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Qualitative Theory of Dynamical Systems

Some Results About Global Asymptotic Stability

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Abstract We study the global asymptotic stability of the origin for the continuous and discrete dynamical system associated to polynomial maps in $\mathbb{R}^n$ (especially when $n = 3$) of the form $F = \lambda I + H$, with $F(0) = 0$, where λ is a real number, I the identity map, and H a map with nilpotent Jacobian matrix JH. We distinguish the cases when the rows of JH are linearly dependent over $\mathbb{R}$ and when they are linearly independent over $\mathbb{R}$. In the linearly dependent case we find non-linearly triangularizable vector fields F for which the origin is globally asymptotically stable singularity (respectively fixed point) for continuous (respectively discrete) systems generated by F. In the independent continuous case, we present a family of maps that have orbits escaping to infinity. Finally, in the independent discrete case, we show a large family of vector fields that have a periodic point of period 3.

Keywords Polynomial vector fields · Global attractor · Markus–Yamabe conjectures

1 Introduction

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map with $F(0) = 0$. Then the origin is a singular point of the differential system

$$\dot{x} = F(x), \tag{1}$$

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and a fixed point of the dynamics of iterations of F

$$x^{(m+1)} = F(x^{(m)}), \quad x^{(0)} \in \mathbb{R}^n. \tag{2}$$

We call the continuous (respectively discrete) dynamical system generated by F to the dynamical system associated to (1) (resp. (2)).

In this article we discuss the global asymptotic stability of the origin for both systems but restricted to a special family of polynomial maps F in $\mathbb{R}^n$, focussing on $n = 3$.

We let $\phi(t, x)$ denote the solution of (1) with initial condition $\phi(0, x) = x$. We say that the origin is a *globally asymptotically stable singularity* of the continuous dynamical system generated by F if for each $x \in \mathbb{R}^n$, we have that the solution $\phi(t, x)$ of (1) is defined for all $t > 0$ and tends to the origin as t tends to infinity.

We say that the origin is a *globally asymptotically stable fixed point* of the discrete dynamical system generated by F if the sequence $x^{(m)}$ of (2) tends to the origin as m tends to infinity, for any $x^{(0)} \in \mathbb{R}^n$.

Our set of vector fields $\mathcal{N}(\lambda, n)$, that depends on a real number λ and a positive integer n, consists of the polynomial maps in $\mathbb{R}^n$ of the form $F = \lambda I + H$, with $F(0) = 0$, where I is the identity map and H has nilpotent Jacobian matrix at every point.

Polynomial maps H defined on $\mathbb{R}^n$ and on $\mathbb{C}^n$ with nilpacobian matrix at every point have been extensively studied from the algebraic geometry viewpoint (see for example [5]). In this paper we make use of some aspects of this theory.

Note that for $F \in \mathcal{N}(\lambda, n)$, the Jacobian matrix JF at each $x \in \mathbb{R}^n$ has all its eigenvalues equal to λ. Therefore, a map $F = \lambda I + H$ in $\mathcal{N}(\lambda, n)$ satisfies the hypotheses of the Markus–Yamabe Conjecture (MYC) (resp. of the Discrete Markus–Yamabe Conjecture (DMYC)) if and only if $\lambda < 0$ (resp. $|\lambda| < 1$). The MYC was established by Markus and Yamabe in 1960 (see [8]) and the DMYC was formulate by LaSalle in 1976 (see [7]). Its precise statements are the following.

The Markus–Yamabe Conjecture (MYC). Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map with $F(0) = 0$. If for any $x \in \mathbb{R}^n$ all the eigenvalues of the Jacobian of F at x have negative real part, then the origin is a global attractor of the system (1).

The Discrete Markus–Yamabe Conjecture (DMYC). Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map with $F(0) = 0$. If for any $x \in \mathbb{R}^n$ all the eigenvalues of the Jacobian of F at x have modulus less than one, then the origin is a global attractor of the discrete dynamical system (2) generated by F.

It is known that the MYC (resp. the DMYC) is true when $n \leq 2$ (resp. $n = 1$) and false when $n \geq 3$ (resp. $n \geq 2$). Notwithstanding, both conjectures are true for triangular maps in any dimension. The continuous case was proved by Markus and Yamabe [8], and the discrete case by Cima et al. [4]. In the case of polynomial maps, the DMYC is also true when $n = 2$ (see [4]), though both conjectures are false when $n \geq 3$. In [2], Cima et al. give an example of a pair of polynomial maps, of which one satisfies the MYC hypotheses and the other the DMYC hypotheses, having both systems orbits that escape to infinity. Further in [3], Cima et al. obtain a family of polynomial counterexamples containing the preceding pair. These counterexamples of [3] are, basically, vector fields $F = \lambda I + H$ in $\mathcal{N}(\lambda, 3)$ where H is a quasi–homogeneous vector field of degree one. We give examples of vector fields in $\mathcal{N}(\lambda, 3)$ which are linearly triangularizable (that is, triangular after a linear change of coordinates) in [6]. For these maps, the MYC (resp. the DMYC) is true when $\lambda < 0$ (resp. $|\lambda| < 1$). Further, [6] contains a family of counterexamples to the MYC which generalizes that of Cima–Gasull–Mañosas. The examples and counterexamples $F = \lambda I + H \in \mathcal{N}(\lambda, n)$ of above have one common characteristic, namely the rows of JH are linearly dependent over $\mathbb{R}$.

The parper is organized as follows. In Sect. 2 we consider the linearly dependent case for $n = 3$. We say that a map $F = \lambda I + H \in \mathcal{N}(\lambda, 3)$ is *linearly dependent* if there exist $(\alpha, \beta, \gamma) \in \mathbb{R}^3 - \{0, 0, 0\}$ such that $\alpha P + \beta Q + \gamma R \equiv 0$, where $H = (P, Q, R)$. We study the global asymptotic stability of the origin for the continuous and discrete dynamical system generated by maps $F = \lambda I + H \in \mathcal{N}(\lambda, 3)$ which are linearly dependent. We give a normal form for these maps (see Proposition 2.1) and characterize those elements which are linearly triangularizable (see Theorem 2.4). The normal form depends on a polynomial $f(t)$ with coefficients in $\mathbb{R}[z]$. In the case $f(t)$ is a polynomial of degree one, we show the global asymptotic stability of the origin for the continuous and discrete cases (see Theorems 2.5 and 2.6). We thus obtain a family of non–linearly triangularizable maps in $\mathcal{N}(\lambda, 3)$ for which the origin is globally asymptotically stable. To our knowledge, there are no examples as the preceding one in the literature. The section concludes showing that, for a linearly dependent map $F \in \mathcal{N}(\lambda, 3)$, in order for the origin not to be a globally asymptotically stable singularity (resp. fixed point) the continuous (resp. discrete) system must have at least one orbit which escapes to infinity (see Theorem 2.7).

In Sect. 3we deal with $F \in \mathcal{N}(\lambda, 3)$ which are not linearly dependent. These maps be called linearly independent. We state the Dependence Problem and the Generalized Dependence Problem introduced by van den Essen in [5, Chapter 7], among others, and we obtain a family of examples $F_{n,r} = \lambda I + H_{n,r}$ in $\mathcal{N}(\lambda, n)$ which are linearly independent for any dimension $n \geq 3$, with rk $JH_{n,r} = r \geq 2$. When $n \geq 3$ and $r = 2$, we show that for these maps the origin is not globally asymptotically stable singularity (see Theorem 3.2). Subsequently, we consider maps $F = \lambda I + H \in \mathcal{N}(\lambda, 3)$, where $H(x, y, z) = (u(x, y, z), v(x, y, z), h(u(x, y, z), v(x, y, z)))$. A large class of these maps H were characterized by M. Chamberland and A. van den Essen in [1]. The characterization depends on a polynomial map $g(t)$. In the case $g(t)$ is a polynomial of degree less than or equal to two, we show that the continuous system generated by $F = \lambda I + H$, with $\lambda < 0$ have orbits that escape to infinity (see Theorem 3.5). On the other hand, in the discrete case, for $|\lambda| < 1$, these maps have a periodic point of period three (see Theorem 3.8). Therefore the origin is not a globally asymptotically stable fixed point.

2 The Linearly Dependent Case

This section is devoted to maps $F = \lambda I + H$ in $\mathcal{N}(\lambda, 3)$ where the component of H are linearly dependent over $\mathbb{R}$. Since $F(0) = 0$, this condition is equivalent to that rows of the Jacobian matrix JH being linearly dependent over $\mathbb{R}$. The first result of this section establish a normal form for this type of maps. For a proof, see for example [1, Corollary 1.1].

Proposition 2.1 *Let $F = \lambda I + (S, U, V) \in \mathcal{N}(\lambda, 3)$ linearly dependent. Then there exists a $T \in Gl_3(\mathbb{R})$ such that $T_* F = \lambda I + (P, Q, 0)$ where*

$$\begin{aligned} P(x, y, z) &= -b(z)\, f(a(z)\, x + b(z)\, y) + c(z) \quad \text{and} \\ Q(x, y, z) &= a(z)\, f(a(z)\, x + b(z)\, y) + d(z) \end{aligned} \tag{3}$$

with $a, b, c, d \in \mathbb{R}[z]$ and $f \in \mathbb{R}[z][t]$.

Remark 2.2 In the normal form (3) we may assume $f(0) = 0$ by modifying the polynomials $c(z)$ and $d(z)$ if necessary.

An interesting question about the maps satisfying the hypotheses of the MYC or the DMYC concerns the injectivity.

Proposition 2.3 *If $\lambda \neq 0$ then any $F \in \mathcal{N}(\lambda, 3)$ linearly dependent is injective.*

Proof The Proposition results from the normal form (3). □

Recall that a map $F : \mathbb{R}^n \to \mathbb{R}^n$ is *triangular* if it has the form

$$F(x_1, x_2, \ldots, x_n) = (F_1(x_1), F_2(x_1, x_2), \ldots, F_n(x_1, x_2, \ldots, x_n)).$$

Our next result establishes conditions under which maps of the form $F = \lambda I + (P, Q, 0) \in \mathcal{N}(\lambda, 3)$, with (P, Q) as in Proposition 2.1, are linearly triangularizable (triangular after a linear change of coordinates).

Theorem 2.4 *Let $F = \lambda I + H \in \mathcal{N}(\lambda, 3)$ where*

$$H(x, y, z) = f(a(z)\, x + b(z)\, y)\, (-b(z), a(z), 0) + (c(z), d(z), 0)$$

with $\lambda \in \mathbb{R}$, $a, b, c, d \in \mathbb{R}[z]$, $f \in \mathbb{R}[z][t]$. Then X is linearly triangularizable if and only if either f is constant or $\{a, b\}$ are linearly dependent over $\mathbb{R}$.

Proof When f depends only on z, the result is clear. In what follows, we will assume that the degree of f with respect to t is greater than zero. If $\{a, b\}$ are linearly dependent over $\mathbb{R}$, then there exists $(\alpha, \beta) \in \mathbb{R}^2 - \{(0, 0)\}$ such that $\alpha\, a(z) + \beta\, b(z) = 0$, for all $z \in \mathbb{R}$. Assume $\beta \neq 0$. Then $b(z) = \delta\, a(z)$, with $\delta = -\frac{\alpha}{\beta}$. Consider the linear isomorphism $T(x, y, z) = (z, x + \delta\, y, y)$. Then

$$T_*(F)(u, v, w) = \lambda\, (u, v, w) + (0, c(u) + \delta\, d(u), a(u)\, f(a(u)\, v) + d(u)),$$

which is triangular.

Now suppose that there exists a linear isomorphism M such that

$$M_*(F)(u, v, w) = \lambda\, (u, v, w) + (A(v, w), B(w), 0).$$

Assume that $[M] = (m_{ij})_{1\le i,j\le 3}$ is the matrix of M with respect to the canonical basis of $\mathbb{R}^3$. We have

$$m_{31}\,[-f(t)\,b(z)+c(z)] \;+\; m_{32}\,[f(t)\,a(z)+d(z)] \equiv 0$$

where $t = a(z)\,x + b(z)\,y$. Then

$$m_{31}\,[-f(0)\,b(z)+c(z)] \;+\; m_{32}\,[f(0)\,a(z)+d(z)] \equiv 0$$

and, therefore,

$$(f(t)-f(0))\,[-m_{31}\,b(z) \;+\; m_{32}\,a(z)] \equiv 0.$$

If $(m_{31}, m_{3,2}) \neq (0,0)$, the proof is complete. If $(m_{31}, m_{3,2}) = (0,0)$, we may assume that $m_{33} = 1$ and $\det[M] = 1$. Thus the matrix of M^{-1} with respect to the canonical basis of $\mathbb{R}^3$ is

$$[M^{-1}] = \begin{pmatrix} m_{22} & -m_{12} & \tilde{m}_{13} \\ -m_{21} & m_{11} & \tilde{m}_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

with $\tilde{m}_{13} = -m_{13}\,m_{22} + m_{12}\,m_{23}$ and $\tilde{m}_{23} = m_{13}\,m_{21} - m_{11}\,m_{23}$. Then

$$t = a(w)\,[m_{22}\,u - m_{12}\,v + \tilde{m}_{13}\,w] \;+\; b(w)\,[-m_{21}\,u + m_{11}\,v + \tilde{m}_{23}\,w]$$

and

$$B(w) = m_{21}\,[-f(t)\,b(w)+c(w)] \;+\; m_{22}\,[f(t)\,a(w)+d(w)].$$

Differentiating the preceding expression with respect to u we obtain

$$0 = f'(t)\,[m_{22}\,a(w) - m_{21}\,b(w)]^2$$

and so $\{a, b\}$ are linearly dependent over $\mathbb{R}$, which completes the proof. □

The next two results assert that, in the linearly dependent case, the origin is a globally asymptotically stable singularity when the degree of the polynomial $f(t)$ is one.

Theorem 2.5 *Let $F = \lambda\,I + H \in \mathcal{N}(\lambda, 3)$ where*

$$H(x,y,z) = g(z)\,(a(z)\,x + b(z)\,y)\,(-b(z), a(z), 0) \;+\; (c(z), d(z), 0)$$

with $\lambda < 0$, $a, b, c, d, g \in \mathbb{R}[z]$. Then the origin is a globally asymptotically stable singularity for the differential system $\dot{x} = F(x)$.

Proof Note that $(x(t), y(t), z(t))$ is a solution of the differential system $\dot{x} = F(x)$ if and only if $z(t) = z_0\, e^{\lambda t}$ and $(x(t), y(t))$ is a solution of the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda - A(t)B(t)G(t) & -B(t)^2G(t) \\ A(t)^2G(t) & \lambda + A(t)B(t)G(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} C(t) \\ D(t) \end{pmatrix}$$

where $(A, B, C, D, G)(t) = (a, b, c, d, g)(z_0\, e^{\lambda t})$. Since the origin is a locally asymptotically stable singularity , there is a basis of solutions of the linear system consisting of solutions which tend to the origin as t tends to $+\infty$. Therefore, the origin is a globally asymptotically stable singularity. □

Theorem 2.6 *Let* $F = \lambda\, I + H \in \mathcal{N}(\lambda, 3)$ *where*

$$H(x, y, z) = g(z)\,(a(z)\,x + b(z)\,y)\,(-b(z), a(z), 0) + (c(z), d(z), 0)$$

with $0 < \lambda < 1$*,* $a, b, c, d, g \in \mathbb{R}[z]$*. Then the origin is a globally asymptotically stable fixed point of the discrete dynamical system (2) generated by* F*.*

Proof Without loss of generality, we may assume that $c(z) \equiv d(z) \equiv 0$. In fact, the polynomials $c(z)$ and $d(z)$ may be eliminated by applying the coordinate change $T(u, v, w) = (u + m(w), v + n(w), w)$ where

$$\begin{pmatrix} m(w) \\ n(w) \end{pmatrix} = \frac{-1}{(1-\lambda)^2}\, [(1-\lambda)\, I + g(w)\, M(w)] \begin{pmatrix} c(w) \\ d(w) \end{pmatrix}$$

with

$$M(w) = \begin{pmatrix} -a(w)b(w) & -b(w)^2 \\ a(w)^2 & a(w)b(w) \end{pmatrix}.$$

So we assume

$$H(x, y, z) = g(z)\,(a(z)\,x + b(z)\,y)\,(-b(z), a(z), 0).$$

Therefore,

$$F(x, y, z) = (A(z) \begin{pmatrix} x \\ y \end{pmatrix}, \lambda\, z)$$

where

$$A(z) = \begin{pmatrix} \lambda - a(z)b(z)g(z) & -b(z)^2g(z) \\ a(z)^2g(z) & \lambda + a(z)b(z)g(z) \end{pmatrix}.$$

Thus it suffices to prove that, for any $(x, y) \in \mathbb{R}^2$, we have

$$\lim_{n\to\infty} A(\lambda^n z)A((\lambda^{n-1} z) \cdots A(\lambda z)A(z) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let $\mathcal{M}_2$ be the normal vector space of the 2×2 real matrices $A = (a_{ij})$ endowed with the norm $\| A\| = 2 \max |a_{ij}|$. Considering $\mathbb{R}^2$ endowed with the norm $\| (x, y)\| = \max\{|x|, |y|\}$, we have

$$\| A(x, y)\| \leq \| A\| \, \| (x, y)\| \quad \text{and} \quad \| AB\| \leq \| A\| \, \| B\| .$$

A simple computation yields

$$A(\lambda^n z)A((\lambda^{n-1}z)\cdots A(\lambda z)A(z) =$$

$$\begin{pmatrix} \lambda^n - na_0b_0g_o\lambda^{n-1} + r_{11}(z) & -nb_0^2g_0\lambda^{n-1} + r_{12}(z) \\ na_0^2g_0\lambda^{n-1} + r_{21}(z) & \lambda^n + na_0b_0g_0\lambda^{n-1} + r_{22}(z) \end{pmatrix}$$

where $r_{ij}(0) = 0$ and $(a_0, b_0, g_0) = (a, b, g)(0)$.

Fix $N \in \mathbb{N}$ so that $2N\lambda^{N-1} \max\{a_0^2g_0, b_0^2g_0, |a_0b_0g_0|\} < 1$. Let $B(z) = A(\lambda^{N-1}z)A((\lambda^{N-2}z)\cdots A(\lambda z)A(z)$. Consider $0 < |z| < z_0$ such that $\| B(z)\| \leq K < 1$. Then, for $n = kN - 1$, we have

$$\left\| A(\lambda^n z)\cdots A(z)\begin{pmatrix} x \\ y \end{pmatrix}\right\| = \left\| B(\lambda^{(k-1)N}z)\cdots B(z)\begin{pmatrix} x \\ y \end{pmatrix}\right\|$$
$$\leq K^k \, \| (x, y)\| \to 0 \quad \text{if} \quad k \to \infty$$

which completes the proof. □

Our next result shows that, for a linearly dependent $F \in \mathcal{N}(\lambda, 3)$, in order for the origin not to be a globally asymptotically stable singularity (resp. fixed point) the continuous (resp. discrete) system must have at least one orbit which escapes to infinity.

Theorem 2.7 *Let $\lambda \in \mathbb{R}$, and let $F = \lambda I + H \in \mathcal{N}(\lambda, 3)$ linearly dependent. If $\lambda < 0$ (resp. $|\lambda| < 1$) and the origin is not a globally asymptotically stable singularity (resp. fixed point) for the differential system $\dot{x} = F(x)$ (resp. for the discrete dynamical system (2) generated by F), then the differential system $\dot{x} = F(x)$ (resp. the discrete dynamical system generated by F) has orbits which escape to infinity.*

Proof We may assume that $H = (P, Q, 0)$ where

$$P(x, y, z) = -b(z)\, f(a(z)\, x + b(z)\, y) \; + \; c(z) \quad \text{and}$$
$$Q(x, y, z) = a(z)\, f(a(z)\, x + b(z)\, y) \; + \; d(z)$$

with $a, b, c, d \in \mathbb{R}[z]$ and $f \in \mathbb{R}[z][t]$.

Consider the case $\lambda < 0$. Let $\gamma(t) = (x(t), y(t), z(t))$ be a solution of the system $\dot{x} = F(x)$. We denote the omega-limit set of γ by $\omega(\gamma)$. Since $z(t) = z(0)\, e^{\lambda t}$, we have $\omega(\gamma) \subset W_\infty$, where W_∞ is the extended plane $\{z = 0\} \cup \{\infty\}$. If the orbit $\gamma(t)$ is bounded, then $\omega(\gamma) = \{0\}$ and we obtain the Theorem. The proof is analogous for the discrete dynamical system generated by F in the case $|\lambda| < 1$. □

Thus we are led to posing the following:

Question 1 *Do there exist linearly independent maps in* $\mathcal{N}(\lambda, 3)$*, with the degree of* $f(t)$ *greater than one, for which the origin is globally asymptotically stable either in the continuous case, or the discrete case, or both?*

3 The Linearly Independent Case

In this section we consider maps $F \in \mathcal{N}(\lambda, 3)$ where the rows of JH are linearly independent over $\mathbb{R}$. We begin with some algebraic preliminaries extracted from [5, Chapter 7] and [1]. The study of the Jacobian Conjecture for polynomial maps of the form $I + H$ where I is the identity map and H a homogeneous map of degree 3, with JH nilpotent, led various authors to the following problem. Let κ be a field of characteristic zero.

Dependence Problem. Let $d \in \mathbb{N}$, with $d \geq 1$, and let $H = (H_1, \ldots, H_n) : \kappa^n \to \kappa^n$ be a homogeneous polynomial map of degree d such that JH is nilpotent. Does it follow that $H_1, \ldots, H_n$ are linearly dependent over κ?

The attempt to solve it by induction led to consider the more general problem:

Generalized Dependence Problem. Let $H = (H_1, \ldots, H_n) : \kappa^n \to \kappa^n$ be a polynomial map such that JH is nilpotent. Are the rows of JH linearly dependent over κ?

The answer to this question turned out to be "yes" if $n \leq 2$ and "no" if $n \geq 3$. More precisely, van den Essen showed the following (see [5, Theorem 7.1.7]).

Theorem 3.1 (i) *If* JH *is nilpotent and rk* $JH \leq 1$*, then the rows of* JH *are linearly dependent over* κ *(here rk is the rank as an element of* $M_n(\kappa(X))$*).*

(ii) *Let* $r \geq 2$*. Then, for any dimension* $n \geq r + 1$*, there exists a polynomial map* $H_{n,r} : \kappa^n \to \kappa^n$ *such that* $JH_{n,r}$ *is nilpotent, rk* $JH_{n,r} = r$*, and the rows of* $JH_{n,r}$ *are linearly independent over* κ*.*

The example is the following. Let $a \in \mathbb{R}[x_1]$ with $\deg a = r$ and $f(x_1, x_2) = x_2 - a(x_1)$. Then $H_{n,r} = (H_1, \ldots, H_n)$ where

$$
\begin{aligned}
H_1(x_1, \ldots, x_n) &= f(x_1, x_2),\\
H_i(x_1, \ldots, x_n) &= x_{i+1} + \frac{(-1)^i}{(i-1)!}\, a^{(i-1)}(x_1)\,(f(x_1, x_2))^{i-1}, \text{ if } 2 \leq i \leq r,\\
H_{r+1}(x_1, \ldots, x_n) &= \frac{(-1)^{r+1}}{r!}\, a^{(r)}(x_1)\,(f(x_1, x_2))^r, \text{ and}\\
H_j(x_1, \ldots, x_n) &= (f(x_1, x_2))^{j-1}, \text{ if } r+1 < j \leq n
\end{aligned}
$$

is a polynomial map satisfying assertion (ii). (See [5, Proposition 7.1.9]). For $r = 2$ and $n \geq 3$, the components of $H_{n,2}$ are

$$
\begin{aligned}
H_1(x_1, \ldots, x_n) &= x_2 - a\,x_1 - b\,x_1^2,\\
H_2(x_1, \ldots, x_n) &= x_3 + (a + 2\,b\,x_1)\,(x_2 - a\,x_1 - b\,x_1^2),
\end{aligned} \tag{4}
$$

$$H_3(x_1, \dots, x_n) = -b\,(x_2 - a\,x_1 - b\,x_1^2)^2, \text{ and for } j \ge 4$$
$$H_j(x_1, \dots, x_n) = (x_2 - a\,x_1 - b\,x_1^2)^{j-1},$$

with $b \neq 0$.

Theorem 3.2 *Let $F_{n,2} = \lambda\, I + (H_1, \dots, H_n)$, with H_i as in* (4) *.*

(a) *If $\lambda < 0$ then the system $\dot{x} = F_{n,2}(x)$ has orbits that escape to infinity.*
(b) *If $-1 < \lambda < 1$ then the discrete dynamical system generate by $F_{n,2}$ has a periodic orbit of period three.*

Proof (a) It suffices to prove the theorem in the case $n = 3$. Thus assume $n = 3$ and put $X = F_{3,2}$. If $(u, v, w) = \phi(x_1, x_2, x_3) = b\,(x_1, x_3 - \lambda\, b\, x_1^2, \lambda\, x_1 + x_2 - a\, x_1 - b\, x_1^2)$, and $\phi^*(X) = Y$ then

$$Y(u, v, w) \;=\; (w, \lambda\, v - w^2, 2\,\lambda\, w + v - \lambda^2\, u).$$

To find orbits of Y that escape to infinity, consider the coordinate change

$$(s, q, p) \;=\; \frac{1}{v}(1, u, w).$$

If Z is the vector field Y in the new coordinates, then $W = (W_1, W_2, W_3) = s\,Z$ is defined by

$$W(s, q, p) \;=\; (-s(\lambda\, s - p^2), s(p - \lambda\, q) + q\, p^2, s(\lambda\, p + 1 - \lambda^2\, q) + p^3).$$

For $s \neq 0$, the orbits of Z and W are the same. Moreover, for $s > 0$ (resp. $s < 0$), the orbits of Z and W have the same (resp. inverse) orientation. Over the plane $s = 0$, the vector field W is radially repeller outside of a line of singular points, namely the line $p = 0$. For $s > 0$, we have $W_1 > 0$ and, therefore, there are no orbits there with $\omega-$limit set contained at $s = 0$. For $s < 0$, we must find orbits of W with $\alpha-$limit set contained at $s = 0$.

Consider the numbers

$$A = 2\,\lambda, \quad s_0 = \frac{1}{512\lambda^3}, \quad p_0 = -\frac{1}{8\,\lambda}, \quad q_0 = \frac{11}{16\,\lambda^2}$$

and the set

$$P_A = \{(s, q, p) : as - p^2 \le 0, s_0 \le s \le 0, 0 \le q \le q_0, 0 \le p \le p_0\}.$$

We have the following:

(1) Over the set $P_A \cap \{(s, q, p) : A\, s - p^2 = 0\}$, the vector field W points outward from the set P_A. In fact, if $(s, q, p) \in P_A$ and $A\, s - p^2 = 0$, then

$$\begin{aligned} A\,W_1 - 2\,p\,W_3 &= -\frac{p^3}{A}\,[p\,(A+3\lambda)\ +\ 2\,(1-\lambda^2\,q)] \\ &\geq -\frac{p^3}{A}\,[p_0\,(A+3\lambda)\ +\ 2\,(1-\lambda^2\,q_0)] = 0. \end{aligned}$$

(2) Over the set $P_A \cap \{(s, q, p_0) : s < 0\}$, the vector field W points outward from the set P_A. In fact, if $p = p_0$, then

$$\begin{aligned} W_3 &= s\,(\lambda\,p_0 + 1 - \lambda^2\,q) + p_0^3 \\ &\geq s\,(\lambda\,p_0 + 1) + p_0^3\ =\ \frac{7}{8}\,s - \frac{1}{8^3\,\lambda^3} \\ &\geq \frac{7}{8}\,s_0 - \frac{1}{8^3\,\lambda^3}\ =\ \frac{-1}{8^4\,\lambda^3} > 0. \end{aligned}$$

(3) Over the set $P_A \cap \{(s, 0, p)\}$, the vector field W points outward from the set P_A. In fact, if $q = 0$, then

$$W_2\ =\ s\,p \leq 0.$$

(4) Over the set $P_A \cap \{(s, q_0, p)\}$, the vector field W points outward from the set P_A. In fact, if $q = q_0$, then

$$W_2\ =\ s\,p - q_0\,(\lambda s - p^2)\ =\ (\lambda\,s - p^2)\,[\frac{sp}{\lambda\,s - p^2} - q_0] \geq 0$$

because

$$\lambda\,s - p^2\ <\ A\,s - p^2 \leq A\,s_0 - \frac{p_0^2}{4}\ =\ 0$$

and

$$h(s, p)\ =\ \frac{sp}{\lambda\,s - p^2} \leq h(s_0, p) \leq h(s_0, \frac{p_0}{2})\ =\ \frac{1}{16\lambda^2}\ <\ q_0.$$

Thus any orbit $\gamma(t)$ of W, with $\gamma(0)$ an interior point of P_A, has α-limit set contained in the line $s = p = 0$. Clearly, any (of these) orbit corresponds to an orbit of our initial vector field X that escapes to infinity. This completes the proof in this case.

(b) When $n = 3$ the system $F_{3,2}$ correspond to a particular case of (10) and the result follows of Theorem 3.8. Thus assume $n > 3$. Observe that

$$F_{n,2}(x_1, \ldots, x_n) = (F_{3,2}(x_1, x_2, x_3), \lambda x_4 + f(x_1, x_2)^3, \ldots, \lambda x_n + f(x_1, x_2)^{n-1})$$

where $f(x_1, x_2) = x_2 - a\,x_1 - b\,x_1^2$.

This implies that the third iterated of $F_{n,2}$ is of the form

$$F_{n,2}^3(x_1, \ldots, x_n) = (F_{3,2}^3(x_1, x_2, x_3), \lambda^3 x_4 + g_4(x_1, x_2, x_3), \ldots, \lambda^3 x_n + g_n(x_1, x_2, x_3)).$$

Then the point $(\overline{x_1}, \ldots, \overline{x_n})$, where $(\overline{x_1}, \overline{x_2}, \overline{x_3})$ is a periodic point of period three of $F_{3,2}$, and $\overline{x_j} = \frac{1}{1-\lambda^3}\, g_j(\overline{x_1}, \overline{x_2}, \overline{x_3})$, for $4 \leq j \leq n$, is a periodic point of period three of $F_{n,2}$. The proof is now complete. □

Observe that the example $H_{3,2}$ has the special form

$$H_{3,2}(x, y, z) = (u(x, y), v(x, y, z), h(u(x, y))). \tag{5}$$

In [1] it proved that a large class of polynomial maps $H = (H_1, H_2, H_3)$ of the form

$$H(x, y, z) = (u(x, y, z), v(x, y, z), h(u(x, y, z), v(x, y, z))). \tag{6}$$

with JH nilpotent and such that H_1, H_2, H_3 are linearly independent, reduce through a linear coordinate change, to a map of the form

$$G(x, y, z) = (g(t), v_1\, z - (b_1 + 2v_1\alpha\, x)\, g(t), \alpha\, g(t)^2) \tag{7}$$

with $t = y + b_1\, x + v_1\alpha\, x^2$ and $v_1\alpha \neq 0$, and $g \in \mathbb{R}[t]$ with $g(0) = 0$ and $\deg_t g(t) \geq 1$. More specifically, it has the following theorem that resume the results of [1].

Theorem 3.3 *Let* $H(x, y, z) = (u(x, y, z), v(x, y, z), h(u(x, y, z), v(x, y, z)))$. *Assume that* $H(0) = 0, h'(0) = 0$, *and the components of* H *are linearly independent over* $\mathbb{R}$. *Let* $A = \frac{\partial v}{\partial x}\frac{\partial u}{\partial z} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$ *and* $B = \frac{\partial v}{\partial y}\frac{\partial u}{\partial z} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}$. *If JH is nilpotent and* $\deg_z(uA) \neq \deg_z(vB)$ *then there exists* $T \in GL_3(\mathbb{R})$ *such that* THT^{-1} *is of the form (7).*

Remark 3.4 (1) By Theorem 3.3, any map

$$F = \lambda\, I + (u(x, y, z), v(x, y, z), h(u(x, y, z), v(x, y, z))) \in \mathcal{N}(\lambda, 3)$$

under the condition $\deg_z(uA) \neq \deg_z(vB)$, modulus a linear change of coordinates, has the form

$$F(x, y, z) = \lambda\, (x, y, z) + (0, v_1\, z, 0) + g(t)\, (1, -(b_1 + 2v_1\alpha\, x), \alpha\, g(t)) \tag{8}$$

with $t = y + b_1\, x + v_1\alpha\, x^2$ and $v_1\alpha \neq 0$, and $g \in \mathbb{R}[t]$ with $g(0) = 0$ and $\deg_t g(t) \geq 1$.

(2) Any map

$$F = \lambda\, I + (u(x, y, z), v(x, y, z), h(u(x, y, z), v(x, y, z))) \in \mathcal{N}(\lambda, 3)$$

under the condition $\deg_z(uA) \neq \deg_z(vB)$ is injective.

(3) For $n = 3$, the map $F_{3,2}$ of Theorem 3.2 up to a linear change of coordinates is of the form (8) with $g(t)$ a polynomial of degree one. Therefore, the origin is not a globally asymptotically singularity (resp. fixed point) for the continuous (resp. discrete) dynamical system generated by $F_{3,2}$.

In consequence we establish the following:

Question 2 *Do there exist linearly independent maps in $\mathcal{N}(\lambda, 3)$ of the form (8) for which the origin is globally asymptotically stable for the corresponding continuous and/or discrete system?*

3.1 The Continuous Case

Our next result gives a negative answer to Question 2, in the continuous case, when the degree of $g(t)$ is less than or equal to two. First observe that, by applying the coordinate change

$$\begin{aligned}(u, v, w) &= \phi(x, y, z)\\ &= (\lambda\, x + g(t), t, v_1\, z + \lambda\, v_1\, \alpha\, x^2)\end{aligned}$$

where $t = y + b_1\, x + v_1 \alpha\, x^2$, to the vector field (8) we obtain

$$\phi_*(F)(u, v, w) \;=\; \lambda\,(u, v, w) \;+\; (g'(v)(\lambda\, v + w), w, \alpha\, v_1\, u^2). \tag{9}$$

Theorem 3.5 *Consider a map $F \in \mathcal{N}(\lambda, 3)$, with $\lambda < 0$, of the form (8) where $g(t) = A_1\, t + A_2\, \frac{t^2}{2}$. Then F has orbits that escape to infinity.*

Proof In the case $A_2 = 0$, making the linear change of coordinates

$$(u, v, w) = \phi(x, y, z) = \frac{1}{m}(x, my, mv_1 z)$$

with $m = A_1$, the vector field $F - \lambda\, I$ has the form (4). The result now follows from Theorem 3.2.
Next consider the case $A_2 \neq 0$. Then we may assume

$$F(x, y, z) = \lambda(x, y, z) + (g'(y)(\lambda y + z), z, v_1 \alpha x^2).$$

To find orbits of F that escape to infinity, we first make the coordinate change

$$(u, v, w) \;=\; \frac{1}{z}(x, y, 1).$$

If Y is the vector field F in the new coordinates, then $Z = w\, Y$ is defined by

$$Z(u, v, w) \;=\; (-\beta u^3 + (A_1\, w + A_2\, v)(\lambda\, v + 1), -\beta u^2 v + w, -w(\lambda w + \beta u^2))$$

with $\beta = v_1\, \alpha$. For $w \neq 0$, the vector fields Y and Z have the same orbits. Moreover, for $w > 0$ (resp. $w < 0$), the orbits of Y and Z have the same (resp. inverse) orientation.

Then we apply the blow-up

$$(s, q, p) = \left(u, \frac{v}{u^3}, \frac{w}{u^5}\right).$$

If Y_1 is the vector field Y in the new coordinates, then $Y_1 = s^2 Z_1$ where

$$Z_1(s, q, p) = A(s, q, p)\,(s, -3q, -5p) + (0, p - \beta q, -p(\beta + \lambda p s^3))$$

with $A(s, q, p) = -\beta + (A_1 p s^2 + A_2 q)(\lambda q s^3 + 1)$.

The singularities of Z_1 over $s = 0$ are

$$(0, 0, 0), \quad \left(0, \frac{2\beta}{3A_2}, 0\right), \quad \text{and} \quad \left(0, \frac{4\beta}{5A_2}, \frac{8\beta^2}{25A_2}\right).$$

The Jacobian matrix of Z_1 at $(0, \frac{4\beta}{5A_2}, \frac{8\beta^2}{25A_2})$ has eigenvalues

$$\mu_1 = -\frac{\beta}{5}, \mu_2 = -\frac{2\beta}{5}, \quad \text{and} \quad \mu_3 = -2\beta.$$

In the case $\beta > 0$ (resp. $\beta < 0$), this singularity is an attractor (resp. repeller) of the vector field Z_1. Given an initial condition $(s(0), q(0), p(0))$ sufficiently close to the singularity, with $s(0)p(0) > 0$ (resp. $s(0)p(0) < 0$) for $\beta > 0$ (resp. $\beta < 0$), we obtain an orbit of the original vector field F that escapes positively to infinity. □

3.2 The Discrete Case

In this subsection we prove that, in the discrete case, the answer to Question 2 is negative for any $g(t) \in \mathbb{R}[t]$ with $g(0) = 0$ and $deg_t g(t) \geq 1$.

For $|\lambda| < 1$, consider

$$\begin{aligned} F(x, y, z) = \lambda\,(x, y, z) &+ (0, v_1 z, 0) \\ &+ g(t)\,(1, -(b_1 + 2v_1\alpha\, x), \alpha\, g(t)) \end{aligned} \tag{10}$$

with $t = y + b_1 x + v_1\alpha\, x^2$ and $v_1\alpha \neq 0$, and $g(t) \in \mathbb{R}[t]$ with $g(0) = 0$ and $deg_t g(t) \geq 1$.

Lemma 3.6 *The set of fixed points of F is reduced to the origin. If $g(t) = At$ then the unique periodic point of period two of F is the origin.*

Lemma 3.7 *If $-1 < \lambda < 1$ and $g(t) = A\,t$, with $A \neq 0$, then F has a periodic point of period three $(x_0, y_0, z_0) \neq (0, 0, 0)$. Furthermore, the eigenvalues of $DF^3(x_0, y_0, z_0)$ are all different from 1.*

Proof Calculations involve MATHEMATICA prove that the point (x_0, y_0, z_0) with

$$x_0 = \frac{(1+\lambda+\lambda^2)\,(1+4\lambda^2+\lambda^4)}{A\,\beta\,(1-\lambda)^3},$$
$$y_0 = -\frac{1+\lambda+\lambda^2}{A^2\,\beta\,(1-\lambda)^6}\,[\lambda\,(1+\lambda+\lambda^2)\,(4+\lambda+8\lambda^2+11\lambda^3+4\lambda^4+7\lambda^5+\lambda^7) + A\,b_1\,(1-\lambda)^3\,(1+4\lambda^2+\lambda^4)]$$
$$z_0 = \frac{(1+\lambda+\lambda^2)^3\,(1+3\lambda^2+4\lambda^3+3\lambda^4+\lambda^6)}{v_1\,A^2\,\beta\,(1-\lambda)^5},$$

where $\beta = v_1\alpha$, is a periodic point of period three of F. On the other hand, we prove that the characteristic polynomial of $DF^3(x_0, y_0, z_0)$ is

$$p(x) = -\lambda^9 - \lambda\,(8+44\,\lambda+104\,\lambda^2+164\,\lambda^3+164\,\lambda^4+113\,\lambda^5+44\,\lambda^6 +8\,\lambda^7-4\,\lambda^8)\,x + (-4+8\,\lambda+44\,\lambda^2+113\,\lambda^3+164\,\lambda^4+164\,\lambda^5 +104\,\lambda^6+44\,\lambda^7+8\,\lambda^8)\,x^2+x^3,$$

and

$$p(1) = 3\,(\lambda-1)^3\,(1+\lambda+\lambda^2)^3 \neq 0.$$

□

Theorem 3.8 *For* $|\lambda| < 1$*, consider*

$$F(x, y, z) = \lambda\,(x, y, z)\ +\ (0, v_1\,z, 0) + g(t)\,(1, -(b_1+2v_1\alpha\,x), \alpha\,g(t))$$

with $t = y+b_1\,x+v_1\alpha\,x^2$ *and* $v_1\alpha \neq 0$*, and* $g(t) \in \mathbb{R}[t]$ *with* $g(0) = 0$ *and* $g'(0) \neq 0$*. Then there exists* $(x_0, y_0, z_0) \neq (0, 0, 0)$ *which is a periodic point of period 3 of* F*.*

Proof Assume $g(t) = A\,t + A_2\,t^2 + \cdots + A_k\,t^k$, with $A \neq 0$. When $(A_2, \ldots, A_k) = (0, \ldots, 0)$ we denote the corresponding map F by F_0. Therefore, F_0 has a periodic point of period three $(x_0, y_0, z_0) \neq (0, 0, 0)$ and the eigenvalues of $DF_0^3(x_0, y_0, z_0)$ are all different from 1. Consider the map $G : \mathbb{R}^{k-1} \times \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$G(A_2, \ldots, A_k, x, y, z)\ =\ F^3(x, y, z)\ -\ (x, y, z).$$

Observe that $G(0, \ldots, 0, x, y, z) = F_0^3(x, y, z) - (x, y, z)$, for all $(x, y, z) \in \mathbb{R}^3$. Then $G(0, \ldots, 0, x_0, y_0, z_0) = (0, 0, 0)$ and $D_2\,G(0, \ldots, 0, x_0, y_0, z_0)$ is invertible. From implicit function theorem, there exist $\varepsilon > 0$ such that, for all $(A_2, \ldots, A_k)$ with $\max\{|A_2|, \ldots, |A_k|\} < \varepsilon$ the map $F(x, y, z)$ has a periodic point of period three. For the general case, observe that if $a \in \mathbb{R} - \{0\}$ and $T(x, y, z) = a^{-1}\,(x, y, z)$, then

$$T(F(T^{-1}(u, v, w) = \lambda\,(u, v, w) \;+\; (0, v_1\,w, 0) \\ + \tilde{g}(t)\,(1, -(b_1 + 2v_1\tilde{\alpha}\,u), \tilde{\alpha}\,\tilde{g}(t))$$

with $\tilde{\alpha} = \alpha\,a$ and

$$\tilde{g}(t) \;=\; a^{-1}\,g(a\,t) \;=\; A\,t + A_2\,a\,t^2 + \cdots + A_k\,a^{k-1}\,t^k.$$

For $|a|$ sufficiently small, the map $T \circ F \circ T^{-1}$ has a non vanished periodic point of period three, and then, the map F also. □

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