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EXISTENCE AND MULTIPLICITY OF SOLUTIONS WITH PRESCRIBED PERIOD FOR A SECOND ORDER QUASILINEAR O.D.E.*

MANUEL A. DEL PINO
School of Mathematics, University of Minnesota, Minneapolis, MN 55455, U.S.A.

RAÚL F. MANÁSEVICH
Departamento de Matemáticas, F.C.F.M., Universidad de Chile, Casilla 170, Correo 3, Santiago, Chile

and

ALEJANDRO E. MURÚA
Division of Applied Mathematics, Brown University, Providence, RI 02912, U.S.A.

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1. INTRODUCTION

IN THIS paper we are concerned with the existence and multiplicity of T-periodic solutions for the nonlinear ordinary differential equation

$$(\phi_p(u'))' + f(t, u) = 0, \tag{1.1}$$

where $\phi_p: \mathbb{R} \to \mathbb{R}$ is given by $\phi_p(s) = |s|^{p-2}s$, $p > 1$, and $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic in t, $T > 0$. (′) denotes d/dt.

By a solution of (1.1) in an interval $I \subset \mathbb{R}$ we mean a function $u: I \to \mathbb{R}$ such that u and $\phi_p(u')$ are continuously differentiable on I and satisfy (1.1).

We note that when $p = 2$, the nonlinear operator $\phi_p(u')'$ reduces to the linear operator u''.

In Section 2 we will define the *Fucik spectrum* for the operator $\phi_p(u')'$ under T-periodic conditions and briefly study some of its properties. We denote this spectrum by S. We will see that $S = \bigcup_{k=0}^{\infty} C_k$, where

$$C_0 = \{(\alpha, \beta) \in \mathbb{R}^2 \mid \alpha = 0 \text{ or } \beta = 0\} \tag{1.2}$$

and for $k \in \mathbb{N}$, C_k is the curve in the first quadrant given by

$$C_k = \left\{(\alpha, \beta) \in \mathbb{R}^2 \mid \alpha^{-1/p} + \beta^{-1/p} = \frac{T}{k\pi_p}\right\}. \tag{1.3}$$

In (1.3), π_p is given by

$$\pi_p = 2(p-1)^{1/p} \int_0^1 (1 - t^p)^{-1/p}\, dt. \tag{1.4}$$

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The results of Sections 4 and 5 will be formulated in terms of the interaction of the nonlinearity f with S.

In Section 4 we will prove that if for $s > 0$ sufficiently large and for all $t \in \mathbb{R}$, the pair

$$\left(\frac{f(t,-s)}{\phi_p(-s)}, \frac{f(t,s)}{\phi_p(s)}\right) \tag{1.5}$$

lies either in a closed rectangle included in the open region bounded by C_k and C_{k+1}, for some $k \in \{0\} \cup \mathbb{N}$, or in the open third quadrant, then (1.1) possesses at least one T-periodic solution.

This result is related with that of [2], for $p = 2$, and extends to the periodic case that of [1] obtained for the Dirichlet boundary value problem.

In Section 5 we let $f(t, 0) = 0$, for all $t \in \mathbb{R}$, and look for nontrivial T-periodic solutions of (1.1). To describe our result let U_k, V_k denote the two open components of $\mathbb{R}^2 \backslash C_k$, $k \in \mathbb{N}$, and let $i, j \in \mathbb{N}$ be such that $i \leq j$. Under certain additional conditions on f which imply that for small positive s and all $t \in \mathbb{R}$ the pair (1.5) stays in a closed rectangle contained in U_i, (V_j) and that for large positive s and all $t \in \mathbb{R}$ the pair (1.5) stays in a closed rectangle contained in V_j, (U_i), we will prove that (1.1) has at least $2(j - i + 1)$ nontrivial T-periodic solutions.

The above result roughly says that if the pair (1.5) *crosses* n curves C_k, $k \in \mathbb{N}$, as s increases from 0 to ∞, then there will be at least $2n$ nontrivial T-periodic solutions of (1.1).

The proof of the results of Section 4 will make use of degree theory and those of Section 5 of the Poincaré–Birkhoff theorem as stated, for instance, in [5]. We remark that the results of both sections depend heavily on a Sturmian comparison result which will be proved in Section 3.

2. REVIEW OF THE FUCIK SPECTRUM FOR $\phi_p(u')'$

We define the Fucik spectrum for $\phi_p(u')'$ under T-periodic conditions to be the set S consisting of all the pairs $(\alpha, \beta) \in \mathbb{R}^2$ such that the equation

$$(\phi_p(u'))' + \beta\phi_p(u^+) - \alpha\phi_p(u^-) = 0 \tag{2.1}$$

possesses nontrivial T-periodic solutions.

Integrating (2.1) from 0 to T we obtain immediately that a necessary condition on (α, β) for (2.1) to have a nontrivial T-periodic solution is that α and β be nonnegative. It is also elementary to observe that C_0 is the unique subset of S associated with nonnull constant solutions of (2.1). Thus, we can assume α and β positive in the rest of this section.

Let us consider the initial value problem

$$(\phi_p(u'))' + \phi_p(u) = 0 \tag{2.2}$$

$$u(0) = 0, \qquad u'(0) = 1. \tag{2.3}$$

As in [5], we denote by $\sin_p t$ the unique solution, defined in the whole real line, to the above initial value problem. We have that $\sin_p t$ is a $2\pi_p$-periodic solution which vanishes at $t = k\pi_p$, $k \in \mathbb{Z}$.

Let us define $w_0 \colon [0, \pi_p(\alpha^{-1/p} + \beta^{-1/p})] \to \mathbb{R}$ by

$$w_0(t) = \begin{cases} \beta^{-1/p} \sin_p \beta^{1/p} t & \text{if } t \in [0, \pi_p \beta^{-1/p}) \\ -\alpha^{-1/p} \sin_p \alpha^{1/p}(t - \pi_p \beta^{-1/p}) & \text{if } t \in [\pi_p \beta^{-1/p}, \pi_p(\alpha^{-1/p} + \beta^{-1/p})]. \end{cases} \tag{2.4}$$

Clearly, w_0 can be extended to a C^1 $\pi_p(\alpha^{-1/p} + \beta^{-1/p})$-periodic function $w: \mathbb{R} \to \mathbb{R}$, which solves (2.1)–(2.3) on the whole real line. From proposition A2 in the Appendix we have that w is in fact the unique solution to this initial value problem. Furthermore w will be T-periodic if and only if, for some $k \in \mathbb{N}$,

$$k\pi_p(\alpha^{-1/p} + \beta^{-1/p}) = T, \tag{2.5}$$

i.e. if and only if $(\alpha, \beta) \in C_k$, $k \in \mathbb{N}$.

We thus have the following proposition.

PROPOSITION 2.1. Equation (2.1) possesses a nontrivial T-periodic solution if and only if $(\alpha, \beta) \in C_k$ for some $k \in \{0\} \cup \mathbb{N}$.

Proof. We only have to show that if for $\alpha > 0, \beta > 0$ there is a nontrivial T-periodic solution u of (2.1), then $(\alpha, \beta) \in C_k$ for some $k \in \mathbb{N}$. So let $u: \mathbb{R} \to \mathbb{R}$ be such a T-periodic solution. Integrating (2.1) from 0 to T, we obtain that there must be a $t_0 \in [0, T]$ such that $u(t_0) = 0$. Since (2.1) is an autonomous equation, uniqueness of the initial value problem for (2.1) yields that $u(t) = dw(t - t_0)$ for some constant $d \neq 0$. Thus $w(t - t_0)$ is T-periodic and hence $(\alpha, \beta) \in C_k$, for some $k \in \mathbb{N}$. This finishes the proof of the proposition. ■

Note. In the particular case $\alpha = \beta = \lambda$, equation (2.1) reduces to

$$(\phi_p(u'))' + \lambda\phi_p(u) = 0. \tag{2.6}$$

From proposition 2.1, (2.6) possesses a nontrivial solution if and only if $\lambda = \lambda_n$, where

$$\lambda_n = \left(\frac{2\pi_p n}{T}\right)^p \qquad n = 0, 1, 2, 3, \ldots. \tag{2.7}$$

In this case we say that λ_n is an eigenvalue of (2.6).

3. A STURMIAN COMPARISON RESULT

Let us consider the equation

$$(\phi_p(u'))' + b(t)\phi_p(u^+) - a(t)\phi_p(u^-) = 0 \tag{3.1}$$

where $t \in [0, T]$ and $a, b \in L^\infty(0, T)$.

By a solution of this equation we mean a function $u \in C^1[0, T]$ such that $\phi_p(u')$ is absolutely continuous in $[0, T]$ and satisfies (3.1) a.e. in $(0, T)$.

The argument in proposition A2 of the Appendix implies uniqueness for the initial value problem associated to (3.1). In particular the zeros of any nontrivial solution of (3.1) must be simple, so that the following definition makes sense.

Definition 3.1. Let u be a nontrivial solution of (3.1) and let k denote the number of zeros of u in $(0, T)$ (k may be zero). We define the rotation of u in $[0, T]$ as

$$\eta(u) = k\pi - \lim_{\varepsilon \to 0^+}\left(\arctan\frac{u'}{u}(T - \varepsilon) - \arctan\frac{u'}{u}(\varepsilon)\right). \tag{3.2}$$

Geometrically, $\eta(u)$ can be interpreted as the total angle the vector $(u(t), u'(t))$ describes in the (u, u') plane as t goes from 0 to T. Positive rotations are defined to be clockwise.

Note. Sometimes we will write $\eta(u(\alpha, \beta))$ to explicitly show the dependency of the rotation on the initial conditions (α, β) of u.

Next we will prove a Sturm-type comparison theorem for equations of the form (3.1).

THEOREM 3.2. Let $\underline{a}, \bar{a}, \underline{b}, \bar{b} \in L^\infty(0, T)$ be such that

$$\underline{a} < \underline{b} \quad \text{and} \quad \bar{a} < \bar{b}, \quad \text{a.e. in } (0, T). \tag{3.3}$$

Let u and v be respectively nontrivial solutions of

$$(\phi_p(u'))' + \bar{a}(t)\phi_p(u^+) - \underline{a}(t)\phi_p(u^-) = 0 \tag{3.4}$$

$$(\phi_p(v'))' + \bar{b}(t)\phi_p(v^+) - \underline{b}(t)\phi_p(v^-) = 0. \tag{3.5}$$

If $u(0) = v(0)$ and $u'(0) = v'(0)$, then

$$\eta(u) < \eta(v). \tag{3.6}$$

Proof. We do the proof for the case u has zeros in $(0, T)$. Thus let $t_1 < t_2 < \cdots < t_k$ denote the zeros of u and $s_1 < s_2 < \cdots < s_l$ the zeros of v in the interval $(0, T)$.

We show first that $s_1 < t_1$. We do this by contradiction, i.e. we assume $s_1 \geq t_1$. Without loss of generality we can take $u(t) > 0$, $v(t) > 0$, $t \in (0, t_1)$.

Setting $w_a(t) = \phi_p((u'(t))/(u(t)))$, $w_b(t) = \phi_p((v'(t))/(v(t)))$ $t \in (0, t_1)$, from (3.4) and (3.5) it follows that w_a and w_b respectively satisfy

$$w_a' + (p-1)|w_a|^{p'} + \bar{a}(t) = 0 \tag{3.7}$$

$$w_b' + (p-1)|w_b|^{p'} + \bar{b}(t) = 0, \tag{3.8}$$

$t \in (0, t_1)$. From here, letting $z(t) = (w_b(t) - w_a(t))$ and $h(t) = p\int_0^1 \phi_{p'}(sw_a(t) + (1-s)w_b(t))\,ds$, we have that z satisfies

$$z' + h(t)z = (\bar{a}(t) - \bar{b}(t)) < 0 \qquad \text{a.e. in } (0, t_1). \tag{3.9}$$

Equation (3.9) implies that there must exist a $t^* \in (0, t_1)$ such that $z(t^*) \neq 0$. Suppose first that $z(t^*) > 0$. From (3.9) it follows that $z(t) > 0$ for all $t \in (0, t^*)$. The definition of $h(t)$ then yields $h(t) > p\phi_{p'}(w_a(t)) = p(u'(t))/(u(t))$ for all $t \in (0, t^*)$. Substituting this inequality in (3.9) and multiplying the resultant expression by $u^p(t)$ we obtain that

$$(u^p(t)z(t))' < 0. \tag{3.10}$$

Integrating this expression from 0 to t^* and using the fact that $\lim_{t\to 0^+} u^p(t)z(t) = 0$ we find that $u^p(t^*)z(t^*) < 0$, which is impossible. Thus, let us suppose that $z(t^*) < 0$. Again from (3.9) we obtain that $z(t) < 0$ for all $t \in (t^*, t_1)$. Reasoning as above and using the fact that $\lim_{t\to t_1} u^p(t)z(t) = 0$ we find that $-u^p(t^*)z(t^*) > 0$, which is again impossible. Thus we have that the assumption $s_1 \geq t_1$ leads to a contradiction and therefore we must have $s_1 < t_1$.

Next, from a Sturm comparison theorem for equations of the form

$$(\phi_p(x'))' + \alpha(t)\phi_p(x) = 0,$$

see [3], it follows that v must vanish in each of the intervals (t_i, t_{i+1}), $i = 1, \ldots, k-1$. This shows that $k \leq l$ and that $s_i \leq t_i$ for $i = 1, \ldots, k$.

Now we continue with the proof of the theorem. We set for convenience $s_0 = t_0 = 0$. From the distribution property of the zeros of u and v we have just proved, we obtain that

$$\eta(v) - \eta(u) = (l - k)\pi + \lim_{\varepsilon \to 0^+} \left(\arctan \frac{u'}{u}(T - \varepsilon) - \arctan \frac{v'}{v}(T - \varepsilon) \right). \tag{3.11}$$

If $l > k$, (3.6) follows from (3.11) and the fact that the second term on the right-hand side of (3.11) must be greater than $-\pi$. Thus we suppose that $k = l$. In this case u and v have the same sign, say the positive sign, in the interval $(t_l, T]$. Defining $z(t)$ as above and reasoning as before, it is easy to see that

$$\lim_{t \to t_l^+} u^p(t) z(t) = 0 \tag{3.12}$$

and hence that $z(t) \leq 0$ for all $t \in (t_l, T)$. From the definition of $z(t)$ we then obtain

$$\eta(v) - \eta(u) \geq 0.$$

Now, $\eta(v) = \eta(u)$ implies that $\lim_{t \to T_-} u^p(t) z(t) = 0$ and this in turn that $z(t) \geq 0$, for all $t \in (t_l, T]$. As before, we obtain a contradiction and thus (3.6) follows. ■

Note. If $(\alpha, \beta) \in C_k$ and u is a nontrivial T-periodic solution of (2.1) then

$$\eta(u) = 2\pi k.$$

4. AT LEAST ONE T-PERIODIC SOLUTION OF (1.1)

In this section we will prove the following theorem.

THEOREM 4.1. Suppose there are real numbers $\alpha^+, \alpha^-, \beta^+, \beta^-$ and $s_0 > 0$ such that for all $s > s_0$ and for all $t \in \mathbb{R}$ we have

$$\alpha^- \leq \frac{f(t, -s)}{\phi_p(-s)} \leq \alpha^+ \qquad \beta^- \leq \frac{f(t, s)}{\phi_p(s)} \leq \beta^+ \tag{4.1}$$

and that at least one of the following alternatives holds true:

(i) $$\alpha^{\pm} > 0 \qquad \beta^{\pm} > 0 \tag{4.2}$$

and for some $k \in \mathbb{N}$

$$\frac{T}{(k+1)\pi_p} < (\alpha^+)^{-1/p} + (\beta^+)^{-1/p} \leq (\alpha^-)^{-1/p} + (\beta^-)^{-1/p} < \frac{T}{k\pi_p}; \tag{4.3}$$

(ii) $$\alpha^{\pm} > 0, \qquad \beta^{\pm} > 0 \qquad \text{and} \qquad \frac{T}{\pi_p} < (\alpha^+)^{-1/p} + (\beta^-)^{-1/p}; \tag{4.4}$$

(iii) $$\alpha^+ < 0 \qquad \text{and} \qquad \beta^+ < 0. \tag{4.5}$$

Then (1.1) possesses at least one T-periodic solution.

Geometrically, the hypotheses of theorem 4.1 imply that the closed rectangle $[\alpha^-, \alpha^+] \times [\beta^-, \beta^+]$ does not intercept the Fucik spectrum and is included in either the first or the third quadrant. We observe that if this rectangle is included in either the second or the fourth quadrant, then a T-periodic solution of (1.1) may not exist, as the examples

$$(\phi_p(u'))' - |u|^{p-1} - 1 = 0$$

and

$$(\phi_p(u'))' + |u|^{p-1} + 1 = 0$$

respectively show. (Just integrate along a period.)

In order to prove this theorem we need some lemmas.

LEMMA 4.2. For each $h \in L^q(0, T)$, $q > 1$, the boundary value problem

$$-(\phi_p(u'))' + \phi_p(u) = h \tag{4.6}$$

$$u(0) = u(T) \qquad u'(0) = u'(T), \tag{4.7}$$

$p > 1$, has a unique solution $R_p(h) \in C^1[0, T]$. Furthermore the operator R_p transforms weak convergency in L^q into strong convergency in $C^1[0, T]$.

The proof of this lemma is similar to one given in Section 2 of [5] and hence it will not be given here.

As an immediate consequence of this lemma we obtain that the operator R_p seen as an operator from $C[0, T]$ into $C^1[0, T]$ is completely continuous. Thus the problem of searching for T-periodic solutions for (1.1) is equivalent to finding solutions in $C^1[0, T]$ of the equation

$$u = R_p(\phi_p(u) + F(u)) \tag{4.8}$$

where $F: C^1[0, T] \to C[0, T]$ is defined by $F(u)(t) = f(t, u(t))$.

We note that the right-hand side of (4.8) defines a completely continuous operator from $C^1[0, T]$ into itself.

LEMMA 4.3. Suppose that $(\alpha, \beta) \in \mathbb{R}^2 \backslash S$ and $\alpha\beta > 0$. Define the operator $T_{\alpha,\beta}: C^1[0, T] \to C^1[0, T]$ by

$$T_{\alpha,\beta}(u) = R_p(\phi_p(u) + \beta\phi_p(u^+) - \alpha\phi_p(u^-)). \tag{4.9}$$

Then, for each $r > 0$ the Leray Schauder degree $d(I - T_{\alpha,\beta}, B(0, r), 0)$ is well defined and different from zero.

Proof. From the definition of $T_{\alpha,\beta}$ and proposition 2.1, it follows that the unique solution of the equation

$$u - T_{\alpha,\beta}(u) = 0 \tag{4.10}$$

is the trivial one. Hence the above degree is well defined and independent of r.

Next, since $\alpha\beta > 0$ there is a pair $(\lambda, \lambda) \in \mathbb{R}^2 \backslash S$ and a continuous curve $(\alpha(\tau), \beta(\tau))$, $\tau \in [0, 1]$ whose image is in $\mathbb{R}^2 \backslash S$ and is such that $(\alpha(0), \beta(0)) = (\alpha, \beta)$, $(\alpha(1), \beta(1)) = (\lambda, \lambda)$.

From the invariance property of the degree under compact homotopies it follows that the degree $d(I - T_{\alpha(\tau),\beta(\tau)}, B(0, r), 0)$ is constant for $\tau \in [0, 1]$. Also from (4.9) it follows that the operator $T_{\lambda,\lambda}$ is odd and thus from Borsuk's theorem that $d(I - T_{\lambda,\lambda}, B(0, r), 0) \neq 0$. Clearly this fact implies the lemma. ■

We can now prove theorem 4.1.

Proof of theorem 4.1. Let us set $\bar{\alpha} = (\alpha^- + \beta^-)/2$, $\bar{\beta} = (\alpha^+ + \beta^+)/2$ and consider the homotopy

$$Q(\tau, u) = R_p(\phi_p(u) + (1 - \tau)F(u) + \tau(\bar{\alpha}\phi_p(u^+) - \bar{\beta}\phi_p(u^-))), \tag{4.11}$$

where $u \in C^1[0, T]$ and $\tau \in [0, 1]$. We claim that there is an $r > 0$ such that for all $\tau \in [0, 1]$ the equation

$$u = Q(\tau, u) \tag{4.12}$$

does not have a solution $u \in C^1[0, T]$ such that $\|u\|_1 \geq r$. Here and henceforth $\| \ \|_1$ denotes the usual norm in $C^1[0, T]$. Assume this is not true. Then there exist sequences $\{u_n\}_{n=1}^{\infty}$ in $C^1[0, T]$ and $\{\tau_n\}_{n=1}^{\infty}$ in $[0, T]$ with $\|u_n\|_1 \to \infty$, $\tau_n \to \bar{\tau}$ such that

$$u_n = Q(\tau_n, u_n). \tag{4.13}$$

Dividing both sides of (4.13) by $\|u_n\|_1$, noting that $tR_p(u) = R_p(t^{p-1}u)$ for $t > 0$ and setting $\hat{u}_n = u_n/\|u_n\|_1$ we obtain

$$\hat{u}_n = R_p\left(\phi_p(\hat{u}_n) + (1 - \tau_n)\frac{F(u_n)}{\|u_n\|_1^{p-1}} + \tau_n\frac{(\bar{\alpha}\phi_p(u_n^+) - \bar{\beta}\phi_p(u_n^-))}{\|u_n\|_1}\right). \tag{4.14}$$

Now from the hypotheses on f it follows that the sequence $h_n = F(u_n)/\|u_n\|^{p-1}$ is bounded in $C[0, T]$ and hence the argument of R_p in (4.14) is bounded in $C[0, T]$. Since R_p is a completely continuous operator we can assume, passing to a subsequence if necessary, that $\hat{u}_n \to \hat{u}$ in $C^1[0, T]$. Moreover reasoning as in [2] or [4], we can also assume that $h_n \rightharpoonup h$ in L^q where h is given by $h(t) = a(t)\phi_p(\hat{u}^+) - b(t)\phi_p(\hat{u}^-)$ with

$$\alpha^- \leq a(t) \leq \alpha^+ \qquad \beta^- \leq b(t) \leq \beta^+ \qquad \text{a.e. in } [0, T]. \tag{4.15}$$

Letting $n \to \infty$ in (4.14) yields

$$\hat{u} = R_p(\phi_p(\hat{u}) + \bar{a}\phi_p(\hat{u}^+) - \bar{b}\phi_p(\hat{u}^-)), \tag{4.16}$$

where $\bar{a}(t) = \bar{\tau}\bar{\alpha} + (1 - \bar{\tau})a(t)$ and $\bar{b}(t) = \bar{\tau}\bar{\beta} + (1 - \bar{\tau})b(t)$.

From (4.15) we obtain that

$$\alpha^- \leq \bar{a}(t) \leq \alpha^+ \qquad \beta^- \leq \bar{b}(t) \leq \beta^+ \qquad \text{a.e. in } [0, T] \tag{4.17}$$

and from (4.16) that $\hat{u}$ satisfies

$$(\phi_p(\hat{u}'))' + \bar{a}(t)\phi_p(\hat{u}^+) - \bar{b}(t)\phi_p(\hat{u}^-) = 0 \qquad \text{a.e. } t \in [0, T] \tag{4.18}$$

$$\hat{u}(0) = \hat{u}(T) \qquad \hat{u}'(0) = \hat{u}'(T). \tag{4.19}$$

Next, let us assume that condition (i) is satisfied. Then there exist pairs $(\alpha_k, \beta_k) \in C_k$, $(\alpha_{k+1}, \beta_{k+1}) \in C_{k+1}$ such that

$$\alpha_k < \alpha^- \leq \alpha^+ < \alpha_{k+1} \qquad \beta_k < \beta^- \leq \beta^+ < \beta_{k+1}. \tag{4.20}$$

Let v_k, v_{k+1} be respectively T-periodic solutions of

$$(\phi_p(v_j'))' + \beta_j\phi_p(v_j^+) - \alpha_j\phi_p(v_j^-) = 0 \qquad j = k, k + 1, \tag{4.21}$$

such that

$$v_k(0) = v_{k+1}(0) = \hat{u}(0) \qquad v_k'(0) = v_{k+1}'(0) = \hat{u}'(0). \tag{4.22}$$

From theorem 3.2 we obtain that

$$\eta(v_k) < \eta(\hat{u}) < \eta(v_{k+1}) \tag{4.23}$$

and then from the note at the end of the last section that

$$2\pi k < \eta(\hat{u}) < 2(k + 1)\pi. \tag{4.24}$$

Since $\hat{u}$ satisfies T-periodic boundary conditions its corresponding rotation $\eta(\hat{u})$ must be of the form $2\pi l$ where l is a nonnegative integer. Thus (4.24) is a contradiction and the claim is true if condition (i) is satisfied.

Now suppose that condition (ii) or (iii) holds. Then there exists a pair $(\alpha_1, \beta_1) \in C_1$ such that

$$\alpha^+ < \alpha_1 \qquad \beta^+ < \beta_1. \tag{4.25}$$

From (4.25) and theorem 3.2 of last section we obtain

$$\eta(\hat{u}) < 2\pi \tag{4.26}$$

which is again a contradiction.

Thus there must exist $r > 0$ such that for all $\tau \in [0, 1]$ equation (4.12) does not have a solution with $\|u\|_1 = r$.

From the invariance of the degree under compact homotopies and lemma 4.3 it follows that

$$d(I - Q(0, \cdot), B(0, r), 0) \neq 0. \tag{4.27}$$

This implies the existence of a solution of equation (4.12) in the ball $B(0, r)$ and therefore the existence of a T-periodic solution of (1.1). ■

We now have the following corollary.

COROLLARY 4.4. Let λ_k, $k \in \mathbb{N}$, be an eigenvalue of (2.6). Suppose there exist real numbers α and β such that

$$\lambda_k < \alpha \leq \frac{f(t, x)}{\phi_p(x)} \leq \beta < \lambda_{k+1}, \tag{4.28}$$

for all $t \in \mathbb{R}$ and all $|x|$ sufficiently large. Then (2.1) has at least a T-periodic solution.

5. NONTRIVIAL T-PERIODIC SOLUTIONS OF (1.1)

In this section we will assume that f in equation (1.1) satisfies the conditions of Section 1 and that $f(t, 0) = 0$, for all $t \in \mathbb{R}$. Furthermore we will suppose that f satisfies the growth restriction

$$|f(t, x)| \leq C|x|^{p-1}, \qquad \text{for all } x \in \mathbb{R}, t \in \mathbb{R}, \tag{5.1}$$

and that $f(t, \phi_{p'}(x))$ is locally Lipschitz in x. Also, if $p > 2$, we will assume that $f(t, x) \neq 0$ for all $x \neq 0$. These conditions ensure, in particular, existence, uniqueness and extendibility for the solutions to the I.V.P. associated to (1.1). This is proved in the Appendix.

THEOREM 5.1. Suppose there are pairs of real numbers $(\alpha_i, \beta_i) \in C_i$, $(\alpha_j, \beta_j) \in C_j$, for $i \in \mathbb{N}$, $j \in \mathbb{N}$, $i \leq j$, and positive real numbers ε, m, M, such that the pair $(\alpha_i - \varepsilon, \beta_i - \varepsilon)$ belongs to V_i, the pair $(\alpha_j + \varepsilon, \beta_j + \varepsilon) \in C_j$, belongs to U_{j+1} and such that for all $t \in \mathbb{R}$ one of the following hypotheses (i) or (ii) holds true:

(i)
$$\frac{f(t, -s)}{\phi_p(-s)} \leq \alpha_i - \varepsilon, \qquad \frac{f(t, s)}{\phi_p(s)} \leq \beta_i - \varepsilon, \qquad \text{if } s \geq M \tag{5.2}$$

and

$$\frac{f(t, -s)}{\phi_p(-s)} \geq \alpha_j + \varepsilon, \qquad \frac{f(t, s)}{\phi_p(s)} \geq \beta_j + \varepsilon, \qquad \text{if } s \leq m; \tag{5.3}$$

(ii)
$$\frac{f(t, -s)}{\phi_p(-s)} \geq \alpha_j + \varepsilon, \qquad \frac{f(t, s)}{\phi_p(s)} \geq \beta_j + \varepsilon, \qquad \text{if } s \geq M \tag{5.4}$$

and

$$\frac{f(t, -s)}{\phi_p(-s)} \leq \alpha_i - \varepsilon, \qquad \frac{f(t, s)}{\phi_p(s)} \leq \beta_i - \varepsilon, \qquad \text{if } s \leq m. \tag{5.5}$$

Then in both cases where (i) holds or (ii) holds, we have that (1.1) possesses at least $2(j - i + 1)$ nontrivial T-periodic solutions.

Note. From (5.1), (5.4) and (5.5) we have that $C \geq \max\{\alpha_j + \varepsilon, \beta_j + \varepsilon\}$. Then it is easy to see that for $s > 0$ and small the pair (1.5) must be in a closed rectangle contained in $U_i(V_j)$ and for $s > 0$ and large the pair (1.5) must be in a closed rectangle contained in $V_j(U_i)$.

To prove theorem 5.1 we need the following lemma.

LEMMA 5.2. Under the conditions of theorem 5.1 there exist two positive numbers δ and Δ such that

(a) if (i) is satisfied then

$$0 < \sqrt{\alpha^2 + \beta^2} \leq \delta \Rightarrow \eta(u(\alpha, \beta)) > 2j\pi, \tag{5.6}$$

and

$$\sqrt{\alpha^2 + \beta^2} \geq \Delta \Rightarrow \eta(u(\alpha, \beta)) < 2i\pi; \tag{5.7}$$

(b) if (ii) is satisfied then

$$0 < \sqrt{\alpha^2 + \beta^2} \leq \delta \Rightarrow \eta(u(\alpha, \beta)) < 2i\pi \tag{5.8}$$

and

$$\sqrt{\alpha^2 + \beta^2} \geq \Delta \Rightarrow \eta(u(\alpha, \beta)) > 2j\pi. \tag{5.9}$$

Proof. We only prove (a) since the proof of (b) is similar. Let $u(t) \equiv u(t, \alpha, \beta)$ denote the solution of (1.1) with initial conditions (α, β). From the fact that $u \equiv 0$ is a solution of (1.1) and from continuity under initial conditions it follows that there is a $\delta > 0$ such that

$$0 < \sqrt{\alpha^2 + \beta^2} < \delta \Rightarrow \sup_{t \in [0, T]} |u(t, \alpha, \beta)| < m. \tag{5.10}$$

We then choose α and β such that $0 < \sqrt{\alpha^2 + \beta^2} < \delta$. We have that u satisfies

$$(\phi_p(u'))' + \bar{b}(t)\phi_p(u^+) - \underline{b}(t)\phi_p(u^-) = 0 \tag{5.11}$$

$t \in [0, T]$, where

$$\bar{b}(t) = \begin{cases} \dfrac{f(t, u(t))}{\phi_p(u(t))} & \text{if } u(t) > 0 \\ \beta_j + \varepsilon & \text{if } u(t) \leq 0 \end{cases} \tag{5.12}$$

$$\underline{b}(t) = \begin{cases} \dfrac{f(t, u(t))}{\phi_p(u(t))} & \text{if } u(t) < 0 \\ \alpha_j + \varepsilon & \text{if } u(t) \geq 0. \end{cases} \tag{5.13}$$

From (5.1), (5.2), (5.11) and (5.12) it follows that $\bar{b}$ and $\underline{b} \in L^\infty(0, T)$ and that $\bar{b} \geq \beta_j + \varepsilon$, $\underline{b} \geq \alpha_j + \varepsilon$.

Now let us consider the differential equation

$$(\phi_p(v'))' + \beta_j\phi_p(u^+) - \alpha_j\phi_p(u^-) = 0 \tag{5.14}$$

for $t \in [0, T]$. From the note of Section 3 we have that

$$\eta(v(\alpha, \beta)) = 2j\pi, \tag{5.15}$$

for any pair $(\alpha, \beta) \in C_j$.

Thus, applying theorem 3.2 to (5.10) and (5.14) and calling on (5.14) we obtain that (5.6) follows.

Next we show the existence of a Δ such that (5.7) is satisfied.

Suppose such a Δ does not exist. Then there exists a sequence of pairs of real numbers (α_n, β_n), $n \in \mathbb{N}$ such that $\sqrt{\alpha_n^2 + \beta_n^2} \to +\infty$ as $n \to \infty$ and $\eta(u_n) \geq 2i\pi$. Here $u_n(t) = u(t, \alpha_n, \beta_n)$, $n \in \mathbb{N}$. Let us set $\hat{u}_n = u_n/\|u_n\|_1$. From (5.1) we obtain that $f(t, u_n)/\|u_n\|_1^{p-1}$ is uniformly bounded on $[0, T]$. Thus, the fact that u_n is a solution of (1.1) for the initial conditions $\alpha = \alpha_n$, $\beta = \beta_n$ and the Ascoli–Arzela theorem imply that $\{\hat{u}_n\}_{n=1}^\infty$ possesses a convergent subsequence in $C^1[0, T]$ which we again denote by $\{\hat{u}_n\}_{n=1}^\infty$. Thus $\lim_{n\to\infty} \hat{u}_n = \hat{u}$ with $\|\hat{u}\|_1 = 1$. Again, reasoning as in [2] or [4] and passing to a subsequence if necessary, which we again denote by $\{\hat{u}_n\}_{n=1}^\infty$, we obtain that $h_n = f(t, u_n)/\|u_n\|_1^{p-1}$, $n \in \mathbb{N}$, converges weakly in $L^q(0, T)$, as $n \to \infty$, to h of the form

$$h(t) = \bar{a}(t)\phi_p(u^+) - \underline{a}(t)\phi_p(u^-), \tag{5.16}$$

where $\bar{a}$, $\underline{a} \in L^\infty$, and are such that

$$\bar{a}(t) \leq \beta_i - \varepsilon \qquad \underline{a}(t) \leq \alpha_i - \varepsilon \qquad \text{a.e. on } [0, T]. \tag{5.17}$$

From (1.1), we have that

$$\phi_p(\hat{u}_n'(t)) = \phi_p(\hat{u}_n'(0)) - \int_0^t h_n(s)\,\mathrm{d}s, \tag{5.18}$$

$n \in \mathbb{N}$. Thus, letting $n \to \infty$ in (5.18) it follows that

$$\phi_p(\hat{u}'(t)) = \phi_p(\hat{u}'(0)) - \int_0^t h(s)\,\mathrm{d}s. \tag{5.19}$$

Thus $\hat{u}$ satisfies

$$(\phi_p(\hat{u}'))' + \bar{a}(t)\phi_p(\hat{u}^+) - \underline{a}(t)\phi(\hat{u}^-) = 0. \tag{5.20}$$

Since $\eta(u_n) = \eta(\hat{u}_n) \geq 2i\pi$, for all $n \in \mathbb{N}$ it follows that $\eta(\hat{u}) \geq 2i\pi$. On the other hand if we consider (5.20) and the comparison equation

$$(\phi_p(v'))' + \beta_i |v|^{p-2}v^+ - \alpha_i |v|^{p-2}v^- = 0, \tag{5.21}$$

we conclude from theorem 3.2 that

$$\eta(\hat{u}) < 2i\pi. \tag{5.22}$$

This is a contradiction and hence (5.7) follows. Thus (a) is proved. ■

Next let us write (1.1) as the equivalent first order system

$$v' = -f(t, u) \qquad u' = \phi_{p'}(v). \tag{5.23}$$

Let P: $\mathbb{R}^2 \to \mathbb{R}^2$ denote the Poincaré map obtained by following the solutions of (5.23) for time T. We have that P is a well-defined area preserving homeomorphism which satisfies $P(0, 0) = (0, 0)$.

Let A be the open region bounded by the circles a_δ and a_Δ defined by

$$a_\delta = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = \delta^2\}$$

and

$$a_\Delta = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = \Delta^2\},$$

and let $\bar{P}$ be the restriction of $\bar{P}$ to A.

The proof of theorem 5.1 will consist of showing that $\bar{P}$ has at least $2(j - i + 1)$ nontrivial different fixed points. To do this we will use the Poincaré-Birkhoff theorem.

We apply polar coordinate transformation $u(t) = R(t)\cos\Theta(t)$, $v(t) = R(t)\sin\Theta(t)$ to (5.23) to obtain

$$\begin{aligned} R' &= -f(t, R\cos\Theta) + R^{p'-1}\phi_{p'}(\sin\Theta)\cos\Theta \\ \Theta' &= -\frac{f(t, R\cos\Theta)}{R}\cos\Theta - R^{p'-2}|(\sin\Theta)|^{p'}. \end{aligned} \tag{5.24}$$

We have that (5.23) and (5.24) are equivalent problems and thus (5.24) possesses a unique solution $(R(t, r, \theta), \Theta(t, r, \theta))$ for each pair of initial conditions $(r, \theta) \in \mathbb{R}^2$, $r > 0$.

Let $H = \{(r, \theta) \mid r > 0, \theta \in \mathbb{R}\}$ and let T be the mapping from H into itself defined by

$$T(r, \theta) = (R(2\pi, r, \theta), \Theta(2\pi, r, \theta)). \tag{5.25}$$

We have that T satisfies

$$T(r, \theta + 2\pi) = T(r, \theta) + (0, 2\pi). \tag{5.26}$$

Let (u, v) be a solution of (5.23) and let (R, Θ) be the corresponding solution of (5.24). The following relationship between $\Theta(0) - \Theta(T)$ and the rotation $\eta(u)$ of u is easily seen

$$\Theta(0) - \Theta(T) = 2n\pi(<2n\pi)(>2n\pi) \qquad \text{if and only if } \eta(u) = 2n\pi(<2n\pi)(>2n\pi).$$

Then, from lemma 5.2 (a) we obtain

$$\Theta(0, \delta, \theta) - \Theta(T, \delta, \theta) > 2j\pi \tag{5.27}$$

for any $\theta \in \mathbb{R}$, and

$$\Theta(0, \Delta, \theta) - \Theta(T, \Delta, \theta) < 2i\pi \tag{5.28}$$

for any $\theta \in \mathbb{R}$. Next let $q \in \mathbb{Z}$ and define $T_q: H \to H$ by

$$T_q(r, \theta + 2\pi) = T(r, \theta) + (0, 2q\pi). \tag{5.29}$$

Each mapping T_q is a homeomorphism of H preserving the area $r\,\mathrm{d}r\,\mathrm{d}\theta$ and satisfying the periodicity condition

$$T_q(r, \theta + 2\pi) = T_q(r, \theta) + (0, 2\pi). \tag{5.30}$$

Clearly every fixed point of T_q leads to a T-periodic solution of (5.24) and hence of (5.23).

Proof of theorem 5.1. Let us set

$$T_q(r, \theta) = (R_q(r, \theta), \Theta_q(r, \theta)), \tag{5.31}$$

$q \in \mathbb{Z}$. From (5.25) and (5.29) we obtain that $R_q(r, \theta) = R(r, \theta)$ and that

$$\Theta_q(r, \theta) = \Theta(r, \theta) + 2q\pi, \tag{5.32}$$

$q \in \mathbb{Z}$. From (5.27) and (5.32) we have that

$$\Theta_q(\delta, \theta) - \theta < 2\pi(q - j) \tag{5.33}$$

and from (5.28) and (5.32)

$$\Theta_q(\Delta, \theta) - \theta > 2\pi(q - i), \tag{5.34}$$

for $q \in \mathbb{Z}$. Thus for $q = i, \ldots, j$ we have that

$$\Theta_q(\delta, \theta) - \theta < 0 \tag{5.35}$$

and

$$\Theta_q(\Delta, \theta) - \theta > 0. \tag{5.36}$$

Let $H_{\delta,\Delta} = \{(r, \theta) \mid \delta < r < \Delta, \theta \in \mathbb{R}\}$. Then from the Poincaré–Birkhoff theorem, in the version proved in [5], it follows that for each $q = i, \ldots, j$ the mapping T_q from $H_{\delta,\Delta}$ onto its image possesses two fixed points, different modulo $(0, 2\pi)$, leading to two T-periodic solutions of (1.1) with rotation $2q\pi$. This shows the existence of at least $2(j - i + 1)$ nontrivial T-periodic solutions of (1.1) and hence the theorem.

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APPENDIX

In this appendix we will briefly study the existence, extendibility and uniqueness of the solutions to the I.V.P.

$$\begin{gathered}(\phi_p(u'))' + f(t, u) = 0 \\ u(t_0) = \alpha, \qquad u'(t_0) = \beta,\end{gathered} \tag{A1}$$

where $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(t, 0) = 0$ for all $t \in \mathbb{R}$.

PROPOSITION A1. Suppose there is a $C > 0$ such that

$$|f(t, x)| < C|x|^{p-1} \tag{A2}$$

for all $t \in x$, for all $x \in \mathbb{R}$, then (A1) has a solution defined in $\mathbb{R}$.

Proof. For simplicity we set $t_0 = 0$. We first show local existence, i.e. existence for (A1) in an interval $[-\delta, \delta]$. Let us denote by C_δ the Banach space of all continuous functions $x\colon [-\delta, \delta] \to \mathbb{R}$, with corresponding sup-norm $\|x\|_\delta$. Problem (A1) is equivalent to the fixed-point problem

$$u = G(u) \tag{A3}$$

where $G\colon C_\delta \to C_\delta$ is given by

$$G(u)(t) = \alpha + \int_0^t \phi_{p'}\left[\phi_p(\beta) - \int_0^\tau f(s, u)\, ds\right] d\tau. \tag{A4}$$

From (A4) we have that G is a completely continuous operator and that

$$\|G(u)\|_\delta \le |\alpha| + \delta|\phi_p(\beta) + C\delta\|u\|_\delta^{p-1}|^{p'-1}. \tag{A5}$$

Hence, taking $r \ge \max\{\phi_p(\beta), |\alpha|/(1 - 2^{p'-1}\delta)\}$, δ small, (A5) yields that

$$G(\overline{B_{C_\delta}(0, r)}) \subset \overline{B_{C_\delta}(0, r)}. \tag{A6}$$

The existence of a solution of (A3) follows then from Schauder's fixed point theorem and thus (A1) has a solution u defined in $[-\delta, \delta]$.

Next let us prove that u can be extended to $[0, +\infty)$ and hence to $\mathbb{R}$.

Suppose that u is defined in $[0, a)$ and define $W(t) \equiv \int_0^t (|u'(s)|^p + |u(s)|^p)\, ds$ for $t \in [0, a)$. Integrating the first of (A1) from 0 to $t \in [0, a)$, using that

$$|u(t)|^p \le 2^p\left(|\alpha|^p + a^{p-1}\int_0^t |u'(s)|^p\, ds\right), \tag{A7}$$

and Holder's inequality we obtain that

$$W(t) \le A e^{Bt}, \qquad t \in [0, a), \tag{A8}$$

where A and B are constants. Equation (A8) and a standard argument imply that the local solution of (A1) can be extended to $[0, +\infty)$. ■

PROPOSITION A2. If f is like in proposition A1 and furthermore $f(t, \phi_{p'}(x))$ is locally Lipschitz in x and for $p > 2$, $f(t, x) \ne 0$ for all $t \in \mathbb{R}$ and $x \ne 0$, then the solution u of (A1) is unique.

Proof. Clearly we only have to prove local uniqueness. We break the proof into four different cases.

(1) $\alpha = 0$, $\beta = 0$. Suppose $\|u\|_\delta \ne 0$. From (A1) and (A2) we obtain

$$\|u'\|_\delta \le (C\delta)^{1/(p-1)}\|u\|_\delta. \tag{A9}$$

Thus, $u(t) = \int_0^t u'(s)\, ds$, (A10) and δ small yield that

$$\|u\|_\delta(1 - C^{1/(p-1)}\delta^{p'}) \le 0 \tag{A10}$$

which is a contradiction. Hence $\|u\|_\delta = 0$.

(2) $\alpha \neq 0$, $\beta = 0$. Suppose $u_i: [-\delta, \delta] \to \mathbb{R}$, $u_i(0) = \alpha$, $u_i'(0) = 0$, are two solutions of (A1). It is easily seen from (A1) that for $t \in [0, \delta]$

$$\|u_2 - u_1\|_\delta \le \int_0^t \left|\phi_{p'}\left(\int_0^\tau f(s, u_2(s))\,\mathrm{d}s - \phi_{p'}\left(\int_0^\tau f(s, u_1(s))\,\mathrm{d}s\right)\right)\right| \mathrm{d}\tau. \tag{A11}$$

If $p' > 2$ (equivalently $1 < p < 2$), $\phi_{p'}$ is differentiable at zero. Thus, the fact that $f(t, \phi_{p'}(x))$ is locally Lipschitz in x and (A1) yield

$$\|u_2 - u_1\|_\delta \le D\delta^2 \|u_1 - u_2\|_\delta \tag{A12}$$

where D is a constant. For δ small, (A12) implies that $u_2 = u_1$. Hence there is uniqueness for $p' > 2$.

Next assume $1 < p' \le 2$, and let u_1 and u_2 be different solutions of (A1). Note that $\alpha \neq 0$ implies

$$\lim_{\tau \to 0} \left|\frac{\int_0^\tau f(s, u_i(s))\,\mathrm{d}s}{\tau}\right| = |f(t, \alpha)| \neq 0, \tag{A13}$$

$i = 1, 2$. Hence for δ small we have that there is an $r > 0$ such that for $\tau \in [-\delta, \delta]$

$$r \le \left|\frac{\int_0^\tau f(s, u_i(s)\,\mathrm{d}s}{\tau}\right| \tag{A14}$$

$i = 1, 2$, and that $\operatorname{sign} \int_0^\tau f(s, u_1(s))\,\mathrm{d}s/\tau = \operatorname{sign} \int_0^\tau f(s, u_2(s))\,\mathrm{d}s/\tau$.

Since $\phi_{p'}$ is C^1 in any interval not containing zero, there is a constant $C > 0$ such that

$$\|u_1 - u_2\|_\delta \le C \int_0^\delta \tau^{p'-2} \int_0^\tau |f(s, u_2(s)) - f(s, u_1(s))|\,\mathrm{d}\tau. \tag{A15}$$

Calling on the Lipschitzian property of f and using that for $p > 2$, ϕ_p is differentiable, we obtain from (A15)

$$\|u_1 - u_2\|_\delta (1 - \tilde{C}\delta^{p'}) \le 0, \tag{A16}$$

where $\tilde{C}$ is a constant. From (A16) and δ small it follows that $u_2 = u_1$. Hence there is uniqueness for $p' \le 2$.

The rest of the cases, i.e. $\alpha = 0$, $\beta \neq 0$ and $\alpha \neq 0$, $\beta \neq 0$ are proved similarly to (2) and thus we omit their proofs. Thus the proposition follows. ■