TORSION POINTS ON THETA DIVISORS

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ABSTRACT. Using the irreducibility of a natural irreducible representation of the theta group of an ample line bundle on an abelian variety, we derive a bound for the number of n-torsion points that lie on a given theta divisor. We present also two alternate approaches to attacking the case $n = 2$.

1. INTRODUCTION

Let A be a complex abelian variety of dimension g and let Θ be an ample divisor on A that gives a principal polarization $\mathcal{L} := \mathcal{O}_A(\Theta)$ (i.e. $\dim H^0(A, \mathcal{L}) = 1$). We will use the notations (A, Θ) and $(A, \mathcal{L})$ interchangeably. For $n \geq 2$, define

$$\Theta(n) := \#A[n] \cap \Theta,$$

where $A[n]$ is the group of n-torsion points on A. It is well-known that Θ does not contain all n-torsion points; this follows easily, for example, from the irreducibility of the representation of the theta group of $\mathcal{L}^n$ in $H^0(A, \mathcal{L}^n)$ as we will discuss below. It is a classical result, [12] that the evaluation at the n-torsion points, $n \geq 4$, of Riemann's theta function completely determines the abelian variety embedded in $\mathbb{P}^{n^g-1}$. The image is the intersection of all the quadrics containing the image of the n-torsion points. Moreover the structure of $A[2] \cap \Theta$ tells us if the principally polarized abelian variety (A, Θ) is decomposable, [14] or is the Jacobian of an hyperelliptic curve, [11]. Also recently, in [2] it has been proved that (A, Θ) is decomposable if and only if the image of the Gauss map at the smooth points of Θ in $A[2] \cap \Theta$ is contained in a quadric of $\mathbb{P}^{g-1}$.

In [13], a bound is obtained for the number of 2-torsion points on a theta divisor. Indeed, they show that $\Theta(2) \leq 4^g - 2^g$. However, this bound is far from optimal, and in the same paper it is conjectured that the actual bound is $4^g - 3^g$ and is achieved if and only if $(A, \mathcal{L})$ is the polarized product of elliptic curves. One could generalize this and conjecture that for n-torsion points the bound should be $n^{2g} - (n^2 - 1)^g$, with equality if and only if $(A, \mathcal{L})$ is the polarized product of elliptic curves.

Let $\tau \in \mathcal{H}_g$ be a matrix in the Siegel upper-half space, and for $\delta, \epsilon \in \mathbb{R}^g$ and $z \in \mathbb{C}^g$ define the theta function with characteristics

$$\theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (\tau, z) := \sum_{m \in \mathbb{Z}^g} \exp[\pi i (m + \delta)^t \tau (m + \delta) + 2\pi i (m + \delta)^t (z + \epsilon)].$$

When $\delta = \epsilon = 0$ we obtain *Riemann's theta function* $\theta(\tau, z) := \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau, z)$;
the projection of $\{\theta(\tau, \cdot) = 0\}$ to $A_\tau := \mathbb{C}^g / \tau \mathbb{Z}^g + \mathbb{Z}^g$ gives a symmetric
theta divisor (i.e. $-\Theta = \Theta$) that we will denote by Θ_τ. We remark that any
complex principally polarized abelian variety is isomorphic to (A_τ, Θ_τ) for
some $\tau \in \mathcal{H}_g$. If we put $\mathcal{L}_\tau := \mathcal{O}_{A_\tau}(\Theta_\tau)$, it is well-known that the set

$$\left\{ \theta \begin{bmatrix} \delta \\ 0 \end{bmatrix} (n\tau, nz) : \delta \in \frac{1}{n} \mathbb{Z}^g / \mathbb{Z}^g \right\}$$

is a basis for $H^0(A_\tau, \mathcal{L}_\tau^n)$ and the set

$$\left\{ \theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (\tau, nz) : \delta, \epsilon \in \frac{1}{n} \mathbb{Z}^g / \mathbb{Z}^g \right\}$$

is a basis for $H^0(A_\tau, \mathcal{L}_\tau^{n^2})$. . A simple calculation shows that

$$\theta(\tau, z + \tau \delta + \epsilon) = \lambda(z) \theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (\tau, z)$$

for some nowhere vanishing function λ, and it immediately follows that
if $\Theta = \Theta_\tau$, then $\Theta(n)$ is exactly the number of vanishing *theta constants*
$\theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (\tau, 0)$ for $\delta, \epsilon \in \frac{1}{n} \mathbb{Z}^g / \mathbb{Z}^g$. A similar statement holds if Θ is the pull-
back of Θ_τ by a translation (i.e. Θ any theta divisor). If $n = 2$ and
$4\delta^t \epsilon \equiv 1 \pmod 2$, then the associated theta constant vanishes, and so
$\Theta_\tau(2) \geq 2^{g-1}(2^g - 1)$. In fact, this is an equality if $A_\tau \in \mathcal{A}_g \backslash \theta^{\text{null}}$, where
θ^{null} is the divisor consisting of principally polarized abelian varieties such
that one of its symmetric theta divisors has a singularity at a point of order 2.

The goal of this paper is to give a stronger bound for $\Theta(n)$. Our main
theorem gives the following:

Theorem 1.1. *Let (A, Θ) be a complex principally polarized abelian variety.
Then*

$$\Theta(2) \leq 4^g - g2^{g-1} - 2^g$$

and for $n \geq 3$

$$\Theta(n) \leq n^{2g} - (g+1)n^g.$$

We can make this bound better if (A, Θ) is decomposable.

After proving this theorem, we present alternative approaches to attacking
the number $\Theta(2)$. One of these points of view will give a better bound than
the theorem, in fact we get

Proposition 1.2. *Let (A, Θ) be a principally polarized abelian variety. Then*

$$\Theta(2) \leq 4^g - \frac{7^g - 1}{3^g - 1}$$

We observe that the methodologies involved are interesting and different
from the original approach, and we believe they will be more useful in the
future.

In particular in the last approach that could produce the conjectural
bound, a matrix M appears, induced by the Weil pairing between the points

of order two in the abelian variety. This matrix appears also in other fields of mathematics, in coding theory as the matrix associated to the Macwilliams identity for the weight enumerator of the codes, cf.[1] page 103, and in the theory of Borcherds' additive lifting as the matrix associated to a unitary representation of the integral metaplectic group on $\mathbb{C}[\mathbb{F}_2^{2g}]$, cf.[4]

2. A BOUND FOR $\Theta(n)$

Since $\mathcal{L}$ is a principal polarization, we have that

$$A[n] = \{x \in A : t_x^* \mathcal{L}^n \simeq \mathcal{L}^n\},$$

where $t_x : A \to A$ denotes translation by x. Recall that in this case, the *theta group of* $\mathcal{L}^n$ is a certain central extension of $A[n]$ by $\mathbb{G}_m$ which we will denote by $\mathcal{G}_n$:

$$1 \to \mathbb{G}_m \to \mathcal{G}_n \to A[n] \to 0.$$

Let $\varphi_n : A \to \mathbb{P}H^0(A, \mathcal{L})$ be the morphism associated to the linear system $|\mathcal{L}^n|$. The vector space $H^0(A, \mathcal{L}^n)$ is an irreducible representation for the theta group $\mathcal{G}_n$ where $\mathbb{G}_m$ acts by scalar multiplication (see [9, Ch. 4] or [10, Theorem 2, pg. 297]), and we therefore obtain a projective representation

$$\rho : A[n] \to \mathrm{PGL}(H^0(A, \mathcal{L}^n)).$$

Because of the irreducibility of the representation, we notice that there is no proper linear subspace of $\mathbb{P}H^0(A, \mathcal{L}^n)$ that is invariant under the action of $A[n]$. Moreover, we have that

$$\rho(x) \cdot \varphi_n(y) = \varphi_n(x + y)$$

for every $x \in A[n]$ and $y \in A$.

Let $H \subseteq A[n]$ be a maximal isotropic subgroup with respect to the commutator pairing associated to the theta group of $\mathcal{L}^n$. We say that H is *c-isotropic* if it has a complementary isotropic subgroup K. We remark that any maximal isotropic subgroup is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^g$, as is its complementary isotropic subgroup if it exists. Let H be c-isotropic, let $p : A \to A/H =: A_H$ be the natural projection, and let $q : A_H \to A$ be the inverse isogeny. We have a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ p\ } & A_H \\
 & {\scriptstyle n_A}\searrow & \downarrow{\scriptstyle q} \\
 & & A
\end{array}
$$

where n_A denotes multiplication by n on A. By descent theory for abelian varieties, we have that there exists a principal polarization $\mathcal{M}$ on A_H such that $\mathcal{L}^n \simeq p^*\mathcal{M}$ and $q^*\mathcal{L} \equiv \mathcal{M}^n$. We see in this case that $\ker q$ is a maximal c-isotropic subgroup of $A_H[n]$. Let N be a complementary isotropic subspace of $\ker q$.

Define $\Sigma = q^{-1}(\Theta) \in |q^*\mathcal{L}|$ and for $a \in A_H$, define $\Sigma_a := \Sigma + a$. For every $b \in A_H[n]$, fix a section $s_b \in H^0(A_H, q^*\mathcal{L})$ such that $\Sigma_b = \text{div}(s_b)$.

Lemma 2.1. *The set $\{s_b : b \in N\}$ is a basis for $H^0(A_H, q^*\mathcal{L})$.*

Proof. We see that for all $a \in \ker q$ and $b \in N$,
$$\Sigma_{a+b} = \Sigma_b + a = q^{-1}(\Theta + q(b)) = \Sigma_b.$$
This means that for all $a \in \ker q$ and $b \in N$, there exists $\lambda_a \in \mathbb{G}_m$ such that $t_a^* s_b = \lambda_a s_b$. In other words, $A_H[n]$ acts on the projective span of $\{s_b : b \in N\}$ in $\mathbb{P}H^0(A_H, q^*\mathcal{L})$. Since the theta group representation is irreducible, we must have that the above set generates the whole space. Moreover, $\#N = \dim H^0(A_H, q^*\mathcal{L})$, and the result follows. $\qquad\square$

Let $\varphi_H : A_H \to \mathbb{P}H^0(A_H, q^*\mathcal{L})$ be the morphism associated with $|q^*\mathcal{L}|$.

Definition 2.2. *For H a maximal c-isotropic subgroup of $A[n]$, let $c_1 + H, \ldots, c_{n^g} + H$ be its cosets (we will assume $c_1 = 0$). We define the integers*
$$Q_{H,c_i} := \dim_{\mathbb{C}} span\{\varphi_H(q^{-1}(c_i))\}$$
$$Q_H := \sum_{i=1}^{n^g} Q_{H,c_i}$$
$$Q(n) := \max\{Q_H : H \subseteq A[n] \text{ max. } c\text{-isotropic subgroup}\}.$$

We can use these numbers to obtain a bound on the number of n-torsion points lying on Θ.

Proposition 2.3. *Let (A, Θ) be a principally polarized abelian variety and let $n \geq 2$. Then $\Theta(n) \leq n^{2g} - n^g - Q(n)$.*

Proof. We will prove that $\Theta(n) \leq n^{2g} - n^g - Q_H$ for every maximal c-isotropic subgroup $H \subseteq A[n]$. Let $S \subseteq H + c_i$ be a subset with $r \leq Q_{H,c_i}$ elements. We will first prove that Θ does not contain $(H + c_i)\backslash S$. We see that
$$(H + c_i)\backslash S \subseteq \Theta \;\; \Leftrightarrow \;\; q^{-1}((H + c_i)\backslash S) \subseteq \Sigma$$
$$\Leftrightarrow \;\; (A_H[n] + d_i)\backslash(\ker q + t_1 \sqcup \cdots \sqcup \ker q + t_r) \subseteq \Sigma$$
where $q(d_i) = c_i$ and the t_j are chosen so that $S = \{q(t_j) : j = 1, \ldots, r\}$. Assume this occurs. Now for all $b \in N$,
$$(A_H[n] + d_i)\backslash(\ker q + t_1 + b \sqcup \cdots \sqcup \ker q + t_r + b) \subseteq \Sigma_b.$$
It follows that $q^{-1}(c_i) = \ker q + d_i \subseteq \Sigma_b$ for every $b \notin (\ker q + d_i - t_j) \cap N$. We see then there are $n^g - r$ options for b. Using Lemma 2.1, this implies that $\varphi_H(q^{-1}(c_i))$ is contained in a linear subspace of $\mathbb{P}H^0(A_H, q^*\mathcal{L})$ of dimension $r - 1$, a contradiction with the choice of r. Therefore in each coset $c_i + H$, there are at most $n^g - Q_{H,c_i} - 1$ points that lie on Θ. By adding everything up we get the bound we were looking for.
$$\square$$

Remark 2.4. *The proof of the previous proposition is valid over any algebraically closed field of characteristic prime to n and for any theta divisor (i.e. not necessarily symmetric). Moreover, the proposition already gives us a better bound than the one in [13]. Indeed, there can be at most one Q_{H,c_i}*

equal to 0 (this happens when $(A_H, \mathcal{M})$ is the polarized product of elliptic curves), and so $\Theta(2) \leq 4^g - 2^g - (2^g - 1) = 4^g - 2^{g+1} + 1$.

The next proposition shows that when looking for a bound for $\Theta(n)$, we can always assume that Θ is given by the zero set of Riemann's theta function.

Proposition 2.5. *If Θ is Riemann's theta function, then equality holds in Proposition 2.3.*

Proof. Assume that Θ is Riemann's theta function, and so $\Theta = \Theta_\tau$ on A_τ for some $\tau \in \mathcal{H}_g$. Let Λ_τ be the lattice $\tau \mathbb{Z}^g + \mathbb{Z}^g$ and take the maximal c-isotropic subgroup $H = \{\tau\epsilon : \epsilon \in \frac{1}{n}\mathbb{Z}^g/\mathbb{Z}^g\} + \Lambda_\tau$ of $A_\tau[n]$. We have the quotient maps

$$A_\tau \xrightarrow{p} A_H = A_{\tau/n} \xrightarrow{q} A_\tau$$

where $p(z + \Lambda_\tau) = z + \Lambda_{\tau/n}$ and $q(z + \Lambda_{\tau/n}) = nz + \Lambda_\tau$. We see that the cosets of H are precisely $\mu + H$ for $\mu \in \frac{1}{n}\mathbb{Z}^g/\mathbb{Z}^g$, and moreover

$$q^{-1}(\mu + \Lambda_\tau) = \frac{1}{n}\mu + \frac{1}{n}\mathbb{Z}^g + \Lambda_{\tau/n}.$$

Then

$$\varphi_H(q^{-1}(\mu + \Lambda_\tau)) = \left\{ \left[\theta \begin{bmatrix} \delta \\ 0 \end{bmatrix} (\tau, \mu + a) \right]_{\delta \in \frac{1}{n}\mathbb{Z}^g/\mathbb{Z}^g} : a \in \mathbb{Z}^g/n\mathbb{Z}^g \right\}.$$

But $\theta \begin{bmatrix} \delta \\ 0 \end{bmatrix} (\tau, \mu + a) = \exp(2\pi i \delta^t a)\theta \begin{bmatrix} \delta \\ \mu \end{bmatrix} (\tau, 0)$. Therefore,

$$Q_{H,\mu} + 1 = \mathrm{rank}\left(\exp(2\pi i n \delta^t \epsilon)\theta \begin{bmatrix} \delta \\ \mu \end{bmatrix} (\tau, 0) \right)_{\delta, \epsilon \in \frac{1}{n}\mathbb{Z}^g/\mathbb{Z}^g},$$

and so we have

$$n^{2g} - n^g - Q_H = n^{2g} - \sum_{\mu \in \frac{1}{n}\mathbb{Z}^g/\mathbb{Z}^g} \mathrm{rank}\left(\exp(2\pi i n \delta^t \epsilon)\theta \begin{bmatrix} \delta \\ \mu \end{bmatrix} (\tau, 0) \right)_{\delta, \epsilon \in \frac{1}{n}\mathbb{Z}^g/\mathbb{Z}^g}.$$

A quick check shows that the sum above is equal to the number of non-vanishing theta constants, which we know is equal to $n^{2g} - \Theta(n)$. $\square$

We can now obtain an explicit bound for the number of torsion points on a theta divisor.

Theorem 2.6. *Let (A, Θ) be a complex principally polarized abelian variety. Then*

$$\Theta(2) \leq 4^g - g2^{g-1} - 2^g$$

and for $n \geq 3$

$$\Theta(n) \leq n^{2g} - (g+1)n^g.$$

Proof. By the previous proposition, we can take $\Theta = \Theta_\tau$ on A_τ for some $\tau \in \mathcal{H}_g$. Using the notation as in the proof of the previous proposition, we have that

$$q^{-1}(\mu + \Lambda_\tau) = \frac{1}{n}\mu + \frac{1}{n}\mathbb{Z}^g + \Lambda_{\tau/n}$$

for $\mu \in \frac{1}{n}\mathbb{Z}^g/\mathbb{Z}^g$. Therefore each member of $\varphi_H(q^{-1}(\mu+\Lambda_\tau))$ differs from the other by the action of the representation $\rho_H : \ker q \to PGL(H^0(A_H, q^*\mathcal{L}))$. It is known that this action (for this particular subgroup) multiplies the projective coordinates of $\mathbb{P}H^0(A_H, q^*\mathcal{L})$ by nth roots of unity, and so we can estimate $Q_{H,\mu}$ by the number of vanishing coordinates. Using this fact, it is easy to see that Q_{H,c_i} is equal to $n^g - 1 - r$, where r is the number of vanishing coordinates. Therefore $n^{Q_{H,\mu}} \geq \#\varphi_H(q^{-1}(\mu))$.

For $n = 2$, when $\mu = 0$ we have 2^g different points in $\varphi_H(\ker q)$, and so $Q_{H,0} \geq g$.

Let us assume $\Theta_{\tau/2}$ irreducible. When $\mu \neq 0$, we have 2^{g-1} different points, and so $Q_{H,\mu} \geq g - 1$. Adding everything up we get

$$\Theta(2) \leq 4^g - 2^g - g - (g-1)(2^g - 1) = 4^g - g2^g - 1.$$

In the case $\Theta_{\tau/2}$ reducible we proceed in the same way, but now we have less points since the map is not injective on the Kummer variety. The worst case will be when (X, Θ) is a product of elliptic curves. In this case depending on μ we can get in the image 2^k different points, $k = 0, \dots g - 1$. Varying μ this happens exactly $\binom{g}{k}$ times. Hence totally we get

$$\Theta(2) \leq 4^g - \sum_{k=0}^{g} \binom{g}{k}(k+1) = 4^g - g2^{g-1} - 2^g.$$

For $n \geq 3$, we have that φ_H is an embedding, and so there are always n^g points in $\varphi_H(q^{-1}(c_i))$. This means that $Q_{H,c_i} \geq g$. Therefore if $n \geq 3$,

$$\Theta(n) \leq n^{2g} - n^g - gn^g = n^{2g} - (g+1)n^g.$$

$\square$

When Θ is reducible, even more can be said:

Corollary 2.7. *If $(A, \Theta) \simeq \prod_{i=1}^{s}(B_i, \Theta_i)$ and $b_i = \dim B_i$, then*

$$\Theta(2) \leq 4^g - 2^g \prod_{i=1}^{s}\left(\frac{b_i}{2} + 1\right)$$

and for $n \geq 3$

$$\Theta(n) \leq n^{2g} - n^g \prod_{i=1}^{s}(b_i + 1).$$

Proof. In this case, we see that the number of n-torsion points on Θ is equal to $n^{2g} - t$ where t is the number of n-torsion points of the form $(x_1, \dots, x_s)$ such that $x_i \notin \Theta_i$ for all i. Therefore

$$\Theta(2) = 4^g - \prod_{i=1}^{s}(4^{b_i} - \Theta_i(2)) \leq 4^g - \prod_{i=1}^{s}(b_i 2^{b_i - 1} + 2^{b_i}).$$

The same technique can be applied for $n \geq 3$. $\square$

Remark 2.8. *If (X, Θ) is simple (or more generally not 2-isogenous to a product), using the action of the symplectic group we can improve the estimate for $Q_{H,0}$; in fact we can get $Q_{H,0} \geq 2g - 1$. Thus in this case we get*

$$\Theta(2) \leq 2^{2g} - 2^g - g2^g = 2^{2g} - (g+1)2^g.$$

This number fits in the general picture.

3. ALTERNATIVE APPROACHES FOR $n = 2$

3.1. **Alternative approach 1.** The methodology in this section is different from that in the previous one, and there are changes in notation. Assume that Θ is symmetric and irreducible, and define

$$B_n := H^0(A, \mathcal{O}_A(n\Theta)).$$

Let B_n^+ be the eigenspace associated to 1 for the automorphism $(-1)^*$. It is well-known that

$$\dim_{\mathbb{C}} B_n^{\pm} = 2^{g-1}(m^g \pm 1).$$

We will use a few results from [8]. For $n \geq 2$ and $m \geq 3$, the natural map

$$B_n \otimes B_m \to B_{m+n}$$

is surjective. Since $B_2 = B_2^+$, we have that $B_2 \otimes B_m^{\pm} \to B_{m+2}^{\pm}$ is surjective, and therefore

$$\mathrm{Sym}^2(B_2) \otimes B_m^{\pm} \to B_{m+4}^{\pm}$$

is surjective. Let $V_2 \subseteq B_4^+$ be the image of $\mathrm{Sym}^2(B_2)$ in B_4^+. We are interested in a basis of V_2, which is given by all $\theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (\tau, 2z)$ for $\delta, \epsilon \in \frac{1}{2}\mathbb{Z}^g / \mathbb{Z}^g$ and $4\delta^t\epsilon \equiv 0 \pmod 2$ such that $\theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (\tau, 0) \neq 0$ (in this section all theta characteristics will be half-integer characteristics). Let $\mathfrak{n}_g$ be the dimension of V_2. It is clear that

$$\Theta(2) = 4^g - \mathfrak{n}_g,$$

since it is the number of vanishing theta constants. As an immediate consequence of the previous discussion we have

Proposition 3.1.
$$\Theta(2) \leq 4^g - \frac{7^g - 1}{3^g - 1}$$

Proof. We have that the map $\mathrm{Sym}^2(B_2) \otimes B_m^{\pm} \to B_{m+4}^{\pm}$ factors as

$$\mathrm{Sym}^2(B_2) \otimes B_m^{\pm} \longrightarrow B_{m+4}^{\pm}$$
$$V_2 \otimes B_m^{\pm}$$

and since the above arrow is surjective, all the arrows are surjective. Therefore,

$$\mathfrak{n}_g \geq \dim_{\mathbb{C}} B_{m+4}^{\pm} / \dim_{\mathbb{C}} B_m^{\pm} = \frac{(m+4)^g \pm 1}{m^g \pm 1}$$

for $m \geq 3$. The maximum of this function in m is achieved when $m = 3$ and the sign is negative. $\square$

3.2. **Alternative approach 2.** From the addition formula for theta functions with semi-integral characteristics (see [7, Theorem 2, pg. 139] we have

$$\theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (\tau, 0) \theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (\tau, 2z) = \sum_\sigma (-1)^{<2\epsilon, 2\sigma>} \theta \begin{bmatrix} \sigma \\ 0 \end{bmatrix} (2\tau, 2z) \theta \begin{bmatrix} \delta + \sigma \\ 0 \end{bmatrix} (2\tau, 2z).$$

Moreover we can restate this saying that

$$\Theta(2) - 2^{g-1}(2^g - 1) = 2^{g-1}(2^g + 1) - \mathfrak{n}_g$$

is the dimension of the space of quadrics that vanish on the image of the *Kummer variety* $K(A) = A/\pm 1$, via the embedding $K(A) \hookrightarrow |2\Theta| \simeq \mathbb{P}^{2^g - 1}$.

Since the Kummer variety is irreducible and the map is finite, we have that the image of $K(A)$ cannot be contained in any quadric of rank 2 in $\mathbb{P}^{2^g - 1}$. These quadrics form a variety of dimension $2^{g+1} - 1$ in the space of all quadrics in $\mathbb{P}^{2^g - 1}$. Thus we have as a rough estimate:

Lemma 3.2. $\mathfrak{n}_g \geq 2^{g+1} - 1$.

Proof. The space of quadrics containing the image of $K(A)$ does not intersect the above described variety. $\square$

This then gives us the bound

$$\Theta(2) \leq 4^g - 2^{g+1} + 1.$$

This estimate is very rough and a careful analysis could produce better results. For example we know that if Θ is irreducible, the number of vanishing quadrics is equal to $1, 10$ when $g = 3, 4$ respectively, and ≥ 66 when $g = 5$. All these are triangular numbers that could give the dimension of the space of quadrics of bounded rank.

3.3. **Alternative approach 3.** This method is different than the previous approach but gives us the same estimate. We have a short exact sequence

$$0 \to R \to V_2 \otimes B_4^+ \to B_8^+ \to 0$$

where R is the space of relations. Let $W_2 \subseteq B_4^+$ be such that $B_4^+ = V_2 \oplus W_2$; it has as a basis the set of theta functions with even characteristics that correspond to a point of order 2 on Θ. Recall that the Heisenberg group, given as a set

$$H = \mathbb{G}_m \times \mathbb{F}_2^g \times \mathrm{Hom}(\mathbb{F}_2, \mathbb{G}_m)^g,$$

is a non-commutative group that is non-canonically isomorphic to the theta group of 2Θ. Now H acts on B_4^+ and B_8^+ and decomposes these spaces with respect to its characters. Moreover, the characters are in one to one correspondence with the points of order 2 on A. It is known (see [6, Section 2.4]) that for a character χ

$$\dim(B_8^+)_\chi = \begin{cases} 2^g & \text{if } \chi \text{ is trivial} \\ 2^{g-1} & \text{if not} \end{cases}$$

$$\dim(B_4^+)_\chi = \begin{cases} 1 & \text{if } \chi \text{ corresponds to an even characteristic} \\ 0 & \text{if not} \end{cases} .$$

Lemma 3.3. *We have an exact sequence*

$$0 \to R_0 \to \bigoplus_\chi (V_2)_\chi \otimes (V_2)_\chi \to (B_8^+)_0 \to 0,$$

where the subscript 0 refers to the eigenspace corresponding to the trivial character.

Proof. This follows from the surjectivity of $\operatorname{Sym}^2(V_2) \oplus (W_2 \otimes V_2) \to B_8^+$. $\square$

Corollary 3.4. *We have $\mathfrak{n}_g = 2^g + \dim R_0$; or in other words, $\Theta(2) = 4^g - 2^g - \dim R_0$.*

In order to estimate $\Theta(2)$, we need a better grasp on what R_0 or a suitable subspace is. Denote by K_g^+ and K_g^- the sets of isotropic (respectively anisotropic) elements in $\mathbb{F}_2^{2g}$ with respect to the quadratic form

$$\langle X, X \rangle = x_1 x_{g+1} + \cdots + x_g x_{2g},$$

and let k_g^+ and k_g^- be their orders. We introduce the matrix

$$M(g) = M := \left(\exp\left[i\pi \sum_{i=1}^g (m_i n_{g+i} - n_i m_{g+i}) \right] \right)_{m,n \in \mathbb{Z}^{2g}/2\mathbb{Z}^{2g}}.$$

Now M has the decomposition

$$M = \begin{pmatrix} M^+ & N \\ N^t & M^- \end{pmatrix}$$

where M^+ (respectively M^-) is the submatrix of M given by the restriction to $K_g^+ \times K_g^+$ (respectively $K_g^- \times K_g^-$). The following proposition is proven in [3, Lemma 1.1]:

Proposition 3.5. *M has two eigenspaces of dimension k_g^+ and k_g^- with eigenvalues $\pm 2^g$, while $M^\pm$ has eigenspaces of dimension $(1/3)(2^g \pm 1)(2^{g-1} \pm 1)$ and $(1/3)(2^{2g} - 1)$ with eigenvalues $\pm 2^g$ and $\mp 2^{g-1}$. For $X \in \mathbb{C}^{k_g^+}$ and $Y \in \mathbb{C}^{k_g^-}$, we have*

$$M \begin{pmatrix} X \\ Y \end{pmatrix} = 2^g \begin{pmatrix} X \\ Y \end{pmatrix} \iff M^- Y = 2^{g-1} Y = N^t X$$

$$M \begin{pmatrix} X \\ Y \end{pmatrix} = -2^g \begin{pmatrix} X \\ Y \end{pmatrix} \iff M^+ X = -2^{g-1} X = NY$$

$$M^+ X = 2^g X \iff N^t X = 0$$

$$M^- Y = -2^g Y \iff NY = 0$$

$$M^+ X = -2^{g-1} X \quad \text{if} \quad M^+ X - NY = 0$$

$$M^- Y = 2^{g-1} Y \quad \text{if} \quad N^t X - M^- Y = 0$$

If $m = (a, b) \in K_g^+$ for a and b considered as elements of $\{0,1\}^g$, then we use the notation $\theta_m(\tau, z) := \theta \begin{bmatrix} a/2 \\ b/2 \end{bmatrix}(\tau, z)$. The following lemma is also proved in [3]:

Lemma 3.6. *If $X = (v_m)_{m \in K_g^+}$ is a column of N, then $M^+ X = -2^{g-1} X$. Moreover we have*

$$\sum_{m \in K_g^+} v_m \theta_m(\tau, 0)^2 \theta_m(\tau, 2z)^2 = 0$$

where $(v_m)_{m \in K_g^+}$ is a column of N.

Since we have

$$B := NN^t = 2^{g-1}(2^g I - M^+),$$

it is easy to deduce that $\mathrm{rk}(N) = \frac{1}{3}(4^g - 1)$. Thus the columns of N span the whole eigenspace of M^+ with eigenvalue -2^{g-1}. If $S_0 \subset R_0$ is the subspace spanned by these relations, then we have

$$\dim S_0 \leq \frac{1}{3}(4^g - 1).$$

Obviously the dimension of S_0 is $\frac{1}{3}(4^g - 1)$ if there are no theta constants vanishing. If there are theta constants that vanish then the dimension could drop.

Let J be the $k_g^+ \times k_g^+$ diagonal matrix whose entries are 0 or 1 depending on whether or not the theta constant $\theta_m(\tau, 0)$ corresponding to $m \in K_g^+$ vanishes. We see that

$$\dim S_0 = \mathrm{rk}(JN) = \mathrm{rk}(JN(JN)^t) = \mathrm{rk}(JBJ^t)$$

where $B = NN^t$. Now deleting the 0 rows and columns, JBJ^t corresponds to a certain principal submatrix B_r of B of size $r \times r$ where

$$r = \mathfrak{n}_g \geq 2^g + \dim S_0 = 2^g + \mathrm{rk}(JBJ^t).$$

Thus to have an estimate for $\mathfrak{n}_g$, we need to estimate the ranks of principal submatrices of B. We therefore obtain:

Proposition 3.7.

$$\Theta(2) \leq 4^g - 2^g - h_0$$

where $h_0 = \min\{k \geq 2^g + rank(S) : S \text{ principal submatrix of } B \text{ of order } k\}$.

Corollary 3.8.

$$\Theta(2) \leq 4^g - 2^{g+1} + 1$$

Proof. We will show that all principal minors of B of size $s \leq 2^g - 1$ are positive definite. The matrix B_{2^g-1} is semi-positive definite. The entries are equal to $2^g - 1$ along the diagonal and ± 1 out of the diagonal. For every $X \in \mathbb{R}^{2^g-1}$ we have

$$X^t B_{2^g-1} X = \sum_{1 \leq i < j \leq k} (x_i \pm x_j)^2 + \sum_{i=1}^{2^g-1} x_i^2.$$

Thus it is positive definite. $\qquad\square$

Now $\mathrm{Sp}(2g, \mathbb{F}_2)$ acts on the set of characteristics by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} := \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \delta \\ \epsilon \end{pmatrix} + \begin{pmatrix} \mathrm{diag}(c^t d) \\ \mathrm{diag}(a^t b) \end{pmatrix}.$$

This action is double transitive on the set of even (respectively odd) characteristics. Therefore if we want to compute the rank of submatrices of the

matrix B, we can consider only orbits with respect to the action of this group.

The Kronecker product of g times the matrix $M^+(1)$ is a matrix $L(g)$ of order 3^g with eigenvalues $(-1)^k 2^{g-k}$ that have multiplicity $\binom{g}{k} 2^{g-k}$ for $k = 0, \ldots, g$. If we look at the submatrix B_k indexed by all even characteristics $m = \begin{bmatrix} \delta \\ \epsilon \end{bmatrix}$ satisfying $4\delta^t \epsilon = 0$ in $\mathbb{Z}$, then

$$B_k = 2^{g-1}(2^g I_{3^g} - L(g))$$

and has rank $3^g - 2^g$. We see that this implies the well-known result that if (A, Θ) is the product of elliptic curves, then there are 3^g points of order two that are not on Θ.

We finish our analysis by looking at the genus 2 case. Double transitivity of the action of the symplectic group implies that all submatrices of degree 8 of $M^+(2)$ are conjugate via the action of the symplectic group. For one of these matrices, we can prove that the rank is 5. This implies that

$$\mathfrak{n}_2 \geq 9,$$

which is sharp. We therefore conjecture the following that would imply that $\Theta(2) \leq 4^g - 3^g$ for all g:

Conjecture 3.9. *The number h_0 is reached at $L(g)$.*

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APPENDIX TO "TORSION POINTS ON THETA DIVISORS"

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ABSTRACT. In this note we revisit a method used in [1] to give a sharp
bound for the number of 2-torsion points on a theta divisor.

1. INTRODUCTION

Let A be a complex abelian variety and let $\mathcal{L} = \mathcal{O}_A(\Theta)$ be a principal
polarization on A. We set

$$\Theta(n) := \#A[n] \cap \Theta,$$

where $A[n]$ is the group of 2-torsion points on A.

In [5], the authors gave a bound for the number of 2-torsion points on a
theta divisor. This bound has been recently improved in [1] where also a
bound for the n torsion points is given. However, these bounds were not
optimal . In [5], it has been conjectured that the bound for the two torsion
points is $4^g - 3^g$ and is achieved if and only if $(A, \mathcal{L})$ is the polarized product
of elliptic curves. Similarly in [1] this conjecture has been generalized to the
case of n-torsion points. In these cases the bound is $n^{2g} - (n^2 - 1)^g$. The
aim of this note is to prove the first part of the first conjecture and we will
give an estimate for $\Theta(n)$ when n is even , more exacly we will prove the
following:

Theorem 1.1. *Let (A, Θ) be a principally polarized abelian variety. Then*

$$\Theta(2) \leq 4^g - 3^g$$

$$\Theta(2m) \leq m^{2g}(4^g - 3^g)$$

The proofs are consequence of results proved in [3] .

2. THE PROOF

We recall shortly some basic facts. Let $\tau \in \mathcal{H}_g$ be a matrix in the Siegel
upper-half space, and for $\delta, \epsilon \in \mathbb{R}^g$ and $z \in \mathbb{C}^g$ define the theta function with
characteristics

$$\theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (\tau, z) := \sum_{m \in \mathbb{Z}^g} \exp[\pi i (m + \delta)^t \tau (m + \delta) + 2\pi i (m + \delta)^t (z + \epsilon)].$$

When $\delta = \epsilon = 0$ we obtain *Riemann's theta function* $\theta(\tau, z) := \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau, z)$; the projection of $\{\theta(\tau, \cdot) = 0\}$ to $A_\tau := \mathbb{C}^g / \tau \mathbb{Z}^g + \mathbb{Z}^g$ gives the symmetric theta divisor Θ_τ. We set $\mathcal{L}_\tau := \mathcal{O}_{A_\tau}(\Theta_\tau)$. A basis for $H^0(A_\tau, \mathcal{L}_\tau^{n^2})$ is given by

$$\left\{ \theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (\tau, nz) : \delta, \epsilon \in \frac{1}{n} \mathbb{Z}^g / \mathbb{Z}^g \right\}.$$

We shortly recall some basic facts about these functions. The evaluation at $z = 0$ of the above functions vanishes if and only if if the n torsion $\tau \delta + \epsilon$ belongs to Θ_τ.

It is a well known fact that the dual torus $\hat{A}$ parametrizes isomorphism classes of line bundles in $Pic^0(A)$. The principal polarization Θ_τ induces an isomorphism $A \to \hat{A}$ sendindg a point x to the line bundle $L_x = t^* \mathcal{L} \otimes \mathcal{L}^{-1}$ in $\hat{A}$. We recall from [2] the following fundamental result (Theorem 6.8), which also appears in [6, Proposition 1.5] and [4, Proposition 7.2.2].

Theorem 2.1. *Let $\mathcal{M}$ be an ample line bundle on A*

- *For L in a non-empty subset of $\hat{A}$ the multiplication*

$$H^0(A, \mathcal{M}^{\otimes 2} \otimes L \otimes M) \otimes H^0(A, \mathcal{M}^{\otimes 2} \otimes N \otimes M^{-1}) \to H^0(A, \mathcal{M}^{\otimes 4} \otimes L \otimes N)$$

  is surjective for fixed M and N in $\hat{A}$.
- *For L in a non-empty subset of $\hat{A}$ the multiplication*

$$H^0(A, \mathcal{M}^{\otimes 2} \otimes M) \otimes H^0(A, \mathcal{M}^{\otimes 2} \otimes N \otimes L) \to H^0(A, \mathcal{M}^{\otimes 4} \otimes L \otimes M \otimes N)$$

  is surjective for fixed M and N in $\hat{A}$.
- *If $n \geq 2$ and $m \geq 3$ then the multiplication*

$$H^0(A, \mathcal{M}^{\otimes m} \otimes M) \otimes H^0(A, \mathcal{M}^{\otimes n} \otimes N) \to H^0(A, \mathcal{M}^{\otimes n+m} \otimes M \otimes N)$$

  is surjective for arbitrary M and N in $\hat{A}$.

As an immediate consequence we get the following:

Corollary 2.2. *Let $\mathcal{M}$ be an ample line bundle on A. For L in a non-empty subset of $\hat{A}$ the multiplication*

$$H^0(A, \mathcal{M}^{\otimes 2}) \otimes H^0(A, \mathcal{M}^{\otimes 2}) \otimes H^0(A, \mathcal{M}^{\otimes 2} \otimes L) \to H^0(A, \mathcal{M}^{\otimes 6} \otimes L)$$

is surjective.

Proof: It is enough in the second item of the previous theorem to take $M = N = \mathcal{O}_A$ and then apply the third item.

Relatively to the map

$$H^0(A, \mathcal{M}^{\otimes 2}) \otimes H^0(A, (\mathcal{M} \otimes L_{2y})^{\otimes 2}) \to H^0(A, (\mathcal{M} \otimes L_y)^{\otimes 4})$$

we know from [3] that for $\mathcal{M} = \mathcal{L}_\tau$ the dimension of the image, say $m(0, 2y)$ is equal to the number of $\eta \in A[2]$ such that $2y + \eta \notin \Theta$. This is equivalent to the number of non-vanishing

$$\left\{ \theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (\tau, 2y) : \delta, \epsilon \in \frac{1}{2} \mathbb{Z}^g / \mathbb{Z}^g \right\}.$$

In particular we have that the map is surjective once

$$2y \notin \bigcup_{\eta \in A[2]} (\Theta + \eta),$$

i.e. no theta function with half integral characteristic vanishes at the point $2y$. Moreover we have

Proposition 2.3. *For any $y \in A$, we have*

$$3^g \le m(0, 2y) \le 4^g.$$

Proof: The upper bound is obvious. The lower bound is an immediate consequence of Corollary 2.2.

Now we can prove the theorem stated in the introduction.

Proof of Th.1.1. Assuming $y = 0$ we get that

$$\Theta(2) = 4^g - m(0, 0) \le 4^g - 3^g$$

Now let $n = 2m$ even, obviously $A[2m]/A[2] \equiv (\mathbb{Z}/m\mathbb{Z})^{2g}$. Now for each $2y$ in the above class the previous estimate holds. Taking the union on all representatives we get

$$\Theta(n) \le n^{2g} - m^{2g}3^g = m^{2g}(4^g - 3^g)$$

and the theorem is proved.

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