Discrete Applied Mathematics 243 (2018) 140–153

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On distance-preserving elimination orderings in graphs: Complexity and algorithms[☆]

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ARTICLE INFO

Article history:
Received 30 March 2017
Received in revised form 17 January 2018
Accepted 19 February 2018
Available online 21 March 2018

Keywords:
Distance-preserving elimination ordering
Metric graph theory
NP-complete
Exact exponential algorithm
Integer linear programming
Bounded treewidth

ABSTRACT

For every connected graph G, a subgraph H of G is *isometric* if the distance between any two vertices in H is the same in H as in G. A *distance-preserving elimination ordering* of G is a total ordering of its vertex-set $V(G)$, denoted $(v_1, v_2, \ldots, v_n)$, such that any subgraph $G_i = G \setminus (v_1, v_2, \ldots, v_i)$ with $1 \leq i < n$ is isometric. This kind of ordering has been introduced by Chepoi in his study on weakly modular graphs (Chepoi, 1998). We prove that it is NP-complete to decide whether such ordering exists for a given graph — even if it has diameter at most 2. Then, we prove on the positive side that the problem of computing a distance-preserving ordering when there exists one is fixed-parameter-tractable in the treewidth. Lastly, we describe a heuristic in order to compute a distance-preserving ordering when there exists one that we compare to an exact exponential time algorithm and to an ILP formulation for the problem.

1. Introduction

Elimination orderings of a graph are total orderings of its vertex-set. Many interesting graph problems can be specified in terms of the existence of an elimination ordering with some given properties. These range from some practical problems in molecular biology and chemistry [8] to the analysis of graph search algorithms [14], the characterization of some graph classes [10,29], and the study of network clustering methods in social networks [26]. On the computational point of view, vertex ordering characterizations of a given graph class often lead to efficient (polynomial-time) recognition algorithms for the graphs in this class [2,6,15,21,28]. In this work we will consider one specific kind of elimination ordering that is called *distance-preserving elimination ordering* [11]. Precisely, let us remind that a subgraph H of a graph G is *isometric* if the distance between any two vertices in H is the same in H as in G. An elimination ordering $(v_1, v_2, \ldots, v_n)$ of G is distance-preserving if it satisfies that each suffix $(v_i, v_{i+1}, \ldots, v_n)$ with $i < n$ induces an isometric subgraph of G.

Distance-preserving elimination orderings encompass several other elimination orderings studied in the literature [6,7,19,24,25,28], all of which can be computed in polynomial time when they exist. In particular, known refinements of distance-preserving elimination orderings comprise the perfect elimination orderings [28], maximum neighbourhood orderings [6], h-extremal orderings [7], semisimplicial elimination orderings [24], dismantlable orderings [25] and more generally *domination elimination orderings* [19]. The latter orderings characterize chordal graphs, dually chordal graphs,

☆ This work has been supported by ANR project Stint under reference ANR-13-BS02-0007, ANR program "Investments for the Future" under reference ANR-11-LABX-0031-01, and the Inria associated team AlDyNet.
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https://doi.org/10.1016/j.dam.2018.02.007
0166-218X/

homogeneously orderable graphs, cop-win graphs and a subclass of tandem-win graphs [12], respectively, and as above stated they all can be computed in polynomial-time when they exist. However the complexity of deciding whether a distance-preserving elimination ordering exists in a given graph has been left open until this paper. We aim at completing the picture and characterizing the complexity of this problem.

Related work. In [17] it has been proved that every graph with a distance-preserving elimination ordering has a *minimum-size cycle basis* with only triangles and quadrangles, that can be easily computed if a distance-preserving elimination ordering is part of the input. This property has been useful in the study of some tree-likeness invariants of graphs (*e.g.*, in comparing treewidth with treelength). However, the complexity of recognizing graphs with a distance-preserving elimination ordering has been left open in [17]. Prior works [9,11] have focused on the existence of distance-preserving elimination orderings in some well-structured graph classes, *i.e.*, the *weakly modular graphs*. In particular, it has been proved recently in [9] that every breadth-first search ordering of a weakly modular graph is distance-preserving, that allows to compute one such ordering in linear time for a given graph in this class.

On the positive side, above stated refinements of distance-preserving elimination orderings [6,7,19,24,25,28] can all be computed with greedy algorithms when they exist. Indeed, for all these orderings it can be tested in polynomial-time whether a given vertex can be eliminated first. As an example, any dominated vertex can be the starting vertex of some domination elimination ordering (total ordering of the vertex-set where for every suffix, the closed neighbourhood of the first vertex is dominated in the subgraph induced by the suffix). The latter implies that any partial domination elimination ordering can be extended unless the graph does not admit such a total order. A first hint that computing a distance-preserving elimination ordering can be more difficult is that it is not that simple to choose a starting vertex. For instance, consider the wheel W_5 obtained from a cycle C_5 of length five by adding a universal vertex. Every elimination ordering of W_5 where the universal vertex is the last vertex eliminated is distance-preserving. However, if the universal vertex is eliminated first then the cycle C_5 is an isometric subgraph of W_5 that does not admit a distance-preserving elimination ordering. The above problem occurring with C_5 also occurs with *hypercubes*, that can be proved using tools from discrete geometry.[1]

Our contributions. We prove on the negative side that it is NP-complete to decide whether a given graph admits a distance-preserving elimination ordering (Section 3). The latter result may look surprising since as above stated, a broad range of distance-preserving orderings with additional properties can be computed in polynomial time when they exist. Then we show that the problem remains NP-complete even for general graphs with diameter at most two (Section 3.3). Note that in a sense our result is optimal w.r.t. the diameter because complete graphs trivially admit a distance-preserving ordering. Our reduction will show how to encode a 3-SAT formula in a graph whose distance-preserving orderings are in many-to-many correspondence with the satisfying assignments for the formula. This line of work resembles to the one in [31] in order to show that it is NP-complete to recognize collapsible complexes. Our work differs from theirs in that we study orderings with very distinct properties and the "simpler" structure of graphs – w.r.t. complexes – further constrains our gadgets to mimic variables and clauses of the formula.

On a more positive side, we prove in Section 4 that the problem of computing a distance-preserving ordering when there exists one is fixed-parameter-tractable in the treewidth.

Next, we show that a meta-theorem on vertex-orderings [3] can be applied to our problem, that results in an algorithm with $\mathcal{O}^*(2^n)$-time and space complexity, as well as in an algorithm with $\mathcal{O}^*(4^n)$-time and polynomial space complexity. We also propose an Integer Linear Programming formulation which may lead to a better running time in practice. These exact algorithms are described in Section 5 as well as simple heuristic algorithms.

Notations. Graphs in this study are finite, simple (hence without loop nor multiple edges) and unweighted. We refer to [5,20] for standard reference books on graphs (see also [1] for a survey about metric graph theory). Let $(v_1, v_2, \ldots, v_n)$ be an elimination ordering of a graph G, we say that vertex v_i, $1 \le i \le n$, is the ith vertex to be eliminated, and that vertex v_i is eliminated before vertex v_j, denoted $v_i \prec v_j$, if $i < j$.

2. Local characterization

In what follows, we will avoid considering all the distances in the graph at each time a vertex is eliminated. That is, we replace the "global" condition that $G \setminus v$ is isometric by a "local" one implying only the neighbours of v. The following characterization will explain how to do so.

Lemma 1. *Let $G = (V, E)$ and $v \in V$, the subgraph $G \setminus v$ is isometric if and only if every two non-adjacent neighbours of vertex v have at least two common neighbours in G (including v).*

Proof. If $G \setminus v$ is isometric, then let $x, y \in N_G(v)$ be non-adjacent. Since $d_{G \setminus v}(x, y) = d_G(x, y) = 2$, x and y have another common neighbour than vertex v. Conversely, suppose that every two non-adjacent neighbours of vertex v have at least

[1] More precisely, for the special case of an n-dimensional hypercube, the distance-preserving orderings are equivalent to the so-called "shellable orderings" as defined in [32]. In particular, if every partial distance-preserving ordering of the n-dimensional hypercube could be extended, then it would imply that its dual, the n-dimensional octahedron, is extendably shellable, that is known to be false for $n \ge 12$ [22].

two common neighbours in G. In particular, every of them have at least one common neighbour in $G \setminus v$. Then, for every two non-adjacent $x, y \in N_G(v)$ the subpath (x, v, y) can be substituted in any shortest-path of G with the subpath (x, u, y) of $G \setminus v$, where u denotes a common neighbour of x, y. This proves that $G \setminus v$ is an isometric subgraph. □

By using Lemma 1, one obtains the following characterization of distance-preserving elimination orderings. It can be seen as a reformulation of the characterization given in [11, Lemma 3.2] in terms of pseudopeakless functions.

Corollary 2. *An elimination ordering $\prec$ of $G = (V, E)$ is distance-preserving if and only if for every $u, v \in V$ at distance $d_G(u, v) = 2$, there is $w \in N_G(u) \cap N_G(v)$ such that $u \prec w$ or $v \prec w$.*

Proof. Let $(v_1, v_2, \ldots, v_n)$ be the elimination ordering we consider. For every $0 \le i < n$, define $G_i = G \setminus \{v_1, \ldots, v_{i-1}\}$ (in particular $G_0 = G$). On the one direction, suppose that $\prec$ is distance-preserving. Let $v_i, v_j \in V$ satisfy $d_G(v_i, v_j) = 2$ with $i < j$. Since $\prec$ is distance-preserving, G_i is an isometric subgraph of G. Hence, since $v_i, v_j \in V(G_i)$ and $d_G(v_i, v_j) = 2$, there exists $w \in N_G(v_i) \cap N_G(v_j)$ such that $w \in V(G_i)$, *i.e.*, $v_i \prec w$. On the other direction, suppose that $\prec$ is not distance-preserving. Let $i \ge 0$ be the least index such that G_i is an isometric subgraph of G but $G_{i+1} = G_i \setminus v_i$ is not. By Lemma 1, there exist $x, y \in N_{G_i}(v_i)$ nonadjacent such that $N_{G_i}(x) \cap N_{G_i}(y) \cap V(G_{i+1}) = \emptyset$. In particular, there does not exist any $w \in N_G(x) \cap N_G(y)$ such that $x \prec w$ or $y \prec w$ (else, $w \in V(G_{i+1})$). □

Finally, it may be easier sometimes to group vertices into subsets whose vertices can be eliminated in an arbitrary way. On such occasions, we will base on the following consequence of Lemma 1.

Corollary 3. *Let G be a graph, $S \subset V(G)$ satisfy that for every $v \in S$, every two non-adjacent neighbours of vertex v have a common neighbour in $G \setminus S$. Then, for any $S' \subseteq S$, the subgraph $G \setminus S'$ is isometric.*

Proof. By contradiction, let $S' \subseteq S$ falsify the corollary with S' being of minimum size w.r.t. this property. Let $v \in S'$, $S'' := S' \setminus v$. The subgraph $G \setminus S''$ is isometric by the minimality of S'. Furthermore, by the hypothesis every two non-adjacent neighbours of v have a common neighbour in $G \setminus S$, hence in $G \setminus S'$ so, $G \setminus S'$ is isometric by Lemma 1. This contradicts the fact that S' falsifies the corollary. □

3. Hardness results

The purpose of this section is to prove the following result.

Theorem 4. *Deciding whether a given graph G admits a distance-preserving elimination ordering is NP-complete, already if G has diameter at most two.*

Note that since the all-pairs-shortest-paths in a graph can be computed in polynomial-time then it easily follows that the problem is in NP and so, we will only prove the NP-hardness. We will first prove that deciding whether a given graph G admits a distance-preserving elimination ordering is NP-hard, already if G has diameter at most five. This first part of the proof is involved and it is based on a technical reduction from 3-SAT, the standard NP-complete problem [13]. Then, we will show how to lower the diameter to two (Section 3.3).

3.1. Main reduction

Given a formula Φ with n variables and m clauses of exactly three literals each, the 3-SAT problem aims at deciding whether there exists a boolean assignment of the variables which makes the formula true. In case it does, then the formula Φ is said satisfiable. We will construct a graph G_Φ from an arbitrary formula Φ so that there is a distance-preserving elimination ordering of G_Φ if and only if Φ is satisfiable. This will prove the NP-hardness of our problem. To achieve the result, assume w.l.o.g. that no literal and its negation can be contained in the same clause of Φ (else, any such clause could be removed from Φ), and every variable appears both positively and negatively in the clauses of Φ (else, any clause containing either this variable or its negation could also be removed from Φ). Let us denote by $x_1, x_2, \ldots, x_n$ the n variables, and by $C_1, C_2, \ldots, C_m$ the m clauses of Φ. The graph G_Φ is defined as follows.

Variable gadget. For every variable x_i, $1 \le i \le n$, let us add in G_Φ an induced quadrangle $(x_i, y_i, \bar{x}_i, \bar{y}_i)$ (*i.e.*, a cycle with four vertices). For every $1 \le j \le m$, if x_i is in the jth clause of the formula then four more vertices $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ are added and made adjacent to vertex x_i. Similarly if $\bar{x}_i$ is in the jth clause of the formula then four more vertices $\bar{a}_{ij}, \bar{b}_{ij}, \bar{c}_{ij}, \bar{d}_{ij}$ are added and made adjacent to vertex $\bar{x}_i$ (this is clearly defined because no clause contains both literals $x_i, \bar{x}_i$ by the hypothesis). We refer to Fig. 1 for an illustration.

To better understand the role played by the quadrangle $(x_i, y_i, \bar{x}_i, \bar{y}_i)$ in our reduction, we make the following observation that captures well the difficulty of the problem. Indeed, every vertex in a quadrangle can be chosen as the starting vertex of a distance-preserving ordering. However, the vertex diametrically opposed cannot be chosen as the second vertex to be eliminated. We will make use of a similar trick in our reduction so as to mimic a truth table with variable gadgets, ensuring that the second vertex to be eliminated in $x_i, \bar{x}_i$ must be eliminated after one of each pair $x_{i'}, \bar{x}_{i'}$ has already been eliminated for any $1 \le i' \le n$.

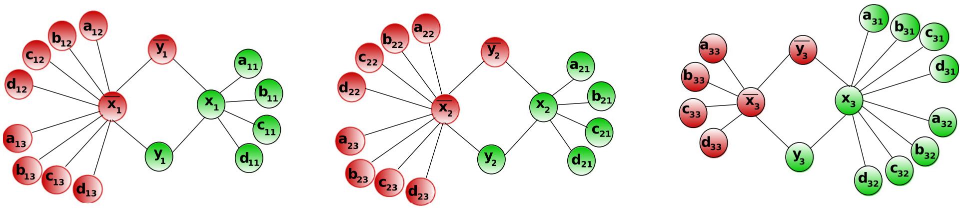


Fig. 1. The three variable gadgets for the formula $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\bar{x_1} \vee \bar{x_2} \vee \bar{x_3}) \wedge (\bar{x_1} \vee \bar{x_2} \vee x_3)$.

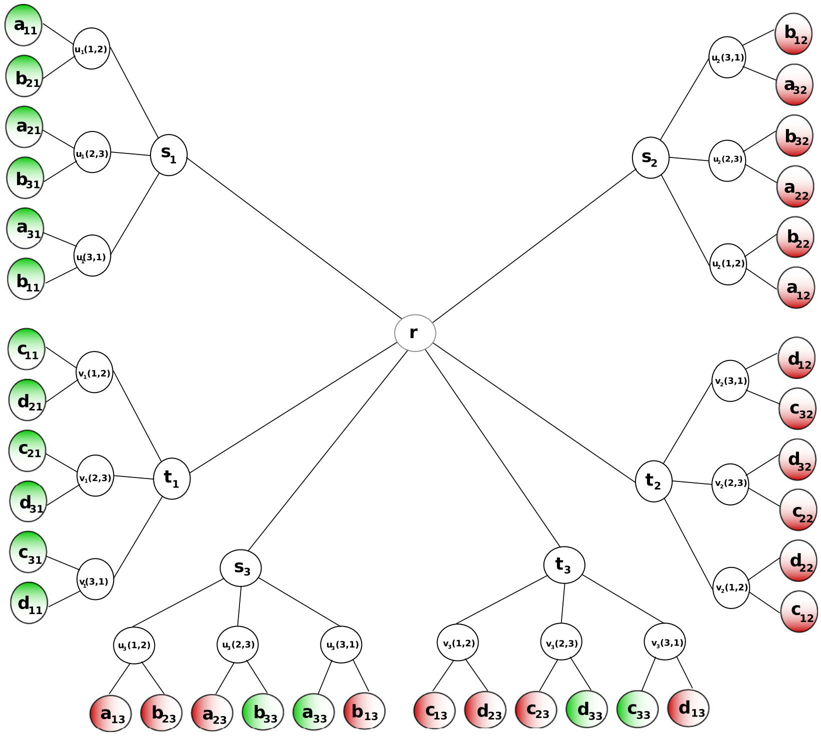


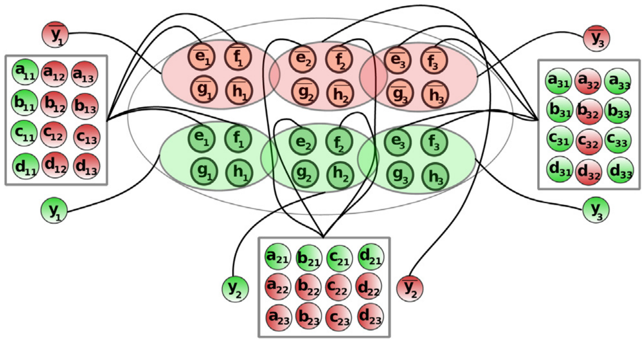
Fig. 2. The clause tree for the formula $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\bar{x_1} \vee \bar{x_2} \vee \bar{x_3}) \wedge (\bar{x_1} \vee \bar{x_2} \vee x_3)$.

Clause tree. Second, a rooted tree of depth two with $8m + 1$ vertices is added in G_Φ. More precisely, the tree is rooted at some newly added vertex r_Φ that has $2m$ children denoted by $s_1, t_1, s_2, t_2, \ldots, s_m, t_m$. Informally, for every $1 \le j \le m$ both nodes s_j, t_j represent the jth clause of Φ. Moreover let $C_j = l_p \vee l_q \vee l_r$ with $p < q < r$ and $l_i \in \{x_i, \bar{x}_i\}$ for every $i \in \{p, q, r\}$. Then, the internal node s_j has three children denoted by $u_j(p, q)$, $u_j(q, r)$ and $u_j(r, p)$, similarly the internal node t_j has three children denoted by $v_j(p, q)$, $v_j(q, r)$ and $v_j(r, p)$. Finally, let us describe how the clause tree is linked to the variable gadgets. Precisely, any leaf node $u_j(p, q)$ is made adjacent to the pair of vertices a_{pj}, b_{qj}, and in the same way any leaf node $v_j(p, q)$ is made adjacent to the pair of vertices c_{pj}, d_{qj}. We refer to Fig. 2 for an illustration.

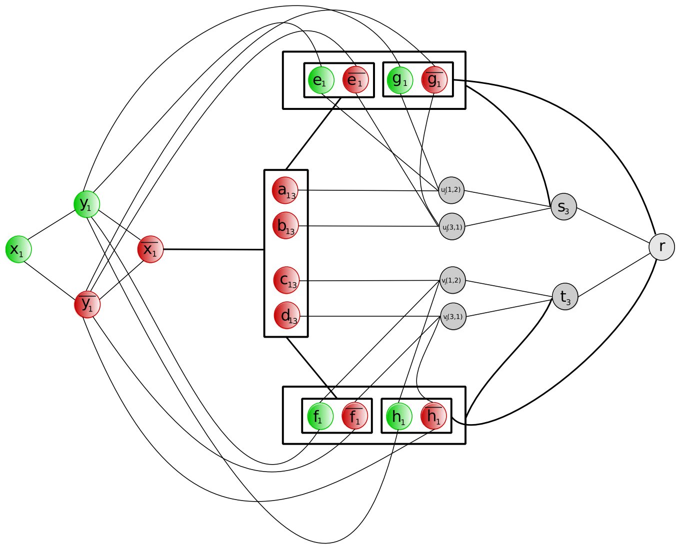
Our reduction will ensure that r_Φ is the unique common neighbour of s_j, t_j in G_Φ. Consequently, by Corollary 2 in any distance-preserving ordering of G_Φ one of s_j, t_j will need to precede vertex r_Φ. We will show that this implies that the jth clause of Φ is satisfied.

Literal clique. The final and most technical part of our reduction is to construct a clique of G_Φ with $8n$ vertices so as to ensure that a distance-preserving ordering exists if Φ is satisfiable. For every $1 \le i \le n$, the clique contains four vertices denoted by e_i, f_i, g_i, h_i (related to variable x_i). In the same way there are four vertices denoted by $\bar{e}_i, \bar{f}_i, \bar{g}_i, \bar{h}_i$ (related to the negated variable $\bar{x}_i$).

This clique is connected to variable gadgets as follows. Vertex y_i (in the ith variable gadget) is made adjacent to each of the four vertices e_i, f_i, g_i, h_i, and in the same way vertex $\bar{y}_i$ is made adjacent to each of the four vertices $\bar{e}_i, \bar{f}_i, \bar{g}_i, \bar{h}_i$. For every $1 \le j \le m$ such that one of $x_i, \bar{x}_i$ is a literal of C_j, the four vertices a_{ij}, b_{ij}, c_{ij} and d_{ij} are made adjacent to each of the four vertices e_i, f_i and $\bar{e}_i, \bar{f}_i$.



(a) Adjacency relations between vertices from the variable gadgets and those from the literal clique.



(b) Adjacency relations w.r.t. literal $\bar{x_1}$ and clause $C_3 = \bar{x_1} \vee \bar{x_2} \vee x_3$.

Fig. 3. The literal clique, for the formula $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\bar{x_1} \vee \bar{x_2} \vee \bar{x_3}) \wedge (\bar{x_1} \vee \bar{x_2} \vee x_3)$.

Then, the clique is connected to the clause tree as follows. For any $1 \le j \le m$, let $C_j = l_p \vee l_q \vee l_r$ with $p < q < r$ and $l_i \in \{x_i, \bar{x_i}\}$ for every $i \in \{p, q, r\}$, then:

- the three vertices $u_j(p, q), u_j(q, r), u_j(r, p)$ are, respectively, made adjacent to the 4-tuples of vertices: (e_p, g_p and $\bar{e_q}, \bar{g_q}$); (e_q, g_q and $\bar{e_r}, \bar{g_r}$); (e_r, g_r and $\bar{e_p}, \bar{g_p}$);
- similarly, the three vertices $v_j(p, q), v_j(q, r), v_j(r, p)$ are, respectively, made adjacent to the 4-tuples of vertices: (f_p, h_p and $\bar{f_q}, \bar{h_q}$); (f_q, h_q and $\bar{f_r}, \bar{h_r}$); (f_r, h_r and $\bar{f_p}, \bar{h_p}$);
- last, vertex s_j is made adjacent to the twelve vertices e_i, g_i and $\bar{e_i}, \bar{g_i}$ with $i \in \{p, q, r\}$; similarly, vertex t_j is made adjacent to the twelve vertices f_i, h_i and $\bar{f_i}, \bar{h_i}$ with $i \in \{p, q, r\}$.

Let $\mathcal{E} = \bigcup_{1\le i\le n}\{e_i, \bar{e_i}\}$, $\mathcal{F} = \bigcup_{1\le i\le n}\{f_i, \bar{f_i}\}$, $\mathcal{G} = \bigcup_{1\le i\le n}\{g_i, \bar{g_i}\}$ and $\mathcal{H} = \bigcup_{1\le i\le n}\{h_i, \bar{h_i}\}$ partition the clique. The root vertex r_Φ of the clause tree is made adjacent to every vertex in $\mathcal{G} \cup \mathcal{H}$. We refer to Fig. 3 for a partial illustration.

The resulting graph G_Φ has diameter at most five. Indeed, all vertices but the $x_i, \bar{x_i}$ with $1 \le i \le n$ are adjacent to the literal clique, therefore it is a 2-distance dominating clique. We will show later how to lower the diameter (Section 3.3). Note that several vertices play almost identical roles in the reduction. This redundancy is necessary in order to ensure that most pairs of vertices that are at distance two in G_Φ only have one common neighbour. Indeed, the latter will impose necessary conditions on an elimination ordering of G_Φ to be distance-preserving.

3.2. Proof of correctness

We are now ready to prove that it is NP-hard to decide whether a given graph G admits a distance-preserving elimination ordering. We divide the proof in two propositions, as follows.

Proposition 5. *If Φ is satisfiable, then G_Φ admits a distance-preserving ordering.*

Proof. Let us fix a boolean assignment of the variables x_i satisfying Φ, that exists by the hypothesis. In particular, let $\{l_i, \bar{l_i}\} = \{x_i, \bar{x_i}\}$ be such that l_i is true, let $V_0 = \{\bar{l_i} \mid 1 \le i \le n\}$ and let $V_1 = \{l_i \mid 1 \le i \le n\}$. Now, consider the following partition of the vertex-set of G_Φ into eleven subsets S_k, with $1 \le k \le 11$. Let $G_0 := G_\Phi$, and let $G_k := G_{k-1} \setminus S_k$ for every $1 \le k < 11$. We will exhibit from the partition a distance-preserving ordering of G_Φ. Precisely, we will prove that for every $1 \le k \le 11$ and for any total ordering S'_k of S_k, the elimination ordering $S'_1, S'_2, \dots, S'_{11}$ is distance-preserving.

The partition is defined as follows:

- The variable gadgets are partitioned into five subsets S_1, S_2, S_7, S_8, S_9. Furthermore, $S_1 = V_1 = \{l_i \mid 1 \le i \le n\}$, $S_8 = V_0 = \{\bar{l_i} \mid 1 \le i \le n\}$, $S_7 = \bigcup_{1 \le i \le n} \{y_i, \bar{y_i}\}$. The subsets S_2, S_9 contain the vertices $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ that are, respectively, adjacent to a vertex of V_1, V_0.
- The clause tree is partitioned into four subsets S_3, S_4, S_5, S_{10}. Furthermore, $S_5 = \{r_\Phi\}$, $S_4 = \{s_1, t_1, s_2, t_2, \dots, s_m, t_m\}$. The subset S_3 contains the vertices $u_j(p, q), v_j(p, q)$ such that the jth clause is satisfied by one of $l_p, l_q \in V_1$; similarly, the subset S_{10} contains the vertices $u_j(p, q), v_j(p, q)$ such that the jth clause is neither satisfied by l_p nor l_q.
- The literal clique is partitioned into two subsets $S_6 = \mathcal{G} \cup \mathcal{H}$ and $S_{11} = \mathcal{E} \cup \mathcal{F}$.

In what follows, we will prove that for every $1 \le k \le 11$, the pair $\langle G_{k-1}, S_k \rangle$ satisfies the sufficient condition of Corollary 3. The latter will prove, as claimed above, that for every $1 \le k \le 11$ and for any total ordering S'_k of S_k, the elimination ordering $S'_1, S'_2, \dots, S'_{11}$ is distance-preserving.

- Let $S_1 = V_1 = \{l_i \mid 1 \le i \le n\}$. Let $l_i \in S_1$. Neighbours of l_i in G_0 are $y_i, \bar{y_i}$ and every of $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ such that $l_i \in C_j$. Let $\alpha, \beta \in N_{G_0}(l_i)$ be non-adjacent. There are four subcases.
  - if $\{\alpha, \beta\} = \{y_i, \bar{y_i}\}$ then $\bar{l_i}$ is a common neighbour of α, β;
  - if one of α, β is equal to y_i and the other is amongst $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ for some j, then e_i, f_i are common neighbours of α, β;
  - similarly, if one of α, β is equal to $\bar{y_i}$ and the other is amongst $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ for some j, then $\bar{e_i}, \bar{f_i}$ are common neighbours of α, β;
  - if α is amongst $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ for some j, and β is amongst $a_{ij'}, b_{ij'}, c_{ij'}, d_{ij'}$ for some j', then e_i, f_i and $\bar{e_i}, \bar{f_i}$ are common neighbours of α, β.

  Therefore, in all cases α, β have a common neighbour in G_1, and Corollary 3 applies. In other words, $G_1 = G_\Phi \setminus S_1$ is isometric and for every total ordering S'_1 of S_1, for any prefix S''_1 of S'_1, $G_\Phi \setminus S''_1$ is isometric.
- Let S_2 contain every $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ such that clause C_j is satisfied by l_i. Let $w \in S_2$. There exist $j \le m$, $p < q < r \le n$ such that neighbours of w in G_1 are composed of $e_p, f_p, \bar{e_p}, \bar{f_p}$ and one of $u_j(p, q), u_j(r, p), v_j(p, q)$ or $v_j(r, p)$. Let $\alpha, \beta \in N_{G_1}(w)$ be non-adjacent. Note that w has only one neighbour in G_1 that is not in the literal clique. Consequently, one of α, β is amongst $u_j(p, q), u_j(r, p), v_j(p, q), v_j(r, p)$. Since the latter four vertices have some neighbour in the literal clique by construction, therefore α, β have a common neighbour in G_2 and Corollary 3 applies. In other words, $G_2 = G_1 \setminus S_1$ is isometric and for every total ordering S'_2 of S_2, for any prefix S''_2 of S'_2, $G_1 \setminus S''_2$ is isometric.
- Let S_3 contain $u_j(p, q), v_j(p, q)$ for every $j \le m$ and $p, q \le n$ such that one of l_p, l_q satisfies clause C_j. Let $w \in S_3$. There exist $j \le m$, $p, q \le n$ such that either $w = u_j(p, q)$ or $w = v_j(p, q)$. Two cases thus need to be distinguished:
  - Case $w = u_j(p, q)$. In particular, the neighbours of w in G_2 are $s_j, e_p, g_p, \bar{e_q}, \bar{g_q}$ and at most one amongst a_{pj}, b_{qj}. Furthermore, let $\alpha, \beta \in N_{G_2}(w)$ be non-adjacent. If $a_{pj} \in N_{G_2}(w)$ then $\alpha, \beta \in N_{G_2}[e_p]$, otherwise $\alpha, \beta \in N_{G_2}[\bar{e_q}]$.
  - Case $w = v_j(p, q)$. In particular, the neighbours of w in G_2 are $t_j, f_p, h_p, \bar{f_q}, \bar{h_q}$ and at most one amongst c_{pj}, d_{qj}. Furthermore, let $\alpha, \beta \in N_{G_2}(w)$ be non-adjacent. If $c_{pj} \in N_{G_2}(w)$ then $\alpha, \beta \in N_{G_2}[f_p]$, otherwise $\alpha, \beta \in N_{G_2}[\bar{f_q}]$.

  In both cases, any two non-adjacent neighbours α, β of w have a common neighbour in G_3 and so, Corollary 3 applies. In other words, $G_3 = G_2 \setminus S_3$ is isometric and for every total ordering S'_3 of S_3, for any prefix S''_3 of S'_3, $G_2 \setminus S''_3$ is isometric.
- Let $S_4 = \{s_1, t_1, s_2, t_2, \dots, s_m, t_m\}$ be the vertices representing each clause. Let $w \in S_4$. Clearly, there exists $j \le m$ such that either $w = s_j$ or $w = t_j$. Furthermore, by the choice of a boolean assignment satisfying Φ, there exists $l_p \in S_1$ satisfying C_j. Up to cyclic permutation of the indices for the variables, this implies by construction $u_j(p, q), u_j(r, p), v_j(p, q), v_j(r, p) \in S_3$ for some $q, r > p$. Two cases need to be distinguished.
  - Case $w = s_j$. In particular, the neighbours of w in G_3 are r_Φ, the twelve vertices $e_i, g_i, \bar{e_i}, \bar{g_i}$ with $i \in \{p, q, r\}$, and possibly $u_j(q, r)$. Recall that r_Φ is adjacent to all the vertices of $\mathcal{G} \cup \mathcal{H}$, furthermore $u_j(q, r)$ is adjacent to e_q, g_q and $\bar{e_r}, \bar{g_r}$. Therefore, $N_{G_3}(w) \subseteq N[g_q] \cap N[\bar{g_r}]$. In this situation for any two non-adjacent neighbours α, β of w in G_3 we have $\alpha, \beta \in N_{G_3}[g_q]$ (resp., $\alpha, \beta \in N_{G_3}[\bar{g_r}]$).

    Let us point out that in the full graph G, the two vertices $u_j(p, q)$ and $u_j(r, p)$ are also neighbours of $w = s_j$ in G. Furthermore, by construction w is the unique common neighbour of $u_j(p, q)$ and $u_j(r, p)$ in G. Hence, it is crucial that since Φ is satisfiable, and so, C_j is satisfied by some literal l_p, the two vertices $u_j(p, q)$ and $u_j(r, p)$ are eliminated in S_3.

– Case $w = t_j$. In particular, the neighbours of w in G_3 are r_Φ, the twelve vertices $f_i, h_i, \bar{f}_i, \bar{h}_i$ with $i \in \{p, q, r\}$, and possibly $v_j(q, r)$. Recall that r_Φ is adjacent to all the vertices of $\mathcal{G} \cup \mathcal{H}$, furthermore $v_j(q, r)$ is adjacent to f_q, h_q and $\bar{f}_r$, $\bar{h}_r$. Therefore, $N_{G_3}(w) \subseteq N[h_q] \cap N[\bar{h}_r]$. In this situation for any two non-adjacent neighbours α, β of w in G_3 we have $\alpha, \beta \in N_{G_3}[h_q]$ (resp., $\alpha, \beta \in N_{G_3}[\bar{h}_r]$).

As before, let us point out that in the full graph G, the two vertices $v_j(p, q)$ and $v_j(r, p)$ are also neighbours of $w = t_j$ in G. Furthermore, by construction w is the unique common neighbour of $v_j(p, q)$ and $v_j(r, p)$ in G. Hence, it is crucial that since Φ is satisfiable, and so, C_j is satisfied by some literal l_p, the two vertices $v_j(p, q)$ and $v_j(r, p)$ are eliminated in S_3.

In both cases, any two non-adjacent neighbours α, β of w have a common neighbour in G_4. By Corollary 3, $G_4 = G_3 \setminus S_4$ is isometric and for every total ordering S_4' of S_4, for any prefix S_4'' of S_4', $G_3 \setminus S_4''$ is isometric.

- Let $S_5 = \{r_\Phi\}$. By construction the neighbourhood of r_Φ in the full graph G is equal to $S_4 \cup \mathcal{G} \cup \mathcal{H}$. Note that for every $j \le m$, r_Φ is the unique common neighbour of s_j and t_j in G, hence $G \setminus r_\Phi$ is not isometric. However, since all vertices in S_4 have been eliminated at this step, r_Φ is simplicial in G_4, *i.e.*, its neighbourhood $N_{G_4}(r_\Phi) = \mathcal{G} \cup \mathcal{H}$ induces a complete subgraph. It is thus straightforward that Corollary 3 applies. In other words, $G_5 = G_4 \setminus r_\Phi$ is isometric.
- Let $S_6 = \mathcal{G} \cup \mathcal{H}$. Let $w \in S_6$. There are four cases to be considered.

  – If $w = g_i$ for some i, then neighbours of g_i in G_5 are those in the literal clique, vertex y_i and every $u_j(i, q) \notin S_3$. Therefore, $N_{G_5}[w] \subseteq N_{G_5}[e_i]$;
  – if $w = \bar{g}_i$ for some i, then neighbours of $\bar{g}_i$ in G_5 are those in the literal clique, vertex $\bar{y}_i$ and every $u_j(p, i) \notin S_3$. Therefore, $N_{G_5}[w] \subseteq N_{G_5}[\bar{e}_i]$;
  – if $w = h_i$ for some i, then neighbours of h_i in G_5 are those in the literal clique, vertex y_i and every $v_j(i, q) \notin S_3$. Therefore, $N_{G_5}[w] \subseteq N_{G_5}[f_i]$;
  – else, $w = \bar{h}_i$ for some i, hence neighbours of $\bar{h}_i$ in G_5 are those in the literal clique, vertex $\bar{y}_i$ and every $v_j(p, i) \notin S_3$. Therefore, $N_{G_5}[w] \subseteq N_{G_5}[\bar{f}_i]$.

  Since, $e_i, \bar{e}_i, f_i, \bar{f}_i \in V(G_6)$, therefore Corollary 3 applies. In other words, $G_6 = G_5 \setminus S_6$ is isometric and for every total ordering S_6' of S_6, for any prefix S_6'' of S_6', $G_5 \setminus S_6''$ is isometric.
- Let S_7 contain $y_i, \bar{y}_i$ for every $1 \le i \le n$. Let $w \in S_7$. There is some i such that neighbours of w in G_6 are vertex $\bar{l}_i$ and either e_i, f_i (if $w = y_i$) or $\bar{e}_i, \bar{f}_i$ (if $w = \bar{y}_i$). Moreover, recall that we assume the existence of some $1 \le j \le m$ such that $\bar{l}_i$ appears in clause C_j. Indeed, all variables are assumed to appear positively and negatively in the clauses of Φ. In particular, by construction $a_{ij}, b_{ij}, c_{ij}, d_{ij} \notin S_2$ and so, $a_{ij}, b_{ij}, c_{ij}, d_{ij} \in V(G_6)$. The latter four vertices are adjacent to every of $\bar{l}_i$, e_i, f_i and $\bar{e}_i, \bar{f}_i$ by construction of G_Φ. As a result, for any $\alpha, \beta \in N_{G_6}(w)$ non-adjacent, α, β have a common neighbour in G_7 and so, Corollary 3 applies. In other words, $G_7 = G_6 \setminus S_7$ is isometric and for every total ordering S_7' of S_7, for any prefix S_7'' of S_7', $G_6 \setminus S_7''$ is isometric.
- Let $S_8 = V_0 = \{\bar{l}_i \mid 1 \le i \le n\}$. Let $\bar{l}_i \in S_8$. Neighbours of $\bar{l}_i$ in G_7 are those $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ such that $\bar{l}_i$ appears in C_j. Every such neighbour is adjacent to the 4-tuple $e_i, f_i, \bar{e}_i, \bar{f}_i$ of the literal clique, hence Corollary 3 applies. In other words, $G_8 = G_7 \setminus S_8$ is isometric and for every total ordering S_8' of S_8, for any prefix S_8'' of S_8', $G_7 \setminus S_8''$ is isometric.
- Let S_9 contain every $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ such that $\bar{l}_i$ appears in C_j. The proof for this case is similar as for S_2. Let $w \in S_9$. There are $j \le m,\ p < q < r \le n$ such that neighbours of w in G_8 are $e_p, f_p, \bar{e_p}, \bar{f_p}$ and at most one of $u_j(p, q)$, $u_j(r, p)$, $v_j(p, q)$ or $v_j(r, p)$. Let $\alpha, \beta \in N_{G_8}(w)$ be non-adjacent. Necessarily, one of α, β must be one of $u_j(p, q)$, $u_j(r, p)$, $v_j(p, q)$, $v_j(r, p)$ because any other neighbour of w is in the literal clique. Furthermore, $u_j(p, q)$, $u_j(r, p)$, $v_j(p, q)$, $v_j(r, p)$ are, respectively, adjacent to e_p, e_r, f_p, f_r in the literal clique, that are part of $\mathcal{E} \cup \mathcal{F}$ and so, have not been eliminated with S_6. Therefore, α, β have a common neighbour in G_9 and so, Corollary 3 applies. In other words, $G_9 = G_8 \setminus S_9$ is isometric and for every total ordering S_9' of S_9, for any prefix S_9'' of S_9', $G_8 \setminus S_9''$ is isometric.
- Let S_{10} contain every $u_j(p, q)$, $v_j(p, q)$ such that $\bar{l}_p$, $\bar{l}_q$ appear in C_j. Equivalently, those are all of $u_j(p, q)$, $v_j(p, q)$ but the ones already in S_3. Let $w \in S_{10}$. There exist j, p, q such that neighbours of w in G_9 are either $e_p, \bar{e}_q$ (if $w = u_j(p, q)$) or $f_p, \bar{f}_q$ (if $w = v_j(p, q)$). As a result, vertex w is simplicial. It thus follows that Corollary 3 trivially applies. In other words, $G_{10} = G_9 \setminus S_{10}$ is isometric and for every total ordering S_{10}' of S_{10}, for any prefix S_{10}'' of S_{10}', $G_9 \setminus S_{10}''$ is isometric.
- Finally, let $S_{11} = \mathcal{E} \cup \mathcal{F}$, this is a clique and so, the vertices in S_{11} can be eliminated sequentially while leaving a sequence of isometric subgraphs.

To sum up, one obtains a distance-preserving ordering of G_Φ by sequentially eliminating vertices in S_1 then in S_2 and so on until S_{11}, in an arbitrary way. □

Proposition 6. *If G_Φ admits a distance-preserving elimination ordering, then Φ is satisfiable.*

Proof. Let $\prec$ be a distance-preserving ordering of G_Φ. For every $1 \le j \le m$ we claim that there is $1 \le i \le n$ such that some $l_i \in \{x_i, \bar{x}_i\}$ satisfies clause C_j, and $l_i \prec r_\Phi$. Then, we will prove that this implies a boolean assignment of the variables satisfying Φ by showing that $r_\Phi \prec \bar{l}_i$, where $\{l_i, \bar{l}_i\} = \{x_i, \bar{x}_i\}$.

To prove the claim, first observe that for every $1 \le j \le m$, r_Φ is the unique common neighbour of s_j, t_j in G_Φ. By Corollary 2, it implies $s_j \prec r_\Phi$ or $t_j \prec r_\Phi$. So, assume $s_j \prec r_\Phi$ (the case $t_j \prec r_\Phi$ is symmetrical to this one). Let $u_j(p, q)$, $u_j(q, r)$, $u_j(r, p)$

be the three children of s_j in the clause tree. Note that the latter three vertices pairwise share s_j as their unique common neighbour in G_Φ. Consequently, by Corollary 2 (applied twice) at least two of them must be eliminated before s_j. W.l.o.g., let $u_j(p, q)$ be eliminated before s_j. In such case, note that $u_j(p, q)$ is the unique common neighbour of a_{pj}, b_{qj} by construction of G_Φ. Therefore, by Corollary 2, $a_{pj} \prec u_j(p, q)$ or $b_{qj} \prec u_j(p, q)$. Suppose by symmetry that $a_{pj} \prec u_j(p, q)$. Let $l_p \in \{x_p, \bar{x_p}\}$ appear in C_j. Since l_p and $u_j(p, q)$ share a_{pj} as their unique common neighbour and $a_{pj} \prec u_j(p, q)$, by Corollary 2 $l_p \prec a_{pj} \prec r_\Phi$, that finally proves the claim.

To conclude let us prove for every $1 \leq i \leq n$, there is $l_i \in \{x_i, \bar{x}_i\}$ such that either $l_i \prec r_\Phi \prec \bar{l}_i$ or $r_\Phi \prec l_i \prec \bar{l}_i$. If so, then let us consider any boolean assignment of the variables satisfying for every $1 \leq i \leq n$, l_i is assigned true if $l_i \prec r_\Phi$ (note that if $r_\Phi \prec l_i \prec \bar{l}_i$, then x_i can be valuated in an arbitrary way). Since by the above claim, for every $1 \leq j \leq m$, there is $l_i \prec r_\Phi$ satisfying clause C_j, therefore any such assignment satisfies the formula Φ. By way of contradiction, suppose $l_i \prec \bar{l}_i \prec r_\Phi$ with $\{l_i, \bar{l}_i\} = \{x_i, \bar{x}_i\}$ for some $1 \leq i \leq n$. Since $y_i, \bar{y}_i$ share $x_i, \bar{x}_i$ as their only two common neighbours in G_Φ, by Corollary 2 $y_i \prec \bar{l}_i$ or $\bar{y}_i \prec \bar{l}_i$. Suppose by symmetry $y_i \prec \bar{l}_i$. Then, since y_i is the unique common neighbour between $\bar{l}_i$ and g_i, h_i, we have by Corollary 2 that $g_i \prec y_i$ and $h_i \prec y_i$. However, we claim that the combination of $g_i \prec y_i \prec r_\Phi$ and $h_i \prec y_i \prec r_\Phi$ contradicts the fact that $\prec$ is distance-preserving. Indeed, g_i, h_i are the only two common neighbours of r_Φ and y_i, so, this contradicts Corollary 2. $\square$

3.3. Reduction to graphs with diameter at most two

As stated before, the graph G_Φ resulting from our reduction in Section 3.1 has diameter at most five. In this section, we improve the result by lowering the diameter to two, thereby proving Theorem 4.

We base on the local view of Corollary 2, which states that in order to obtain a distance-preserving ordering of G it is necessary and sufficient to ensure that vertices at distance two in G still have a common neighbour in the graph at each time a vertex is eliminated. This motivates the following Definition 7 — to embed any graph G into a graph G' with diameter at most two such that any two vertices at distance two in G have the same set of common neighbours in G and G'.

Definition 7. Let G be a connected graph with n vertices, let $\mathcal{H} = \{\{u, v\} \mid u, v \in V(G) \text{ and } d_G(u, v) \geq 3\}$ and let $p = |\mathcal{H}|$. The graph G' is obtained from G by adding a clique Z of $n + p$ vertices, defined as follows.

For every vertex $v \in V(G)$, there is $z_v \in Z$ that is adjacent to v in $V(G)$.

For every $u, v \in V(G)$ such that $d_G(u, v) \geq 3$, *i.e.*, $\{u, v\} \in \mathcal{H}$, there is $z_{uv} \in Z$ that is adjacent to u, v in $V(G)$.

Lemma 8. *For any connected graph G, let G' be as in Definition 7, G' has diameter at most two.*

Proof. Let $u, v \in V(G')$. If $u \in Z$ or $v \in Z$ then $d_{G'}(u, v) \leq 2$ because either $u, v \in Z$ are adjacent or, w.l.o.g., $u \in Z$ and $z_v \in Z$ is a common neighbour of u, v in G' by Definition 7. Else, $u, v \in V(G)$ and so, $d_{G'}(u, v) \leq d_G(u, v)$ because G is an induced subgraph of G'. Moreover, if $d_G(u, v) \geq 3$ then by Definition 7 there is $z_{uv} \in Z$ adjacent to u, v in G', therefore $d_{G'}(u, v) = 2$. $\square$

Lemma 9. *For any connected graph G, let G' be as in Definition 7, G admits a distance-preserving ordering if and only if G' admits one.*

Proof. Let $(v_1, v_2, \ldots, v_n)$ be a distance-preserving ordering of G. For every $1 \leq i < n$, let $G_i := G \setminus (v_1, \ldots, v_i)$ be an isometric subgraph of G, let G'_i be the subgraph of G' induced by $V(G_i) \cup Z$ (by convention, $G_0 := G$, $G'_0 := G'$). We claim that for every $1 \leq i < n$, G'_i is an isometric subgraph of G'. Note that if the claim holds, then $(v_1, v_2, \ldots, v_n)$ can be completed into a distance-preserving ordering of G' as follows: vertices $v_1, v_2, \ldots, v_n$ are sequentially eliminated, then vertices of the clique Z are eliminated in an arbitrary way.[2] To prove the claim, by Lemma 1 it suffices to prove that any two $x, y \in N_{G'_{i-1}}(v_i)$ non-adjacent share a common neighbour in G'_i. If $x, y \in V(G_{i-1})$, then by Lemma 1 they share a common neighbour in G_i, hence in G'_i. Else, one of x, y is in Z, w.l.o.g. say $x \in Z$ and so, $z_y \in Z$ is a common neighbour of x, y in G'_i.

Conversely, let G' admit a distance-preserving ordering. Let $\prec$ be a distance-preserving elimination ordering of G', and let us consider the restriction $(v_1, v_2, \ldots, v_n)$ of the total ordering $\prec$ to the vertices of G. We claim that it is a distance-preserving elimination ordering of G. By contradiction, let i be the least index such that $G_i := G \setminus (v_1, v_2, \ldots, v_i)$ is not an isometric subgraph of G (by convention, $G_0 := G$). Let j be such that v_i is the jth vertex to be eliminated in G' w.r.t. $\prec$, and let G'_j be obtained from G' by removing the j first vertices to be eliminated in G' w.r.t. $\prec$. Note that G'_j is an isometric subgraph of G' because $\prec$ is distance-preserving by the hypothesis. Moreover, since $(v_1, v_2, \ldots, v_n)$ is assumed not to be distance-preserving, then by Lemma 1, there exist $x, y \in N_{G_{i-1}}(v_i)$ non-adjacent whose unique common neighbour in the subgraph G_{i-1} is v_i. In such case, $d_G(x, y) = 2$, therefore x, y have no common neighbour in the clique Z by Definition 7. However, $V(G_i) \subseteq V(G'_j) \subseteq V(G_i) \cup Z$ by construction, therefore x, y have no common neighbour in G'_j, that contradicts the fact that G'_j is an isometric subgraph of G' by Lemma 1. $\square$

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[2] In fact, if vertices $z_{v_1}, z_{v_2}, \ldots, z_{v_n}$ are the last removed in Z then one obtains a breadth-first search ordering rooted at z_{v_n}. This proves that the problem of deciding whether there exists a breadth-first search ordering that is distance-preserving is NP-complete.

Altogether, we can now prove our main result as follows.

Proof of Theorem 4. The problem is in NP. In order to prove the NP-hardness, let Φ be any instance for 3-SAT. The graph G_Φ, described in Section 3.1, can be constructed from Φ in polynomial time. Furthermore, by the combination of Propositions 5 and 6, G_Φ admits a distance-preserving ordering if and only if Φ is satisfiable. Finally, let G'_Φ be obtained from G_Φ as defined in Definition 7. By Lemma 8, G'_Φ has diameter at most two, furthermore by Lemma 9, G'_Φ admits a distance-preserving ordering if and only if G_Φ admits one, that is if and only if Φ is satisfiable. Since 3-SAT is NP-complete [13], this proves the hardness and so, the result. $\Box$

4. A polynomial case

In this section, we prove that the problem of computing a distance-preserving ordering when there exists one is fixed-parameter-tractable in the treewidth.

A *tree-decomposition* $(T, \mathcal{X})$ of a graph $G = (V, E)$ is a pair consisting of a tree T and of a family $\mathcal{X} = (X_t)_{t \in V(T)}$ of subsets of V indexed by the nodes of T and satisfying:

(i) $\bigcup_{t \in V(T)} X_t = V$;
(ii) for any edge $e = \{u, v\} \in E$, there exists $t \in V(T)$ such that $u, v \in X_t$;
(iii) for any $v \in V$, $\{t \in V(T) \mid v \in X_t\}$ induces a subtree, denoted by T_v, of T.

The sets X_t are called *the bags* of the decomposition. Furthermore, the *width* of $(T, \mathcal{X})$ is equal to $\max_{t \in V(T)} |X_t| - 1$, and the *treewidth* of G is the minimum possible width of its tree-decompositions.

It is well-known that many NP-hard problems are fixed-parameter tractable (FPT) in the treewidth [18]. Furthermore, the existence of distance-preserving orderings has been proved useful in the comparative study of treewidth with some other properties of the tree-decompositions of graphs [17]. We prove that it can be decided in polynomial-time whether a given bounded treewidth graph admits a distance-preserving ordering, and if so, one such ordering can also be computed in polynomial-time. More precisely, we prove in what follows that the problem is FPT with the treewidth as parameter.

Theorem 10. *For every $G = (V, E)$ with treewidth at most k, it can be decided whether a distance-preserving ordering exists in time $2^{2^{\mathcal{O}(k)}} \cdot n^{\mathcal{O}(1)}$. Furthermore, if it is the case, then a distance-preserving ordering for G can also be computed within the same amount of time.*

Proof. For simplicity, we will work on a specific kind of tree-decompositions, called nice tree-decompositions. A tree-decomposition $(T, \mathcal{X})$ is *nice* if T is rooted in some node $r \in V(T)$, any node of T has at most two children and, for any $t \in V(T)$,

- either t is a leaf of T and $|X_t| = 1$ (*Leaf Node*);
- or t has one child u and there exists $v \in V$ such that $X_t = X_u \setminus \{v\}$ (*Forget Node*);
- or t has one child u and there exists $v \in V$ such that $X_t = X_u \cup \{v\}$ (*Introduced Node*);
- or t has two children u and w and $X_u = X_w = X_t$ (*Join Node*).

In what follows, let $(T, \mathcal{X})$ be a nice tree-decomposition of width $\mathcal{O}(k)$. It can be computed in time $2^{\mathcal{O}(k)} n$ [4]. For every $t \in V(T)$, let T_t be the subtree rooted at node t and let $V_t = \bigcup_{u \in T_t} X_u$. We aim at computing all the orderings on V_t that can be extended to a distance-preserving ordering of G. In order to do so, we will represent an ordering on V_t as follows:

- its subordering $\prec_t$ on X_t;
- the collection $\mathcal{C}_t$ of pairs $(N(v) \cap X_t, pos_v)$ for every $v \in V_t \setminus X_t$, where pos_v is the number of neighbours in $N(v) \cap X_t$ preceding vertex v;
- finally, a set $\mathcal{P}_t$ of pairs $x, y \in X_t$ at distance two in G such that both x and y are preceded by all their common neighbours in V_t.

Note that for any *fixed* vertex $v \in V_t$, there are $2^{\mathcal{O}(k)}$ possibilities for $N(v) \cap X_t$ and $\mathcal{O}(k)$ possibilities for pos_v. In particular, since $\mathcal{C}_t$ can be any subset of a set with $\mathcal{O}(k)2^{\mathcal{O}(k)}$ elements, there are $2^{\mathcal{O}(k)2^{\mathcal{O}(k)}}$ possibilities for $\mathcal{C}_t$. Overall there are $k! \cdot 2^{\mathcal{O}(k)2^{\mathcal{O}(k)}} \cdot \mathcal{O}(k^2) = 2^{2^{\mathcal{O}(k)}}$ possible representations.

Intuitively, we aim at computing for every node $t \in V(T)$ the suborderings $\prec_t$ of V_t that could be potentially extended to a distance-preserving elimination ordering of G. In order to do so, let $\prec$ be any distance-preserving elimination ordering of G, let $t \in V(T)$ and let $\prec_t$ be the subordering of $\prec$ constrained to V_t. By Corollary 2, for every $x, y \in V_t$ at distance two in G, there exists a common neighbour $z \in N_G(x) \cap N_G(y)$ such that either $x \prec z$ or $y \prec z$. Furthermore, if $z \in V_t$ then we have either $x \prec_t z$ or $y \prec_t z$, otherwise since $z \notin V_t$ we have by the properties of a tree-decomposition that $x, y \in X_t$. Hence, we will consider a representation to be valid at node $t \in V(T)$ if it represents an ordering $\prec'_t$ of V_t with the following property: for every $x, y \in V_t$ at distance two in G, there exists a common neighbour $z \in N_G(x) \cap N_G(y)$ such that either $\{x, y\} \in \mathcal{P}_t$ and $z \in V \setminus V_t$, or $z \in V_t$ and one of x or y precedes z w.r.t. $\prec'_t$.

For every $t \in V(T)$, the following algorithm will compute *all* the valid representations at node t. Let us observe that for every subordering $\prec_t$ of V_t and for any child $u \in V(T)$ of t, if $\prec_t$ has a valid representation at node t then its restriction $\prec_u$ to V_u also has a valid representation at node u. We will use this observation in what follows in order to compute the valid representations at every node by dynamic programming. Furthermore, if $\prec$ is a distance-preserving ordering of G then as proved above, for every $t \in V(T)$ its restriction $\prec_t$ to V_t has a valid representation at node t. Conversely, by Corollary 2 a valid representation at the root is equivalent to the existence of a distance-preserving ordering of G. Therefore, the valid representations at the root are exactly the representations of distance-preserving orderings of G, and so, the following algorithm is correct.

- **Case of a Leaf Node.** In this situation, $V_t = X_t = \{v\}$ for some $v \in V$. So, there is a unique valid representation $(\prec_t = (v), \mathcal{C}_t = \emptyset, \mathcal{P}_t = \emptyset)$.
- **Case of a Forget Node.** Let $u \in V(T)$ be the unique child of node t and let $v \in V$ be such that $X_t = X_u \setminus \{v\}$. Consider any valid representation at node u.

  If there is a pair $\{x, v\}$ containing v in $\mathcal{P}_u$ then we claim that it cannot be extended to a valid representation at node t. Indeed, since $v \in X_u \setminus X_t$ it has no neighbour in $V \setminus V_t$ (by Property (ii) of tree-decompositions). Therefore, given any subordering on V_t that is mapped to this representation, the two vertices v and x are eliminated after all their common neighbours in *any* extension of this subordering to a total ordering on V. The latter falsifies the characterization of distance-preserving orderings given in Corollary 2, that proves the claim.

  Else, there is no pair of $\mathcal{P}_u$ containing v. In this situation, the representation can be transformed into a valid representation at node t by taking the restriction of $\prec_u$ to X_t and by constructing $\mathcal{C}_t$ as follows. First let us add the pair $(N(v) \cap X_t, pos_v)$ in $\mathcal{C}_t$, that can be easily computed from $\prec_u$ (recall that pos_v is the number of neighbours of v in X_u that are preceding v w.r.t. $\prec_u$). Then for every pair $(N, p) \in \mathcal{C}_u$, either v is among the p first neighbours in N w.r.t. $\prec_u$, in which case let us add $(N \setminus v, p-1)$ in $\mathcal{C}_t$, or let us add $(N \setminus v, p)$ in $\mathcal{C}_t$.
- **Case of an Introduced Node.** Let $u \in V(T)$ be the unique child of node t and let $v \in V$ be such that $X_t = X_u \cup \{v\}$. Consider any valid representation at node u. We consider the $\mathcal{O}(k)$ possible ways to insert v w.r.t. $\prec_u$, in order to obtain the subordering $\prec_t$. For every $\prec_t$, we need to consider all vertices in V_t that are at distance two from v. We distinguish between two subcases.

  - First, let $Y_u \subseteq X_u$ contain all the vertices x of X_u that are at distance two from v. For every $x \in Y_u$, we check whether there exists a common neighbour z such that either $z \in X_u$ and it is preceded by one of x or v (this can be checked with $\prec_t$), or $z \notin V_t$. If no such vertex z exists (that means that all common neighbours are in V_t and they all preceed x and v in the current ordering) then we claim that it is not possible to extend to a valid representation at node t. Indeed, let us fix an arbitrary extension $\prec$ of $\prec_t$ to a total ordering on V. Suppose by way of contradiction that $\prec$ is distance-preserving. By Corollary 2, there exists $z' \in N(v) \cap N(x)$ such that $x \prec z'$ or $v \prec z'$. Furthermore, $z' \in V_t$ (else, we could choose $z = z'$, that is a contradiction). By Property (ii) of tree-decompositions, v has no neighbours in $V_t \setminus X_u$, and so, $z' \in X_u$. However, since $\prec$ is an extension of $\prec_t$, the latter implies that $x \prec_t z'$ or $v \prec_t z'$. Hence we could choose $z = z'$, that is a contradiction. Therefore, the claim is proved.

    Otherwise, there exists a common neighbour z as defined above. In this situation, we will need to add the pair $\{x, v\}$ in $\mathcal{P}_t$ if and only if all possible choices for z are in $V \setminus V_t$. Note that after iterating on all the vertices of Y_u, we will also need to complete $\mathcal{P}_t$ with the pairs $\{x, y\} \in \mathcal{P}_u$ such that x and y have a common neighbour in $V \setminus V_t$ and they are preceded by all their common neighbours in $V_t = V_u \cup \{v\}$. Furthermore, we need to check that for all the pairs $\{x, y\} \in \mathcal{P}_u \setminus \mathcal{P}_t$, vertex v is a common neighbour of x and y such that either $x \prec_t v$ or $y \prec_t v$ (otherwise, we cannot extend to a valid representation and $\prec_t$ can be discarded).
  - Second, let us consider all vertices $x \in V_t \setminus X_u$ that are at distance two from v. Note that since by Property (ii) of tree-decompositions v has no neighbours in $V_t \setminus X_u$, we have that for every $x \in V_t \setminus X_u$, x is at distance two from v if and only if $(N(x) \cap X_u) \cap N(v) \neq \emptyset$. Precisely, all the common neighbours of v and x are in X_u (and in X_t). So, let us consider all the pairs $(N, p) \in \mathcal{C}_u$ such that $N \cap N(v) \neq \emptyset$ (intuitively, this corresponds to a vertex $x \in V_t \setminus X_u$ that has a common neighbour with v).

    For every such pair (N, p), let us define N^+ as the subset obtained from N by removing its p first vertices w.r.t. $\prec_u$. Similarly, let $N^+(v)$ be the vertices of $N(v) \cap X_u$ that are preceded by v w.r.t. $\prec_t$. We check whether either $N^+(v) \cap N \neq \emptyset$ or $N^+ \cap N(v) \neq \emptyset$. Intuitively, the former corresponds to the case where v preceeds one common neighbour of x and v, and the latter corresponds to the case where x preceeds one common neighbour of x and v, with x being such that $(N(x) \cap X_u, pos_x) = (N, p)$. If the test fails then we claim that we cannot extend to a valid representation at node t (and so, the current subordering $\prec_t$ can be discarded).

    Indeed, let $\prec$ be any extension of $\prec_t$ to a total ordering on V. Suppose by way of contradiction that $\prec$ is distance-preserving. Let $x \in V_t \setminus X_u$ be such that $(N(x) \cap X_u, pos_x) = (N, p)$. Note that x and v are at distance two. So, by Corollary 2, there exists a common neighbour $z \in X_u$ that is preceded by at least one of v or x. Furthermore, let us denote by $N^+(x) = N^+$ the subset of $N(x) \cap X_u$ obtained by removing its pos_x first neighbours in X_u w.r.t. $\prec_t$. We get that either v precedes z, and so, $N^+(v) \cap N(x) \neq \emptyset$, or x precedes z, and so, $N^+(x) \cap N(v) \neq \emptyset$. The latter contradicts that neither $N^+(v) \cap N \neq \emptyset$ nor $N^+ \cap N(v) \neq \emptyset$, therefore the claim is proved.

    Conversely, let us point out that if for every $(N, p) \in \mathcal{C}_u$ such that $N \cap N(v) \neq \emptyset$, either $N^+(v) \cap N \neq \emptyset$ or $N^+ \cap N(v) \neq \emptyset$, then the following holds for every $x \in V_t \setminus X_u$ at distance two from v: either $N^+(v) \cap N(x) \neq \emptyset$, and so, there exists a common neighbour z preceded by v, or $N^+(x) \cap N(v) \neq \emptyset$, and so, there exists a common neighbour z preceded by x.

Note that the collection $\mathcal{C}_u = \mathcal{C}_t$ is not modified.

- **Case of a Join Node.** Let u, w be the two children nodes of t. Recall that $X_u = X_w = X_t$. Consider any valid representation at node u, and any valid representation at node w. They can be merged into a valid representation at node t only if $\prec_u = \prec_w$. If so, let $\prec_t = \prec_u$, let $\mathcal{P}_t = \mathcal{P}_u \cap \mathcal{P}_w$ and let $\mathcal{C}_t = \mathcal{C}_u \cup \mathcal{C}_w$.

  In order to decide whether this can be extended into a valid representation at node t, we need to consider all the pairs of vertices in V_t at distance two in G that are neither both contained in V_u nor both contained in V_w. More precisely, we need to consider all the pairs of vertices $v_u \in V_u \setminus X_u,\ v_w \in V_w \setminus X_w$ at distance two in G. Notice that since by Property (ii) of tree-decompositions, there cannot be an edge between $V_u \setminus X_u$ and $V_w \setminus X_w$, the pairs $v_u \in V_u \setminus X_u,\ v_w \in V_w \setminus X_w$ that need to be considered are exactly those such that $(N(v_u) \cap X_u) \cap (N(v_w) \cap X_w) \neq \emptyset$. Hence, let us consider all the pairs $(N_u, p_u) \in \mathcal{C}_u,\ (N_w, p_w) \in \mathcal{C}_w$ such that $N_u \cap N_w \neq \emptyset$.

  For every two pairs $(N_u, p_u) \in \mathcal{C}_u,\ (N_w, p_w) \in \mathcal{C}_w$ such that $N_u \cap N_w \neq \emptyset$, let N_u^+ be the subset obtained from N_u by removing its p_u first vertices w.r.t. $\prec_t = \prec_u$. Intuitively, N_u^+ corresponds to the neighbours of some vertex $v_u \in V_u \setminus X_u$ that are in $X_u = X_t$ and preceded by v_u. Similarly, let N_w^+ be the subset obtained from N_w by removing its p_w first vertices w.r.t. $\prec_t = \prec_w$. We check whether either $N_u^+ \cap N_w \neq \emptyset$ or $N_w^+ \cap N_u \neq \emptyset$. If it is not the case then we claim that we cannot extend to a valid representation at node t (and so, the current subordering $\prec_t$ can be discarded).

  Indeed, let $\prec$ be any extension of $\prec_t$ to a total ordering on V. Suppose by way of contradiction that $\prec$ is distance-preserving. Let $v_u \in V_u \setminus X_u,\ v_w \in V_w \setminus X_w$ be such that $(N(v_u) \cap X_u, pos_{v_u}) = (N_u, p_u)$ and $(N(v_w) \cap X_w, pos_{v_w}) = (N_w, p_w)$. Since $N(v_u) \cap N(v_w) = N_u \cap N_w \neq \emptyset$, v_u and v_w are at distance two. Therefore, by Corollary 2, there exists a common neighbour $z \in X_t$ that is preceded by at least one of v_u or v_w. Furthermore, let us denote by $N^+(v_u) = N_u^+$ the subset of $N(v_u) \cap X_u$ obtained by removing its pos_{v_u} first neighbours in X_u w.r.t. $\prec_t = \prec_u$; similarly, let us denote by $N^+(v_w) = N_w^+$ the subset of $N(v_w) \cap X_w$ obtained by removing its pos_{v_w} first neighbours in X_w w.r.t. $\prec_t = \prec_w$. We get that either v_u precedes z, and so, $N^+(v_u) \cap N(v_w) \neq \emptyset$, or v_w precedes z, and so, $N^+(v_w) \cap N(v_u) \neq \emptyset$. The latter contradicts that neither $N_u^+ \cap N_w \neq \emptyset$ nor $N_w^+ \cap N_u \neq \emptyset$, therefore the claim is proved.

  Conversely, let us point out that if for every $(N_u, p_u) \in \mathcal{C}_u,\ (N_w, p_w) \in \mathcal{C}_w$ such that $N_u \cap N_w \neq \emptyset$, either $N_u^+ \cap N_w \neq \emptyset$ or $N_w^+ \cap N_u \neq \emptyset$, then the following holds for every $v_u \in V_u \setminus X_u,\ v_w \in V_w \setminus X_w$ at distance two in G: either $N^+(v_u) \cap N(v_w) \neq \emptyset$, and so, there exists a common neighbour z preceded by v_u, or $N^+(v_w) \cap N(v_u) \neq \emptyset$, and so, there exists a common neighbour z preceded by v_w. $\square$

5. Exact algorithms and heuristics

The purpose of the section is to describe algorithms in order to compute a distance-preserving ordering for a given graph G when it exists. Exhaustive-search on all possible vertex-orderings of the graph would require $\mathcal{O}^*(n!) = 2^{\mathcal{O}(n \log n)}$-time,[3] and the algorithm parameterized by treewidth that we have presented in Section 4 has huge constants which makes it rather impractical.

In this section, we describe exact and heuristic algorithms that can effectively be used to decide if a graph has a distance-preserving ordering, and return one when it exists.

5.1. Exact exponential time algorithm

A meta-theorem for computing vertex-orderings in graphs with given properties was proved in [3]. It bases on dynamic programming. Here, we prove that the theorem of [3] also applies to distance-preserving orderings. For any elimination ordering $(v_1, v_2, \ldots, v_n)$ of a graph $G = (V, E)$ and for any $1 \leq i \leq n$, let $V_{i+1} = \{v_{i+1}, v_{i+2}, \ldots, v_n\} = \{u \in V \mid v_i \prec u\}$.

Theorem 11 (*[3]*). *Let f be a polynomial time computable function mapping each* 3*-tuple, consisting of a graph $G = (V, E)$, a vertex set $S \subseteq V$, and a vertex $v \in V$ to an integer.*

Then we can compute in $\mathcal{O}^*(2^n)$-time and space, or in $\mathcal{O}^*(4^n)$-time and polynomial-space, the following values for a given graph $G = (V, E)$:

- $\min_{\prec} \max_{v_i \in V} f(G, V_{i+1}, v_i)$;
- $\min_{\prec} \sum_{v_i \in V} f(G, V_{i+1}, v_i)$.

Corollary 12. *The problem of deciding whether a given graph admits a distance-preserving elimination ordering can be solved in $\mathcal{O}^*(2^n)$-time and space, or in $\mathcal{O}^*(4^n)$-time and polynomial-space.*

[3] The notation $\mathcal{O}^*(f(n))$ is for a complexity $f(n) \cdot n^{\mathcal{O}(1)}$.

Proof. Let the function f map every 3-tuple (G, S, v) to the number of pairs $x, y \in S \cap N_G(v)$ of nonadjacent vertices with no common neighbour in S. Given a graph $G = (V, E)$ our aim is to compute an elimination ordering $(v_1, v_2, \ldots, v_n)$ of G that minimizes $\max_{1 \le i < n} f(G, V_{i+1}, v_i)$, with $V_{i+1} = \{v_{i+1}, v_{i+2}, \ldots, v_n\}$. Indeed, by Corollary 2, G admits a distance-preserving elimination ordering if and only if there is one such ordering such that for every $1 \le i \le n$, $f(G, V_{i+1}, v_i) = 0$, *i.e.*, $\max_{1 \le i < n} f(G, V_{i+1}, v_i) = 0$. By Theorem 11 and since f is polynomial-time computable an ordering that minimizes $\max_{1 \le i < n} f(G, V_{i+1}, v_i)$ can be computed in $\mathcal{O}^*(2^n)$-time and space, or in $\mathcal{O}^*(4^n)$-time and polynomial-space. □

5.2. Integer linear programming

Integer linear programming (ILP) formulations have been proved useful in practical computation of vertex orderings [8,16]. For completeness, we hence propose an ILP formulation that fits to our problem. Like in [16], total ordering on the vertices is expressed through n^2 binary variables $x_{v,i}$, each denoting whether vertex $v \in V$ is amongst the i first vertices to be eliminated.

$$\sum_{v \in V} x_{v,i} = i \qquad \forall\ 1 \le i \le n \tag{1}$$

$$x_{v,i} \le x_{v,i+1} \qquad \forall\ v \in V,\ \forall\ 1 \le i < n \tag{2}$$

In order to ensure that the total ordering is distance-preserving, we impose that for all pairs of vertices $u, v \in V$ at distance two in G, at least one of u or v must be eliminated before some of their common neighbours w. It can be expressed as follows:

$$\sum_{w \in N_G(u) \cap N_G(v)} x_{w,i} \le x_{u,i} + x_{v,i} + (|N_G(u) \cap N_G(v)| - 1) \qquad \forall\ u, v \text{ s.t. } \mathrm{d}_G(u, v) = 2,\ \forall\ 1 \le i \le n \tag{3}$$

The correctness of our formulation directly follows from Corollary 2.

5.3. Heuristics

In this section, we present three heuristics to decide whether a graph admits a distance-preserving ordering. Then, we propose two ways to generate graphs admitting distance-preserving orderings.

Heuristic `Greedy_Pruning`. The first heuristic, very naive, attempts to find a distance preserving ordering greedily. Precisely, given a graph G, it computes the set C of all vertices v such that $G \setminus v$ is an isometric subgraph of G. Note that, by Lemma 1, this can be done by checking only the vertices at distance at most two for every vertex in G. Once the set C of candidates has been computed, one vertex v is randomly chosen in it (this will be the first vertex of the tried ordering) and the process goes on $G \setminus v$. If $C = \emptyset$, the process stops and returns that no distance preserving ordering has been found. If G has no more vertices, the algorithm returns the found ordering. The pseudo-code of this heuristic is presented in Algorithm 1.

Algorithm 1 `Greedy_Pruning`

Require: A graph $G = (V, E)$, a layout L of a subset $S \subseteq V$ of the vertices of G
$H := G[V \setminus S]$
if $V(H) = \emptyset$ **then**
  return L
$C := \emptyset$
for all $v \in V \setminus S$ **do**
  if $H \setminus v$ is an isometric subgraph of H **then**
    $C := C \cup \{v\}$
if $C = \emptyset$ **then**
  return "No ordering found"
Let $v \in C$ randomly chosen
return `Greedy_Pruning`$(G, L \odot v)$

Heuristic `Greedy_Reverse_Pruning`. The second heuristic attempts to build the ordering starting from its last vertex. Precisely, it guesses the last vertex (all vertices of the graph G may be considered as last vertex). From the current vertex, the algorithm tries to guess its predecessor in the ordering. Precisely, assuming that the algorithm has already computed a partial layout $(v_{i+1}, \ldots, v_n)$ of a set $S = \{v_{i+1}, \ldots, v_n\} \subseteq V$, it aims at finding a vertex v_i such that $G[S]$ is an isometric subgraph of $G[\{v_i\} \cup S]$. For this purpose, it computes the set C of all vertices $v \in N(S) = \{v \in V(G) \setminus S \mid \exists u \in S,\ \{u, v\} \in E(G)\}$ that satisfies this property. By Lemma 1, a vertex $v \in N(S)$ is added to C if any two non-adjacent neighbours $x, y \in N(v) \cap S$ have another common neighbour in S. Once the set C of candidates has been computed, if it is empty, then the process stops and returns that no distance preserving ordering has been found. Otherwise, there are two variants of the heuristic:

- In the first one, one vertex v is randomly chosen in C.
- In the second case, one vertex v is randomly chosen in the set of the vertices of C that have maximum degree in S.

In both cases, the chosen vertex v is added as first vertex of the current layout. The pseudo-code of this heuristic (second variant) is presented in Algorithm 2. For the first variant, the only difference is that Line 9 must be replaced by "Let $C^* := C$".

Algorithm 2 `Greedy_Reverse_Pruning`

Require: A graph $G = (V, E)$, a layout L of a subset $S \subseteq V$ of the vertices of G
1: **if** $S = V$ **then**
2:     **return** L
3: $C := \emptyset$
4: **for all** $v \in N(S) = \{v \in V(G) \setminus S \mid \exists u \in S,\ \{u, v\} \in E(G)\}$ **do**
5:     **if** $\forall x, y \in N_G(v) \cap S, N(x) \cap N(y) \cap S \neq \emptyset$ or $\{x, y\} \in E$ **then**
6:         $C := C \cup \{v\}$
7: **if** $C = \emptyset$ **then**
8:     **return** "No ordering found"
9: Let $C^* \subseteq C$ be the set of the vertices in C with maximum degree in S
10: Let $v \in C^*$ randomly chosen
11: **return** `Greedy_Reverse_Pruning` $(G, v \odot L)$

The intuition behind the fact that it seems preferable to take a vertex with maximum degree in S (Line 9 of Algorithm 2) is clear since it will maximize the number of pairs of vertices already having a common neighbour in S (which is a required condition to compute C on Line 5 of Algorithm 2).

Graph generation. To generate graphs with distance preserving ordering (in order to test the heuristics), we propose the following two algorithms.

- The first algorithm (INC) creates a graph by adding the vertices one by one. Precisely, assuming that a graph G (admitting a distance-preserving ordering) has already been created, the algorithm adds a new vertex as follows. First a vertex $x \in V(G)$ is randomly chosen. Then, the algorithm randomly chooses a set $X \subseteq \{w \in V(G) \mid dist(x, w) \leq 2\}$ with the property that any two non-adjacent vertices in X have a common neighbour is G. Finally, a new vertex v is added to G by making v adjacent to every vertex in $X \cup \{x\}$. The cardinality of X can additionally be bounded.
- The second algorithm (AUG) aims at augmenting a given graph into a super-graph of it admitting a distance-preserving ordering. For this purpose, the algorithm starts from a given graph $G = (V, E)$ and first computes a random ordering $L = (v_1, \ldots, v_n)$ of V. Then, we aim at adding edges to G in order to make L a distance-preserving ordering of the resulting super-graph of G. Precisely, the algorithm considers the vertices one by one from v_1 to v_n. When considering v_i, if $G_{i+1} = G[\{v_{i+1}, \ldots, v_n\}]$ is an isometric subgraph of G_i, then no edges are added. Otherwise, by Lemma 1, this means that two non-adjacent neighbours $x, y \in V(G_{i+1}) \cap N(v_i)$ of v_i have no common neighbour in G_{i+1}. Hence, the algorithm adds edges between every such a pair of vertices.

We have used these generators to perform basic experiments on a standard laptop, using the *Sagemath* open-source mathematical software [30] to implement the algorithms and IBM Ilog CPLEX [23] to solve the ILP formulations. Our first observation is that the ILP formulation is generally able to decide if a graph with up to 50 nodes has a distance-preserving ordering in a few minutes (we also tried Erdős–Rényi and Barabási–Albert random graphs). However, it can hardly be used for larger graphs due to excessive running time. Our second observation is that the `Greedy_Pruning` heuristic is not effective at all. It is able to find a distance-preserving ordering on very few small graphs (less than 20 nodes) only. The `Greedy_Reverse_Pruning` heuristic guided by the maximum degree is much more efficient. We have executed it on graphs generated by the INC generator (100 n-node graphs, for each $n \in \{20, 30, \ldots, 100\}$). The heuristic has been able to confirm that more than 96% of these graphs have a distance-preserving ordering. Also, this heuristic appears to be particularly efficient on dense graphs. Precisely, we performed many experiments on Erdős–Rényi random graphs (100 n-node graphs, for each $n \in \{100, \ldots, 200\}$ and $p \in \{0.1, 0.2, \ldots, 0.5\}$) and our heuristic returns that more than 99% of them actually have a distance-preserving ordering when the probability is high ($p \geq 0.3$). The latter supports a recent conjecture from [27]. Further experimental and theoretical investigations are needed to determine the minimum probability upon which Erdős–Rényi random graphs have a distance-preserving ordering asymptotically almost surely. We let this interesting question as an open problem for future research.

Acknowledgements

We wish to thank the referees for their careful reading of the first version of this manuscript, their useful comments, and for pointing out the relationship between the distance-preserving orderings of hypercubes and the shellable orderings of cross-polytopes.

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