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# CUBOS DINÁMICOS DIRECCIONALES PARA $\mathbb{Z}^d$-SISTEMAS MINIMALES

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CUBOS DINÁMICOS DIRECCIONALES PARA $\mathbb{Z}^d$-SISTEMAS MINIMALES

En 2005, B. Host y B. Kra [24] probaron la convergencia de algunos promedios ergódicos múltiples introduciendo para cada $d \in \mathbb{N}$ un factor que caracteriza el comportamiento de estos promedios. En 2010, B. Host, B. Kra y A. Maass [25] estudiaron una contraparte topológica de estos factores para sistemas dinámicos topológicos $(X, T)$, donde $T : X \to X$ es un homeomorfismo de $X$ en sí mismo. En este trabajo introdujeron la estructura de cubos topológicos, denotada por $\mathbf{Q}^{[d]}(X, T)$, y probaron un teorema de estructura para sistemas transitivos con la propiedad de "completación única de la última coordinada de un punto en $\mathbf{Q}^{[d]}(X, T)$". Este teorema de estructura se puede ver como el análogo topológico del teorema de estructura ergódico probado en [24]. Además, introdujeron la relación regionalmente proximal de orden $d$, denotada $\mathbf{RP}^{[d]}(X, T)$, y mostraron en el caso minimal distal que la relación es de equivalencia y que ${}^{X}\!/_{\mathbf{RP}^{[d]}(X,T)}$ es el factor maximal con la propiedad de completación única en $\mathbf{Q}^{[d]}({}^{X}\!/_{\mathbf{RP}^{[d]}(X,T)})$.

En 2014, S. Donoso y W. Sun [7] estudiaron una variante de los cubos topológicos para un sistema minimal $(X, S, T)$, donde $S$ y $T$ son dos homeomorfismos que conmutan. Esta nueva estructura se motiva en la búsqueda de factores característicos para promedios ergódicos múltiples con transformaciones que conmutan. Los autores prueban un teorema de estructura para sistemas minimales con la propiedad de "completación única de la última coordinada de un punto en $\mathbf{Q}_{S,T}(X)$". Introducen además la relación $(S, T)$-regionalmente proximal, denotada por $\mathcal{R}_{S,T}(X)$, que es una variante más débil de la relación regionalmente proximal de primer orden para acciones de $\mathbb{Z}^2$. Finalmente, en el caso distal prueban que la relación $(S, T)$-regionalmente proximal es una relación de equivalencia y que ${}^{X}\!/_{\mathcal{R}_{S,T}(X)}$ es el factor maximal con la propiedad de completación única en $\mathbf{Q}_{S,T}({}^{X}\!/_{\mathcal{R}_{S,T}(X)})$.

En esta tesis generalizamos el concepto de cubos topológicos para sistemas minimales $(X, T_1, \ldots, T_d)$, donde $T_1, \ldots, T_d$ son $d$ homeomorfismos que conmutan, así como la relación $(T_1, \ldots, T_d)$-regionalmente proximal introducidas en [7]. En primer lugar demostramos un teorema estructural para sistemas minimales distales con la propiedad de completación única. Luego, para cada $i \in \{1, \ldots, d\}$ definimos la clase $\mathsf{Z}_0^{e_i}$, que corresponde a la clase de sistemas dinámicos donde la acción $T_i$ es la identidad y describimos, para cada sistema dinámico $(X, T_1, \ldots, T_d)$, su factor $\mathsf{Z}_0^{e_i}$-maximal. Adicionalmente estudiamos las propiedades de los conjuntos de recurrencia para sistemas minimales distales con la propiedad de completación única para la clase de cubos desarrollada en esta tesis.

In 2005, B. Host and B. Kra [24] proved the convergence of some multiple ergodic averages by introducing for each $d \in \mathbb{N}$ a factor that characterizes the behavior of these averages. In 2010, B. Host, B. Kra and A. Maass [25] studied a topological counterpart of these factors for topological dynamic systems $(X,T)$, where $T\colon X \to X$ is a homeomorphism from $X$ to itself. In this work they introduced the structure of topological cubes, denoted by $\mathbf{Q}^{[d]}(X,T)$, and they proved a structure theorem for transitive systems with the property of "unique completion of the last coordinate of a point in $\mathbf{Q}^{[d]}(X,T)$". This structure theorem can be seen as the topological analog of the purely ergodic structure theorem proved in [24]. In addition, they introduced the regionally proximal relation of order $d$, denoted $\mathbf{RP}^{[d]}(X,T)$, and showed in the minimal distal case that the relation is an equivalence relation and that ${}^{X}/_{\mathbf{RP}^{[d]}(X,T)}$ is the maximal factor with the unique completion property in $\mathbf{Q}^{[d]}({}^{X}/_{\mathbf{RP}^{[d]}(X,T)})$.

In 2014, S. Donoso and W. Sun [7] studied a variant of the topological cubes for a minimal system $(X,S,T)$, where $S$ and $T$ are two commuting homeomorphisms. This new structure is motivated in the search of characteristic factors for multiple ergodic averages with commuting transformations. The authors prove a structure theorem for minimal systems with the property of "unique completion of the last coordinate of a point in $\mathbf{Q}_{S,T}(X)$". They also introduce the relation $(S,T)$-regionally proximal, denoted by $\mathcal{R}_{S,T}(X)$, which is a weaker variant of the regionally proximal relation of order one for $\mathbb{Z}^2$-actions. Finally, in the distal case they proved that the relation $(S,T)$-regionally proximal is an equivalence relation and that ${}^{X}/_{\mathcal{R}_{S,T}(X)}$ is the maximal factor with the unique completion property in $\mathbf{Q}_{S,T}({}^{X}/_{\mathcal{R}_{S,T}(X)})$.

In this thesis we generalize the concept of topological cubes for minimal systems $(X,T_1,\dots,T_d)$, where $T_1,\dots,T_d$ are $d$ commuting homeomorphisms, as well as the relation $(T_1,\dots,T_d)$-regionally proximal introduced in [7]. First, we prove a structural theorem for distal minimal systems with the closing parallelepiped property. Then, for each $i \in \{1,\dots,d\}$ we define the class $\mathsf{Z}_0^{e_i}$, which corresponds to the class of dynamical systems where the action $T_i$ is the identity and we describe, for each dynamical system $(X,T_1,\dots,T_d)$, its maximal $\mathsf{Z}_0^{e_i}$-factor. Additionally, we studied the properties of recurrence sets for distal minimal systems with the closing parallelepiped property for the class of cubes developed in this thesis.

*«Introduce a little anarchy, upset the established order,*
*and everything becomes chaos. I'm an agent of chaos,*
*and you know the thing about chaos? It's fair.»*
*The Joker.*

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# Contents

# Introduction

In 1977 H. Furstenberg [15] proved Szemerédi's theorem using ergodic methods. Since this proof, mathematicians started to ask about the $L^2$ convergence of certain averages, called nowadays as multiple ergodic averages. Given a measure-preserving system $(X, \mathcal{B}, \mu, T)$, $d \geq 1$ an integer and $f_1, \ldots, f_d \in L^\infty(X, \mathcal{B}, \mu)$, we ask about of the convergence of the multiple ergodic averages

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) \cdots f_d(T^{dn} x).$$

After approximately 30 years of efforts, this problem was finally solved in 2005 in [24, 37]. The authors proved the convergence using the strategy of characteristic factors. Namely, they defined for each $d \in \mathbb{N}$ a factor $\mathcal{Z}_d$ characterizing the behavior of these averages and in addition they proved that these factors can be endowed with a structure of a nilmanifold. Later, in 2008 T. Tao proved the general commutative case, i.e., where the transformations are not only the powers of a single transformation but they are $d$ commuting transformations [33]. However, until now there is not an idea of which is the structure of the characteristic factors in the general commutative case. His proof does not belong to ergodic theory and does not use characteristic factors. Instead he studied the average in a finitary setting.

In 2010, B. Host, B. Kra, and A. Maass studied a topological counterpart of these characteristic factors for topological dynamical system $(X, T)$, where $X$ is a compact metric space and $T\colon X \to X$ is a homeomorphism from $X$ to itself [25]. They introduced a cube structure, denoted by $\mathbf{Q}^{[d]}(X, T)$ and proved a structure theorem for transitive systems with the property of "completion of the last coordinate of a point in $\mathbf{Q}^{[d]}(X, T)$ in a unique way" on these structures. This condition is referred in this thesis as the *closing parallelepiped property.* The structure theorem for topological dynamical systems can be viewed as an analog of the purely ergodic structure theorem of [24] in the study of a topological counterpart of the characteristic factors introduced by B. Host and B. Kra. In this work the authors introduced the regionally proximal relation of order $d$, $\mathbf{RP}^{[d]}(X, T)$, and provided some relations between $\mathbf{RP}^{[d]}(X, T)$ and $\mathbf{Q}^{[d+1]}(X, T)$. They also proved that $\mathbf{RP}^{[d]}(X, T)$ is an equivalence relation for minimal distal systems and that the quotient space $X/\mathbf{RP}^{[d]}(X,T)$ is the maximal factor with the closing parallelepiped property. Later in 2012, S. Shao and X. Ye [32] proved the same two results for general minimal systems, using a structure theorem for minimal systems and the enveloping semigroup. Finally, Shao and Ye also introduced the regionally proximal relation for general abelian groups and extended the results proved in [25] and [32] in this general context.

In 2014, Donoso and Sun [7] studied a variant of the cube structure for $\mathbb{Z}^2$-minimal systems

$(X, S, T)$, where $X$ is a compact metric space and $S, T : X \to X$ are homeomorphisms from $X$ to itself, in an effort to study a topological counterpart of the characteristic factors for the general commutative case, motivated by Host's construction in [23]. Motivated by the ideas in [25] they proved a structure theorem for systems with the property of "completion of the last coordinate of a point in $\mathbf{Q}_{S,T}(X)$ in a unique way" or closing parallelepiped property. In this work, Donoso and Sun introduced the $(S, T)$-regionally proximal relation, $\mathcal{R}_{S,T}(X)$, which is a weaker variant of the regionally proximal relation of order one $\mathbf{RP}^{[1]}(X, \langle S, T\rangle)$, but for $\mathbb{Z}^2$-actions, associated to the cube structure $\mathbf{Q}_{S,T}(X)$. Finally, in the distal case they proved that the $(S, T)$-regionally proximal relation is an equivalence relation and that $X/\mathcal{R}_{S,T}(X)$ is the maximal factor with the closing parallelepiped property.

The main purpose of this thesis is to extend Donoso and Sun's results for $\mathbb{Z}^d$-minimal systems $(X, T_1, \ldots, T_d)$, where $X$ is a compact metric space and $T_1, \ldots, T_d \colon X \to X$ are commuting homeomeomorphisms from $X$ to itself. We introduce a cube structure for $\mathbb{Z}^d$-minimal actions, denoted by $\mathbf{Q}_{T_1,\ldots,T_d}(X)$, and the $(T_1, \ldots, T_d)$-regionally proximal relation, denoted by $\mathcal{R}_{T_1,\ldots,T_d}(X)$. Both are generalizations of the definitions given in [7]. We prove a structure theorem for systems verifying the closing parallelepiped property in $\mathbf{Q}_{T_1,\ldots,T_d}(X)$ for $\mathbb{Z}^d$-minimal distal systems. By doing so we will to develop techniques for studying the topological counterpart of the characteristic factors in the commutative case.

This thesis is organized in the following way. In Chapter 1 the basic theory of Ergodic Theory and Topological Dynamics is introduced. We provide a complete self-contained survey of the necessary results to understand the work developed in this thesis. In Chapter 2 we present a survey about the historical results of the different cube structures starting with the ergodic motivations, called the multiple ergodic averages. Then we view the Host-Kra-Maass's work which is the first work about the topological counterpart of characteristic factors, and we finished with Donoso and Sun's in the commutative case with 2 commuting transformations. In Chapter 3 we present the directional dynamical cubes for topological dynamical systems $(X, T_1, \ldots, T_d)$ with $d$ commuting transformations. We start this chapter with some general properties of this cube structure and then we introduce the $(T_1, \ldots, T_d)$-regionally proximal relation associated to it. We end this chapter introducing the classes $\mathsf{Z}_0^{e_i}$, which correspond to systems $(X, T_1, \ldots, T_d)$ where the action $T_i$ is trivial, i.e., is the identity and we compute the maximal $\mathsf{Z}_0^{e_i}$-factor for any topological dynamical system. In Chapter 4 we prove a structure theorem for $\mathbb{Z}^d$-minimal distal systems with the closing parallelepiped property. We start with the description of the cube structures and the $(T_1, \ldots, T_d)$-regionally proximal relation for distal systems. We then give the proof of the structure theorem and we prove that the $(T_1, \ldots, T_d)$-regionally proximal relation is an equivalence relation. We end this chapter studying the sets of return times for this kind of systems and we give an explicit example of a system with closing parallelepiped property. In Chapter 5 we give a family of examples of $\mathbb{Z}^d$-minimal distal systems with the closing parallelepiped property.

# Chapter 1

# Background on topological dynamics

In this chapter we introduce the basics of ergodic theory and topological dynamics. We start by providing some basic definitions and results, with the purpose of introducing the vocabulary and notation to be used in this thesis. We then focus in the topological setting providing some definitions and important results related to this thesis that concern equicontinuous, distal, proximal and weakly mixing systems. We finish this chapter with a brief survey of the elementary properties of the enveloping semigroup of a topological dynamical system introduced by Ellis, which is an extremely useful tool in the study of dynamical systems. We refer to [31] and [36] for results in ergodic theory, and [1] and [12] for the results in topological dynamics.

## 1.1 Basic definitions

### 1.1.1 Measure-preserving systems

A *measure-preserving system* is a 4-tuple $(X,\mathcal{B},\mu,G)$, where $(X,\mathcal{B},\mu)$ is a probability space and $G$ is a countable group of measurable and measure-preserving transformations acting on $X$, i.e.,

$$(\forall A\in\mathcal{B})(\forall g\in G)\quad \mu(g^{-1}A)=\mu(A).$$

In some cases, and if there is no confusion, we refer to a measure-preserving system $(X,\mathcal{B},\mu,G)$ simply by $X$. In this thesis we always consider $G$ as a discrete group.

If $(X,\mathcal{B},\mu)$ is a probability space and $T:X\to X$ is a measurable, invertible and measure-preserving transformation, we use $(X,\mathcal{B},\mu,T)$ to denote the measure-preserving system $(X,\mathcal{B},\mu,\{T^n\colon n\in\mathbb{Z}\})$.

If $T_1,\ldots,T_d\colon X\to X$ are $d$ measurable, invertible and measure-preserving commuting transformations, we write $(X,\mathcal{B},\mu,T_1,\ldots,T_d)$ to denote the measure-preserving system $(X,\mathcal{B},\mu,\{T_1^{n_1}\cdots T_d^{n_d}\colon n_1,\ldots,n_d\in\mathbb{Z}\})$. The transformations $T_1,\ldots,T_d$ span a $\mathbb{Z}^d$-action, but we stress that we will consider this action with a given set of generators.

Some classical notions of special measure-preserving systems are the following.

**Definition 1.1.** Let $(X, \mathcal{B}, \mu, G)$ be a measure-preserving system. We say that the system is *ergodic* if for all $A \in \mathcal{B}$ we have that

$$\left[(\forall g \in G)\ \mu(g^{-1}A\Delta A) = 0\right] \implies \mu(A) = 0 \ \vee \ \mu(A) = 1.$$

We now recall the notions of factor and conjugacy in the measure-theoretic framework.

**Definition 1.2.** Let $(X, \mathcal{B}, \mu, G)$ and $(Y, \mathcal{D}, \nu, G)$ be measure-preserving systems. We say that $X$ is a *factor* of $Y$ if there exists a measure-preserving map $\pi \colon Y \to X$ such that $\pi \circ g = g \circ \pi$ for all $G$. We also say that $\pi$ is the *factor map* and $Y$ is an *extension* of $X$. If $\pi$ is a bi-measurable bijection, we say that $\pi$ is an *isomorphism* and that $X$ and $Y$ are *isomorphic*.

As mentioned in [16, Chapter 5] and [18, Chapter 2], a factor of a measure-preserving system is determined, up to isomorphism, by a $G$-invariant sub-$\sigma$-algebra. Thus, given a factor map $\pi : Y \to X$ we can define a conditional expectation operator from $L^2(Y, \mathcal{D}, \nu)$ to $L^2(X, \mathcal{B}, \mu)$.

**Definition 1.3.** Let $\pi : Y \to X$ be a factor map between two measure-preserving systems $(X, \mathcal{B}, \mu, G)$ and $(Y, \mathcal{D}, \nu, G)$ and let $f \in L^2(Y, \mathcal{D}, \nu)$. The *conditional expectation* of $f$ with respect to $X$ is the function $\mathbb{E}(f|X) \in L^2(X, \mathcal{B}, \mu)$ defined by the equation

$$\int_X \mathbb{E}(f|X) \cdot g d\mu = \int_Y f \cdot g \circ \pi d\nu, \ \forall g \in L^2(X, \mathcal{B}, \mu).$$

Another special class of measure-preserving systems are the weakly mixing ones. For the definition we have to consider the Koopman representation of a measure-preserving system. Let $(X, \mathcal{B}, \mu, G)$ be a measure-preserving system. The *Koopman representation* of $(X, \mathcal{B}, \mu, G)$ is the representation $\kappa$ of $G$ on $L^2(X, \mathcal{B}, \mu)$ given by $\kappa(g)(f)(x) = f(g^{-1}x)$.

**Definition 1.4.** A measure-preserving system $(X, \mathcal{B}, \mu, G)$ is *weakly mixing* if the constant functions are the unique eigenvectors of the Koopman representation.

**Proposition 1.5.** *A weakly mixing measure-preserving system is ergodic.*

**Definition 1.6.** A measure-preserving system $(X, \mathcal{B}, \mu, G)$ is *isometric* if every function $f \in L^2(X, \mathcal{B}, \mu)$ is compact, i.e., the set $\{f \circ g \colon g \in G\}$ has a compact closure in the norm topology of $L^2(X, \mathcal{B}, \mu)$.

An important theorem about weakly mixing systems is the following.

**Theorem 1.7.** *An ergodic system $(X, \mathcal{B}, \mu, G)$ is weakly mixing if and only if it admits no nontrivial isometric factors.*

### 1.1.2 Topological dynamical systems

A *topological dynamical system* is a pair $(X, G)$, where $X$ is a compact metric space and $G$ is a group of homeomorphisms of the space $X$ into itself. In this thesis we always consider $G$ as a discrete and countable group.

If $X$ is a compact metric space and $T : X \to X$ is a homeomorphism of $X$ to itself, we use $(X, T)$ to denote the topological dynamical system $(X, \{T^n : n \in \mathbb{Z}\})$.

If $T_1, \dots, T_d : X \to X$ are $d$ commuting homeomorphisms of $X$ to itself, we write $(X, T_1, \dots, T_d)$ instead of $(X, \{T_1^{n_1} \cdots T_d^{n_d} : n_1, \dots, n_d \in \mathbb{Z}\})$. As before, the transformations $T_1, \dots, T_d$ span a $\mathbb{Z}^d$-action, but we stress that we consider this action with a given set of generators.

For a point $x \in X$ we define its *orbit* as the set $\mathcal{O}(x, G) = \{gx : g \in G\}$. If $A \subseteq X$, we say that $A$ is $G$-invariant if $\{gx : g \in G,\ x \in A\} \subseteq A$.

**Definition 1.8.** Let $(X, G)$ be a topological dynamical system. A subset $K \subseteq X$ is called a *minimal set* if $K$ is closed, non-empty, $G$-invariant and has no proper closed non-empty invariant subsets. That is, if $N \subseteq M$ is closed and $G$-invariant, then $N = \emptyset$ or $N = M$.

If $(X, G)$ is a topological dynamical system and $K \subseteq X$ is a minimal set, then we say that $(K, G)$ is a *minimal system*. We have the following result.

**Proposition 1.9.** *Let $(X, G)$ be a topological dynamical system. $X$ is minimal if and only if for all $x \in X$, $\overline{\mathcal{O}(x, G)} = X$.*

Sometimes a system is not minimal, but there exist some points whose orbits are dense in the space.

**Definition 1.10.** Let $(X, G)$ be a topological dynamical system. The system is said to be *transitive* if there exists $x \in X$ such that $\overline{\mathcal{O}(x, G)} = X$.

A subset $\Gamma$ of $G$ is said to be (left) *syndetic* if there is a compact subset $K \subseteq G$ (finite in this case) such that $G = \Gamma K = \{ak : a \in \Gamma,\ k \in K\}$.

**Definition 1.11.** Let $(X, G)$ be a topological dynamical system and $x \in X$. We say that $x$ is an *almost periodic point* if for every neighborhood $U$ of $x$ there exists a syndetic subset $\Gamma$ of $G$ such that $\{gx : g \in \Gamma\} \subseteq U$. The system is said to be *pointwise almost periodic* if every point $x \in X$ is an almost periodic point.

**Theorem 1.12.** *Let $(X, G)$ be a topological dynamical system. Then, $x \in X$ is an almost periodic point if and only if $\overline{\mathcal{O}(x, G)}$ is a minimal set.*

An almost periodic point is also called *minimal point*. Like in the measure-theoretic setting we recall the definitions of factor and conjugacy in the topological dynamics framework.

**Definition 1.13.** Let $(X, G)$ and $(Y, G)$ be two topological dynamical systems. A *factor map* from $Y$ to $X$ is a continuous and onto map $\pi : Y \to X$ such that $\pi(gy) = g\pi(y)$ for all $y \in Y$ and $g \in G$. If $\pi$ is a factor map from $Y$ to $X$ we say that $X$ is a *factor* of $Y$ and that $Y$ is an *extension* of $X$. If $\pi$ is a bi-continuous bijection, we say that $\pi$ is an *isomorphism* and that $X$ and $Y$ are *isomorphic* or *conjugate*.

Observe that if $\pi : Y \to X$ is a factor map, then $R_\pi = \{(y, y') \in Y \times Y : \pi(y) = \pi(y')\}$ is a closed and $G$-invariant equivalence relation. Conversely, if $(Y, G)$ is a topological dynamical system and $R$ is a closed and $G$-invariant equivalence relation in $Y$, then the quotient space ${}^{Y}\!/_{R}$ is a factor of $Y$.

Let $(X, G)$ and $(Y, G)$ be two topological dynamical systems. We have that $X \times Y$ is always an extension of both systems $X$ and $Y$ using the projection to the coordinates as factor maps. But we can construct other extensions of $X$ and $Y$ from $X \times Y$.

**Definition 1.14.** Let $k \geq 1$ be an integer and $(X_1, G), (X_2, G), \ldots, (X_k, G)$ be $k$ topological dynamical systems. A *joining* between $(X_1, G), \ldots, (X_k, G)$ is a closed subset $Z \subseteq X_1 \times \cdots \times X_k$ which is invariant under the diagonal action $g \times \cdots \times g$ ($k$ times) for all $g \in G$ and projects onto the each factor.

Suppose the systems $X_1, \ldots, X_k$ are extensions of a common system $W$ and denote by $\pi_i \colon X_i \to W$ the associated factor maps. We say that a joining $Z \subseteq X_1 \times \cdots \times X_k$ is *relatively independent* with respect to $W$ if for every $i \in \{1, \ldots, k\}$, every $(x_1, \ldots, x_k) \in Z$ and every $x_i, x_i' \in X_i$ with $\pi_i(x_i) = \pi_i(x_i')$ we have

$$(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_k) \in Z.$$

That is, we can freely change a point in a coordinate of the joining by any other point whose projection on the factor $W$ is the same.

The properties of these extensions can be reviewed in [14].

**Remark 1.15.** In this thesis, for finite product metrics we always consider the maximum metric, i.e, if $(X, \rho)$ is a metric space, the metric used in $X \times X$ is

$$\rho((x_1, x_2), (x_3, x_4)) = \max\{\rho(x_1, x_3), \rho(x_2, x_4)\}.$$

A first result about factors and minimality is the following.

**Proposition 1.16.** *Let $\pi : Y \to X$ be a factor map between two topological dynamical systems $(X, G)$ and $(Y, G)$. If $K$ is a minimal subset of $Y$, then $\pi(K)$ is a minimal subset of $X$. In particular, a factor of a minimal system is also a minimal system.*

## 1.2 Equicontinuous Systems

**Definition 1.17.** A topological dynamical system $(X, G)$ is *equicontinuous* if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $d(x, x') < \delta$, then $d(gx, gy) < \varepsilon$ for all $g \in G$.

**Lemma 1.18.** *If $(X, G)$ is equicontinuous, then it is pointwise almost periodic, i.e., every point has a minimal orbit.*

It follows from the previous lemma that an equicontinuous system is minimal if and only if it has a dense orbit.

**Lemma 1.19.** *If $(X_i, G)_{i \in I}$ is an arbitrary collection of equicontinuous systems, then the product system $\left(\prod_{i \in I} X_i, G\right)$ is also equicontinuous.*

Another important result about equicontinuous systems is the following.

**Proposition 1.20.** *A factor of an equicontinuous system is equicontinuous.*

Any topological dynamical system $(X, G)$ has a maximal equicontinuous factor $(X_{eq}, G)$, i.e., $X_{eq}$ is an equicontinuous factor of $X$, and if $Z$ is another equicontinuous factor of $X$, then $Z$ is a factor of $X_{eq}$. The maximal equicontinuous factor is given by the *regionally proximal relation.*

**Definition 1.21.** Let $(X, G)$ be a topological dynamical system. Two points $x, y \in X$ are said to be *regionally proximal* if for all $\varepsilon > 0$ there exist $x', y' \in X$ and $g \in G$ such that $d(x, x') < \varepsilon$, $d(y, y') < \varepsilon$ and $d(gx', gy') < \varepsilon$. We define the *regionally proximal relation* by

$$\mathbf{RP}(X, G) = \{(x, y) \in X \times X \colon x, y \text{ are regionally proximal}\}.$$

The relation $\mathbf{RP}(X, G)$ is $G$-invariant, closed, symmetric and reflexive. But in general this relation may not be transitive. In the case that $G$ is abelian and $X$ is a minimal system we have that $\mathbf{RP}(X, G)$ is an equivalence relation. An important theorem of Gottschalk and Ellis [13] is the characterization of the maximal equicontinuous factor of a topological dynamical system.

**Theorem 1.22.** *[13] Let $(X, G)$ be a topological dynamical system and let $S_{eq}$ be the smallest closed and $G$-invariant equivalence relation containing $\mathbf{RP}(X, G)$. Then ${}^{X}/_{S_{eq}}$ is the maximal equicontinuous factor of $X$.*

It follows that,

**Theorem 1.23.** *The topological dynamical system $(X, G)$ is equicontinuous if and only if $\mathbf{RP}(X, G) = \Delta_X := \{(x, x) \colon x \in X\}$, the diagonal of $X \times X$.*

## 1.3 Distal Systems

**Definition 1.24.** If $(X, G)$ is a topological dynamical system, then $x$ and $y$ in $X$ are said to be *proximal* if and only if $\inf_{g \in G} d(gx, gy) = 0$. If $x$ and $y$ are not proximal they are said to be *distal*.

We write $\mathbf{P}(X, G) = \{(x, y) \in X \times X |\ x \text{ and } y \text{ are proximal}\}$ for the proximal relation in $(X, G)$. It is easy to see that $\mathbf{P}(X, G)$ is a reflexive symmetric $G$-invariant relation, but in general it is not transitive or closed.

The system $(X, G)$ is said to be *proximal* if every pair is proximal and it is called *distal* if there are no non-trivial proximal pairs, i.e., $\mathbf{P}(X, G) = \Delta_X$. It is easy to see that equicontinuous systems are distal, and in the same spirit of Theorem 1.22, Ellis and Gottschalk proved the following result.

**Theorem 1.25.** *[13] Let $(X, G)$ be a topological dynamical system and $P_{eq}$ the smallest closed and $G$-invariant equivalence relation containing $\mathbf{P}(X, G)$. Then, ${}^{X}/{}_{P_{eq}}$ is the maximal distal factor of $X$.*

Interestingly, any point of a system is proximal to a minimal point as stated in the next result.

**Theorem 1.26.** *Let $(X, G)$ be a topological dynamical system and let $x \in X$. Then, there exists an almost periodic point $x^*$ which is proximal to $x$.*

This result has the following useful corollaries.

**Corollary 1.27.** *Let $(X, G)$ be a topological dynamical system. If $(X, G)$ is distal, then it is pointwise almost periodic.*

**Corollary 1.28.** *A distal system is minimal if and only if it is transitive.*

From a structural point of view the next two results provide the main properties of distal systems that we will use in the sequel.

**Lemma 1.29.** *Let $(X_i, G)_{i \in I}$ be a family of topological dynamical systems. Then, the product system $\left(\prod_{i \in I} X_i, G\right)$ is distal if and only if for every $i \in I$ the system $(X_i, G)$ is distal.*

**Theorem 1.30.** *Let $(X, G)$ be a topological dynamical system. Then, the following properties are equivalent:*

*(1) $(X, G)$ is distal.*

*(2) $(X^a, G)$ is distal, for all cardinal numbers $a \geq 1$.*

*(3)* $(X^a, G)$ *is distal, for some cardinal number* $a \geq 1$.

*(4)* $(X^a, G)$ *is pointwise almost periodic, for all cardinal numbers* $a \geq 1$.

*(5)* $(X^a, G)$ *is pointwise almost periodic, for some cardinal number* $a \geq 2$.

We will also need the following results concerning distality and factor maps.

**Proposition 1.31.** *Let* $(X, G)$ *and* $(Y, G)$ *be two topological dynamical systems and let* $\pi : Y \to X$ *be a factor map. Then,* $\pi \times \pi(\mathbf{P}(Y, G)) \subseteq \mathbf{P}(X, G)$. *If* $Y$ *is minimal, then* $\pi \times \pi(\mathbf{P}(Y, G)) = \mathbf{P}(X, G)$. *In addition, a factor of a distal system is distal.*

**Theorem 1.32.** *Let* $(X, G)$ *and* $(Y, G)$ *be distal minimal system, and let* $\pi \colon Y \to X$ *be a factor map. Then the factor map* $\pi$ *is open, meaning that the image of an open set is open.*

## 1.4 Topological Weakly Mixing Systems

At the opposite of distal systems are weakly mixing systems

**Definition 1.33.** The topological dynamical system $(X, G)$ is (topologically) *weakly mixing* if the product system $(X \times X, G)$, where the action of $G$ is the diagonal action, is transitive, i.e., for every four non-empty open sets $U_i$, $i \in \{1, \ldots, 4\}$,

$$N(U_1 \times U_3, U_2 \times U_4) = N(U_1, U_2) \cap N(U_3, U_4) \neq \emptyset,$$

where we define for two sets $A, B \subseteq X$

$$N(A, B) = \{g \in G : gA \cap B \neq \emptyset\}.$$

Equivalently, a topological dynamical system $(X, G)$ is weakly mixing if every non-empty open invariant subset $U$ of $X \times X$ is dense in $X \times X$. It is easy to see that if $(X, G)$ is weakly mixing then $\mathbf{RP}(X, G) = X \times X$, so $(X, G)$ has no non-trivial equicontinuous factor. In fact, in the class of minimal systems which admit an invariant measure this property implies that the system is topologically weak mixing.

**Theorem 1.34.** *Let* $(X, G)$ *be a minimal topological dynamical system which admits an invariant measure. Then, the following properties are equivalent:*

*(1)* $(X, G)$ *is weakly mixing.*

*(2)* $(X, G)$ *has no non-trivial equicontinuous factor (i.e.,* $S_{eq} = X \times X$*).*

*(3)* $(X, G)$ *has no non-trivial distal factor.*

*(4)* $\mathbf{RP}(X, G) = X \times X$.

*(5) The proximal relation* $\mathbf{P}(X, G)$ *is dense in* $X \times X$.

## 1.5 The Enveloping Semigroup of a Topological Dynamical System

If $X$ is a compact metric space we denote by $X^X$ the collection of all maps from $X$ to itself. We endow this set with the product topology or the topology of pointwise convergence. By Tychonoff's theorem, $X^X$ is compact and Hausdorff. We have that $X^X$ has a semigroup structure defined by composition: if $\xi, \eta \in X^X$ then $\xi\eta \in X^X$.

Now we consider a topological dynamical systems $(X, G)$ and we see $G$ as a subset of $X^X$.

**Definition 1.35.** Let $(X, G)$ be a topological dynamical system. We define the *enveloping semigroup* of $(X, G)$ as $E(X, G) = \overline{G}$ the closure of $G$ in $X^X$.

We have that $E(X, G)$ is compact and Hausdorff, usually non-metrizable. Also, the maps $E(X, G) \to E(X, G)$ $p \mapsto pq$ and $p \mapsto gp$ are continuous for all $q \in E(X, G)$ and $g \in G$. We have that $(X^X, G)$ is a topological dynamical system and $(E(X, G), G)$ is a subsystem. In general, the elements of $E(X, G)$ are neither one to one nor onto nor continuous. As a system $(E(X, G), G)$ is transitive, since $\overline{\mathcal{O}(e, G)} = E(X, G)$, where $e$ is the identity in $G$, but in general $(E(X, G), G)$ is not minimal.

Algebraic properties of the enveloping semigroup have a precise translation into dynamical properties of the system. For example we have the following theorem.

**Theorem 1.36.** *A topological dynamical system is distal if and only if its enveloping semigroup is a group.*

In relation to factors maps the enveloping semigroup behaves as stated in the following result.

**Theorem 1.37.** *Let $(X, G)$ and $(Y, G)$ be topological dynamical systems and let $\pi : Y \to X$ be a factor map. Then, there exists a unique continuous semigroup homomorphism $\theta : E(Y, G) \to E(X, G)$ such that $\pi(py) = \theta(p)\pi(y)$ for all $x \in Y$ and $p \in E(Y, G)$.*

A *left ideal* in a enveloping semigroup $E(X, G)$ is a non-empty subset $I \subseteq E(X, G)$ such that $E(X, G)I \subseteq I$. A *minimal left ideal* is one which does not properly contain a left ideal. Observe that a left ideal is also a semigroup. Moreover, if $I$ is a minimal left ideal in $E(X, G)$ and $K$ is a left ideal in the semigroup $I$, then it is easy to see that $K = I$. An idempotent in a enveloping semigroup $E(X, G)$ is an element $u \in E(X, G)$ such that $u^2 = u$. We denote by $J(E(X, G))$ the set of idempotents in the semigroup $E(X, G)$.

We can introduce a quasi-order $<$ on the set $J(E(X, G))$ by defining $v < u$ if and only if $vu = v$. If $v < u$ and $u < v$ we say that $u$ and $v$ are *equivalent* and we write $u \sim v$. An idempotent $u \in J(E(X, G))$ is *minimal* if $v \in J(E(X, G))$ and $v < u$ implies $u < v$.

**Lemma 1.38.** *[11, Lemma 4.4 and Proposition 4.5] Let $I$ be a left ideal of a semigroup $E(X, G)$ and let $u \in J(E(X, G))$. Then, there exists an idempotent $v$ in $Iu$ such that $v < u$.*

*Also, an idempotent is minimal if and only if it is contained in some minimal left ideal.*

The enveloping semigroup is extremely useful in studying proximality. The main connection is reflected in the following theorem.

**Theorem 1.39.** *Let $(X, G)$ be a topological dynamical system and let $x, y \in X$. Then, the following properties are equivalent.*

*(1) $x$ and $y$ are proximal.*

*(2) $px = py$, for some $p \in E(X, G)$.*

*(3) There is a minimal left ideal $I$ in $E(X, G)$ such that $px = py$, for all $p \in I$.*

Finally we have the following theorem which characterizes the minimality of a point $x$ in a topological dynamical system $(X, G)$ using idempotents of its enveloping semigroup.

**Theorem 1.40.** *Let $(X, G)$ be a topological dynamical system and let $I$ be a minimal left ideal in $E(X, G)$. We have that,*

*(1) If $x \in X$, then $Ix$ is a minimal subset of $X$.*

*(2) If $x \in X$ and $v$ is an idempotent in $E(X, G)$, then $(x, vx) \in \mathbf{P}(X, G)$.*

*(3) Let $x \in X$. The following properties are equivalent:*

  *(a) $x$ is an almost periodic point.*

  *(b) $x \in Ix$.*

  *(c) $ux = x$, for some $u \in J(I)$.*

  *In particular, if $(X, G)$ is a minimal system, then for every $x \in X$, there exists $u \in E(X, G)$ idempotent such that $ux = x$.*

# Chapter 2

# Nilfactors and dynamical cubes

In this chapter we introduce the different motivations of the study of directional dynamical cubes in the topological setting. We start with the measure-theoretic setting, stating the Furstenberg's proof of Szemerédi's theorem that led to the study of the norm convergence of some multiple ergodic averages. The considered problems were open for nearly 30 years until Bernard Host and Bryna Kra solved in 2005 the case of one transformation and three years later Terence Tao solved the case of a group generated by finitely many commuting transformations. We then summarize the work of B. Host, B. Kra and A. Maass where they studied a topological counterpart of the characteristic factors introduced by B. Host and B. Kra in 2005. In this work they introduced the notion of dynamical cube and proved an important structure theorem for topological dynamical systems. We end this chapter with a work of Sebastian Donoso and Wenbo Sun with a variant of dynamical cubes for minimal $\mathbb{Z}^2$-actions and an associated structure theorem. This last result is the main motivation of this work.

## 2.1 Multiple ergodic averages

An important connection between ergodic theory, additive combinatorics and number theory started in the 70's with Furstenberg's proof of Szemerédi's theorem via the following ergodic theorem.

**Theorem 2.1.** *[15] Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system and let $A \in \mathcal{B}$ with positive measure. Then, for every $d \geq 1$,*

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-n}A \cap \ldots T^{-dn}A) > 0.$$

So, it is natural to ask about the convergence of these averages, and more generally, about the convergence in $L^2(X, \mathcal{B}, \mu)$ of the multiple ergodic averages

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) \cdots f_d(T^{dn} x),$$

where $f_1, \ldots, f_d \in L^\infty(X, \mathcal{B}, \mu)$. The case $d = 1$ is the standard ergodic theorem of von Neumann. If one assumes that $T$ is weakly mixing, Furstenberg proved in [15] that for every $d \geq 1$ the limit always exists and is constant. However, without the assumption of weakly mixing one can easily show that the limit need not to be constant. Multiple ergodic averages are those for which even if the system is ergodic, the limit is not necessarily constant. This is the case for $d \geq 2$.

Due to Theorem 1.7, the absence of weak mixing implies the existence of "group rotation" factors of the system $(X, \mathcal{B}, \mu, T)$. This also implies the existence of more complex factors called *measure distal systems*. It can be shown that the behavior of some ergodic averages can be reduced to the study of the average in an appropriate distal factor, as those we are showing below. When this is possible, we shall say that the factor is a *characteristic factor* for the average.

After approximately 30 years of efforts, the convergence of the following ergodic averages was finally stated in [24, 37]. The authors proved,

**Theorem 2.2.** *[24] Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system and let $d \geq 1$ be an integer. For $f_i \in L^\infty(X, \mathcal{B}, \mu)$, $1 \leq i \leq d$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x) \cdots f_d(T^{dn} x) \tag{2.1}$$

*exists in $L^2(X, \mathcal{B}, \mu)$.*

In the proof the authors define for each integer $d \geq 1$ the factor $\mathcal{Z}_d$, which are characteristic for these averages. They also proved that these factors can be endowed with the structure of a nilmanifold: they are measurably isomorphic to an inverse limit of ergodic rotations on nilmanifolds. The main ingredient in the proof is the notion of measure-theoretical cube structure and their properties.

Observe that the multiple ergodic averages for commuting transformations, i.e., when we change the transformations $T, T^2, \ldots, T^d$ in (2.1) for commuting transformations $T_1, \ldots, T_d$ was obtained by Tao in [33] using finitary ergodic methods, by Towsner in [34] using non-standard analysis and by Austin in [3] and Host in [23] using more conventional ergodic methods. But until now we do not know the precise structure of the characteristic factors. For instance, it is still open if nilsystems have a role to play there.

## 2.2 Nilfactors and dynamical cubes for $\mathbb{Z}$-actions

### 2.2.1 Nilmanifolds and nilsystems

We start with the definition of $d$-step nilmanifolds.

Let $G$ be a group. For $g, h \in G$ we write $[g, h] = ghg^{-1}h^{-1}$ for the commutator of $g$ and $h$, and for $A, B \subseteq G$ we write $[A, B]$ for the subgroup spanned by $\{[a, b] \colon a \in A, b \in B\}$. The commutator subgroups $G_j, j \geq 1$, are defined inductively by setting $G_1 = G$ and

$G_{j+1} = [G_j, G]$. Let $d \geq 1$ be an integer. We say that $G$ is $d$-step nilpotent if $G_{d+1}$ is the trivial subgroup.

Let $G$ be a $d$-step nilpotent Lie group and $\Gamma$ be a discrete cocompact subgroup of $G$. The compact manifold $X = {}^{G}/_{\Gamma}$ is called a *$d$-step nilmanifold*. The group $G$ acts on $X$ by left translations and we write this action by $(g, x) \mapsto gx$. The Haar measure $\mu$ of $X$ is the unique probability measure on $X$ invariant under this action. Let $\tau \in G$ and $T$ be the transformation $x \mapsto \tau x$ of $X$. Then, $(X, \mathcal{B}, \mu, T)$ is called a *$d$-step nilsystem*.

Nilmanifolds were first introduced and studied by Mal'cev [29] in 1949. But recently its importance has grown in ergodic theory and additive combinatorics in the study of multiple ergodic averages [24], in the structure analysis of measurable and topological systems [24, 25] and in the analysis of certain patterns in a subset of the integers [19]. Here we will talk about applications of nilsystems in topological dynamics, searching for a structure theorem for minimal systems. The structure theorem for topological dynamical systems can be viewed as an analog of the purely ergodic structure theorem of [24] in the study of a topological counterpart of the characteristic factors introduced by Host and Kra.

### 2.2.2 Topological cubes and the regionally proximal relation of order $d$

Let $X$ be a set, let $d \geq 2$ be an integer, and write $[d] = \{1, 2, \ldots, d\}$. We view $\{0,1\}^d$ in one of two ways, either as a sequence $\varepsilon = \varepsilon_1 \ldots \varepsilon_d$ of 0's and 1's written without commas or parentheses; or as a subset of $[d]$. A subset $\varepsilon$ corresponds to the sequence $\varepsilon_1 \ldots \varepsilon_d \in \{0,1\}^d$ such that $i \in \varepsilon$ if and only if $\varepsilon_i = 1$ for $i \in [d]$.

We denote $X^{2^d}$ by $X^{[d]}$. A point $\mathbf{x} \in X^{[d]}$ can be written in one of two equivalent ways, depending on the context:

$$\mathbf{x} = (x_\varepsilon \colon \varepsilon \in \{0,1\}^d) = (x_\varepsilon \colon \varepsilon \subseteq [d]).$$

For $x \in X$, we write $x^{[d]} = (x, x, \ldots, x) \in X^{[d]}$ and the diagonal of $X^{[d]}$ is $\Delta^{[d]} = \{x^{[d]} \colon x \in X\}$.

A point $\mathbf{x} \in X^{[d]}$ can be decomposed as $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$ with $\mathbf{x}', \mathbf{x}'' \in X^{[d-1]}$, where $\mathbf{x}' = (x_{\varepsilon 0} \colon \varepsilon \in \{0,1\}^{d-1})$ and $\mathbf{x}'' = (x_{\varepsilon 1} \colon \varepsilon \in \{0,1\}^{d-1})$. We can also isolate the first coordinate, writing $X_*^{[d]} = X^{2^d - 1}$ and then writing a point $\mathbf{x} \in X^{[d]}$ as $\mathbf{x} = (x, \mathbf{x}_*)$, where $\mathbf{x}_* = (x_\varepsilon \colon \varepsilon \subseteq [d],\ \varepsilon \neq \emptyset) \in X_*^{[d]}$.

The *faces* of dimension $r$ of a point $\mathbf{x} \in X^{[d]}$ are defined as follows. Let $J \subseteq [d]$ with $|J| = d - r$ and $\xi \in \{0,1\}^{d-r}$. The elements $(x_\varepsilon \colon \varepsilon \in \{0,1\}^d, \varepsilon_J = \xi)$ of $X^{[r]}$ are called *faces* of dimension $r$ of $\mathbf{x}$, where $\varepsilon_J = (\varepsilon_i \colon i \in J)$.

Identifying $\{0,1\}^d$ with the set of vertices of the Euclidean unit cube, an Euclidean isometry of the unit cube permutes the vertices of the cube and thus the coordinates of a point $\mathbf{x} \in X^{[d]}$. These permutations are the *Euclidean permutations* of $X^{[d]}$.

Let $(X, T)$ be a topological dynamical system and $d$ an integer. We define $\mathbf{Q}^{[d]}(X, T)$ to be the closure in $X^{[d]}$ of the elements of the form

$$(T^{\mathbf{n} \cdot \varepsilon} x \colon \varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{0,1\}^d),$$

where $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $x \in X$. As an illustration, $\mathbf{Q}^{[2]}(X,T)$ is the closure in $X^{[2]}$ of the set

$$\{(x, T^n x, T^m x, T^{n+m} x) \colon x \in X, n.m \in \mathbb{Z}\},$$

and $\mathbf{Q}^{[3]}(X,T)$ is the closure in $X^{[3]}$ of the set

$$\{(x, T^n x, T^m x, T^{n+m} x, T^p x, T^{n+p} x, T^{m+p} x, T^{n+m+p} x) \colon x \in X, n, m, p \in \mathbb{Z}\}.$$

It is important to observe that $\mathbf{Q}^{[d]}(X,T)$ is invariant under the Euclidean permutations of $X^{[d]}$.

As mentioned before, the cube structure for topological dynamical systems was introduced in [25] as the topological counterpart of the theory of measure-theoretical cubes developed in [24].

The following structure theorem relates the notion of cube and nilsystems.

**Theorem 2.3.** *[25, Theorem 1.2] Assume that $(X,T)$ is a transitive topological dynamical system and let $d \geq 1$ be an integer. The following properties are equivalent:*

*(1) If $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{Q}^{[d+1]}(X,T)$ have $2^{d+1} - 1$ coordinates in common, then $\boldsymbol{x} = \boldsymbol{y}$.*

*(2) If $x, y \in X$ are such that $(x, y, \ldots, y) \in \mathbf{Q}^{[d+1]}(X,T)$, then $x = y$.*

*(3) $X$ is an inverse limit of minimal $d$-step nilsystems.*

*A transitive system satisfying either of the equivalent properties above is called a $d$-*step nilsystem *or a* system of order *$d$.*

The cube structure $\mathbf{Q}^{[d+1]}(X,T)$ also allows us to build the maximal factor of order $d$ for a topological dynamical system $(X,T)$. Let $(X,T)$ be a topological dynamical system and $d \geq 1$ be an integer. A pair $(x,y) \in X \times X$ is said to be *regionally proximal of order $d$* if for any $\delta > 0$ there exist $x', y' \in X$ and $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ such that $\rho(x, x') < \delta$, $\rho(y, y') < \delta$ and

$$\rho(T^{\mathbf{n}\cdot\varepsilon} x', T^{\mathbf{n}\cdot\varepsilon} y') < \delta$$

for every nonempty subset $\varepsilon \subseteq [d]$, where $\rho$ is the metric on $X$.

The set of regionally proximal pairs of order $d$ is denoted by $\mathbf{RP}^{[d]}(X,T)$, and is called *the regionally proximal relation of order $d$.* We remark that when $d = 1$, $\mathbf{RP}^{[1]}(X,T)$ is nothing but the regionally proximal relation $\mathbf{RP}(X,T)$ defined in Chapter 1.

It is easy to see that $\mathbf{RP}^{[d]}(X,T)$ is a closed and invariant relation for all $d \in \mathbb{N}$. We also have that

$$\mathbf{P}(X,T) \subseteq \ldots \mathbf{RP}^{[d+1]}(X,T) \subseteq \mathbf{RP}^{[d]}(X,T) \subseteq \ldots \subseteq \mathbf{RP}^{[2]}(X,T) \subseteq \mathbf{RP}^{[1]}(X,T) = \mathbf{RP}(X,T).$$

**Lemma 2.4.** *[25, Lemma 3.3] Let $(X,T)$ be a minimal system and let $d \geq 1$ be an integer. Take $x, y \in X$. Then, $(x,y) \in \mathbf{RP}^{[d]}(X,T)$ if and only if there exists $\boldsymbol{a}_* \in X_*^{[d]}$ such that $(x, \boldsymbol{a}_*, y, \boldsymbol{a}_*) \in \mathbf{Q}^{[d+1]}(X)$.*

Host, Kra and Maass [25] proved the following theorems in the minimal distal case, then Shao and Ye [32] proved the same theorems for general minimal systems.

**Theorem 2.5.** *[25, 32] Let $(X, T)$ be a minimal topological dynamical system and let $d \in \mathbb{N}$. Then,*

*(1) $\mathbf{RP}^{[d]}(X, T)$ is an equivalence relation.*

*(2) The quotient of $X$ under $\mathbf{RP}^{[d]}(X, T)$ is the maximal $d$-step nilfactor of $X$.*

In particular, $(X, T)$ is a system of order $d$ if and only if the regionally proximal relation of order $d$ coincides with the diagonal relation. Furthermore, it is easy to see that a system of order $d$ is distal.

**Theorem 2.6.** *[32, Theorem 6.4] Let $\pi : (Y, T) \to (X, T)$ be a factor map between the minimal systems $(X, T)$ and $(Y, T)$ and let $d \in \mathbb{N}$. Then, $\pi \times \pi(\mathbf{RP}^{[d]}(Y, T)) = \mathbf{RP}^{[d]}(X, T)$.*

The study of nilsystems took another course since the work of Omar Antolin Camarena and Balázs Szegedy [6] with a more abstract definition of a nilsystem, and the works of Yonatan Gutman, Freddie Mannes and Peter Varjú in [20, 21, 22] concerning the structure theory of nilspaces. But in the topological and measure-theoretical setting it is still open the existence of cubes structures for a finitely generated commutative group action.

## 2.3 Dynamical cubes for $\mathbb{Z}^2$-actions

In 2014, Donoso and Sun [7] studied a variant of the cube structure defined in the previous section in an effort to study a topological counterpart of the characteristic factors in the general commutative case, motivated by Host's construction in [23]. Given a compact metric space $X$ and two commuting homeomorphisms $S, T \colon X \to X$ they introduced the space of dynamical cubes $\mathbf{Q}_{S,T}(X)$ as

$$\mathbf{Q}_{S,T}(X) = \overline{\{(x, S^n x, T^m x, S^n T^m x) \colon x \in X,\ n, m \in \mathbb{Z}\}} \subseteq X^4.$$

Another cube structures related to $\mathbf{Q}_{S,T}(X)$ are the following

$$\begin{aligned} \mathbf{Q}_S(X) &= \overline{\{(x, S^n x) \colon x \in X, n \in \mathbb{Z}\}}, \\ \mathbf{Q}_T(X) &= \overline{\{(x, T^m x) \colon x \in X, m \in \mathbb{Z}\}}. \end{aligned}$$

With the same idea of the work of Host, Kra and Maass in [25] they proved a structure theorem for systems with the property "completion of the last coordinate of a point in $\mathbf{Q}_{S,T}(X)$ in a unique way" or also called *closing parallelepiped property*. This relation served to study product structures inherent to the system $(X, S, T)$ and in particular to identify a "product behaviour". A product system is one of the form $(Y \times W, \sigma \times \mathrm{id}, \mathrm{id} \times \tau)$, where $(Y, \sigma)$ and $(W, \tau)$ are topological dynamical systems.

Associated to this new cube structure they defined a relation in $X$ called the $(S, T)$-*regionally proximal relation* in the following way

**Definition 2.7.** Let $(X, S, T)$ be a minimal system with commuting transformations $S$ and $T$. We define

$$\begin{aligned}
\mathcal{R}_S(X) &= \{(x, y) \in X \times X : (x, y, a, a) \in \mathbf{Q}_{S,T}(X) \text{ for some } a \in X\}, \\
\mathcal{R}_T(X) &= \{(x, y) \in X \times X : (x, b, y, b) \in \mathbf{Q}_{S,T}(X) \text{ for some } b \in X\}, \\
\mathcal{R}_{S,T}(X) &= \mathcal{R}_S(X) \cap \mathcal{R}_T(X).
\end{aligned}$$

The authors proved the following structure theorem for systems $(X, S, T)$.

**Theorem 2.8.** *[7, Theorem 1.1] Let $(X, S, T)$ be a minimal system with commuting transformations $S$ and $T$. The following properties are equivalent:*

*(1) $(X, S, T)$ is a factor of a product system.*

*(2) If $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{Q}_{S,T}(X)$ have three coordinates in common, then $\boldsymbol{x} = \boldsymbol{y}$.*

*(3) $\mathcal{R}_S(X) = \Delta_X$.*

*(4) $\mathcal{R}_T(X) = \Delta_X$.*

*(5) $\mathcal{R}_{S,T}(X) = \Delta_X$.*

For the proof of this structure theorem they introduced the so called topological magic extensions, motivated by Host's work [23]. In this extension B. Host found a characteristic factor that looks like the Cartesian product of single transformations.

A minimal system $(X, S, T)$ with commuting transformations $S$ and $T$ is called a *magic system* if $\mathcal{R}_S(X) \cap \mathcal{R}_T(X) = \mathbf{Q}_S(X) \cap \mathbf{Q}_T(X)$. Notice that in principle the relations $\mathbf{Q}_S(X)$ and $\mathbf{Q}_T(X)$ are much easier to compute than $\mathcal{R}_S(X)$ and $\mathcal{R}_T(X)$. The term magic reflects then that computing $\mathcal{R}_S(X) \cap \mathcal{R}_T(X)$ is not that complicated.

**Lemma 2.9.** *[7, Proposition 3.10] Let $(X, S, T)$ be a minimal system with commuting transformations $S$ and $T$. Then $(X, S, T)$ admits a minimal magic extension, i.e., it has an extension which is a minimal magic system.*

Additionally, in the distal case, they proved another related result.

**Theorem 2.10.** *[7] Let $(X, S, T)$ be a minimal distal system with commuting transformations $S$ and $T$. Then*

*(1) $\mathbf{Q}_S(X)$, $\mathbf{Q}_T(X)$ and $\mathcal{R}_{S,T}(X)$ are closed equivalence relations on $X$.*

*(2) $(X/\mathcal{R}_{S,T}(X), S, T)$ is the maximal factor of $(X, S, T)$ having a product extension.*

# Chapter 3

# Directional dynamical cubes for $d$ commuting transformations

In this chapter we present the notion of directional dynamical cubes for a topological dynamical system $(X, T_1, \ldots, T_d)$ with $d$ commuting transformations. This is a generalization of the dynamical cubes introduced by S. Donoso and W. Sun in [7] whose main properties were discussed in Chapter 2. We start the chapter with some general properties of the cube structure and then we introduce the $(T_1, \ldots, T_d)$-regionally proximal relation associated with the cube structure. We end the chapter introducing the classes $\mathsf{Z}_0^{e_i}$, which correspond to systems $(X, T_1, \ldots, T_d)$, where the action $T_i$ is trivial, i.e., is the identity, and we compute the maximal $\mathsf{Z}_0^{e_i}$-factor for any topological dynamical system. This notion will be used to describe systems where our cube structure has the closing parallelepiped property.

## 3.1 Notation

Let $d \geq 2$ be an integer and consider $T_1, \ldots, T_d \colon X \to X$, $d$ commuting homeomorphisms of $X$. As was mentioned in Chapter 1, we write $(X, T_1, \ldots, T_d)$ to denote the topological dynamical system $(X, \{T_1^{n_1} \cdots T_d^{n_d} \colon \ n_1, \ldots, n_d \in \mathbb{Z}\})$. The transformations $T_1, \ldots, T_d$ span a $\mathbb{Z}^d$-action. Throughout this thesis we always use $G \cong \mathbb{Z}^d$ to denote the group generated by $T_1, \ldots, T_d$.

**Definition 3.1.** Let $d \geq 2$ be an integer and $(X, T_1, \ldots, T_d)$ be a topological dynamical system with commuting transformations $T_1, \ldots, T_d$. For $j \in [d]$, the *$j$-th face transformation* $T_j^{[d]} \colon X^{[d]} \to X^{[d]}$ is defined for every $\mathbf{x} \in X^{[d]}$ and every $\varepsilon \subseteq [d]$ by:

$$(T_j^{[d]}\mathbf{x})_\varepsilon = \begin{cases} T_j x_\varepsilon & \text{if } j \in \varepsilon, \\ x_\varepsilon & \text{if } j \notin \varepsilon. \end{cases}$$

The *face group of dimension $d$* is the group $\mathcal{F}_{T_1,\ldots,T_d}$ of transformations of $X^{[d]}$ generated by the face transformations. Let $G$ be the group generated by the transformations $T_1, \ldots, T_d$. We have that $G \cong \mathbb{Z}^d$. We write $G_{[d]}^{\Delta} = \{g^{[d]} \colon g \in G\}$. Let $\mathcal{G}_{T_1,\ldots,T_d}$ denote the subgroup of $G^{[d]}$ generated by $\mathcal{F}_{T_1,\ldots,T_d}$ and $G_{[d]}^{\Delta}$.

Using the same notation of Section 2.2, for every $j \in [d]$ we define the following transformations

$$
\begin{array}{rccl}
\Phi_j\colon & \{0,1\}^d & \to & \{0,1\}^d \\
 & \varepsilon & \mapsto & \Phi_i(\varepsilon) = \left\{ \begin{array}{cl} \varepsilon_k & \text{if } k \neq j, \\ 1-\varepsilon_k & \text{if } k = j. \end{array} \right.
\end{array}
$$

$$
\begin{array}{rccl}
\Psi_j^0\colon & \{0,1\}^{d-1} & \to & \{0,1\}^d \\
 & \varepsilon & \mapsto & \Psi_0^j(\varepsilon) = \varepsilon_1\varepsilon_2 \ldots \varepsilon_{j-1} 0 \varepsilon_j \ldots \varepsilon_{d-1}.
\end{array}
$$

$$
\begin{array}{rccl}
\Psi_j^1\colon & \{0,1\}^{d-1} & \to & \{0,1\}^d \\
 & \varepsilon & \mapsto & \Psi_0^j(\varepsilon) = \varepsilon_1\varepsilon_2 \ldots \varepsilon_{j-1} 1 \varepsilon_j \ldots \varepsilon_{d-1}.
\end{array}
$$

Let $X$ be a set. From now on, when we consider a point $\mathbf{x} \in X^8$ we will represent it as in Figure 3.1.

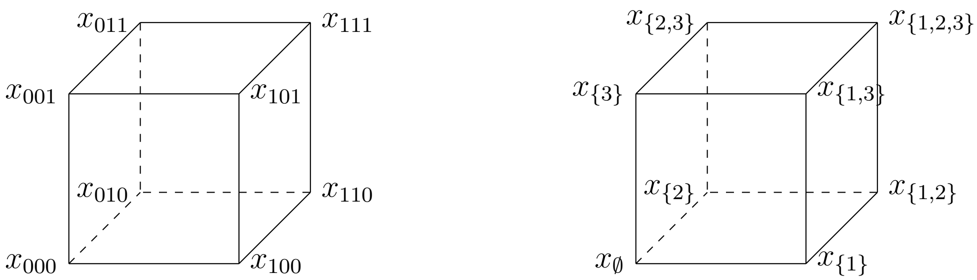



**Figure 3.1:** Representations of a point in $X^8$.

Thus, we have that the face permutations $\Phi_1, \Phi_2, \Phi_3$ in the case $d = 3$ illustrate the following Euclidean permutations when applied to the coordinates of a point in $X^8$ as in the Figure 3.2.

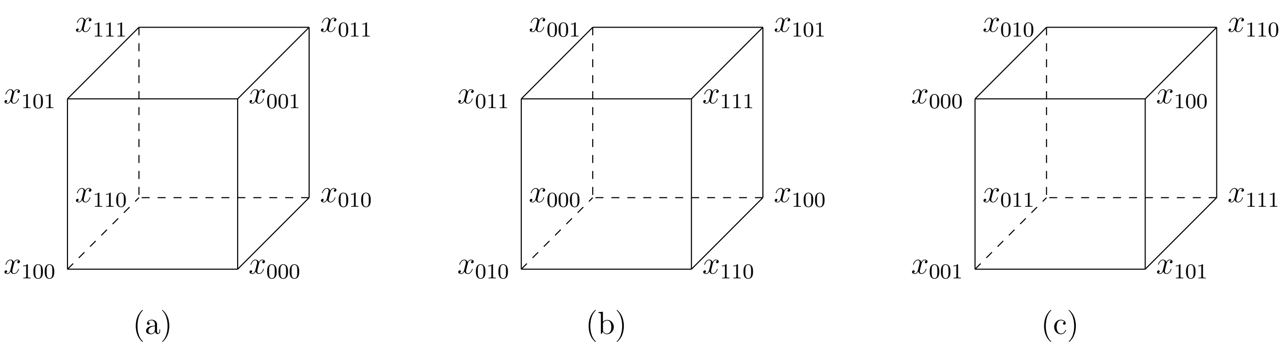



**Figure 3.2:** The figures (a), (b) and (c) represent $\Phi_1(x)$, $\Phi_2(x)$ and $\Phi_3(x)$, where $x$ is the point of Figure 3.1 in the case $d = 3$.

The maps $\Psi_j^0$ and $\Psi_j^1$ are used to replicate a face. Let $d \geq 2$ be an integer, $j \in [d]$ and $\mathbf{x} \in X^{[d-1]}$. If we define $\mathbf{y} \in X^{[d]}$ by $y_\varepsilon = x_\eta$ if $\varepsilon = \Psi_j^0(\eta) \;\vee\; \varepsilon = \Psi_j^1(\eta)$, we have that the face that is determined by fixing the value of $j$ in the coordinates is equal to $\mathbf{x}$, i.e.,

$(y_\varepsilon\colon \varepsilon(j)=0), (y_\varepsilon\colon \varepsilon(j)=0) \in X^{[d-1]}$ and

$$(y_\varepsilon\colon \varepsilon(j)=0) = (y_\varepsilon\colon \varepsilon(j)=1) = \mathbf{x}.$$

## 3.2 Directional dynamical cubes for $d$ commuting transformations

### 3.2.1 Directional dynamical cubes

We introduce the notion of directional dynamical cubes for a system with $d$ commuting transformations and we study its basic properties.

**Definition 3.2.** Let $d \geq 2$ be an integer and let $(X, T_1, \ldots, T_d)$ be a topological dynamical system with commuting transformations $T_1, \ldots, T_d$. We define,

$$\mathbf{Q}_{T_1,\ldots,T_d}(X) = \overline{\big\{(T_1^{n_1\varepsilon_1}\cdots T_d^{n_d\varepsilon_d}x)_{\varepsilon\in\{0,1\}^d}\colon x \in X, \mathbf{n} = (n_1,\ldots,n_d) \in \mathbb{Z}^d\big\}} \subseteq X^{[d]}.$$

We define the same structure for $\{j_1,\ldots,j_k\} \subseteq [d]$ as

$$\mathbf{Q}_{T_{j_1},\ldots,T_{j_k}}(X) = \overline{\big\{(T_{j_1}^{n_1\varepsilon_1}\cdots T_{j_k}^{n_k\varepsilon_k}x)_{\varepsilon\in\{0,1\}^k}\colon x \in X, \mathbf{n} = (n_1,\ldots,n_k) \in \mathbb{Z}^k\big\}}.$$

For example, if $k = 1$ and $j \in [d]$ we have

$$\mathbf{Q}_{T_j}(X) = \overline{\big\{(x, T_j^n x)\colon x \in X, n \in \mathbb{Z}\big\}}.$$

For $x_0 \in X$, we define

$$\mathbf{K}_{T_1,\ldots,T_d}^{x_0} = \overline{\Big\{(T_1^{n_1\varepsilon_1}\ldots T_d^{n_d\varepsilon_d}x_0)_{\varepsilon\in\{0,1\}^d\setminus\{\vec{0}\}}\colon \mathbf{n} = (n_1,\ldots,n_d) \in \mathbb{Z}^d\Big\}}.$$

As an example, for $d = 3$ we have

$$\begin{array}{ll}
Q_{T_1,T_2,T_3}(X) & = \overline{\{(x, T_1^n x, T_2^m x, T_1^n T_2^m x, T_3^p x, T_1^n T_3^p x, T_2^m T_3^p x, T_1^n T_2^m T_3^p x) : x \in X,\ n, m, p \in \mathbb{Z}\}},\\
Q_{T_1,T_2}(X) & = \overline{\{(x, T_1^n x, T_2^m x, T_1^n T_2^m x) : x \in X,\ n, m \in \mathbb{Z}\}},\\
Q_{T_1,T_3}(X) & = \overline{\{(x, T_1^n x, T_3^p x, T_1^n T_3^p x) : x \in X,\ n, p \in \mathbb{Z}\}},\\
Q_{T_2,T_3}(X) & = \overline{\{(x, T_2^m x, T_3^p x, T_2^m T_3^p x) : x \in X,\ m, p \in \mathbb{Z}\}},\\
Q_{T_1}(X) & = \overline{\{(x, T_1^n x) : x \in X,\ n \in \mathbb{Z}\}},\\
Q_{T_2}(X) & = \overline{\{(x, T_2^m x) : x \in X,\ m \in \mathbb{Z}\}},\\
Q_{T_3}(X) & = \overline{\{(x, T_3^p x) : x \in X,\ p \in \mathbb{Z}\}},\\
K_{T_1,T_2,T_3}^{x_0} & = \overline{\{(T_1^n x_0, T_2^m x_0, T_1^n T_2^m x_0, T_3^p x_0, T_1^n T_3^p x_0, T_2^m T_3^p x_0, T_1^n T_2^m T_3^p x_0) : n, m, p \in \mathbb{Z}\}}.
\end{array}$$

This generalizes the definitions given by Donoso and Sun in [7] for two commuting transformations. In general we have that

$$\mathbf{K}_{T_1,\ldots,T_d}^{x_0} \neq \mathbf{Q}_{T_1,\ldots,T_d}(x_0) = \{\mathbf{a}_* \in X_*^{[d]}\colon (x, \mathbf{a}_*) \in \mathbf{Q}_{T_1,\ldots,T_d}(X)\},$$

but with an identical proof of Lemma 4.5 in [17] we have that there exists a $G_\delta$-dense subset $A \subseteq X$ such that $\mathbf{K}^{x_0}_{T_1,\dots,T_d} = \mathbf{Q}_{T_1,\dots,T_d}(x_0)$ for every $x_0 \in A$. We say that $x_0 \in X$ is a *continuity point* if $\mathbf{K}^{x_0}_{T_1,\dots,T_d} = \mathbf{Q}_{T_1,\dots,T_d}(x_0)$. We start with some basic properties of these cubes structures.

**Proposition 3.3.** *Let $d \geq 2$ be an integer and $(X, T_1, \dots, T_d)$ be a topological dynamical system with commuting transformations $T_1, \dots, T_d$. Then,*

*(1) $x^{[d]} \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ for every $x \in X$.*

*(2) $\mathbf{Q}_{T_1,\dots,T_d}(X)$ is invariant under $\mathcal{G}_{T_1,\dots,T_d}$.*

*(3) (Face permutation invariance)Let $\boldsymbol{x} \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ and $j \in [d]$. If $\boldsymbol{y} \in X^{[d]}$ is defined as*

$$y_\varepsilon = x_{\Phi_j(\varepsilon)},\ \forall \varepsilon \in \{0,1\}^d,$$

*then $\boldsymbol{y} \in \mathbf{Q}_{T_1,\dots,T_d}(X)$.*

*(4) (Projection) Let $\boldsymbol{x} \in \mathbf{Q}_{T_1,\dots,T_d}(X)$, $\{j_1, \dots, j_k\} \subseteq [d]$ and $\xi \in \{0,1\}^{d-k}$. Then, we have that*

$$(x_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon_{[d]\setminus\{j_1,\dots,j_k\}} = \xi) \in \mathbf{Q}_{T_{j_1},\dots,T_{j_k}}(X).$$

*(5) (Duplication) Let $\{j_1, \dots, j_k\} \subseteq [d]$ and $\boldsymbol{x} = (x_\eta \colon \eta \in \{0,1\}^k) \in \mathbf{Q}_{T_{j_1},\dots,T_{j_k}}(X)$. We have that if $\boldsymbol{y} \in X^{[d]}$ is defined such that for $\varepsilon \in \{0,1\}^d$,*

$$y_\varepsilon = x_\eta \iff \forall \ell \in \{1, \dots, k\},\ \varepsilon_{i_\ell} = \eta_\ell,$$

*then $\boldsymbol{y} \in \mathbf{Q}_{T_1,\dots,T_d}(X)$.*

*(6) $(x, y) \in \mathbf{Q}_{T_j}(X) \Leftrightarrow (y, x) \in \mathbf{Q}_{T_j}(X)$, for all $x, y \in X$ and for all $j \in [d]$.*

This proposition shows the basic structural properties that have these cubes structures. Property (3) shows that some Euclidean permutations leave invariant $\mathbf{Q}_{T_1,\dots,T_d}(X)$. However, if $d = 2$, and $(x_0, x_1, x_2, x_3) \in \mathbf{Q}_{T_1,T_2}(X)$, we may have that $(x_0, x_2, x_1, x_3) \notin \mathbf{Q}_{T_1,T_2}(X)$. But, we can assure that $(x_0, x_2, x_1, x_3) \in \mathbf{Q}_{T_2,T_1}(X)$. Roughly speaking, this lack of symmetry makes the problem harder because we do not expect to get the strong algebraic consequences that one obtains in the case $T_i = T^i$ like in [24]. Property (4) shows that the projection of the faces of a cube is a cube of the dimension of the face. Finally property (5) shows the lifting property of cubes in order to obtain a cube with a higher dimension.

**Proof.** (1) Take $x \in X$ and $\vec{0} \in \mathbb{Z}^d$ in the definition.

(2) It is direct to see that $\mathbf{Q}_{T_1,\dots,T_d}(X)$ is invariant under $G^{\Delta}_{[d]}$ and $\mathcal{F}_{T_1,\dots,T_d}(X)$.

(3) Let $(x^i)_{i\in\mathbb{N}} \subseteq X$ and $(\mathbf{n}(i))_{i\in\mathbb{N}} \subseteq \mathbb{Z}^d$ such that

$$\forall \varepsilon \in \{0,1\}^d, \quad x_\varepsilon = \lim_{i\to\infty} T_1^{n_1(i)\varepsilon_1} \cdots T_d^{n_d(i)\varepsilon_d} x^i.$$

Let $y^i = T_j^{n_j(i)} x^i$ and $\mathbf{m}(i) = (n_1(i), \dots, -n_j(i), \dots, n_d(i)) \in \mathbb{Z}^d$. For $\varepsilon \in \{0,1\}^d$ we have

$$\begin{aligned} y_\varepsilon &= \lim_{i\to\infty} T_1^{m_1(i)\varepsilon_1} \cdots T_j^{m_j(i)\varepsilon_j} \cdots T_d^{m_d(i)\varepsilon_d} y^i, \\ &= \lim_{i\to\infty} T_1^{n_1(i)\varepsilon_1} \cdots T_{j-1}^{n_{j-1}(i)\varepsilon_{j-1}} T_j^{n_j(1-\varepsilon_j)} T_{j+1}^{n_{j+1}(i)\varepsilon_{j+1}} \cdots T_d^{n_d(i)\varepsilon_d} x^i, \\ &= x_{\Phi_j(\varepsilon)}. \end{aligned}$$

We conclude that $\mathbf{y} \in \mathbf{Q}_{T_1,\dots,T_d}(X)$.

(4) Let $(x^i)_{i\in\mathbb{N}} \subseteq X$ and $(\mathbf{n}(i))_{i\in\mathbb{N}} \subseteq \mathbb{Z}^d$ such that

$$\forall \varepsilon \in \{0,1\}^d, \quad x_\varepsilon = \lim_{i\to\infty} T_1^{n_1(i)\varepsilon_1} \cdots T_d^{n_d(i)\varepsilon_d} x^i.$$

Now, consider $\eta \in \{0,1\}^d$ such that

$$\eta_\ell = \begin{cases} 0 & \ell \in \{j_1, \dots, j_k\}, \\ \eta_\ell & \ell \notin \{j_1, \dots, j_k\}, \end{cases}$$

and put $y^i = \mathbf{x}^i_\eta$. If $\varepsilon \in \{0,1\}^d$ is such that $\varepsilon_{[d]\setminus\{j_1,\dots,j_k\}} = \xi$, then

$$\begin{aligned} x_\varepsilon &= \lim_{i\to\infty} T_1^{n_1(i)\varepsilon_1} \cdots T_d^{n_d(i)\varepsilon_d} x^i, \\ &= \lim_{i\to\infty} T_{j_1}^{n_{j_1}(i)\varepsilon_{j_1}} \cdots T_{j_k}^{n_{j_k}(i)\varepsilon_{j_k}} y^i. \end{aligned}$$

We conclude that $(x_\varepsilon \colon \varepsilon \in \{0,1\}^d, \varepsilon_{[d]\setminus\{j_1,\dots,j_k\}} = \xi) \in \mathbf{Q}_{T_{j_1},\dots,T_{j_k}}(X)$.

(5) Let $(x^i)_{i\in\mathbb{N}} \subseteq X$ and $(\mathbf{n}(i))_{i\in\mathbb{N}} \subseteq \mathbb{Z}^d$ such that

$$\forall \varepsilon \in \{0,1\}^d, \quad x_\varepsilon = \lim_{i\to\infty} T_1^{n_1(i)\varepsilon_1} \cdots T_d^{n_d(i)\varepsilon_d} x^i.$$

Then, there exists $\mathbf{m} \in \mathbb{Z}^k$ such that

$$\forall \eta \in \{0,1\}^k, \quad x_\eta = \lim_{i\to\infty} T_{j_1}^{m_{j_1}(i)\varepsilon_{j_1}} \cdots T_{j_k}^{m_{j_k}(i)\varepsilon_{j_k}} x^i.$$

If we use $\mathbf{n}(i) \in \mathbb{Z}^d$ such that

$$n_p(i) = \begin{cases} m_{j_\ell} & \text{if } p = j_\ell, \text{ for some } j_\ell \in \{j_1, \dots, j_k\}, \\ 0 & \text{if } p \notin \{j_1, \dots, j_k\}, \end{cases}$$

we conclude the statement.

(6) This follows directly from definitions.

$\square$

It is easy to see that the relation $\mathbf{Q}_{T_j}(X)$ is symmetric, reflexive, closed and $T_j$-invariant. In [17] Eli Glasner proved that $\mathbf{Q}_{T_j}[x] = \{y \in X : (x,y) \in \mathbf{Q}_{T_j}(X)\}$ is a transitive set for a $G_\delta$-dense subset of $X$. But in general this relation is not an equivalence relation (for an example see [35]).

By Proposition 3.3 (2) we have that $(\mathbf{Q}_{T_1,\dots,T_d}(X), \mathcal{G}_{T_1,\dots,T_d})$ is a topological dynamical system. Moreover, we have

**Proposition 3.4.** *Let $d \geq 2$ be an integer and $(X, T_1, \dots, T_d)$ be a minimal system with commuting transformations $T_1, \dots, T_d$. Then, $(\mathbf{Q}_{T_1,\dots,T_d}(X), \mathcal{G}_{T_1,\dots,T_d})$ is a minimal system. Furthermore, if $(X, T_1, \dots, T_d)$ is distal, then $(\mathbf{Q}_{T_1,\dots,T_d}(X), \mathcal{G}_{T_1,\dots,T_d})$ is also distal.*

**Proof.** The proof is similar to Proposition 3.4 in [7]. Let $E(\mathbf{Q}_{T_1,\dots,T_d}(X), \mathcal{G}_{T_1,\dots,T_d})$ be the enveloping semigroup of the system $(\mathbf{Q}_{T_1,\dots,T_d}(X), \mathcal{G}_{T_1,\dots,T_d})$. For every $\varepsilon \in \{0,1\}^d$, let $\pi_\varepsilon \colon \mathbf{Q}_{T_1,\dots,T_d}(X) \to X$ be the projection onto the $\varepsilon$-th coordinate and let $\pi_\varepsilon^* \colon E(\mathbf{Q}_{T_1,\dots,T_d}(X), \mathcal{G}_{T_1,\dots,T_d}) \to E(X,G)$ be the respective semigroup homomorphism.

Let $u \in E(\mathbf{Q}_{T_1,\dots,T_d}, G^\Delta_{[d]})$ denote a minimal idempotent for the system $(\mathbf{Q}_{T_1,\dots,T_d}, G^\Delta_{[d]})$. We show that $u$ is also a minimal idempotent in $E(\mathbf{Q}_{T_1,\dots,T_d}(X), \mathcal{G}_{T_1,\dots,T_d})$. By Theorem 1.40, it suffices to show that if $v \in E(\mathbf{Q}_{T_1,\dots,T_d}(X), \mathcal{G}_{T_1,\dots,T_d})$ with $vu = v$, then $uv = u$. Projecting onto the corresponding coordinates, we deduce that $\pi^*_\varepsilon(vu) = \pi^*_\varepsilon(v)\pi^*_\varepsilon(u) = \pi^*_\varepsilon(v)$ for every $\varepsilon \in \{0,1\}^d$. The projection of a minimal idempotent of $E(\mathbf{Q}_{T_1,\dots,T_d}(X), G^\Delta_{[d]})$ is a minimal idempotent in $E(X,G)$. Then, we have that $\pi^*_\varepsilon(u)\pi^*_\varepsilon(v) = \pi^*_\varepsilon(u)$ for every $\varepsilon \in \{0,1\}^d$. Since $\mathbf{Q}_{T_1,\dots,T_d}(X) \subseteq X^{[d]}$ we view the elements of $E(\mathbf{Q}_{T_1,\dots,T_d}(X), \mathcal{G}_{T_1,\dots,T_d})$ as vectors of dimension $2^d$, then the projections to the coordinates determine an element of $E(\mathbf{Q}_{T_1,\dots,T_d}(X), \mathcal{G}_{T_1,\dots,T_d})$. Hence, we have that $uv = u$. Therefore, we conclude that $u$ is a minimal idempotent in $E(\mathbf{Q}_{T_1,\dots,T_d}(X), \mathcal{G}_{T_1,\dots,T_d})$.

Now, let $x \in X$. Since $X$ is minimal, by Theorem 1.40 there exists a minimal idempotent $u \in E(X,G)$ such that $ux = x$. Consider $u^{[d]} \in E(\mathbf{Q}_{T_1,\dots,T_d}(X), G^\Delta_{[d]})$. We have that $u^{[d]}x^{[d]} = x^{[d]}$, so by Theorem 1.40 $x^{[d]}$ is a minimal point in $X^{[d]}$ (so in $\mathbf{Q}_{T_1,\dots,T_d}(X)$) under the action $G^\Delta_{[d]}$. We observe that the point $x^{[d]}$ is minimal under the action $\mathcal{G}_{T_1,\dots,T_d}$ since $u^{[d]}$ is also a minimal idempotent in $E(\mathbf{Q}_{T_1,\dots,T_d}(X), \mathcal{G}_{T_1,\dots,T_d})$. As $\overline{\mathcal{O}(x^{[d]}, \mathcal{G}_{T_1,\dots,T_d})} = \mathbf{Q}_{T_1,\dots,T_d}(X)$, we conclude that $(\mathbf{Q}_{T_1,\dots,T_d}(X), \mathcal{G}_{T_1,\dots,T_d})$ is a minimal system.

Now, if $(X, T_1, \dots, T_d)$ is distal, then $(X^{[d]}, \mathcal{G}_{T_1,\dots,T_d})$ is also distal. Additionally, since $\mathbf{Q}_{T_1,\dots,T_d}(X)$ is invariant under $\mathcal{G}_{T_1,\dots,T_d}$ we also obtain that $(\mathbf{Q}_{T_1,\dots,T_d}(X), \mathcal{G}_{T_1,\dots,T_d})$ is distal. $\square$

From previous result we can conclude that for every $\{j_1, \dots, j_k\} \subseteq [d]$ the system $\mathbf{Q}_{T_{j_1,\dots,j_k}}(X)$ is minimal under the action of the group $\mathcal{G}_{T_{j_1},\dots,T_{j_k}}$. Particularly, for every transformation $T_j$ the system $\mathbf{Q}_{T_j}(X)$ is minimal under the action generated by $g \times g$ for $g \in G$ and $\mathrm{id} \times T_j$. However, the system $(\mathbf{K}^{x_0}_{T_1,\dots,T_d}, \mathcal{F}^{x_0}_{T_1,\dots,T_d})$ is not necessarily minimal, despite of Theorem 3.1 in [32], because the minimality of this system implies the minimality of $\overline{\mathcal{O}(x_0, T_j)}$ for every $j \in [d]$.

**Proposition 3.5.** *Let $d \geq 2$ be an integer and let $\pi\colon Y \to X$ be a factor map between two minimal systems $(Y, T_1, \ldots, T_d)$ and $(X, T_1, \ldots, T_d)$ with commuting transformations $T_1, \ldots, T_d$. Then,*

$$\pi^{[d]}(\mathbf{Q}_{T_1,\ldots,T_d}(Y)) = \mathbf{Q}_{T_1,\ldots,T_d}(X).$$

**Proof.** It is easy to see that $\pi^{[d]}(\mathbf{Q}_{T_1,\ldots,T_d}(Y)) \subseteq \mathbf{Q}_{T_1,\ldots,T_d}(X)$ and by minimality we have the equality.

Let $\mathbf{x} \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$, $(x^i)_{i\in\mathbb{N}} \subseteq X$ and $(\mathbf{n}(i))_{i\in\mathbb{N}} \subseteq \mathbb{Z}^d$ such that

$$\forall \varepsilon \in \{0,1\}^d, \quad x_\varepsilon = \lim_{i\to\infty} T_1^{n_1(i)\varepsilon_1} \cdots T_d^{n_d(i)\varepsilon_d} x^i.$$

We take an arbitrary $y^i \in \pi^{-1}(x^i)$. By compactness, we can assume that $y^i \to y$ and for all $\varepsilon \in \{0,1\}^d$ we can assume that

$$\lim_{i\to\infty} T_1^{n_1(i)\varepsilon_1} \cdots T_d^{n_d(i)\varepsilon_d} y^i = y_\varepsilon.$$

Now, by continuity of $\pi$, we have that $\pi(y^i) \to \pi(y) = x$ and that for $\varepsilon \in \{0,1\}^d$

$$\pi(y_\varepsilon) = \pi\left(\lim_{i\to\infty} T_1^{n_1(i)\varepsilon_1} \cdots T_d^{n_d(i)\varepsilon_d} y^i\right) = \lim_{i\to\infty} T_1^{n_1(i)\varepsilon_1} \cdots T_d^{n_d(i)\varepsilon_d} \pi\left(y^i\right) = x_\varepsilon.$$

Then, $\mathbf{y} \in \mathbf{Q}_{T_1,\ldots,T_d}(Y)$ and $\pi^{[d]}(\mathbf{y}) = \mathbf{x} \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$. □

**Remark.** Previous result is also true for every subset of $[d]$, i.e., if $\{j_1, \ldots, j_k\} \subseteq [d]$, then

$$\pi^{[k]}\left(\mathbf{Q}_{T_{j_1},\ldots,T_{j_k}}(Y)\right) = \mathbf{Q}_{T_{j_1},\ldots,T_{j_k}}(X).$$

Particularly, $\pi \times \pi(\mathbf{Q}_{T_j}(X)) = \mathbf{Q}_{T_j}(Y)$, where $j \in [d]$.

### 3.2.2 The $(T_1, \ldots, T_d)$-regionally proximal relation

We define a relation in $X$ associated with this cube structure as in Chapter 2. We introduce the $(T_1, \ldots, T_d)$-*regionally proximal relation* which generalizes the definition given by Donoso and Sun for two commuting transformations in [7].

**Definition 3.6.** Let $d \geq 2$ be an integer and let $(X, T_1, \ldots, T_d)$ be a topological dynamical system with commuting transformations $T_1, \ldots, T_d$. For $x, y \in X$, $\mathbf{a}_* \in X_*^{[d-1]}$ and $j \in [d]$, we define $\mathbf{z}(x, y, \mathbf{a}_*, j) \in X^{[d]}$ as the point such that

$$z_\varepsilon = \begin{cases} x & \text{if } \varepsilon = \emptyset, \\ y & \text{if } \varepsilon = \{j\}, \\ (\mathbf{a}_*)_\eta & \text{if } \varepsilon = \Psi_j^0(\eta) \ \vee \ \varepsilon = \Psi_j^1(\eta). \end{cases}$$

We define the $T_j$-regionally proximal relation as

$$\mathcal{R}_{T_j}(X) = \left\{(x,y) \in X \times X \colon \exists \mathbf{a}_* \in X_*^{[d-1]},\ \mathbf{z}(x,y,\mathbf{a}_*,j) \in \mathbf{Q}_{T_1,\dots,T_d}(X)\right\}.$$

Finally, we define the $(T_1,\dots,T_d)$-regionally proximal relation as

$$\mathcal{R}_{T_1,\dots,T_d}(X) = \bigcap_{j=1}^{d} \mathcal{R}_{T_j}(X).$$

As an example, for $d=3$, the relations are defined as follows

$$\begin{aligned}
\mathcal{R}_{T_1}(X) &= \{(x,y) \in X \times X \colon (x,y,a,a,b,b,c,c) \in \mathbf{Q}_{T_1,T_2,T_3}(X) \text{ for some } a,b,c \in X\},\\
\mathcal{R}_{T_2}(X) &= \{(x,y) \in X \times X \colon (x,a,y,a,b,c,b,c) \in \mathbf{Q}_{T_1,T_2,T_3}(X) \text{ for some } a,b,c \in X\},\\
\mathcal{R}_{T_3}(X) &= \{(x,y) \in X \times X \colon (x,a,b,c,y,a,b,c) \in \mathbf{Q}_{T_1,T_2,T_3}(X) \text{ for some } a,b,c \in X\}.
\end{aligned}$$

The representations of these relations are the following,

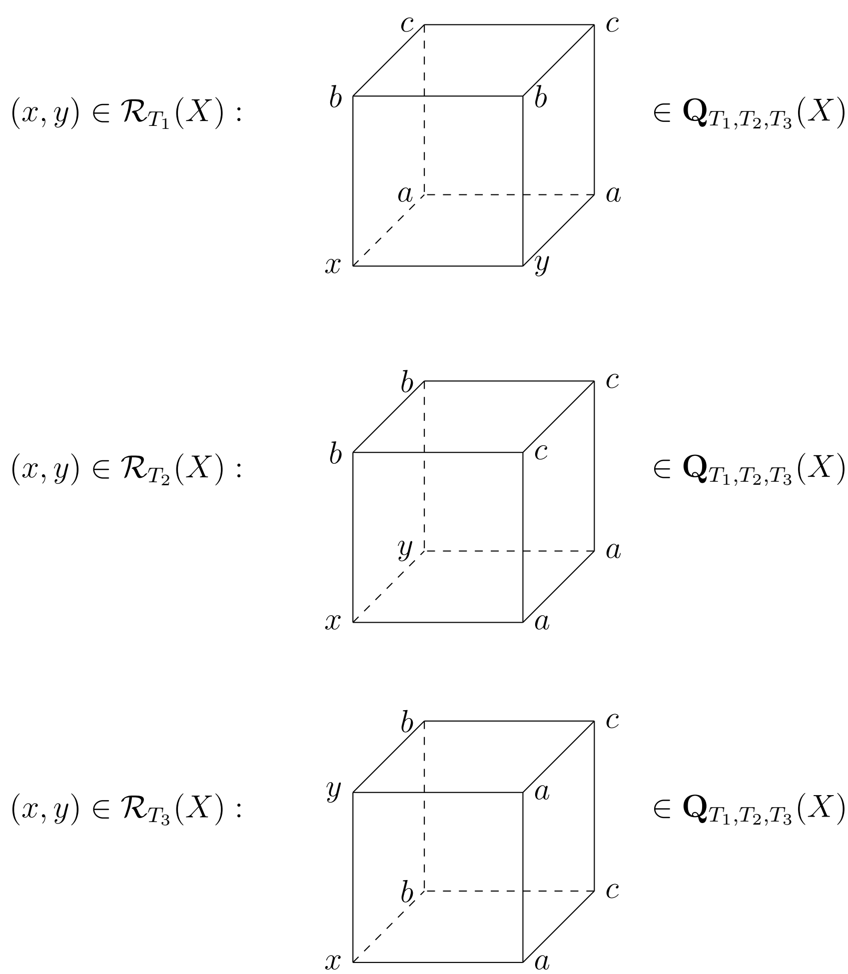



**Figure 3.3:** Representation of the relations $\mathcal{R}_{T_1}(X)$, $\mathcal{R}_{T_2}(X)$ and $\mathcal{R}_{T_3}(X)$ in the case $d=3$.

**Remark 3.7.** We remark that $(x, y) \in \mathcal{R}_{T_d}(X)$ if and only if there exists $\mathbf{a}_* \in X_*^{[d-1]}$ such that $(x, \mathbf{a}_*, y, \mathbf{a}_*) \in \mathbf{Q}_{T_1,\dots,T_d}(X)$.

It is easy to see that the relations are reflexive, symmetric, closed and invariant under $G$. In the next chapter we prove that these relations are transitive in the distal case, but we do not know if these relations are transitive in the general minimal case. The next proposition follows from the definition. We introduce this notion because it is useful in order to describe systems with the closing parallelepiped property.

**Proposition 3.8.** *Let $d \geq 2$ be an integer and let $\pi\colon Y \to X$ be a factor map between two minimal systems $(Y, T_1, \dots, T_j)$ and $(X, T_1, \dots, T_d)$ with commuting transformations $T_1, \dots, T_d$. Then, $\pi^{[d]}(\mathcal{R}_{T_j}(Y)) \subseteq \mathcal{R}_{T_j}(X)$, where $j \in [d]$. In particular,*

$$\pi^{[d]}(\mathcal{R}_{T_1,\dots,T_d}(Y)) \subseteq \mathcal{R}_{T_1,\dots,T_d}(X).$$

We will study minimal systems $(X, T_1, \dots, T_d)$ with commuting transformations $T_1, \dots, T_d$ which have the following property:

**Definition 3.9.** Let $d \geq 2$ be an integer and let $(X, T_1, \dots, T_d)$ be a minimal system with commuting transformations $T_1, \dots, T_d$. We say that $X$ has the *closing parallelepiped property* if $\mathbf{x}, \mathbf{y} \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ have $2^d - 1$ coordinates in common, then $\mathbf{x} = \mathbf{y}$.

**Proposition 3.10.** *Let $d \geq 2$ be an integer and let $(X, T_1, \dots, T_d)$ be a minimal system with commuting transformations $T_1, \dots, T_d$ which has the closing parallelepiped property. Then, for every $j \in [d]$ we have that $\mathcal{R}_{T_j}(X) = \Delta_X$.*

**Proof.** Let $x, y \in X$ be such that $(x, y) \in \mathcal{R}_{T_j}(X)$. Then, by definition, there exists $\mathbf{a}_* \in X_*^{[d]}$ such that $\mathbf{z}(x, y, \mathbf{a}_*, j) \in \mathbf{Q}_{T_1,\dots,T_d}(X)$. By Proposition 3.3 (4), we have that $(x, \mathbf{a}_*) \in \mathbf{Q}_{T_1,\dots,T_{j-1},T_j,\dots,T_d}(X)$. Define $\mathbf{w} \in X^{[d]}$ as

$$w_\varepsilon = \begin{cases} x & \text{if } \varepsilon = \emptyset, \\ x & \text{if } \varepsilon = \{j\}, \\ (\mathbf{a}_*)_\eta & \text{if } \varepsilon = \Psi_j^0(\eta) \ \vee \ \varepsilon = \Psi_j^1(\eta). \end{cases}$$

By Proposition 3.3 (5), we have that $\mathbf{w} \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ and therefore the closing parallelepiped property implies that $x = y$. $\square$

The proof of Proposition 3.10 in the case $d = 3$ and $j = 1$ can be illustrated as follows:

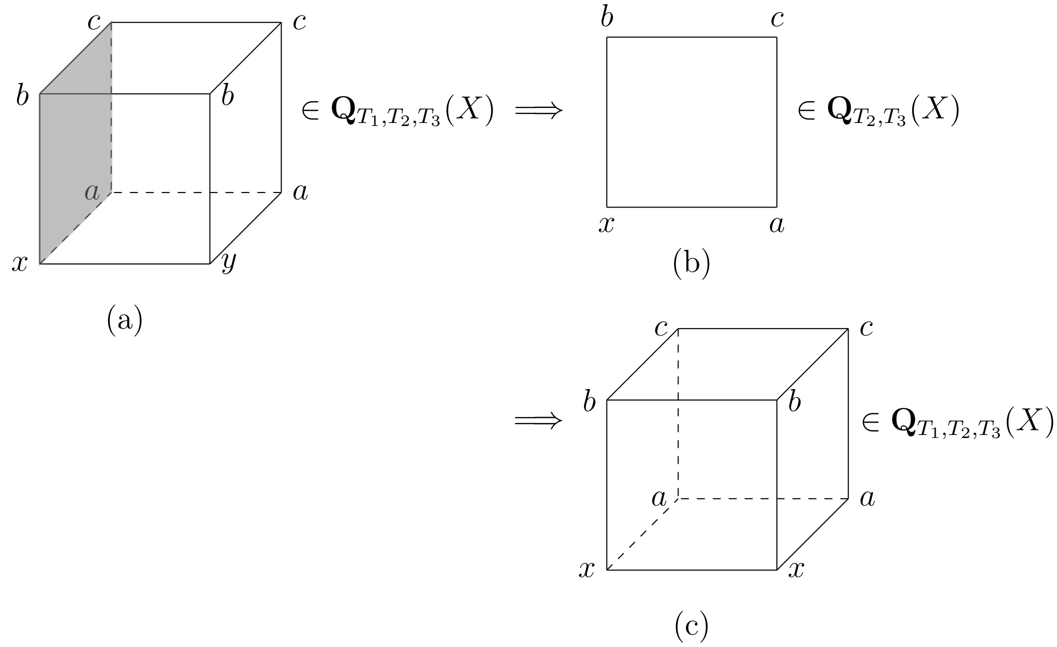



**Figure 3.4:** Illustration of the proof of Proposition 3.10 in the case $d = 3$ and $j = 1$. We have the existence of a cube like in (a) because $(x, y) \in \mathcal{R}_{T_1}(X)$. By Proposition 3.3 (4), the cube in (b) belongs to $\mathbf{Q}_{T_2,T_3}(X)$. Finally, by Proposition 3.3 (5), the cube in (c) belongs to $\mathbf{Q}_{T_1,T_2,T_3}(X)$.

## 3.3 The classes $\mathsf{Z}_0^{e_j}$

In this section we define for every $j \in [d]$ the class of dynamical systems $(X, T_1, \ldots, T_d)$ where the action $T_j$ is the identity, denoted by $\mathsf{Z}_0^{e_j}$. This notation is analogous as the one used by Austin in [2] for the measure-theoretical setting. Then we compute the maximal $\mathsf{Z}_0^{e_j}$-factor for any system $(X, T_1, \ldots, T_d)$.

Consider the following classes of topological dynamical systems. For every $j \in [d]$ define

$$\mathsf{Z}_0^{e_j} = \{(X, T_1, \ldots, T_d) \colon T_j \text{ is the identity on } X\}.$$

We remark that these classes satisfy the following properties:

- The trivial system (one point) belongs to $\mathsf{Z}_0^{e_j}$.
- $\mathsf{Z}_0^{e_j}$ is productive, i.e., any product of systems in $\mathsf{Z}_0^{e_j}$ is also in $\mathsf{Z}_0^{e_j}$.
- $\mathsf{Z}_0^{e_j}$ is hereditary, i.e., a closed invariant subsystem of any system in $\mathsf{Z}_0^{e_j}$ is in $\mathsf{Z}_0^{e_j}$.
- $\mathsf{Z}_0^{e_j}$ is closed under isomorphism.

Using a Zorn's Lemma argument it can be proved that for every system $(X, T_1, \ldots, T_d)$ there exists a maximal $\mathsf{Z}_0^{e_j}$-factor [1, Chapter 9]. In addition, this factor can be characterized.

For a minimal system $(X, T_1, \ldots, T_d)$ with commuting transformations $T_1, \ldots, T_d$, we define $\sigma_{T_j}(X)$ as the smallest closed and $T_j$-invariant equivalence relation which contains the relation $\mathbf{Q}_{T_j}(X)$.

**Lemma 3.11.** *Let $(X, T_1, \dots, T_d)$ be a minimal system with commuting transformations $T_1, \dots, T_d$ and take $j \in [d]$. Then, ${}^{X}/_{\sigma_{T_j}(X)}$ is the maximal $\mathsf{Z}_0^{e_j}$-factor of $(X, T_1, \dots, T_d)$.*

**Proof.** Let $\pi : X \to {}^{X}/_{\sigma_{T_j}(X)}$. We have that ${}^{X}/_{\sigma_{T_j}(X)} \in \mathsf{Z}_0^{e_j}$. Indeed, if $x, x' \in X$ are such that $x' \in \mathcal{O}(X, T_j)$, then $\pi(x) = \pi(x')$, because $T_j[x] = [T_j x]$ and since $(x, T_j x) \in \sigma_{T_j}(X)$, then $T_j[x] = [x]$. So $T_j$ acts trivially on ${}^{X}/_{\sigma_{T_j}(X)}$. Now, let $Z$ be a factor of $X$ in $\mathsf{Z}_0^{e_j}$ and $\pi' \colon X \to Z$ be the factor map. We define $R_{\pi'} = \{(x, y) \in X \colon \pi'(x) = \pi'(y)\}$. We have to prove that $\sigma_{T_j(X)} \subseteq R_{\pi'}$. In fact, since $R_{\pi'}$ is closed and $T_j$-invariant equivalence relation, it suffices to prove that $\mathbf{Q}_{T_j}(X) \subseteq R_{\pi'}$. Fix $x \in X$. We have that

$$T_j \circ \pi = \pi' \circ T_j = \pi' \circ \mathrm{id}_Z \,.$$

Thus $\{(x, T_j^n x)\}_{n \in \mathbb{N}} \subseteq R_{\pi'}$. Since $R_\pi$ is closed we conclude that $\mathbf{Q}_{T_j}(X) \subseteq R_{\pi'}$. $\square$

# Chapter 4

# The structure theorem for minimal distal systems with the closing parallelepiped property

In this chapter we prove the following structure theorem for minimal distal systems, which is the main result in this thesis:

**Theorem 4.1.** *Let $d \geq 2$ be an integer and $(X, T_1, \ldots, T_d)$ be a minimal distal system with commuting transformations $T_1, \ldots, T_d$. Then, the following statements are equivalent:*

*(1) $X$ has the closing parallelepiped property, i.e., if $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$ have $2^d - 1$ coordinates in common, then $\boldsymbol{x} = \boldsymbol{y}$.*

*(2) $\mathcal{R}_{T_1,\ldots,T_d}(X) = \Delta_X$.*

*(3) $X$ has a minimal distal extension $(Y, \hat{T}_1, \ldots, \hat{T}_d)$ which is a joining of the systems $Y/\mathbf{Q}_{\hat{T}_j}$ for $j \in [d]$ and is relatively independent with respect to the systems $Y/\mathbf{Q}_{\hat{T}_{j_1}}/\mathbf{Q}_{\hat{T}_{j_2}}$, where $j_1, j_2 \in [d]$ with $j_1 \neq j_2$.*

Rougly speaking, the extension $Y$ of point (3) can be constructed using $\mathbb{Z}^{d-1}$-minimal distal actions. We start with some important properties of dynamical cubes and the $(T_1, \ldots, T_d)$-regionally proximal relation in minimal distal systems. We then give a proof of the structure theorem for distal systems. We finish the chapter studying a property of the systems with the closing parallelepiped property, which corresponds to the sets of recurrence induced by these systems.

## 4.1 Directional dynamical cubes for minimal distal systems

In this section we present the main properties of directional dynamical cubes for minimal distal systems.

### 4.1.1 The $\mathsf{Z}_0^{e_j}$-maximal factor for distal systems

An important consequence of the distality of a system is the following.

**Proposition 4.2.** *Let $d \geq 2$ be an integer, $(X, T_1, \ldots, T_d)$ be a minimal distal system with commuting transformations $T_1, \ldots, T_d$ and $j \in [d]$ be fixed. Then, $\mathbf{Q}_{T_j}(X)$ is a closed and invariant equivalence relation of $X$ for every $j \in [d]$.*

**Proof.** The proof is similar to the one of Lemma 5.2 in [7]. We only need to prove transitivity. Let $(x, y), (y, z) \in \mathbf{Q}_{T_j}(X)$. Pick any $a \in X$. Then, $(a, a) \in \mathbf{Q}_{T_j}(X)$. By Proposition 3.4, there exists a sequence $(g_n)_{n\in\mathbb{N}} = ((g_n', g_n''))_{n\in\mathbb{N}} \in \mathcal{G}_{T_j}$ such that $g_n(x, y) = (g_n'x, g_n''y)$ converges to $(a, a)$, where $\mathcal{G}_{T_j}$ is the group generated by $\mathrm{id} \times T_j$ and $g \times g$, $g \in G$. We can assume, by compactness, that $g_n''z \to u$ and thus $(g_n''y, g_n''z) \to (a, u) \in \mathbf{Q}_{T_j}(X)$. Now, we have that $g_n(x, z) = (g_n'x, g_n''z) \to (a, u)$ and this point belongs to the closed orbit of $(x, z)$ under $\mathcal{G}_{T_j}$. By distality this orbit is minimal and so it follows that $(x, z)$ is in the closed orbit of $(a, u)$ and thus $(x, z) \in \mathbf{Q}_{T_j}(X)$. $\square$

A similar argument can be used to prove the next result.

**Lemma 4.3** (Gluing Lemma)**.** *Let $d \geq 2$ be an integer and $(X, T_1, \ldots, T_d)$ be a minimal distal system with commuting transformations $T_1, \ldots, T_d$. Consider $\boldsymbol{x} = (\boldsymbol{x}', \boldsymbol{x}''), \boldsymbol{y} = (\boldsymbol{y}', \boldsymbol{y}'') \in X^{[d]}$ with $\boldsymbol{x}', \boldsymbol{x}'', \boldsymbol{y}', \boldsymbol{y}'' \in X^{[d-1]}$ and $\boldsymbol{x}'' = \boldsymbol{y}'$. If $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$, then $\boldsymbol{z} = (\boldsymbol{x}', \boldsymbol{y}'') \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$.*

**Proof.** Let $\mathbf{x}', \mathbf{x}'', \mathbf{y}'' \in X^{[d-1]}$ be such that $(\mathbf{x}', \mathbf{x}''), (\mathbf{x}'', \mathbf{y}'') \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$. Pick any $a \in X$. Then, $(a^{[d-1]}, a^{[d-1]}) \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$. By Proposition 3.4, there exists a sequence $(g_n)_{n\in\mathbb{N}} = ((g_n', g_n''))_{n\in\mathbb{N}} \in \mathcal{G}_{T_1,\ldots,T_d}$ such that $g_n(\mathbf{x}', \mathbf{x}'') = (g_n'\mathbf{x}', g_n''\mathbf{x}'') \to (a^{[d-1]}, a^{[d-1]})$. We can assume, by compactness, that $g_n''\mathbf{y}'' \to \mathbf{u}$ and thus $(g_n''\mathbf{x}'', g_n''\mathbf{y}'') \to (a^{[d-1]}, \mathbf{u}) \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$. Now, we have that $g_n(\mathbf{x}', \mathbf{y}'') = (g_n'\mathbf{x}', g_n''\mathbf{y}'') \to (a^{[d-1]}, \mathbf{u})$, and this point belongs to the closed orbit of $(\mathbf{x}', \mathbf{y}'')$ under $\mathcal{G}_{T_1,\ldots,T_d}$. By distality this orbit is minimal and so it follows that $(\mathbf{x}', \mathbf{y}'')$ is in the closed orbit of $(a^{[d-1]}, \mathbf{u})$ and thus $(\mathbf{x}', \mathbf{y}'') \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$. $\square$

**Remark.** Given two points $\mathbf{x}, \mathbf{y} \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$ we can use the Gluing Lemma when we have $(x_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon(j) = 0) = (y_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon(j) = 1)$ for some $j \in [d]$. Namely, we can apply a sequence of Euclidean permutations until we get two points $\mathbf{w} = (\mathbf{w}, \mathbf{w}'')$, $\mathbf{z} = (\mathbf{z}', \mathbf{z}'') \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$ such that $\mathbf{z}' = (x_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon(j) = 0)$ and $\mathbf{w}'' = (y_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon(j) = 1)$. We note that the Euclidean permutations that are needed for the aforementioned procedure are different from those in Proposition 3.3 (3). In the next figure we illustrate this comment.

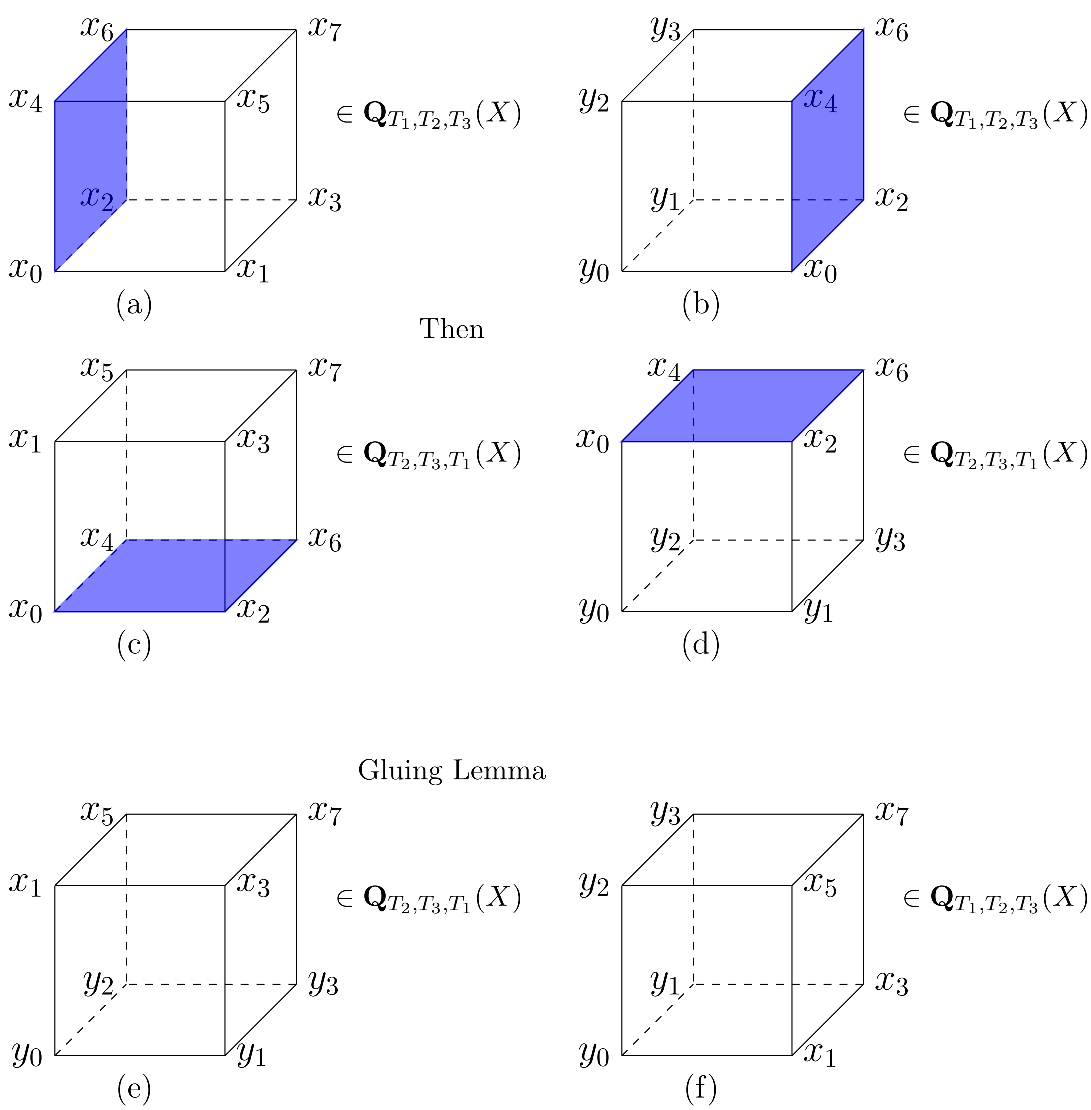



**Figure 4.1:** Illustration of Gluing Lemma. In (a) and (b) we have two cubes in $\mathbf{Q}_{T_1,T_2,T_3}(X)$ with two equal faces. With an Euclidean permutation we have that the cubes (c) and (d) are in $\mathbf{Q}_{T_1,T_2,T_3}(X)$ satisfying the hypothesis for Lemma 4.3. By the Gluing Lemma we have the existence of the cube (e) in $\mathbf{Q}_{T_1,T_2,T_3}(X)$ and finally we apply the inverse of the Euclidean permutation to obtain a cube in $\mathbf{Q}_{T_1,T_2,T_3}(X)$.

Using Lemma 3.11 and Proposition 4.2 we get the following result.

**Corollary 4.4.** *Let $d \geq 2$ be an integer and $(X, T_1, \ldots, T_d)$ be a minimal distal system with commuting transformations $T_1, \ldots, T_d$. Then, for $j \in [d]$, $X/\mathbf{Q}_{T_j}(X)$ is the maximal $\mathsf{Z}_0^{e_j}$-factor.*

**Proof.** By Proposition 4.2 we have that $\sigma_{T_j}(X) = \mathbf{Q}_{T_j}(X)$. We conclude with Lemma 3.11. $\square$

### 4.1.2 The system $(\mathbf{K}^{x_0}_{T_1,\ldots,T_d}, \mathcal{F}^{x_0}_{T_1,\ldots,T_d})$ for distal systems

Let $d \geq 2$ be an integer. For a minimal distal system $(X, T_1, \ldots, T_d)$ with commuting transformations $T_1, \ldots, T_d$ and $x_0 \in X$, we consider the system $(\mathbf{K}^{x_0}_{T_1,\ldots,T_d}, \mathcal{F}^{x_0}_{T_1,\ldots,T_d})$, defined in Section 3.2, based in Donoso and Sun's work [7]. This system has a decomposition on factors when we project to the coordinates where the corresponding transformations act as the identity, represented in the following commutative diagram:

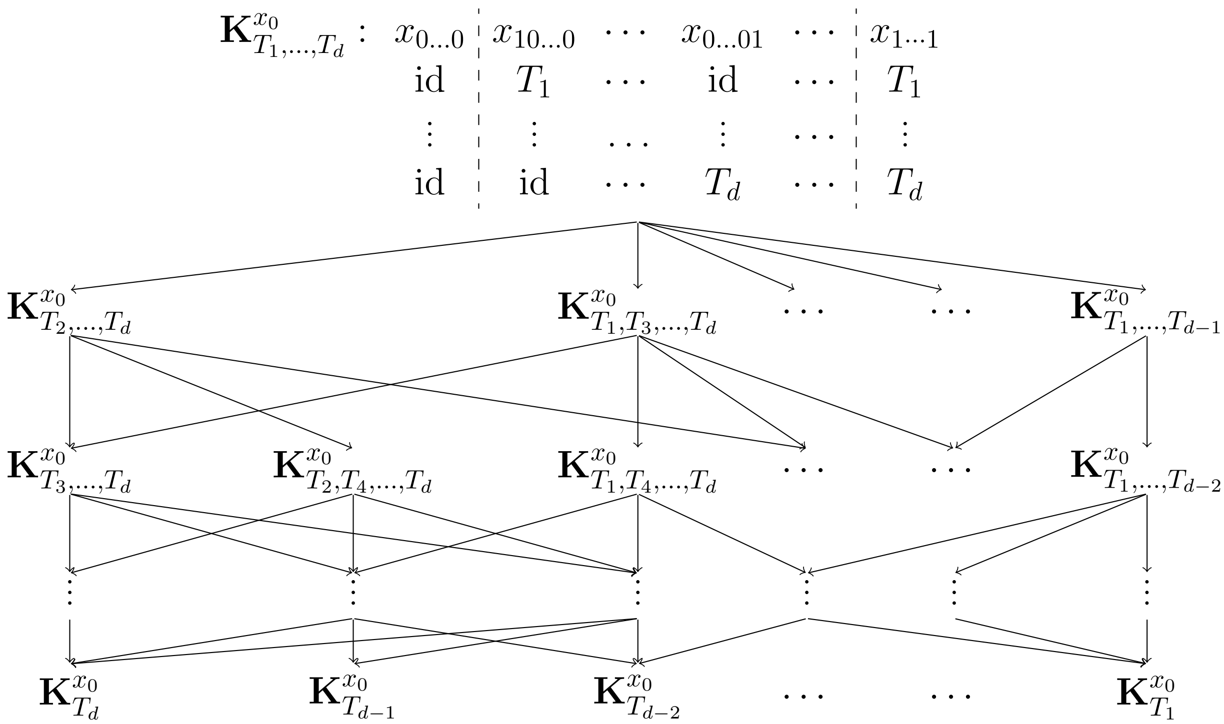



**Figure 4.2:** Decomposition of the system $(\mathbf{K}^{x_0}_{T_1,\ldots,T_d}, \mathcal{F}^{x_0}_{T_1,\ldots,T_d})$ using the projection on the coordinates where the transformations act as the identity.

Here, $\mathbf{K}^{x_0}_{T_1,\ldots,T_{j-1},T_{j+1},\ldots,T_d}$ corresponds to the projection onto the coordinates where the action $T_j$ acts trivially, i.e., in the coordinates where $j \notin \varepsilon$. We have that elements of $\mathbf{K}^{x_0}_{T_1,\ldots,T_d}$ have $2^d - 1$ coordinates. Indeed, $\mathbf{x} \in \mathbf{K}^{x_0}_{T_1,\ldots,T_d}$, then $(x_0, \mathbf{x}) \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$. So, in systems with the closing parallelepiped property we have that, if two points $\mathbf{x}, \mathbf{y} \in \mathbf{K}^{x_0}_{T_1,\ldots,T_d}$ have the same projections onto the systems $\mathbf{K}^{x_0}_{T_1,\ldots,T_{j-1},T_{j+1},\ldots,T_d}$ for all $j \in [d]$, then $\mathbf{x} = \mathbf{y}$. In fact, the system $\mathbf{K}^{x_0}_{T_1,\ldots,T_{j-1},T_{j+1},\ldots,T_d}$ has the coordinates in the face $\varepsilon(j) = 0$ and the last coordinate is a function of the other coordinates. For example, for 3 transformations we have the following diagram:

$$\mathbf{K}^{x_0}_{T_1,T_2,T_3}: \begin{array}{c|cccccc|c} x_{000} & x_{100} & x_{010} & x_{110} & x_{001} & x_{101} & x_{011} & x_{111} \\ \mathrm{id} & T_1 & \mathrm{id} & T_1 & \mathrm{id} & T_1 & \mathrm{id} & T_1 \\ \mathrm{id} & \mathrm{id} & T_2 & T_2 & \mathrm{id} & \mathrm{id} & T_2 & T_2 \\ \mathrm{id} & \mathrm{id} & \mathrm{id} & \mathrm{id} & T_3 & T_3 & T_3 & T_3 \end{array}$$

$$\mathbf{K}^{x_0}_{T_2,T_3}: \begin{array}{ccc} x_{010} & x_{001} & x_{011} \\ \mathrm{id} & \mathrm{id} & \mathrm{id} \\ T_2 & \mathrm{id} & T_2 \\ \mathrm{id} & T_3 & T_3 \end{array} \quad \mathbf{K}^{x_0}_{T_1,T_3}: \begin{array}{ccc} x_{100} & x_{001} & x_{101} \\ T_1 & \mathrm{id} & T_1 \\ \mathrm{id} & \mathrm{id} & \mathrm{id} \\ \mathrm{id} & T_3 & T_3 \end{array} \quad \mathbf{K}^{x_0}_{T_1,T_2}: \begin{array}{ccc} x_{100} & x_{010} & x_{110} \\ T_1 & \mathrm{id} & T_1 \\ \mathrm{id} & T_2 & T_2 \\ \mathrm{id} & \mathrm{id} & \mathrm{id} \end{array}$$

$$\mathbf{K}^{x_0}_{T_3}: \begin{array}{c} x_{001} \\ \mathrm{id} \\ \mathrm{id} \\ T_3 \end{array} \qquad \mathbf{K}^{x_0}_{T_2}: \begin{array}{c} x_{010} \\ \mathrm{id} \\ T_2 \\ \mathrm{id} \end{array} \qquad \mathbf{K}^{x_0}_{T_1}: \begin{array}{c} x_{100} \\ T_1 \\ \mathrm{id} \\ \mathrm{id} \end{array}$$

**Figure 4.3:** Illustration of the decomposition of $(\mathbf{K}^{x_0}_{T_1,T_2,T_3}, \mathcal{F}^{x_0}_{T_1,T_2,T_3})$.

In this way, we have that in a system with the closing parallelepiped property, a point $\mathbf{x} \in \mathbf{K}^{x_0}_{T_1,\dots,T_d}$ can be deduced by its projections on the systems $\mathbf{K}^{x_0}_{T_1,\dots,T_{j-1},T_{j+1},\dots,T_d}$ for every $j \in [d]$.

By the previous discussion, we have that in a with the closing parallelepiped property the system $\mathbf{K}^{x_0}_{T_1,\dots,T_d}$ can be viewed as a joining of the systems $\mathbf{K}^{x_0}_{T_1,\dots,T_{j-1},T_{j+1},\dots,T_d}$ for every $j \in [d]$. We also remark that for a minimal system $(X, T_1, \dots, T_d)$ with commuting transformations $T_1, \dots, T_d$, $j \in [d]$ and $x_0 \in X$ we have that $\mathbf{K}^{x_0}_{T_1,\dots,T_{j-1},T_{j+1},\dots,T_d} \in \mathsf{Z}_0^{e_j}$. We will prove that $\mathbf{K}^{x_0}_{T_1,\dots,T_{j-1},T_{j+1},\dots,T_d}$ is the $\mathsf{Z}_0^{e_j}$-maximal factor in the distal case.

**Proposition 4.5.** *Let $d \geq 2$ be an integer, $(X, T_1, \dots, T_d)$ be a minimal distal system with commuting transformations $T_1, \dots, T_d$ and take $x_0 \in X$. Then, $\mathbf{K}^{x_0}_{T_1,\dots,T_{j-1},T_{j+1},\dots,T_d}$ is isomorphic to $\mathbf{K}^{x_0}_{T_1,\dots,T_d} / \mathbf{Q}_{T_j^{[d]}}(\mathbf{K}^{x_0}_{T_1,\dots,T_d})$ for any $j \in [d]$.*

**Proof.** Let $\mathbf{x}, \mathbf{y} \in \mathbf{K}^{x_0}_{T_1,\dots,T_d}$ be such that

$$(x_\varepsilon \colon \varepsilon \in \{0,1\}^d,\, \varepsilon(j) = 0) = (y_\varepsilon \colon \varepsilon \in \{0,1\}^d,\, \varepsilon(j) = 0).$$

We have to show that $(\mathbf{x}, \mathbf{y}) \in \mathbf{K}^{x_0}_{T_1,\dots,T_d}$. Let $\delta > 0$, $\pi \colon \mathbf{K}^{x_0}_{T_1,\dots,T_d} \to \mathbf{K}^{x_0}_{T_1,\dots,T_{j-1},T_{j+1},\dots,T_d}$ be the factor map. By the openness of $\pi$, we can find $0 < \delta' < \delta$ such that

$$B((x_\varepsilon \colon \varepsilon \in \{0,1\}^d,\, \varepsilon(j) = 0), \delta') \subseteq \pi(B(\mathbf{x}, \delta)) \cap \pi(B(\mathbf{y}, \delta)) \tag{4.1}$$

By definition of $\mathbf{K}^{x_0}_{T_1,\dots,T_d}$, there exists $\mathbf{n}\in\mathbb{Z}^d$ such that

$$\rho((T_1^{n_1\varepsilon_1}\dots T_d^{n_d\varepsilon_d}x_0)_{\varepsilon\in\{0,1\}^d\setminus\{\vec{0}\}},\mathbf{x})<\delta'.$$

Call $\mathbf{z}=(T_1^{n_1\varepsilon_1}\dots T_d^{n_d\varepsilon_d}x_0)_{\varepsilon\in\{0,1\}^d\setminus\{\vec{0}\}}$. Then, we have that

$$\rho((z_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=0),(x_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=0))<\delta'.$$

Thus, by (4.1), there exists $\mathbf{z}^1\in\mathbf{K}^{x_0}_{T_1,\dots,T_d}$ such that

$$(z^1_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=0)=(z_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=0)\ \wedge\ \rho(\mathbf{z}^1,\mathbf{y})<\delta.$$

Let $0<\delta''<\delta'$ be such that $\delta'+\delta''<\delta$ and for every $\mathbf{u},\mathbf{v}\in\mathbf{K}^{x_0}_{T_1,\dots,T_d}$ we have

$$\rho(\mathbf{u},\mathbf{v})<\delta''\implies\rho((T_j^{[d]})^{n_j}\mathbf{u},(T_j^{[d]})^{n_j}\mathbf{v})<\delta'.$$

By definition of $\mathbf{K}^{x_0}_{T_1,\dots,T_d}$, there exists $\mathbf{n}''\in\mathbb{Z}^d$ such that

$$\rho((T_1^{n'_1\varepsilon_1}\dots T_d^{n'_d\varepsilon_d}x_0)_{\varepsilon\in\{0,1\}^d\setminus\{\vec{0}\}},\mathbf{z}^1)<\delta''.$$

We call $\mathbf{z}^2=(T_1^{n'_1\varepsilon_1}\dots T_d^{n'_d\varepsilon_d}x_0)_{\varepsilon\in\{0,1\}^d\setminus\{\vec{0}\}}$. We define

$$\mathbf{z}^3=(T_j^{[d]})^{n_j-n'_j}\mathbf{z}^2=(T_1^{n'_1\varepsilon_1}\dots T_{j-1}^{n'_{j-1}\varepsilon_{j-1}}T_j^{n_j\varepsilon_j}T_{j+1}^{n'_{j+1}\varepsilon_{j+1}}\dots T_d^{n'_d\varepsilon_d}x_0)_{\varepsilon\in\{0,1\}^d\setminus\{\vec{0}\}}.$$

We have

$$\begin{aligned}\rho(\mathbf{z}^3,\mathbf{x})\ &\le\rho((z^3_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=0),(z_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=0))\\ &\quad+\rho((z_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=0),(x_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=0))\\ &\quad+\rho((z^3_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=1),(z_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=1))\\ &\quad+\rho((z_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=1),(x_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=1))\\ &\le\delta''+\delta'+\delta'+\delta'\\ &\le 3\delta,\end{aligned}$$

and

$$\begin{aligned}\rho((T_j^{[d]})^{n'_j-n_j}\mathbf{z}^3,\mathbf{y})\ &\le\rho((((T_j^{[d]})^{n'_j-n_j}z^3)_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=0),(z^1_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=0))\\ &\quad+\rho((z^1_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=0),(y_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=0))\\ &\quad+\rho((((T_j^{[d]})^{n'_j-n_j}z^3)_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=1),(z^1_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=1))\\ &\quad+\rho((z^1_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=1),(y_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=1))\\ &\le\delta''+\delta'+\delta''+\delta\\ &\le 3\delta.\end{aligned}$$

□

By Proposition 3.5, if $(Z,T_1,\dots,T_d) \in \mathsf{Z}_0^{e_j}$ for some $j \in [d]$ and is a factor of $(\mathbf{K}^{x_0}_{T_1,\dots,T_d}, T_1^{[d]},\dots,T_d^{[d]})$, then, by Corollary 4.4, there exists a map from $\mathbf{K}^{x_0}_{T_1,\dots,T_d}\big/\mathbf{Q}_{T_j^{[d]}(\mathbf{K}^{x_0}_{T_1,\dots,T_d})}$ to $Z$. As a consequence, we have proved the following result.

**Corollary 4.6.** *Let $d \geq 2$ be an integer, let $(X,T_1,\dots,T_d)$ be a minimal distal system with commuting transformations $T_1,\dots,T_d$ and $x_0 \in X$. Then, for every $j \in [d]$ the system $\mathbf{K}^{x_0}_{T_1,\dots,T_{j-1},T_{j+1},\dots,T_d} \cong {}^{\mathbf{K}^{x_0}_{T_1,\dots,T_d}}\!/\!_{\mathbf{Q}_{T_j^{[d]}}(\mathbf{K}^{x_0}_{T_1,\dots,T_d})}$ is the maximal $\mathsf{Z}_0^{e_j}$-factor of $\mathbf{K}^{x_0}_{T_1,\dots,T_d}$.*

With an identical proof we obtain the following proposition.

**Proposition 4.7.** *Let $d \geq 2$ be an integer, $(X,T_1,\dots,T_d)$ be a minimal distal system with commuting transformations $T_1,\dots,T_d$, $x_0 \in X$, $\{j_1,\dots,j_\ell\} \subseteq [d]$ and $k \in \{1,\dots,\ell\}$. Then, the system $\mathbf{K}^{x_0}_{T_{j_1},\dots,T_{j_{k-1}},T_{j_{k+1}}\dots T_{j_\ell}}$ is the maximal $\mathsf{Z}_0^{e_{j_k}}$-factor of $\mathbf{K}^{x_0}_{T_{j_1},\dots,T_{j_\ell}}$.*

A direct consequence from this proposition is,

**Corollary 4.8.** *Let $d \geq 2$ be an integer, $(X,T_1,\dots,T_d)$ be a minimal distal system with commuting transformations $T_1,\dots,T_d$, $x_0 \in X$, $\{j_1,\dots,j_\ell\} \subseteq [d]$ and $k_1,k_2 \in \{1,\dots,\ell\}$. Then,*

$$ {}^{\mathbf{K}^{x_0}_{T_{j_1},\dots,T_{j_\ell}}}\!/\!_{\mathbf{Q}_{T_{j_{k_1}}}}\Big/\mathbf{Q}_{T_{j_{k_2}}} \cong {}^{\mathbf{K}^{x_0}_{T_{j_1},\dots,T_{j_\ell}}}\!/\!_{\mathbf{Q}_{T_{j_{k_2}}}}\Big/\mathbf{Q}_{T_{j_{k_1}}}. $$

Now, let $(X,T_1,\dots,T_d)$ be a minimal distal system with commuting transformations $T_1,\dots,T_d$. Let $H \leq \langle T_1,\dots T_d\rangle$ be a subgroup. We define

$$ \mathbf{Q}_H(X) = \overline{\{(x,hx)\colon x \in X,\ h \in H\}} \subseteq X^2. $$

In particular, if $\{j_1,\dots,j_k\} \subseteq [d]$, we have

$$ \mathbf{Q}_{\langle T_{j_1},\dots,T_{j_k}\rangle}(X) = \overline{\{(x,hx)\colon x \in X,\ h \in \langle T_{j_1},\dots,T_{j_k}\rangle\}}. $$

Be careful and do not get confused: the sets $\mathbf{Q}_{\langle T_{j_1},\dots,T_{j_k}\rangle}(X)$ and $\mathbf{Q}_{T_{j_1},\dots,T_{j_k}}(X)$ are different. We denote by $\mathsf{Z}_0^{e_{j_1}} \wedge \mathsf{Z}_0^{e_{j_2}}$ the intersection between the classes $\mathsf{Z}_0^{e_{j_1}}$ and $\mathsf{Z}_0^{e_{j_2}}$, that is, a system belongs to $\mathsf{Z}_0^{e_{j_1}} \wedge \mathsf{Z}_0^{e_{j_2}}$ if both transformations $j_1$, $j_2$ act trivially.

Similarly as stated in Proposition 4.2 one proves that the relation $\mathbf{Q}_{\langle T_{j_1},T_{j_2}\rangle}(X)$ is a closed and invariant equivalence relation, and by a proof analogous to that of Lemma 4.4 we have that ${}^X\!/\!_{\mathbf{Q}_{\langle T_{j_1},T_{j_2}\rangle}(X)}$ is the maximal $\mathsf{Z}_0^{e_{j_1}} \wedge \mathsf{Z}_0^{e_{j_2}}$-factor of $X$. We conclude that ${}^{X/\mathbf{Q}_{T_{j_1}}(X)}\!/\!_{\mathbf{Q}_{T_{j_2}}(X)}$ is a factor of ${}^X\!/\!_{\mathbf{Q}_{\langle T_{j_1},T_{j_2}\rangle}(X)}$. On the other hand, ${}^X\!/\!_{\mathbf{Q}_{\langle T_{j_1},T_{j_2}\rangle}(X)}$ is a factor of ${}^X\!/\!_{\mathbf{Q}_{T_{j_1}}(X)}$, which belongs to $\mathsf{Z}_0^{e_{j_2}}$. Thus, ${}^X\!/\!_{\mathbf{Q}_{\langle T_{j_1},T_{j_2}\rangle}(X)}$ is a factor of ${}^{X/\mathbf{Q}_{T_{j_1}}(X)}\!/\!_{\mathbf{Q}_{T_{j_2}}(X)}$. Then, we have the following result,

**Proposition 4.9.** *Let $d \geq 2$ be an integer and $(X, T_1, \ldots, T_d)$ be a minimal distal system with commuting transformations $T_1, \ldots, T_d$. If $j_1, j_2 \in [d]$ with $j_1 \neq j_2$, then* ${}^{X/\mathbf{Q}_{T_{j_1}}(X)}\big/\mathbf{Q}_{T_{j_2}}(X) \cong X/\mathbf{Q}_{\langle T_{j_1}, T_{j_2}\rangle}(X)$.

We can generalize the previous result for any subset of $[d]$ in the following way.

**Lemma 4.10.** *Let $d \geq 2$ be an integer and $(X, T_1, \ldots, T_d)$ be a minimal distal system with commuting transformations $T_1, \ldots, T_d$. Let $\{j_1, \ldots, j_k\} \subseteq [d]$. Then, $\mathbf{Q}_{\langle T_{j_1}, \ldots, T_{j_k}\rangle}(X)$ is a closed and invariant equivalence relation such that*

$$ {}^{X/\mathbf{Q}_{T_{j_1}}}\Big/ {}^{\cdots}/\mathbf{Q}_{T_{j_k}} \cong X/\mathbf{Q}_{\langle T_{j_1}, \ldots, T_{j_k}\rangle}(X). $$

*Finally, $X/\mathbf{Q}_{\langle T_{j_1}, \ldots, T_{j_k}\rangle}(X)$ is the maximal $\mathsf{Z}_0^{e_{j_1}} \wedge \ldots \wedge \mathsf{Z}_0^{e_{j_k}}$-factor of $(X, T_1, \ldots, T_d)$.*

### 4.1.3 The $(T_1, \ldots, T_d)$-regionally proximal relation for distal systems

In this section we study the properties of the $(T_1, \ldots, T_d)$-regionally proximal relation for distal systems that will be used in the proof of Theorem 4.1.

We start with the following proposition, which was proved in the case $d = 2$ by Donoso and Sun [7].

**Proposition 4.11.** *Let $d \geq 2$ be an integer and $(X, T_1, \ldots, T_d)$ be a distal system with commuting transformations $T_1, \ldots, T_d$. Consider $x, y \in X$. The following statements are equivalent:*

*(1) $(x, y_*^{[d]}) \in \mathbf{Q}_{T_1, \ldots, T_d}(X)$.*

*(2) There exists $\boldsymbol{a}_* \in X_*^{[d]}$ such that $(x, \boldsymbol{a}_*), (y, \boldsymbol{a}_*) \in \mathbf{Q}_{T_1, \ldots, T_d}(X)$.*

*(3) For every $\boldsymbol{a}_* \in X_*^{[d]}$, $(x, \boldsymbol{a}_*) \in \mathbf{Q}_{T_1, \ldots, T_d}(X)$ if and only if $(y, \boldsymbol{a}_*) \in \mathbf{Q}_{T_1, \ldots, T_d}(X)$.*

*(4) $(x, y) \in \mathcal{R}_{T_1, \ldots, T_d}(X)$.*

*(5) There exists $j \in [d]$ such that $(x, y) \in \mathcal{R}_{T_j}(X)$.*

**Proof.** (1) $\implies$ (4). Let $j \in [d]$ and $\mathbf{a}_* = y^{[d-1]} \in X^{[d-1]}$, then $\mathbf{z}(x, y, \mathbf{a}_*, j) = (x, y_*^{[d]}) \in \mathbf{Q}_{T_1, \ldots, T_d}(X)$.

(4) $\implies$ (5). It follows directly from the definition.

(5) $\implies$ (1). The idea of the proof is to construct, in two stages, a sequence of points $\mathbf{z}^1, \ldots, \mathbf{z}^d \in \mathbf{Q}_{T_1, \ldots, T_d}(X)$ so that we increase the number of times that $y$ appears as a coordinate of $\mathbf{z}^i$, using Lemma 4.3 until we get $\mathbf{z}^d = (x, y_*^{[d]}) \in \mathbf{Q}_{T_1, \ldots, T_d}(X)$.

In Stage 1 we construct the points $\mathbf{z}^1, \ldots, \mathbf{z}^j \in \mathbf{Q}_{T_1, \ldots, T_d}(X)$ such that for $2 \leq k \leq j$ we have that the point $\mathbf{z}^j$ satisfies the following properties:

- For every $\varepsilon \subseteq [k]$, $\varepsilon \neq \emptyset$, $z_\varepsilon^k = z_{\varepsilon \cup \{j\}}^k = y$ and $z_\emptyset^k = x$.

- For every $\varepsilon \subseteq [d]$ with $\varepsilon \not\subseteq [k]$ and every $1 \leq \ell < k$ we have that $z^k_\varepsilon = z^k_{\Phi_\ell(\varepsilon)}$. It is easy to see that this is equivalent to the following property. Let $\varepsilon_1, \varepsilon_2 \subseteq [d]$ with $\varepsilon_1, \varepsilon_2 \not\subseteq [k]$ and $\varepsilon_1(r) = \varepsilon_2(r)$ for every $r \geq k$, then $z^k_{\varepsilon_1} = z^k_{\varepsilon_2}$.
- For every $\varepsilon \subseteq [d]$, $\varepsilon \neq \emptyset$, we have that $z^k_\varepsilon = z^k_{\Phi_j(\varepsilon)}$.

This Stage is not necessary for the case $j = 1$.

In Stage 2 we have the point $\mathbf{z}^j \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ satisfying the following properties:

- For every $\varepsilon \subseteq [k]$, $\varepsilon \neq \emptyset$, $z^k_\varepsilon = y$ and $z^k_\emptyset = x$.
- For every $\varepsilon \subseteq [d]$ with $\varepsilon \not\subseteq [k]$ and every $1 \leq \ell \leq k$ we have that $z^k_\varepsilon = z^k_{\Phi_\ell(\varepsilon)}$.

Finally we have that $\mathbf{z}^d = (x, y^d_*) \in \mathbf{Q}_{T_1,\dots,T_d}(X)$. This Stage is not necessary for the case $j = d$.

Take $x, y \in X$ such that $(x, y) \in \mathcal{R}_{T_j}(X)$ for some $j \in [d]$. Then, there exists $\mathbf{a}_* \in X^{[d-1]}$ for which $\mathbf{z}(x, y, \mathbf{a}_*, j) \in \mathbf{Q}_{T_1,\dots,T_d}(X)$. We call $\mathbf{z}^1 = \mathbf{z}(x, y, \mathbf{a}_*, j)$.

*Stage 1. Construction of the points* $\mathbf{z}^1, \dots, \mathbf{z}^j \in \mathbf{Q}_{T_1,\dots,T_d}(X)$.

Assume $j \neq 1$. Consider the face $(z^1_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon(j) = 1)$ of $\mathbf{z}^1$. We have $(z^1_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon(j) = 1) \in \mathbf{Q}_{T_1,\dots,T_{j-1},T_{j+1},\dots,T_d}(X)$ by Proposition 3.3 (4). We view $(z^1_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon(j) = 1)$ as a point in $X^{[d-1]}$, then we can write

$$(z^1_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon(j) = 1) = (u^1_\eta \colon \eta \in \{0,1\}^{d-1})$$

where $u^1_\eta = z^1_\varepsilon$ if and only if $\varepsilon = \Psi^1_j(\eta)$. Then, by Proposition 3.3 (5), there exists a point $\mathbf{v}^1 \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ with $v^1_\alpha = u^1_\eta$ if and only if $\alpha = \Psi^0_j(\eta)$ or $\alpha = \Psi^1_j(\eta)$m i.e., we duplicate $\mathbf{u}^1$. In particular, $v^1_\emptyset = v^1_{\{j\}} = y$. Let $\varepsilon \subseteq [d]$, $\varepsilon \neq \emptyset$. If $j \in \varepsilon$, take $\eta$ such that $\varepsilon = \Psi^1_j(\eta)$. Then, we have that $v^1_\varepsilon = u^1_\eta = z^1_\varepsilon$. If $j \notin \varepsilon$, take $\eta$ such that $\varepsilon = \Psi^1_j(\eta)$. Then, we have that

$$v^1_\varepsilon = u^1_\eta = (a_*)_\eta = z^1_{\Phi_j(\varepsilon)} = z^1_\varepsilon.$$

Thus, we have that for every $\varepsilon \subseteq [d]$, $\varepsilon \neq \emptyset$, $v^1_\varepsilon = z^1_\varepsilon$. In particular, $(z^1_\varepsilon \colon \varepsilon \in \{0,1\}^d \colon \varepsilon(1) = 1) = (v^1_\alpha \colon \alpha \in \{0,1\}^d, \alpha(1) = 1)$. By Lemma 4.3, there exists a point $\mathbf{z}^2 \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ such that $(z^2_\varepsilon \colon \varepsilon \in \{0,1\}^d \colon \varepsilon(1) = 0) = (z^1_\varepsilon \colon \varepsilon \in \{0,1\}^d \colon \varepsilon(1) = 0)$ and $(z^2_\varepsilon \colon \varepsilon \in \{0,1\}^d \colon \varepsilon(1) = 1) = (v^1_\alpha \colon \varepsilon \in \{0,1\}^d \colon \alpha(1) = 0)$ with $z^2_\varepsilon = v^1_\alpha$ if and only if $\varepsilon = \Phi_1(\alpha)$. In particular, we have $z^2_\emptyset = x$, $z_{\{1\}} = z^2_{\{j\}} = z^2_{\{1,j\}} = y$ and for every $\varepsilon \subseteq [d]$, $\varepsilon \neq \emptyset, \{1\}$, $z^2_\varepsilon = z^2_{\Phi_1(\varepsilon)}$. Since $\mathbf{z}^1$ satisfies that $z^1_\varepsilon = z^1_{\Phi_j(\varepsilon)}$ for every $\varepsilon \subseteq [d]$, $\varepsilon \neq \emptyset, \{j\}$, we have that $z^2_\varepsilon = z^2_{\Phi_j(\varepsilon)}$.

Now, assume we have built $\mathbf{z}^k \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ for $2 \leq k < j$ such that the following properties hold:

- For every $\varepsilon \subseteq [k]$, $\varepsilon \neq \emptyset$, $z^k_\varepsilon = z^k_{\varepsilon \cup \{j\}} = y$ and $z^k_\emptyset = x$.
- For every $\varepsilon \subseteq [d]$ with $\varepsilon \not\subseteq [k]$ and every $1 \leq \ell < k$ we have that $z^k_\varepsilon = z^k_{\Phi_\ell(\varepsilon)}$. It is easy to see that this is equivalent to the following property. Let $\varepsilon_1, \varepsilon_2 \subseteq [d]$ with $\varepsilon_1, \varepsilon_2 \not\subseteq [k]$ and $\varepsilon_1(r) = \varepsilon_2(r)$ for every $d \geq r \geq k$, then $z^k_{\varepsilon_1} = z^k_{\varepsilon_2}$.
- For every $\varepsilon \subseteq [d]$, $\varepsilon \neq \emptyset, \{j\}$, we have that $z^k_\varepsilon = z^k_{\Phi_j(\varepsilon)}$.

Consider the face $(z^k_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=1)\ \in \mathbf{Q}_{T_1,\dots,T_{j-1},T_{j+1},\dots,T_d}(X)$. If we denote $(z^k_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(j)=1) = (u^k_\eta\colon \eta\in\{0,1\}^{d-1})$ with $z^k_\varepsilon = u^k_\eta$ if and only if $\varepsilon=\Psi^1_j(\eta)$. Then, by Proposition 3.3 (5), there exists $\mathbf{v}^k\in\mathbf{Q}_{T_1,\dots,T_d}(X)$, where $v^k_\alpha=u^k_\eta$ if and only if $\alpha=\Psi^0_j(\eta)$ or $\alpha=\Psi^1_j(\eta)$. Let $\varepsilon\subseteq[d]$, $\varepsilon\neq\emptyset$. If $j\in\varepsilon$, take $\eta$ such that $\varepsilon=\Psi^1_j(\eta)$. Then, we have that $v^k_\varepsilon=u^k_\eta=z^k_\varepsilon$. If $j\notin\varepsilon$, take $\eta$ such that $\varepsilon=\Psi^0_j(\eta)$. Then, we have that $v^k_\varepsilon=u^k_\eta=z^k_{\Phi_j(\varepsilon)}=z^k_\varepsilon$. Thus, we have that for every $\varepsilon\subseteq[d]$, $\varepsilon\neq\emptyset$, $v^k_\varepsilon=z^k_\varepsilon$. In particular, $(z^k_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(k)=1)=(v^k_\alpha\colon \alpha\in\{0,1\}^d,\ \alpha(k)=1)$. By Lemma 4.3, there exists $\mathbf{z}^{k+1}\in\mathbf{Q}_{T_1,\dots,T_d}(X)$ such that $(z^{k+1}_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(k)=0)=(z^k_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(k)=0)$ and $(z^{k+1}_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(k)=1)=(v^k_\alpha\colon \alpha\in\{0,1\}^d,\ \alpha(k)=0)$ with $z^{k+1}_\varepsilon=v^k_\alpha$ if and only if $\varepsilon=\Phi_k(\alpha)$.

If $\varepsilon\subseteq[k+1]$, $\varepsilon\neq\emptyset$, then $z^{k+1}_\varepsilon=z_{\varepsilon\cup\{j\}}=y$ and $z^{k+1}_\emptyset=x$.

Let $\varepsilon\subseteq[d]$ with $\varepsilon\not\subseteq[k]$ and $1\leq\ell<k$. If $k\notin\varepsilon$ we have that $z^{k+1}_\varepsilon=z^k_\varepsilon=z^k_{\Phi_\ell(\varepsilon)}=z^{k+1}_{\Phi_\ell(\varepsilon)}$. If $k\in\varepsilon$ we have that $z^{k+1}_\varepsilon=v^k_{\Phi_k(\varepsilon)}=z^k_{\Phi_k(\varepsilon)}=z^k_{\Phi_\ell(\Phi_k(\varepsilon))}=z^k_{\Phi_k(\Phi_\ell(\varepsilon))}=v^k_{\Phi_k(\Phi_\ell)(\varepsilon)}=z^{k+1}_{\Phi_\ell(\varepsilon)}$.

Let $\varepsilon\subseteq[d]$, $\varepsilon\neq\emptyset,\{j\}$. If $k\notin\varepsilon$ we have that $z^{k+1}_{\Phi_j(\varepsilon)}=z^k_{\Phi_j(\varepsilon)}=z^k_\varepsilon=z^{k+1}_\varepsilon$. If $k\in\varepsilon$ we have that $z^{k+1}_{\Phi_j(\varepsilon)}=v^k_{\Phi_k(\Phi_j(\varepsilon))}=z^k_{\Phi_k(\Phi_j(\varepsilon))}=z^k_{\Phi_j(\Phi_k(\varepsilon))}=z^k_{\Phi_k(\varepsilon)}=v^k_{\Phi_k(\varepsilon)}=z^{k+1}_\varepsilon$.

We proceed inductively until $k=j-1$, where we use the face $\varepsilon(j-1)=1$ in Lemma 4.3 to get the point $\mathbf{z}^j\in\mathbf{Q}_{T_1,\dots,T_d}(X)$ with the following properties:

- For every $\varepsilon\subseteq[j]$, $\varepsilon\neq\emptyset$, $z^j_\varepsilon=y$ and $z^j_\emptyset=x$.
- For every $\varepsilon\subseteq[d]$ with $\varepsilon\not\subseteq[j]$ and every $1\leq\ell<j$ we have that $z^j_\varepsilon=z^j_{\Phi_\ell(\varepsilon)}$. It is easy to see that this is equivalent to the following property. Let $\varepsilon_1,\varepsilon_2\subseteq[d]$ with $\varepsilon_1,\varepsilon_2\not\subseteq[j]$ and $\varepsilon_1(r)=\varepsilon_2(r)$ for every $d\geq r\geq j$, then $z^j_{\varepsilon_1}=z^j_{\varepsilon_2}$.
- For every $\varepsilon\subseteq[d]$, $\varepsilon\neq\emptyset,\{j\}$, we have that $z^j_\varepsilon=z^j_{\Phi_j(\varepsilon)}$.

Then, for $\varepsilon_1,\varepsilon_2\subseteq[d]$ with $\varepsilon_1,\varepsilon_2\not\subseteq[j]$ and for every $d\geq r>j$, $\varepsilon_1(r)=\varepsilon_2(r)$.

The point $\mathbf{z}^j$ also satisfies that if $\varepsilon\subseteq[j]$, $\varepsilon\neq\emptyset$, $z^j_\varepsilon=y$ and $z^j_\emptyset=x$.

*Stage 2. Construction of the points $\mathbf{z}^{j+1},\dots,\mathbf{z}^d\in\mathbf{Q}_{T_1,\dots,T_d}(X)$.*

Assume $j\neq d$. We proceed by induction in the following way. Suppose that for $j\leq k<d$, the point $\mathbf{z}^k\in\mathbf{Q}_{T_1,\dots,T_d}(X)$ satisfies the following properties:

- For every $\varepsilon\subseteq[k]$, $\varepsilon\neq\emptyset$, $z^k_\varepsilon=y$ and $z^k_\emptyset=x$.
- For every $\varepsilon\subseteq[d]$ with $\varepsilon\not\subseteq[k]$ and every $1\leq\ell\leq k$ we have that $z^k_\varepsilon=z^k_{\Phi_\ell(\varepsilon)}$.

Consider the face $(z^k_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(k)=1)\ \in\mathbf{Q}_{T_1,\dots,T_{k-1},T_{k+1},\dots,T_d}(X)$. Denote $(z^k_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(k)=1)=(u^k_\eta\colon \eta\in\{0,1\}^{d-1})$ with $z^k_\varepsilon=u^k_\eta$ if and only if $\varepsilon=\Psi^1_k(\eta)$. Then, by Proposition 3.3 (5), there exists $\mathbf{v}^k\in\mathbf{Q}_{T_1,\dots,T_d}(X)$, where $v^k_\alpha=u^k_\eta$ if and only if $\alpha=\Psi^0_k(\eta)$ or $\alpha=\Psi^1_k(\eta)$. Let $\varepsilon\subseteq[d]$, $\varepsilon\neq\emptyset$. If $k\in\varepsilon$, take $\eta$ such that $\varepsilon=\Psi^1_k(\eta)$. Then, we have that $v^k_\varepsilon=u^k_\eta=z^k_\varepsilon$. If $k\notin\varepsilon$, take $\eta$ such that $\varepsilon=\Phi^0_k(\eta)$. Then, we have that $v^k_\varepsilon=u^k_\eta=z^k_{\Phi_k(\varepsilon)}=z^k_\varepsilon$. Thus, we have that for every $\varepsilon\subseteq[d]$, $\varepsilon\neq\emptyset$, $v^k_\varepsilon=z^k_\varepsilon$. In particular, $(z^k_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(k+1)=1)=(v^k_\alpha\colon \alpha\in\{0,1\}^d,\ \alpha(k+1)=1)$. By Lemma 4.3, there exists $\mathbf{z}^{k+1}\in\mathbf{Q}_{T_1,\dots,T_d}(X)$ such that $(z^{k+1}_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(k+1)=0)=(z^k_\varepsilon\colon \varepsilon\in\{0,1\}^d,\ \varepsilon(k+1)=0)$ and

$(z_\varepsilon^{k+1}\colon \varepsilon \in \{0,1\}^d,\ \varepsilon(k+1)=1) = (v_\alpha^k\colon \alpha \in \{0,1\}^d,\ \alpha(k+1)=0)$ with $z_\varepsilon^{k+1} = v_\alpha^k$ if and only if $\varepsilon = \Phi_{k+1}(\alpha)$.

Let $\varepsilon \subseteq [k+1]$, $\varepsilon \neq \emptyset$. We distinguish three cases:

- If $\varepsilon = \emptyset$, then $z_\emptyset^{k+1} = z_\emptyset^k = x$.
- If $k+1 \notin \varepsilon$, $z_\varepsilon^{k+1} = z_\varepsilon^k = y$.
- If $k+1 \in \varepsilon$, $z_\varepsilon^{k+1} = v_{\Phi_{k+1}(\varepsilon)}^k$. Now, we note that $\Phi_{k+1}(\varepsilon) \subseteq [k]$. Then, $v_{\Phi_{k+1}(\varepsilon)}^k = y$. Thus, $z_\varepsilon^{k+1} = y$.

Let $\varepsilon \subseteq [d]$ with $\varepsilon \not\subseteq [k+1]$ and $1 \leq \ell \leq k+1$. We have the following cases:

- If $k+1 \notin \varepsilon$ and $1 \leq \ell \leq k$, then, $z_\varepsilon^{k+1} = z_\varepsilon^k = z_{\Phi_\ell(\varepsilon)}^k = z_{\Phi_\ell(\varepsilon)}^{k+1}$.
- If $k+1 \notin \varepsilon$ and $\ell = k+1$, then, $z_{\Phi_\ell(\varepsilon)}^{k+1} = v_\varepsilon^{k+1} = z_\varepsilon^k$.
- If $k+1 \in \varepsilon$ and $1 \leq \ell \leq k$, then

$$z_{\Phi_\ell(\varepsilon)}^{k+1} = v_{\Phi_{k+1}(\Phi_\ell(\varepsilon))}^k = z_{\Phi_{k+1}(\Phi_\ell(\varepsilon))}^k = z_{\Phi_\ell(\Phi_{k+1}(\varepsilon))}^k = z_{\Phi_{k+1}(\varepsilon)}^k = v_{\Phi_{k+1}(\varepsilon)}^k = z_\varepsilon^{k+1}.$$

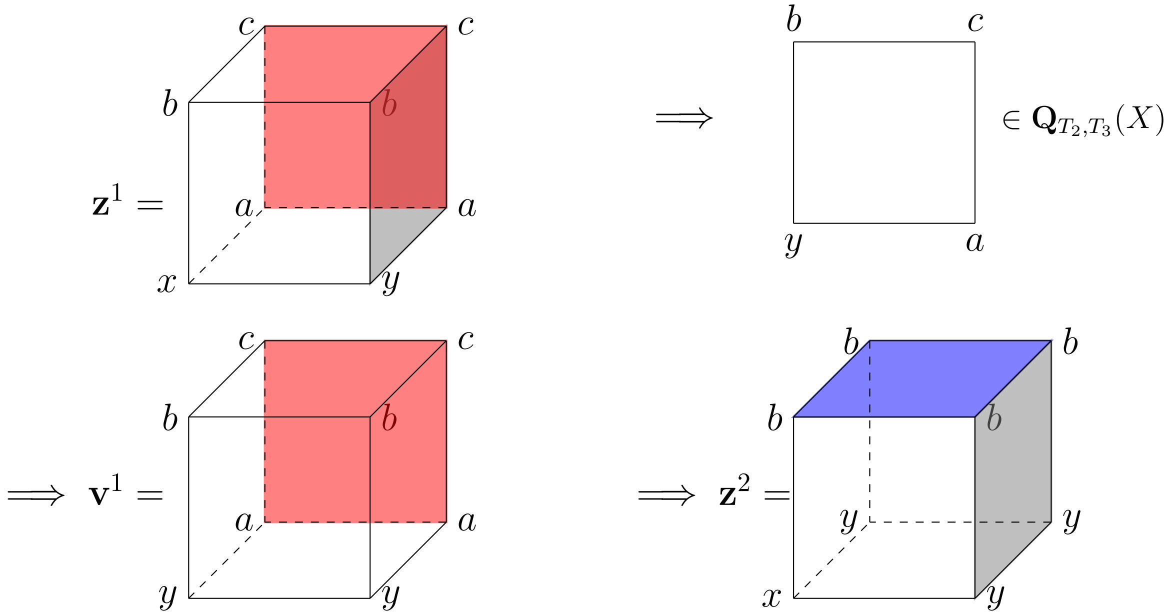
- If $k+1 \in \varepsilon$ and $\ell = k+1$, then, $z_{\Phi_{k+1}(\varepsilon)}^{k+1} = z_{\Phi_{k+1}(\varepsilon)}^k = v_{\Phi_{k+1}(\varepsilon)}^k = z_\varepsilon^k$.

With this we prove that $\mathbf{z}^{k+1}$ satisfies the following properties:

- For every $\varepsilon \subseteq [k+1]$, $\varepsilon \neq \emptyset$, $z_\varepsilon^{k+1} = y$ and $z_\emptyset^{k+1} = x$.
- For every $\varepsilon \subseteq [d]$ with $\varepsilon \not\subseteq [k+1]$ and every $1 \leq \ell \leq k+1$ we have that $z_\varepsilon^{k+1} = z_{\Phi_\ell(\varepsilon)}^{k+1}$.

We proceed inductively until $k = d-1$, where we use the face $\varepsilon(d) = 1$ and we have $\mathbf{z}^d \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ such that for every $\varepsilon \subseteq [d]$, $\varepsilon \neq \emptyset$, $z_\varepsilon^d = y$ and $z_\emptyset^{k+1} = x$, i.e., $\mathbf{z}^d = (x, y_*^{[d]}) \in \mathbf{Q}_{T_1,\dots,T_d}(X)$.

In the case $d=3$ the proof of (5) $\implies$ (1) can be illustrated in the following diagram.



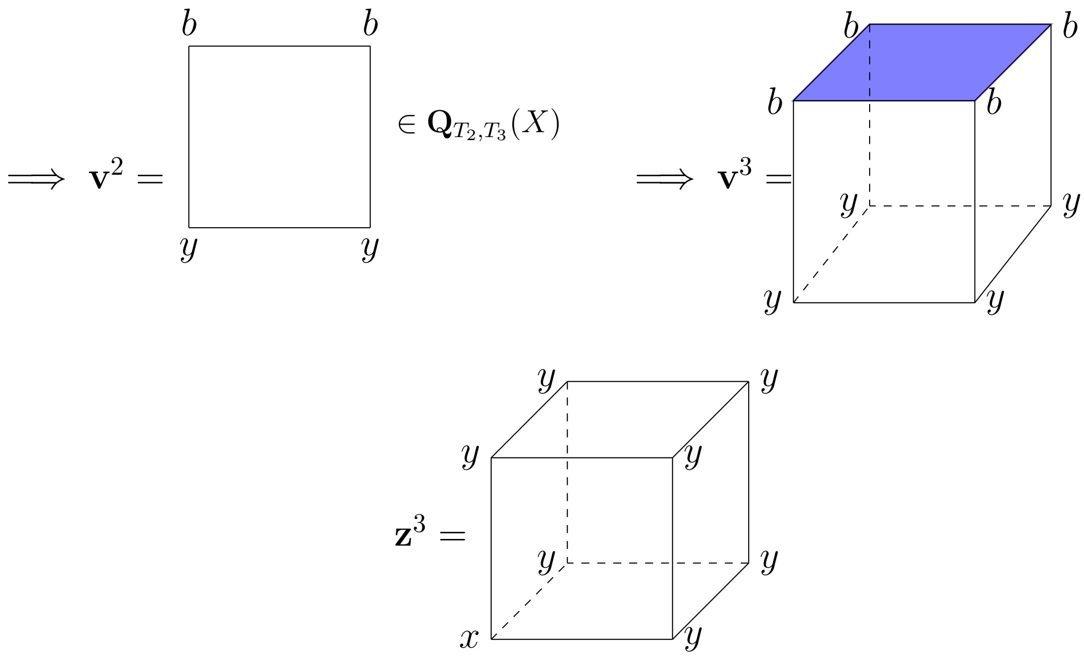



**Figure 4.4:** Illustration of the proof (5) $\implies$ (1) of Proposition 4.11 for the cases $d = 3$ and $j = 1$.

(1) $\implies$ (3). Let $\mathbf{a}_*$ be such that $\mathbf{z} = (x, \mathbf{a}_*) \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$. We prove that $(y, \mathbf{a}_*) \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$. The idea of the proof is to construct two sequences $\mathbf{z}^1, \ldots, \mathbf{z}^{d-1} \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$ and $\mathbf{v}^1, \ldots, \mathbf{v}^{d+1} \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$ such that for every $1 \leq k \leq d$ and every $\varepsilon \subseteq [d]$, the value of $z^k_\varepsilon$ is equal to $z_\eta$ if and only if $\eta \subseteq [k]$ and for every $j \in [k]$, $\varepsilon(j) = \eta(j)$, i.e., the coordinates of $\mathbf{z}^k$ depend only on the subsets of $[k]$ with respect to the coordinates of $\mathbf{z}$. To do this, by Proposition 3.3 (4), we project $\mathbf{z} \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$ in $\mathbf{Q}_{T_1,\ldots,T_k}(X)$ using the subset $[k] \subseteq [d]$ and $\xi = \vec{0} \in \{0,1\}^{d-k}$. We get that

$$\left(z_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon_{[d]\setminus[k]} = \vec{0}\right) = \left(z_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon \subseteq [k]\right) \in \mathbf{Q}_{T_1,\ldots,T_k}(X).$$

We consider $\mathbf{z}^k$ to be a lifting of $\left(z_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon \subseteq [k]\right) \in \mathbf{Q}_{T_1,\ldots,T_k}(X)$ to $\mathbf{Q}_{T_1,\ldots,T_d}(X)$ using Proposition 3.3 (5).

For the construction of the sequence $\mathbf{v}^1, \ldots, \mathbf{v}^{d+1} \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$, we know that property (1) is equivalent to property (5). Then, by (5), we have that $\mathbf{v}^1 = (y, x^d_*) \in \mathbf{Q}_{T_1,\ldots,T_k}(X)$. We have that for $1 \leq d \leq k$, $(v^k_\varepsilon \colon \varepsilon(k) = 1) = (z^k_\varepsilon \colon \varepsilon(k) = 0)$ and, by Lemma 4.3, we have the existence of the point $\mathbf{v}^{k+1} \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$ such that $(v^{k+1}_\varepsilon \colon \varepsilon(k) = 0) = (v^k_\varepsilon \colon \varepsilon(k) = 0)$ and $(v^{k+1}_\varepsilon \colon \varepsilon(k) = 1) = (z^k_\varepsilon \colon \varepsilon(k) = 1)$. With this we finally have that $v^{d+1} = (y, \mathbf{a}_*)$.

We remark that $(v^1_\varepsilon \colon \varepsilon(1) = 1) = (z^1_\varepsilon \colon \varepsilon(1) = 0) = x^{[d-1]}$. By Lemma 4.3, we have the existence of a point $\mathbf{v}^2 \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$ such that

$$v^2_\varepsilon = \begin{cases} v^1_\varepsilon & \text{if } \varepsilon(1) = 0, \\ z^1_\varepsilon & \text{if } \varepsilon(1) = 1. \end{cases}$$

Let $\varepsilon \subseteq [d]$. If $\varepsilon = \emptyset$, we have that $v^2_\emptyset = v^1_\emptyset = y$. If $\varepsilon(1) = 0$, then $v^2_\varepsilon = v^1_\varepsilon = x = z^1_\varepsilon$. If $\varepsilon(1) = 1$, we have that $v^2_\varepsilon = z^1_\varepsilon$. Thus, for every $\varepsilon \subseteq [d]$, $\varepsilon \neq \emptyset$, $v^2_\varepsilon = z^1_\varepsilon$.

Now assume that for all $1 \leq k < d-1$ we have constructed using Lemma 4.3 inductively $\mathbf{v}^{k+1} \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$, with $v^{k+1}_{\emptyset} = y$ and $v^{k+1}_{\varepsilon} = z^k_{\varepsilon}$ for all $\varepsilon \subseteq [d]$, $\varepsilon \neq \emptyset$. Let $\varepsilon \subseteq [d]$ with $\varepsilon(k) = 0$. Then, $z^{k+1}_{\varepsilon} = z^k_{\varepsilon}$ and $v^k_{\Phi_{k+1}(\varepsilon)} = z^k_{\Phi_{k+1}(\varepsilon)} = z^k_{\varepsilon}$. Thus, $(z^{k+1}_{\varepsilon} \colon \varepsilon(k+1) = 0) = (v^{k+1}_{\varepsilon} \colon \varepsilon(k+1) = 1)$. By Lemma 4.3, we have the existence of the point $\mathbf{v}^{k+2} \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$ such that

$$v^{k+2}_{\varepsilon} = \begin{cases} v^{k+1}_{\varepsilon} & \text{if } \varepsilon(k+1) = 0, \\ z^{k+1}_{\varepsilon} & \text{if } \varepsilon(k+1) = 1. \end{cases}$$

Thus $v^{k+2}_{\emptyset} = y$. Let $\varepsilon \subseteq [d]$ with $\varepsilon \neq \emptyset$. If $\varepsilon(k+1) = 1$, then $v^{k+2}_{\varepsilon} = z^{k+1}_{\varepsilon}$. If $\varepsilon(k+1) = 0$, then

$$v^{k+2}_{\varepsilon} = v^{k+1}_{\varepsilon} = z^k_{\varepsilon} = z^{k+1}_{\varepsilon}.$$

We conclude that $v^{k+2}_{\varepsilon} = z^{k+1}_{\varepsilon}$ for all $\varepsilon \subseteq [d]$, $\varepsilon \neq \emptyset$. Inductively, we proceed until $k = d-2$, where we obtain a point $\mathbf{v}^d \in \mathbf{Q}_{T_1,\ldots,T_d}$ using Lemma 4.3 with $\mathbf{v}^{d-1}$ and $\mathbf{z}^{d-1}$, and $v^d_{\emptyset} = y$ and $v^d_{\varepsilon} = z^{d-1}_{\varepsilon}$ for all $\varepsilon \subseteq [d]$, $\varepsilon \neq \emptyset$.

Finally, we observe that $(v^d_{\varepsilon} \colon \varepsilon(d) = 1) = (z_{\varepsilon} \colon \varepsilon(d) = 0)$. Using Lemma 4.3 with $\mathbf{v}^d$ and $\mathbf{z}$ we get the existence of the element $\mathbf{v}^{d+1} \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$ such that

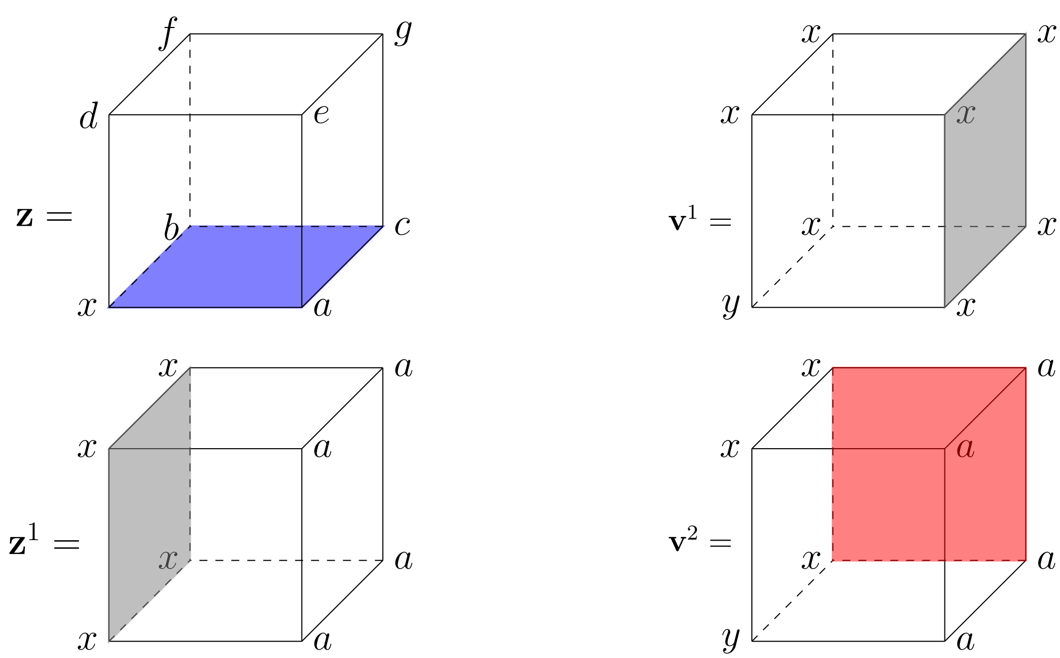
$$v^{d+1}_{\varepsilon} = \begin{cases} v^d_{\varepsilon} & \text{if } \varepsilon(d) = 0, \\ z_{\varepsilon} & \text{if } \varepsilon(d) = 1. \end{cases}$$

Let $\varepsilon \subseteq [d]$. If $\varepsilon = \emptyset$, then $v^{d+1}_{\varepsilon} = y$. Now assume that $\varepsilon \neq \emptyset$. If $\varepsilon(d) = 1$, then $v^{d+1}_{\varepsilon} = z_{\varepsilon}$. If $\varepsilon(d) = 0$ we have that

$$v^{d+1}_{\varepsilon} = v^d_{\varepsilon} = z^{d-1}_{\varepsilon} = z_{\varepsilon}.$$

We conclude that $\mathbf{v}^{d+1} = (y, \mathbf{a}_*) \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$.

In the case $d = 3$ the proof of (1) $\implies$ (3) can be illustrated in the following diagram.



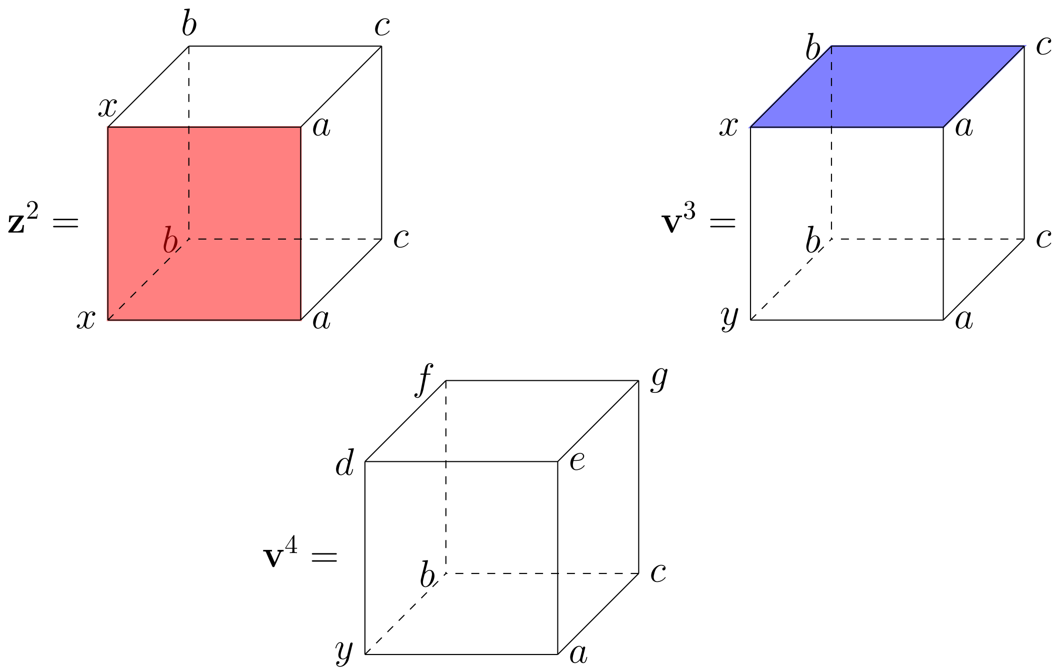



**Figure 4.5:** Illustration of the proof of (1) $\implies$ (3) for the case $d = 3$.

(3) $\implies$ (2) It follows directly. Since $x^{[d]} \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$, then $(y, x_*^{[d]}) \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$.

(2) $\implies$ (1) Suppose that $\mathbf{u} = (x, \mathbf{a}_*) = (\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{v} = (y, \mathbf{a}_*) = (\mathbf{x}_3, \mathbf{x}_2) \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$ for some $\mathbf{a}_* \in X_*^{[d]}$. Then, $(u_\varepsilon \colon \varepsilon(d) = 1) = (v_\varepsilon \colon \varepsilon(d) = 1)$. By Proposition 3.3 (3) (using $j = d$) and Lemma 4.3 we have the existence of a point $\mathbf{z} \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$ such that

$$z_\varepsilon = \begin{cases} u_\varepsilon & \varepsilon(d) = 0, \\ v_{\Phi_d(\varepsilon)} & \varepsilon(d) = 1. \end{cases}$$

Then, $\mathbf{z} = (\mathbf{x}_1, \mathbf{x}_3)$, i.e., there exists $b_* \in X_*^{d-1}$ such that $\mathbf{z} = (x, b_*, y, b_*) \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$. Thus, by Remark 3.7, $(x, y) \in \mathcal{R}_{T_d}(X)$ and from the proof of (5) $\implies$ (1), we conclude that $(x, y, \ldots, y) \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$. $\square$

We use Proposition 4.11 to prove the following lemma.

**Lemma 4.12.** *Let $d \geq 2$ be an integer, $(X, T_1, \ldots, T_d)$ be a minimal distal system with commuting transformations $T_1, \ldots, T_d$, and $x_0 \in X$ a continuity point. Suppose that $X$ has the closing parallelepiped property. Then, $\mathbf{K}^{x_0}_{T_1,\ldots,T_d}$ is relatively independent with respect to their maximal $\mathsf{Z}_0^{e_{j_1}} \wedge \mathsf{Z}_0^{e_{j_2}}$-factors, for all $j_1, j_2 \in [d]$ with $j_1 \neq j_2$.*

**Proof.** Let $\mathbf{x}, \mathbf{y} \in X_*^{[d]}$ be such that:

1. $\mathbf{x} \in \mathbf{K}^{x_0}_{T_1,\ldots,T_d}$.

2. For every $j_1 \in [d]$, $(y_\varepsilon \colon \varepsilon \in \{0,1\}^d \setminus \{\vec{0}\},\ \varepsilon(j_1) = 0) \in \mathbf{K}^{x_0}_{T_1,\ldots,T_{j_1-1},T_{j_1+1},\ldots,T_d}$.

3. For every $j_1, j_2 \in [d]$, $j_1 \neq j_2$,

$$(x_\varepsilon \colon \varepsilon \in \{0,1\}^d \backslash \{\vec{0}\}, \varepsilon(j_1)=0,\ \varepsilon(j_2)=0) = (y_\varepsilon \colon \varepsilon \in \{0,1\}^d \backslash \{\vec{0}\}, \varepsilon(j_1)=0,\ \varepsilon(j_2)=0),$$

i.e., the projection of $\mathbf{x}$ and $\mathbf{y}$ onto the coordinates given by the maximal $\mathsf{Z}_0^{e_{j_1}} \wedge \mathsf{Z}_0^{e_{j_2}}$-factor are equal.

We want to prove that $\mathbf{y} \in \mathbf{K}^{x_0}_{T_1,\dots,T_d}$. By (3), the only coordinates $\varepsilon \subseteq [d]$, $\varepsilon \neq \emptyset$ such that $x_\varepsilon \neq y_\varepsilon$ are the coordinates $\varepsilon = [d]$ and $\varepsilon = [d] \setminus \{j_1\}$ for some $j_1 \in [d]$. For the proof we construct a sequence of points $\mathbf{x}^0, \mathbf{x}^1 \dots, \mathbf{x}^d \in \mathbf{K}^{x_0}_{T_1,\dots,T_d}$ such that $\mathbf{x}^0 = \mathbf{x}$ and for every $j_1 \in [d]$ we have

$$x_\varepsilon^{j_1-1} = \begin{cases} x_\varepsilon & \varepsilon \subseteq [d],\ \varepsilon \neq [d]\setminus\{\ell\},\ \text{for some } \ell \in [d], \\ y_\varepsilon & \varepsilon = [d]\setminus\{r\},\ \text{for every } r \in \{1,\dots,j_1-1\}, \\ a^{j_1} & \varepsilon = [d], \end{cases}$$

for some $a^{j_1-1} \in X$. Since $X$ has the closing parallelepiped property, $a^{j_1}$ is unique. Thus, $\mathbf{x}^d = \mathbf{y} \in \mathbf{K}^{x_0}_{T_1,\dots,T_d}$. For this, we project $\mathbf{x}^{j_1-1}$, $\mathbf{y}$ in $\mathbf{K}^{x_0}_{T_1,\dots,T_{j_1-1},T_{j_1+1},\dots,T_d}$. We observe that $(x_\varepsilon^{j_1-1} \colon \varepsilon \in \{0,1\}^d \setminus \{\vec{0}\}, \varepsilon(j_1)=0)$, $(y_\varepsilon \colon \varepsilon \in \{0,1\}^d \setminus \{\vec{0}\}, \varepsilon(j_1)=0)$ only differ in the coordinate $[d]\setminus\{j_1\}$.

**Claim.** *If we replace the coordinate $\varepsilon = [d]\setminus\{j_1\}$ of $\boldsymbol{x}^{j_1-1}$ by the same coordinate of $\boldsymbol{y}$ we obtain a point $\boldsymbol{x}^{j_1} \in \mathbf{K}^{x_0}_{T_1,\dots,T_d}$.*

**Proof of the Claim.** Consider the system $(X, T_1, \dots, T_{j_1-1}, T_{j_1+1}, \dots, T_d)$. Since the points $(x_\varepsilon^{j_1-1} \colon \varepsilon \in \{0,1\}^d, \varepsilon(j_1)=0)$, $(y_\varepsilon \colon \varepsilon \in \{0,1\}^d, \varepsilon(j_1)=0)$ differ only in one coordinate, by Proposition 4.11 we have that

$$(x^{j_1-1}_{[d]\setminus\{j_1\}}, \dots, x^{j_1-1}_{[d]\setminus\{j_1\}}, y_{[d]\setminus\{j_1\}}) \in \mathbf{Q}_{T_1,\dots,T_{j_1-1},T_{j_1+1},\dots,T_d}(X). \tag{4.2}$$

Now, by Proposition 3.3 (4), we have that $(x^{j_1-1}_{[d]\setminus\{j_1\}}, x^{j_1-1}_{[d]}) \in \mathbf{Q}_{T_{j_1}}(X)$. If we define $\mathbf{v}^1 \in X^{[d]}$ as

$$v^1_\varepsilon = \begin{cases} x_{[d]\setminus\{j_1\}} & \text{if } \varepsilon(j_1)=0, \\ x^{j_1-1}_{[d]} & \text{if } \varepsilon(j_1)=1, \end{cases}$$

then, by Proposition 3.3 (5), $\mathbf{v}^1 \in \mathbf{Q}_{T_1,\dots,T_d}(X)$. The idea of the proof is to construct a sequence of points $\mathbf{u}^1, \dots, \mathbf{u}^d$ in $\mathbf{Q}_{T_1,\dots,T_d}(X)$ such that $\mathbf{u}^1$ satisfies

$$u^1_\varepsilon = \begin{cases} x^{j_1-1}_{[d]\setminus\{j_1\}} & \text{if } \varepsilon(j_1)=0\ \wedge \exists j_2 \neq j_1,\ \varepsilon(j_2)=0, \\ y_{[d]\setminus\{j_1\}} & \text{if } \varepsilon = [d]\setminus\{j_1\}, \\ x^{j_1-1}_{[d]} & \text{if } \varepsilon(j_1)=1\ \wedge \exists j_2 \neq j_1,\ \varepsilon(j_2)=0, \\ a & \text{if } \varepsilon = [d], \end{cases}$$

for some $a \in X$. For this we use Proposition 3.3 (5). We construct the rest of the sequence $\mathbf{u}^2, \dots, \mathbf{u}^d \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ from $\mathbf{u}^1$ satisfying: $u^k_{[d]\setminus\{j_1\}} = y_{[d]\setminus\{j_1\}}$, $u^k_{[d]} = a$ and for $\varepsilon \subseteq [d]$ with $\varepsilon \neq [d]$ and $\varepsilon \neq [d]\setminus\{j_1\}$, the value $u^k_\varepsilon$ depends only in the elements of

$\{j_1, d, d-1, \ldots, d-k+2\}$ (with respect to the coordinates of $\mathbf{x}^{j_1-1}$) in the following way. If $\eta \subseteq [d]$, we have that

$$u^k_\varepsilon = x^{j_1-1}_\eta \iff \begin{array}{c} \forall \ell \in \{j_1, d, d-1, \ldots, d-k+2\},\ \varepsilon(\ell) = \eta(\ell), \\ \wedge \\ \forall \ell \in [d] \setminus \{j_1, d, d-1, \ldots, d-k+2\}, \eta(\ell) = 1. \end{array} \tag{4.3}$$

Here, we separate the proof of the Claim in three cases.

Case 1. If $j_1 = 1$ we proceed inductively until $k = d$.

Case 2. If $j_1 \neq 1$ and $j_1 \neq d$, we proceed like in Case 1 until $k = d - j_1 + 2$ and we obtain the point $\mathbf{u}^{d-j_1+1} \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$. Here we have that the coordinates of $\mathbf{u}^{d-j_1+1}$ depend on the subset $\{j_1, j_1+1, \ldots, d\}$. For the rest of the sequence, i.e., for $d - j_1 + 2 \leq k \leq d$ we proceed inductively like in the previous case, but with a slight difference.

Case 3. If $j_1 = d$ we proceed like in the second part of Case 2.

In the three cases we will finish with $\mathbf{u}^d \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$ such that $u^d_{[d]\setminus\{j_1\}} = y_{[d]\setminus\{j_1\}}$ and $u^d_{[d]} = a$, and the rest of the coordinates depend on the subset $[d]$ like in (4.3), i.e., for every $\varepsilon \in \{0,1\}^d$, $\varepsilon \neq [d]$, $\varepsilon \neq [d] \setminus \{j_1\}$, $u^d_\varepsilon = x^{j_1-1}_\varepsilon$ and thus $u^d_\varepsilon$ and $x^{j_1-1}_\varepsilon$ differ just in the coordinate $\varepsilon = [d] \setminus \{j_1\}$ in $\mathbf{K}^{x_0}_{T_1,\ldots,T_d}$. Then we choose $\mathbf{u}^d$ to be $\mathbf{x}^{j_1}$.

Now to formalize the proof, we divide the construction of the sequence in two stages. In Stage 1 we construct $\mathbf{u}^1 \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$, and in Stage 2 we construct the rest of the sequence $\mathbf{u}^2, \ldots, \mathbf{u}^d \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$.

*Stage 1. Construction of the point* $\mathbf{u}^1 \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$.

We consider two cases, if $\mathbf{Q}_{T_{j_1}}(x_{[d]\setminus\{j_1\}}) = \overline{\mathcal{O}(x^{j_1-1}_{[d]\setminus\{j_1\}}, T_{j_1})}$ and if $\mathbf{Q}_{T_{j_1}}(x^{j_1-1}_{[d]\setminus\{j_1\}}) \supsetneq \overline{\mathcal{O}(x^{j_1-1}_{[d]\setminus\{j_1\}}, T_{j_1})}$. By Lemma 4.5 of [17], we have a $G_\delta$-dense subset of $X$ such that $\mathbf{Q}_{T_{j_1}}(x) = \overline{\mathcal{O}(x, T_{j_1})}$.

Case 1: $\mathbf{Q}_{T_{j_1}}(x^{j_1-1}_{[d]\setminus\{j_1\}}) = \overline{\mathcal{O}(x_{[d]\setminus\{j_1\}}, T^{j_1-1}_{j_1})}$. Let $(n_i)_{i\in\mathbb{N}} \subseteq \mathbb{Z}$ be a sequence such that

$$T^{n_i}_{j_1} x_{[d]\setminus\{j_1\}} \to x^{j_1-1}_{[d]}.$$

By compactness we assume that

$$T^{n_i}_{j_1} y^{j_1-1}_{[d]\setminus\{j_1\}} \to a.$$

Thus, by Proposition 3.3 (5), we have that there exists $\mathbf{u}^1 \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$ such that

$$u^1_\varepsilon = \begin{cases} x^{j_1-1}_{[d]\setminus\{j_1\}} & \text{if } \varepsilon(j_1) = 0\ \wedge \exists j_2 \neq j_1,\ \varepsilon(j_2) = 0, \\ y_{[d]\setminus\{j_1\}} & \text{if } \varepsilon = [d] \setminus \{j_1\}, \\ x^{j_1-1}_{[d]} & \text{if } \varepsilon(j_1) = 1\ \wedge \exists j_2 \neq j_1,\ \varepsilon(j_2) = 0, \\ a & \text{if } \varepsilon = [d]. \end{cases}$$

Case 2: $\mathbf{Q}_{T_{j_1}}(x^{j_1-1}_{[d]\setminus\{j_1\}}) \supsetneq \overline{\mathcal{O}(x^{j_1-1}_{[d]\setminus\{j_1\}}, T_{j_1})}$. We define the following projection maps.

- $\phi_1$ is the projection from $\mathbf{K}^{x_0}_{T_1,\dots,T_d}$ onto the coordinates $\varepsilon \in \{0,1\}^d$ where $\varepsilon(j_1) = 0$ and the coordinate $\varepsilon = [d]$, which we call $\mathbf{K}^{x_0}_{\{T_1,\dots,T_{j_1-1},T_{j_1+1},\dots,T_d\}\cup[d]}$.
- $\phi_2$ is the projection from $\mathbf{K}^{x_0}_{\{T_1,\dots,T_{j_1-1},T_{j_1+1},\dots,T_d\}\cup[d]}$ onto $\mathbf{K}^{x_0}_{T_1,\dots,T_{j_1-1},T_{j_1+1},\dots,T_d}$.
- $\phi_3$ is the projection from $\mathbf{K}^{x_0}_{T_1,\dots,T_{j_1-1},T_{j_1+1},\dots,T_d}$ onto the coordinates $\varepsilon \in \{0,1\}^d$ where $\varepsilon(j_1) = 0$ and there exists $j_2 \in [d]$, $j_2 \neq j_1$, such that $\varepsilon(j_2) = 0$, which we call $\mathbf{K}^{x_0}_{\{T_1,\dots,T_{j_1-1},T_{j_1+1},\dots,T_d\}\setminus([d]\setminus\{j_1\})}$.

Let $\delta > 0$. By Theorem 1.32 we consider $0 < \delta' < \delta$ such that

$$B\left(\left(x^{j_1-1}_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon(j_1) = 0\right), \delta'\right) \subseteq \phi_2\left(B\left(\left(x^{j_1-1}_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon(j_1) = 0, \vee\ \varepsilon = d\right)\right), \delta\right), \tag{4.4}$$

and $0 < \delta'' < \delta'$ such that

$$\begin{aligned} &B((x^{j_1-1}_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon(j_1) = 0 \wedge\ \exists j_2 \in [d], j_2 \neq j_1, \varepsilon(j_2) = 0), \delta'') \\ &\subseteq \phi_3(B((x^{j_1-1}_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon(j_1) = 0), \delta')). \end{aligned} \tag{4.5}$$

From now, a point $(a_\varepsilon \colon \varepsilon \in \{0,1\}^d, \varepsilon(j_1) = 0) \in \mathbf{K}^{x_0}_{T_1,\dots,T_{j_1-1},T_{j_1+1},\dots,T_d}$ will be written as $(a'_\eta \colon \eta \in \{0,1\}^{d-1})$, with the correspondance $a_\varepsilon = a'_\eta$ if and only if $\varepsilon = \Psi^0_{j_1}(\eta)$.

Let $(a'_\eta \colon \eta \in \{0,1\}^{d-1}) \in \mathbf{K}^{x_0}_{T_1,\dots,T_{j_1-1},T_{j_1+1},\dots,T_d}$ be such that $\mathbf{Q}_{T_{j_1}}(a'_{[d-1]}) = \mathbf{Q}_{T_{j_1}}(a_{[d]\setminus\{j_1\}}) = \overline{\mathcal{O}(a_{[d]\setminus\{j_1\}}, T_{j_1})}$ and

$$\rho((a'_\eta \colon \eta \in \{0,1\}^{d-1}), ((x^{j_1-1}_\eta)' \colon \eta \in \{0,1\}^{d-1})) < \delta''.$$

Thus, by Remark 1.15,

$$\rho((a'_\eta \colon \eta \in \{0,1\}^{d-1}, \eta \neq [d-1]), ((x^{j_1-1}_\eta)' \colon \eta \in \{0,1\}^{d-1}, \eta \neq [d-1])) < \delta''.$$

Then, by (4.5), there exists $(b'_\eta \colon \eta \in \{0,1\}^{d-1})$ such that

$$\rho((b'_\eta \colon \eta \in \{0,1\}^{d-1}), (y'_\eta \colon \eta \in \{0,1\}^{d-1})) < \delta',$$

and $(a'_\eta \colon \eta \in \{0,1\}^{d-1},\ \eta \neq [d-1]) = (b'_\eta \colon \eta \in \{0,1\}^{d-1},\ \eta \neq [d-1])$. Therefore, $(a'_\eta \colon \eta \in \{0,1\}^{d-1})$ and $(b'_\eta \colon \eta \in \{0,1\}^{d-1})$ correspond to Case 1. By (4.4), there exists $a_{[d]} \in X$ with $a_{[d]} \in \overline{\mathcal{O}(a_{[d]\setminus\{j_1\}}, T_{j_1})}$ and

$$\rho((a_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon(j_1) = 0, \vee\ \varepsilon = [d]), (x^{j_1-1}_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon(j_1) = 0, \vee\ \varepsilon = [d])) < \delta. \tag{4.6}$$

Now, let $\mathbf{w} \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ constructed in Case 1 for the point $(a'_\eta \colon \eta \in \{0,1\}^{d-1}) \in \mathbf{K}^{x_0}_{T_1,\dots,T_{j_1-1},T_{j_1+1},\dots,T_d}$, i.e.,

$$w_\varepsilon = \begin{cases} a_{[d]\setminus\{j_1\}} & \text{if } \varepsilon(j_1) = 0\ \wedge \exists j_2 \neq j_1,\ \varepsilon(j_2) = 0, \\ b_{[d]\setminus\{j_1\}} & \text{if } \varepsilon = [d] \setminus \{j_1\}, \\ a_{[d]} & \text{if } \varepsilon(j_1) = 1\ \wedge \exists j_2 \neq j_1,\ \varepsilon(j_2) = 0, \\ u_\delta & \text{if } \varepsilon = [d], \end{cases}$$

for some $u_\delta \in X$. If $\delta \to 0$, by compactness, we can assume that $u_\delta \to a$. Thus we have that there exists $\mathbf{u}^1 \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ such that

$$u^1_\varepsilon = \begin{cases} x^{j_1-1}_{[d]\setminus\{j_1\}} & \text{if } \varepsilon(j_1)=0 \ \wedge \exists j_2 \neq j_1,\ \varepsilon(j_2)=0, \\ y_{[d]\setminus\{j_1\}} & \text{if } \varepsilon = [d]\setminus\{j_1\}, \\ x^{j_1-1}_{[d]} & \text{if } \varepsilon(j_1)=1 \ \wedge \exists j_2 \neq j_1,\ \varepsilon(j_2)=0, \\ a & \text{if } \varepsilon=[d]. \end{cases}$$

*Stage 2. Construction of the points* $\mathbf{u}^2,\dots,\mathbf{u}^d \in \mathbf{Q}_{T_1,\dots,T_d}(X)$.

From now assume $j_1 \neq d$ and consider the face $(u^1_\varepsilon \colon \varepsilon(d)=0)$. We have two different values, $x^{j_1-1}_{[d]\setminus\{j_1\}}$ if $\varepsilon(j_1)=0$ and $x^{j_1-1}_{[d]}$ if $\varepsilon(j_1)=1$. Since $(x_0,\mathbf{x}^{j_1-1}) \in \mathbf{Q}_{T_1,\dots,T_d}(X)$, using Proposition 4.11 (4) and (5) with the subset $\{j_1,d\}$ there exists $\mathbf{v}^2 \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ such that

$$v^2_\varepsilon = \begin{cases} x^{j_1-1}_{[d]\setminus\{d,j_1\}} & \text{if } \varepsilon(j_1)=0 \ \wedge \ \varepsilon(d)=0, \\ x^{j_1-1}_{[d]\setminus\{d\}} & \text{if } \varepsilon(j_1)=1 \ \wedge \ \varepsilon(d)=0, \\ x^{j_1-1}_{[d]\setminus\{j_1\}} & \text{if } \varepsilon(j_1)=0 \ \wedge \ \varepsilon(d)=1, \\ x^{j_1-1}_{[d]} & \text{if } \varepsilon(j_1)=1 \ \wedge \ \varepsilon(d)=1. \end{cases}$$

Hence, $(v^2_\varepsilon \colon \varepsilon \in \{0,1\}^d, \varepsilon(d)=1) = (u^1_\varepsilon \colon \varepsilon \in \{0,1\}^d, \varepsilon(d)=0)$. By Lemma 4.3, there exists $\mathbf{u}^2 \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ such that

$$u^2_\varepsilon = \begin{cases} x^{j_1-1}_{[d]\setminus\{d,j_1\}} & \text{if } \varepsilon(j_1)=0 \ \wedge \ \varepsilon(d)=0, \\ x^{j_1-1}_{[d]\setminus\{d\}} & \text{if } \varepsilon(j_1)=1 \ \wedge \ \varepsilon(d)=0, \\ x^{j_1-1}_{[d]\setminus\{j_1\}} & \text{if } \varepsilon(j_1)=0 \ \wedge \ \varepsilon(d)=1,\ \varepsilon\neq[d]\setminus\{j_1\}, \\ x^{j_1-1}_{[d]} & \text{if } \varepsilon(j_1)=1 \ \wedge \ \varepsilon(d)=1,\ \varepsilon\neq[d], \\ y_{[d]\setminus\{j_1\}} & \varepsilon=[d]\setminus\{j_1\}, \\ a & \varepsilon=[d]. \end{cases}$$

Now assume that we have $\mathbf{u}^k \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ for $2 \leq k < d-j_1+1$ such that for $\varepsilon \in \{0,1\}^d$ the value $u^k_\varepsilon$ depends only in the elements of $\{j_1,d,d-1,\dots,d-k+2\}$ (with respect to the coordinates of $\mathbf{x}^{j_1-1}$), except $\varepsilon=[d]$ and $\varepsilon=[d]\setminus\{j_1\}$, where $u^k_{[d]\setminus\{j_1\}} = y_{[d]\setminus\{j_1\}}$ and $u^k_{[d]}=a$. We consider the face $(u^k_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon(d-k+1)=0)$. Since $\mathbf{x}^{j_1-1} \in \mathbf{Q}_{T_1,\dots,T_d}(X)$, then using Proposition 3.3 (4) and (5) with $\{j_1,d,d-1,\dots,d-k+1\} \subseteq [d]$, there exists $\mathbf{v}^{k+1} \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ such that its coordinates depend only in the set $\{j_1,d,d-1,\dots,d-k+1\}$ (with respect to the coordinates of $\mathbf{x}^{j_1-1}$), thus $(v^{k+1}_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon(d-k+1)=1) = (u^k_\varepsilon \colon \varepsilon \in \{0,1\}^d,\ \varepsilon(d-k+1)=0)$. Then, by Lemma 4.3, there exists $\mathbf{u}^{k+1} \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ such that for every $\varepsilon \in \{0,1\}^d$

$$u^{k+1}_\varepsilon = \begin{cases} v^{k+1}_\varepsilon & \text{if } \varepsilon(d-k+1)=0, \\ u^k_\varepsilon & \text{if } \varepsilon(d-k+1)=1. \end{cases}$$

By construction, we have that the coordinates of $\mathbf{u}^{k+1}$ only depend on the set $\{j_1,d,d-1,\dots,d-k+1\}$ (with respect to the coordinates of $\mathbf{x}^{j_1-1}$), except $\varepsilon=[d]\setminus\{j_1\}$

and $\varepsilon = [d]$, where we have $u^{k+1}_{[d]\setminus\{j_1\}} = y_{[d]\setminus\{j_1\}}$ and $u_{[d]} = a$. Here we have to separate in three cases, $j_1 = 1$, $j_1 = d$ and $j_1 \neq 1, d$. The only difference in these three cases is just the behavior of the set $\{j_1, \dots, d, d-1, \dots d-k+2\}$ but the procedure is the same.

- If $j_1 = 1$ we proceed inductively until $k = d$.
- If $j_1 \neq 1$ and $j_1 \neq d$, we proceed like the previous case until $k = d - j_1 + 2$ and obtain the point $\mathbf{u}^{d-j_1+1} \in \mathbf{Q}_{T_1,\dots,T_d}(X)$. Here we have that the coordinates of $\mathbf{u}^{d-j_1+1}$ depend on the subset $\{j_1, j_1+1, \dots, d\}$, so the difference is that for $\mathbf{u}^{d-j_1+1}$ we use the face $\varepsilon(j_1 - 1) = 0$ in Lemma 4.3 instead of $\varepsilon(j_1 - 1) = 0$ and for the rest of the sequence, i.e., $d - j_1 + 1 \leq k < d$, we use the face $\varepsilon(d-k) = 0$ until $k = d-1$ where we use the face $\varepsilon(1) = 0$ and we get the point $\mathbf{u}^d \in \mathbf{Q}_{T_1,\dots,T_d}(X)$.
- If $j_1 = d$, for $2 \leq k < d$ we use the face $\varepsilon(d-k) = 0$ until $k = d-1$ where we use the face $\varepsilon(1) = 0$ and we get the point $\mathbf{u}^d \in \mathbf{Q}_{T_1,\dots,T_d}(X)$.

Finally, for $\mathbf{u}^d$ we have that its coordinates depend on the entire set $[d]$ (with respect to $\mathbf{x}^{j_1-1}$) except the coordinates $\varepsilon = [d] \setminus \{j_1\}$ and $\varepsilon = [d]$. Thus, $u^d_\varepsilon = x^{j_1-1}_\varepsilon$ for all $\varepsilon \in \{0,1\}^d$, $\varepsilon \neq [d] \setminus \{j_1\}$, $\varepsilon \neq [d]$, $u^d_{[d]\setminus\{j_1\}} = y_{[d]\setminus\{j_1\}}$ and $u^d_{[d]} = a$. $\square$

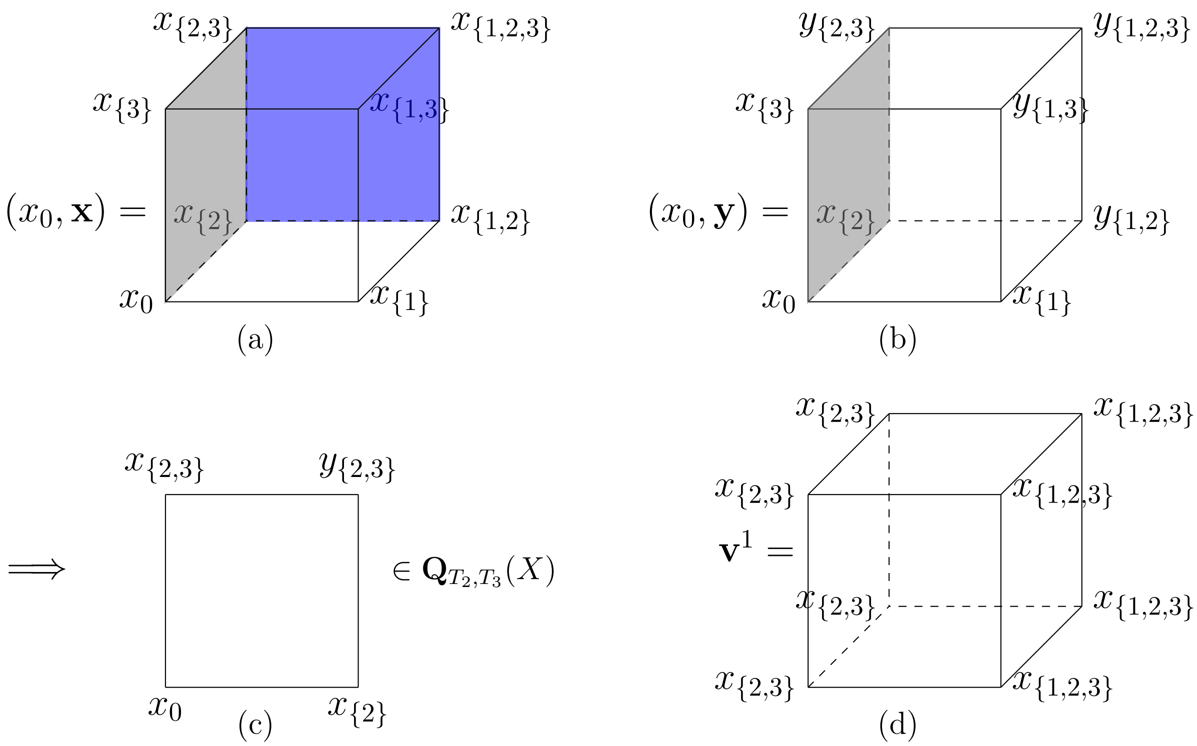
To conclude, first we use the claim for $1 \leq j_1 \leq d$ to construct the point $\mathbf{x}^{j_1} \in \mathbf{K}^{x_0}_{T_1,\dots,T_d}$. Finally, for $\mathbf{x}^d \in \mathbf{K}^{x_0}_{T_1,\dots,T_d}$

$$x^d_\varepsilon = \begin{cases} x_\varepsilon & \varepsilon \subseteq [d],\ \varepsilon \neq [d] \setminus \{\ell\} \text{for every } \ell \in [d], \\ y_\varepsilon & \varepsilon = [d] \setminus \{r\},\ \text{for some } r \in [d], \\ a^d & \varepsilon = [d], \end{cases}$$

which allow to conclude that $\mathbf{x}^d = y \in \mathbf{K}^{x_0}_{T_1,\dots,T_d}$.

$\square$

In the case $d = 3$ the proof of previous lemma can be illustrated in the following diagram. In such diagram we assume we can prove the relatively independence with respect to the factors of $\mathbf{K}^{x_0}_{T_2,T_3}$.


$(x_0, \mathbf{x}) =$
$x_{\{2,3\}}$
$x_{\{1,2,3\}}$
$x_{\{3\}}$
$x_{\{1,3\}}$
$x_{\{2\}}$
$x_{\{1,2\}}$
$x_0$
$x_{\{1\}}$
(a)
$(x_0, \mathbf{y}) =$
$y_{\{2,3\}}$
$y_{\{1,2,3\}}$
$x_{\{3\}}$
$y_{\{1,3\}}$
$x_{\{2\}}$
$y_{\{1,2\}}$
$x_0$
$x_{\{1\}}$
(b)
$\Longrightarrow$
$x_{\{2,3\}}$
$y_{\{2,3\}}$
$\in \mathbf{Q}_{T_2,T_3}(X)$
$x_0$
$x_{\{2\}}$
(c)
$\mathbf{v}^1 =$
$x_{\{2,3\}}$
$x_{\{1,2,3\}}$
$x_{\{2,3\}}$
$x_{\{1,2,3\}}$
$x_{\{2,3\}}$
$x_{\{1,2,3\}}$
$x_{\{2,3\}}$
$x_{\{1,2,3\}}$
(d)

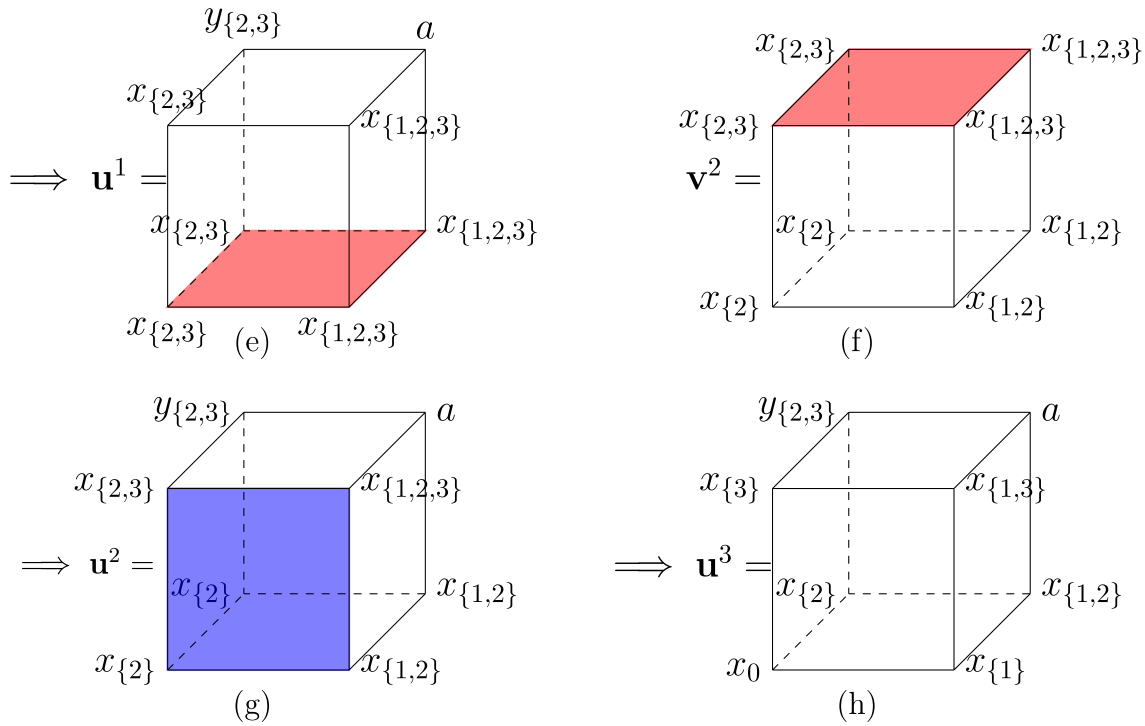



**Figure 4.6:** Illustration of the proof of the claim in previous proof for the case $d = 3$ and with respect to the factors of $\mathbf{K}^{x_0}_{T_2,T_3}$. In (a) we have the point $(x, \mathbf{x}) \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$. In (b) we have the point $(x, \mathbf{y})$ such that the projection to $\mathbf{K}^{x_0}_{T_2,T_3}$ is $(x_0, x_{\{2\}}, x_{\{3\}}, y_{\{2,3\}})$. Then, by Proposition 3.3 (4) the point in (c) belongs to $\mathbf{Q}_{T_2,T_3}(X)$. Additionally, the point in (d) belongs to $\mathbf{Q}_{T_1,T_2,T_3}(X)$ by Proposition 3.3 (4) and (5) using the point $\mathbf{z}$. Finally, for Cases 1 and 2 of Stage 1 we have the existence of the point (e) in $\mathbf{Q}_{T_1,\ldots,T_d}(X)$. Using Proposition 3.3 (4) and (5) described in the proof we have that the point (f) is in $\mathbf{Q}_{T_1,T_2,T_3}(X)$. Using Lemma 4.3 with $\mathbf{u}^1$ and $\mathbf{v}^2$ we have the existence of the point (g) in $\mathbf{Q}_{T_1,T_2,T_3}(X)$. Finally, using Lemma 4.3 with the points $\mathbf{u}^2$ and $\mathbf{z}$ we have the existence of the point (h) in $\mathbf{Q}_{T_1,T_2,T_3}(X)$.

## 4.2 Proof of Theorem 4.1

We can prove that $\mathcal{R}_{T_1,\ldots,T_d}(X)$ is an equivalence relation in the distal case.

**Theorem 4.13.** *Let $d \geq 2$ be an integer and $(X, T_1, \ldots, T_d)$ be a distal system with commuting transformations $T_1, \ldots, T_d$. Then, $\mathcal{R}_{T_1,\ldots,T_d}(X)$ is a closed and invariant equivalence relation on $X$.*

**Proof.** It suffices to prove the transitivity of $\mathcal{R}_{T_1,\ldots,T_d}(X)$. Let $(x,y),(y,z) \in \mathcal{R}_{T_1,\ldots,T_d}(X)$. By Proposition 4.11 (1) we have that $(y, z_*^{[d]}) \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$. Now, by Proposition 4.11 $(x, z_*^{[d]}) \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$ and thus $(x,z) \in \mathcal{R}_{T_1,\ldots,T_d}(X)$. □

We also have the following property, which allows us to lift the relation $\mathcal{R}_{T_1,\ldots,T_d}(X)$ by a factor map and to prove our main theorem.

**Theorem 4.14.** *Let $\pi\colon Y \to X$ be a factor map between topological dynamical systems $(X, T_1, \dots, T_d)$ and $(Y, T_1, \dots, T_d)$ with commuting transformations $T_1, \dots, T_d$. If $(X, T_1, \dots, T_d)$ is distal, then $\pi \times \pi(\mathcal{R}_{T_1,\dots,T_d}(Y)) = \mathcal{R}_{T_1,\dots,T_d}(X)$.*

**Proof.** The proof is similar to Theorem 6.4 in [32]. By Proposition 3.8 we have that $\pi \times \pi(\mathcal{R}_{T_1,\dots,T_d}(Y)) \subseteq \mathcal{R}_{T_1,\dots,T_d}(X)$. Let $(y_1, y_2) \in \mathcal{R}_{T_1,\dots,T_d}(X)$. First, by Proposition 3.3 (5), there exist a sequence $(y^i)_{i\in\mathbb{N}} \subseteq Y$ and a sequence $\mathbf{n}(i) \subseteq \mathbb{Z}^d$ such that

$$\lim_{i\to\infty} T_1^{n_1(i)\varepsilon_1} \cdots T_d^{n_d(i)\varepsilon_d} y^i = \begin{cases} y_1 & \varepsilon \neq [d], \\ y_2 & \varepsilon = [d]. \end{cases}$$

Let $(x^i)_{i\in\mathbb{N}}$ in $X$ be such that $\pi(x^i) = y^i$. By compactness we can assume there exist $y_1 \in X$ and $\mathbf{a}_* \in X_*^{[d]}$ such that

$$\lim_{i\to\infty} T_1^{n_1(i)\varepsilon_1} \cdots T_d^{n_d(i)\varepsilon_d} x^i = \begin{cases} x_1 & \varepsilon = \emptyset, \\ (a_*)_\varepsilon & \varepsilon \neq \emptyset. \end{cases}$$

If $\mathbf{x} = (x_1, \mathbf{a}_*)$, then $\pi^{[d]}(\mathbf{x}) = (y_1, \dots, y_1, y_2)$. Let $\mathbf{x_I} = (x_\varepsilon \colon \varepsilon(d) = 0)$ and $\mathbf{x_{II}} = (x_\varepsilon \colon \varepsilon(d) = 1)$. We have that $\mathbf{x}_I \in \mathbf{Q}_{T_1,\dots,T_{d-1}}(X)$. By minimality, there exist sequences $g^i \in G$ and $\mathbf{n}^1(i) \subseteq \mathbb{Z}^{d-1}$ with

$$\forall \eta \in \{0,1\}^{d-1}, \lim_{i\to\infty} T_1^{n_1^1(i)\eta_1} \cdots T_{d-1}^{n_{d-1}^1(i)\eta_{d-1}} g^i(\mathbf{x_I})_\eta = x_1.$$

By compactness, we can assume that

$$\forall \eta \in \{0,1\}^{d-1}, \lim_{i\to\infty} T_1^{n_1^1(i)\eta_1} \cdots T_{d-1}^{n_{d-1}^1(i)\eta_{d-1}} g^i(\mathbf{x_{II}})_\eta = (\mathbf{x}'_{\mathbf{II}})_\eta.$$

If we define $\mathbf{m}^1(i) = (\mathbf{n}^1(i), 0) \in \mathbb{Z}^d$, then

$$\lim_{i\to\infty} T_1^{m_1^1(i)\varepsilon_1} \cdots T_d^{m_d^1(i)\varepsilon_d} g^i(\mathbf{x})_\varepsilon = \begin{cases} x_1 & \text{if } \varepsilon(d) = 0, \\ (\mathbf{x}'_{\mathbf{II}})_\eta & \text{if } \varepsilon = \Psi_d^1(\eta). \end{cases}$$

Let $\mathbf{x}^1 = (x_1^{[d-1]}, \mathbf{x}'_{\mathbf{II}})$. We observe that

$$\pi^{[d]}(\mathbf{x}^1) = (y_1, \dots, y_1, y_3),$$

where $y_3 = \lim\limits_{i\to\infty} T_1^{n_1^1(i)} \cdots T_{d-1}^{n_{d-1}^1(i)} g^i y_2$. Hence, $(y_1, y_3) \in \overline{\mathcal{O}((y_1, y_2), G_{[2]}^\Delta)}$.

Now assume we have produced points $\mathbf{x^j} \in \mathbf{Q}_{T_1,\dots,T_d}(X)$ for $1 \le j \le d$ with $\pi^{[d]}(\mathbf{x}^j) = \mathbf{y}^j$ such that $x_\varepsilon^j = x_1$ if there exists some $k$ with $d - j + 1 \le k \le d$ and $\varepsilon(k) = 0$, $y_{[d]}^j = y_{j+2}$, $y_\varepsilon^j = y_1$ for all $\varepsilon \neq [d]$, and $(y_1, y_{j+2}) \in \overline{\mathcal{O}((y_1, y_{j+1}), G_{[2]}^\Delta)}$.

Let $\mathbf{x}^j_{\mathbf{I}} = (x^j_\varepsilon \colon \varepsilon \in \{0,1\}^d, \varepsilon(d-j) = 0)$ and $\mathbf{x}^j_{\mathbf{II}} = (x^j_\varepsilon \colon \varepsilon \in \{0,1\}^d, \varepsilon(d-j) = 1)$. By minimality, there exist a sequence $g^i_j \in G$ and $\mathbf{n}^{j+1}(i) \subseteq \mathbb{Z}^{d-1}$ such that

$$\forall \eta \in \{0,1\}^d, \lim_{i\to\infty} T_1^{n_1^{j+1}(i)\eta_1} \cdots T_{d-j-1}^{n_{d-j-1}^{j+1}(i)\eta_{d-j-1}} T_{d-j+1}^{n_{d-j}^{j+1}(i)\eta_{d-j}} \cdots T_d^{n_{d-1}^{j+1}(i)\eta_{d-1}} g^i_j(\mathbf{x_{II}})_\eta = x_1.$$

By compactness, we can assume that

$$\forall \eta \in \{0,1\}^d, \lim_{i\to\infty} T_1^{n_1^{j+1}(i)\eta_1} \cdots T_{d-j-1}^{n_{d-j-1}^{j+1}(i)\eta_{d-j-1}} T_{d-j+1}^{n_{d-j}^{j+1}(i)\eta_{d-j}} \cdots T_d^{n_{d-1}^{j+1}(i)\eta_{d-1}} g^i_j(\mathbf{x_{II}})_\eta = (\mathbf{x}^{j'}_{\mathbf{II}})_\eta.$$

Consider $\mathbf{m}^{j+1}(i) = (n_1^{j+1}(i), \ldots, n_{d-j-1}^{j+1}(i), 0, n_{d-j}^{j+1}(i), \ldots, n_{d-1}^{j+1}(i)) \subseteq \mathbb{Z}^d$. If we define $\mathbf{x}^{j+1} \in X^{[d]}$ as

$$\mathbf{x}^{j+1}_\varepsilon = \begin{cases} x_1 & \varepsilon(d-j) = 0, \\ (\mathbf{x}^{j'}_{\mathbf{II}})_\eta & \varepsilon = \Psi^1_{d-j}(\eta), \end{cases}$$

then

$$\lim_{i\to\infty} T_1^{m_1^{j+1}(i)\varepsilon_1} \cdots T_d^{m_d^{j+1}(i)\varepsilon_d} g^i_j \mathbf{x}^j_\varepsilon = \mathbf{x}^{j+1}_\varepsilon.$$

Let $\mathbf{y}^{j+1} = \pi^{[d]}(\mathbf{x}^{j+1})$. We have that $y^{j+1}_\varepsilon = y_1$ for all $\varepsilon \neq [d]$ and $y^{j+1}_{[d]} = y_{j+3}$. We have that $(y_1, y_{j+3}) \in \overline{\mathcal{O}((y_1, y_{j+1}), G^\Delta_{[2]})}$.

Inductively we get $\mathbf{x}^1, \ldots, \mathbf{x}^d$ and $\mathbf{y}^1, \ldots, \mathbf{y}^d$ such that for all $j \in [d]$, $\pi^{[d]}(\mathbf{x}^j) = \mathbf{y}^j$.

For $\mathbf{x}^d$, we have that $x^d_\varepsilon = x_1$ if there exists some $k$ with $1 \leq k \leq d$ such that $\varepsilon(k) = 0$. That means there is some $x_2 \in X$ such that

$$\mathbf{x}^d = (x_1, \ldots, x_1, x_2).$$

By Proposition 4.11, $(x_1, x_2) \in \mathcal{R}_{T_1,\ldots,T_d}(Y)$. We observe that $\pi(x_2) = y_{d+2}$. By distality and minimality we have that $(y_1, y_{d+2}) \in \overline{\mathcal{O}((y_1, y_2), G^\Delta_{[2]})}$, then there exists $g^i_{d+1} \in G$ such that $(g^i_{d+1}y_1, g^i_{d+1}y_{d+2}) \to (y_1, y_2)$. By compactness we can assume that $(g^i_{d+1}x_1, g^i_{d+1}x_2) \to (z_1, z_2)$. Then, as $\mathcal{R}_{T_1,\ldots,T_d}(X)$ is closed and invariant $(z_1, z_2) \in \mathcal{R}_{T_1,\ldots,T_d}(Y)$ and $\pi \times \pi(z_1, z_2) = (y_1, y_2)$. □

We are now ready to prove our main theorem.

**Proof of Theorem 4.1.** (1) $\implies$ (2). This follows from Proposition 3.10.

(2) $\implies$ (1). Suppose that $(X, T_1, \ldots, T_d)$ does not verify the closing parallelepiped property, then there exist $x, y \in X$ with $x \neq y$ and $\mathbf{a}_* \in X^{[d]}_*$ such that $(x, \mathbf{a}_*)$, $(y, \mathbf{a}_*) \in \mathbf{Q}_{T_1,\ldots,T_d}(X)$. By Proposition 4.11, we have that $(x, y) \in \mathcal{R}_{T_1,\ldots,T_d}(X)$. Then, $x = y$, which is a contradiction.

(1) $\implies$ (3). This is a consequence of Lemma 4.12.

(3) $\implies$ (1). If $(X, T_1, \ldots, T_d)$ has the closing parallelepiped property, by Lemma 4.12 we consider $\mathbf{K}^{x_0}_{T_1,\ldots,T_d}$ for a fixed $x_0 \in X$.

Let $(Y,T_1,\dots,T_d)$ be an extension of $X$ which is a joining of the maximal $\mathsf{Z}_0^{e_j}$-factors for $j\in[d]$. Then, $Y\subseteq \prod\limits_{j\in[d]} Y/\mathbf{Q}_{\hat{T}_j}$ is invariant under the transformations $\hat{T}_1,\dots,\hat{T}_d$. Consider the action $\hat{T}_j$ on $Y$. Let $\vec{y}=(y_1,\dots,y_d)\in Y$. As the component $j$ of $\vec{y}$ is an element of $Y/Q_{T_j}$, this component is invariant under the action of $\hat{T}_j$, i.e.,

$$\hat{T}_j\vec{y}=(T_jy_1,\dots,T_jy_{j-1},y_j,T_jy_{j+1},\dots,T_jy_d)\,.$$

We will see that $Y$ has the closing parallelepiped property in $\mathbf{Q}_{\hat{T}_1,\dots,\hat{T}_j}(Y)$. Indeed, let $\mathbf{y}\in\mathbf{Q}_{\hat{T}_1,\dots,\hat{T}_j}(Y)$, then there exists $(\vec{y}^i)_{i\in\mathbb{N}}\subseteq\mathbb{N}$ and $(\mathbf{n}(i))_{i\in\mathbb{N}}\subseteq\mathbb{Z}^d$ such that

$$\forall\varepsilon\in\{0,1\}^d,\ \vec{y}_\varepsilon=\lim_{i\to\infty}\hat{T}_1^{n_1(i)\varepsilon_1}\cdots\hat{T}_d^{n_d(i)\varepsilon_d}\vec{y}^i.$$

We now study the component $\varepsilon=[d]$ of this element. We have that,

$$\vec{y}_{[d]}=\lim_{i\to\infty}\hat{T}_1^{n_1(i)}\cdots\hat{T}_d^{n_d(i)}\vec{y}^i.$$

So, for every $j\in[d]$

$$\begin{aligned}(\vec{y}_{[d]})_j\ &=\lim_{i\to\infty}T_1^{n_1(i)}\cdots T_d^{n_d(i)}(\vec{y}^i)_j,\\ &=\lim_{i\to\infty}T_1^{n_1(i)}\cdots T_{j-1}^{n_{j-1}(i)}T_{j+1}^{n_{j+1}(i)}\cdots T_d^{n_d(i)}(\vec{y}^i)_j,\\ &=\lim_{i\to\infty}\hat{T}_1^{n_1(i)\eta_1}\cdots T_d^{n_d(i)\eta_d}(\vec{y}^i)_j,\end{aligned}$$

where $\eta=[d]\setminus\{j\}$. Thus, the component $j$ of $\vec{y}_{[d]}$ corresponds to the component $j$ of $\vec{y}_{[d]\setminus\{j\}}$. Hence the last component of the elements of the cube $\mathbf{Q}_{\hat{T}_1,\dots,\hat{T}_d}(Y)$ is a function of the rest of the coordinates. From this, $Y$ has the closing parallelepiped property. Since (1) is equivalent with (2) we have that $\mathcal{R}_{\hat{T}_1,\dots,\hat{T}_d}(Y)=\Delta_Y$. By Theorem 4.14 we have that $\mathcal{R}_{T_1,\dots,T_d}(X)=\Delta_X$. We then conclude that $X$ also has the closing parallelepiped property. □

The following corollary is proved implicitly in Theorem 4.1.

**Corollary 4.15.** *Let $\pi\colon(Y,T_1,\dots,T_d)\to(X,T_1,\dots,T_d)$ be a factor map between minimal distal systems $(X,T_1,\dots,T_d)$ and $(Y,T_1,\dots,T_d)$ with commuting transformations $T_1,\dots,T_d$. If $Y$ has the closing parallelepiped property, then $X$ has it too. In particular, having the closing parallelepiped property is an invariant under factor maps in the class of minimal distal systems.*

**Remark.** Observe that if $(\vec{y}_1,\vec{y}_2)\in\mathbf{Q}_{T_j}(Y)$, then $(\vec{y}_1)_j=(\vec{y}_2)_j$. This implies that the extension in the corollary also satisfies

$$\bigcap_{i=1}^d\mathbf{Q}_{T_i}(Y)=\Delta_Y.$$

**Remark.** Consider the case $d = 2$. We have that $Y$ is a joining between ${}^{Y}/_{\mathbf{Q}_{T_1}(Y)}$ and ${}^{Y}/_{\mathbf{Q}_{T_2}(Y)}$. Since the next level is the trivial system, by the relatively independence of these systems we conclude that $Y \cong {}^{Y}/_{\mathbf{Q}_{T_1}(Y)} \times {}^{Y}/_{\mathbf{Q}_{T_2}(Y)}$, i.e., $Y$ is a product system, generalizing the result of Donoso and Sun in [7] for the distal case.

Using Theorem 4.1 and Theorem 4.14 we get the following corollary.

**Corollary 4.16.** *Let $d \geq 2$ be an integer and $(X, T_1, \dots, T_d)$ be a minimal distal system with commuting transformations $T_1, \dots, T_d$. Then, $({}^{X}/_{\mathcal{R}_{T_1,\dots,T_d}(X)}, T_1, \dots, T_d)$ has the closing parallelepiped property. Moreover, this system is the maximal factor with this property, i.e., any other factor of $X$ with the closing parallelepiped property factorizes through it.*

**Proof.** Observe that if $(Z, T_1, \dots, T_d)$ is a factor of $(X, T_1, \dots, T_d)$ such that $\mathcal{R}_{T_1,\dots,T_d}(Z) = \Delta_Z$, then, by Theorem 4.14, $\pi \times \pi(\mathcal{R}_{T_1,\dots,T_d}(X)) = \mathcal{R}_{T_1,\dots,T_d}(Z) = \Delta_Z$. That is, there exists a factor map from $({}^{X}/_{\mathcal{R}_{T_1,\dots,T_d}(X)}, T_1, \dots, T_d)$ to $(Z, T_1, \dots, T_d)$. It remains to prove that $\mathcal{R}_{T_1,\dots,T_d}({}^{X}/_{\mathcal{R}_{T_1,\dots,T_d}(X)}) = \Delta_{{}^{X}/_{\mathcal{R}_{T_1,\dots,T_d}(X)}}$. Let $\pi \colon X \to {}^{X}/_{\mathcal{R}_{T_1,\dots,T_d}(X)}$ be the quotient map and $(y_1, y_2) \in \mathcal{R}_{T_1,\dots,T_d}({}^{X}/_{\mathcal{R}_{T_1,\dots,T_d}(X)})$. By Theorem 4.14, there exists $(x_1, x_2) \in \mathcal{R}_{T_1,\dots,T_d}(X)$ with $\pi(x_1) = y_1$ and $\pi(x_2) = y_2$. But $y_1 = \pi(x_1) = \pi(y_1) = y_2$, so $\mathcal{R}_{T_1,\dots,T_d}({}^{X}/_{\mathcal{R}_{T_1,\dots,T_d}(X)})$ coincides with the diagonal of ${}^{X}/_{\mathcal{R}_{T_1,\dots,T_d}(X)}$. $\square$

## 4.3 Recurrence in systems with the closing parallelepiped property

In this section we study sets of return times for distal systems with the closing parallelepiped property. In particular, we get a characterization of minimal distal systems with this property using return time ideas.

We define the sets of return times for $\mathbb{Z}^d$-minimal systems in the following way.

**Definition 4.17.** Let $d \geq 2$ be an integer and $(X, T_1, \dots, T_d)$ be a minimal system with commuting transformations $T_1, \dots, T_d$. Consider $x \in X$ and an open neighborhood $U$ of $x$. We define the *set of return times* $N_{T_1,\dots,T_d}(x, U) = \{(n_1, \dots, n_d) \in \mathbb{Z}^d \colon\ T_1^{n_1} \cdots T_d^{n_d} x \in U\}$.

A subset $A$ of $\mathbb{Z}^d$ is a set of return times for a minimal distal system if there exists a minimal distal system $(X, T_1, \dots, T_d)$, $x \in U$ and an open neighborhood $U$ of $x$ such that $N_{T_1,\dots,T_d}(x, U) \subseteq A$.

We can characterize $\mathbb{Z}^d$ sets of return times for distal systems via the closing parallelepiped property. For this we consider the following definition

**Definition 4.18.** Let $d \geq 2$ be an integer, $B_1, \dots, B_d \subseteq \mathbb{Z}^{d-1}$. We define the $d$-joining of $B_1, \dots, B_d$ as the set

$$B = \{(n_1, \dots, n_d) \in \mathbb{Z}^d \colon \forall i \in [d],\ (n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_d) \in B_i\} \subseteq \mathbb{Z}^d.$$

We remark that the 2-joining of $B_1, B_2 \subseteq \mathbb{Z}$ is the Cartesian product $B_1 \times B_2$. Now we have the following theorem

**Theorem 4.19.** *Let $d \geq 2$ be an integer. A subset $B \subseteq \mathbb{Z}^d$ is a set of return times for a minimal distal with the closing parallelepiped property if and only if $B$ contains a $d$-joining of $d$ subsets $B_1, \dots, B_d \subseteq \mathbb{Z}^d$.*

**Proof.** Let $(X, T_1, \dots, T_d)$ be a minimal distal system with an extension $(Y, \hat{T}_1, \dots, \hat{T}_d)$ like in Theorem 4.1, $x \in X$ and $U$ be an open neighborhood of $x$. Denote by $\pi\colon Y \to X$ the factor map. Let $(y_1, \dots, y_d) \in Y$ such that $\pi(y_1, \dots, y_d) = x$ and $\tilde{U}$ be a neighborhood of $(y_1, \dots, y_d)$ in $Y$ such that $\pi(\tilde{U}) \subseteq U$. Then, we have that $N_{\hat{T}_1,\dots,\hat{T}_d}((y_1, \dots, y_d), \tilde{U}) \subseteq N_{T_1,\dots,T_d}(x, U)$. Now, we consider the factor map $\pi_i\colon Y \to Y_i$, the projection onto the $i$-th coordinate. Since all the systems that we are considering are minimal and distal, then the map $\pi_i$ is open, by Theorem 1.32. So $\pi(\tilde{U})$ is open in $Y_i$ and we have that $N_{\hat{T}_1,\dots,\hat{T}_d}((y_1, \dots, y_d), \tilde{U}) = N_{T_1,\dots,T_d}(y_i, \pi_i(\tilde{U}))$. But, in $Y_i$ the action $T_i$ is the identity, and the action on $Y_i$ is an action of $\mathbb{Z}^{d-1}$. So we have that $(n_1, \dots, n_d) \in N_{\hat{T}_1,\dots,\hat{T}_d}((y_1, \dots, y_d), \tilde{U})$ if and only if $\forall i \in [d],\ (n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_d) \in N_{T_1,\dots,T_{i-1},T_{i+1},\dots,T_d}(y_i, \pi_i(\tilde{U}))$.

Conversely, let $B_i \subseteq \mathbb{Z}^{d-1}$ be a set of return times for a minimal distal system and for $i \in [d]$ let $(Y_i, T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_d)$ be a minimal distal system. Consider $y_i \in Y_i$ and an open neighborhood $U_i$ of $y_i$ such that

$$N_{T_1,\dots,T_{i-1},T_{i+1},\dots,T_d}(y_i, U_i) \subseteq B_i.$$

We define $B = \{\mathbf{n} \in \mathbb{Z}^d\colon\ \forall i \in [d],\ (n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_d) \in B_i\}$. We prove that $B$ is a set of return times for a minimal distal system with the closing parallelepiped property.

For every $Y_i$ we consider the action $T_i$ as the identity, and we set $Y = \prod_{i=1}^{d} Y_i$, $\mathbf{y} = (y_1, \dots, y_d)$ and $Z = \overline{\mathcal{O}(\mathbf{y}, T_1, \dots, T_d)} \subseteq Y$, which is minimal and distal. As $T_i$ acts trivially in the $i$-th coordinate on $Z$, then $Z$ has the closing parallelepiped property. Now, we consider $U = \left(\prod_{i=1}^{d} U_i\right) \cap Z$, which is an open neighborhood of $\mathbf{y}$ in $Z$. Then, $N_{T_1,\dots,T_d}(\mathbf{y}, U) \subseteq B$ and

$$N_{T_1,\dots,T_d}(\mathbf{y}, U) = \{\mathbf{n} \in \mathbb{Z}^d\colon \forall i \in [d],\ (n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_d) \in N_{T_1,\dots,T_{i-1},T_{i+1},\dots,T_d}(y_i, U_i)\}.$$

□

We denote by $\mathcal{B}_{T_1,\dots,T_d}$ the family generated by sets of return times arising from $\mathbb{Z}^d$-minimal distal systems with the closing parallelepiped property, and by $\mathcal{B}^*_{T_1,\dots,T_d}$ the family of subsets of $\mathbb{Z}^d$ which have non-empty intersection with every set in $\mathcal{B}_{T_1,\dots,T_d}$.

**Lemma 4.20.** *Let $d \geq 2$ be an integer and $(X, T_1, \dots, T_d)$ be a minimal distal system with commuting transformations $T_1, \dots, T_d$. Suppose $(x, y) \in \mathcal{R}_{T_1,\dots,T_d}(X)$. Let $(Z, T_1, \dots, T_d)$ be a minimal distal system with $\mathcal{R}_{T_1,\dots,T_d}(Z) = \Delta_Z$ and let $J$ be a joining between $X$ and $Z$. Then, for $z_0 \in Z$ we have that $(x, z_0) \in J$ if and only if $(y, z_0) \in J$.*

**Proof.** The proof is similar to the proof of Lemma 6.19 in [7], which is an adaptation of the proof of Theorem 3.5 in [27]. Let $W = Z^Z$ and $T_1^Z, \dots, T_d^Z \colon W \to W$ be the corresponding commuting transformations. Let $\omega^* \in W$ be the point satisfying $\omega^*(z) = z$ for all $z \in Z$ and $Z_\infty = \overline{\mathcal{O}(\omega^*, G^Z)}$, where $G^Z$ is the group generated by $T_1^Z, \dots, T_d^Z$. Then, $Z_\infty$ is minimal and distal. So for any $\omega \in Z_\infty$, there exists $p \in E(Z, G)$ such that $\omega(z) = p\omega^*(z) = p(z)$ for any $z \in Z$. Since $(Z, T_1, \dots, T_d)$ is minimal and distal, $E(Z, G)$ is a group, so $p \colon Z \to Z$ is surjective. Thus, there exists $z_\omega \in Z$ such that $\omega(z_\omega) = z_0$.

Take a minimal subsystem $(A, T_1 \times T_1^Z, \dots, T_d \times T_d^Z)$ of the product system $(X \times Z_\infty, T_1 \times T_1^Z, \dots, T_d \times T_d^Z)$. Let $\pi_X \colon A \to X$ be the natural coordinate projection. Then, $\pi_X$ is a factor map between two distal minimal systems. By Theorem 4.14, there exist $\omega^1, \omega^2 \in W$ such that $((x, \omega^1), (y, \omega^2)) \in \mathcal{R}_{\hat{T}_1, \dots, \hat{T}_d}(A)$, where $\hat{T}_i = T_i \times T_i^Z$, for $i \in [d]$.

Let $z_1 \in Z$ be such that $\omega^1(z_1) = z_0$. Denote by $\pi \colon A \to X \times Z$, $\pi(u, \omega) = (u, \omega(z_1))$ for $(u, \omega) \in A$, $u \in X$ and $\omega \in W$. Consider the projection $B = \pi(A)$. Then, $(B, T_1 \times T_1, \dots, T_d \times T_d)$ is a minimal distal subsystem of $(X \times Z, T_1 \times T_1, \dots, T_d \times T_d)$ and since $\pi(x_0, \omega^1) = (x, z_0) \in B$ we have that $J$ contains $B$. Suppose that $\pi(x, \omega^2) = (x, z_2)$. Then, $((x, z_0), (y, z_2)) \in \mathcal{R}_{T_1 \times T_1, \dots, T_d \times T_d}(B)$ and we conclude that $(z_0, z_2) \in \mathcal{R}_{T_1, \dots, T_d}(Z)$. Since $\mathcal{R}_{T_1, \dots, T_d}(Z) = \Delta_Z$ we have that $z_0 = z_2$ and thus $(y, z_0) \in B \subseteq J$. $\square$

**Lemma 4.21.** *Let $d \geq 2$ be an integer and $(X, T_1, \dots, T_d)$ be a minimal distal system with commuting transformations $T_1, \dots T_d$. Then, for $x, y \in X$, $(x, y) \in \mathcal{R}_{T_1, \dots, T_d}(X)$ if and only if $N_{T_1, \dots, T_d}(x, U) \in \mathcal{B}^*_{T_1, \dots, T_d}$ for any open neighborhood $U$ of $y$.*

**Proof.** The proof is similar to Theorem 6.20 in [7]. Suppose $N(x, U) \in \mathcal{B}^*_{T_1, \dots, T_d}$ for any open neighborhood $U$ of $y$. Since $X$ is distal, $\mathcal{R}_{T_1, \dots, T_d}(X)$ is an equivalence relation. Let $\pi$ be the projection map $\pi \colon X \to Y = X/\mathcal{R}_{T_1, \dots, T_d}(X)$. By Proposition 4.16 we have that $\mathcal{R}_{T_1, \dots, T_d}(Y) = \Delta_Y$. Since $(X, T_1, \dots, T_d)$ is distal, then the factor map $\pi$ is open and $\pi(U)$ is an open neighborhood of $\pi(y)$. Particularly, $N_{T_1, \dots, T_d}(x, U) \subseteq N_{T_1, \dots, T_d}(\pi(x), \pi(U))$. Let $V$ be an open neighborhood of $\pi(x)$. By hypothesis, we have that $N_{T_1, \dots, T_d}(x, U) \cap N_{T_1, \dots, T_d}(\pi(x), \pi(U)) \neq \emptyset$, which implies that $N_{T_1, \dots, T_d}(\pi(x), \pi(U)) \cap N_{T_1, \dots, T_d}(\pi(x), V) \neq \emptyset$. This implies that $\pi(U) \cap V \neq \emptyset$. But this holds for every $V$, then we have that $\pi(x) \in \overline{\pi(U)} = \pi(\overline{U})$. Finally, since this fact holds for every $U$ we conclude that $\pi(x) = \pi(y)$. This shows that $(x, y) \in \mathcal{R}_{T_1, \dots, T_d}(X)$.

Conversely, suppose that $(x, y) \in \mathcal{R}_{T_1, \dots, T_d}(X)$. Let $U$ be an open neighborhood of $y$ and $A$ be a $\mathcal{B}^*_{T_1, \dots, T_d}$ set. Then, there exist a minimal distal system $(Z, T_1, \dots, T_d)$ with $\mathcal{R}_{T_1, \dots, T_d}(Z) = \Delta_Z$, an open set $V \subseteq Z$ and $z_0 \in V$ such that $N_{T_1, \dots, T_d}(z_0, V) \subseteq A$. Let $J$ be the orbit closure of $(x, z_0)$ under $T_i \times T_i$ for $i \in [d]$. By distality we have that $(J, T_1 \times T_1, \dots, T_d \times T_d)$ is a minimal system and $(x, z_0) \in J$. By Theorem 4.14 we have that $(y, z_0) \in J$ and particularly there exist sequences $(\mathbf{n}^i)_{i \in \mathbb{N}} \subseteq \mathbb{Z}^d$ such that $(T_1^{n_1^i} \cdots T_d^{n_d^i} x, T_1^{n_1^i} \cdots T_d^{n_d^i} z_0) \to (y, z_0)$. This implies that $N_{T_1, \dots, T_d}(x, U) \cap N_{T_1, \dots, T_d}(z_0, V) \neq \emptyset$ and the proof is finished. $\square$

We get the following characterization of the closing parallelepiped property for minimal distal systems.

**Corollary 4.22.** *Let $d \geq 2$ be an integer and $(X, T_1, \ldots, T_d)$ be a minimal distal system with commuting transformations $T_1, \ldots, T_d$. Then, $(X, T_1, \ldots, T_d)$ has the closing parallelepiped property if and only if for every $x \in X$ and every open neighborhood $U$ of $x$, $N_{T_1,\ldots,T_d}(x, U)$ contains the d-joining of $d$ sets of return times for $\mathbb{Z}^{d-1}$-distal systems.*

**Proof.** We only need to prove one implication. Let us suppose that there exists $(x, y) \in \mathcal{R}_{T_1,\ldots,T_d}(X) \setminus \Delta_X$ and let $U, V$ be open neighborhoods of $x$ and $y$ respectively such that $U \cap V = \emptyset$. By assumption $N_{T_1,\ldots,T_d}(x, U)$ is a $\mathcal{B}_{T_1,\ldots,T_d}$ set, and by Lemma 4.21 $N_{T_1,\ldots,T_d}(x, V)$ has nonempty intersection with $N_{T_1,\ldots,T_d}(x, U)$. This implies that $U \cap V \neq \emptyset$, a contradiction. We conclude that $\mathcal{R}_{T_1,\ldots,T_d}(X) = \Delta_X$ and therefore $(X, T_1, \ldots, T_d)$ has the closing parallelepiped property. $\square$

# Chapter 5

# Examples of systems with the closing parallelepiped property

In this chapter we present a family of examples of systems with the closing parallelepiped property for each $d \geq 2$.

## 5.1 Affine transformations in the torus.

Let $r \geq 1$, $\alpha \in \mathbb{T}^r$ and $A$ be a $r \times r$ *unipotent* integer matrix, i.e., $(A - I)^p = 0$ for some $p \in \mathbb{N}$. A basic result about unipotent matrices is the following.

**Proposition 5.1.** *Let $A$ be a square matrix. Then, $A$ is unipotent if and only if its characteristic polynomial, $p_A(t)$, is a power of $t-1$. Equivalently, $A$ is unipotent if all its eigenvalues are 1.*

Let $T\colon \mathbb{T}^r \to \mathbb{T}^r$ be the affine transformation $x \mapsto Ax + \alpha$. Let $H$ be the group of transformations of $\mathbb{T}^r$ generated by $A$ and the translations of $\mathbb{T}^r$. That is, every element $h \in H$ is a map $x \mapsto A^i x + \beta$ for some $i \in \mathbb{Z}$ and $\beta \in \mathbb{T}^r$. The group $H$ acts transitively on $\mathbb{T}^r$ and we may identify this space with ${}^{H}/{}_{\Gamma}$, where $\Gamma$ is the stabilizer of 0, which consists of the powers of $A$. The system $(\mathbb{T}^r, \mu^{\oplus r}, T)$ is called an *affine nilsystem* (here $\mu$ is the Haar measure on $\mathbb{T}$). Properties such as transitivity, minimality and ergodicity are equivalent for systems in this class and this can be checked by looking at the rotation induced by $\alpha$ on the projection ${}^{\mathbb{T}^r}/{}_{\ker(A-I)}$ [30].

We consider different affine transformations $T_i\colon \mathbb{T}^r \to \mathbb{T}^r$, $x \mapsto A_i x + \alpha_i$, where $A_i$ is an unipotent matrix for every $i \in \{1, \ldots, d\}$. We can still regard the system $(\mathbb{T}^r, T_1, \ldots, T_d)$ as a nilsystem (described in Chapter 2) as long as the matrices commute. Let $G$ be the group of transformations of $\mathbb{T}^r$ generated by the matrices $A_1, \ldots, A_d$ and the translations of $\mathbb{T}^r$. Then, every element $g \in G$ is a map $x \mapsto A(g)x + \beta(g)$, where $A(g) = A_1^{m_1} \cdots A_d^{m_d}$, $m_1, \ldots, m_d \in \mathbb{Z}$ and $\beta(g) \in \mathbb{T}^r$.

A simple computation shows that if $g_1, g_2 \in G$, then the commutator $[g_1, g_2]$ is the map $x \mapsto x + (A(g_1) - I)\beta(g_2) - (A(g_2) - I)\beta(g_1)$ and thus it is a translation of $\mathbb{T}^r$. On the other hand, if $g \in G$ and $\beta \in \mathbb{T}^d$, then $[g, \beta]$ is the translation $x \mapsto x + (A(g) - I)\beta$. It follows that the iterated commutator $[\cdots[[g_1, g_2], g_3]\cdots g_k]$ belongs to $\mathbb{T}^r$ and is contained in the image of $(A(g_3) - I)\cdots(A(g_k) - I)$. If $k$ is large enough, this product is trivial. So $G$ is a nilpotent Lie group. The torus $\mathbb{T}^r$ can be identified with ${}^{G}\!/\!_{\Gamma}$, where $\Gamma$ is the stabilizer of 0, which is the group generated by the matrices $A_1, \ldots, A_d$. We refer to $(\mathbb{T}^r, T_1, \ldots, T_d)$ as an affine nilsystem with $d$ transformations. It is worth noting that the transformations $T_i$ and $T_j$ commute if and only if $(A_i - I)\alpha_j = (A_j - I)\alpha_i$ in $\mathbb{T}^r$.

By a theorem from Leibman [28] we get:

**Proposition 5.2.** *Let $(\mathbb{T}^r, T_1, \ldots, T_d)$ be an affine nilsystem with $d$ transformations. Then, the properties of transitivity, minimality, ergodicity and unique ergodicity under the action of $\langle T_1, \ldots, T_d\rangle$ are equivalent.*

We consider some conditions on the commuting transformations $T_1, \ldots, T_d$ such that $(\mathbb{T}^r, T_1, \ldots, T_d)$ has the closing parallelepiped property. We start by presenting the examples but the proofs will be given in subsequent sections. First, we consider the case $d = 2$. We have the following lemma.

**Lemma 5.3.** *Let $(\mathbb{T}^r, T_1, T_2)$ be an affine nilsystem with 2 commuting transformations, with $T_i x = A_i x + \alpha_i$ for $i \in \{1, 2\}$. Then, we have that for every $n, m \in \mathbb{Z}$,*

$$T_1^n T_2^m x = T_1^n x + T_2^m x - x,$$

*if and only if the following conditions hold*

$$(A_1 - I)(A_2 - I) = 0. \tag{5.1}$$

$$(A_1 - I)\alpha_2 = (A_2 - I)\alpha_1 = 0. \tag{5.2}$$

*In particular, if conditions (5.1) and (5.2) are satisfied we have that*

$$\mathbf{Q}_{T_1,T_2}(X) = \overline{\{(x, T_1^n x, T_2^x, T_1^n x + T_2^m x - x) \colon x \in \mathbb{T}^r,\ n, m \in \mathbb{Z}\}},$$

*and thus $(\mathbb{T}^r, T_1, T_2)$ has the closing parallelepiped property.*

**Example.** Consider the following matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 3 & 1 & 4 \\ 0 & 0 & 1 & 6 & 3 & 6 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the first matrix, the eigenspace is given by,

$$W_1(A_1) = \left\langle \left\{ \begin{pmatrix} 1\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\-2\\2\\1 \end{pmatrix} \right\} \right\rangle.$$

For the second one, the eigenspace is given by,

$$W_1(A_2) = \left\langle \left\{ \begin{pmatrix} 1\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\-1\\-1\\1 \end{pmatrix} \right\} \right\rangle.$$

It is easy to see that $(A_1-I)(A_2-I)=0$ and we can choose $\alpha_1 \in W_1(A_2)$ and $\alpha_2 \in W_1(A_1)$ such that $(\mathbb{T}^6, T_1, T_2)$ has the closing parallelepiped property.

We can generalize the conditions (5.1) and (5.2) in the following way.

**Lemma 5.4.** *Let $(\mathbb{T}^r, T_1, \dots, T_d)$ be an affine nilsystem with $d$ commuting transformations, with $T_i x = A_i x + \alpha_i$ for $i \in [d]$. Then, we have that for every $\boldsymbol{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$,*

$$T_1^{n_1} \cdots T_d^{n_d} x = (-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} \bigcirc_{\substack{k \in I \subseteq [d] \\ |I|=i}} T_k^{n_k} x$$

*if and only if the following conditions hold:*

$$\prod_{i=1}^{d} (A_i - I) = 0, \tag{5.3}$$

$$\forall j \in [d],\ \prod_{\substack{i=1 \\ i \neq k}}^{d} (A_i - I)\alpha_j = 0. \tag{5.4}$$

*In particular, if conditions* (5.3) *and* (5.4) *hold the system $(\mathbb{T}^r, T_1, \dots, T_d)$ has the closing parallelepiped property.*

## 5.2 Proof of Lemma 5.3 and 5.4

We start with the following proposition which gives a description of the iterates of an affine transformation.

**Proposition 5.5.** *Let $r \geq 1$ be an integer, $\alpha \in \mathbb{T}^r$, $A$ a $r \times r$ unipotent integer matrix and $T\colon \mathbb{T}^r \to \mathbb{T}^r$ be the affine transformation $x \mapsto Ax + \alpha$. Then, for every $n \in \mathbb{Z}$ and $x \in \mathbb{T}^r$ we have that*

$$T^n x = A^n x + \operatorname{sgn}(n) \sum_{k=1}^{|n|-1} A^{\operatorname{sgn}(n)k} \alpha_1 + \beta(n),$$

*where,*

$$\beta(n) = \begin{cases} \alpha & \text{if } n \geq 1, \\ A^{-n}\alpha & \text{if } n \leq -1, \\ 0 & \text{if } n = 0, \end{cases}$$

*and*

$$\operatorname{sgn}(n) = \begin{cases} 1 & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \\ -1 & \text{if } n \leq 1. \end{cases}$$

**Proof.** Let $x \in \mathbb{T}^d$. We observe that

$$\begin{aligned} Tx &= Ax + \alpha, \\ T^2 x &= T(Ax + \alpha) \quad = A^2 x + A\alpha + \alpha. \end{aligned}$$

Suppose that for some $n \geq 1$ we have proved that $T^n x = A^n x + \sum_{k=0}^{n-1} A^k \alpha$. Then,

$$\begin{aligned} T^{n+1} x &= T\left(A^n x + \sum_{k=0}^{n-1} A^k \alpha\right), \\ &= A\left(A^n x + \sum_{k=0}^{n-1} A^k \alpha\right) + \alpha, \\ &= A^{n+1} x + \sum_{k=1}^{n} A^k \alpha + \alpha, \\ &= A^{n+1} x + \sum_{k=0}^{n} A^k \alpha. \end{aligned}$$

Hence,

$$\forall n \geq 1,\ T^n x = A^n x + \sum_{k=0}^{n-1} A^k \alpha.$$

For the case $n \leq -1$ we observe that

$$x = T^n(T^{-n} x) = A^n(T^{-n} x) + \sum_{k=0}^{n-1} A^k \alpha.$$

Therefore,

$$\begin{aligned} T^{-n}x &= A^{-n}x - A^{-n}\sum_{k=0}^{n-1} A^k\alpha, \\ &= A^{-n}x - \sum_{k=0}^{n-1} A^{k-n}\alpha, \\ &= A^{-n}x - \sum_{k=1}^{n} A^{-k}\alpha. \end{aligned}$$

Finally,

$$\forall n \in \mathbb{Z},\ T^n x = A^n x + \operatorname{sgn}(n)\sum_{k=1}^{|n|-1} A^{\operatorname{sgn}(n)k}\alpha + \beta(n).$$

□

First we focus on the case of two commuting transformations. The following propositions give algebraic consequences of conditions (5.1) and (5.2).

**Proposition 5.6.** *Let $r \geq 1$ be an integer, $A_1, A_2$ be two $r \times r$ unipotent integer matrices and $\alpha_1, \alpha_2 \in \mathbb{T}^r$ be such that condition (5.1) holds and the transformations $x \mapsto A_i x + \alpha_i$, $i \in \{1,2\}$, commute. Then, we have that for every $n, m \in \mathbb{Z}$,*

$$\begin{aligned} A_2^m\alpha_1 &= mA_2\alpha_1 - (m-1)\alpha_1, \\ A_1^n\alpha_2 &= nA_1\alpha_2 - (n-1)\alpha_2. \end{aligned}$$

**Proof.** We observe that

$$(A_1 - I)^2\alpha_2 = (A_1 - I)(A_1 - I)\alpha_2 = (A_1 - I)(A_2 - I)\alpha_2 = 0.$$

Then,

$$A_1^2\alpha_2 = 2A_1\alpha_2 - \alpha_2.$$

By induction we prove that

$$\forall n \geq 0,\ A_1^n\alpha_2 = nA_1\alpha_2 - (n-1)\alpha_2.$$

We observe that

$$A_1(-A_1\alpha_2 + 2\alpha_2) = -A_1^2\alpha_2 + 2A_1\alpha_2 = -2A_1\alpha_2 + \alpha_2 + 2A_1\alpha_2 = \alpha_2.$$

Hence, $A_1^{-1}\alpha_2 = -A_1\alpha_2 + 2\alpha_2$. Now, assume that the following formula works for some $n \geq 1$,

$$A_1^{-n}\alpha_2 = -nA_1\alpha_2 + (n+1)\alpha_2.$$

Then, by the induction hypothesis,

$$\begin{aligned} A_1(-(n+1)A_1\alpha_2 + (n+2)\alpha_2) &= -(n+1)A_1^2\alpha_2 + (n+2)A_1\alpha_2, \\ &= -(n+1)(2A_1\alpha_2 - \alpha_2) + (n+2)A_1\alpha_2, \\ &= -2(n+1)A_1\alpha_2 + (n+1)\alpha_2 + (n+2)A_1\alpha_2, \\ &= -nA_1\alpha_2 + (n+1)\alpha_2, \\ &= A_1^{-n}\alpha_2. \end{aligned}$$

Hence, $-(n+1)A_1\alpha_2 + (n+2)\alpha_2 = A_1^{-1}(A_1^{-n}\alpha_2) = A_1^{-(n+1)}\alpha_2$. Thus, we conclude that for every $n \in \mathbb{Z}$

$$A_1^{-n}\alpha_2 = -nA_1\alpha_2 + (n+1)\alpha_2.$$

The argument for $A_2$ and $\alpha_1$ is the same. $\square$

**Proposition 5.7.** *Let $r \geq 1$ be an integer, $A_1, A_2$ be two $r \times r$ commuting unipotent integer matrices such that condition* (5.1) *hold. Then, for every $n, m \in \mathbb{Z}$ we have that*

$$A_1^n A_2^m = A_1^n + A_2^m - I.$$

**Proof.** First we consider the cases $n, m \geq 1$. Assume that for some $n \geq 1$, $A_1^n A_2 = A_1^n + A_2 - I$. We have that

$$A_1^{n+1}A_2 = A_1^{n+1} + A_1A_2 - A_1 = A_1^{n+1} + A_1 + A_2 - I - A_1 = A_1^{n+1} + A_2 - I.$$

Then, by induction, we have that for every $n \geq 1$

$$A_1^n A_2 = A_1^n + A_2 - I.$$

Also, by induction we can prove that

$$\forall n, m \geq 1,\ A_1^n A_2^m = A_1^n + A_2^m - I.$$

Let $m \geq 1$. We have that,

$$A_1^n = A_1^n A_2^m + I - A_2^m.$$

Then, $A_1^n A_2^{-m} = A_1^n + A_2^{-m} - I$. Finally, we get that

$$\forall n, m \in \mathbb{Z},\ A_1^n A_2^m = A_1^n + A_2^m - I.$$

$\square$

**Proof of Lemma 5.3.** Let $x \in \mathbb{T}^r$ and $n, m \in \mathbb{Z}$. We compute an expression for $T_1^n T_2^m x$. By Proposition 5.5, we have that

$$\begin{aligned}
T_1^n T_2^m x &= T_1^n \left( A_2^m x + \operatorname{sgn}(m) \sum_{j=1}^{|m|-1} A_2^{\operatorname{sgn}(m)j}\alpha_2 + \beta_2(m) \right), \\
&= A_1^n \left( A_2^m x + \operatorname{sgn}(m) \sum_{j=1}^{|m|-1} A_2^{\operatorname{sgn}(m)j}\alpha_2 + \beta_2(m) \right) \\
&\quad + \operatorname{sgn}(n) \sum_{k=1}^{|n|-1} A_1^{\operatorname{sgn}(n)k}\alpha_1 + \beta_1(n), \\
&= A_1^n A_2^m x + \operatorname{sgn}(m) A_1^n \sum_{j=1}^{|m|-1} A_2^{\operatorname{sgn}(m)j}\alpha_2 + A_1^n\beta_2(m) \\
&\quad + \operatorname{sgn}(n) \sum_{k=1}^{|n|-1} A_1^{\operatorname{sgn}(n)k}\alpha_1 + \beta_1(n).
\end{aligned}$$

Now, by Proposition 5.7,

$$\begin{aligned}
T_1^n T_2^m x \ &= A_1^n x + A_2^m x - x + \operatorname{sgn}(m) A_1^n \sum_{j=1}^{|m|-1} A_2^{\operatorname{sgn}(m)j} \alpha_2 + A_1^n \beta_2(m) \\
&\quad + \operatorname{sgn}(n) \sum_{k=1}^{|n|-1} A_1^{\operatorname{sgn}(n)k} \alpha_1 + \beta_1(n),
\end{aligned}$$

$$\begin{aligned}
&= -x + T_1^n x + A_2^m x + \operatorname{sgn}(m) A_1^n \sum_{j=1}^{|m|-1} A_2^{\operatorname{sgn}(m)j} \alpha_2 + A_1^n \beta_2(m), \\
&= -x + T_1^n x + A_2^m x + \operatorname{sgn}(m) \sum_{j=1}^{|m|-1} A_1^n A_2^{\operatorname{sgn}(m)j} \alpha_2 + A_1^n \beta_2(m), \\
&= -x + T_1^n x + A_2^m x + \operatorname{sgn}(m) \sum_{j=1}^{|m|-1} (A_1^n + A_2^{\operatorname{sgn}(m)j} - I) \alpha_2 + A_1^n \beta_2(m), \\
&= -x + T_1^n x + A_2^m x + \operatorname{sgn}(m)(|m|-1) A_1^n \alpha_2 \\
&\quad + \operatorname{sgn}(m) \sum_{j=1}^{|m|-1} A_2^{\operatorname{sgn}(m)j} \alpha_2 - \operatorname{sgn}(m)(|m|-1)\alpha_2 + A_1^n \beta_2(m).
\end{aligned}$$

Then, by Proposition 5.6,

$$\begin{aligned}
T_1^n T_2^m x \ &= -x + T_1^n x + A_2^m x + \operatorname{sgn}(m)(|m|-1)(n A_1 \alpha_2 - (n-1)\alpha_2) \\
&\quad + \operatorname{sgn}(m) \sum_{j=1}^{|m|-1} A_2^{\operatorname{sgn}(m)j} \alpha_2 - \operatorname{sgn}(m)(|m|-1)\alpha_2 + n A_1 \beta_2(m) - (n-1)\beta_2(m), \\
&= -x + T_1^n x + T_2^m x + \operatorname{sgn}(m)(|m|-1) n A_1 \alpha_2 - \operatorname{sgn}(m)(|m|-1) n \alpha_2 \\
&\quad + n A_1 \beta_2(m) - n \beta_2(m), \\
&= -x + T_1^n + T_2^m x + n(A_1 - I)(\operatorname{sgn}(m)(|m|-1)\alpha_2 + \beta_2(m)). \\
&= -x + T_1^n + T_2^m x + n C(m)(A_1 - I)\alpha_2,
\end{aligned}$$

where $C(m)$ is a matrix which depends on $m$. If $\alpha_2$ is an eigenvector of $A_1$ (and $\alpha_1$ is an eigenvector of $A_2$) we have that

$$T_1^n T_2^m x = -x + T_1^n x + T_2^m x.$$

So,

$$\mathbf{Q}_{T_1,T_2}(X) = \overline{\{(x, T_1^n x, T_2^x, T_1^n x + T_2^m x - x) \colon x \in \mathbb{T}^r,\ n, m \in \mathbb{Z}\}},$$

and thus $(\mathbb{T}^r, T_1, T_2)$ has the closing parallelepiped property.

Conversely, assume that for every $n, m \in \mathbb{Z}$ and $x \in X$ we have that

$$T_1^n T_2^m x = T_1^n x + T_2^n x - x.$$

In particular,

$$\begin{aligned}
T_1 T_2 x \quad &= T_1 x + T_2 x - x, \\
T_1(A_2 x + \alpha_2) \quad &= A_1 x + \alpha_1 + A_2 x + \alpha_2 - x, \\
A_1(A_2 x + \alpha_2) + \alpha_1 \quad &= A_1 x + \alpha_1 + A_2 x + \alpha_2 - x, \\
A_1 A_2 x + A_1 \alpha_2 + \alpha_1 \quad &= A_1 x + \alpha_1 + A_2 x + \alpha_2 - x, \\
A_1 A_2 x + A_1 \alpha_2 \quad &= A_1 x + A_2 x + \alpha_2 - x.
\end{aligned}$$

If $x = 0$ we have that

$$A_1\alpha_2 = \alpha_2 \implies (A_1 - I)\alpha_2 = 0.$$

Then, $(A_2 - I)\alpha_1 = (A_1 - I)\alpha_2 = 0$ and condition (5.2) is satisfied. We conclude that for every $x \in \mathbb{T}^r$,

$$A_1A_2x = A_1x + A_2x - x.$$

This implies that

$$A_1A_2 = A_1 + A_2 - I \implies (A_1 - I)(A_2 - I) = 0.$$

□

Now we consider the affine nilsystem $(\mathbb{T}^r, T_1, \ldots, T_d)$. We remark that Condition (5.3) can be rewritten as

$$\sum_{i=0}^{d}(-1)^{i+1} \prod_{\substack{k\in I\subseteq[d] \\ |I|=i}} A_k = 0, \tag{5.5}$$

since the two expressions, the left side of equations (5.8) and (5.5), are equal up to an eventually change of sign.

Indeed, suppose that for some $d \in \mathbb{N}$ the two formulas are equalup to a change of sign. Then,

$$\begin{aligned}
\prod_{i=1}^{d+1}(A_i - I) = 0 \quad & \Leftrightarrow (A_{d+1} - I)\prod_{i=1}^{d}(A_i - I) = 0, \\
& \Leftrightarrow (A_{d+1} - I)\left(\sum_{i=0}^{d}(-1)^{i+1} \prod_{\substack{k\in I\subseteq[d] \\ |I|=i}} A_k\right) = 0, \\
& \Leftrightarrow A_{d+1}\sum_{i=0}^{d}(-1)^{i+1} \prod_{\substack{k\in I\subseteq[d] \\ |I|=i}} A_k - \sum_{i=0}^{d}(-1)^{i+1} \prod_{\substack{k\in I\subseteq[d] \\ |I|=i}} A_k = 0, \\
& \Leftrightarrow \sum_{i=0}^{d}(-1)^{i+1} \prod_{\substack{k\in I\cup\{d+1\} \\ I\subseteq[d] \\ |I|=i}} A_k - \sum_{i=0}^{d}(-1)^{i+1} \prod_{\substack{k\in I\subseteq[d] \\ |I|=i}} A_k = 0, \\
& \Leftrightarrow \sum_{i=1}^{d+1}(-1)^{i} \prod_{\substack{k\in I\subseteq[d+1] \\ |I|=i,\ d+1\in I}} A_k - \sum_{i=0}^{d}(-1)^{i+1} \prod_{\substack{k\in I\subseteq[d] \\ |I|=i}} A_k = 0, \\
& \Leftrightarrow \sum_{i=1}^{d+1}(-1)^{i} \prod_{\substack{k\in I\subseteq[d+1] \\ |I|=i,\ d+1\in I}} A_k + \sum_{i=0}^{d+1}(-1)^{i} \prod_{\substack{k\in I\subseteq[d+1] \\ |I|=i,\ d+1\notin I}} A_k = 0, \\
& \Leftrightarrow \sum_{i=1}^{d+1}(-1)^{i} \prod_{\substack{k\in I\subseteq[d+1] \\ |I|=i}} A_k = 0, \\
& \Leftrightarrow \sum_{i=1}^{d+1}(-1)^{i+1} \prod_{\substack{k\in I\subseteq[d+1] \\ |I|=i}} A_k = 0,
\end{aligned} \tag{5.6}$$

with the convention $\prod_{k\in\emptyset} A_k = I$. We get the following proposition.

**Proposition 5.8.** *Let $r \geq 1$ be an integer, $A_1, \dots, A_d$ be $r \times r$ commuting unipotent integer matrices which satisfy condition (5.3). Then, for every $\boldsymbol{n} \in \mathbb{Z}^d$ we have*

$$\prod_{i=1}^{d} A_i^{n_i} = (-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} \prod_{\substack{k\in I\subseteq[d] \\ |I|=i}} A_k^{n_k}.$$

**Proof.** By (5.6) we have that

$$\begin{aligned}
A_1^2 \prod_{i=2}^{d} A_i &= A_1 \left( (-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} \prod_{\substack{k\in I\subseteq[d] \\ |I|=i}} A_k \right), \\
&= (-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} A_1 \prod_{\substack{k\in I\subseteq[d] \\ |I|=i}} A_k, \\
&= (-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} A_1 \prod_{\substack{k\in I\subseteq[d] \\ |I|=i,\ I\neq\{2,\dots,d\}}} A_k + (-1)^d(-1)^d \prod_{i=1}^{d} A_i, \\
&= (-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} A_1 \prod_{\substack{k\in I\subseteq[d] \\ |I|=i,\ I\neq\{2,\dots,d\}}} A_k + (-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} \prod_{\substack{k\in I\subseteq[d] \\ |I|=i}} A_k.
\end{aligned} \tag{5.7}$$

Let $I \subseteq [d]$ with $0 \leq |I| \leq d-1$, $1 \notin I$ and $I \neq \{2, \dots, d\}$. Then,

$$A_1 \prod_{k\in I} A_k = \prod_{k\in I\cup\{1\}} A_k.$$

Hence, we remark that for $i \in \{0, 1, \dots, d-1\}$ and $I \subseteq [d]$ such that $1 \notin I$

$$A_1 \prod_{\substack{k\in I\subseteq[d] \\ |I|=i,\ I\neq\{2,\dots,d\}}} A_k = \prod_{\substack{k\in I\subseteq[d] \\ |I|=i+1,\ I\neq[d]}} A_k.$$

The two previous expressions appear in the sums of (5.7), but with different sign. So they cancel each other. Therefore, in the first sum, only appear the subsets such that $1 \in I$ and in the second sum the subsets such that $1 \notin I$. Then,

$$\begin{aligned}
A_1^2 \prod_{i=2}^{d} A_i &= (-1)^d \sum_{i=1}^{d-1} (-1)^{i+1} A_1 \prod_{\substack{k\in I\subseteq[d] \\ |I|=i,\ I\neq\{2,\dots,d\},\ 1\in I}} A_k + (-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} \prod_{\substack{k\in I\subseteq[d] \\ |I|=i,\ 1\notin I}} A_k, \\
&= (-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} \prod_{\substack{k\in I\subseteq[d] \\ |I|=i}} A_k^{\gamma_k},
\end{aligned}$$

where

$$\gamma_k = \begin{cases} 2 & \text{if } k = 1, \\ 1 & \text{if } k \neq 1. \end{cases}$$

Using a similar argument as in the case $d = 2$, then by induction over the integer vectors $\mathbf{n} \in \mathbb{Z}^d$ with positive entries and finally extending the formula for all $\mathbf{n} \in \mathbb{Z}^d$ we get that

$$\forall \mathbf{n} \in \mathbb{Z}^d, \ \prod_{i=1}^{d} A_i^{n_i} \ = (-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} \prod_{\substack{k \in I \subseteq [d] \\ |I| = i}} A_k^{n_k}.$$

$\square$

Now we compute the expression for $T_1^{n_1} \cdots T_d^{n_d} x$. For simplification, we only consider integer vectors $\mathbf{n} \in \mathbb{Z}^d$ with positive entries.

**Proposition 5.9.** *Let $(\mathbb{T}^r, T_1, \ldots, T_d)$ be an affine nilsystem with $d$ commuting transformations, with $T_i x = A_i x + \alpha_i$ for $i \in [d]$. Then, we have that for every $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$,*

$$T_1^{n_1} \cdots T_d^{n_d} x = \prod_{i=1}^{d} A_i^{n_i} x + \sum_{i=1}^{d} \prod_{j=1}^{d-i} A_j^{n_j} \sum_{k_{d-i+1}=0}^{n_{d-i+1}-1} A_{d-i+1}^{k_{d-i+1}} \alpha_{d-i+1}.$$

**Proof.** The case $d = 1$ follows from Proposition 5.5. For $d = 2$

$$T_1^{n_1} T_2^{n_2} x = A_1^{n_1} A_2^{n_2} x + A_1^{n_1} \sum_{j=0}^{n_2-1} A_2^j \alpha_2 + \sum_{k=0}^{n_1-1} A_1^k \alpha_1,$$

and

$$\prod_{i=1}^{2} A_i^{n_i} x + \sum_{i=1}^{2} \prod_{j=1}^{2-i} A_j \sum_{k_{2-i+1}=0}^{n_{2-i+1}-1} A_{2-i+1}^{k_{2-i+1}} \alpha_{2-i+1} \ = A_1^{n_1} A_2^{n_2} x + A_1^{n_1} \sum_{k_2=0}^{n_2-1} A_2^{k_2} \alpha_2 + \sum_{k_1=0}^{n_1-1} A_1^{k_1} \alpha_1.$$

Using the induction hypothesis we have that

$$\begin{aligned}
T_1^{n_1} \cdots T_{d+1}^{n_{d+1}} x \ &= T_1 \left( \prod_{i=1}^{d} A_{i+1}^{n_{i+1}} x + \sum_{i=1}^{d} \prod_{j=1}^{d-i} A_{j+1}^{n_{j+1}} \sum_{k_{d-i+1+1}=0}^{n_{d-i+1+1}-1} A_{d-i+1+1}^{k_{d-i+1+1}} \alpha_{d-i+1+1} \right), \\
&= T_1 \left( \prod_{i=2}^{d+1} A_i^{n_i} x + \sum_{i=1}^{d} \prod_{j=2}^{d+1-i} A_j^{n_j} \sum_{k_{d+2-i}=0}^{n_{d+2-i}-1} A_{d+2-i}^{k_{d+2-i}} \alpha_{d+2-i} \right), \\
&= A_1^{n_1} \left( \prod_{i=2}^{d+1} A_i^{n_i} x + \sum_{i=1}^{d} \prod_{j=2}^{d+1-i} A_j^{n_j} \sum_{k_{d+2-i}=0}^{n_{d+2-i}-1} A_{d+2-i}^{k_{d+2-i}} \alpha_{d+2-i} \right) + \sum_{k_1=0}^{n_1-1} A_1^{k_1} \alpha_1, \\
&= \prod_{i=1}^{d+1} A_i^{n_i} x + \sum_{i=1}^{d} \prod_{j=1}^{d+1-i} A_j^{n_j} \sum_{k_{d+2-i}=0}^{n_{d+2-i}-1} A_{d+2-i}^{k_{d+2-i}} \alpha_{d+2-i} + \sum_{k_1=0}^{n_1-1} A_1^{k_1} \alpha_1, \\
&= \prod_{i=1}^{d+1} A_i^{n_i} x + \sum_{i=1}^{d+1} \prod_{j=1}^{d+1-i} A_j^{n_j} \sum_{k_{d+2-i}=0}^{n_{d+2-i}-1} A_{d+2-i}^{k_{d+2-i}} \alpha_{d+2-i}.
\end{aligned}$$

$\square$

By Propositions 5.8 and 5.9 we have that,

$$\begin{aligned}
T_1^{n_1}\cdots T_d^{n_d}x &= \prod_{i=1}^{d} A_i^{n_i}x + \sum_{i=1}^{d}\prod_{j=1}^{d-i} A_j^{n_j}\sum_{k_{d-i+1}=0}^{n_{d-i+1}-1} A_{d-i+1}^{k_{d-i+1}}\alpha_{d-i+1},\\
&= (-1)^d\sum_{i=0}^{d-1}(-1)^{i+1}\prod_{\substack{k\in I\subseteq[d]\\|I|=i}} A_k^{n_k}x + \sum_{i=1}^{d}\prod_{j=1}^{d-i} A_j^{n_j}\sum_{k_{d-i+1}=0}^{n_{d-i+1}-1} A_{d-i+1}^{k_{d-i+1}}\alpha_{d-i+1},\\
&= (-1)^d\sum_{i=0}^{d-1}(-1)^{i+1}\prod_{\substack{k\in I\subseteq[d]\\|I|=i}} A_k^{n_k}x + \prod_{j=1}^{d-1} A_j^{n_j}\sum_{k_d=0}^{n_d-1} A_d^{k_d}\alpha_d\\
&\quad+\sum_{i=2}^{d}\prod_{j=1}^{d-i} A_j^{n_j}\sum_{k_{d-i+1}=0}^{n_{d-i+1}-1} A_{d-i+1}^{k_{d-i+1}}\alpha_{d-i+1}.
\end{aligned}$$

Now, by Condition (5.4),

$$\begin{aligned}
T_1^{n_1}\cdots T_d^{n_d}x &= (-1)^d\sum_{i=0}^{d-1}(-1)^{i+1}\prod_{\substack{k\in I\subseteq[d]\\|I|=i}} A_k^{n_k}x + \sum_{k_d=0}^{n_d-1}\left((-1)^d\sum_{i=0}^{d-1}(-1)^{i+1}\prod_{\substack{j\in I\subseteq[d]\\|I|=i}} A_j^{m_j}\right)\alpha_d,\\
&\quad+\sum_{i=2}^{d}\prod_{j=1}^{d-i} A_j^{n_j}\sum_{k_{d-i+1}=0}^{n_{d-i+1}-1} A_{d-i+1}^{k_{d-i+1}}\alpha_{d-i+1}
\end{aligned}$$

where

$$m_j = \begin{cases} n_j & \text{if } j\neq d,\\ k_d & \text{if } j=d.\end{cases}$$

Thus,

$$\begin{aligned}
T_1^{n_1}\cdots T_d^{n_d}x &= (-1)^d\sum_{i=0}^{d-1}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d]\\|I|=i}} A_k^{n_k}x + \sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=i}} A_j^{m_j}\alpha_d\right)\\
&\quad+\sum_{i=2}^{d}\prod_{j=1}^{d-i} A_j^{n_j}\sum_{k_{d-i+1}=0}^{n_{d-i+1}-1} A_{d-i+1}^{k_{d-i+1}}\alpha_{d-i+1}.
\end{aligned} \tag{5.8}$$

The first expression in previous equality is equal to

$$\begin{aligned}
&= (-1)^d\sum_{i=0}^{d-1}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d]\\|I|=i}} A_k^{n_k}x + \sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=i,d\notin I}} A_j^{m_j}\alpha_d + \sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=i,d\in I}} A_j^{m_j}\alpha_d\right),\\
&= (-1)^d\sum_{i=0}^{d-1}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d]\\|I|=i}} A_k^{n_k}x + n_d\prod_{\substack{j\in I\subseteq[d]\\|I|=i,d\notin I}} A_j^{n_j}\alpha_d + \sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=i,d\in I}} A_j^{m_j}\alpha_d\right).
\end{aligned}$$

In the case $i=d-1$ in the first sum appears

$$\begin{aligned}
&\prod_{\substack{k\in I\subseteq[d]\\|I|=d-1}} A_k^{n_k}x + n_d \prod_{\substack{j\in I\subseteq[d]\\|I|=d-1,d\notin I}} A_j^{n_j}\alpha_d + \sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=d-1,d\in I}} A_j^{m_j}\alpha_d\\
=&\prod_{\substack{k\in I\subseteq[d]\\|I|=d-1}} A_k^{n_k}x + n_d \prod_{j=1}^{d-1} A_j^{n_j}\alpha_d + \sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=d-1,d\in I}} A_j^{m_j}\alpha_d,\\
=&\prod_{\substack{k\in I\subseteq[d]\\|I|=d-1}} A_k^{n_k}x + n_d(-1)^{d-1}\sum_{i=0}^{d-2}(-1)^{i+1}\prod_{\substack{k\in I\subseteq[d-1]\\|I|=i}} A_k\alpha_d + \sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=d-1,d\in I}} A_j^{m_j}\alpha_d.
\end{aligned}$$

So we have that

$$\begin{aligned}
T_1^{n_1}\cdots T_d^{n_d}x =& (-1)^d\sum_{i=0}^{d-2}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d]\\|I|=i}} A_k^{n_k}x + n_d\prod_{\substack{j\in I\subseteq[d]\\|I|=i,d\notin I}} A_j^{n_j}\alpha_d + \sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=i,d\in I}} A_j^{m_j}\alpha_d\right)\\
&+\sum_{i=2}^{d}\prod_{j=1}^{d-i}A_j^{n_j}\sum_{k_{d-i+1}=0}^{n_{d-i+1}-1}A_{d-i+1}^{k_{d-i+1}}\alpha_{d-i+1} + \prod_{\substack{k\in I\subseteq[d]\\|I|=d-1}} A_k^{n_k}x\\
&+n_d(-1)^{d-1}\sum_{i=0}^{d-2}(-1)^{i+1}\prod_{\substack{k\in I\subseteq[d-1]\\|I|=i}} A_k\alpha_d + \sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=d-1,d\in I}} A_j^{m_j}\alpha_d,\\
=& (-1)^d\sum_{i=0}^{d-2}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d]\\|I|=i}} A_k^{n_k}x + \sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=i,d\in I}} A_j^{m_j}\alpha_d\right) + \prod_{\substack{k\in I\subseteq[d]\\|I|=d-1}} A_k^{n_k}x\\
&+\sum_{i=2}^{d}\prod_{j=1}^{d-i}A_j^{n_j}\sum_{k_{d-i+1}=0}^{n_{d-i+1}-1}A_{d-i+1}^{k_{d-i+1}}\alpha_{d-i+1} + \sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=d-1,d\in I}} A_j^{m_j}\alpha_d\\
&+n_d(-1)^{d-1}\sum_{i=0}^{d-2}(-1)^{i+1}\prod_{\substack{k\in I\subseteq[d-1]\\|I|=i}} A_k\alpha_d + n_d(-1)^d\sum_{i=0}^{d-2}(-1)^{i+1}\prod_{\substack{j\in I\subseteq[d]\\|I|=i,d\notin I}} A_j^{n_j}\alpha_d.
\end{aligned}\tag{5.9}$$

From here we deduce that,

$$\begin{aligned}
T_1^{n_1}\cdots T_d^{n_d}x =& (-1)^d\sum_{i=0}^{d-2}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d]\\|I|=i}} A_k^{n_k}x + \sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=i,d\in I}} A_j^{m_j}\alpha_d\right) + \prod_{\substack{k\in I\subseteq[d]\\|I|=d-1}} A_k^{n_k}x\\
&+\sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=d-1,d\in I}} A_j^{m_j}\alpha_d + \sum_{i=2}^{d}\prod_{j=1}^{d-i}A_j^{n_j}\sum_{k_{d-i+1}=0}^{n_{d-i+1}-1}A_{d-i+1}^{k_{d-i+1}}\alpha_{d-i+1}.
\end{aligned}$$

Now we are ready to prove Lemma 5.4. For illustration we first present an example of Equation (5.10) in the proof of Lemma 5.4.

**Example.** Consider the case $d = 2$. We have that

$$
\begin{aligned}
-x + A_1^{n_1}x + A_2^{n_2}x + \sum_{k_2=0}^{n_2-1} A_2^{k_2}\alpha_2 + \sum_{k_1=0}^{n_2-1} A_1^{k_1}\alpha_1 &= -x + A_1^{n_1}x + \sum_{k_1=0}^{n_2-1} A_1^{k_1}\alpha_1 \\
&\quad + A_2^{n_2}x + \sum_{k_2=0}^{n_2-1} A_2^{k_2}\alpha_2, \\
&= -x + T_1^{n_1}x + T_2^{n_2}x.
\end{aligned}
$$

Consider now the case $d = 3$. We have that the left side of Equation 5.10 is equal to

$$x - T_1^{n_1}x - T_2^{n_2}x - T_3^{n_3}x + T_1^{n_1}T_2^{n_2}x + T_1^{n_1}T_3^{n_3}x + T_2^{n_2}T_3^{n_3}x,$$

and the right side of Equation (5.10) is equal to

$$
\begin{aligned}
& -(-x + A_1^{n_1}x + A_2^{n_2}x + A_3^{n_3}x + \sum_{k_3=0}^{n_3-1} A_3^{k_3}\alpha_3) + A_1^{n_1}A_2^{n_2}x + A_1^{n_1}A_3^{n_3}x + A_2^{n_2}A_3^{n_3}x \\
& + \sum_{k_3=0}^{n_3-1} A_1^{n_1}A_3^{k_3}\alpha_3 + \sum_{k_3=0}^{n_3-1} A_2^{n_2}A_3^{k_3}\alpha_3 + A_1^{n_1}\sum_{k_2=0}^{n_2-1} A_2^{k_2}\alpha_2 + \sum_{k_1=0}^{n_1-1} A_1^{k_1}\alpha_1 \\
= \;& -\left(-x + A_1^{n_1}x + A_2^{n_2}x + A_3^{n_3}x - A_1^{n_1}A_2^{n_2}x - A_2^{n_2}A_3^{n_3}x - \sum_{k_3=0}^{n_3-1} A_2^{n_2}A_3^{k_3}\alpha_3 - \sum_{k_2=0}^{n_2-1} A_2^{k_2}\alpha_2\right) \\
& + A_1^{n_1}A_3^{n_3}x - \sum_{k_2=0}^{n_2-1} A_2^{k_2}\alpha_2 - \sum_{k_3=0}^{n_3-1} A_3^{k_3}\alpha_3 + \sum_{k_3=0}^{n_3-1} A_1^{n_1}A_3^{k_3}\alpha_3 + A_1^{n_1}\sum_{k_2=0}^{n_2-1} A_2^{k_2}\alpha_2 + \sum_{k_1=0}^{n_1-1} A_1^{k_1}\alpha_1, \\
= \;& -(-x + A_1^{n_1}x + A_2^{n_2}x + A_3^{n_3}x - A_1^{n_1}A_2^{n_2}x - A_1^{n_1}A_3^{n_3}x - T_2^{n_2}T_3^{n_3}x) \\
& - \sum_{k_2=0}^{n_2-1} A_2^{k_2}\alpha_2 - \sum_{k_3=0}^{n_3-1} A_3^{k_3}\alpha_3 + \sum_{k_3=0}^{n_3-1} A_1^{n_1}A_3^{k_3}\alpha_3 + A_1^{n_1}\sum_{k_2=0}^{n_2-1} A_2^{k_2}\alpha_2 + \sum_{k_1=0}^{n_1-1} A_1^{k_1}\alpha_1, \\
= \;& -\left(-x + A_1^{n_1}x + A_2^{n_2}x + \sum_{k_2=0}^{n_2-1} A_2^{k_2}\alpha_2 + A_3^{n_3}x - A_1^{n_1}A_2^{n_2}x - A_1^{n_1}A_3^{n_3}x - T_2^{n_2}T_3^{n_3}x\right) \\
& - \sum_{k_3=0}^{n_3-1} A_3^{k_3}\alpha_3 + \sum_{k_3=0}^{n_3-1} A_1^{n_1}A_3^{k_3}\alpha_3 + A_1^{n_1}\sum_{k_2=0}^{n_2-1} A_2^{k_2}\alpha_2 + \sum_{k_1=0}^{n_1-1} A_1^{k_1}\alpha_1, \\
= \;& -(-x + A_1^{n_1}x + T_2^{n_2}x + A_3^{n_3}x - A_1^{n_1}A_2^{n_2}x - A_1^{n_1}A_3^{n_3}x - T_2^{n_2}T_3^{n_3}x) \\
& - \sum_{k_3=0}^{n_3-1} A_3^{k_3}\alpha_3 + \sum_{k_3=0}^{n_3-1} A_1^{n_1}A_3^{k_3}\alpha_3 + A_1^{n_1}\sum_{k_2=0}^{n_2-1} A_2^{k_2}\alpha_2 + \sum_{k_1=0}^{n_1-1} A_1^{k_1}\alpha_1, \\
= \;& -(-x + A_1^{n_1}x + T_2^{n_2}x + A_3^{n_3}x - A_1^{n_1}A_2^{n_2}x - T_1^{n_1}T_3^{n_3}x - T_2^{n_2}T_3^{n_3}x) \\
& - \sum_{k_3=0}^{n_3-1} A_3^{k_3}\alpha_3 + A_1^{n_1}\sum_{k_2=0}^{n_2-1} A_2^{k_2}\alpha_2, \\
= \;& -\left(-x + A_1^{n_1}x + T_2^{n_2}x + A_3^{n_3}x + \sum_{k_3=0}^{n_3-1} A_3^{k_3}\alpha_3 - A_1^{n_1}A_2^{n_2}x - T_1^{n_1}T_3^{n_3}x - T_2^{n_2}T_3^{n_3}x\right) \\
& + A_1^{n_1}\sum_{k_2=0}^{n_2-1} A_2^{k_2}\alpha_2, \\
= \;& x - A_1^{n_1}x - T_2^{n_2}x - T_3^{n_3}x + A_1^{n_1}A_2^{n_2}x + A_1^{n_1}\sum_{k_2=0}^{n_2-1} A_2^{k_2}\alpha_2 + T_1^{n_1}T_3^{n_3}x + T_2^{n_2}T_3^{n_3}x, \\
= \;& x + T_1^{n_1}x - T_2^{n_2}x - T_3^{n_3}x + T_1^{n_1}T_2^{n_2}x + T_1^{n_1}T_3^{n_3}x + T_2^{n_2}T_3^{n_3}x.
\end{aligned}
$$

**Proof of Lemma 5.4.** Suppose that for some $d \geq 2$ the following identity holds,

$$
\begin{aligned}
(-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} \bigcirc_{\substack{k\in I\subseteq[d]\\|I|=i}} T_k^{n_k} x = \;& (-1)^d \sum_{i=0}^{d-2} (-1)^{i+1} \left( \prod_{\substack{k\in I\subseteq[d]\\|I|=i}} A_k^{n_k} x + \sum_{k_d=0}^{n_d-1} \prod_{\substack{j\in I\subseteq[d]\\|I|=i,d\in I}} A_j^{m_j} \alpha_d \right) \\
& + \prod_{\substack{k\in I\subseteq[d]\\|I|=d-1}} A_k^{n_k} x + \sum_{k_d=0}^{n_d-1} \prod_{\substack{j\in I\subseteq[d]\\|I|=d-1,d\in I}} A_j^{m_j} \alpha_d \\
& + \sum_{i=2}^{d} \prod_{j=1}^{d-i} A_j^{n_j} \sum_{k_{d-i+1}=0}^{n_{d-i+1}-1} A_{d-i+1}^{k_{d-i+1}} \alpha_{d-i+1}.
\end{aligned}
\tag{5.10}
$$

We now extend the identity to $d+1$. Indeed,

$$
\begin{aligned}
& (-1)^{d+1} \sum_{i=0}^{d-1} (-1)^{i+1} \left( \prod_{\substack{k\in I\subseteq[d+1]\\|I|=i}} A_k^{n_k} x + \sum_{k_{d+1}=0}^{n_{d+1}-1} \prod_{\substack{j\in I\subseteq[d+1]\\|I|=i,d+1\in I}} A_j^{m_j} \alpha_{d+1} \right) + \prod_{\substack{k\in I\subseteq[d+1]\\|I|=d}} A_k^{n_k} x \\
& + \sum_{k_{d+1}=0}^{n_{d+1}-1} \prod_{\substack{j\in I\subseteq[d+1]\\|I|=d,d+1\in I}} A_j^{m_j} \alpha_{d+1} + \sum_{i=2}^{d+1} \prod_{j=1}^{d+1-i} A_j^{n_j} \sum_{k_{d+2-i}=0}^{n_{d+2-i}-1} A_{d+2-i}^{k_{d+2-i}} \alpha_{d+2-i} \\
= \;& (-1)^{d+1} \sum_{i=0}^{d-2} (-1)^{i+1} \left( \prod_{\substack{k\in I\subseteq[d+1]\\|I|=i}} A_k^{n_k} x + \sum_{k_{d+1}=0}^{n_{d+1}-1} \prod_{\substack{j\in I\subseteq[d+1]\\|I|=i,d+1\in I}} A_j^{m_j} \alpha_{d+1} \right) \\
& + (-1)^{d+1}(-1)^d \left( \prod_{\substack{k\in I\subseteq[d+1]\\|I|=d-1}} A_k^{n_k} x + \sum_{k_{d+1}=0}^{n_{d+1}-1} \prod_{\substack{j\in I\subseteq[d+1]\\|I|=d-1,d+1\in I}} A_j^{m_j} \alpha_{d+1} \right) \\
& + \prod_{\substack{k\in I\subseteq[d+1]\\|I|=d}} A_k^{n_k} x + \sum_{k_{d+1}=0}^{n_{d+1}-1} \prod_{\substack{j\in I\subseteq[d+1]\\|I|=d,d+1\in I}} A_j^{m_j} \alpha_{d+1} + \sum_{i=2}^{d+1} \prod_{j=1}^{d+1-i} A_j^{n_j} \sum_{k_{d+2-i}=0}^{n_{d+2-i}-1} A_{d+2-i}^{k_{d+2-i}} \alpha_{d+2-i}, \\
= \;& (-1)^{d+1} \sum_{i=0}^{d-2} (-1)^{i+1} \left( \prod_{\substack{k\in I\subseteq[d+1]\\|I|=i}} A_k^{n_k} x + \sum_{k_{d+1}=0}^{n_{d+1}-1} \prod_{\substack{j\in I\subseteq[d+1]\\|I|=i,d+1\in I}} A_j^{m_j} \alpha_{d+1} \right) \\
& - \left( \prod_{\substack{k\in I\subseteq[d+1]\\|I|=d-1}} A_k^{n_k} x + \sum_{k_{d+1}=0}^{n_{d+1}-1} \prod_{\substack{j\in I\subseteq[d+1]\\|I|=d-1,d+1\in I}} A_j^{m_j} \alpha_{d+1} \right) \\
& + \prod_{\substack{k\in I\subseteq[d+1]\\|I|=d}} A_k^{n_k} x + \sum_{k_{d+1}=0}^{n_{d+1}-1} \prod_{\substack{j\in I\subseteq[d+1]\\|I|=d,d+1\in I}} A_j^{m_j} \alpha_{d+1} + \sum_{i=2}^{d+1} \prod_{j=1}^{d+1-i} A_j^{n_j} \sum_{k_{d+2-i}=0}^{n_{d+2-i}-1} A_{d+2-i}^{k_{d+2-i}} \alpha_{d+2-i},
\end{aligned}
$$

$$\begin{aligned}
&= -B_d + (-1)^{d+1}\sum_{i=0}^{d-2}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d+1]\\|I|=i,\ d+1\in I}} A_k^{n_k}x + \sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\|I|=i,d+1\in I}} A_j^{m_j}\alpha_{d+1}\right)\\
&\quad -(-1)^{d+1}\sum_{i=0}^{d-2}(-1)^{i+1}\sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=i,d\in I}} A_j^{m_j}\alpha_d - \prod_{\substack{k\in I\subseteq[d+1]\\|I|=d-1}} A_k^{n_k}x + \prod_{\substack{k\in I\subseteq[d+1]\\|I|=d}} A_k^{n_k}x\\
&\quad + \sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\|I|=d,d+1\in I}} A_j^{m_j}\alpha_{d+1} - \sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\|I|=d-1,d+1\in I}} A_j^{m_j}\alpha_{d+1}\\
&\quad + \sum_{i=2}^{d+1}\prod_{j=1}^{d+1-i} A_j^{n_j}\sum_{k_{d+2-i}=0}^{n_{d+2-i}-1} A_{d+2-i}^{k_{d+2-i}}\alpha_{d+2-i},
\end{aligned}$$

where $B_d = (-1)^d\sum_{i=0}^{d-2}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d]\\|I|=i}} A_k^{n_k}x + \sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=i,d\in I}} A_j^{m_j}\alpha_d\right)$. Now we have that previous equation is equal to

$$\begin{aligned}
&= -B_d - C_d + (-1)^{d+1}\sum_{i=0}^{d-2}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d+1]\\|I|=i,\ d+1\in I}} A_k^{n_k}x + \sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\|I|=i,d+1\in I}} A_j^{m_j}\alpha_{d+1}\right)\\
&\quad -(-1)^{d+1}\sum_{i=0}^{d-2}(-1)^{i+1}\sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=i,d\in I}} A_j^{m_j}\alpha_d - \prod_{\substack{k\in I\subseteq[d+1]\\|I|=d-1,\ d+1\in I}} A_k^{n_k}x + \prod_{\substack{k\in I\subseteq[d+1]\\|I|=d}} A_k^{n_k}x\\
&\quad + \sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\|I|=d,d+1\in I}} A_j^{m_j}\alpha_{d+1} - \sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\|I|=d-1,d+1\in I}} A_j^{m_j}\alpha_{d+1}\\
&\quad + \sum_{i=2}^{d+1}\prod_{j=1}^{d+1-i} A_j^{n_j}\sum_{k_{d+2-i}=0}^{n_{d+2-i}-1} A_{d+2-i}^{k_{d+2-i}}\alpha_{d+2-i},\\
&= -B_d - C_d + (-1)^{d+1}\sum_{i=0}^{d-1}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d+1]\\|I|=i,\ d+1\in I}} A_k^{n_k}x + \sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\|I|=i,d+1\in I}} A_j^{m_j}\alpha_{d+1}\right)\\
&\quad -(-1)^{d+1}\sum_{i=0}^{d-2}(-1)^{i+1}\sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=i,d\in I}} A_j^{m_j}\alpha_d + \prod_{\substack{k\in I\subseteq[d+1]\\|I|=d}} A_k^{n_k}x\\
&\quad + \sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\|I|=d,d+1\in I}} A_j^{m_j}\alpha_{d+1} + \sum_{i=2}^{d+1}\prod_{j=1}^{d+1-i} A_j^{n_j}\sum_{k_{d+2-i}=0}^{n_{d+2-i}-1} A_{d+2-i}^{k_{d+2-i}}\alpha_{d+2-i},
\end{aligned}$$

$$
\begin{aligned}
= \;& -B_d - C_d - D_d + (-1)^{d+1}\sum_{i=1}^{d-1}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d+1]\\ |I|=i,\ d+1\in I}} A_k^{n_k}x + \sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\ |I|=i, d+1\in I}} A_j^{m_j}\alpha_{d+1}\right)\\
& -(-1)^{d+1}\sum_{i=1}^{d-1}(-1)^{i+1}\sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\ |I|=i, d\in I}} A_j^{m_j}\alpha_d\\
& +\prod_{\substack{k\in I\subseteq[d+1]\\ |I|=d}} A_k^{n_k}x + \sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\ |I|=d, d+1\in I}} A_j^{m_j}\alpha_{d+1} + \sum_{i=2}^{d+1}\prod_{j=1}^{d+1-i}A_j^{n_j}\sum_{k_{d+2-i}=0}^{n_{d+2-i}-1}A_{d+2-i}^{k_{d+2-i}}\alpha_{d+2-i}\\
= \;& -B_d - C_d - D_d + (-1)^{d+1}\sum_{i=1}^{d}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d+1]\\ |I|=i,\ d+1\in I}} A_k^{n_k}x + \sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\ |I|=i, d+1\in I}} A_j^{m_j}\alpha_{d+1}\right)\\
& -(-1)^{d+1}\sum_{i=1}^{d-1}(-1)^{i+1}\sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\ |I|=i, d\in I}} A_j^{m_j}\alpha_d + \prod_{\substack{k\in I\subseteq[d]\\ |I|=d}} A_k^{n_k}x\\
& +\sum_{i=2}^{d+1}\prod_{j=1}^{d+1-i}A_j^{n_j}\sum_{k_{d+2-i}=0}^{n_{d+2-i}-1}A_{d+2-i}^{k_{d+2-i}}\alpha_{d+2-i}\\
= \;& -B_d - C_d - D_d + (-1)^{d+1}\sum_{i=1}^{d}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d+1]\\ |I|=i,\ d+1\in I}} A_k^{n_k}x + \sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\ |I|=i, d+1\in I}} A_j^{m_j}\alpha_{d+1}\right)\\
& -(-1)^{d+1}\sum_{i=1}^{d-1}(-1)^{i+1}\sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\ |I|=i, d\in I}} A_j^{m_j}\alpha_d + \prod_{k=1}^{d}A_k^{n_k}x + \sum_{i=2}^{d+1}\prod_{j=1}^{d+1-i}A_j^{n_j}\sum_{k_{d+2-i}=0}^{n_{d+2-i}-1}A_{d+2-i}^{k_{d+2-i}}\alpha_{d+2-i}.
\end{aligned}
$$

But with a simple change of variable we have that

$$
\prod_{k=1}^{d}A_k^{n_k}x + \sum_{i=2}^{d+1}\prod_{j=1}^{d+1-i}A_j^{n_j}\sum_{k_{d+2-i}=0}^{n_{d+2-i}-1}A_{d+2-i}^{k_{d+2-i}}\alpha_{d+2-i} \quad = T_1^{n_1}\cdots T_d^{n_d}x.
$$

Thus expression 5.10 is equal to

$$
\begin{aligned}
= \;& -B_d - C_d - D_d + (-1)^{d+1}\sum_{i=1}^{d}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d+1]\\ |I|=i,\ d+1\in I}} A_k^{n_k}x + \sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\ |I|=i, d+1\in I}} A_j^{m_j}\alpha_{d+1}\right)\\
& -(-1)^{d+1}\sum_{i=1}^{d-1}(-1)^{i+1}\sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\ |I|=i, d\in I}} A_j^{m_j}\alpha_d + T_1^{n_1}\cdots T_d^{n_d}x,\\
= \;& -B_d - C_d - D_d + (-1)^{d+1}\sum_{i=1}^{d}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d+1]\\ |I|=i,\ d+1\in I}} A_k^{n_k}x + \sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\ |I|=i, d+1\in I}} A_j^{m_j}\alpha_{d+1}\right)\\
& -(-1)^{d+1}\sum_{i=1}^{d-1}(-1)^{i+1}\sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\ |I|=i, d\in I}} A_j^{m_j}\alpha_d + T_1^{n_1}\cdots T_d^{n_d}x,
\end{aligned}
$$

$$
\begin{aligned}
= \quad & (-1)^{d+1}\sum_{i=0}^{d-1}(-1)^{i+1}\bigcirc_{\substack{k\in I\subseteq[d]\\|I|=i}} T_k^{n_k}x + T_1^{n_1}\cdots T_d^{n_d}x\\
& +(-1)^{d+1}\sum_{i=1}^{d}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d+1]\\|I|=i,\ d+1\in I}} A_k^{n_k}x + \sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\|I|=i,d+1\in I}} A_j^{m_j}\alpha_{d+1}\right)\\
& -(-1)^{d+1}\sum_{i=1}^{d-1}(-1)^{i+1}\sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=i,d\in I}} A_j^{m_j}\alpha_d + \sum_{i=2}^{d}\prod_{j=1}^{d-i}A_j^{n_j}\sum_{k_{d-i+1}=0}^{n_{d-i+1}-1}A_{d-i+1}^{k_{d-i+1}}\alpha_{d-i+1}.
\end{aligned}
$$

Now we will show that

$$
\begin{aligned}
(-1)^{d+1}\sum_{i=1}^{d}(-1)^{i+1}\bigcirc_{\substack{k\in I\subseteq[d+1]\\|I|=i,\ d+1\in I}} T_k^{n_k}x = \quad & (-1)^{d+1}\sum_{i=1}^{d}(-1)^{i+1}\prod_{\substack{k\in I\subseteq[d+1]\\|I|=i,\ d+1\in I}} A_k^{n_k}x\\
& +(-1)^{d+1}\sum_{i=1}^{d}(-1)^{i+1}\sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\|I|=i,d+1\in I}} A_j^{m_j}\alpha_{d+1}\\
& -(-1)^{d+1}\sum_{i=1}^{d-1}(-1)^{i+1}\sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=i,d\in I}} A_j^{m_j}\alpha_d\\
& +\sum_{i=2}^{d}\prod_{j=1}^{d-i}A_j^{n_j}\sum_{k_{d-i+1}=0}^{n_{d-i+1}-1}A_{d-i+1}^{k_{d-i+1}}\alpha_{d-i+1}.
\end{aligned}
\tag{5.11}
$$

We remark that

$$
\begin{aligned}
(-1)^{d+1}\sum_{i=1}^{d}(-1)^{i+1}\bigcirc_{\substack{k\in I\subseteq[d+1]\\|I|=i,\ d+1\in I}} T_k^{n_k}x & = (-1)^{d+1}\sum_{i=1}^{d}(-1)^{i+1}\bigcirc_{\substack{k\in I\subseteq[d]\\|I|=i-1}} T_k^{n_k}(T_{d+1}^{n_{d+1}}x),\\
& = (-1)^{d+1}\sum_{i=0}^{d-1}(-1)^{i+2}\bigcirc_{\substack{k\in I\subseteq[d]\\|I|=i}} T_k^{n_k}(T_{d+1}^{n_{d+1}}x),\\
& = (-1)^{d}\sum_{i=0}^{d-1}(-1)^{i+1}\bigcirc_{\substack{k\in I\subseteq[d]\\|I|=i}} T_k^{n_k}(T_{d+1}^{n_{d+1}}x).
\end{aligned}
$$

Therefore, expression (5.11) is equal to

$$
\begin{aligned}
= \quad & (-1)^{d}\sum_{i=0}^{d-2}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d]\\|I|=i}} A_k^{n_k}T_{d+1}^{n_{d+1}}x + \sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=i,d\in I}} A_j^{m_j}\alpha_d\right) + \prod_{\substack{k\in I\subseteq[d]\\|I|=d-1}} A_k^{n_k}T_{d+1}^{n_{d+1}}x\\
& +\sum_{k_d=0}^{n_d-1}\prod_{\substack{j\in I\subseteq[d]\\|I|=d-1,d\in I}} A_j^{m_j}\alpha_d + \sum_{i=2}^{d}\prod_{j=1}^{d-i}A_j^{n_j}\sum_{k_{d-i+1}=0}^{n_{d-i+1}-1}A_{d-i+1}^{k_{d-i+1}}\alpha_{d-i+1},
\end{aligned}
$$

$$
\begin{aligned}
= \quad & (-1)^d \sum_{i=0}^{d-2} (-1)^{i+1} \left( \prod_{\substack{k\in I\subseteq[d]\\|I|=i}} A_k^{n_k} \left( A_{d+1}^{n_{d+1}} x + \sum_{k_{d+1}=0}^{n_{d+1}-1} A_{d+1}^{k_{d+1}} \alpha_{d+1} \right) + \sum_{k_d=0}^{n_d-1} \prod_{\substack{j\in I\subseteq[d]\\|I|=i, d\in I}} A_j^{m_j} \alpha_d \right) \\
& + \prod_{\substack{k\in I\subseteq[d]\\|I|=d-1}} A_k^{n_k} \left( A_{d+1}^{n_{d+1}} x + \sum_{k_{d+1}=0}^{n_{d+1}-1} A_{d+1}^{k_{d+1}} \alpha_{d+1} \right) + \sum_{k_d=0}^{n_d-1} \prod_{\substack{j\in I\subseteq[d]\\|I|=d-1, d\in I}} A_j^{m_j} \alpha_d, \\
& + \sum_{i=2}^{d} \prod_{j=1}^{d-i} A_j^{n_j} \sum_{k_{d-i+1}=0}^{n_{d-i+1}-1} A_{d-i+1}^{k_{d-i+1}} \alpha_{d-i+1}, \\
= \quad & (-1)^d \sum_{i=0}^{d-2} (-1)^{i+1} \left( \prod_{\substack{k\in I\subseteq[d]\\|I|=i}} A_k^{n_k} A_{d+1}^{n_{d+1}} x + \prod_{\substack{k\in I\subseteq[d]\\|I|=i}} A_k^{n_k} \sum_{k_{d+1}=0}^{n_{d+1}-1} A_{d+1}^{k_{d+1}} \alpha_{d+1} + \sum_{k_d=0}^{n_d-1} \prod_{\substack{j\in I\subseteq[d]\\|I|=i, d\in I}} A_j^{m_j} \alpha_d \right) \\
& + \prod_{\substack{k\in I\subseteq[d]\\|I|=d-1}} A_k^{n_k} A_{d+1}^{n_{d+1}} x + \prod_{\substack{k\in I\subseteq[d]\\|I|=d-1}} A_k^{n_k} \sum_{k_{d+1}=0}^{n_{d+1}-1} A_{d+1}^{k_{d+1}} \alpha_{d+1} + \sum_{k_d=0}^{n_d-1} \prod_{\substack{j\in I\subseteq[d]\\|I|=d-1, d\in I}} A_j^{m_j} \alpha_d \\
& + \sum_{i=2}^{d} \prod_{j=1}^{d-i} A_j^{n_j} \sum_{k_{d-i+1}=0}^{n_{d-i+1}-1} A_{d-i+1}^{k_{d-i+1}} \alpha_{d-i+1}, \\
= \quad & (-1)^d \sum_{i=0}^{d-2} (-1)^{i+1} \left( \prod_{\substack{k\in I\subseteq[d+1]\\|I|=i+1, d+1\in I}} A_k^{n_k} x + \sum_{k_{d+1}=0}^{n_{d+1}-1} \prod_{\substack{j\in I\subseteq[d+1]\\|I|=i+1,\ d+1\in I}} A_j^{m_j} \alpha_{d+1} + \sum_{k_d=0}^{n_d-1} \prod_{\substack{j\in I\subseteq[d]\\|I|=i, d\in I}} A_j^{m_j} \alpha_d \right) \\
& + \prod_{\substack{k\in I\subseteq[d]\\|I|=d,\ d+1\in I}} A_k^{n_k} x + \sum_{k_{d+1}=0}^{n_{d+1}-1} \prod_{\substack{k\in I\subseteq[d+1]\\|I|=d,\ d+1\in I}} A_j^{m_j} \alpha_{d+1} + \sum_{k_d=0}^{n_d-1} \prod_{\substack{j\in I\subseteq[d]\\|I|=d-1, d\in I}} A_j^{m_j} \alpha_d \\
& + \sum_{i=2}^{d} \prod_{j=1}^{d-i} A_j^{n_j} \sum_{k_{d-i+1}=0}^{n_{d-i+1}-1} A_{d-i+1}^{k_{d-i+1}} \alpha_{d-i+1}, \\
= \quad & (-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} \left( \prod_{\substack{k\in I\subseteq[d+1]\\|I|=i+1, d+1\in I}} A_k^{n_k} x + \sum_{k_{d+1}=0}^{n_{d+1}-1} \prod_{\substack{j\in I\subseteq[d+1]\\|I|=i+1,\ d+1\in I}} A_j^{m_j} \alpha_{d+1} \right) \\
& + (-1)^d \sum_{i=0}^{d-2} (-1)^{i+1} \sum_{k_d=0}^{n_d-1} \prod_{\substack{j\in I\subseteq[d]\\|I|=i, d\in I}} A_j^{m_j} \alpha_d + \sum_{k_d=0}^{n_d-1} \prod_{\substack{j\in I\subseteq[d]\\|I|=d-1, d\in I}} A_j^{m_j} \alpha_d \\
& + \sum_{i=2}^{d} \prod_{j=1}^{d-i} A_j^{n_j} \sum_{k_{d-i+1}=0}^{n_{d-i+1}-1} A_{d-i+1}^{k_{d-i+1}} \alpha_{d-i+1}. \\
= \quad & (-1)^{d+1} \sum_{i=1}^{d} (-1)^{i+1} \left( \prod_{\substack{k\in I\subseteq[d+1]\\|I|=i, d+1\in I}} A_k^{n_k} x + \sum_{k_{d+1}=0}^{n_{d+1}-1} \prod_{\substack{j\in I\subseteq[d+1]\\|I|=i,\ d+1\in I}} A_j^{m_j} \alpha_{d+1} \right) \\
& + (-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} \sum_{k_d=0}^{n_d-1} \prod_{\substack{j\in I\subseteq[d]\\|I|=i, d\in I}} A_j^{m_j} \alpha_d + \sum_{i=2}^{d} \prod_{j=1}^{d-i} A_j^{n_j} \sum_{k_{d-i+1}=0}^{n_{d-i+1}-1} A_{d-i+1}^{k_{d-i+1}} \alpha_{d-i+1}, \\
= \quad & (-1)^{d+1} \sum_{i=1}^{d} (-1)^{i+1} \left( \prod_{\substack{k\in I\subseteq[d+1]\\|I|=i, d+1\in I}} A_k^{n_k} x + \sum_{k_{d+1}=0}^{n_{d+1}-1} \prod_{\substack{j\in I\subseteq[d+1]\\|I|=i,\ d+1\in I}} A_j^{m_j} \alpha_{d+1} \right) \\
& - (-1)^{d+1} \sum_{i=0}^{d-1} (-1)^{i+1} \sum_{k_d=0}^{n_d-1} \prod_{\substack{j\in I\subseteq[d]\\|I|=i, d\in I}} A_j^{m_j} \alpha_d + \sum_{i=2}^{d} \prod_{j=1}^{d-i} A_j^{n_j} \sum_{k_{d-i+1}=0}^{n_{d-i+1}-1} A_{d-i+1}^{k_{d-i+1}} \alpha_{d-i+1}.
\end{aligned}
$$

Finally we have that

$$\begin{aligned}
& (-1)^{d+1}\sum_{i=0}^{d-1}(-1)^{i+1}\left(\prod_{\substack{k\in I\subseteq[d+1]\\|I|=i}} A_k^{n_k}x+\sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\|I|=i,d+1\in I}} A_j^{m_j}\alpha_{d+1}\right)+\prod_{\substack{k\in I\subseteq[d+1]\\|I|=d}} A_k^{n_k}x\\
& +\sum_{k_{d+1}=0}^{n_{d+1}-1}\prod_{\substack{j\in I\subseteq[d+1]\\|I|=d,d+1\in I}} A_j^{m_j}\alpha_{d+1}+\sum_{i=2}^{d+1}\prod_{j=1}^{d+1-i} A_j^{n_j}\sum_{k_{d+2-i}=0}^{n_{d+2-i}-1} A_{d+2-i}^{k_{d+2-i}}\alpha_{d+2-i}\\
= \; & (-1)^{d+1}\sum_{i=0}^{d-1}(-1)^{i+1}\bigcirc_{\substack{k\in I\subseteq[d]\\|I|=i}} T_k^{n_k}x+T_1^{n_1}\cdots T_d^{n_d}x+(-1)^{d+1}\sum_{i=1}^{d}(-1)^{i+1}\bigcirc_{\substack{k\in I\subseteq[d+1]\\|I|=i,\ d+1\in I}} T_k^{n_k}x,\\
= \; & (-1)^{d+1}\sum_{i=0}^{d}(-1)^{i+1}\bigcirc_{\substack{k\in I\subseteq[d]\\|I|=i}} T_k^{n_k}x+(-1)^{d+1}\sum_{i=1}^{d}(-1)^{i+1}\bigcirc_{\substack{k\in I\subseteq[d+1]\\|I|=i,\ d+1\in I}} T_k^{n_k}x,\\
= \; & (-1)^{d+1}\sum_{i=0}^{d}(-1)^{i+1}\bigcirc_{\substack{k\in I\subseteq[d+1]\\|I|=i}} T_k^{n_k}x.
\end{aligned}$$

We conclude that if $(\mathbb{T}^r, T_1, \ldots, T_d)$ satisfies conditions (5.3) and (5.4), then

$$T_1^{n_1}\cdots T_d^{n_d}x=(-1)^d\sum_{i=0}^{d-1}(-1)^{i+1}\bigcirc_{\substack{k\in I\subseteq[d]\\|I|=i}} T_k^{n_k}x$$

and thus the system has the closing parallelepiped property since the last coordinate of $\mathbf{Q}_{T_1,\ldots,T_d}(X)$ is a function of the rest of the coordinates.

Conversely, assume that for every $\mathbf{n}=(n_1,\ldots,n_d)\in\mathbb{Z}^d$ and $x\in X$ we have that

$$T_1^{n_1}\cdots T_d^{n_d}x=(-1)^d\sum_{i=0}^{d-1}(-1)^{i+1}\bigcirc_{\substack{k\in I\subseteq[d]\\|I|=i}} T_k^{n_k}x.$$

In particular,

$$\begin{aligned}
T_1\cdots T_d x &= (-1)^d\sum_{i=0}^{d-1}(-1)^{i+1}\bigcirc_{\substack{k\in I\subseteq[d]\\|I|=i}} T_k x,\\
\prod_{i=1}^{d} A_i x+\sum_{i=1}^{d}\prod_{j=1}^{d-i} A_j\alpha_{d-i+1} &= (-1)^d\sum_{i=0}^{d-1}(-1)^{i+1}\bigcirc_{\substack{k\in I\subseteq[d]\\|I|=i}} T_k x,\\
\prod_{i=1}^{d} A_i x+\sum_{i=1}^{d}\prod_{j=1}^{d-i} A_j\alpha_{d-i+1} &= (-1)^d\sum_{i=0}^{d-1}(-1)^{i+1}\sum_{\substack{I\subseteq[d]\\|I|=i}}\bigcirc_{k\in I} T_k x.
\end{aligned}$$

From now we assume the subsets $I = \{\ell_1, \ldots, \ell_i\}$ are given in increasing order. Then,

$$\begin{aligned}\prod_{i=1}^{d} A_i x + \sum_{i=1}^{d} \prod_{j=1}^{d-i} A_j \alpha_{d-i+1} &= (-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} \sum_{\substack{I\subseteq[d]\\|I|=i}} \prod_{k\in I} A_k x \\ &+ (-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} \sum_{\substack{I\subseteq[d]\\|I|=i}} \sum_{r=1}^{k} \prod_{j\in\{\ell_1,\ldots,\ell_{k-r}\}} A_j \alpha_{\ell_{k-r+1}}.\end{aligned}$$

If $x = 0$ we have that

$$\sum_{i=1}^{d} \prod_{j=1}^{d-i} A_j \alpha_{d-i+1} = (-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} \sum_{\substack{I\subseteq[d]\\|I|=i}} \sum_{r=1}^{i} \prod_{j\in\{\ell_1,\ldots,\ell_{i-r}\}} A_j \alpha_{\ell_{i-r+1}},$$

which is equivalent to

$$\sum_{i=0}^{d} (-1)^{i+1} \sum_{\substack{I\subseteq[d]\\|I|=i}} \sum_{r=1}^{i} \prod_{j\in\{\ell_1,\ldots,\ell_{i-r}\}} A_j \alpha_{\ell_{i-r+1}} = 0. \tag{5.12}$$

We separate the left side of (5.12) in the following sums,

$$\begin{aligned}\sum_{i=0}^{d} (-1)^{i+1} \sum_{\substack{I\subseteq[d]\\|I|=i}} \sum_{r=1}^{i} \prod_{j\in I=\{\ell_1,\ldots,\ell_{i-r}\}} A_j \alpha_{\ell_{i-r+1}} &= \sum_{i=0}^{d} (-1)^{i+1} \sum_{\substack{I\subseteq[d]\\|I|=i,\, d\in I}} \prod_{j\in\{\ell_1,\ldots,\ell_{i-r}\}} A_j \alpha_d \\ &+ \sum_{i=1}^{d-1} (-1)^{i+1} \sum_{\substack{I\subseteq[d]\\|I|=i,\, d\notin I}} \sum_{r=1}^{i} \prod_{j\in\{\ell_1,\ldots,\ell_{i-r}\}} A_j \alpha_{\ell_{i-r+1}} \\ &+ \sum_{i=2}^{d} (-1)^{i+1} \sum_{\substack{I\subseteq[d]\\|I|=i,\, d\in I}} \sum_{r=2}^{i} \prod_{j\in\{\ell_1,\ldots,\ell_{i-r}\}} A_j \alpha_{\ell_{i-r+1}}.\end{aligned} \tag{5.13}$$

Let $1 \leq i < d$ and $I \subseteq [d]$ with $|I| = i$ and $d \notin I$. We consider $J = I \cup \{d\}$. We observe that if $r \in \{2, \ldots, i+1\}$, then $i + 2 - r < i + 1$. Therefore $\ell_{i+1-r+1} \neq d$. Then, for $J$ in the third sum we have

$$\sum_{r=2}^{i+1} \prod_{j\in\{\ell_1,\ldots,\ell_{i+1-r}\}} A_j \alpha_{\ell_{i+1-r+1}} = \sum_{r=1}^{i} \prod_{j\in\{\ell_1,\ldots,\ell_{i-r}\}} A_j \alpha_{\ell_{i-r+1}},$$

which is the expression for $I$ in the second sum. Since the map $I \mapsto I \cup \{d\}$ is a bijection when $|I| < d$, we have that the last two sums on (5.13) cancel each other. Finally, equation (5.12) is equivalent to

$$\sum_{i=0}^{d} (-1)^{i+1} \sum_{\substack{I\subseteq[d]\\|I|=i,\, d\in I}} \prod_{j\in\{\ell_1,\ldots,\ell_{i-r}\}} A_j \alpha_d = 0 \iff \prod_{i=1}^{d-1} A_j \alpha_d = 0.$$

We have deduced that condition (5.4) holds. We conclude that for every $x \in \mathbb{T}^r$,

$$\prod_{i=1}^{d} A_i x = (-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} \sum_{\substack{I \subseteq [d] \\ |I|=i}} \prod_{k \in I} A_k x.$$

Then,

$$\prod_{i=1}^{d} A_i = (-1)^d \sum_{i=0}^{d-1} (-1)^{i+1} \sum_{\substack{I \subseteq [d] \\ |I|=i}} \prod_{k \in I} A_k \implies \prod_{i=1}^{d} (A_i - I) = 0,$$

and condition 5.3 holds. □

# Perspectives

The main result of this thesis is the introduction of a new dynamical cubic structure for a minimal $\mathbb{Z}^d$-action on a compact metric space and, for a minimal distal system with the closing parallelepiped property, the proof of a structure theorem. These results extend the same results in the case of a $\mathbb{Z}^2$-action.

As we mentioned before, the systems with the closing parallelepiped property are the topological versions of the ones that control the behavior of multiple ergodic averages in the commuting case. It is a hard and open question to give a precise description of them, and it seems that nilsystems play a role in it. Some progress in this direction was made by T. Austin in [4, 5]. The work we have done here contributes to the understanding of such systems, giving a *partial* structure theorem for topological systems. We do not know if we can further improve the relatively independence result we have to get nilsystems involved in the picture.

In particular, we would to determine if the closing parallelepiped property is preserved under factor maps. This property was proved for two commuting transformations and even in that case the proof is non trivial. Some ideas to tackle these open problems would be to find a topological counterpart of the *magic systems* introduced by B. Host in [23], as was done in [7], and some tricky use of the enveloping semigroup.

An aspect that would be nice to see further developed in the future is the use of these cube structures to understand the group of automorphisms of a dynamical system, in particular, of some symbolic systems and tilings. In [7] S. Donoso and W. Sun used the cube structure $\mathbf{Q}_{S,T}(X)$ to understand the group of automorphisms of the Robinson tiling.

Another problem we would like to tackle is to study the maximal nilsystem factor of a system $(X, T_1, \ldots, T_d)$ with commuting transformations $T_1, \ldots, T_d$, and if there exists a relation with the factors defined in this thesis. Let $H$ be the group spanned by the transformations $T_{i,j} = T_i^{-1}T_j$. We have that in the factor ${}^{X}/_{\mathbf{Q}_H(X)}$ the transformations $T_1, \ldots, T_d$ are equal, so if the factor has the closing parallelepiped property, then this factor is a nilsystem, but this factor is not necessarily the maximal nilsystem factor of $X$. It is a hard and open question to give a precise description of them.

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