geq 2)$. Suppose that Assumptions 1, 2 and 3 hold. Then for each $y_0 \in X$ satisfying $y_0 \in C(0, y_0)$, there exists a Lipschitz continuous solution for problem*

$$(\mathcal{S}_1) \qquad \begin{cases} (y^*)'(t) \in -N_{C(t,J^*y^*(t))}(J^*y^*(t)) - F(t, J^*y^*(t)) & a.e. \ \ t \in [0,T], \\ y^*(0) = Jy_0. \end{cases}$$

Proof. For given positive integer n, let us define the step-size by $\mu_n := n/T$. Let $t_{n,i} := i\mu_n$ and $I_{n,i} := [t_{n,i}, t_{n,i+1})$. By using Moreau's catching-up algorithm, we approximate the system $(\mathcal{S}_1)$ as follows

$$\begin{aligned} y_{n,0} &:= y_0 \in C(0,y_0), \ \ z^*_{n,0} \in F(t_0, y_0) \cap k_F(1 + \|y_0\|)\mathbb{B}^*; \\ z^*_{n,i} &\in F(t_{n,i}, y_{n,i}) \cap k_F(1 + \|y_{n,i}\|)\mathbb{B}^*, \\ y_{n,i+1} &:= \pi_{C(t_{n,i+1}, y_{n,i})}\big(Jy_{n,i} - \mu_n z^*_{n,i}\big) \text{ for } 0 \leq i \leq n-1; \\ \text{For } t \in I_{n,i} : & \\ y^*_n(t) &:= Jy_{n,i} + \frac{t - t_{n,i}}{\mu_n}(Jy_{n,i+1} - Jy_{n,i}); \\ y_n(t) &:= J^*y^*_n(t); \\ z^*_n(t) &:= z^*_{n,i}; \ \ \theta_n(t) := t_{n,i}; \ \ \eta_n(t) := t_{n,i+1}. \end{aligned} \tag{14}$$

Using Corollary 5, we have

$$\begin{aligned} a_X\|Jy_{n,i+1}-Jy_{n,i}+\mu_n z^*_{n,i}\| &\leq V_*^{1/2}(J^*Jy_{n,i+1}, Jy_{n,i}-\mu_n z^*_{n,i}) \\ &= V_*^{1/2}(y_{n,i+1}, Jy_{n,i}-\mu_n z^*_{n,i}) \\ &= V^{1/2}(Jy_{n,i}-\mu_n z^*_{n,i}, y_{n,i+1}) \\ &= d^V_{C(t_{n,i+1},y_{n,i})}(Jy_{n,i}-\mu_n z^*_{n,i}) - d^V_{C(t_{n,i},y_{n,i-1})}(Jy_{n,i}) \\ &\leq \mu_n(\lambda_1+\lambda_2\|z^*_{n,i}\|)+\lambda_3(\|Jy_{n,i}-Jy_{n,i-1}\|). \end{aligned} \tag{15}$$

Hence,

$$\begin{aligned} \|Jy_{n,i+1}-Jy_{n,i}\| &\leq \mu_n\Big(\frac{\lambda_1}{a_X}+(\frac{\lambda_2}{a_X}+1)\|z^*_{n,i}\|\Big)+\frac{\lambda_3}{a_X}\|Jy_{n,i}-Jy_{n,i-1}\| \\ &\leq \mu_n\Big(\frac{\lambda_1}{a_X}+(\frac{\lambda_2}{a_X}+1)k_F(1+\|y_{n,i}\|)\Big)+\frac{\lambda_3}{a_X}\|Jy_{n,i}-Jy_{n,i-1}\| \\ &\leq \mu_n(\alpha_1+\alpha_2\|y_{n,i}\|)+\alpha_3\|Jy_{n,i}-Jy_{n,i-1}\|, \end{aligned}$$

where $\alpha_1=\frac{\lambda_1+\lambda_2 k_F}{a_X}+k_F, \alpha_2=(\frac{\lambda_2}{a_X}+1)k_F, \alpha_3=\frac{\lambda_3}{a_X}$. By induction, one has

$$\|Jy_{n,i+1}-Jy_{n,i}\| \leq \mu_n\alpha_1\sum_{j=0}^{i-1}\alpha_3^j+\mu_n\alpha_2\sum_{j=0}^{i-1}\alpha_3^j\|y_{n,i-j}\|+\alpha_3^i\|Jy_{n,1}-Jy_0\|.$$

Note that, similarly as in (15), one has

$$\begin{aligned} \|Jy_{n,1}-Jy_0+\mu_n z^*_0\| &\leq \frac{1}{a_X}V^{1/2}(Jy_0-z^*_0, y_{n,1}) = \frac{1}{a_X}d^V_{C(t_{n,1},y_{n,0})}(Jy_{n,0}-\mu_n z^*_{n,0}) \\ &= \frac{1}{a_X}(d^V_{C(t_{n,1},y_0)}(Jy_{n,0}-\mu_n z^*_0)-d^V_{C(0,y_0)}(Jy_0)) \\ &\leq \mu_n(\frac{\lambda_1}{a_X}+\frac{\lambda_2}{a_X}\|z^*_0\|). \end{aligned}$$

Thus,

$$\|Jy_{n,1}-Jy_{n,0}\| \leq \mu_n(\alpha_1+\alpha_2\|z^*_0\|).$$

Consequently,

$$\begin{aligned} \|Jy_{n,i+1}-Jy_{n,i}\| &\leq \mu_n\Big(\alpha_1\sum_{j=0}^{i-1}\alpha_3^j+\alpha_2\sum_{j=0}^{i-1}\alpha_3^j\|y_{n,i-j}\|+\alpha_3^i(\alpha_1+\alpha_2\|z^*_0\|)\Big) \\ &\leq \mu_n(\beta_1+\alpha_2\sum_{j=0}^{i-1}\alpha_3^j\|y_{n,i-j}\|), \end{aligned} \tag{16}$$

where $\beta_1=\frac{\alpha_1}{1-\alpha_3}+\alpha_1+\alpha_2\|z^*_0\|$ since $\alpha_3<1$. So

$$\|y_{n,i+1}\| \leq \|y_{n,i}\|+\mu_n(\beta_1+\alpha_2\sum_{j=0}^{i-1}\alpha_3^j\|y_{n,i-j}\|). \tag{17}$$

By induction, one has

$$\begin{aligned} \|y_{n,i+1}\| &\leq \beta_1 T+\|y_{n,1}\|+\mu_n\alpha_2\{\|y_{n,i}\|+\|y_{n,i-1}\|(1+\alpha_3) \\ &+ \ldots+\|y_1\|(1+\alpha_3+\ldots+\alpha_3^{i-1})\} \\ &\leq \beta_2+\mu_n\beta_3\sum_{j=1}^{i}\|y_{n,j}\|, \end{aligned}$$

where $\beta_2 := \beta_1 T + \|y_0\| + \alpha_1 + \alpha_2\|z_0^*\|$ and $\beta_3 := \frac{\alpha_2}{1-\alpha_3}$. Using the discrete Gronwall's inequality in Lemma 7, one obtains that the sequence $(y_{n,i})$ is bounded by

$$M_1 := \beta_2 e^{\beta_3 T}. \tag{18}$$

We have,

$$(y_n^*)'(t) = \frac{Jy_{n,i+1} - Jy_{n,i}}{\mu_n},\ t \in I_{n,i}.$$

From (16) one deduces that

$$\|\frac{Jy_{n,i+1} - Jy_{n,i}}{\mu_n}\| \leq \beta_1 + \beta_3 M_1 \leq \max\{\beta_1 + \beta_3 M_1, M_1\} := M. \tag{19}$$

Thus $y_n^*(\cdot)$ is Lipschitz continuous with the same Lipschitz constant M. Note that $Jy_{n,i}, Jy_{n,i+1} \in J\big(C(t, M\mathbb{B})\big) \cap M\mathbb{B}^* \subset K$. Hence

$$y_n^*(t) = \frac{t_{n,i+1} - t}{\mu_n} Jy_{n,i} + \frac{t - t_{n,i}}{\mu_n} Jy_{n,i+1} \in K.$$

Consequently, for each $t \geq 0$, the set $\Omega^*(t) := \{y_n^*(t) : n \geq 1\}$ is relatively compact in X^*.
Using the Arzelà-Ascoli Theorem (see, e.g., [6]), there exist a subsequence of $(y_n^*)_{n\in\mathbb{N}}$, still denoted by itself and some $y^* \in C(0,T;X^*)$ such that $y_n^* \to y^*$ uniformly on $[0,T]$ and $(y_n^*)'$ converges weakly to $(y^*)'$ in $L^1(0,T;X^*)$. Hence $J^*y_n^*$ converges to J^*y^* since J^* is uniformly continuous on bounded sets. In addition, we have

$$\lim_{n\to\infty} \theta_n(t) = \lim_{n\to\infty} \eta_n(t) = t,$$

and

$$\lim_{n\to\infty} y_n^*(\theta_n(t)) = \lim_{n\to\infty} y_n^*(\eta_n(t)) = y^*(t).$$

Since $y_n(\eta_n(t)) \in C(\eta_n(t), y_n(\theta_n(t)))$ and

$$\begin{aligned} d^V_{C(t,J^*y^*(t))}(y^*(t)) &= d^V_{C(t,J^*y^*(t))}(y^*(t)) - d^V_{C(\eta_n(t),y_n(\theta_n(t)))}(Jy_n(\eta_n(t))) \\ &\leq \lambda_1|t - \eta_n(t)| + \lambda_2\|y^*(t) - y_n^*(\eta_n(t))\| \\ &+ \lambda_3\|y^*(t) - y_n^*(\theta_n(t))\| \to 0 \text{ as } n \to +\infty. \end{aligned}$$

Hence $J^*y^*(t) \in C(t, J^*y^*(t))$ since $C(t, J^*y^*(t))$ is closed. By using Proposition 2 and (14), it is easy to see that

$$\frac{Jy_{n,i+1} - Jy_{n,i}}{\mu_n} + z_{n,i}^* \in -N_{C(t_{n,i+1},y_{n,i})}(y_{n,i+1}).$$

On the other hand $z_n^*(t) \in F(\theta_n(t), y_n(\theta_n(t))) \cap k_F(1 + \|y_n(\theta_n(t))\|)\mathbb{B}^*$. Let $R := k_F(1+M) + M$. Then by using Proposition 3, one has for a.e. $t \in [0,T]$ that

$$\begin{aligned} (y_n^*)'(t) + z_n^*(t) &\in -N_{C(\eta_n(t),y_n(\theta_n(t)))} y_n(\eta_n(t)) \cap R\mathbb{B}^* \\ &= -R\partial d_{C(\eta_n(t),y_n(\theta_n(t)))}(y_n(\eta_n(t))). \end{aligned} \tag{20}$$

Note that $(z_n^*)_{n\in\mathbb{N}}$ is bounded by R, thus there exist a subsequence still denoted by itself and some $z^* \in L^1(0,T;X^*)$ such that z_n^* converges weakly to z^* in $L^1(0,T;X^*)$. We will prove that

$$(y^*)'(t) + z^*(t) \in -N_{C(t,J^*y^*(t))}(J^*y^*(t)) \ \ a.e.\ t \in [0,T].$$

Using Mazur's lemma and the fact that $(y_n^*)' + z_n^*$ converges weakly to $(y^*)' + z^*$, one has

$$(y^*)'(t) + z^*(t) \in \bigcap_n \overline{\text{co}}\{(y_i^*)'(t) + z_i^*(t) : i \geq n\} \text{ for } a.e.\ t \in [0,T]. \tag{21}$$

Fix $t \in [0,T]$ satisfying (21) and $\xi \in X$. The relation (21) implies that

$$\begin{aligned} \langle (y^*)'(t) + z^*(t), \xi \rangle &\leq \inf_n \sup_{i \geq n} \langle (y_i^*)'(t) + z_i^*(t), \xi \rangle \\ &\leq \limsup_n \delta(-R\partial d_{C(\eta_n(t), y_n(\theta_n(t)))}(y_n(\eta_n(t))); \xi), \end{aligned}$$

thanks to (20) where $\delta_C(\cdot)$ denotes the support function to the set C. Let us recall that the convex weak* compact valued mapping $\partial d_{C(t,x)}(y)$ is scalarly upper semicontinuous, that is, for every $\xi \in X$, the function $\delta(\partial d_{C(t,x)}(y); \xi)$ is upper semicontinuous on $[0,T] \times X \times X$ due to Assumption 1 and Proposition I. 17 [23]. Thus

$$\langle (y^*)'(t) + z^*(t), \xi \rangle \leq \delta(-R\partial d_{C(t,J^*y^*(t))}(J^*y^*(t)); \xi).$$

Since the set-valued mapping $t \mapsto \partial d_{C(t,J^*y^*(t))}(J^*y^*(t))$ is measurable and convex weak* compact valued and $J^*y^*(t) \in C(t, J^*y^*(t))$, it follows that (see, e.g., [23, 12])

$$(y^*)'(t) + z^*(t) \in -R\partial d_{C(t,J^*y^*(t))}(J^*y^*(t)) \in -N_{C(t,J^*y^*(t))}(J^*y^*(t)).$$

Similarly, we have $z^*(t) \in F(t, J^*y^*(t))$ due to the upper-semicontinuity with convex, weak* compact values of F. In conclusion, one obtains

$$(y^*)'(t) \in -N_{C(t,J^*y^*(t))}(J^*y^*(t)) - F(t, J^*y^*(t)),$$

and the proof is completed. □ □

Example 10. Let $\Omega \in \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with Lipschitz boundary. Let $X := L^p(\Omega)$, $2 \leq p < +\infty$, then X is a p-uniformly convex and 2-uniformly smooth Banach space (see e.g., [5]) with $X^* = L^q(\Omega)$, where q is the conjugate number of p, i. e., $1/p + 1/q = 1$. The duality mapping J can be computed explicitly as follows [4]

$$Jx = \|x\|_{L^p}^{2-p} |x|^{p-2} x \in X^*, \quad \forall x \in X. \tag{22}$$

Let the set-valued mappings $C : [0,T] \times X \rightrightarrows X$ and $F : [0,T] \times X \rightrightarrows X^*$ satisfy Assumptions 1, 2, 3. We consider the following parabolic quasi-variational inequality: find an absolutely continuous function $y^* : [0,T] \to X^*$ such that there exists a function $z^* \in L^1(0,T; X^*)$ so that for almost all $t \in [0,T]$, we have $J^*y^*(t) \in C(t, J^*y^*(t))$, $z^*(t) \in F(t, J^*y^*(t))$ and

$$\begin{cases} \big\langle (y^*)'(t) + z^*(t), v - J^*y^*(t) \big\rangle \geq 0, \ \ \forall v \in C(t, J^*y^*(t)). \\ \\ y^*(0) = Jy_0, \ y_0 \in C(0, y_0). \end{cases} \tag{23}$$

It is easy to see that the evolution quasi-variational inequality problem (23) can be recast into the form $(\mathcal{S}_1)$. By using Theorem 9, we assert that there exists at least one Lipschitz continuous solution for problem (23).

Remark 11. (i) Theorem 9 deals with the perturbed state-dependent sweeping processes in uniformly convex and q-uniformly smooth Banach spaces ($q \geq 2$). In the recent paper [10], the authors considered the unperturbed state-dependent sweeping processes in p-uniformly convex and q-uniformly smooth Banach spaces ($p, q > 2$) and left the perturbed problem as an open question.
(ii) We note that the moving set C is allowed to be unbounded while in all previous works in Banach spaces the boundedness of C is essential for the technical reasons, see e.g., [8, 9, 10].
(iii) The compactness assumption here (Assumption 2) is also better than the one used in [8, 9] since it needs to check only in a fixed ball. The perturbation term F only need to satisfy the weak linear growth condition.
(iv) We may expect to replace the constant k_F in Assumption 3 by some function $k_F(\cdot) \in L^1(0,T;\mathbb{R}^+)$. However, in this case, the sequence $(y_{n,i})$ may be unbounded. Thus we have to assume the boundedness of the moving set C from the beginning.

If there is no perturbation ($F \equiv 0$), the Lipschitz assumption on the function $(t,x,y) \mapsto d^V_{C(t,x)}(y)$ can be replaced by the Lipschitz continuity of the function $(t,x,y) \mapsto d_{C(t,x)}(y)$ which is easier to deal with since it only involves the usual distance function on X. Clearly, the function $(t,x,y) \mapsto d_{C(t,x)}(y)$ is also easier to check than the function $(t,x,y) \mapsto d^{q/p}_{C(t,x)}(y)$, which is used in [10]. Furthermore, in this case, our compactness assumption is also improved. Theorem 3 is in this sense.

Assumption 1′ *There exist nonnegative real numbers λ_1, λ_3 with $\lambda_3 < \frac{a_X}{b_X}$ (a_X and b_X are defined in (8) and (7) respectively) such that for all $t_1, t_2 \geq 0$ and $x_1, x_2, y \in X$, one has*

$$|d_{C(t_1,x_1)}(y) - d_{C(t_2,x_2)}(y)| \leq \lambda_1|t_1 - t_2| + \lambda_3\|Jx_1 - Jx_2\|. \tag{24}$$

Assumption 2′ *For any $t \in [0,T]$ and any set $A \subset M\mathbb{B}$ with $\gamma(A) > 0$, one has*

$$\gamma(C(t,A) \cap (M+1)\mathbb{B}) < \gamma(A), \tag{25}$$

where γ is the Kuratowski measure of noncompactness and M is a constant depending only on initial data defined in (31).

Theorem 12. *Let X be a uniformly convex and q-uniformly smooth Banach space ($q \geq 2$) and $F \equiv 0$. Suppose that Assumptions $1'$ and $2'$ hold. Then for each initial condition, there exists a Lipschitz continuous solution of $(\mathcal{S}_1)$ defined on $[0,T]$.*

Proof. Using similar arguments as in the proof of Theorem 9 with $z^*_{n,i} = 0$ and noting that the estimation in (15) can be improved as follows

$$\begin{aligned} a_X\|Jy_{n,i+1} - Jy_{n,i}\| &\leq V_*^{1/2}(J^*Jy_{n,i+1}, Jy_{n,i}) = V_*^{1/2}(y_{n,i+1}, Jy_{n,i}) \\ &= V^{1/2}(Jy_{n,i}, y_{n,i+1}) = d^V_{C(t_{n,i+1},y_{n,i})}(Jy_{n,i}) \\ &\leq b_X d_{C(t_{n,i+1},y_{n,i})}(y_{n,i}) \text{ (see (9) in Proposition 5)} \\ &= b_X\big(d_{C(t_{n,i+1},y_{n,i})}(y_{n,i}) - d_{C(t_{n,i},y_{n,i-1})}(y_{n,i})\big) \\ &\leq b_X(\mu_n\lambda_1 + \lambda_3\|Jy_{n,i} - Jy_{n,i-1}\|). \end{aligned} \tag{26}$$

Hence,

$$\|Jy_{n,i+1} - Jy_{n,i}\| \leq \mu_n\alpha_1 + \alpha_3\|Jy_{n,i} - Jy_{n,i-1}\|, \tag{27}$$

where $\alpha_1 := \frac{b_X \lambda_1}{a_X}$ and $\alpha_3 := \frac{b_X \lambda_3}{a_X} < 1$. Similarly as in (16), one has

$$\|Jy_{n,i+1} - Jy_{n,i}\| \leq \mu_n \beta_1, \tag{28}$$

where $\beta_1 := \frac{\alpha_1}{1-\alpha_3} + \alpha_1$. Consequently,

$$\|y_{n,i+1}\| \leq \|y_{n,i}\| + \mu_n \beta_1 \leq \ldots \leq \|y_0\| + (i+1)\mu_n \beta_1 \leq \|y_0\| + \beta_1 T \leq M, \tag{29}$$

and

$$\|\frac{Jy_{n,i+1} - Jy_{n,i}}{\mu_n}\| \leq \beta_1 \leq M, \tag{30}$$

where

$$M := \max\{\beta_1, \|y_0\| + \beta_1 T\}. \tag{31}$$

It remains to check that for each $t \geq 0$, the set $\Omega^*(t) := \{y_n^*(t) : n \geq 1\}$ is relatively compact in X^* under Assumption $2'$. Suppose to the contrary that there exists some $t_0 \in [0, T]$ such that $\Omega^*(t_0)$ is not relatively compact. Then $\Omega(t_0) := \{J^* y_n^*(t_0) = y_n(t_0) : n \geq 1\}$ is also not relatively compact in X since J is uniformly continuous on bounded sets. By using Assumption $2'$, there exists some $0 < \sigma < 1$ such that

$$\gamma(\Omega(t_0)) - \gamma(C(t_0, \Omega(t_0)) \cap (M+1)\mathbb{B}) \geq 3\sigma. \tag{32}$$

One can find some positive integer i such that $t_0 \in I_{n,i}$. Observe that

$$\begin{aligned} y_{n,i+1} \in C(t_{n,i+1}, y_{n,i}) \quad &\subset \quad C(t_0, y_n(t_0)) + (\lambda_1 \mu_n + \lambda_3 \|Jy_n(t_0) - Jy_{n,i}\|)\mathbb{B} \\ &\leq \quad C(t_0, y_n(t_0)) + (M+1)\max\{\lambda_1, \lambda_3\}\mu_n \mathbb{B}. \end{aligned}$$

On the other hand,

$$y_n(t_0) - y_{n,i+1} = J^* y_n^*(t_0) - J^* J y_{n,i+1},$$

and

$$\|y_n^*(t_0) - Jy_{n,i+1\|} = \|\frac{t_0 - t_{n,i+1}}{\mu_n}(Jy_{n,i+1} - Jy_{n,i})\| \leq M_1 \mu_n \to 0 \text{ as } n \to +\infty.$$

Since J^* is uniformly continuous on bounded sets, there exists $n_0 > 0$, such that for all $n \geq n_0$, one has

$$\|y_n(t_0) - y_{n,i+1}\| \leq \sigma/2.$$

Thus,

$$y_n(t_0) \in C(t_0, y_n(t_0)) + ((M+1)\max\{\lambda_1, \lambda_3\}\mu_n + \sigma/2)\mathbb{B} \subset C(t_0, \Omega(t_0)) + \sigma\mathbb{B} \;\; \forall n \geq n_1,$$

where $n_1 \geq n_0$ is chosen such that $(M+1)\max\{\lambda_1, \lambda_3\}\mu_n \leq \sigma/2$. Since $y_n(t_0)$ is also bounded by M, the last inclusion implies that

$$y_n(t_0) \in (C(t_0, \Omega(t_0)) \cap (M+\sigma)\mathbb{B}) + \sigma\mathbb{B} \subset (C(t_0, \Omega(t_0)) \cap (M+1)\mathbb{B}) + \sigma\mathbb{B} \;\; \forall n \geq n_1.$$

Hence,

$$\begin{aligned} \gamma(\Omega(t_0)) \quad &= \quad \gamma(\{y_n(t_0) : n \geq n_1\}) \leq \gamma\big((C(t_0, \Omega(t_0)) \cap (M+1)\mathbb{B}) + \sigma\mathbb{B}\big) \\ &\leq \quad \gamma(C(t_0, \Omega(t_0)) \cap (M+1)\mathbb{B}) + \gamma(\sigma\mathbb{B}) \\ &\leq \quad \gamma(C(t_0, \Omega(t_0)) \cap (M+1)\mathbb{B}) + 2\sigma, \end{aligned}$$

which is a contradiction. $\square$ $\square$

4 Second-order Unbounded perturbed State-Dependent Sweeping Processes

Let us consider now the second order unbounded state-dependent sweeping processes with perturbation $(\mathcal{S}_2)$.

Assumption 4 The set-valued map $C : [0,T] \times X \rightrightarrows X$ is $(\lambda_1, \lambda_2, \lambda_3)$-Lipschitz continuous where $\lambda_1, \lambda_2, \lambda_3$ are nonnegative real number, in the sense that for all $t_1, t_2 \in [0,T]$, $x_1, x_2 \in X$ and $y_1, y_2 \in X^*$, one has

$$|d^V_{C(t_1,x_1)}(y_1) - d^V_{C(t_2,x_2)}(y_2)| \le \lambda_1 |t_1 - t_2| + \lambda_2 \|y_1 - y_2\| + \lambda_3 \|Jx_1 - Jx_2\|. \tag{33}$$

Assumption 5 There exists a convex compact set $K \subset X^*$ such that

$$J\big(C(t, M\mathbb{B})\big) \cap M\mathbb{B}^* \subset K, \tag{34}$$

where M is a constant depending only on the initial data defined in (43).
Assumption 6 The set-valued mapping $F : [0,T] \times X^* \times X^* \rightrightarrows X^*$ is upper semi-continuous with convex weak* compact values in X^* and satisfies the following weak linear growth condition, i.e., there exists $k_F > 0$ such that for all $t \ge 0$ and $x, y \in X^*$, one has

$$F(t,x,y) \cap k_F(1 + \|x\| + \|y\|)\mathbb{B}^* \neq \emptyset. \tag{35}$$

Theorem 13. *Let X be a uniformly convex and q-uniformly smooth Banach space $(q \ge 2)$. Suppose that Assumptions 4, 5 and 6 hold. Then for $y_0, u_0 \in X$ satisfying $u_0 \in J(C(0, y_0))$, there exists at least one solution which belongs to $C^{1,1}(0,T;X^*)$ of problem*

$$(\mathcal{S}_2) \quad \begin{cases} (y^*)''(t) \in -N_{C(t,J^*y^*(t))}(J^*(y^*)'(t)) - F(t, y^*(t), (y^*)'(t)) \ \ a.e.\ t \in [0,T], \\ y^*(0) = Jy_0, \ (y^*)'(0) = Ju_0. \end{cases}$$

Proof. We define μ_n, $t_{n,i}$, $I_{n,i}$ as in Theorem 9 and consider the following discretization scheme

$$\begin{array}{rcl} y_{n,0} &=& y_0, \ \ u_{n,0} = y_0 \in C(0, y_0)\ , \\ z^*_{n,0} &\in& F(t_0, Jy_0, Ju_0) \cap k_F(1 + \|y_0\| + \|u_0\|)\mathbb{B}^*; \\ Jy_{n,i+1} &=& Jy_{n,i} + \mu_n Ju_{n,i}\ , \\ z^*_{n,i} &\in& F(t_{n,i}, Jy_{n,i}, Ju_{n,i}) \cap k_F(1 + \|y_{n,i}\| + \|u_{n,i}\|)\mathbb{B}^*; \\ u_{n,i+1} &=& \pi\big(C(t_{n,i+1}, y_{n,i+1}); Ju_{n,i} - \mu_n z^*_{n,i}\big) \text{ for } 0 \le i \le n-1; \end{array} \tag{36}$$

For $t \in I_{n,i}$:

$$\begin{array}{rcl} y^*_n(t) &=& Jy_{n,i} + \dfrac{Jy_{n,i+1} - Jy_{n,i}}{\mu_n}(t - t_{n,i})\ , \ \ y_n(t) = J^* y^*_n(t); \\ u^*_n(t) &=& Ju_{n,i} + \dfrac{Ju_{n,i+1} - Ju_{n,i}}{\mu_n}(t - t_{n,i})\ , \ \ u_n(t) = J^* u^*_n(t); \\ z^*_n(t) &=& z^*_{n,i}\ , \quad \theta_n(t) = t_{n,i}; \quad \eta_n(t) = t_{n,i+1}. \end{array}$$

Using Corollary 6, we have

$$\begin{aligned} a_X\|Ju_{n,i+1}-Ju_{n,i}+\mu_n z^*_{n,i}\| &\le V_*^{1/2}(J^*Ju_{n,i+1}, Ju_{n,i}-\mu_n z^*_{n,i}) \\ &= V_*^{1/2}(u_{n,i+1}, Ju_{n,i}-\mu_n z^*_{n,i}) \\ &= V^{1/2}(Ju_{n,i}-\mu_n z^*_{n,i}, u_{n,i+1}) \\ &= d^V_{C(t_{n,i+1},y_{n,i+1})}(Ju_{n,i}-\mu_n z^*_{n,i}) - d^V_{C(t_{n,i},y_{n,i})}(Ju_{n,i}) \\ &\le \mu_n(\lambda_1+\lambda_2\|z^*_{n,i}\|)+\lambda_3(\|Jy_{n,i+1}-Jy_{n,i}\|) \\ &= \mu_n(\lambda_1+\lambda_2\|z^*_{n,i}\|+\lambda_3\|u_{n,i}\|). \end{aligned} \tag{37}$$

Hence,

$$\begin{aligned} \|u_{n,i+1}\| &\le \|u_{n,i}\| + \frac{\mu_n}{a_X}\big(\lambda_1+(\lambda_2+a_X)\|z^*_{n,i}\|+\lambda_3\|u_{n,i}\|\big) \\ &\le \|u_{n,i}\| + \frac{\mu_n}{a_X}\big(\lambda_1+(\lambda_2+a_X)k_F(1+\|y_{n,i}\|+\|u_{n,i}\|)+\lambda_3\|u_{n,i}\|\big), \end{aligned} \tag{38}$$

where the second inequality comes from the weak linear growth condition (35). On the other hand, due to (36), one has

$$\|y_{n,i+1}\| \le \|y_{n,i}\| + \mu_n\|u_{n,i}\|. \tag{39}$$

From (38) and (39), it follows that

$$\|u_{n,i+1}\|+\|y_{n,i+1}\| \le \|u_{n,i}\|+\|y_{n,i}\|+\mu_n\beta(1+\|u_{n,i}\|+\|y_{n,i}\|), \tag{40}$$

where

$$\beta := \max\{\frac{\lambda_1+(\lambda_2+a_X)k_F}{a_X}, \frac{(\lambda_2+a_X)k_F+\lambda_3}{a_X}+1\}.$$

By induction, one has

$$\|u_{n,i+1}\|+\|y_{n,i+1}\| \le \beta T+\|u_0\|+\|y_0\|+\mu_n\beta\sum_{k=0}^{i}(\|u_{n,k}\|+\|y_{n,k}\|). \tag{41}$$

Using the discrete Gronwall's inequality in Lemma 7, it is easy to deduces that

$$\|u_{n,i}\|+\|y_{n,i}\| \le (\beta T+\|u_0\|+\|y_0\|)e^{\beta T} =: M_1 \ \text{ for } \ i=0,1,\dots,n. \tag{42}$$

Furthermore, from (37) and (38) we deduce

$$\frac{\|Ju_{n,i+1}-Ju_{n,i}\|}{\mu_n} \le \beta(1+\|u_{n,i}\|+\|y_{n,i}\|) \le \beta(1+M_1) =: M. \tag{43}$$

As a result, the sequences of functions $(u_n),(y_n),(u^*_n),(y^*_n)$ are bounded and $u^*_n(\cdot)$ is Lipschitz continuous with the same constant M. Note that $Ju_{n,i}, Ju_{n,i+1} \in J\big(C(t,M\mathbb{B})\big)\cap M\mathbb{B}^* \subset K$, where K is defined in Assumption 5. Hence

$$u^*_n(t) = \frac{t_{n,i+1}-t}{\mu_n}Ju_{n,i}+\frac{t-t_{n,i}}{\mu_n}Ju_{n,i+1} \in K.$$

Thus, for each $t \ge 0$, the set $\Omega^*(t) := \{u^*_n(t) : n \ge 1\}$ is relatively compact in X^*. By using the Arzelà-Ascoli Theorem, there exist a subsequence of $(u^*_n)_{n\in\mathbb{N}}$, still denoted by itself and some

$u^* \in C(0,T;X^*)$ such that $u_n^* \to u^*$ uniformly on I and $(u_n^*)'$ converges weakly to $(u^*)'$ in $L^1(0,T;X^*)$. Clearly, u^* is also Lipschitz continuous with constant M. Let us define the functions

$$y^*(t) = Jy_0 + \int_0^t u^*(s)ds, \quad t \in [0,T], \tag{44}$$

and for $t \in I_{n,i}$

$$\tilde{u}_n^*(t) = Ju_{n,i}. \tag{45}$$

Thus

$$y_n^*(t) = Jy_0 + \int_0^t \tilde{u}_n^*(s)ds, \quad \forall t \in [0,T].$$

One has,

$$\begin{aligned}
\sup_{t\in I} \|y_n^*(t) - y^*(t)\| &\leq \sup_{t\in I} \int_0^t \|\tilde{u}_n^*(s) - u^*(s)\| ds \\
&\leq \int_0^T (\|\tilde{u}_n^*(s) - u_n^*(s)\| + \|u_n^*(s) - u^*(s)\|) ds \\
&\leq MT\mu_n + \int_0^T \|u_n^*(s) - u^*(s)\| ds \to 0 \text{ as } n \to +\infty.
\end{aligned}$$

Thus y_n^* converges to y^* uniformly and $(y^*)' = u^*$. From the construction and Proposition 2, we deduce that

$$\frac{Ju_{n,i+1} - Ju_{n,i}}{\mu_n} + z_{n,i}^* \in -N(C(t_{n,i+1}, y_{n,i+1}); u_{n,i+1}). \tag{46}$$

Let $R := M + k_F(1+2M)$. Using Proposition 3 and (46), one obtains for *a.e.* $t \in [0,T]$ that

$$\begin{aligned}
(u_n^*)'(t) + z_n^*(t) &\in -N(C(\eta_n(t), y_n(\eta_n(t))); u_n(\eta_n(t))) \cap R\mathbb{B}^* \\
&= -R\partial d_{C(\eta_n(t), y_n(\eta_n(t)))}(u_n(\eta_n(t))). \qquad (47)
\end{aligned}$$

Note that

$$\lim_{n\to\infty} \theta_n(t) = \lim_{n\to\infty} \eta_n(t) = t, \lim_{n\to\infty} u_n(\eta_n(t)) = \lim_{n\to\infty} J^* u_n^*(\eta_n(t)) = J^* u^*(t),$$

and

$$\lim_{n\to\infty} y_n(\eta_n(t)) = \lim_{n\to\infty} J^* y_n^*(\eta_n(t)) = J^* y^*(t).$$

We have $u_n(\eta_n(t)) \in C(\eta_n(t), y_n(\eta_n(t)))$ and

$$\begin{aligned}
d^V_{C(t,J^*y^*(t))}(u^*(t)) &= d^V_{C(t,J^*y^*(t))}(u^*(t)) - d^V_{C(\eta_n(t), y_n(\eta_n(t)))}(Ju_n(\eta_n(t))) \\
&\leq \lambda_1 |t - \eta_n(t)| + \lambda_2 \|u^*(t) - u_n^*(\eta_n(t))\| \\
&+ \lambda_3 \|y^*(t) - y_n^*(\eta_n(t))\| \to 0 \text{ as } n \to +\infty.
\end{aligned}$$

Thus $J^* u^*(t) \in C(t, J^* y^*(t))$ since $C(t, J^* y^*(t))$ is closed. Furthermore (z_n^*) is bounded by $k_F(1+2M)$. Thus there exist a subsequence, denoted by itself and some $z^* \in L^1(0,T;X^*)$ such that z_n^* converges weakly to z^* in $L^1(0,T;X^*)$. As a consequence, $(u_n^*)'(t) + z_n^*(t)$ converges weakly to $u' + z^*$ in $L^1(0,T;X^*)$. From Mazur's Lemma, we deduce that

$$(u^*)'(t) + z^*(t) \in \bigcap_n \overline{\text{co}}\{(u_i^*)'(t) + z_i^*(t) : i \geq n\} \text{ for } a.e.\ t \in [0,T]. \tag{48}$$

Fix t satisfying (48) and any $\xi \in X$. The inclusion (48) implies that

$$\begin{aligned}\langle (u^*)'(t) + z^*(t), \xi\rangle &\leq \inf_n \sup_{i\geq n} \langle (u_i^*)'(t) + z_i^*(t), \xi\rangle \\ &\leq \limsup_n \delta(-R\partial d_{C(\eta_n(t), y_n(\theta_n(t)))}(u_n(\eta_n(t))); \xi), \\ &\leq \delta(-R\partial d_{C(t, J^*y^*(t))}(J^*u^*(t)); \xi),\end{aligned}$$

Therefore,

$$(u^*)'(t) + z^*(t) \in -N_{C(t,J^*y^*(t))}(J^*u^*(t)). \tag{49}$$

Similarly, one can prove that $z^*(t) \in F(t, y^*(t), u^*(t))$. In conclusion one has

$$(y^*)''(t) \in -N_{C(t,J^*y^*(t))}(J^*(y^*)'(t)) - F(t, y^*(t), (y^*)'(t)) \;\; a.e. \;\; t \in [0, T],$$

which completes the proof of Theorem 13. $\square$ $\square$

5 Conclusion

In this paper, we studied the existence of solutions for a class of unbounded first and second order sweeping processes with perturbation in uniformly convex and q-uniformly smooth Banach spaces. The boundedness assumption on the moving sets, which is essential in previous works, is unnecessary here. In addition, the compactness assumption is also improved. It is also interesting to consider the nonconvex (for example, prox-regular) sweeping sets in the same settings of reflexive Banach spaces. The major difficulty is that the extension of the notion of prox-regular sets to Banach spaces is not so obvious. There are only few works in this direction and we plan to undertake it in a future research project.

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