Discrete Applied Mathematics 245 (2018) 194–201

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Weighted antimagic labeling[☆,☆☆]

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ARTICLE INFO

Article history:
Received 29 December 2015
Received in revised form 5 May 2017
Accepted 10 May 2017
Available online 12 June 2017

Keywords:
Graph labeling
Antimagic labeling
Complete bipartite graph
Weighted antimagic labeling

ABSTRACT

A graph $G = (V, E)$ is *weighted-k-antimagic* if for each $w : V \to \mathbb{R}$, there is an injective function $f : E \to \{1, \ldots, |E| + k\}$ such that the following sums are all distinct: for each vertex u, $\sum_{v:uv\in E} f(uv) + w(u)$. When such a function f exists, it is called a (w, k)*-antimagic labeling* of G. A connected graph G is *antimagic* if it has a $(w_0, 0)$-antimagic labeling, for $w_0(u) = 0$, for each $u \in V$.

In this work, we prove that all the complete bipartite graphs $K_{p,q}$, are weighted-0-antimagic when $2 \le p \le q$ and $q \ge 3$. Moreover, an algorithm is proposed that computes in polynomial time a $(w, 0)$-antimagic labeling of the graph. Our result implies that if H is a complete partite graph, with $H \ne K_{1,q}, K_{2,2}$, then any connected graph G containing H as a spanning subgraph is antimagic.

1. Introduction

A connected graph $G = (V, E)$ with m edges and n vertices is *antimagic* if there exists a bijective function $f : E \to \{1, \ldots, m\}$, such that the following sums are all different: for each vertex u, $\sum_{e\in E(u)} f(e)$, where $E(u)$ is the set of edges incident to vertex u. Hartsfield and Ringel conjectured that every connected graph with at least two edges is antimagic [7].

It is easy to see that a graph with n vertices and maximum degree $n - 1$ is antimagic. It is also easy to see that cycles and paths are antimagic. Less obvious, in [2], it was proved that the class of antimagic graphs contains every complete partite graph, except K_2, and every graph with n vertices and maximum degree $n - 2$. This latter result was improved in [14], where it was proved that every graph with maximum degree at least $n - 3$ is antimagic as well. Cartesian products of various graphs, as path and cycles, have also been proved to be antimagic [4,11,12]. More general, in [2], it was also proved that there is a constant c such that any graph with n vertices and minimum degree at least $c \log n$ is antimagic. This result is obtained by applying Lovász's Local Lemma and, in fact, shows the existence of a much more general kind of labeling which we discuss later. In [5], it was proved that regular bipartite graphs are antimagic. This result was extended in [6] to regular graphs of odd degree, and recently, proved for all regular graphs [3]. However, the conjecture is still open even for trees, where the best result, proved in [10], shows that trees with at most one vertex of degree two are antimagic.

In order to gain more intuition about the conjecture, it is natural to explore some variations. Among the ideas considered so far, in [8], the following notion was considered. Given an integer k, a graph $G = (V, E)$ is *weighted-k-antimagic* if for any weight function $w : V \to \mathbb{R}$, there is an injective function $f : E \to \{1, \ldots, |E| + k\}$, such that the following sums are all

☆ An extended abstract of this work will appear in the Proceeding of LAGOS 2015, Electronic Notes in Discrete Mathematics.
☆☆ Partially supported Basal program PBF 03 and Núcleo Milenio Información y Coordinación en Redes ICM/FIC RC130003. The second author is also partially supported by Fondecyt Regular 1160975.
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http://dx.doi.org/10.1016/j.dam.2017.05.006

different: for each vertex u, $w(u) + \sum_{e\in E(u)} f(e)$. Such a function f is called a (w, k)-*antimagic labeling*. Clearly, if a graph is weighted-0-antimagic, then it is antimagic.

The proof of Theorem 2 in [2] actually shows that there is a constant c such that every graph with n vertices and minimum degree at least $c \log n$ is weighted-0-antimagic. Besides this latter result only a few more families are known to be weighted-0-antimagic. In [8], by using the Combinatorial Nullstellensatz Theorem [1], it was proved that any graph on n vertices having a 2-factor with each cycle of length 3, is weighted-0-antimagic, if $n = 3^l$, for some integer l. Later, in [9], this result was extended to any odd prime instead of 3.

On the other hand, not every graph is weighted-0-antimagic, as one can easily see, by considering the complete bipartite graph $K_{1,n-1}$ [13]. Then, it is natural to ask if there is a constant k such that every graph is weighted-k-antimagic. A partial answer to this question was given in [13] where it was shown that $K_{1,n-1}$ is weighted-2-antimagic, and it is weighted-1-antimagic, when n is odd. Moreover, it was also proved that a path on n vertices, with n prime, is weighted-1-antimagic.

The main characteristic of all the proofs, of the previously mentioned results, is that they are non-constructive as they are based on the Combinatorial Nullstellensatz Theorem [1]. For instance, for each weight function w there is a $(w, 1)$-antimagic labeling of $K_{1,12}$ and we do not know how to construct it.

Our contribution

The purpose of this work is to present an algorithmic approach to deal with weighted-k-antimagic graphs. This approach is new in this context and allows to generalize some known results, together with providing an explicit construction of the labelings. Even though some proofs appearing in the previously mentioned works on antimagic labeling are constructive, it is not clear how the associated algorithms can be transferred to the context of weighted-k-antimagic labeling. For instance, from the proof of Theorem 1.3 in [2], it is possible to deduce the existence of a procedure to construct a $(w, 0)$-antimagic labeling of $K_{p,q}$, but only for some specific w, those which are zero in the vertices of the smaller independent set.

In Section 2, we prove that there exists an algorithm for arbitrary weights.

Result 1. *For each $2 \le p \le q$ and $q \ge 3$, the graph $K_{p,q}$ is weighted-0-antimagic. Moreover, there exists an algorithm which on input $w \in \mathbb{R}^{p+q}$ runs in polynomial time and returns a $(w, 0)$-antimagic labeling of $K_{p,q}$.*

Result 1 is tight. In [13], it was noticed that $K_{1,n-1}$ is not weighted-0-antimagic. On the other hand, $K_{2,2}$ is not weighted-0-antimagic either. In fact, let $\{x_1, x_2\}$ and $\{y_1, y_2\}$ be the independent sets of $K_{2,2}$. Let w be the weight function given by $w(x_1) = w(x_2) = 0$ and $w(y_1) = w(y_2) = 1$. For the sake of contradiction, let us assume that $f : \{x_1y_1, x_2y_2, x_2y_1, x_2y_2\} \to \{1, 2, 3, 4\}$ is a $(w, 0)$-antimagic labeling of $K_{2,2}$. W.l.o.g., we can assume that $f(x_1y_1) = 1$. Hence, since $1 + 4 = 2 + 3$, we are forced to set $f(x_2y_2) = 4$. When $f(x_1y_2) = 3$ the vertex sum at vertex x_1 is $0 + 1 + 3 = 4$ and the sum at vertex y_1 is $1 + 1 + 2 = 4$. Otherwise, when $f(x_1y_2) = 2$, the sum at vertex x_2 is $0 + 3 + 4 = 7$ and the sum at vertex y_2 is $1 + 2 + 4 = 7$. Therefore, there is no $(w, 0)$-labeling of $K_{2,2}$.

From the graph theoretical point of view, the correctness of our algorithm implies that every complete bipartite graph with the adequate size is weighted-0-antimagic. To the best of our knowledge, this result is new.

From this, it follows that a large class of graphs are weighted-0-antimagic, thus antimagic. In fact, in Section 3, we prove the following result which is a generalization of Theorem 1.3 in [2] from antimagic labeling to weighted-0-antimagic labeling.

Result 2. *Let H be an arbitrary complete partite graph with at least five vertices and $H \ne K_{1,n-1}$. Then, any graph containing H as a spanning subgraph is weighted-0-antimagic. Moreover, given a weight function w, a $(w, 0)$-antimagic labeling can be computed in polynomial time.*

The ideas used in our algorithm allow us to give a short algorithmic proof of a generalization of the previously cited result about graphs having universal vertices [13]. More precisely, our result clarifies how far from being weighted-1-antimagic the graph $K_{1,n-1}$ is, by giving a complete characterization of those weights w for which a $(w, 0)$-antimagic labeling of $K_{1,n-1}$ does not exist. This information allows us to prove that the only graph with a universal vertex which is not weighted-1-antimagic is the graph $K_{1,n-1}$, when n is even.

Result 3. *Each graph G on n vertices having a universal vertex is weighted-1-antimagic, unless $G = K_{1,n-1}$ and n is even. Given a weight function w, a $(w, 1)$-antimagic labeling can be constructed in polynomial time.*

Surprisingly, despite its simplicity, our technique easily works if we replace the set $\{1, \dots, m\}$ by any set of m consecutive positive integers.

We can further extend previous result to any set of given integers. The proof of this result requires to enhance our algorithmic techniques to consider new difficulties not appearing in the current setting. We present this result in a forthcoming paper.

2. Antimagic algorithm

In this section, we present an algorithm, called Antimagic, which receives a weight function w and construct a $(w, 0)$-antimagic labeling of $K_{p,q}$.

For the sake of clarity, we present our algorithm in the language of matrices. Let $[p]$ denote the set $\{1, \ldots, p\}$, for $p \in \mathbb{N}$. It is clear that there is a one-to-one correspondence between edge labeling of $K_{p,q}$ using integers in $[pq]$ and $p \times q$ matrices filled with integers in $[pq]$. Given a matrix $A = (a_{ij})$ with p rows and q columns, and $d = (i, j) \in [p] \times [q]$, we call a_{ij} *the value of A at position d*; we denote it by $A(d)$. We denote by $r(A) \in \mathbb{R}^p$ and $c(A) \in \mathbb{R}^q$ the column and row vectors given by $r(A)_i = \sum_{j\in[q]} a_{ij}$, for each $i \in [p]$, and $c(A)_j = \sum_{i\in[p]} a_{ij}$, for each $j \in [q]$, respectively.

By using this terminology, $K_{p,q}$ is weighted-0-antimagic if and only if for each $a \in \mathbb{R}^p$ and $b \in \mathbb{R}^q$ we can find a $p \times q$ matrix A filled with all the integers in $[pq]$ and such that the sequence

$$S(A, a, b) := r(A)_1 + a_1, \ldots, r(A)_p + a_p, c(A)_1 + b_1, \ldots, c(A)_q + b_q,$$

has no repeated values. When such a matrix A exists we call it an *(a, b)-antimagic* matrix.

Without loss of generality, in the rest of this paper, we shall assume that a and b are *non-decreasing* vectors, that is, $a_1 \le \cdots \le a_p$ and $b_1 \le \cdots \le b_q$. We say that a matrix A with p rows and q columns, filled with all the integers in $[pq]$ is *increasing*, if $r(A)$ and $c(A)$ are *increasing* vectors, that is, $r(A)_1 < \cdots < r(A)_p$ and $c(A)_1 < \cdots < c(A)_q$, respectively.

Given a and b as before, and A an increasing matrix, we define vectors $r^a(A)$ and $c^b(A)$ by $r^a(A) = r(A) + a$ and $c^b(A) = c(A) + b$. Since a and b are non-decreasing vectors and, $r(A)$ and $c(A)$ are increasing vectors, the vectors $r^a(A)$ and $c^b(A)$ are increasing as well. In this situation, the sequence $S(A, a, b)$ has repeated values if and only if there are indices $i \in [p]$ and $j \in [q]$ such that $r^a(A)_i = c^b(A)_j$. We call such a pair (i, j) a *collision* of $S(A, a, b)$. Hence, an increasing matrix A is an (a, b)-antimagic matrix if and only if $S(A, a, b)$ has no collisions.

One important property of the set of all collisions is that it is totally ordered by the component-wise order of $\mathbb{R}^2$ that we denote by $\le_2$. That is to say, if (i, j) and (k, l) are two different collisions of $S(A, a, b)$, then either $i < k$ and $j < l$, or $k < i$ and $l < j$.

We call a collision (s, t) with $s < p$ and $t < q$ an *inner* collision. Otherwise, we say that it is a *boundary* collision. An inner collision (s, t) is *r-modifiable* if

$$r^a(A)_{s+1} \le c^b(A)_{t+1} \tag{1}$$

and it is *c-modifiable* when

$$c^b(A)_{t+1} < r^a(A)_{s+1}. \tag{2}$$

Local transformations

Given a matrix A of size $p \times q$ filled with distinct elements from the set $[pq]$, we define two local transformations.

Let $d = (s, t) \in [p] \times [q]$. When $s < p$ we define a *row flip of A at d* as the matrix obtained from A by interchanging its values at positions d and $d + e_r$, where $e_r = (1, 0)$; we denote it by *r-flip*(A, d). Similarly, if $t < q$, then the matrix obtained from A by interchanging its values at positions d and $d+e_c$, where $e_c = (0, 1)$, is called a *column flip of A at d* and it is denoted by *c-flip*(A, d).

It is immediate that for $A' = r\text{-flip}(A, d)$ we have $c(A') = c(A)$. Moreover, $r(A')$ and $r(A)$ differ only at coordinates s and $s + 1$, and

$$r(A)_{s+1} - r(A')_{s+1} = r(A')_s - r(A)_s = A(d + e_r) - A(d). \tag{3}$$

Similarly, if $A' = c\text{-flip}(A, d)$, then $r(A') = r(A)$. Moreover, $c(A')$ and $c(A)$ differ only at coordinates t and $t + 1$, and

$$c(A)_{t+1} - c(A')_{t+1} = c(A')_t - c(A)_t = A(d + e_c) - A(d). \tag{4}$$

Let $d = (s, t)$ be an inner collision. When A satisfies

$$0 < 2(A(d + e_r) - A(d)) < r^a(A)_{s+1} - r^a(A)_s \tag{5}$$

we say that A is *r-feasible* at d. Similarly, when

$$0 < 2(A(d + e_c) - A(d)) < c^b(A)_{t+1} - c^b(A)_t \tag{6}$$

we say that A is *c-feasible* at d.

We shall prove in the next lemma that under certain conditions, by applying row or column flips, we can reduce the number of collisions.

Lemma 1. *Let $a \in \mathbb{R}^p$, $b \in \mathbb{R}^q$ be non-decreasing vectors and let A be an increasing matrix. Let us assume that the smallest (w.r.t $\le_2$) collision d of $S(A, a, b)$ is an inner collision.*

1. *If d is r-modifiable and A is r-feasible at d, then $A' = r$-flip(A, d) is an increasing matrix.*
2. *If d is c-modifiable and A is c-feasible at d, then $A' = c$-flip(A, d) is an increasing matrix.*

In both cases, the set of collisions of $S(A', a, b)$ is a proper subset of the set of collisions of $S(A, a, b)$.

Proof. We give the proof only for the first case, as the second one can be proved in a similar manner.

As A is r-feasible at d, from (3) and the left hand inequality of (5) we have that

$$r^a(A)_s < r^a(A')_s = r^a(A)_s + A(d+e_r) - A(d),$$

$$r^a(A')_{s+1} = r^a(A)_{s+1} - (A(d+e_r) - A(d)) < r^a(A)_{s+1}$$

and by the right hand inequality of (5):

$$\begin{aligned} r^a(A')_s &= r^a(A)_s + (A(d+e_r) - A(d)) \\ &< r^a(A)_{s+1} - (A(d+e_r) - A(d)) \\ &= r^a(A')_{s+1} \end{aligned}$$

which implies

$$r^a(A)_s < r^a(A')_s < r^a(A')_{s+1} < r^a(A)_{s+1}.$$

Since $c^b(A) = c^b(A')$ and $r(A)_i = r(A')_i$, for each $i \neq s, s+1$, we deduce that A' is an increasing matrix.

Moreover, since $c^b(A')_t = r^a(A)_s < r^a(A')_s$, we see that (s, t) is not a collision of $S(A', a, b)$ and since A is r-modifiable, then $r^a(A')_{s+1} < r^a(A)_{s+1} \leq c^b(A')_{t+1}$, which means that no new collision appears. Finally, as we have $r^a(A')_i = r^a(A)_i$, for each $i \neq s, s+1$, and $c^b(A) = c^b(A')$, each collision of $S(A', a, b)$ is a collision of $S(A, a, b)$. $\square$

Observation 1. *Let A and A' be as in the previous lemma. Let us assume that $S(A, a, b)$ and $S(A', a, b)$ have collision and let $d = (s, t)$ and (s', t') be the smallest ones, respectively. From previous lemma, we deduce that*

1. *If d is r-modifiable and A is r-feasible at d, then $s' > s+1$ and $t' > t$.*
2. *If d is c-modifiable and A is c-feasible at d, then $s' > s$ and $t' > t+1$.*

Moreover, in both cases, if A is r-feasible (resp. c-feasible) at (i, j) with $i \geq s'$ and $j \geq t'$, then also A' is r-feasible (resp. c-feasible) at (i, j).

Now, we prove our first result:

Theorem 1. *For each $3 \leq p \leq q$, the graph $K_{p,q}$ is weighted-0-antimagic.*

Proof. We prove that given $a \in \mathbb{R}^p$ and $b \in \mathbb{R}^q$, two non-decreasing vectors, there is an (a, b)-antimagic matrix A. To ease the presentation, we denote by $S(A)$ the sequence $S(A, a, b)$.

The matrix A is constructed iteratively starting with the matrix A_0 given by $A_0(i, j) = (i-1)q + j$, for each $(i, j) \in [p] \times [q]$.

It is easy to check that A_0 is an increasing matrix. Hence, the conclusion is direct when $S(A_0)$ has no collision. Let us assume that $S(A_0)$ has at least one collision.

Notice that for each $i \in [p-1]$, $r(A_0)_{i+1} - r(A_0)_i = q^2$ and for each $j \in [q]$, $A_0(i+1, j) - A_0(i, j) = q$. As $q \geq 3$, we have that $q^2 > 2q$ which implies that A_0 is r-feasible at any inner collision of $S(A_0)$. On the other hand, for each $j \in [q-1]$, $c(A_0)_{j+1} - c(A_0)_j = p$ and for each $i \in [p]$, $A_0(i, j+1) - A_0(i, j) = 1$. Since $p \geq 3$, A_0 is c-feasible at any inner collision of $S(A_0)$.

Let A_i be an increasing matrix obtained at step i. If $S(A_i)$ has no collisions, then A_i is (a, b)-antimagic. Otherwise, we eliminate its smallest inner collision $d = (s, t)$. We define $A_{i+1} = r\text{-flip}(A_i)$ if d is r-modifiable or $A_{i+1} = c\text{-flip}(A_i)$ if d is c-modifiable. From Lemma 1 and Observation 1, we know that A_{i+1} is an increasing matrix such that the smallest collision (s', t') of $S(A_{i+1})$, if it exists, satisfies $s' > s+1$ and $t' > t$, when $A_{i+1} = r\text{-flip}(A_i)$, and $s' > s$ and $t' > t+1$, otherwise. We obtain a finite sequence of increasing matrices $A_0, A_1, \ldots, A_l$ such that for each $i < l$, $S(A_{i+1})$ has less collisions than $S(A_i)$. Since the set of collision of $S(A_0)$ is finite, then $S(A_l)$ either has no collision or it has exactly one boundary collision. In the first case, A_l is an (a, b)-antimagic matrix and the proof is finished. Otherwise, let $d = (s, t)$ be the boundary collision of $S(A_l)$. From Observation 1 we know that no r-flip was applied at row $p-1$ and no c-flip was applied at column $q-1$. Hence, the values of A_l in the last row and in the last column are those of A_0. In particular, $r^a(A_l)_p = r^a(A_0)_p$ and $c^b(A_l)_q = c^b(A_0)_q$.

If $s < p$ and $t = q$, let $A = r\text{-flip}(A_l, d)$. By Observation 1, A_l coincides with A_0 in column q and in all rows with index at least s, then

$$A_l(s+1, q) - A_l(s, q) = A_0(s+1, q) - A_0(s, q) = q$$

and $r(A_l)_{s+1} - r(A_l)_s = q^2$. These facts and $q \geq 3$ imply that

$$\begin{aligned} c^b(A)_q &= r^a(A_l)_s \\ &< r^a(A_l)_s + q = r^a(A)_s \\ &< r^a(A_l)_s + q^2 - q = r^a(A)_{s+1}. \end{aligned}$$

Hence, A is an increasing matrix. Since $c^b(A)_q$ is the maximum value of $c^b(A)$, the sequence $S(A)$ does not have any collision. Therefore, A is an (a, b)-antimagic matrix.

If $s = p$ and $t < q$, let $A = c$-flip(A_l, d). As before, it can be shown that A is an (a, b)-antimagic matrix.

Finally, when $(s, t) = (p, q)$, we still have two subcases. We first consider the case $r^a(A_l)_{p-1} \leq c^b(A_l)_{q-1}$. In this case, let $A = c$-flip$(A_l, (p, q-1))$. From Observation 1, we know that the smallest collision (s', t') of $S(A_{l-1})$ satisfies $s' \leq p-1$, $t' \leq q-1$ and $(s', t') \neq (p-1, q-1)$. Hence, $A_l(p, q-1) = A_0(p, q-1)$ and $A_l(p, q) = A_0(p, q) = A_0(p, q-1) + 1$. Hence,

$$r^a(A)_{p-1} \leq c^b(A_l)_{q-1} < c^b(A)_{q-1} = c^b(A_l)_{q-1} + 1.$$

Moreover, from the definition of flips, we know that $c^b(A_l)_{q-1} \leq c^b(A_0)_{q-1}$ (strict inequality holds when a c-flip was applied at some collision in column $q-2$). This implies that $c^b(A_l)_q - c^b(A_l)_{q-1} \geq c^b(A_0)_q - c^b(A_0)_{q-1} = q \geq 3$. Therefore,

$$c^b(A_l)_{q-1} + 1 < c^b(A)_q = c^b(A_l)_q - 1 < r^a(A)_p.$$

Thus, A is an (a, b)-antimagic matrix.

When $r^a(A_l)_{p-1} > c^b(A_l)_{q-1}$, one can prove with a similar argument that $A = r$-flip$(A_l, (p-1, q))$ is an (a, b)-antimagic matrix. □

From the proof of Theorem 1, we can describe an algorithm for the case $q \geq 3$. The algorithm, called Antimagic, receives as input two non-decreasing vectors $a \in \mathbb{R}^p$ and $b \in \mathbb{R}^q$, and computes an increasing matrix A with p rows and q columns such that the sequence $S(A, a, b)$ has no collision (see Fig. 1).

Algorithm 1 Antimagic(case $p \geq 3$)

Require: Two non-decreasing vectors $a \in \mathbb{R}^p$ and $b \in \mathbb{R}^q$, $3 \leq p \leq q$.
Ensure: A – an (a, b)-antimagic matrix.
  Initialization: $A(i, j) = (i-1)q + j$, for each $(i, j) \in [p] \times [q]$.
  while the sequence $S(A, a, b)$ has a collision **do**
    find $d = (s, t)$ the smallest, w.r.t. $\leq_2$, collision in $S(A, a, b)$
    if d is an inner collision **then**
      if $r^a(A)_{s+1} \leq c^b(A)_{t+1}$ **then** $A \leftarrow r$-flip(A, d).
      else $A \leftarrow c$-flip(A, d).
    else if $s < p$ and $t = q$, **then** $A \leftarrow r$-flip(A, d).
    else if $s = p$ and $t < q$, **then** $A \leftarrow c$-flip(A, d).
      else if $r^a(A)_{p-1} \leq c^b(A)_{q-1}$ **then** $A \leftarrow c$-flip$(A, d - e_c)$
      else $A \leftarrow r$-flip$(A, d - e_r)$.
  end while
  return A

Let us discuss the complexity of the algorithm. The direct implementation of the algorithm takes time proportional to pq as it has to manage matrices of size $p \times q$. We notice that the only relevant information in order to obtain the (a, b)-antimagic matrix is the set of flips that the algorithm performs in the initial matrix A_0. Hence, one can obtain a linear time version of the algorithm by only using the values of $r^a(A_0)$ and $c^b(A_0)$, if we allow to represent the output in terms of a set of flips.

Initially, we compute $r^a(A_0)_i = a_i + q(q+1)/2 + (i-1)q^2$ for each $i \in [p]$ and $c^b(A_0)_j = b_j + p(1 + q(p-1)/2) + (j-1)p$, for each $j \in [q]$ which can be done in time $O(p+q)$. It is clear that by checking the values $r^a(A_0)$ and $c^b(A_0)$, we can determine all the collision of $S(A_0)$ and also those where the Antimagic algorithm performs some flips, together with the type of the flips. Obviously, we can extend the algorithm to arbitrary inputs $a \in \mathbb{R}^p$ and $b \in \mathbb{R}^q$ by first sorting each of them. This adds time $O(p \log p + q \log q)$ to the complexity of the algorithm.

We now consider the case $p = 2$.

Lemma 2. *Let $q \geq 3$. Let A_0, A_1 and A_2 be the matrices defined by $A_0(i, j) = 2(j-1) + i$, for each $(i, j) \in [2] \times [q]$, $A_1 = r$-flip$(A_0, (1, 1))$ and $A_2 = r$-flip$(A_1, (1, 2))$. Then, for every two non-decreasing vectors $a \in \mathbb{R}^2$ and $b \in \mathbb{R}^q$, at least one of these matrices is an (a, b)-antimagic matrix.*

Proof. We assume that neither A_0 nor A_1 are (a, b)-antimagic and we prove that A_3 is (a, b)-antimagic. Let j and l are integers in $[q]$ such that $c^b(A_0)_j \in \{r^a(A_0)_1, r^a(A_0)_2\}$ and $c^b(A_1)_l \in \{r^a(A_1)_1, r^a(A_1)_2\}$. From the definition of A_0 we have that $r^a(A_0)_2 - r^a(A_0)_1 \geq q \geq 3$. Since $r^a(A_1)_1 = r^a(A_0)_1 + 1$ and $r^a(A_1)_2 = r^a(A_0)_2 - 1$, we have $r^a(A_1)_1 < r^a(A_1)_2$ and $l \neq j$. We also obtain from the definition of A_0 that consecutive values of $c^b(A_0)$ differs by at least 4. Hence, $c^b(A_0)_j = r^a(A_0)_1$ if and only if $c^b(A_0)_l = r^a(A_1)_2$. When $c^b(A_0)_j = r^a(A_0)_1$, we have that $c^b(A_0)_l = r^a(A_1)_2$ and $j < l$. Hence,

$$c^b(A_0)_j < r^a(A_2)_1 = r^a(A_0) + 2 < c^b(A_0)_{j+1}$$

and

$$c^b(A_0)_{l-1} < r^a(A_2)_2 = r^a(A_1)_2 - 1 < c^b(A_0)_l.$$

Therefore, A_2 is (a, b)-antimagic. In the case of $c^b(A_0)_j = r^a(A_0)_2$, we can proceed in a similar way. □

In this case, we need a simpler algorithm.

	0	30	35	126	140	161	252	
126	1	2	3	4	5	6	7	154
126	8	9	10	11	12	13	14	203
126	15	16	17	18	19	20	21	252
126	22	23	24	25	26	27	28	301
126	29	30	31	32	33	34	35	350
126	36	37	38	39	40	41	42	399
126	43	44	45	46	47	48	49	448
	154	191	203	301	322	350	448	

	0	30	35	126	140	161	252	
126	2	1	3	4	5	6	7	154
126	8	9	10	11	12	13	14	203
126	15	16	17	18	19	20	21	252
126	22	23	24	25	26	27	28	301
126	29	30	31	32	33	34	35	350
126	36	37	38	39	40	41	42	399
126	43	44	45	46	47	48	49	448
	155	190	203	301	322	350	448	

	0	30	35	126	140	161	252	
126	2	1	3	4	5	6	7	154
126	8	9	17	11	12	13	14	210
126	15	16	10	18	19	20	21	245
126	22	23	24	25	26	27	28	301
126	29	30	31	32	33	34	35	350
126	36	37	38	39	40	41	42	399
126	43	44	45	46	47	48	49	448
	155	190	203	301	322	350	448	

	0	30	35	126	140	161	252	
126	2	1	3	4	5	6	7	154
126	8	9	17	11	12	13	14	210
126	15	16	10	18	19	20	21	245
126	22	23	24	26	25	27	28	301
126	29	30	31	32	33	34	35	350
126	36	37	38	39	40	41	42	399
126	43	44	45	46	47	48	49	448
	155	190	203	302	321	350	448	

	0	30	35	126	140	161	252	
126	2	1	3	4	5	6	7	154
126	8	9	17	11	12	13	14	210
126	15	16	10	18	19	20	21	245
126	22	23	24	26	25	27	28	301
126	29	30	31	32	33	41	35	357
126	36	37	38	39	40	34	42	392
126	43	44	45	46	47	48	49	448
	155	190	203	302	321	350	448	

	0	30	35	126	140	161	252	
126	2	1	3	4	5	6	7	154
126	8	9	17	11	12	13	14	210
126	15	16	10	18	19	20	21	245
126	22	23	24	26	25	27	28	301
126	29	30	31	32	33	41	35	357
126	36	37	38	39	40	34	49	399
126	43	44	45	46	47	48	42	441
	155	190	203	302	321	350	448	

Fig. 1. Five iteration of ANTIMAGIC algorithm. The input vectors are $a = (126, 126, 126, 126, 126, 126)$ and $b = (0, 30, 35, 126, 140, 161, 252)$. The initial matrix A_0 has five collisions: $(1, 1), (2, 3), (4, 4), (5, 6)$ and $(7, 7)$. The (a, b)-antimagic matrix is obtained from A_0 by applying c-flips at $(1, 1)$ and $(4, 4)$, and r-flips at $(2, 3), (5, 6)$ and $(6, 7)$.

Algorithm 2 ANTIMAGIC-2 (case $p = 2$)

Require: Two non-decreasing vectors $a \in \mathbb{R}^2$ and $b \in \mathbb{R}^q$, $3 \le q$.
Ensure: A – an (a, b)-antimagic matrix.
Initialization: $A(i, j) = 2(j - 1) + i$, for each $(i, j) \in [2] \times [q]$.

if $S(A, a, b)$ has a collision **then** $A \leftarrow r$-flip$(A, (1, 1))$.
  if $S(A, a, b)$ has a collision **then** $A \leftarrow r$-flip$(A, (1, 2))$.
return A

3. Antimagic graphs

From the results of the previous section, we obtain the following corollary which for further use we present in terms of graphs.

Corollary 1. *Given two integers p, q with $2 \le p \le q$ and $q \ge 3$, the complete bipartite graph $K_{p,q}$ is weighted-0-antimagic. Moreover, given a function w, a $(w, 0)$-antimagic labeling can be computed in polynomial time.*

In [13], the authors made the following observation.

Observation 2. *If a graph contains a weighted-k-antimagic spanning subgraph, then it is weighted-k-antimagic.*

We use Observation 2 and Corollary 1 to show the following result.

Theorem 2. *Let H be an arbitrary complete partite graph with $n \ge 5$ vertices and $H \ne K_{1,n-1}$. Then, any graph containing H as a spanning subgraph is weighted-0-antimagic. Moreover, given a weight function w, a $(w, 0)$-antimagic labeling can be computed in polynomial time.*

Proof. From Observation 2, it is enough to prove that any complete partite graph $H \ne K_{1,n-1}$ with at least five vertices is weighted-0-antimagic. One can see that a such graph H always contains a spanning complete bipartite graph $K_{p,q}$, with $2 \le p \le q$ and $q \ge 3$. From Corollary 1, we get the conclusion. □

We already mentioned that $K_{1,m}$ is not weighted-0-antimagic and that it is weighted-k-antimagic, where $k \in \{1, 2\}$ is such that $k + m$ is odd.

In terms of matrices, this means that there are $a \in \mathbb{R}$ and $b \in \mathbb{R}^m$ such that no (a, b)-antimagic matrix of size $1 \times m$ exists. By using the ideas given in ANTIMAGIC algorithm, we give a complete characterization of those a and b preventing the existence of (a, b)-antimagic matrices of size $1 \times m$.

This provides a new proof of Theorem 13 in [13]. This proof is shorter than the original one and, additionally, it provides an effective procedure to obtain the desired antimagic matrix.

We say that a matrix A filled with coefficients in $[m + k]$ is an (a, b, k)-*antimagic matrix* if its coefficients are all distinct and $S(A, a, b)$ has no repetitions.

Theorem 3 (*Theorem 13, in [13]*). *Given $a \in \mathbb{R}$ and $b \in \mathbb{R}^m$ a non-decreasing vector, there is an (a, b, k)-antimagic matrix, where $k \in \{1, 2\}$ is such that $m + k$ is odd.*

Proof. Let $d_0 = (1, \dots, m)$. If d_0 is not an $(a, b, 0)$-antimagic matrix, then there is a unique i such that $b_i + i = a + m(m+1)/2$. Let p be the largest integer such that $b_i = b_{i+p}$ and let $\alpha := a + m(m+1)/2$.

Given $t \in \mathbb{N}$ and $k \in \{1, 2\}$, let $d^{t,k}$ denote the matrix of size $1 \times m$ given by

$$d^{t,k} = (1, \dots, t, t+2, \dots, m, m+k).$$

Let α' be the sum of the values of $d^{t,k}$. Then, $\alpha' = \alpha + m + k - (t+1)$. We prove that there are t and k such that $d^{t,k}$ is (a, b, k)-antimagic.

When $p = m-1$, we have that $i = 1$, $\alpha = b_1 + 1$ and b is constant. Let $t = \lceil m/2 \rceil$ and $k \in \{1, 2\}$ such that $m + k$ is odd. Then, $m + k - 1 = 2t$. Hence, $\alpha' = b_1 + 1 + t$. Therefore, $d^{t,k}$ is an (a, b, k)-antimagic matrix since

$$d^{t,k}_t + b_t = t + b_1 < \alpha' < t + 2 + b_1 = d^{k,t}_{t+1} + b_{t+1}.$$

If $p < m - 1 \le 2p + i$, then let $t = m - p - 2$ and $k = 1$. Then, $\alpha' = \alpha + m - t = \alpha + p + 2$. Since, $2p + i \ge m - 1$ we have that $i + p \ge t + 1$. Then,

$$d^{t,k}_{i+p} + b_{i+p} = i + p + 1 + b_i = \alpha + p + 1 < \alpha'.$$

If $i + p = m$, then we conclude that $d^{t,k}_m + b_m < \alpha'$. Otherwise, by the definition of p, we get that

$$\alpha + p + 2 = \alpha' < d^{t,k}_{p+i+1} + b_{p+i+1} = i + p + 2 + b_{p+i+1}.$$

The last case is when $2p + i < m - 1$. Let $t = m - p - 1$ and $k = 1$. Then, $i + p < m - p - 1 = t$. Hence,

$$d^{t,k}_{i+p} + b_{i+p} = i + p + b_i = \alpha + p < \alpha + p + 1 = \alpha + m - t = \alpha'.$$

As $t \ge i + p + 1$, from the definition of p, we get that

$$\alpha + p + 1 = \alpha' < d^{t,k}_{i+p+1} + b_{i+p+1} = i + p + 1 + b_{i+p+1}. \quad \square$$

Observation 3. *From the proof of Theorem 3, it is clear that given $a \in \mathbb{R}$ and $b \in \mathbb{R}^m$, we can construct an (a, b, k)-antimagic matrix in polynomial time.*

Observation 4. *We notice that in the proof of Theorem 3, the only case where k is forced to be 2 is when $p = m - 1$ and m is odd. This situation happens if and only if a is constant and $a + m(m+1)/2 = b_1$. Hence, the only vectors preventing the existence of an $(a, b, 1)$-antimagic matrix is when a and b satisfy $b_i = m(m+1)/2 + a$, for each $i \in [m]$.*

Observation 5. *Theorem 3 remains valid if we use any set of $m+k$ consecutive integers instead of the set $[m+k]$, in the following sense: given a set I of $m+k$ consecutive integers, a non-decreasing vector $b \in \mathbb{R}^m$ and $a \in \mathbb{R}$, then there is a $1 \times m$ matrix A filled with integers in I and such that $S(A, a, b)$ has no repetitions unless $b_i = m(m+1)/2 + a$, for each $i \in [m]$. In fact, given I, a and b, we can find a' and b' such that by adding a constant to each entry of an (a', b', k)- antimagic matrix, we obtain the desired matrix A.*

The information given in Observations Observations 4 and 5 leads us to the following generalization of Theorem 3. As in the previous cases, from its proof, we can obtain an efficient algorithm that, given a weight function, w computes a $(w, 1)$-antimagic labeling.

Let G be a graph on $n \ge 3$ vertices and of maximum degree $n - 1$. From Theorem 13 in [12], we know that G is weighted-1-antimagic when n is odd, and it is weighted-2-antimagic when n is even. In the next theorem, we improve this conclusion by showing that G is weighted-1-antimagic unless $G = K_{1,n-1}$ and n is even.

Theorem 4. *Each connected graph G on $n \ge 3$ vertices having $K_{1,n-1}$ as a spanning subgraph is weighted-1-antimagic unless $G = K_{1,n-1}$ and n is even.*

Proof. Let $K_{1,n-1}$ be a spanning subgraph of G and let x be the vertex of degree $n-1$ of $K_{1,n-1}$. Let $w : V(G) \to \mathbb{N}$ be a weight function. We assume that $w(y_1) \le \cdots \le w(y_{n-1})$, where $y_1, \dots, y_{n-1}$ are the leaves of $K_{1,n-1}$.

Let E be the set of edges of G and let m be its cardinality. Let g be any labeling from the set of edges of $G - x$ to $[m+1] \setminus [n-1]$. Then, g defines partial sums at the leaves of $K_{1,n-1}$ that we denote by

$$b_i(g) = w(y_i) + \sum_{v \ne x, y_i v \in E} g(y_i v),$$

for each $i \in [n-1]$.

We apply Theorem 3 and Observation 4 to $K_{1,n-1}$ with weight function $b = b(g) \in \mathbb{R}^{n-1}$ and $a = w(x)$. When $b(g)$ is non-constant or it is constant and n is odd, there exists an $(b(g), b, 1)$-antimagic matrix d. Then, we can extend g to E by $g(xy_i) = d_i$, for each $i \in [n-1]$, and to obtain a $(w, 1)$-antimagic labeling of G.

We can proceed in a similar way when $b(g)$ is constant, n is even and $G-x$ has at least two edges. In fact, by interchanging the values of g in two edges of $G-x$, we get a new labeling g' such that the vector $b(g')$ is non-constant.

It remains to consider the case when $b(g)$ is constant, $G-x$ has at most one edge and n is even, thus $n \geq 4$. Since $G \neq K_{1,n-1}$, the graph $G-x$ has exactly one edge $e = y_j y_l$ and $g(e) = m+1 = n+1$. Let g'' be the labeling assigning to e the value 1. Since $n \geq 4$, there exists at least one vertex y_i isolated in $G-x$. Then, $b(g'')_i = b(g)_i = w(y_i) = b(g)_j = b(g'')_j + n$. Then, $a(g'')$ is non-constant. By Observation 5, we can use the set $[m+1] \setminus \{1\}$ in the proof of Theorem 3 to obtain an $(b(g''), b, 1)$-antimagic matrix. As before, from this matrix, we get a $(w, 1)$-antimagic labeling of G. $\square$

It is still possible that each G with n vertices and maximum degree $n-1$ is weighted-0-antimagic, unless $G = K_{1,n-1}$. In order to analyze this possibility, it would be worth to characterize those $b \in \mathbb{R}^{n-1}$ and $a \in \mathbb{R}$ for which an $(a, b, 0)$-antimagic matrix of $K_{1,n-1}$ does not exist. We leave this as an open problem.

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