Minisuperspace example of non-Lagrangian quantization

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We present the construction of a Hamiltonian description for a (non-Lagrangian) Bianchi type-V cosmological model using the solutions of the equations of motion as a starting point. We study the quantization of the model and compare our results to those found using a more standard approach.

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I. INTRODUCTION

One of us (S.H.) has recently been studying nonstandard Hamiltonian dynamics in classical mechanics [1]. The study is based on work of Feynman reported by Dyson [2] and its extension by Hojman and Shepley [3]. Feynman's original work showed that Poisson-bracket relations place strong constraints on the types of forces allowed in physical systems. Hojman and Shepley generalized Feynman's work and are able to make a consistent Poisson-bracket Hamiltonian theory for any system whose equations of motion come from a variational principle. Finally, Hojman has extended the formalism to any first-order equations of motion for a set of coordinates x^a of the form $x^a = f^a(x^b)$, whether or not they come from a variational principle.

A review of the classical theory can be given very quickly, since the equations $\dot{x}^a = f^a(x^b)$ are supposed to represent Hamilton's equations, where usually half of the x^a are generalized momenta p^a. All that is necessary in principle for a Hamiltonian theory is a function $H(x^b)$ which gives

$$\dot{x}^a = \{x^a, H\} = f^a \ . \tag{1.1}$$

In order to define the Poisson brackets $\{\ \}$, it is necessary to consider $\{x^a, x^b\}$. These brackets have the form

$$\{x^a, x^b\} = J^{ab} \ , \quad J^{ab} = -J^{ba} \ . \tag{1.2}$$

The matrix J^{ab} must be allowed to have a more general form than the usual symplectic structure matrix, where $J^{ab}=0$ except when one of the x^a is a momentum variable and x^b is its canonically conjugate coordinate. In this formulation all that is required is that J^{ab} be antisymmetric, and satisfy the Jacobi identity

$$J^{ab}{}_{,d}J^{dc} + J^{bc}{}_{,d}J^{da} + J^{ca}{}_{,d}J^{db} = 0 \tag{1.3}$$

that is necessary for the consistency of the equations of motion. Often, the requirement of nonsingularity is added, but it is by no means necessary in order to construct a Hamiltonian description.

Classically the formalism we have outlined above is reasonable in the sense that it defines a "Hamiltonian" that gives the first-order form of the equations of motion. The main question that remains is whether, for any system of equations of the form $\dot{x}^a = f^a$, a J^{ab} exists that casts them into the Hamiltonian form (1.1). In any case where there are N x^a $(a=1,\ldots,N)$, it can be shown that there exists one way of constructing an antisymmetric J^{ab} which obeys the Jacobi identity if one knows N constants of the motion, C_i, $(N-1)$ of which do not depend explicitly on time, that is, knows them as explicit functions of the coordinates (a fairly strong requirement, equivalent to knowing the full classical solution). The preceding requirement is sufficient to be able to reduce the equations of motion to Hamiltonian form. It is, of course, not necessary for constructing the Hamiltonian theory.

This J^{ab} may be constructed by summing elements of the basic form

$$J_1{}^{ab} = \lambda(x^c)\epsilon^{ab\mu_1\cdots\mu_{N-2}}C_{1,\mu_1}\cdots C_{N-2,\mu_{N-2}} \ , \tag{1.4}$$

where $C_{,\mu} \equiv \partial C/\partial x^\mu$, $\epsilon^{ab\mu_1\cdots\mu_{N-2}}$ is the N-index Levi-Civita symbol, and $\lambda(x^c)$ is a function of the coordinates to be explained below. This J^{ab} satisfies the Jacobi identity. The $C_1,\ldots,C_{N-1}$ are time-independent constants of the motion. The Hamiltonian is defined by $H=C_{N-1}$, along with $C_N = t + d_N$, where d_N is time independent. This can always be achieved by a change of coordinates. It is easy to realize that $\lambda(x^c)$ may always be chosen so that $J_1{}^{ab}\partial H/\partial x^b = f^a$.

Since there is considerable freedom in this formalism in selecting the Hamiltonian H, one can choose a series of functions H that appear at first sight to be untenable as Hamiltonians when seen from the point of view of familiar canonical theory. One example is the two-dimensional harmonic oscillator, where the equations of motion can be written in the form

$$\dot{x} = u, \quad \dot{y} = v, \quad \dot{u} = -x, \quad \dot{v} = -y \ . \tag{1.5}$$

A possible Hamiltonian is the angular momentum $P_\theta = xv - yu$ which would usually be associated with an attempt to choose $\theta = \arctan(y/x)$ as "time." There exists a procedure based on a parametrized time that allows one to do just this (even though the choice is quite unphysical in most cases), but in the present case the nonstandard canonical approach allows us both to choose P_θ as our Hamiltonian *and* retain the Newtonian time t as our temporal marker. In the Appendix we compare these

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two approaches to the problem of the two-dimensional harmonic oscillator.

Even if we may regard the classical systems represented by the nonstandard Hamiltonian approach as unobjectionable, the goal of this paper is the quantization of a system using this method. Here a number of serious questions arise: (1) Does a particular choice of Hamiltonian lead to physically reasonable quantum behavior? (2) Are the solutions to the different Hamiltonians in any sense unitarily equivalent? (3) Since the Poisson-bracket relation implied by (1.2) is not the usual one, how may we realize the operators associated with the variables in a reasonable way? We will not attempt to answer these questions in general here. For the system we have chosen, we will take H to be as much like a standard Hamiltonian as possible, and in our case our solution is as physically reasonable as any other for Bianchi models. That is, there are serious questions about all quantum cosmology solutions, as we will discuss further when we have defined our system.

One of us (S.H.) has been studying a variety of nonstandard Lagrangian and Hamiltonian formulations for a number of years. One of the original goals of this study was to apply these methods to the quantization of the gravitational field. As a first attempt at this we would like to study a minisuperspace model for simplicity. Since the new Hamiltonian formalism is applicable to equations of motion that are not derivable from a variational principle, we felt that an excellent example to begin with is one in which it is possible to write the equations of motion in such a way that they are not the result of varying any action. Probably the simplest minisuperspace that has this property is a Bianchi type-V metric with $g_{0i}=0$ and the logarithm of the radius of the Universe taken as an internal time.

A general Bianchi type-V metric has the form

$$ds^2=-g_{00}(t)dt^2+g_{0i}(t)\sigma^i dt+g_{ij}(t)\sigma^i\sigma^j \; , \tag{1.6}$$

where the one-forms σ^i are $\sigma^1=e^{-z}dx$, $\sigma^2=e^{-z}dy$, $\sigma^3=dz$. If we parametrize $g_{ij}(t)$ as

$$g_{ij}(t)=e^{2\alpha(t)}e^{2\beta_{ij}(t)} \; , \tag{1.7}$$

with

$$\beta_{ij}(t)=\mathrm{diag}\{\beta_+ +\sqrt{3}\beta_-\sqrt{3}\beta_-,\beta_+ -\sqrt{3}\beta_-,-2\beta_+\}, \tag{1.8}$$

we can now make a change of coordinates where $t'=\alpha(t)$ and put $g_{0i}=0$. The metric becomes

$$ds^2=-g_{00}(\alpha)d\alpha^2+e^{2\alpha}e^{2\beta_{ij}(\alpha)}\sigma^i\sigma^j \; . \tag{1.9}$$

We now parametrize g_{00} by $g_{00}\equiv e^{6\alpha}/p_\alpha^2$, define $p_\pm \equiv p_\alpha d\beta_\pm/d\alpha$, so that the Einstein equations in the coordinate basis (α,x,y,z) become

$$G^\alpha{}_\alpha=3e^{-6\alpha}\{e^{4(\alpha+\beta_+)}-p_\alpha^2+p_+^2+p_-^2\}=0 \; , \tag{1.10}$$

$$G^\alpha{}_z=-6e^{-6\alpha}p_\alpha p_+=0 \; , \tag{1.11}$$

$$\begin{aligned} G^k{}_k &\equiv \tfrac{1}{3}(G^x{}_x+G^y{}_y+G^z{}_z) \\ &= e^{-6\alpha}\{e^{4(\alpha+\beta_+)}-2p_\alpha\dot p_\alpha+3(p_\alpha^2-p_+^2-p_-^2)\}=0 \; , \end{aligned} \tag{1.12}$$

$$\tfrac{1}{3}(G^x{}_x+G^y{}_y-2G^z{}_z)=2p_\alpha e^{-6\alpha}\dot p_+ \; , \tag{1.13}$$

$$G^x{}_x-G^y{}_y=2\sqrt{3}p_\alpha e^{-6\alpha}\dot p_- \; , \tag{1.14}$$

where an overdot denotes $d/d\alpha$.

These equations reduce to the set

$$C\equiv e^{4(\alpha+\beta_+)}-p_\alpha^2+p_+^2+p_-^2=0 \; , \tag{1.15}$$

$$p_+=0 \; , \tag{1.16}$$

$$e^{4(\alpha+\beta_+)}-2p_\alpha\dot p_\alpha+3(p_\alpha^2-p_+^2-p_-^2)=0 \; , \tag{1.17}$$

$$\dot p_+=0 \; , \tag{1.18}$$

$$\dot p_-=0 \; , \tag{1.19}$$

$$\dot\beta_\pm=\frac{p_\pm}{p_\alpha} \; . \tag{1.20}$$

These equations do not come directly from a variational principle. They have the form of the equations expected from the Arnowitt-Deser-Misner (ADM) Hamiltonian formulation where the metric (1.9) is simply inserted into the ADM action (with the momenta p_α, $p_\pm$ suitably defined), but Eq. (1.18) is given incorrectly by this straightforward procedure. In any case, we only plan to take (1.15)–(1.20) as a set of minisuperspace Einstein equations for the six variables $(\alpha,\beta_\pm,p_\alpha,p_\pm)$ that does not come from a variational principle and construct (and solve) the quantum equations for a quantum cosmology using the nonstandard Hamiltonian techniques discussed above.

In order to avoid the most vexing problems associated with this method, we will choose a Hamiltonian that is as close as possible to the ADM Hamiltonian, that is, the Hamiltonian achieved by taking some combination of the metric variables as an internal time. In any case, we are quantizing in minisuperspace, the space of the "configuration variables" α, $\beta_\pm$, *not* spacetime. There is enough ambiguity about the interpretation of quantum gravity so that the "wave function of the Universe" found by these nonstandard techniques is as worthy as any other of being considered a serious candidate for a state function for Bianchi type-V models.

II. THE HAMILTONIAN FORMULATION OF THE BIANCHI TYPE-V EQUATIONS OF MOTION

Given the set of equations (1.15)–(1.20), we can use the formalism discussed in the Introduction to construct the Hamiltonian formulation. Equations (1.15) and (1.16) are constraints, while the other five are dynamical equations. Notice that $\dot C=3C$, so one is only allowed to assume that $C=0$ on shell. We can, however, set $p_+=0$ without loss of generality.

The full set of equations of motion are

$$\dot p_+=0 \; , \quad \dot p_-=0 \; , \quad \dot\beta_+=\frac{p_+}{p_\alpha} \; , \quad \dot\beta_-=\frac{p_-}{p_\alpha} \; ,$$

$$\dot p_\alpha=\frac{e^{4(\alpha+\beta_+)}+3(p_\alpha^2-p_+{}^2-p_-{}^2)}{2p_\alpha} \; . \tag{2.1}$$

For our purposes it will be more convenient to work in the variable set

$$\{p_+, p_-, \gamma, \beta_-^{(0)}, p_\alpha\} , \tag{2.2}$$

where $\gamma \equiv \alpha + \beta_+$ and $\beta_-^{(0)}$ is the constant of motion corresponding to the value of β_- at $\alpha=0$. Since β_- does not appear explicitly in the equations of motion, this change of variables can be made without major revisions of the form of the equations of motion. These equations now read

$$\dot{p}_+ = 0 , \quad \dot{p}_- = 0 , \quad \dot{\gamma} = 1 , \quad \dot{\beta}_-^{(0)} = 0 ,$$
$$\dot{p}_\alpha = \frac{e^{4\gamma} + 3(p_\alpha^2 - p_-{}^2)}{2p_\alpha} , \tag{2.3}$$

while

$$C = e^{4\gamma} - p_\alpha^2 + p_-{}^2 = 0 . \tag{2.4}$$

In order to develop the Hamiltonian form of these equations mentioned in the Introduction, we need a set of five constants of the motion corresponding to the five dynamical variables $\{p_+, p_-, \gamma, \beta_-^{(0)}, p_\alpha\}$. It is obvious that $p_\pm$, γ_0, and $\beta_-^{(0)}$ are constants, where γ_0 is minus the value of γ at $\alpha=0$, that is, $\gamma_0 = \alpha - \gamma$. A fifth constant is

$$C_0 = e^{-3\gamma} C . \tag{2.5}$$

As we will show below, the choice of C_0 as our Hamiltonian allows us to find a quantum minisuperspace solution that is as close as possible to similar solutions for other Bianchi types. This sidesteps the difficulty of possible nonequivalence of the quantum systems achieved by choosing other constants of the motion as our Hamiltonian. In the next section we will briefly discuss the problems associated with the choice of other Hamiltonians. For the Hamiltonian we have chosen, the construction of J^{ab} is straightforward.

Since in this new variable set p_+ has zero Poisson brackets with all other variables, it is a Casimir function (or operator in the quantum version) for the system. This means that, in constructing J^{ab}, derivatives of p_+ will appear in every term, and we need only take variable pairs of the other four constants to make up J^{ab}. That is, J^{ab} has the form

$$J_{ab} = \lambda(x^c) \epsilon^{abcde} p_{+,c} \{ p_{-,d} \beta_-{}^{(0)}{}_{,e} + \gamma_{0,d} C_{0,e} \} . \tag{2.6}$$

If we set $\lambda = e^{3\gamma}/2p_\alpha$, and use $H = C_0 = e^{-3\gamma}(e^{4\gamma} - p_\alpha^2 + p_-{}^2)$, J^{ab} becomes

$$J^{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{e^{3\gamma}}{2p_\alpha} \\ 0 & 1 & 0 & 0 & \dfrac{p_-}{p_\alpha} \\ 0 & 0 & \dfrac{e^{3\gamma}}{2p_\alpha} & -\dfrac{p_-}{p_\alpha} & 0 \end{bmatrix} , \tag{2.7}$$

where the sequence of rows and columns is p_+, p_-, γ, $\beta_-^{(0)}$, p_α. It is easy to show that $J^{ab}\partial H/\partial x^b = f^a$, for the f^a given by (2.3). A straightforward calculation shows that J^{ab} given by (2.7), satisfies the Jacobi identity.

III. CLASSICAL AND QUANTUM SOLUTIONS

A. Classical solutions

The classical solution to the equations of motion is well known, but presenting it in the context of our variable set will be useful in helping to understand the quantum problem. The equations $\dot{p}_+ = 0$ and $p_+ = 0$ are consistent, and $\dot{p}_- = 0$ implies that $p_- = p_-^{(0)} = \text{const}$. The constraint $p_+ = 0$ gives us $\beta_+ = \beta_+{}^{(0)}$, and the constraint $C=0$ can be solved for p_α to give

$$p_\alpha = \pm \left[(p_-^{(0)})^2 + e^{4\alpha} e^{4\beta_+^{(0)}} \right]^{1/2} . \tag{3.1}$$

Finally, the equation for β_-, $\dot{\beta}_- = p_-^{(0)}/p_\alpha$ can be solved by quadratures to give

$$\beta_- = \delta \pm p_-^{(0)} \int \frac{d\alpha}{\left[(p_-^{(0)})^2 + e^{4\alpha} e^{4\beta_+^{(0)}} \right]^{1/2}} , \tag{3.2}$$

where $\delta = \text{const}$. The integration can be done, and defining $\beta_-^{(0)} = \beta_-(\alpha=0)$ we find

$$\beta_- = \beta_-^{(0)} \pm \alpha \mp \tfrac{1}{2} \ln\{ [(p_-^{(0)})^2 + e^{4\alpha} e^{4\beta_+^{(0)}}]^{1/2} + p_-^{(0)} \}$$
$$\pm \tfrac{1}{2} \ln\{ [(p_-^{(0)})^2 + e^{4\beta_+^{(0)}}]^{1/2} + p_-^{(0)} \} . \tag{3.3}$$

In terms of our new configuration variable set $\gamma_0(\alpha)$, $\beta_-^{(0)}(\alpha)$ the solution is

$$\gamma_0 = \beta_+^{(0)} , \tag{3.4}$$

$$\beta_-^{(0)} = \beta_- \mp \alpha \pm \tfrac{1}{2} \ln\{ [(p_-^{(0)})^2 + e^{4\alpha} e^{4\beta_+^{(0)}}]^{1/2} + p_-^{(0)} \}$$
$$\mp \tfrac{1}{2} \ln\{ [(p_-^{(0)})^2 + e^{4\beta_+^{(0)}}]^{1/2} + p_-^{(0)} \} . \tag{3.5}$$

An exact solution of this type is not very helpful in seeing exactly what behavior is indicated by (3.3). If we were able to take $p_\alpha = -H, H$ the Hamiltonian of the system, and take the positive sign for the square root in (3.1), one can think of (3.1) as the "relativistic" version of a particle moving in a time-dependent (α-dependent) "potential" $e^{4\alpha} e^{4\beta_+}$.

This "potential" represents roughly a very steep wall at some point where $\alpha + \beta_+ = \text{const}$. Near the cosmological singularity $\alpha \to -\infty$ this "wall" is located near $\beta_+ = +\infty$. As we move away from the singularity the wall moves to the left, and as α becomes very large, the wall position approaches $\beta_+ = -\infty$. Because of the constraint, the particle represented by the "Hamiltonian" (3.1) does not interact in the usual way with the wall; that is, it does not "reflect" from the wall in any way because it is constrained to move on a $\beta_+ = \text{const}$ line at $\beta_+ = \beta_+^{(0)}$. This means that the wall begins far to the right of the $\beta_+^{(0)}$ line, and as α grows, approaches and passes the particle trajectory line and eventually ends far to the left of it. If we were to try to interpret this in the usual way, the particle would begin in a conventional way above the potential, but as time progressed would end up "underneath" the potential which would lead to imaginary solutions for $p_-^{(0)}$. However, from (3.1) we can see that there is no problem, since all the quantities under the radical are positive definite. What does happen is that near the singularity ($\alpha \to -\infty$) the potential term is very small and $p_\alpha \cong \pm p_-^{(0)}$. This means that the particle moves with velocity, $\dot{\beta}_- = \pm 1$. As α becomes large and positive the potential term dominates and $p_\alpha \cong \pm e^{2\alpha} e^{2\beta_+^{(0)}}$. From

$$\frac{d\beta_-}{d\alpha} = \frac{p_-^{(0)}}{p_\alpha} \cong \pm p_-^{(0)} e^{-2\alpha} e^{-2\beta_+^{(0)}} \tag{3.6}$$

we see that $\beta_- \sim B \mp \frac{1}{2} p_-^{(0)} e^{-2\alpha} e^{-2\beta_+^{(0)}}$, B=const, so that as $\alpha \to +\infty$, β_- approaches a constant whose value depends on $p_-^{(0)}$, $\beta_-^{(0)}$, and $\beta_+^{(0)}$. In our variable set, the solution only differs by having γ_0 in place of $\beta_+^{(0)}$, and we solve for $\beta_+^{(0)}$ as a function of β_-, α, $p_-^{(0)}$, and γ_0.

B. Quantum solutions

In order to pass to the quantum theory we must convert the variables p_+, p_-, γ, $\beta_-^{(0)}$, and p_α into the operators $\hat{p}_+$, $\hat{p}_-$, $\hat{\gamma}$, $\hat{\beta}_-^{(0)}$, and $\hat{p}_\alpha$ and the Poisson-bracket relations $\{x^a, x^b\} = J^{ab}$ into commutators. The operators $\hat{\gamma}$ and $\hat{\beta}_-^{(0)}$ will be taken to be multiplication operators on the state function $\Psi(\gamma, \beta_-; \alpha)$, where since α is our time variable it has a privileged status in the state function. In principle we must realize all of the rest of the operator set as operators (such as derivative or matrix operators) on the family of state functions Ψ that obey the relevant commutation relations.

In this case, however, our interest is in the functional form of Ψ and we can use a series of relations valid for our Hamiltonian that allow us to find Ψ without realizing all of the operators explicitly.

Since α is time, our Hamiltonian $H = C_0$ will imply the Schrödinger equation $\hat{H}\Psi = i\partial\Psi/\partial\alpha$. However, the constraint $C_0 = 0$ gives $\hat{H}\Psi = 0$, so that Ψ has no *explicit* dependence on α. It is not totally independent of α as we will see below. From the form of J^{ab} and the equation $\dot{\gamma} = 1 = \{\gamma, H\}$, we can see that in the quantum version $[\hat{\gamma}, \hat{H}] = i$, or that we can realize $\hat{H}$ as $-i\partial/\partial\gamma$. From the constraint $C_0 = 0$ we get the equation

$$\hat{C}_0 \Psi = -i \frac{\partial \Psi}{\partial \gamma} = 0 , \tag{3.7}$$

so Ψ also has no *explicit* dependence on γ. In our variable set, then,

$$\Psi = \Psi(\beta_-^{(0)}) , \quad \Psi \text{ arbitrary} . \tag{3.8}$$

Remembering that the partial derivatives mean derivatives holding *chosen* quantities constant, we can see that Ψ can depend on α and β_- through the definition of $\beta_-^{(0)}$ in terms of these variables, and we can finally write

$$\Psi = \Psi(\beta_- \mp \alpha \pm \tfrac{1}{2} \ln[(p_-^{(0)2} + e^{4\alpha} e^{4\beta_-^{(0)}})^{1/2} + p_-^{(0)}] \mp \tfrac{1}{2} \ln[(p_-^{(0)2} + e^{4\beta_+^{(0)}})^{1/2} + p_-^{(0)}]) . \tag{3.9}$$

where Ψ is again arbitrary.

If we consider the plane-wave solution $\Psi = e^{ik\beta_-^{(0)}}$, we can see that, for α large and negative,

$$\Psi \sim e^{ik(\beta_- \mp \alpha)} , \tag{3.10}$$

and, for α large and positive,

$$\Psi \sim e^{ik\beta_-} , \tag{3.11}$$

that is, the plane wave moves initially as a relativistic plane wave with velocity $\dot{\beta}_- = \pm 1$, and at large positive α it becomes independent of time so that the dynamics freezes, giving us the analogue of the classical behavior.

Up to this point we have been treating the system of equations (1.15)–(1.20) as if they were completely divorced from a variational principle, which, in general, they are. However, if the constraint $p_+ = 0$ is used to reduce β_+ to a constant, $\beta_+^{(0)}$, and the result is introduced into the equation set, Bianchi type-V models are one of the only class-B Bianchi models that allow a straightforward, variational principle. The resulting equation set comes from

$$I = \int [p_- \dot{\beta}_- + p_\alpha \dot{\alpha} - N e^{-3\alpha}(p_-^2 - p_\alpha^2 + e^{4\alpha} e^{4\beta_-^{(0)}})] d\tau . \tag{3.12}$$

We can now compare the solution to the Wheeler-DeWitt equation for this reduced action with the solution (3.9). There is a problem of factor ordering due to the factor $e^{-3\alpha}$ in the constraint obtained by varying N. For our purposes it is sufficient to divide by $e^{-3\alpha}$ classically and write the equation as

$$\frac{\partial^2 \Psi}{\partial \alpha^2} - \frac{\partial^2 \Psi}{\partial \beta_-{}^2} + e^{4\alpha} e^{4\beta_-^{(0)}} \Psi = 0 . \tag{3.13}$$

This is separable, and for $\Psi = A(\alpha) e^{ik\beta_-}$,

$$\frac{d^2 A}{d\alpha^2} + (k^2 + e^{4\alpha} e^{4\beta_-^{(0)}}) A = 0 \tag{3.14}$$

which has the solution

$$A = Z_{i(k/2)}(\tfrac{1}{2} e^{2\alpha} e^{2\beta_-^{(0)}}) , \tag{3.15}$$

where $Z_{i(k/2)}$ is a Bessel function. While this is not exactly equal to our previous solution, note that for $\alpha \to -\infty$ we find

$$\Psi \sim e^{ik(\beta_- \pm \alpha)} , \tag{3.16}$$

which coincides with our former result, while, for $\alpha \to +\infty$,

$$\Psi \sim e^{ik\beta_-} e^{-\alpha} . \tag{3.17}$$

There are a number of reasons why this large $-\alpha$ form for Ψ differs from our previous solution: (1) we have taken only the simplest possible $\Psi(\beta_-^{(0)})$, and other functions may more closely approximate this behavior; (2) the Wheeler-DeWitt equation is second order, and square-root techniques that would give a true Schrödinger equation in α drastically change the form of Ψ; and (3) the probability density associated with our solution will be $\Psi^*\Psi$, consistent with the Schrödinger-equation form our true Hamiltonian has provided, while the probability density to be associated with the Ψ given above is still debatable. In any case, our nonstandard solution more closely approximates the classical behavior [at least directly; it is possible to construct a Fourier transformation on the eigensolutions represented by (3.15) that may have different behavior] in that the plane wave has the asymptotic behavior for $\alpha \to \pm\infty$ that we might expect, while (3.17) implies that $\Psi \to 0$ for large α. In any case, our Ψ, as we mentioned in the Introduction, has as much claim to be the true "wave function of the Universe" as any other, given the present state of knowledge in quantum gravity.

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APPENDIX

For the case of the two-dimensional harmonic oscillator of Sec. II, we have two possible ways of making $P_\theta = xv - yu$, where x, y, u, v are defined in that section, a Hamiltonian for the system. The standard way would be to write the action for the system in terms of a parametrized time τ;

$$I = \int \left[p_r \dot{r} + p_\theta \dot{\theta} + p_t \dot{t} - N \left[\frac{1}{r} p_r r p_r + \frac{{p_\theta}^2}{r^2} - p_t + r^2 \right] \right] d\tau \ , \tag{A1}$$

where we have inserted 1 in the form $r(1/r)$ into the p_r term in the constraint generated by varying N in order to achieve the correct quantum equations in polar coordinates. In this action, τ is completely arbitrary. It is possible in this formalism to choose $-p_\theta$ as a Hamiltonian by choosing $\theta(\tau)$ as the time τ. In that case

$$I = \int (p_r \dot{r} + p_t \dot{t} - H) d\tau \ , \tag{A2}$$

where $H = -p_\theta$, and H is obtained by solving the δN constraint as

$$-p_\theta = H = \pm \left[\frac{1}{r} p_r r p_r - p_t + r^2 \right]^{1/2} . \tag{A3}$$

There are a number of objections to this quantity as the Hamiltonian for so simple a system as the two-dimensional oscillator (although in gravitation a similar formalism may be correct). They are (1) the unconventional square-root Hamiltonian, closer to relativistic form than the usual Newtonian Hamiltonian, (2) the variable θ ranges over $(0, 2\pi)$ with identification of 0 and 2π, which implies closed lines in time. Despite these physical objections, the mathematical procedure is straightforward, so there is no objection to taking p_θ (or $-p_\theta$) as our Hamiltonian.

In the nonconventional scheme outlined in Sec. I, the equations of motion for the two-dimensional oscillator in polar coordinates have the form

$$\dot{r} = v_r \ , \tag{A4}$$

$$\dot{\theta} = \frac{p_\theta}{r^2} \ , \tag{A5}$$

$$\dot{v}_r = \frac{{p_\theta}^2}{r^3} - r \ , \tag{A6}$$

$$\dot{p}_\theta = 0 \ . \tag{A7}$$

We may take $p_\theta = H$ if we write (for $x^1 = r$, $x^2 = \theta$, $x^3 = p_r$, $x^4 = p_\theta$)

$$\dot{x}^a = J^{ab} \frac{\partial H}{\partial x^b} \tag{A8}$$

for

$$J^{ab} = \begin{bmatrix} 0 & -\frac{1}{r} & -\frac{p_\theta}{r^2} & v_r \\ \frac{1}{r} & 0 & 0 & \frac{p_\theta}{r^2} \\ \frac{p_\theta}{r^2} & 0 & 0 & \frac{{p_\theta}^2}{r^3} - r \\ -v_r & -\frac{p_\theta}{r^2} & -\frac{{p_\theta}^2}{r^2} + r & 0 \end{bmatrix} . \tag{A9}$$

This matrix J^{ab} satisfies the Jacobi identity.

We have shown first that taking p_θ as our Hamiltonian is by no means strange in that the well-known parametrized time formalism allows it, but the nonconventional formalism allows us to make this choice with a number of advantages: (1) We no longer need worry about the square-root Hamiltonian; and, most importantly, (2) we are allowed to retain the well-behaved Newtonian time t. All this is at the cost of the unconventional commutation relations among the variables x^a.

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