

Spin-wave-theory analytic solution of a Heisenberg model with long-range interactions on a Bethe lattice

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An analytic solution for the Heisenberg Hamiltonian with arbitrarily long-range, ferromagnetic and/or anti-ferromagnetic, interactions on a Bethe lattice is obtained in the semiclassical approximation ($S \rightarrow \infty$). The density of states of the excitations is evaluated in the one-spin-wave approximation. The stability of a general helical ground state configuration, against variations in the magnitude and sign of the long-range interactions, is investigated for several illustrative examples, generating a wealth of sharp and smooth parametric transitions. [S0163-1829(97)04918-7]

I. INTRODUCTION

Exact solutions of simple models are a valuable asset in theoretical physics. On the other hand, in the description and understanding of magnetism, the Heisenberg Hamiltonian has played a central role. It was independently put forward by Heisenberg¹ and Dirac² in 1926. However, and in spite of the permanent interest it has attracted³⁻⁵ and the large amount of work devoted to this model ever since, only a few analytic results have been obtained in almost seven decades. In fact, for the ferromagnetic (FM) case (parallel spin alignment) only in one dimension has the whole excitation spectrum been obtained.⁶ In two and three dimensions analytic results are limited to the ground state and to one- and two-magnon excitations.⁶ For the antiferromagnetic (AFM) case analytic solutions have only been found in one dimension. Very recently an interesting analytic result, for the infinite spin (i.e., classical limit) of the two-dimensional Heisenberg model was reported by Curély.⁷ Certainly, abundant numerical and approximate results can be found in the literature.

In the semiclassical approximation, where noninteger quasicontinuous values for the spin projection are accepted, the excitations are known as spin waves, and were introduced by Bloch⁸ and Slater⁹ (in other words, one expands about the $S \rightarrow \infty$ limit). Over the years a large body of results¹⁰ has disclosed the somewhat surprising fact that spin wave theory is remarkably accurate, especially in two and higher dimensions,¹¹ even in the extreme quantum limit.¹²

However, it is important to point out that the well-known analytic results, as well as most of the numeric ones, are limited to nearest-neighbor interactions. Two main causes explain this state of affairs: (i) in most physical systems next-nearest-neighbor (NNN) interactions are rather small; and (ii) the difficulties that have to be faced in the treatment of longer range interactions, are quite formidable. Actually, a whole new area of interest is opened merely by incorporating competitive NNN interactions, i.e., couplings of varying sign depending on distance. The presence of competitive interactions lies at the focal point of a variety of interesting phe-

nomena peculiar to magnetic systems, like spin glasses and modulated phases. They imply the impossibility of finding a magnetic structure in which all the pairwise interactions are satisfied: the well-known phenomenon named frustration does emerge, which is also related to Anderson's "resonating valence bond" state,³ which lately has attracted so much attention.

Moreover, there are actual physical systems of interest in which long range magnetic interactions play a decisive role. Certainly, the rare earths and their metallic alloys, which are well described by a Ruderman-Kittel¹³ (RKKY) long range oscillatory interaction, fall into this category. The competition between the ferromagnetic (FM) and antiferromagnetic (AFM) (alternating sign) of the interactions leads to complex ordered configurations for these systems, as well as for the spin glasses.

On the other hand, already in 1959 Yoshimori¹⁴ suggested a helical configuration as the ground state for a system with competitive first- and second-neighbor magnetic interactions. Almost simultaneously such ordering was observed experimentally in the MnAu₂ compound by Herpin, Mériel, and Villain.¹⁵ In addition competitive long range interactions have attracted much interest in the context of the anisotropic NNN Ising model,¹⁶⁻²⁴ known under the acronym of ANNNI. While this model is simpler than our Heisenberg one, the configurations it yields and especially the resulting thermodynamics are of considerable complexity.²⁵⁻²⁷

At this point it is worth mentioning that several of the above-mentioned results were obtained for a Bethe lattice topology.^{23,26,27} Most often the solution on a Bethe lattice is much simpler than on a regular lattice, but still equivalent to Bethe-Peierls theory,²⁸ with which it is identical if the crystal lattice has the same coordination number. However, not much work on frustrated systems has been carried out in the Bethe lattice context, quite likely because of the intricacy of the problem once long-range competitive interactions are incorporated. It is worth mentioning that a Bethe lattice with random sign bonds is not frustrated, due to the absence of closed loops.^{29,30}

Consequently, in order to make progress towards analytic solutions of the Heisenberg model, with long-range interactions and going beyond dimension one, the Bethe lattice geometry constitutes an interesting option. This alternative was explored by Trias and Yndurain,³¹ who investigated the stability of the magnetic ordering of an originally FM semiclassical Heisenberg system. They were able to obtain an analytic solution to the problem, including the one-magnon excitation spectrum. As an illustration they applied their solution to a system with alternating FM and AFM interactions, among odd and even neighbors, respectively (FM with first, third, fifth, etc., and AFM with second, fourth, sixth, etc., neighboring shells).

Here we report on a fully analytic treatment of the same problem, but removing the constraint of using a FM ground state as the initial configuration. Thus, in our treatment we start from a general helical ground state configuration and investigate how it responds to various types of long-range order interactions.

This paper is organized as follows: after this introduction the model and its solution are presented in Sec. II. In Sec. III various illustrative cases are discussed: they range from simple applications to the inclusion of long-range interactions. In Sec. IV we close this paper with a brief summary.

II. MODEL AND SOLUTION

A. The Hamiltonian

Our system is made up of magnetic atoms placed at the nodes of a Bethe lattice. Each atom has a single degree of freedom, associated to its spin. The main peculiarities of the work reported in this paper are (i) the original configuration, i.e., the one adopted as a starting point in the calculation, is not ferromagnetic (FM) but a general helical state; and (ii) the range of the interaction between spins is unrestricted. That is, the spin \vec{S}_j (at site j) interacts, with a coupling strength J_{jk} , with spin \vec{S}_k at site k , which is located a distance L away. L is measured in units of lattice parameters along the Bethe lattice. The Hamiltonian which specifies the system analytically is the Heisenberg Hamiltonian given by

$$H = -\frac{1}{2} \sum_{j,k} J_{jk} \vec{S}_j \cdot \vec{S}_k. \quad (2.1)$$

The summation above is over all (j,k) pairs ($j \neq k$), while the factor $\frac{1}{2}$ compensates for the double counting.

To work out the dynamics of the system a Green's function formalism of the Zubarev³² type is used. Trias and Yndurain³¹ solved the problem formulated above for the special case where the ferromagnetic configuration is adopted as the initial ground state symmetry. They obtained the following expression for the dispersion relation $\omega(k)$, for magnons on a Bethe lattice of coordination c and no restriction on the range of the interaction $J_{0,n}$:

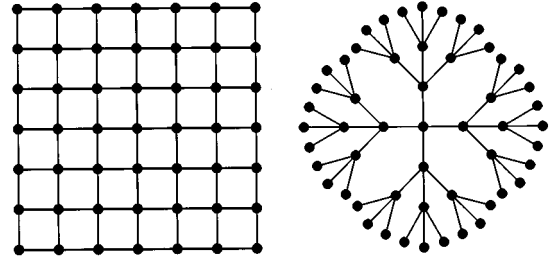


FIG. 1. On the left a square two-dimensional lattice. On the right the equivalent (same coordination $c=4$) Bethe lattice.

$$\omega = \sum_{n=1}^{\infty} J_{0,n} c(c-1)^{n-1} - \sum_{n=1}^{\infty} J_{0,n} (c-1)^{n/2} \times \left[2 \cos(n\phi) + \frac{c-2}{c-1} \frac{\sin[(n-1)\phi]}{\sin(\phi)} \right], \quad (2.2)$$

and for the local density of states

$$D_0(\omega) = \frac{2}{\pi} \frac{(c-1)c \sin^2[\phi(\omega)]}{c^2 - 4(c-1)\cos^2[\phi(\omega)]} \frac{d\phi}{d\omega}, \quad (2.3)$$

where c is the coordination of the Bethe lattice, $\phi = ka$ and a is the lattice parameter. The derivative above is obtained from Eq. (2.2).

While this beautiful analytic solution constitutes, in our opinion, a little exploited piece of work, it has the shortcoming of requiring a ferromagnetic configuration as the launching site for the calculation. This way, one is limited to either FM long-range interactions, or weak competitive antiferromagnetic ones. Only then it is feasible to obtain the excitation spectrum. However, when the competitive interactions grow large enough, the FM configuration becomes unstable, giving rise to a soft magnon which “freezes” in as the new ground state of the system. This novel magnetic order invalidates the excitation spectrum obtained above, which is reflected in the appearance of unphysical negative minima of ω . In what follows below we remove this restriction, thus allowing the use of general helical ground state configurations. In fact, for the linear chain (coordination $c=2$) we do adopt as the starting configuration a general helical structure, in which the angle between a pair of adjacent spins is a constant denoted by θ . Two special cases are worth mentioning as examples: when $\theta=0$ we recover the FM order discussed by Trias and Yndurain,³¹ but, if $\theta=\pi$ it implies the adoption of the Néel state as the starting ground state configuration.

At this stage it is convenient to rewrite the Heisenberg Hamiltonian, introducing the Bethe lattice topology, which is illustrated in Fig. 1, and a label for the successive shells. This way our Hamiltonian takes the form

$$H = -\frac{1}{2} \sum_{n=0}^{\infty} \sum_{p=1}^L J_p D(n) \vec{S}_n \cdot \left[\vec{S}_{n-p} + (c-1)^p \vec{S}_{n+p} + \sum_{m=1}^{p-1} (c-2)(c-1)^{p-m-1} \vec{S}_{n+p-2m} \right]. \quad (2.4)$$

Above n is a shell index, while p establishes the range of the interaction. The p summation incorporates three types of terms. The first one describes the interaction between a spin on shell n and one on shell $(n-p)$. We notice that there is only one nearest neighbor in the backward direction, towards the origin, located in shell $(n-1)$. The second term describes the interaction between a spin on shell n and one in shell $(n+p)$. The factor $(c-1)^p$ takes into consideration that there are $(c-1)$ equivalent nearest neighbors in the forward direction.

The third term corresponds to interactions that follow a path on the lattice which first moves a distance m towards the origin, and then $p-m$ away from it, *without retracing itself*. The factor $(c-2)(c-1)^{p-m-1}$ takes into account the topology, while the symbol Σ' denotes a summation in which the $p=1$ term is excluded. The factor $D(n)$ in Eq. (2.4) is the number of equivalent lattice sites on shell n and is given by

$$D(n) = \begin{cases} 1, & \text{if } n=0, \\ c(c-1)^n, & \text{if } n \neq 0. \end{cases} \quad (2.5)$$

It is important to point out that this Hamiltonian describes correctly the system beyond the L th shell from the origin, as if the Bethe lattice would have no privileged central site. This is essential to derive the right boundary conditions for the Green's functions, as we will do later on.

We now describe our system and fully specify it geometrically. We assume that at each lattice site a local coordinate system is specified, with the x -axis parallel to the links. Thus, two adjacent z -axis, corresponding to two successive local systems, form an angle θ with each other. Each spin points essentially along the local z axes, which is the axis of quantization. The summation in Eq. (2.4) is carried out evaluating each term in its own coordinate system; that is, for a site on shell n we write $\vec{S}_{n\pm p}$ in the local coordinate system of site n . To perform the scalar product in the local coordinate system we use, in Eq. (2.4), the rotation matrices associated with an angle $p\theta$, for a pair of spins separated by a distance p . In component notation each term of Eq. (2.4) reads

$$\vec{S}_m \cdot \vec{S}_n = S_m^z S_n^z + \frac{1}{2} (S_m^+ S_n^- + S_n^+ S_m^-). \quad (2.6)$$

The components of \vec{S}_{m+n} , at one site of shell m , are related to the local components of the vector \vec{S}_{m+n} at another site of shell $m+n$, which is at a distance n from shell m , by the transformation

$$S_{m+n}^+ = \tilde{S}_{m+n}^+ \cos^2\left(\frac{n\theta}{2}\right) + \tilde{S}_{m+n}^- \sin^2\left(\frac{n\theta}{2}\right) - i\tilde{S}_{m+n}^z \sin(n\theta), \quad (2.7a)$$

$$S_{m+n}^- = \tilde{S}_{m+n}^+ \sin^2\left(\frac{n\theta}{2}\right) + \tilde{S}_{m+n}^- \cos^2\left(\frac{n\theta}{2}\right) + i\tilde{S}_{m+n}^z \sin(n\theta), \quad (2.7b)$$

$$S_{m+n}^z = -\frac{i}{2} \tilde{S}_{m+n}^+ \sin(n\theta) + \frac{i}{2} \tilde{S}_{m+n}^- \sin(n\theta) + \tilde{S}_{m+n}^z \cos(n\theta), \quad (2.7c)$$

where $i = \sqrt{-1}$. This way the Hamiltonian of Eq. (2.4) can be written exclusively in terms of local spin operators, in which S^z is essentially equal to S .

To make further progress we relabel \vec{S} by S and write the S^+ , S^- , and S^z operators of Eqs. (2.7) in terms of new a and a^\dagger operators, which obey Bose commutation relations. This is achieved by means of the following Holstein-Primakoff transformation:

$$S_v^+ = \sqrt{2S} \left(1 - \frac{a_v^\dagger a_v}{2S}\right)^{1/2} a_v, \quad (2.8a)$$

$$S_v^- = \sqrt{2S} a_v^\dagger \left(1 - \frac{a_v^\dagger a_v}{2S}\right)^{1/2}, \quad (2.8b)$$

$$S_v^z = S - a_v^\dagger a_v, \quad (2.8c)$$

where the commutation relations obeyed by these a and a^\dagger operators are

$$[a_\nu, a_\mu] = 0, \quad [a_\nu, a_\mu^\dagger] = \delta_{\nu\mu}. \quad (2.9)$$

Within the one magnon approximation one obtains, for the operators of Eqs. (2.8)

$$S_v^+ = \sqrt{2S} a_v, \quad (2.10a)$$

$$S_v^- = \sqrt{2S} a_v^\dagger, \quad (2.10b)$$

$$S_v^z = S - a_v^\dagger a_v. \quad (2.10c)$$

Since in Eq. (2.6) we managed to write the Hamiltonian in terms of the local spin operators, which satisfy $S^z \approx S$, we can use the above approximation to obtain

$$H = -E(\theta) + H_0 + H_1, \quad (2.11)$$

with

$$E(\theta) = \frac{\mathcal{N}}{2} \sum_{p=1}^L J_p S^2 c(c-1)^{p-1} \cos(p\theta), \quad (2.12)$$

where \mathcal{N} corresponds to the total number of lattice sites, and $E(\theta)$ is a nonoperator quantity, which depends exclusively on the initial ground state angle between spins. Thus, it only shifts the energy origin and can be ignored in the calculations of the excitation spectra. The H_0 term of the Hamiltonian is given by

$$\begin{aligned}
H_0 = & -\frac{1}{2} \sum_{n=0}^{\infty} \sum_{p=1}^L J_p D(n) \left\{ iS \sqrt{\frac{S}{2}} \sin(p\theta) \left[a_{n-p}^\dagger - a_{n-p} + a_n - a_n^\dagger + (c-1)^p (a_{n+p}^\dagger - a_{n+p} + a_n - a_n^\dagger) + \sum_{m=1}^{p-1} (c-2) \right. \right. \\
& \times (c-1)^{p-m-1} (a_{n+p-2m}^\dagger - a_{n+p-2m} + a_n - a_n^\dagger) \left. \right] - S \cos(p\theta) \left[a_{n-p}^\dagger a_{n-p} + a_n^\dagger a_n + (c-1)^p (a_{n+p}^\dagger a_{n+p} + a_n^\dagger a_n) \right. \\
& + \sum_{m=1}^{p-1} (c-2)(c-1)^{p-m-1} (a_{n+p-2m}^\dagger a_{n+p-2m} + a_n^\dagger a_n) \left. \right] + S \sin^2\left(\frac{p\theta}{2}\right) \left[a_n a_{n-p} + a_{n-p}^\dagger a_n^\dagger + (c-1)^p \right. \\
& \times (a_n a_{n+p} + a_{n+p}^\dagger a_n^\dagger) + \sum_{m=1}^{p-1} (c-2)(c-1)^{p-m-1} (a_n a_{n+p-2m} + a_{n+p-2m}^\dagger a_n^\dagger) \left. \right] + S \cos^2\left(\frac{p\theta}{2}\right) \left[a_n a_{n-p}^\dagger + a_{n-p} a_n^\dagger \right. \\
& \left. \left. + (c-1)^p (a_n a_{n+p}^\dagger + a_{n+p} a_n^\dagger) + \sum_{m=1}^{p-1} (c-2)(c-1)^{p-m-1} (a_n a_{n+p-2m}^\dagger + a_{n+p-2m} a_n^\dagger) \right] \right\}, \quad (2.13)
\end{aligned}$$

where we have eliminated the tildes in order to simplify the notation. Finally, in the spin-wave approximation the term H_1 can be ignored, since it only contains products of more than two operators, which correspond to magnon-magnon interactions.

Thus, in what follows we compute the excitation spectrum of H_0 and add the contribution of $E(\theta)$ at the end of the calculation.

B. Green's functions and excitation spectrum

In order to obtain the excitation spectrum we use the Zubarev³² Green's function formalism. To treat H_0 it is convenient to define the following pair of functions:

$$G_{j,k}(t-t') = \langle\langle a_j(t), a_k^\dagger(t') \rangle\rangle, \quad (2.14a)$$

$$K_{l,m}(t-t') = \langle\langle a_l^\dagger(t), a_m^\dagger(t') \rangle\rangle, \quad (2.14b)$$

once a time Fourier transform is performed they are found to obey the equations of motion

$$\begin{aligned}
[\omega - \epsilon_j(\theta)] G_{j,k}(\omega) = & \delta_{j,k} - \sum_{p=1}^L V_{p,j} \left\{ \cos^2\left(\frac{p\theta}{2}\right) \left[G_{j-p,k}(\omega) + (c-1)^p G_{j+p,k}(\omega) + \sum_{m=1}^{p-1} (c-2)(c-1)^{p-m-1} G_{j+p-2m,k}(\omega) \right] \right. \\
& \left. + \sin^2\left(\frac{p\theta}{2}\right) \left[K_{j-p,k}(\omega) + (c-1)^p K_{j+p,k}(\omega) + \sum_{m=1}^{p-1} (c-2)(c-1)^{p-m-1} K_{j+p-2m,k}(\omega) \right] \right\}, \quad (2.15a)
\end{aligned}$$

$$\begin{aligned}
[\omega + \epsilon_l(\theta)] K_{l,n}(\omega) = & - \sum_{p=1}^L V_{p,l} \left\{ \sin^2\left(\frac{p\theta}{2}\right) \left[G_{l-p,n}(\omega) + (c-1)^p G_{l+p,n}(\omega) + \sum_{m=1}^{p-1} (c-2)(c-1)^{p-m-1} G_{l+p-2m,n}(\omega) \right] \right. \\
& \left. + \cos^2\left(\frac{p\theta}{2}\right) \left[K_{l-p,n}(\omega) + (c-1)^p K_{l+p,n}(\omega) + \sum_{m=1}^{p-1} (c-2)(c-1)^{p-m-1} K_{l+p-2m,n}(\omega) \right] \right\}, \quad (2.15b)
\end{aligned}$$

where the following definitions have been introduced:

$$V_{p,j} = S J_p D(j), \quad (2.16a)$$

$$\epsilon_j(\theta) = \sum_{p=1}^L V_{p,j} c (c-1)^{p-1} \cos(p\theta). \quad (2.16b)$$

We notice that, as a consequence of the spin wave approximation, Eqs. (2.15) do not mix in higher order Green's functions. Thus, the system of algebraic equations can be solved analytically. To do so we introduce two auxiliary functions Γ and Λ , defined as

$$\Gamma_{j,k}(\omega) = G_{j,k}(\omega) + K_{j,k}(\omega), \quad (2.17a)$$

$$\Lambda_{j,k}(\omega) = G_{j,k}(\omega) - K_{j,k}(\omega), \quad (2.17b)$$

which satisfy the equations of motion

$$\omega\Gamma_{j,k}(\omega) = \delta_{j,k} + \epsilon_j(\theta)\Lambda_{j,k}(\omega) - \sum_{p=1}^L V_{p,j}\cos(p\theta) \left[\Lambda_{j-p,k}(\omega) + (c-1)^p\Lambda_{j+p,k}(\omega) + \sum_{m=1}^{p-1} (c-2)(c-1)^{p-m-1}\Lambda_{j+p-2m,k}(\omega) \right], \quad (2.18a)$$

$$\omega\Lambda_{j,k}(\omega) = \delta_{j,k} + \epsilon_j(\theta)\Gamma_{j,k}(\omega) - \sum_{p=1}^L V_{p,j} \left[\Gamma_{j-p,k}(\omega) + (c-1)^p\Gamma_{j+p,k}(\omega) + \sum_{m=1}^{p-1} (c-2)(c-1)^{p-m-1}\Gamma_{j+p-2m,k}(\omega) \right]. \quad (2.18b)$$

For $j=M \geq L$, and $k=0$, we obtain the following set of algebraic equations:

$$\omega\Gamma_M(\omega) = \epsilon(\theta)\Lambda_M(\omega) - \sum_{p=1}^L V_p\cos(p\theta) \left[\Lambda_{M-p}(\omega) + (c-1)^p\Lambda_{M+p}(\omega) + \sum_{m=1}^{p-1} (c-2)(c-1)^{p-m-1}\Lambda_{M+p-2m}(\omega) \right], \quad (2.19a)$$

$$\omega\Lambda_M(\omega) = \epsilon(\theta)\Gamma_M(\omega) - \sum_{p=1}^L V_p \left[\Gamma_{M-p}(\omega) + (c-1)^p\Gamma_{M+p}(\omega) + \sum_{m=1}^{p-1} (c-2)(c-1)^{p-m-1}\Gamma_{M+p-2m}(\omega) \right]. \quad (2.19b)$$

The equations corresponding to $M < L$ will be used later on to impose appropriate boundary conditions. Also, the following notation was introduced:

$$\Gamma_n(\omega) \equiv \Gamma_{n,0}(\omega), \quad (2.20a)$$

$$\Lambda_n(\omega) \equiv \Lambda_{n,0}(\omega). \quad (2.20b)$$

Since we are only interested in G_{00} we have that

$$V_p \equiv V_{p,0} = SJ_p, \quad (2.21a)$$

$$\epsilon(\theta) \equiv \epsilon_0(\theta) = \sum_{p=1}^L c(c-1)^{p-1} V_p \cos(p\theta). \quad (2.21b)$$

Since Eqs. (2.15) constitute a set of coupled ordinary linear finite difference equations, and following the spirit of the work of Trias and Yndurain,³¹ we use the ansatz

$$\Gamma_M(\omega) = Ax^M(\omega), \quad (2.22a)$$

$$\Lambda_M(\omega) = By^M(\omega). \quad (2.22b)$$

This implies that a constant σ , defined by

$$\sigma = \frac{Ax^M}{By^M}, \quad (2.23)$$

exists and is independent of M , x , and y .

We now define

$$p \equiv x(c-1)^{1/2}, \quad q \equiv y(c-1)^{1/2}, \quad (2.24a)$$

$$Y_n \equiv V_n(c-1)^{n/2}, \quad Z_n(\theta) \equiv V_n \cos(n\theta)(c-1)^{n/2}, \quad (2.24b)$$

$$\tilde{\omega}_1 \equiv \omega - \epsilon(\theta)/\sigma, \quad \tilde{\omega}_2 \equiv \omega - \epsilon(\theta)\sigma. \quad (2.24c)$$

These definitions, in combination with Eqs. (2.22) and Eq. (2.23), allow us to rewrite Eqs. (2.19) in a form closely analogous to Eq. (2.2), which read

$$\tilde{\omega}_1 = -\frac{1}{\sigma} \sum_{n=1}^L Z_n(\theta) \left[q^{-n} + q^n + \frac{c-2}{c-1} \left(\frac{q^{-(n-1)} + q^{(n-1)}}{q^{-1} + q} \right) \right] \equiv \tilde{W}_1(q), \quad (2.25a)$$

$$\tilde{\omega}_2 = -\sigma \sum_{n=1}^L Y_n \left[p^{-n} + p^n + \frac{c-2}{c-1} \left(\frac{p^{-(n-1)} + p^{(n-1)}}{p^{-1} + p} \right) \right] \equiv \tilde{W}_2(p), \quad (2.25b)$$

where $Q(q) = q^L[\tilde{W}_1(q) - \tilde{\omega}_1]$ and $P(p) = p^L[\tilde{W}_2(p) - \tilde{\omega}_2]$ are polynomials of degree $2L$ in the q and p variables, respectively. We denote the roots of these polynomials by q_k and p_k [i.e., $\tilde{W}_1(q_k) = \tilde{\omega}_1$ and $\tilde{W}_2(p_k) = \tilde{\omega}_2$]; they are solutions for the Green's functions of Eqs. (2.22), in combination with the definitions provided in Eq. (2.24a).

The structure of Eqs. (2.25) implies the following symmetry: if p_k is a root $1/p_k$ is also a root, and the same holds for q_k . However, only L of these $2L$ solutions are physically meaningful and they satisfy $|x_k = p_k/(c-1)^{1/2}| < 1$ and $|y_k = q_k/(c-1)^{1/2}| < 1$, so that the Green's functions of Eq. (2.22) remain finite for $M \geq 1$. Consequently, the most general expression for the Green's functions can be built as a linear combination of all the valid solutions $\{x_k\}$ and $\{y_k\}$, respectively, as follows:

$$G_M(\omega) = \frac{1}{2} \sum_{j=1}^L (A_j x_j^M + B_j y_j^M) = \frac{1+\sigma}{2} \sum_{j=1}^L A_j x_j^M. \quad (2.26)$$

The weight factors A_j and B_j are determined from the boundary conditions, which simply specify the requirement that the origin (labeled as 0) be equivalent to any lattice site. An expression for σ is derived in the Appendix and given by

Eq. (A4). Once these constants are determined the residue theorem is used, and the following integral expression for the Green's functions are obtained:

$$\Gamma_0(\omega) = \frac{2c(c-1)}{\pi} \int_0^\pi \frac{\sin^2(\phi)}{c^2 - 4(c-1)\cos^2(\phi)} \times \frac{1}{\bar{\omega}_1 - \bar{W}_1(\phi)} d\phi, \quad (2.27a)$$

$$\Lambda_0(\omega) = \frac{2c(c-1)}{\pi} \int_0^\pi \frac{\sin^2(\phi)}{c^2 - 4(c-1)\cos^2(\phi)} \times \frac{1}{\bar{\omega}_2 - \bar{W}_2(\phi)} d\phi. \quad (2.27b)$$

These can be combined, using Eqs. (2.18), to obtain the Green's function $G_0(\omega)$ in which we are interested in. The procedure to do so is outlined in the Appendix. The result is

$$G_0(\omega) = 2c(c-1) \int_0^\pi \frac{\sin^2(\phi)}{c^2 - 4(c-1)\cos^2(\phi)} \times \frac{1}{\omega - \sqrt{W_1(\phi, \theta)W_2(\phi, \theta)}} \frac{d\phi}{\pi}, \quad (2.28)$$

where, for an arbitrarily long range interaction, $W_1(\phi, \theta)$ y $W_2(\phi, \theta)$ are given by

$$W_1(\phi, \theta) = \sum_{n=1}^{\infty} V_n \cos(n\theta) \left[c(c-1)^{n-1} - (c-1)^{n/2} \times \left(2 \cos(n\phi) + \frac{c-2}{c-1} \frac{\sin[(n-1)\phi]}{\sin(\phi)} \right) \right], \quad (2.29a)$$

$$W_2(\phi, \theta) = \sum_{n=1}^{\infty} V_n \left[\cos(n\theta) c(c-1)^{n-1} - (c-1)^{n/2} \times \left(2 \cos(n\phi) + \frac{c-2}{c-1} \frac{\sin[(n-1)\phi]}{\sin(\phi)} \right) \right]. \quad (2.29b)$$

The local density of states $D_0(\omega)$, at the central site is obtained, using Eq. (2.28), in the usual way

$$D_0(\omega) = \frac{2c(c-1)}{\pi} \frac{\sin^2[\phi(\omega)]}{c^2 - 4(c-1)\cos^2[\phi(\omega)]} \frac{d\phi}{d\omega}, \quad (2.30)$$

where ϕ is implicitly given by

$$\omega_\theta(\phi) = \sqrt{W_1(\phi, \theta)W_2(\phi, \theta)}, \quad \text{with } \phi = ka. \quad (2.31)$$

Equation (2.31) is the dispersion relation $\omega_\theta(k)$ for magnons on a Bethe lattice of coordination c , for an arbitrarily long range interaction, when the ground state is helical with an angle θ between consecutive spins.

The only remaining item in the development of the formalism is to determine the pitch of the helix, or equivalently, the angle θ just mentioned. It is obtained minimizing, respect to θ , the energy per lattice site $\mathcal{E}(\theta, k)$ of the spin wave labeled by k , for a fixed value of k (say $k=0$). \mathcal{E} is finally given by

$$\mathcal{E}(\theta, k) = -\frac{S}{2} \sum_{n=1}^{\infty} V_n c(c-1)^{n-1} \cos(n\theta) + \frac{\omega_\theta(k)}{\mathcal{N}}, \quad (2.32)$$

where the first term on the right hand side is equal to $-E(\theta)$ of Eq. (2.12) divided by the number of lattice sites \mathcal{N} , and thus independent of \mathcal{N} . However, since this number of lattice sites \mathcal{N} is very large, the second term above, which is $\propto \mathcal{N}^{-1}$, vanishes. Replacing the value of θ , which minimizes \mathcal{E} in Eq. (2.31), fully determines the dispersion relation and consequently the physics of the problem at hand.

III. APPLICATIONS

A. Simple examples

Having developed the formalism in the preceding section we now turn to several quite straightforward applications. Their purpose is to illustrate the power and versatility of the method, but for the time being we do not pretend to exhaust all possible options, restricting ourselves to rather simple examples.

The first obvious choice is to impose $\theta=0$ and recover, as expected, the solution of Trias and Yndurain,³¹ provided as Eqs. (2.2) and (2.3). But, one can easily go one step further and consider a linear chain (i.e., a Bethe lattice of coordination $c=2$) with only nearest-neighbor (NN) interactions, but of arbitrary sign and magnitude, like

$$\mathcal{E}(\theta, k=0) = -\frac{1}{2} VS \cos(\theta). \quad (3.1)$$

Minimization yields two possible solutions

$$\theta_{\min} = \begin{cases} 0, & \text{if } V > 0, \\ \pi, & \text{if } V < 0, \end{cases} \quad (3.2)$$

and consequently the following well-known dispersion relations:

$$\omega(k) = \begin{cases} 2V[1 - \cos(ka)] & \text{if } V > 0 \text{ (FM case),} \\ 2|V||\sin(ka)| & \text{if } V < 0 \text{ (AFM case).} \end{cases} \quad (3.3)$$

This result is quite remarkable, since *both* the FM and the AFM solutions are obtained within the same formalism, on the basis of a continuous variation of the angle θ which minimizes the energy, in spite of the fact that one is quadratic (FM) and the other linear (AFM) in k .

B. Next-nearest-neighbor interactions

In a pioneer work Majumdar and Ghosh^{33,34} (hereafter denoted as MG) introduced and studied the stability of the magnetic structure, of an exactly solvable finite one-dimensional chain model with nearest- and next-nearest-neighbor (NNN) interactions. They exactly diagonalized the

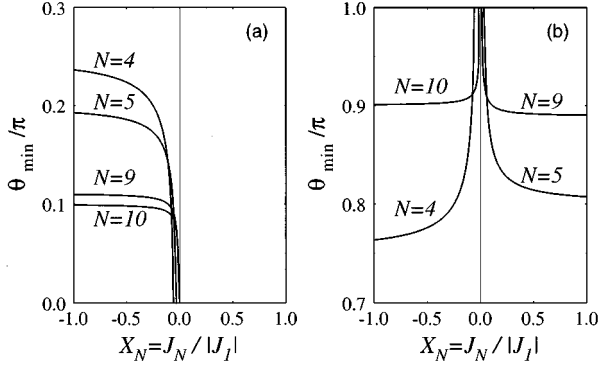


FIG. 2. θ_{\min}/π as a function of $X_N = J_N/|J_1|$, for $N = 4, 5, 9$, and 10 . The first-neighbor interaction J_1 is (a) ferromagnetic ($J_1 > 0$), and (b) antiferromagnetic ($J_1 < 0$).

Hamiltonian for chains of $3 \leq N \leq 8$ and $N = 10$ particles (Refs. 33 and 34, respectively) and suggested that, in the presence of frustration and in the $N \rightarrow \infty$ limit, the ground state belongs to the lowest total spin configuration. Moreover, they were able to find bounds for the ground state energy both for the linear chain and the two-dimensional square lattice, with NN and NNN interactions.

In spite of the fact that the results of MG are obtained for spin $\frac{1}{2}$, while our formalism is rigorously valid in the $S \rightarrow \infty$ limit, it is possible to establish a qualitative comparison of some of the relevant conclusions reached through both approaches. To do so we recall that a Bethe lattice of coordination $c = 2$ is a genuine one-dimensional infinite chain; if in addition we limit the arbitrary range of the interactions to NNN, our problem becomes analogous with MG's, with the caveat stated at the beginning of this paragraph. In this context our formalism, as well as MG's, yields a nonmagnetic ground state, except when all interactions are FM, and also an instability of the FM order against spin waves for $\alpha < -1/4$. However, the most striking feature of our results is that they lend support to the conjecture of MG that $\alpha = 0$ is a critical value for the stability of the FM state, against a transition to a total spin $S = 0$ state in the $N \rightarrow \infty$ limit. The evidence for the preceding statement is apparent from the fact that \mathcal{E} is practically stationary in the neighborhood of $\theta = 0$, for small negative values of α , which implies that a large pitch helical and the FM configuration have nearly the same energy. In other words, the FM state of a linear chain is unstable against a total spin $S = 0$ state, for any $\alpha < 0$, in the $S_{\text{tot}} \rightarrow \infty$ limit and quite likely also in the $S = \frac{1}{2}$ case.

C. Helical systems

The study of simple models of helical structures is of interest because they allow to describe polymers and molecules with this symmetry, like e.g., DNA.³⁵ Previous work on these systems³⁵⁻³⁷ is limited to Ising-type interactions. Below we present an application of the formalism developed in Sec. II, to helical structures described by a Heisenberg Hamiltonian. The systems we study are structures generated by one-dimensional chains of atoms wrapped on the surface of a cylinder of radius r . The positions of the atoms on the cylinder (in cylindrical coordinates) are specified by r , the polar angle ϕ_n , and the coordinate z_q along the z axis. The

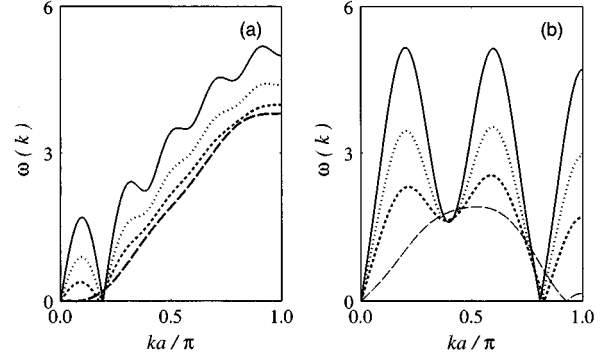


FIG. 3. Magnon dispersion relation of a five-spin per turn helix. For (a) $\text{sgn}(J_1) = +1$ and $X_5 = 0.05, 0.25, 0.5$ and 0.9 ; they correspond to long-dashed, dashed, dotted, and full lines, respectively. For (b) $\text{sgn}(J_1) = -1$ and $X_5 = -0.05, -0.25, -0.5$ and -0.9 ; they correspond to long dash, dashed, dotted and full lines, respectively.

angle $\phi_n = 2\pi n/N$, where N is the number of sites per turn. $z_q = pq + p\phi_n/2\pi$, with integer n and q , and where p is the pitch of the helix.

Each spin is allowed to interact only with its NN, i.e., with its first neighbors along the chain and with those in the next and preceding turn of the helix. They are specified by

$$V_n = \begin{cases} J_1, & \text{if } n=1, \\ J_N, & \text{if } n=N, \\ 0 & \text{otherwise.} \end{cases} \quad (3.4)$$

Obviously, we are most interested in competitive interactions and will concentrate on them later on, but as of now we leave all sign options open. Minimization of the per site energy yields

$$\mathcal{E}(\theta, k=0) = -\frac{1}{2}|J_1|S[\text{sgn}(J_1)\cos(\theta) + X_N\cos(N\theta)], \quad (3.5)$$

where the function $\text{sign}(\xi)$ denotes the sign of ξ , which corresponds to $+$ ($-$) for FM (AFM) interactions. $|J_1|$ is the absolute value of the NN interaction, which is just an energy scaling factor. X_N is the interaction strength ratio defined by

$$X_N = \frac{J_N}{|J_1|}. \quad (3.6)$$

Figure 2 illustrates the behavior of θ_{\min} as a function of X_N , for various values of N . It is noticed that critical values of X_N do exist for which both the FM and the AFM order becomes unstable, giving rise to ground state helical structures of the system. It can be rigorously shown that the critical values of X_N are given by

$$X_N^{\text{crit}} = \begin{cases} -\frac{1}{N^2} & \text{if } \text{sgn}(J_1) = +1, \\ (-1)^{N+1} \frac{1}{N^2} & \text{if } \text{sgn}(J_1) = -1, \end{cases} \quad (3.7)$$

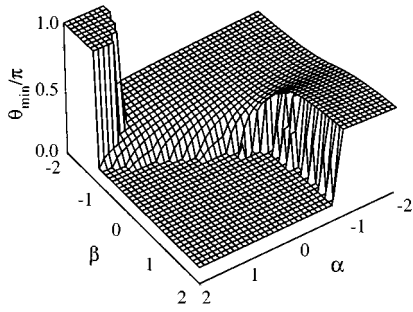


FIG. 4. θ_{\min}/π as a function of the nondimensional second- and third-nearest-neighbor interaction strengths α and β , respectively, for a linear chain with FM first neighbor interaction $J_1 > 0$.

which implies that the ground state of the system adopts helical order for $|X_N| > |X_N^{\text{crit}}|$.

In Fig. 3 we display the magnon dispersion relation for a helix of $N=5$ spins per turn, for both signs of J_1 and several values of X_N . They are stable over the whole interval (i.e., no negative frequency values are found). However, a new feature emerges: ω has a minimum value $\omega=0$ for $ka=\theta_{\min}$ which, within the present formalism, can be rigorously shown to be a general result for systems of coordination $c=2$. On the other hand, these zeroes do not imply a ground state degeneracy, since they correspond to a superposition of an excitation which is commensurate with the ground state. Moreover, the other local minima that are observed in the figure correspond to higher order harmonics, with $ka=n\theta_{\min}$.

The bounds on θ_{\min} imply a tendency towards stability of helical order commensurate with the helix periodicity. Since this feature is found in the absence of anisotropy, which would further enhance it, we conclude that a helix is the most stable configuration once FM or AFM order breaks down.

The above results are examples of many open options. In fact, while the $\text{sgn}(J_1)$ and N characterize the helical system, the parameter X_N , which can be varied externally by stretching or reducing the pitch of the helix, induces interesting transitions. In fact, for values of $|X_N| > |X_N^{\text{crit}}|$ conventional ordering (both FM and AFM) breaks down. Finally, it is worth mentioning that the analytic values we obtain, for the N dependence of X_N^{crit} , are in good agreement with those in the literature.^{38–40}

D. Longer-range interactions

Usually the study of the Heisenberg model faces rather formidable difficulties, both in analytic and numeric treatments.^{41,42} This is especially true when long-range interactions are added, which explains why so few references can be found on the subject. Actually, the notable early exception being the pioneer work of Majumdar and Ghosh^{33,34} mentioned in Sec. III B. Instead, our formalism constitutes a rather simple approach, within the limitations of the Bethe lattice approximation. At this point, and in order to keep the dimensionality of the parameter space manageable, we limit the interactions to first-, second-, and third-nearest-neighbors. In addition, this allows us to compare with the numeric results of Mesías and Vogel⁴³ obtained for the linear

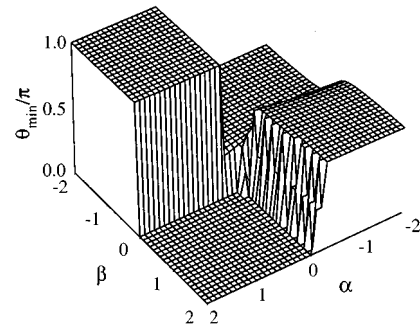


FIG. 5. θ_{\min}/π as a function of the nondimensional second- and third-nearest-neighbor interaction strengths α and β , respectively, for a coordination $c=6$ Bethe lattice with a FM first-neighbor interaction ($J_1 > 0$).

chain ($c=2$) case. Analytically, the interactions V_n of Eqs. (2.4) and (2.16) are specified by

$$V_n = \begin{cases} J_1 & \text{if } n=1, \\ J_2 & \text{if } n=2, \\ J_3 & \text{if } n=3, \\ 0 & \text{if } n \geq 4. \end{cases} \quad (3.8)$$

We readily obtain the angle between consecutive spins, which substituted in expression (2.32) for the energy yields

$$\mathcal{E}(\theta, k=0) = -\frac{1}{2} |J_1| \text{Sc}[\text{sgn}(J_1) \cos(\theta) + \alpha(c-1) \cos(2\theta) + \beta(c-1)^2 \cos(3\theta)], \quad (3.9)$$

where $|J_1|$ again is only an energy scaling factor, which we use to reduce the number of free parameters by means of

$$\alpha = \frac{J_2}{|J_1|}, \quad \beta = \frac{J_3}{|J_1|}. \quad (3.10)$$

This way, the $\text{sgn}(J_1)$, α and β , are the relevant variables, which in combination with θ_{\min} , span the parameter space that determines the physics of the problem. In Fig. 4 we plot θ_{\min} as a function of α and β , for $c=2$ and FM interaction J_1 . FM, AFM and helical order are readily noticed, with both sharp and smooth transitions between them. Similar features are observed in Fig. 5, which corresponds to coordination $c=6$.

These figures imply an interesting result: for up to third NN interactions (nonzero α and β) the ground state is neither FM nor AFM below a universal value $\alpha < -0.5$. The resulting helical order is independent of $\text{sgn}(J_1)$ and depends only weakly on β and c . Furthermore, the pitch of the helix reaches well defined limiting values for large β , given by

$$\theta_{\min} \rightarrow \begin{cases} \frac{\pi}{3} & \text{for } \beta \ll -1, \\ \frac{2\pi}{3} & \text{for } \beta \gg +1. \end{cases} \quad (3.11)$$

These values of θ_{\min} turn out to be independent of c and $\text{sgn}(J_1)$ and can be understood as follows: as $\beta \gg 1$ ($\beta \ll -1$) the dominant interaction is a third neighbor FM

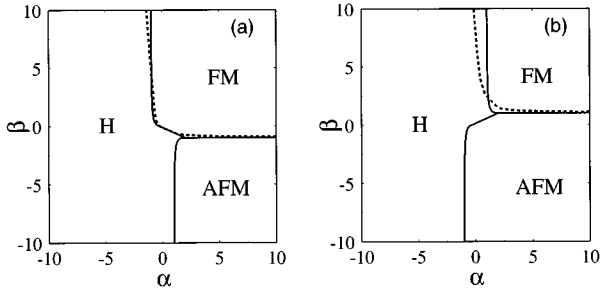


FIG. 6. Magnetic phase diagrams for a linear chain. The full lines are our results and the dashed lines are after Ref. 43. The labels FM, AFM and H correspond to ferro-, antiferro- and helical-magnetic order, respectively. In (a) $\text{sgn}(J_1) > 0$; in (b) $\text{sgn}(J_1) < 0$.

(AFM) coupling. Thus, third neighbors align parallel (anti-parallel) to each other and $3\theta_{\min} = 2\pi$ ($3\theta_{\min} = \pi$).

In order to test the power of our formalism in a rather extreme case we compare our results with the ones obtained by Mesías and Vogel,⁴³ who computed the ground state of a spin $S = \frac{1}{2}$ linear chain with AFM-NN [$\text{sgn}(J_1) = -1$], and arbitrary second and third NN, interactions. They proceeded via a direct diagonalization of the Heisenberg Hamiltonian of a 10-spin linear chain. Consequently, there are two important differences between their study and the present one: (i) the spin $S = \frac{1}{2}$ they adopt contrasts with our spin-wave approximation ($S \rightarrow \infty$); and (ii) their treatment applies to a linear chain subject to periodic boundary conditions, which are extrapolated to obtain results valid for an infinite chain, while we treat an infinite system from the outset. In spite of these caveats our results, which are displayed in Fig. 6, show a quite remarkable agreement with Ref. 43.

E. RKKY interaction

In addition to the successful use of our procedure in the above examples many other possible applications can be explored. As a particularly interesting case, which can easily be tackled in this way, we mention the RKKY interaction.¹³ Basically it is of infinite range, and few results have been reported in this context. Consequently, even the one-dimensional case ($c=2$) is of interest. We have computed for the RKKY interaction

$$V_n = -Jq^4 \frac{\sin(\pi qn) - \pi qn \cos(\pi qn)}{(\pi qn)^4}, \quad (3.12)$$

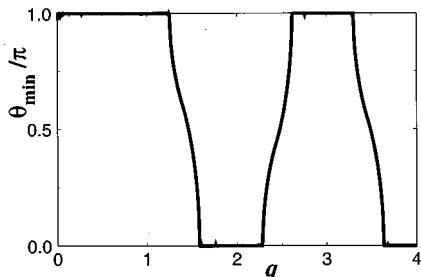


FIG. 7. Stationary values of the pitch of a one-dimensional chain of spins coupled by a RKKY interaction, as a function of the parameter $q = 2k_F a / \pi$.

the equilibrium configuration as a function of the parameter q that characterizes the commensurability of the oscillating RKKY interaction in relation to the lattice periodicity. It is defined as

$$q = \frac{2k_F a}{\pi}, \quad (3.13)$$

where a is the lattice parameter and k_F the Fermi wave number. The stationary values of pitch of the helix θ_{\min} , as a function of q , are displayed in Fig. 7. Wide phase locked regions (both FM and AFM) and sharp transitions, with rather narrow regions separating them, are observed in the illustration. They constitute a clear, and rather surprising, indication that the RKKY interaction favors simple FM and AFM magnetic ordering, even in the semiclassical $S \rightarrow \infty$ limit. A simplified version of the RKKY coupling, in which the periodicity of the interaction was assumed commensurate with a FM lattice, was already investigated by Trias and Yndurain.³¹

IV. SUMMARY

In this paper we have provided an analytic treatment of a Heisenberg Hamiltonian, with arbitrarily long-range interactions on a Bethe lattice, assuming an initial general helical ground state configuration of the system. This could be achieved by restricting ourselves to the semiclassical $S \rightarrow \infty$ limit. The Green's function solution derived above is also valid for other elementary excitations, like electrons and phonons.

In Sec. III several examples were developed, mostly involving frustrated magnetic configurations. First, well known results, like those of Trias and Yndurain,³¹ were recovered in Sec. III A; as an extra bonus, the dispersion relations for both FM and AFM 1D chains with NN interactions were derived within a single formalism. After doing so, we contrasted our results with those obtained, in a pioneer work, by Majumdar and Ghosh,^{33,34} who included NN and NNN interactions. Good qualitative and quantitative agreement was found, in spite of the fact that they treated the extreme quantum limit ($S = 1/2$). Next, in Sec. III C we studied helical systems, with interactions between nearest and far apart neighbors, as well as systems with longer-range interactions. Again we found our results to be in accord with previously published work⁴³ carried out in the extreme quantum limit and including up to third NN interactions. Finally, in Sec. III E, we applied our formalism to a linear chain with magnetic moments coupled by the very long-range RKKY interaction, to obtain interesting new insights.

While we solved the problem for any c value the $c=2$ coordination (linear chain) Bethe lattice was used as the prevalent illustration in the preceding examples, both because of its simplicity and because it could be compared with results available in the existing literature. In all of the cases explored above we found that the formalism is convenient to apply and that it leads to robust and reliable results.

ACKNOWLEDGMENTS

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APPENDIX: DETERMINATION OF $G_0(\omega)$

Our aim is to eliminate the σ dependence of $G_0(\omega)$. To do so we combine the solutions found for the auxiliary functions $\Gamma_0(\omega)$ and $\Lambda_0(\omega)$, given by Eqs. (2.27), with Eqs. (2.17), to obtain

$$G_0(\omega) = c(c-1) \int_0^\pi \frac{\sin^2(\phi)}{c^2 - 4(c-1)\cos^2(\phi)} \times \left(\frac{1}{\tilde{\omega}_1 - \tilde{W}_1} + \frac{1}{\tilde{\omega}_2 - \tilde{W}_2} \right) \frac{d\phi}{\pi}. \tag{A1}$$

However, the σ dependence of $G_0(\omega)$ by way of $\tilde{\omega}_{1,2}$ and $\tilde{W}_{1,2}$ also has to be removed. To achieve this we use the definitions (2.24a) to rewrite Eqs. (2.25) as follows:

$$\tilde{\omega}_1 = \omega - \frac{\epsilon(\theta)}{\sigma} = \tilde{W}_1(\phi, \theta) \equiv -\frac{1}{\sigma} \Omega_1(\phi, \theta), \tag{A2a}$$

$$\tilde{\omega}_2 = \omega - \epsilon(\theta)\sigma = \tilde{W}_2(\phi, \theta) \equiv -\sigma \Omega_2(\phi, \theta), \tag{A2b}$$

where

$$\Omega_1(\phi, \theta) = \sum_{n=1}^L V_n(c-1)^{n/2} \cos(n\theta) \times \left[2 \cos(n\phi) + \frac{c-2}{c-1} \frac{\sin[(n-1)\phi]}{\sin(\phi)} \right], \tag{A3a}$$

$$\Omega_2(\phi, \theta) = \sum_{n=1}^L V_n(c-1)^{n/2} \times \left[2 \cos(n\phi) + \frac{c-2}{c-1} \frac{\sin[(n-1)\phi]}{\sin(\phi)} \right]. \tag{A3b}$$

This way σ is finally determined from Eqs. (2.6) and reads

$$\sigma = \sqrt{\frac{\epsilon(\theta) - \Omega_1(\phi, \theta)}{\epsilon(\theta) - \Omega_2(\phi, \theta)}}. \tag{A4}$$

Now, all what we need to evaluate $G_0(\omega)$ is the bracket in the integral of Eq. (A1); thus, we have to calculate

$$\Xi \equiv \frac{1}{\tilde{\omega}_1 - \tilde{W}_1} + \frac{1}{\tilde{\omega}_2 - \tilde{W}_2} = \frac{\tilde{\omega}_1 - \tilde{\omega}_2 - \tilde{W}_1 - \tilde{W}_2}{\tilde{\omega}_1 \tilde{\omega}_2 - \tilde{\omega}_1 \tilde{W}_2 - \tilde{\omega}_2 \tilde{W}_1 + \tilde{W}_1 \tilde{W}_2}. \tag{A5}$$

Using relations (A2) we obtain

$$\Xi = \frac{2\omega - \{[(\epsilon(\theta) - \Omega_1)\sigma^{-1} + [\epsilon(\theta) - \Omega_2]\sigma]\}}{[\omega - \epsilon(\theta)\sigma^{-1}][\omega - \epsilon(\theta)\sigma] + [\omega\sigma - \epsilon(\theta)]\Omega_2 + [\omega\sigma^{-1} - \epsilon(\theta)]\Omega_1 + \Omega_1\Omega_2}, \tag{A6}$$

which can be rearranged as follows:

$$\Xi = \frac{2\omega - \{(\epsilon(\theta) - \Omega_1)\sigma^{-1} + [\epsilon(\theta) - \Omega_2]\sigma\}}{\omega^2 - \omega\{(\epsilon(\theta) - \Omega_1)\sigma^{-1} + (\epsilon(\theta) - \Omega_2)\sigma\} + [\epsilon(\theta) - \Omega_1][\epsilon(\theta) - \Omega_2]}. \tag{A7}$$

Using Eqs. (A3) we obtain

$$[(\epsilon(\theta) - \Omega_1)\sigma^{-1} + (\epsilon(\theta) - \Omega_2)\sigma] = 2\sqrt{(\epsilon(\theta) - \Omega_1)(\epsilon(\theta) - \Omega_2)}, \tag{A8}$$

which replaced in Eqs. (A7) yields

$$\Xi = \frac{2\omega - 2\sqrt{(\epsilon(\theta) - \Omega_1)(\epsilon(\theta) - \Omega_2)}}{[\omega - \sqrt{(\epsilon(\theta) - \Omega_1)(\epsilon(\theta) - \Omega_2)}]^2} = \frac{2}{\omega - \sqrt{W_1(\phi, \theta)W_2(\phi, \theta)}}, \tag{A9}$$

where

$$W_i(\phi, \theta) = \epsilon(\theta) - \Omega_i(\phi, \theta). \tag{A10}$$

Substitution of Ξ in Eq. (A1) yields the Green's function we seek:

$$G_0(\omega) = 2c(c-1) \int_0^\pi \frac{\sin^2(\phi)}{c^2 - 4(c-1)\cos^2(\phi)} \frac{1}{\omega - \sqrt{W_1(\phi, \theta)W_2(\phi, \theta)}} \frac{d\phi}{\pi}, \quad (\text{A11})$$

with $W_1(\phi, \theta)$ and $W_2(\phi, \theta)$ given by

$$W_1(\phi, \theta) = \sum_{n=1}^L V_n \cos(n\theta) \left\{ c(c-1)^{n-1} - (c-1)^{n/2} \left[2 \cos(n\phi) + \frac{c-2}{c-1} \frac{\sin[(n-1)\phi]}{\sin(\phi)} \right] \right\}, \quad (\text{A12a})$$

$$W_2(\phi, \theta) = \sum_{n=1}^L V_n \left\{ \cos(n\theta) c(c-1)^{n-1} - (c-1)^{n/2} \left[2 \cos(n\phi) + \frac{c-2}{c-1} \frac{\sin[(n-1)\phi]}{\sin(\phi)} \right] \right\}. \quad (\text{A12b})$$

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