

Contents

Introduction	1
1 Background on topological dynamics	3
1.1 Basic definitions	3
1.1.1 Measure-preserving systems	3
1.1.2 Topological dynamical systems	5
1.2 Equicontinuous Systems	7
1.3 Distal Systems	8
1.4 Topological Weakly Mixing Systems	9
1.5 The Enveloping Semigroup of a Topological Dynamical System	10
2 Nilfactors and dynamical cubes	13
2.1 Multiple ergodic averages	13
2.2 Nilfactors and dynamical cubes for \mathbb{Z} -actions	14
2.2.1 Nilmanifolds and nilsystems	14
2.2.2 Topological cubes and the regionally proximal relation of order d	15
2.3 Dynamical cubes for \mathbb{Z}^2 -actions	17
3 Directional dynamical cubes for d commuting transformations	19
3.1 Notation	19
3.2 Directional dynamical cubes for d commuting transformations	21
3.2.1 Directional dynamical cubes	21
3.2.2 The (T_1, \dots, T_d) -regionally proximal relation	25
3.3 The classes $Z_0^{e_j}$	28
4 The structure theorem for minimal distal systems with the closing parallelepiped property	31
4.1 Directional dynamical cubes for minimal distal systems	31
4.1.1 The $Z_0^{e_j}$ -maximal factor for distal systems	32
4.1.2 The system $(\mathbf{K}_{T_1, \dots, T_d}^{x_0}, \mathcal{F}_{T_1, \dots, T_d}^{x_0})$ for distal systems	34
4.1.3 The (T_1, \dots, T_d) -regionally proximal relation for distal systems	38
4.2 Proof of Theorem 4.1	51
4.3 Recurrence in systems with the closing parallelepiped property	55
5 Examples of systems with the closing parallelepiped property	59
5.1 Affine transformations in the torus.	59
5.2 Proof of Lemma 5.3 and 5.4	61

Perspectives	79
Bibliography	83