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A NON VARIATIONAL VERSION OF A MAX–MIN PRINCIPLE

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1. INTRODUCTION

LET H BE a real Hilbert space and $f: H \to \mathbb{R}$ of class C^2. We denote by ∇f and $D^2 f$ the gradient and the Hessian of f respectively. It is known that if there exists a constant $m > 0$ such that $\forall u \in H$ and $\forall w \in H$ we have

$$\langle D^2 f(u)\, w, w\rangle \geq m\|w\|^2 \tag{1}$$

then there exists a unique $u_0 \in H$ such that $\nabla f(u_0) = 0$. Furthermore $f(u_0) = \min_{u \in H} f(u)$. In 1975 Lazer, Landesman & Meyer (L.L.M.) [3] extended these results to the following situation. Let X and Y be two closed subspaces of H, such that $H = X \oplus Y$, X is finite dimensional and X and Y are not necessarily orthogonal. Let $T = \nabla f$, then $T: H \to H$ and is a C^1 mapping. Its Frechet derivative at $u \in H$ is given by $T'(u) = D^2 f(u)$. Let m_1 and m_2 be two positive constants such that $\forall u \in H$, $\forall x \in X$, and $\forall y \in Y$ we have

$$\langle T'(u)x, x\rangle \leq -m_1\|x\|^2 \tag{2}$$

$$\langle T'(u)y, y\rangle \geq m_2\|y\|^2. \tag{3}$$

Under these conditions (L.L.M.) proved that there exists a unique $u_0 \in H$, such that $T(u_0) = 0$ and that u_0 satisfies

$$f(u_0) = \max_{x \in X} \min_{y \in Y} f(x + y). \tag{4}$$

These results were applied in [3] to prove the existence of solutions of boundary value problems.

In [6] assuming that neither X nor Y are finite dimensional the existence of a unique $u_0 \in H$ which satisfies $T(u_0) = 0$ was proved. Furthermore u_0 can be characterized by

$$f(u_0) = \max_{x \in X} \min_{y \in Y} f(x + y) = \min_{y \in Y} \max_{x \in X} f(x + y). \tag{5}$$

In this paper we will consider a C^n mapping $T: H \to H$, $n \geq 1$. Assuming that $T'(u)$, $u \in H$ satisfies (2) and (3) and that

$$\langle T'(u)x, y\rangle = \langle x, T'(u)y\rangle, \tag{6}$$

$\forall u \in H$, $\forall x \in X$, $\forall y \in Y$, we will prove that T is a C^n diffeomorphism. In particular it will follow the existence of a unique $u_0 \in H$, such that $T(u_0) = 0$.

We note that if $T = \nabla f$, then $T'(u)$ is a self-adjoint operator for each $u \in H$. In this case condition (6) is automatically satisfied. Thus our results will include those of (L.L.M.) Since in our case $T'(u)$ is not necessarily a self-adjoint operator, our results provide a non-self-adjoint extension of the (L.L.M.) results.

Condition (6) is also satisfied if X and Y are orthogonal and $T'(u)(X) \subset X$, $T'(u)(Y) \subset Y$, $\forall u \in H$. The extreme case $X = \{0\}$, $Y = H$ is also of interest.

2. THE MAIN RESULT

In this section H will denote a real Hilbert space and X and Y two closed subspaces of H such that $H = X \oplus Y$. We will start this section with some preliminary results.

PROPOSITION 2.1. Let $S: H \to H$ be a bounded linear mapping and let m_1, m_2 be two positive constants such that

$$\langle Sx, x \rangle \leqslant -m_1 \|x\|^2 \tag{7}$$

$\forall x \in X$,

$$\langle Sy, y \rangle \geqslant m_2 \|y\|^2 \tag{8}$$

$\forall y \in Y$, and

$$\langle Sx, y \rangle = \langle x, Sy \rangle, \tag{9}$$

$\forall x \in X$ and $\forall y \in Y$. Then S is an isomorphism onto H. Furthermore if $c = \min\{m_1, m_2\} > 0$, then $\|S^{-1}\| \leqslant 2/c$.

Proof. It is enough to prove that S is a one to one mapping from H onto H, see for instance [2, p. 18]. Let $u \in H$, $u = x + y$, $x \in X$, $y \in Y$. We have

$$\langle Su, y - x \rangle = \langle Sx, y \rangle - \langle Sx, x \rangle + \langle Sy, y \rangle - \langle Sy, x \rangle. \tag{10}$$

From (7), (8) and (9) we obtain that

$$\langle Su, y - x \rangle \geqslant m_1 \|x\|^2 + m_2 \|y\|^2. \tag{11}$$

On the other hand it is clear that

$$\|x \pm y\|^2 \leqslant 2\,[\|x\|^2 + \|y\|^2]. \tag{12}$$

Thus, first applying the Cauchy–Schwartz inequality to the left hand side of (11), taking the square of both sides of the resulting inequality, and then applying (12), we obtain

$$c^2[\|x\|^2 + \|y\|^2] \leqslant 2\,\|Su\|^2. \tag{13}$$

Applying again (12) to the left-hand side of (13) we obtain

$$\frac{c}{2}\|u\| \leqslant \|Su\|. \tag{14}$$

From (14) it follows that S is a one to one mapping. Following a straightforward argument it also follows from (14) that $S(H)$ is a closed subspace of H. We will prove next that $S(H) = H$. To do this let us assume there exists a $z \in S(H)^{\perp}$, $z \neq 0$. Then $\langle z, Su\rangle = 0$, $\forall u \in H$. We have that z can be decomposed as $z = h + k$, where $h \in X$ and $k \in Y$. Take $u = k - h$, then we have

$$0 = \langle z, Su\rangle = \langle h, Sk\rangle - \langle h, Sh\rangle + \langle k, Sk\rangle - \langle k, Sh\rangle. \tag{15}$$

From (7), (8), (9) and (15) it follows that

$$0 = \langle z, Su\rangle \geqslant m_1\|h\|^2 + m_2\|k\|^2, \tag{16}$$

which is a contradiction. Thus $S(H)^{\perp} = \{0\}$ and S is onto H. Finally from (14) it follows that $\|S^{-1}\| \leqslant 2/c$.

The next proposition will be also needed in the proof of our main theorem. The proof of this proposition follows from theorem 1.22 of [7] and from theorem 5.4.4 of [2]. ■

PROPOSITION 2.2. Let E and F be two real Banach spaces and $T: E \to F$ a C^n mapping, $n \geqslant 1$. Let $T'(u)$ denote the Frechet derivative of T at $x \in E$. If
(i) $T'(x)$ is an isomorphism from E onto F for each $x \in E$,
(ii) $\|[T'(x)]^{-1}\| \leqslant m$, $\forall x \in E$, where m is a positive constant,
then T is a C^n diffeomorphism onto F.

The theorem it follows next is our main theorem. Let the spaces H, X and Y have the meaning established at the beginning of this section.

THEOREM 2.1. Let $T: H \to H$ be a C^n mapping, $n \geqslant 1$, and assume there exist two positive constants m_1 and m_2 such that

$$\langle T'(u)x, x\rangle \leqslant -m_1\|x\|^2, \tag{17}$$

$\forall u \in H$, $\forall x \in X$,

$$\langle T'(u)y, y\rangle \geqslant m_2\|y\|^2, \tag{18}$$

$\forall u \in H$, $\forall y \in Y$,

$$\langle T'(u)x, y\rangle = \langle x, T'(u)y\rangle, \tag{19}$$

$\forall u \in H$, $\forall x \in X$, $\forall y \in Y$. Then under these condition we have that T is a C^n diffeomorphism onto H.

Proof. From (17), (18) and (19) and from proposition 2.1. it follows that for each $u \in H$, $T'(u)$ is an isomorphism from H onto H and furthermore that $\|[T'(u)]^{-1}\| \leqslant 2/c$, where $c = \min\{m_1, m_2\}$. Thus, all the conditions of proposition 2.2. are satisfied, so T is a C^n diffeomorphism onto H. ■

3. SOME APPLICATIONS OF THEOREM 2.1

Let $G: \mathbb{R}^n \to \mathbb{R}$ be C^2, $e: \mathbb{R} \to \mathbb{R}^n$ continuous and 2π-periodic and A be a constant symmetric matrix. Let $(,)$ and $| \ |$ denote the euclidean inner product and norm in $\mathbb{R}^n$, respectively. Using Theorem 2.1 we can prove the following

THEOREM 3.1. If there exist an integer $N \geq 0$ and positive constants γ_1 and γ_2 such that

$$N^2 < \gamma_1 \leq \gamma_2 < (N+1)^2 \tag{20}$$

$$\gamma_1 I \leq D^2 G(u) \leq \gamma_2 I \tag{21}$$

$\forall u \in \mathbb{R}^n$, where $D^2G(u)$ denotes the Hessian of G at $u \in \mathbb{R}^n$ and I is the $n \times n$ identity matrix, then the differential equation

$$z'' + Az' + \nabla G(z) = e(t) \tag{22}$$

has a unique 2π-periodic solution.

We note that the case $A = 0$ has been studied in [5] by Lazer & Sanchez.

Proof. Let us say $v \in P_n$ if,
(i) $v : \mathbb{R} \to \mathbb{R}^n$ is 2π-periodic and absolutely continuous, (ii) $\int_0^{2\pi} |v'(t)|^2 \, dt < \infty$. It is known that P_n is a real Hilbert space for the following inner product

$$\langle u, v \rangle = \int_0^{2\pi} [(u', v') + (u, v)] \, dt.$$

The norm induced in P_n by this inner product will be denoted by $\| \ \|$. Let us define now two subspaces of P_n as follows, X consists of the $x \in P_n$ such that $x(t) = (a_0/2) + \sum_{k=1}^{N} (a_k \cos kt + b_k \sin kt)$ and Y consists of the $y \in P_n$ such that $y(t) = \sum_{k=N+1}^{\infty} (a_k \cos kt + b_k \sin kt)$, where a_k, $b_k \in \mathbb{R}^n$ and $\sum_{k=1}^{\infty} (k^2 + 1)(|a_k|^2 + |b_k|^2) < \infty$. Then we have that X and Y are two closed subspaces of P_n such that $P_n = X \oplus Y$.

Next using the Riesz representation theorem let us define a mapping $T : P_n \to P_n$ by

$$\langle T(u), v \rangle = \int_0^{2\pi} [(u', v') - (Au', v) - (\nabla G(u), v)] \, dt \tag{23}$$

$\forall v \in P_n$. We observe that T in (23) is defined implicitly. From (23) and from the fact that G is C^2 it can be proved that T is C^1 and that

$$\langle T'(u)w, v \rangle = \int_0^{2\pi} [(v', w') - (Aw', v) - (D^2G(u)w, v)] \, dt \tag{24}$$

$\forall w, v, u \in P_n$. We note at this point that in general $T'(u)$, $u \in P_n$, is not a self-adjoint operator. Again using the Riesz representation theorem let d be the unique element in P_n such that

$$\langle d, v \rangle = -\int_0^{2\pi} (e(t), v(t)) \, dt. \tag{25}$$

$\forall v \in P_n$. It can be proved that u is a 2π-periodic solution of (22) if and only if u satisfies the operator equation

$$T(u) = d. \tag{26}$$

We will next show that T satisfies the conditions of Theorem 2.1. (for $n = 1$). This in turn will imply that (22) has a unique 2π-periodic solution.

Let $x \in X$, $y \in Y$ and let u be any element in P_n. We have

$$\langle T'(u)x, y\rangle - \langle x, T'(u)y\rangle = -\int_0^{2\pi} (Ax'(t), y(t))\,\mathrm{d}t + \int_0^{2\pi} (x(t), Ay'(t))\,\mathrm{d}t = 0, \tag{27}$$

since Ax' is orthogonal to y and x is orthogonal to Ay' in $L_2[0, 2\pi]$. Thus (19) of Theorem 2.1. is satisfied. Next let us note that for $v \in P_n$ we have

$$\int_0^{2\pi} (Av', v)\,\mathrm{d}t = \tfrac{1}{2}(Av(t), v(t))\Big|_0^{2\pi} = 0. \tag{28}$$

From (24) and (28) and for any $u \in P_n$, any $x \in X$ and any $y \in Y$ we have that

$$\langle T'(u)x, x\rangle = \int_0^{2\pi} \left[(x', x') - (D^2G(u)x, x)\right]\mathrm{d}t \tag{29}$$

and

$$\langle T'(u)y, y\rangle = \int_0^{2\pi} \left[(y', y') - (D^2G(u)y, y)\right]\mathrm{d}t. \tag{30}$$

We also have that for $x \in X$ and $y \in Y$ the following inequalities are true

$$\int_0^{2\pi} |x'(t)|^2\,\mathrm{d}t \leqslant N^2 \int_0^{2\pi} |x(t)|^2\,\mathrm{d}t \tag{31}$$

$$\int_0^{2\pi} |y'(t)|^2\,\mathrm{d}t \geqslant (N+1)^2 \int_0^{2\pi} |y(t)|^2\,\mathrm{d}t. \tag{32}$$

From (20), (21), (29), (30), (31) and (32) it is possible to prove the existence of two positive constants m_1, m_2 such that (17) and (18) of Theorem 2.1. are satisfied. The proof of this is left to the reader. ■

Using similar techniques we can prove the following.

THEOREM 3.2. Let G, A and e as in Theorem 3.1. Suppose there exist two real constant symmetric matrices E and F such that for $u \in \mathbb{R}^n$,

$$E \leqslant D^2G(u) \leqslant F$$

and such that if $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ and $\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_n$ denote the eigenvalues of E and F respectively then there exist integers $N_k \geqslant 0$, $k = 1, 2, \ldots, n$, such that

$$N_k^2 < \lambda_k \leqslant \mu_k < N_{k+1}^2.$$

Then under the above conditions the differential equation (22) has a unique 2π-periodic solution.

The uniqueness and existence of a 2π-periodic solution of (22) for the case $A = 0$ has been proved by Lazer [4] and Ahmad [1].

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