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Potential reconstruction for a class of hyperbolic systems from incomplete measurements

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Abstract

In this article, we study the reconstruction of spatially dependent potentials in n coupled hyperbolic equations in cascade from $n-1$ components of the solution of the system. More precisely, we prove local uniqueness and Lipschitz stability for this inverse problem. The main tool is a Carleman estimate for a cascade system with missing observations.

Keywords: inverse problem, Carleman estimates, potential reconstruction, cascade hyperbolic systems

1. Introduction

1.1. General setting

Let Ω be a smooth open set in $\mathbb{R}^d$ with boundary $\partial\Omega$, $d \geqslant 1$ and $T > 0$. Let us consider the following coupled hyperbolic system in cascade:

$$\begin{cases} \Box u_1 + q_1 u_1 = a_1 u_2 + g_1, & \text{in } \Omega \times (0,T), \\ \Box u_2 + q_2 u_2 = a_2 u_3 + g_2, & \text{in } \Omega \times (0,T), \\ \quad \vdots & \quad \vdots \\ \Box u_{n-1} + q_{n-1} u_{n-1} = a_{n-1} u_n + g_{n-1}, & \text{in } \Omega \times (0,T), \\ \Box u_n + q_n u_n = g_n, & \text{in } \Omega \times (0,T), \\ \partial_t^k u_j(0) = u_j^k, \ k = 0,1, \ j = 1,\dots,n, & \text{in } \Omega, \\ u_j = 0, \ j = 1,\dots,n, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{1.1}$$

Here, $\Box := \partial_t^2 - \Delta$ is the D'Alembertian operator, a_j are non-zero constants, $u_j^k \in L^2(\Omega)$ are the initial conditions and $q_j \in L^2(\Omega)$ are the potentials and $g_j \in L^2(\Omega \times (0,T))$ are the source terms, for every $k = 0, 1$ and $j = 1, \ldots, n$.

It is well known that if $g_j \in L^1\left(0, T; L^2(\Omega)\right)$, $u_j^0 \in H_0^1(\Omega)$ and $u_j^1 \in L^2(\Omega)$, $j = 1, \ldots, n$, system (1.1) is well posed in the sense of Hadamard (see for instance [35]).

Hyperbolic and parabolic systems play an important role in biological, chemical, engineering, mechanical and medical applications. Nevertheless, some components of such models are not accessible in practice. Motivated for this kind of limitations, some natural questions arise: can we observe such systems from incomplete measurements? Can we retrieve information of the inaccessible components of such systems from information of the accesible ones? These questions has been studied recently by several authors for different kind of PDE models, see for instance [1, 5, 15] and the bibliographic discusion below.

In this paper, we are interested in the following inverse problem:

1.1.1. Inverse problem. Is it possible to retrieve the potentials $q_1, \ldots, q_n$ in system (1.1) from incomplete data, that is to say, from a reduced number of measurements of the solution?

In particular, we are interested in the stability of the potentials in terms of the observations of the solution of system (1.1) when the last component is missing.

1.2. Literature

Bukhgeim and Klibanov dealt for the first time with uniqueness issues in inverse problems for the wave equation in [32] using local Carleman estimates. Then, the first results about the stability of inverse problems for hyperbolic equations were obtained using local Carleman estimates (see e.g. [11, 13, 26, 29, 30, 38]). Concerning other inverse problems for the wave equation with a single observation, we refer to [8, 27, 28, 33] and the references therein. In these articles, the authors consider the case of interior or Dirichlet boundary data observation satisfying geometric conditions and they use global Carleman estimates. We refer to [10, 12] for logarithmic stability results when no geometric condition is fulfilled. Let us also mention the work [31] where the authors proved the uniqueness of the inverse problem of recovering a spatial component of the source term of the wave equation from the final observation data.

However, to the best of our knowledge, there exist few works concerning inverse problems for coupled parabolic or hyperbolic systems with incomplete measurements of their components. In the recent work [5], the authors study the reconstruction of the spatial distribution of external forces only from data of one component of a 2 coupled hyperbolic system in cascade. The proof is based on an observability property of such system, following the approach of [37].

Similar inverse problems for linear and semilinear parabolic systems like reaction-diffusion systems has been studied in [14, 15, 18–20]. In these articles, the authors deal with identification and stability of the inverse problem of recovering parameters and initial conditions of such systems from a finite number of measurements of one component using appropriate Carleman estimates for parabolic equations.

Furthermore, hyperbolic–parabolic systems are considered in [24] with different kinds of observations. Another relevant work is [25] for the Stokes system, where the authors give a reconstruction algorithm for a source of the form $F(x, t) = f(x)\sigma(t)$ from incomplete velocity measurements.

Exact controllability properties of hyperbolic systems with a reduced number of controls has been extensively studied and there exist many works published on this topic. In [1], a strategy called two-level energy method is developed to prove positive results in the case of wave-type systems (see also [2, 3, 6], and the references therein). Moreover these results allow to deduce null-controllability results for the heat or the Schrödinger equations satisfying the geometric control condition using the transmutation method.

Furthermore, the literature is also very rich in controllability results for coupled parabolic systems with a reduced numbers of controls in the one or multidimensional setting. We refer to the survey article [7, 23], and the references therein.

Coupled systems are also connected with insensitizing control problems, notion introduced by Lions in [34]. Indeed, these problems are equivalent to the null-controllability of a cascade system. We reference to [21, 36, 4] for some results about this subject in the case of wave-type equations, [16, 17, 22] in the case of parabolic equations and systems.

1.3. Main result

In this article, we will give a Lipschitz stability result for system (1.1) stated in the introduction, from observations in all the components of the system except the last one.

We shall assume some geometrical and time conditions which are classical in the context of control and inverse problems for hyperbolic equations. Let $x_0 \notin \overline{\Omega}$, $\Gamma_0 \subset \partial\Omega$ and $T > 0$ such that:

- **Geometric condition:**

$$\{x \in \partial\Omega;\ (x - x_0)\cdot \nu(x) \geqslant 0\} \subset \Gamma_0 \subset \partial\Omega. \tag{1.2}$$

- **Time condition:**

$$\sup_{x\in\Omega} |x - x_0| < \sqrt{\beta}\, T, \quad \text{for some } \beta \in (0, 1). \tag{1.3}$$

Now, we define the admissible set of the unknown potentials. For a positive number m, set

$$L^\infty_{\leqslant m}(\Omega) = \{p \in L^\infty(\Omega);\ \|p\|_{L^\infty(\Omega)} \leqslant m\}.$$

Let us state the main result of this article:

Theorem 1. *Suppose that $\Gamma_0 \subset \partial\Omega$, $T > 0$ and $x_0 \notin \overline{\Omega}$ satisfy the geometric and time conditions (1.2) and (1.3). Let $\omega \subset \Omega$ such that $\overline{\Gamma_0} \subset \partial\omega \cap \partial\Omega$. Let $(u_1, \ldots, u_n)$ and $(\tilde{u}_1, \ldots, \tilde{u}_n)$ be the solutions of the system (1.1) associated to the potentials $q_1, \ldots, q_n \in L^\infty_{\leqslant m}(\Omega)$ and $\tilde{q}_1, \ldots, \tilde{q}_n \in L^\infty_{\leqslant m}(\Omega)$, respectively, with $m > 0$. Assume that there exists a constant $c > 0$ such that*

$$\|\tilde{u}_j(0)\|^2_{L^2(\Omega)} \geqslant c, \quad \forall j = 1, 2, \ldots, n. \tag{1.4}$$

Furthermore, suppose that

$$\begin{cases} u_j, \tilde{u}_j \in H^3(0, T; H^2(\Omega) \cap H^1_0(\Omega) \cap L^\infty(\Omega)), \quad j = 1, \ldots, n, \\ u_{n-1}, \tilde{u}_{n-1} \in H^4(0, T; H^2(\Omega) \cap H^1_0(\Omega) \cap L^\infty(\Omega)). \end{cases}$$

Then, there exists a constant $C = C(\beta, c, T, \Omega, \omega)$ such that

$$\sum_{j=1}^{n} \|q_j - \tilde{q}_j\|^2_{L^2(\Omega)} \leqslant C \sum_{j=1}^{n-2} \|u_j - \tilde{u}_j\|^2_{H^3(0,T;L^2(\omega))} + C\|u_{n-1} - \tilde{u}_{n-1}\|^2_{H^4(0,T;L^2(\omega))}. \tag{1.5}$$

Remark 1. Let us emphasize that inequality (1.5) establishes the Lipschitz stability of the hyperbolic system (1.1) with incomplete measurements in the sense that u_n is missing. Moreover, notice that the estimate (1.5) does not depend on the observations of the gradients.

Remark 2. Theorem 1 is also valid if we suppose that the coupling coefficients a_j are not constants satisfying

$$a_j(x) \geqslant c > 0, \quad \text{in } \omega',$$

where $\omega' \subset \Omega$ such that $\overline{\Gamma_0} \subset \partial\omega'$ and $\omega' \cap \omega \neq \emptyset$. In other words, the inequality (1.5) holds if the coupled and the observations regions of each components meet.

The main tool of the proof of theorem 1 is a Carleman estimate for a hyperbolic system in cascade where we do not have access to the observations associated to the last component. This inequality depends on a suitable Carleman estimate for the scalar wave equation in the spirit of the work of Imanuvilov and Yamamoto [27] (see also [9]).

1.4. Plan of the paper

We conclude this section by giving the outline of the rest of the paper. Section 2 is devoted to the proof of a new Carleman inequality for a system of wave equations in cascade with observations in all their components except the last one. In section 3, we prove theorem 1 by applying the corresponding Carleman estimate proved in the previous section. Finally, we conclude with some comments in section 4.

2. Carleman estimates

2.1. Technical results

The goal of this section is to prove a Carleman estimate for a system of wave equations in cascade. In order to do that, our starting point is a suitable Carleman estimate for the scalar wave equation. But first, we will give a technical

Lemma 1. *Let $z \in L^2(-T,T;H_0^1(\Omega))$ be a function such that $\Box z + pz \in L^2(\Omega\times(-T,T))$, $\partial_\nu z \in L^2(\partial\Omega\times(-T,T))$ and $z(\pm T) = 0$ in Ω, with $p \in L^\infty(\Omega)$. Let $\gamma \in \mathbb{R}$. Let $\omega_1, \omega_2 \subset \Omega$ be two open sets such that $\overline{\omega_1} \subset \omega_2$.*

(a) If $\tilde\varphi \in C^1([-T,T]\times\overline{\Omega})$, then there exists a constant $C > 0$ such that

$$\int_{-T}^{T}\int_{\omega_1} \mathrm{e}^{2s\tilde\varphi}|\nabla z|^2\mathrm{d}x\mathrm{d}t \leqslant Cs^{\max\{2,\gamma\}}\int_{-T}^{T}\int_{\omega_2}\mathrm{e}^{2s\tilde\varphi}|z|^2\mathrm{d}x\mathrm{d}t + C\int_{-T}^{T}\int_{\omega_2}\mathrm{e}^{2s\tilde\varphi}|\partial_t z|^2\mathrm{d}x\mathrm{d}t + Cs^{-\gamma}\int_{-T}^{T}\int_{\omega_2}\mathrm{e}^{2s\tilde\varphi}|\Box z + pz|^2\mathrm{d}x\mathrm{d}t, \tag{2.1}$$

for all $s \geqslant 1$.

(b) If the function $\tilde\varphi \in C^1([-T,T];C^2(\overline{\Omega}))$ satisfies

$$\inf_{x\in\Omega}|\nabla\tilde\varphi(t)| \geqslant c_0 > 0, \quad \forall t \in [-T,T],$$

then, there exist two positive constants C and s_0 independent of s such that

$$
\begin{aligned}
&s^2\int_{-T}^{T}\int_{\omega_1} e^{2s\tilde{\varphi}}|z|^2\mathrm{d}x\mathrm{d}t+\int_{-T}^{T}\int_{\omega_1} e^{2s\tilde{\varphi}}|\partial_t z|^2\mathrm{d}x\mathrm{d}t\\
&\leqslant Cs^{\max\{0,\gamma-2\}}\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}|\nabla z|^2\mathrm{d}x\mathrm{d}t+Cs^{-\gamma}\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}|\Box z+pz|^2\mathrm{d}x\mathrm{d}t, \qquad (2.2)
\end{aligned}
$$

for all $s\geqslant s_0$.

Remark 3. The principal significance of part (a) on lemma 1 is that it allows to drop the local term of the gradient. This fact plays an important role in some steps of the proof of the Carleman estimate for the wave system in cascade in section 2.2.

Proof of lemma 1. Let us consider a function $\xi\in C^\infty(\Omega,\mathbb{R})$ such that

$$
\begin{cases}
0\leqslant\xi\leqslant 1, & \text{in } \Omega,\\
\xi\equiv 1, & \text{in } \omega_1,\\
\xi\equiv 0, & \text{in } \Omega\setminus\overline{\omega_2}.
\end{cases}
$$

Additionally, we suppose that ξ has the form $\xi=e^{\phi}$ in $\omega_2\setminus\overline{\omega_1}$, for some smooth function ϕ. We have the following identity:

$$
\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi z\partial_t^2 z\mathrm{d}x\mathrm{d}t-\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi z\Delta z\mathrm{d}x\mathrm{d}t+\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi p|z|^2\mathrm{d}x\mathrm{d}t=\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi z(\Box z+pz)\mathrm{d}x\mathrm{d}t. \qquad (2.3)
$$

Integration by parts yields

$$
\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi z\partial_t^2 z\mathrm{d}x\mathrm{d}t=-2s\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\partial_t\tilde{\varphi}\xi z\partial_t z\mathrm{d}x\mathrm{d}t-\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi|\partial_t z|^2\mathrm{d}x\mathrm{d}t, \qquad (2.4)
$$

and

$$
\begin{aligned}
-\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi z\Delta z\mathrm{d}x\mathrm{d}t&=2s\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi z\nabla\tilde{\varphi}\cdot\nabla z\mathrm{d}x\mathrm{d}t\\
&+\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}z\nabla\xi\cdot\nabla z\mathrm{d}x\mathrm{d}t+\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi|\nabla z|^2\mathrm{d}x\mathrm{d}t. \qquad (2.5)
\end{aligned}
$$

Substituting (2.4) and (2.5) into (2.3), we have

$$
\begin{aligned}
&\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi|\nabla z|^2\mathrm{d}x\mathrm{d}t\\
&=2s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\partial_t\tilde{\varphi}z\partial_t z\mathrm{d}x\mathrm{d}t+\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi|\partial_t z|^2\mathrm{d}x\mathrm{d}t+\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi p|z|^2\mathrm{d}x\mathrm{d}t\\
&+\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi z(\Box z+pz)\mathrm{d}x\mathrm{d}t-2s\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi z\nabla\tilde{\varphi}\cdot\nabla z\mathrm{d}x\mathrm{d}t-\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}z\nabla\xi\cdot\nabla z\mathrm{d}x\mathrm{d}t\\
&=J_1+J_2. \qquad (2.6)
\end{aligned}
$$

Here, J_1 is the sum of the first four terms of (2.6) and J_2 is the sum of the fifth and sixth terms of the same equation. Straightforward computations show that

$$|J_1| \leqslant 2\|\xi\|_{L^\infty(\omega_2)} \int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}|\partial_t z|^2 dxdt + \frac{1}{3}s^{-\gamma}\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}|\Box z + pz|^2 dxdt$$
$$+ \left(\|\partial_t\tilde{\varphi}\|^2_{L^\infty(\omega_2\times(-T,T))}s^2 + \frac{3}{4}\|\xi\|_{L^\infty(\omega_2)}s^\gamma + \|p\|_{L^\infty(\omega_2)}\right)\|\xi\|_{L^\infty(\omega_2)}\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}|z|^2 dxdt, \tag{2.7}$$

and

$$|J_2| \leqslant \left(3\|\nabla\tilde{\varphi}\|^2_{L^\infty(\omega_2\times(-T,T))}s^2 + \frac{3}{4}\|\nabla\phi\|_{L^\infty(\omega_2\setminus\overline{\omega_1})}\right)\|\xi\|_{L^\infty(\omega_2)}\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}|z|^2 dxdt$$
$$+ \frac{2}{3}\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi|\nabla z|^2 dxdt. \tag{2.8}$$

Combining (2.7), (2.8) with (2.6) we obtain (2.1), which completes the part (a) of lemma 1. The rest of the proof runs as before but additionally we have to estimate the local term $|z|^2$ by using the weighted Poincaré inequality (see [8], lemma 2.4). □

Now, we introduce the classical Carleman weights for the scalar wave equation. Suppose that Γ_0, x_0 and $T > 0$ satisfy the Geometric and Time condition (1.2) and (1.3). Let $\beta \in (0, T)$. For $(x, t) \in \Omega \times (-T, T)$, we define the following functions:

$$\psi(x,t) = |x - x_0|^2 - \beta t^2 + C_0, \quad \varphi(x,t) = e^{\lambda\psi(x,t)}, \tag{2.9}$$

where $\lambda > 0$ and $C_0 > 0$ is chosen such that $\psi \geqslant 0$ (and therefore $\varphi \geqslant 1$) in $\Omega \times (-T, T)$.

For brevity, we shall use the following notation

$$I(\alpha, v, \Omega) = s^\alpha \int_\Omega e^{2s\varphi(0)}\left(s^2|v(0)|^2 + |\partial_t v(0)|^2 + |\nabla v(0)|^2\right) dx$$
$$+ s^{\alpha+1}\int_{-T}^{T}\int_\Omega e^{2s\varphi}\left(s^2|v|^2 + |\partial_t v|^2 + |\nabla v|^2\right) dxdt.$$

In the remainder of this section, C denotes a generic positive constant which depends at least on Γ_0, T and x_0 and may change from line to line.

Proposition 1. *Assume that Γ_0, T and x_0 satisfy the Geometric condition and Time condition (1.2) and (1.3) and let $p \in L^\infty_{\leqslant m}(\Omega)$ with $m > 0$. Let us consider the Carleman weight functions defined in (2.9). Let $\omega_0 \subset \Omega$ be an open subset such that $\overline{\Gamma_0} \subset \partial\omega_0 \cap \partial\Omega$. Then, there exist two positive constants $C_1 = C_1(\Gamma_0, T, x_0, \omega_2)$ and $s_0 = s_0(\Gamma_0, T, x_0, \omega_2)$ independent of s such that for all $s \geqslant s_0$, we have*

$$I(0, v, \Omega) \leqslant C_1\int_{-T}^{T}\int_\Omega e^{2s\varphi}|\Box v + pv|^2 dxdt + C_1 s\int_{-T}^{T}\int_{\omega_0} e^{2s\varphi}\left(s^2|v|^2 + |\partial_t v|^2\right) dxdt, \tag{2.10}$$

for all $v \in L^2(-T, T; H^1_0(\Omega))$ such that $\Box v + pv \in L^2(\Omega \times (-T, T))$, $\partial_\nu v \in L^2(\partial\Omega \times (-T, T))$ and $v(\pm T) = \partial_t v(\pm T) = 0$ in Ω.

Remark 4. In contrast to the theorem 2.5 in [8] in the case of the Carleman estimate of the scalar wave equation with a single boundary observation, we emphasize that proposition 1 requires the assumptions $z(\pm T) = \partial_t z(\pm T) = 0$ in Ω. This point becomes important if we want to eliminate more components in the inequality (1.5) of theorem 1.

Let us point out that the proof of the proposition 1 is straightforward and many of the ingredients of the proof are already available in the literature (see for instance [9, 27]). Nevertheless, for our purposes, it is convenient to write the Carleman estimate for wave equation under the form of proposition 1. For the sake of completeness, we will give the proof of this result.

Proof of proposition 1. For $s \geqslant 1$, let us define

$$E_s(t) = \frac{1}{2}\int_\Omega e^{2s\varphi(t)}\left(|\partial_t v(t)|^2 + |\nabla v(t)|^2\right) dx, \quad \forall t \in (-T, T).$$

Differentiation with respect to t and integration by parts in space yields

$$\begin{aligned}\frac{dE_s}{dt}(t) =& s\int_\Omega e^{2s\varphi(t)}\partial_t\varphi(t)\left(|\partial_t v(t)|^2 + |\nabla v(t)|^2\right) dx + \int_\Omega e^{2s\varphi(t)}\partial_t v(t)\Box v(t) dx \\ &- 2s\int_\Omega e^{2s\varphi(t)}\partial_t v(t)\nabla v(t)\cdot\nabla\varphi(t) dx, \quad \forall t \in (-T, T).\end{aligned}$$

After integration on $(-T, 0)$ in time we obtain

$$\begin{aligned}E_s(0) =& s\int_{-T}^0\int_\Omega e^{2s\varphi}\partial_t\varphi\left(|\partial_t v|^2 + |\nabla v|^2\right) dxdt + \int_{-T}^0\int_\Omega e^{2s\varphi}\partial_t v\Box v dxdt \\ &- 2s\int_{-T}^0\int_\Omega e^{2s\varphi}\partial_t v\nabla v\cdot\nabla\varphi dxdt,\end{aligned}$$

where we have used $v(-T) = \partial_t v(-T) = 0$ in Ω. Applying Young's inequality and the weighted Poincaré inequality to v (see [8], Lemma 2.4) we obtain

$$\begin{aligned}&\int_\Omega e^{2s\varphi(0)}\left(s^2|v(0)|^2 + |\partial_t v(0)|^2 + |\nabla v(0)|^2\right) dx \\ &\leqslant Cs\int_{-T}^T\int_\Omega e^{2s\varphi}\left(s^2|v|^2 + |\partial_t v|^2 + |\nabla v|^2\right) dxdt + C\int_{-T}^T\int_\Omega e^{2s\varphi}|\Box v + pv|^2 dxdt, \quad \forall s \geqslant s_0.\end{aligned} \tag{2.11}$$

On the other hand, let us recall the classical Carleman estimate for the wave equation with $\lambda = \lambda_0$ fixed applied to v:

$$s\int_{-T}^T\int_\Omega e^{2s\varphi}(s^2|v|^2 + |\partial_t v|^2 + |\nabla v|^2) dxdt \leqslant C\int_{-T}^T\int_\Omega e^{2s\varphi}|\Box v + pv|^2 dxdt + Cs\int_{-T}^T\int_{\Gamma_0} e^{s\varphi}|\partial_\nu v|^2 d\sigma dt. \tag{2.12}$$

Let us consider an open subset $\omega_0' \subset \omega_0$ such that $\overline{\omega_0'} \subset \omega$ and $\partial\omega_0' \cap \partial\Omega \subset \partial\omega_0 \cap \partial\Omega$. Consider the function $\eta \in C^\infty(\overline{\Omega}, \mathbb{R})$ satisfying

$$\begin{cases} 0 \leqslant \eta \leqslant 1, & \text{in } \Omega, \\ \eta \equiv 1, & \text{in } \Omega \setminus \overline{\omega_0'}, \\ \eta = \partial_\nu\eta \equiv 0, & \text{on } \Gamma_0. \end{cases}$$

Replacing v by ηv in (2.12), we have

$$
\begin{aligned}
& s \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} \left(s^2|v|^2 + |\partial_t v|^2 + |\nabla v|^2\right) \mathrm{d}x\mathrm{d}t \\
\leqslant & C \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |\Box v + pv|^2 \mathrm{d}x\mathrm{d}t + C \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} \left(|v|^2 + |\nabla v|^2\right) \mathrm{d}x\mathrm{d}t \\
& + s \int_{-T}^{T} \int_{\omega_0'} e^{2s\varphi} \left(s^2|v|^2 + |\partial_t v|^2 + |\nabla v|^2\right) \mathrm{d}x\mathrm{d}t,
\end{aligned}
\tag{2.13}
$$

where we have used that $\Box(\eta v) = \eta \Box v - \Delta\eta v - 2\nabla\eta \cdot \nabla v$ in $\Omega \times (-T, T)$ and $\nabla\eta \equiv 0$ in $\Omega \setminus \overline{\omega_1}$. Notice that the second term of the right-hand side of (2.13) can be absorbed taking s large enough. Finally, combining the previous estimate obtained with (2.11) and applying the estimate (2.1) with $\tilde{\varphi} = \varphi$, $\omega_1 = \omega_0'$, $\omega_2 = \omega_0$ and $\gamma = 1$, the proof of (2.10) is complete. □

2.2. A new Carleman estimate for a hyperbolic system

The aim of this section is to prove a Carleman estimate for a wave-type system with potentials. In order to formulate our result, let us consider the following system:

$$
\begin{cases}
\Box v_1 + r_1 v_1 = v_2 + h_1, & \text{in } \Omega \times (-T, T), \\
\Box v_2 + r_2 v_2 = v_3 + h_2, & \text{in } \Omega \times (-T, T), \\
\quad \vdots & \quad \vdots \\
\Box v_{n-1} + r_{n-1} v_{n-1} = v_n + h_{n-1}, & \text{in } \Omega \times (-T, T), \\
\Box v_n + r_n v_n = h_n, & \text{in } \Omega \times (-T, T), \\
\partial_t^k v_j(\pm T) = 0,\ k = 0, 1,\ j = 1, \ldots, n, & \text{in } \Omega, \\
v_j = 0,\ j = 1, \ldots, n, & \text{on } \partial\Omega \times (-T, T).
\end{cases}
\tag{2.14}
$$

Here, $r_j \in L^\infty(\Omega)$ are the potentials and $h_j \in L^2(\Omega \times (-T, T))$ are the source terms, for each $j = 1, \ldots, n$.

Now, we are in position to state the Carleman estimate for system (2.14), which is one of the main results of this article:

Theorem 2. *Let us consider the Carleman weights defined in (2.9), where $\Gamma_0 \subset \partial\Omega$, $T > 0$ and $x_0 \notin \overline{\Omega}$ satisfy the geometric and time conditions (1.2) and (1.3). For $m > 0$, suppose that $r_j \in L^\infty_{\leqslant m}(\Omega)$, $j = 1, \ldots, n$, and let $\omega \subset \Omega$ be an open set such that $\overline{\Gamma_0} \subset \partial\omega \cap \partial\Omega$. In addition, consider $h_j \in L^2(\Omega \times (-T, T))$ for each $j = 1, \ldots, n-2$ and $h_{n-1}, h_n \in H^1(-T, T; L^2(\Omega))$ such that*

$$
\begin{cases}
v_j \in H^1\left(-T, T; H^2(\Omega) \cap H_0^1(\Omega)\right),\ j = 1, \ldots, n, \\
v_{n-1} \in H^2(-T, T; H^2(\Omega) \cap H_0^1(\Omega)).
\end{cases}
$$

Furthermore, we choose $1 < \alpha < 2$. Then, there exist two constants $C_2 = C_2(\Gamma_0, \Omega, \omega, T, x_0) > 0$ and $s_0 = s_0(\Gamma_0, \Omega, \omega, T, x_0)$ independent of s such that for all $s \geqslant s_0$, the solution $(v_1, \ldots, v_n)$ of system (2.14) satisfies

$$\sum_{j=1}^{n-1} I(\alpha, v_j, \Omega) + I(0, v_n, \Omega)$$
$$\leqslant C_2 s^{\alpha+1} \sum_{j=1}^{n-2} \int_{-T}^{T} \int_{\omega} e^{2s\varphi} \left(s^2|v_j|^2 + |\partial_t v_j|^2\right) dxdt + C_2 s^{\alpha} \sum_{j=1}^{n-2} \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |h_j|^2 dxdt$$
$$+ C_2 s^3 \int_{-T}^{T} \int_{\omega} e^{2s\varphi} \left(s^5|v_{n-1}|^2 + s^3|\partial_t v_{n-1}|^2 + |\partial_t^2 v_{n-1}|^2\right) dxdt$$
$$+ C_2 \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} \left(s^3|h_{n-1}|^2 + |h_n|^2 + s|\partial_t h_{n-1}|^2 + |\partial_t h_n|^2\right) dxdt. \tag{2.15}$$

Remark 5. We emphasize that the Carleman estimate (2.15) depends only on h_n and $\partial_t h_n$ in the last component.

Proof of theorem 2. Let ω_1 and ω_2 be two subsets of ω be two open sets such that $\overline{\Gamma_0} \subset \partial\omega_j \cap \partial\Omega$ for each $j = 1, 2$ and $\overline{\omega_1} \subset \omega_2$ and $\overline{\omega_2} \subset \omega$.

We start applying the Carleman inequality of proposition 1 to $v_1, \ldots, v_n$ in system (2.14) with $\omega_0 = \omega_1$. We have:

$$\sum_{j=1}^{n-1} I(\alpha, v_j, \Omega) + I(0, v_n, \Omega)$$
$$\leqslant C s^{\alpha} \sum_{j=2}^{n} \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |v_j|^2 dxdt + C s^{\alpha} \sum_{j=1}^{n-1} \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |h_j|^2 dxdt + C \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |h_n|^2 dxdt$$
$$+ C s^{\alpha+1} \sum_{j=1}^{n-1} \int_{-T}^{T} \int_{\omega_1} e^{2s\varphi} (s^2|v_j|^2 + |\partial_t v_j|^2) dxdt + Cs \int_{-T}^{T} \int_{\omega_1} e^{2s\varphi} (s^2|v_n|^2 + |\partial_t v_n|^2) dxdt.$$

Note that the first term of the right-hand side of the inequality above can be absorbed by taking s large enough since $1 < \alpha < 2$. Therefore, we can rewrite this inequality as follows:

$$\sum_{j=1}^{n-1} I(\alpha, v_j, \Omega) + I(0, v_n, \Omega)$$
$$\leqslant C s^{\alpha} \sum_{j=1}^{n-1} \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |h_j|^2 dxdt + C \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |h_n|^2 dxdt$$
$$+ C s^{\alpha+1} \sum_{j=1}^{n-1} \int_{-T}^{T} \int_{\omega_1} e^{2s\varphi} \left(s^2|v_j|^2 + |\partial_t v_j|^2\right) dxdt + Cs \int_{-T}^{T} \int_{\omega_1} e^{2s\varphi} \left(s^2|v_n|^2 + |\partial_t v_n|^2\right) dxdt. \tag{2.16}$$

Now we are going to estimate the local term of v_n and $\partial_t v_n$ in (2.16). To do this, we consider a cut-off function $\xi \in C^\infty(\Omega, \mathbb{R})$ such that

$$\begin{cases} 0 \leqslant \xi \leqslant 1 & \text{in } \Omega, \\ \xi \equiv 1 & \text{in } \omega_1, \\ \xi \equiv 0 & \text{in } \Omega \setminus \overline{\omega_2}. \end{cases}$$

Using the equation of v_{n-1} in (2.14), we see that:

$$s^3\int_{-T}^{T}\int_{\omega_2} \mathrm{e}^{2s\varphi}\xi|v_n|^2\mathrm{d}x\mathrm{d}t = s^3\int_{-T}^{T}\int_{\omega_2} \mathrm{e}^{2s\varphi}\xi v_n\left(\Box v_{n-1}+r_{n-1}v_{n-1}\right)\mathrm{d}x\mathrm{d}t - s^3\int_{-T}^{T}\int_{\omega_2} \mathrm{e}^{2s\varphi}\xi v_n h_{n-1}\mathrm{d}x\mathrm{d}t. \tag{2.17}$$

Let us estimate each term of the equation above. First, by Young's inequality for every $\delta > 0$, there exists a constant $C = C(\delta)$ such that

$$s^3\int_{-T}^{T}\int_{\omega_2} \mathrm{e}^{2s\varphi}\xi h_{n-1}v_n\mathrm{d}x\mathrm{d}t \leqslant \delta s^3\int_{-T}^{T}\int_{\omega_2} \mathrm{e}^{2s\varphi}|v_n|^2\mathrm{d}x\mathrm{d}t + Cs^3\int_{-T}^{T}\int_{\omega_2} \mathrm{e}^{2s\varphi}|h_{n-1}|^2\mathrm{d}x\mathrm{d}t. \tag{2.18}$$

On the other hand, integration by parts yields

$$\begin{aligned} & s^2\int_{-T}^{T}\int_{\omega_2} \mathrm{e}^{2s\varphi}\xi\partial_t^2 v_{n-1}v_n\mathrm{d}x\mathrm{d}t \\ =& s^3\int_{-T}^{T}\int_{\omega_2} \mathrm{e}^{2s\varphi}\xi(4s^2|\partial_t\varphi|^2+2s\partial_t^2\varphi)v_{n-1}v_n\mathrm{d}x\mathrm{d}t + 4s^4\int_{-T}^{T}\int_{\omega_2} \mathrm{e}^{2s\varphi}\xi\partial_t\varphi v_{n-1}\partial_t v_n\mathrm{d}x\mathrm{d}t \\ &+ s^3\int_{-T}^{T}\int_{\omega_2} \mathrm{e}^{2s\varphi}\xi v_{n-1}\partial_t^2 v_n\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} & - s^3\int_{-T}^{T}\int_{\omega_2} \mathrm{e}^{2s\varphi}\xi\Delta v_{n-1}v_n\mathrm{d}x\mathrm{d}t \\ =& 2s^4\int_{-T}^{T}\int_{\omega_2} \mathrm{e}^{2s\varphi}\xi\nabla\varphi\left(v_n\nabla v_{n-1}-v_{n-1}\nabla v_n\right)\mathrm{d}x\mathrm{d}t + s^3\int_{-T}^{T}\int_{\omega_2} \mathrm{e}^{2s\varphi}\nabla\xi\left(v_n\nabla v_{n-1}-v_{n-1}\nabla v_n\right)\mathrm{d}x\mathrm{d}t \\ &- s^3\int_{-T}^{T}\int_{\omega_2} \mathrm{e}^{2s\varphi}\xi v_{n-1}\Delta v_n\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.20}$$

By (2.19) and (2.20) $\Box v_n + r_n v_n = h_n$, we have

$$\begin{aligned} & s^3\int_{-T}^{T}\int_{\omega_2} \mathrm{e}^{2s\varphi}\xi v_n\left(\Box v_{n-1}+r_{n-1}v_{n-1}\right)\mathrm{d}x\mathrm{d}t \\ \leqslant & C\int_{-T}^{T}\int_{\omega_2} \mathrm{e}^{2s\varphi}\left(s^3|h_{n-1}|^2+|h_n|^2\right)\mathrm{d}x\mathrm{d}t \\ &+ Cs^5\int_{-T}^{T}\int_{\omega_2} \mathrm{e}^{2s\varphi}\xi\left(s^2|v_{n-1}|^2+|\partial_t v_{n-1}|^2+|\nabla v_{n-1}|^2\right)\mathrm{d}x\mathrm{d}t + \delta I(0,v_n,\omega_2), \end{aligned} \tag{2.21}$$

for every $\delta > 0$, where we have used the Young inequality. Moreover, by part (a) of lemma 1 applied to v_{n-1}, ω_2 and ω with $\gamma = 3$ one has

$$s^5 \int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}|\nabla v_{n-1}|^2 dxdt \leqslant Cs^2 \int_{-T}^{T}\int_{\omega} e^{2s\varphi}\left(|v_n|^2+|h_{n-1}|^2\right)dxdt + Cs^5\int_{-T}^{T}\int_{\omega} e^{2s\varphi}\left(s^3|v_{n-1}|^2+|\partial_t v_{n-1}|^2\right)dxdt. \quad (2.22)$$

Substituting (2.22) into (2.21) and substituting the obtained estimate into (2.18), we conclude that

$$\begin{aligned} & s^3 \int_{-T}^{T}\int_{\omega_1} e^{2s\varphi}|v_n|^2 dxdt \\ & \leqslant C\int_{-T}^{T}\int_{\omega} e^{2s\varphi}\left(s^3|h_{n-1}|^2+|h_n|^2\right)dxdt \\ & + Cs^8\int_{-T}^{T}\int_{\omega} e^{2s\varphi}|v_{n-1}|^2 dxdt + Cs^5\int_{-T}^{T}\int_{\omega} e^{2s\varphi}|\partial_t v_{n-1}|^2 dxdt + \delta I(0,v_n,\omega), \end{aligned} \quad (2.23)$$

for every $\delta > 0$, where we have included the integral term of $|v_n|^2$ which has in front s^2 in $\delta I(0, v_n, \omega)$ by taking s large enough. In the same manner, we can estimate the local term of $\partial_t v_n$ as follows:

$$s\int_{-T}^{T}\int_{\omega_1} e^{2s\varphi}|\partial_t v_n|^2 dxdt \leqslant C\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\left(s|\partial_t h_{n-1}|^2+|\partial_t h_n|^2\right)dxdt + Cs^3\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\left(s^3|\partial_t v_{n-1}|^2+|\partial_t^2 v_{n-1}|^2\right)dxdt + \delta I(0,v_n,\omega). \quad (2.24)$$

Finally, by substituting (2.23) and (2.24) into (2.16), by taking the Carleman parameter $s \geqslant 1$ large enough and by choosing $\delta > 0$ sufficiently small, we obtain

$$\begin{aligned} & \sum_{j=1}^{n-1} I(\alpha, v_j, \Omega) + I(0, v_n, \Omega) \\ & \leqslant Cs^\alpha \sum_{j=1}^{n-2}\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|h_j|^2 dxdt + C\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\left(s^3|h_{n-1}|^2+|h_n|^2\right)dxdt \\ & + C\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\left(s|\partial_t h_{n-1}|^2+|\partial_t h_n|^2\right)dxdt \\ & + Cs^{\alpha+1}\sum_{j=1}^{n-2}\int_{-T}^{T}\int_{\omega} e^{2s\varphi}\left(s^2|v_j|^2+|\partial_t v_j|^2\right)dxdt \\ & + Cs^3\int_{-T}^{T}\int_{\omega} e^{2s\varphi}\left(s^5|v_{n-1}|^2+s^3|\partial_t v_{n-1}|^2+|\partial_t^2 v_{n-1}|^2\right)dxdt, \end{aligned}$$

which completes the proof of theorem 2. □

3. Proof of theorem 1

The plan of the proof of theorem 1 contains three parts:

Step 1 In the same spirit of the Bukhgeim–Klibanov method, we rewrite appropriately system (1.1) to apply the estimate (2.15) in theorem 2.
Step 2 After applying the Carleman estimate of theorem 2 to the new system, we estimate the residual and source terms.
Step 3 We conclude the proof gathering the estimates of the previous steps and eliminating the small order terms.

- **Step 1: Setting**

For each $j = 1, \dots, n$, let us denote by $y_j = u_j - \tilde{u}_j$, $p_j = q_j$, $f_j = q_j - \tilde{q}_j$ and $R_j = \tilde{u}_j$. Then, following this notation, $y_1, \dots, y_n$ solves:

$$\begin{cases} \Box y_1 + p_1 y_1 = y_2 + f_1 R_1, & \text{in } \Omega \times (0,T), \\ \Box y_2 + p_2 y_2 = y_2 + f_2 R_2, & \text{in } \Omega \times (0,T), \\ \quad \vdots & \quad \vdots \\ \Box y_{n-1} + p_{n-1} y_{n-1} = y_n + f_{n-1} R_{n-1}, & \text{in } \Omega \times (0,T), \\ \Box y_n + p_n y_n = f_n R_n, & \text{in } \Omega \times (0,T), \\ \partial_t^k y(0) = 0, \ k = 0,1, \ j = 1, \dots, n, & \text{in } \Omega, \\ y_j = 0, \ j = 1, \dots, n, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{3.1}$$

For each $j = 1, \dots, n$, we set $w_j = \partial_t^2 y_j$. Then, the new variables solve the following system:

$$\begin{cases} \Box w_1 + p_1 w_1 = w_2 + f_1 \partial_t^2 R_1, & \text{in } \Omega \times (0,T), \\ \Box w_2 + p_2 w_2 = w_3 + f_2 \partial_t^2 R_2, & \text{in } \Omega \times (0,T), \\ \quad \vdots & \quad \vdots \\ \Box w_{n-1} + p_{n-1} w_{n-1} = w_n + f_{n-1} \partial_t^2 R_{n-1}, & \text{in } \Omega \times (0,T), \\ \Box w_n + p_n w_n = f_n \partial_t^2 R_n, & \text{in } \Omega \times (0,T), \\ \partial_t^k w_j(0) = f_j \partial_t^k R_j(0), \ k = 0,1, \ j = 1, \dots, n, & \text{in } \Omega, \\ w_j = 0, \ j = 1, \dots, n, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{3.2}$$

Now, we want to apply theorem 2 to a suitable system. In order to do that, we extend system (3.2) in an even way, setting $w_j(x,t) = w_j(x,-t)$ for all $(x,t) \in \Omega \times (-T,0)$. We also extend the functions R_j, $\partial_t R_j$ and $\partial_t^2 R_j$ in an even way and keep the same notations for the new system.

To be able to apply the Carleman estimate (2.15), the functions w_j must satisfy $\partial_t^k w(\pm T) = 0$ in Ω, for $k = 0, 1$. However, this condition does not hold. To avoid this difficulty, we consider a cut-off function $\theta \in C_c^\infty((-T,T), \mathbb{R})$ defined as follows:

$$\begin{cases} 0 \leqslant \theta \leqslant 1, & \text{in } (-T,T), \\ \theta \equiv 1, & \text{in } (-T+\tau, T-\tau). \end{cases}$$

According to the definition of θ, it is clear that $z_j = \theta w_j$, for $j = 1, \ldots, n$, solves

$$\begin{cases} \Box z_1 + p_1 z_1 = z_2 + F_1, & \text{in } \Omega \times (-T, T), \\ \Box z_2 + p_2 z_2 = z_3 + F_2, & \text{in } \Omega \times (-T, T), \\ \quad \vdots & \quad \vdots \\ \Box z_{n-1} + p_{n-1} z_{n-1} = z_n + F_{n-1}, & \text{in } \Omega \times (-T, T), \\ \Box z_n + p_n z_n = F_n, & \text{in } \Omega \times (-T, T), \\ \partial_t^k z(0) = f_j \partial_t^k R(0), \ k = 0, 1, \ j = 1, \ldots, n, & \text{in } \Omega, \\ \partial_t^k z(\pm T) = 0, \ k = 0, 1, 2, \ j = 1, \ldots, n, & \text{in } \Omega, \\ z_j = 0, \ j = 1, \ldots, n, & \text{on } \partial\Omega \times (-T, T). \end{cases} \tag{3.3}$$

Here, the functions F_j are defined by

$$F_j = \theta f_j \partial_t^2 R_j + \partial_t^2 \theta w_j + 2 \partial_t \theta \partial_t w_j, \quad \text{in } \Omega \times (-T, T),$$

for each $j = 1, \ldots, n$.

- **Step 2: Applying Carleman estimate for hyperbolic systems**

In this step, we denote by C a generic positive constant which depends at least of Γ_0, m, T, ω and x_0 and may change from line to line.

Applying the Carleman estimate of theorem 2 to the system (3.3) with $v_j = z_j$, $r_j = p_j$ and $h_j = F_j$, we see that

$$\begin{aligned} & \sum_{j=1}^{n-1} I(\alpha, z_j, \Omega) + I(0, z_n, \Omega) \\ & \leqslant Cs \sum_{j=1}^{n-2} \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |F_j|^2 \mathrm{d}x\mathrm{d}t + C \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} \left(s^3 |F_{n-1}|^2 + |F_n|^2 \right) \mathrm{d}x\mathrm{d}t \\ & \quad + C \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} \left(s |\partial_t F_{n-1}|^2 + |\partial_t F_n|^2 \right) \mathrm{d}x\mathrm{d}t \\ & \quad + Cs^{\alpha+1} \sum_{j=1}^{n-2} \int_{-T}^{T} \int_{\omega_2} e^{2s\varphi} \left(s^2 |z_j|^2 + |\partial_t z_j|^2 \right) \mathrm{d}x\mathrm{d}t \\ & \quad + Cs^3 \int_{-T}^{T} \int_{\omega_2} e^{2s\varphi} \left(s^5 |z_{n-1}|^2 + s^3 |\partial_t z_{n-1}|^2 + |\partial_t^2 z_{n-1}|^2 \right) \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.4}$$

Note that the assumption (1.4) implies

$$c < |R_j(0)|^2, \quad \forall j = 1, 2, \ldots, n.$$

Then, the following estimate holds:

$$c \int_{\Omega} e^{2s\varphi(0)} |f_j|^2 \mathrm{d}x \leqslant \int_{\Omega} e^{2s\varphi(0)} |f_j R_j(0)|^2 \mathrm{d}x = \int_{\Omega} e^{2s\varphi(0)} |z_j(0)|^2 \mathrm{d}x. \tag{3.5}$$

Hence, summing (3.5) over j, we deduce that

$$\frac{1}{c} \sum_{j=1}^{n-1} I(\alpha, z_j) + \frac{1}{c} I(0, z_n) \geqslant s^{\alpha+2} \sum_{j=1}^{n-1} \int_{\Omega} e^{2s\varphi(0)} |f_j|^2 \mathrm{d}x + s^2 \int_{\Omega} e^{2s\varphi(0)} |f_n|^2 \mathrm{d}x.$$

Now we estimate the global terms of F_j and its derivatives, for each $j = 1, \ldots, n$. By definition,

$$\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|F_j|^2 dxdt \leqslant 2\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\theta f_j\partial_t^2 R_j|^2 dxdt + 2\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|2\partial_t\theta\partial_t w_j + \partial_t^2\theta w_j|^2 dxdt$$
$$\leqslant C\int_{\Omega} e^{2s\varphi(0)}|f_j|^2 dx + 2\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|2\partial_t\theta\partial_t w_j + \partial_t^2\theta w_j|^2 dxdt. \quad (3.6)$$

Now, we focus our attention on estimating the global term of $2\partial_t\theta\partial_t w_j + \partial_t^2\theta w_j$, for $j = 1, \ldots, n$. Notice that if the Time condition (1.3) holds, the Carleman weight ψ defined in (2.9) satisfies

$$\psi(x, \pm T) = |x - x_0|^2 - \beta T^2 + C_0 < C_0, \quad \text{in } \Omega.$$

Then, we choose $\tau > 0$ such that

$$\psi(x, t) \leqslant C_0, \quad \text{in } \Omega \times ([-T, -T+\tau] \cup [T-\tau, T]),$$

and therefore,

$$\varphi(x, t) = e^{\lambda C_0} < e^{\lambda\psi(x,0)} = \varphi(x, 0), \quad \text{in } \Omega \times ([-T, -T+\tau] \cup [T-\tau, T]).$$

Since the derivatives of θ vanish in $[-T+\tau, T-\tau]$ we see that

$$\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|2\partial_t\theta\partial_t w_j + \partial_t^2\theta w_j|^2 dxdt$$
$$\leqslant C\left(\int_{-T}^{-T+\tau} + \int_{T-\tau}^{T}\right) e^{2se^{\lambda C_0}}\int_{\Omega}\left(|\partial_t w_j|^2 + |w_j|^2\right) dxdt.$$

Now, we will estimate the last term of the above inequality. To do this, we will use the following energy estimates of (3.2):

$$\int_{\Omega}|\partial_t w_j(t)|^2 dx + \int_{\Omega}|\nabla w_j(t)|^2 dx \leqslant C\int_{\Omega}|f_j(t)|^2 dx + C\int_{\Omega}|w_{j+1}(t)|^2 dx, \quad \forall t \in (-T, T),$$

for each $j = 1, \ldots, n$ and

$$\int_{\Omega}|\partial_t w_n(t)|^2 dx + \int_{\Omega}|\nabla w_n(t)|^2 dx \leqslant C\int_{\Omega}|f_n|^2 dx, \forall t \in (-T, T),$$

where have used that $R_j \in H^2(-T, T; L^\infty(\Omega))$ for each $j = 1, \ldots, n$. Integrating on $(-T, -T+\tau) \cup (T-\tau, T)$ the estimate above and using the Poincaré inequality to w_j, we see that:

$$\left(\int_{-T}^{-T+\tau} + \int_{T-\tau}^{T}\right) e^{2se^{\lambda C_0}}\int_{\Omega}\left(|\partial_t w_j|^2 + |w_j|^2\right) dxdt \leqslant \int_{\Omega} e^{2s\varphi(0)}|f_j|^2 dx + Ce^{se^{\lambda C_0}}\int_{-T}^{T}\int_{\Omega}|w_{j+1}|^2 dxdt,$$

for each $j = 1, \ldots, n-1$. Furthermore, due to the structure in cascade of system (3.3), we obtain

$$e^{se^{\lambda C_0}}\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|w_{j+1}|^2 dxdt \leqslant Ce^{se^{\lambda C_0}}\sum_{i=j+1}^{n}\int_{\Omega}|f_i|^2 dx \leqslant C\sum_{i=j+1}^{n}\int_{\Omega} e^{2s\varphi(0)}|f_i|^2 dx,$$

for each $j = 2, \ldots, n-1$. Therefore, for every $j = 1, \ldots, n$, we deduce that

$$\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|2\partial_t\theta\partial_t w_j + \partial_t^2\theta w_j|^2 dxdt \leqslant C\sum_{i=j}^{n}\int_{\Omega} e^{2s\varphi(0)}|f_i|^2 dx. \quad (3.7)$$

Substituting (3.7) in (3.6), we see that

$$s^{\alpha}\sum_{j=1}^{n-2}\int_{-T}^{T}\int_{\Omega}e^{2s\varphi}|F_j|^2\mathrm{d}x\mathrm{d}t+\int_{-T}^{T}\int_{\Omega}e^{2s\varphi}\left(s^3|F_{n-1}|^2+|F_n|^2\right)\mathrm{d}x\mathrm{d}t$$
$$\leqslant Cs^{\alpha}\sum_{j=1}^{n-2}\int_{\Omega}e^{2s\varphi(0)}|f_j|^2\mathrm{d}x+C\int_{\Omega}e^{2s\varphi(0)}\left(s^3|f_{n-1}|^2+|f_n|^2\right)\mathrm{d}x. \tag{3.8}$$

In the same manner we can see that

$$\int_{-T}^{T}\int_{\Omega}e^{2s\varphi}\left(s|\partial_tF_{n-1}|^2+|\partial_tF_n|^2\right)\mathrm{d}x\mathrm{d}t\leqslant C\int_{-T}^{T}\int_{\Omega}e^{2s\varphi}\left(s|f_{n-1}|^2+|f_n|^2\right)\mathrm{d}x\mathrm{d}t. \tag{3.9}$$

Thus, substituting (3.8) and (3.9) into (3.4), and taking s large enough, we have

$$s^{\alpha+2}\sum_{j=1}^{n-1}\int_{\Omega}e^{2s\varphi(0)}|f_j|^2\mathrm{d}x+s^2\int_{\Omega}|f_n|^2\mathrm{d}x$$
$$\leqslant Cs^{\alpha+1}\sum_{j=1}^{n-2}\int_{-T}^{T}\int_{\omega_2}e^{2s\varphi}\left(s^2|z_j|^2+|\partial_tz_j|^2\right)\mathrm{d}x\mathrm{d}t$$
$$+Cs^3\int_{-T}^{T}\int_{\omega_2}e^{2s\varphi}\left(s^5|z_{n-1}|^2+s^3|\partial_tz_{n-1}|^2+|\partial_t^2z_{n-1}|^2\right)\mathrm{d}x\mathrm{d}t. \tag{3.10}$$

- **Step 3: Last arrangements and conclusion**

From (3.10), we fix the parameter s and put it into the constant C:

$$\sum_{j=1}^{n}\int_{\Omega}|f_j|^2\mathrm{d}x\leqslant C\sum_{j=1}^{n}\int_{-T}^{T}\int_{\omega_\omega}e^{2s\varphi}(|z_j|^2+|\partial_tz_j|^2)\mathrm{d}x\mathrm{d}t$$
$$+C\int_{-T}^{T}\int_{\omega}\left(|z_{n-1}|^2+|\partial_tz_{n-1}|^2+|\partial_t^2z_{n-1}|^2\right)\mathrm{d}x\mathrm{d}t,$$

where we have used that the Carleman weights defined in (2.9) are bounded. Moreover, by definition of each z_j we see that

$$\sum_{j=1}^{n}\|f_j\|^2_{L^2(\Omega)}\leqslant C\sum_{j=1}^{n-1}\|y_j\|^2_{H^3(-T,T;L^2(\omega))}+\|y_n\|^2_{H^4(-T,T;L^2(\omega))}. \tag{3.11}$$

Finally, replacing $f_j=q_j-\tilde{q}_j$ and $y_j=u_j-\tilde{u}_j$ by (3.11) for each $j=1,\ldots,n$, we conclude the proof of theorem 1.

□

4. Conclusions and further comments

In this article we proved Lipschitz stability result for the reconstruction of the spatially dependent potentials in n coupled hyperbolic system in cascade from $n-1$ components of the solution of the system using suitable global Carleman estimates for a cascade system with missing observations. Let us conclude with some comments and possible extensions of the present results.

First of all, concerning the special structure of the cascade system we considered in this study, notice that in (3.7), the source terms $f_j, \ldots, f_n$ arise in the estimate of F_j, for each $j = 1, \ldots, n$, because of the cascade structure of system (1.1) and the Carleman estimate of proposition 1, see also remark 4. This is the main difficulty to recover the potentials $(q_1, \ldots, q_n)$ with less components of (1.1). Then, the stability of the inverse problem treated in this article with two or more inaccesible components is open.

Regarding relationships of the present work with controllability, let us notice that in the particular case of $h_j = 0$ for each $j = 1, \ldots, n$ in (2.15) and under strong assumptions on the regularity of the solutions of (2.14), one can obtain a Carleman inequality of (2.14) with internal measurements of the first component of the system. To be more precise, for each $j = 1, \ldots, n$, we define α_j such that

$$\begin{cases} \alpha_{j+1} + 1 < \alpha_j < \alpha_{j+1} + 2, & j = 1, \ldots, n-1, \\ \alpha_n = 0. \end{cases}$$

Then, there exist two constants $C = C(\Omega, \omega, T, x_0) > 0$ and $s_0 \geqslant 1$ such that for all $s \geqslant s_0$, the following inequality holds:

$$\sum_{j=1}^{n} I(\alpha_j, v_j) \leqslant C \sum_{j=0}^{2^{n-1}} s^{\beta_j} \int_{-T}^{T} \int_{\omega_\omega} \mathrm{e}^{2s\varphi} |\partial_t^j v_1|^2 \mathrm{d}x\mathrm{d}t,$$

for each solution of system (2.14) and for some positive constants $\beta_j, j = 0, \ldots, 2^{n-1}$. In principle, this would allow to construct a control that would require stronger regularity assumptions.

Finally, let us remark that a slight change in the proof of proposition 1 shows that

$$I(0, v, \Omega) \leqslant C_2 \int_{-T}^{T} \int_{\Omega} \mathrm{e}^{2s\varphi} |\Box v + pv|^2 \mathrm{d}x\mathrm{d}t + C_2 s \int_{-T}^{T} \int_{\omega_2} \mathrm{e}^{2s\varphi} |\nabla v|^2 \mathrm{d}x\mathrm{d}t, \tag{4.1}$$

for all $v \in L^2(-T, T; H_0^1(\Omega))$ such that $\Box v + pv \in L^2(\Omega \times (-T, T)), \partial_\nu v \in L^2(\partial\Omega \times (-T, T))$ and $v(\pm T) = 0$ in Ω. The main ingredient of the proof are the part b) of lemma 1 and the weighted Poincaré inequality (see [8]). Under that form, estimate (4.1) can be used in the study of wave systems with first order coupling terms.

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