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Infimal Convolution and Optimal Time Control Problem I: Fréchet and Proximal Subdifferentials

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Abstract We consider a general minimal time problem with a convex constant dynamics and a lower semicontinuous extended real-valued target function defined on a Banach space. If the target function is the indicator function of a closed set, this problem is a minimal time problem for a target set, studied previously in particular by Colombo, Goncharov and Mordukhovich. We investigate several properties of the Fréchet and proximal subdifferentials for the infimum time function. Also explicit expressions of the above mentioned subdifferentials as well as various directional derivatives are obtained. We provide some examples to show the essentiality of assumptions of our theorems.

Keywords Fréchet subdifferential · Proximal subdifferential · Dini directional derivative · Generalized directional derivative · Minimal time function · Minimal time projection · Infimal convolution

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1 Introduction

Given a non-empty closed convex set G of a Banach space $(X, \|\cdot\|)$ with $0 \in G$ and an extended real-valued function f defined on X, we investigate the Moreau-type infimal convolution

$$T_f(x) := \inf_{y \in X} \left(f(y) + \rho_G(x-y)\right), \tag{1.1}$$

where ρ_G denotes the Minkowski functional of G. Our motivation for such a study is the following. Consider the optimal control problem

$$\text{Minimize } \; t + f\left(\zeta(t;x)\right) \tag{1.2}$$

over all $t \geq 0$ and all solutions $\zeta(\cdot) = \zeta(\cdot; x)$ of the differential inclusion

$$\frac{d\zeta}{dt}(t) \in -G, \quad t \geq 0$$

with the initial condition

$$\zeta(0) = x.$$

As we will see in the next section, the function T_f is exactly the infimum value of this optimal control problem. In the case when f is the indicator function of a closed subset C of the Banach space X, the problem is reduced to a minimal time control problem for a target set C, studied previously in particular by Colombo, Goncharov and Mordukhovich [8]. We also refer to [1, 9, 10, 13, 14, 16, 20, 26, 27, 35]; see, e.g., [12, 22] for studies related to generalized best approximation problems. Some results concerning subdifferentials of the infimum convolution T_f, without refering to the (motivation) optimal control problem (1.2) can also be found in [11, 30, 31, 36]. In addition to subdifferential properties, Nam and Cuong [31] studied also the weak lower semicontinuity of T_f under some conditions on the function f.

Considering a closed set C in a Hilbert space H and a sufficiently regular function $\theta : C \to \mathbb{R}$, Goncharov and Pereira [15, 32] investigated regularity properties of the function $\hat{u}$ defined by

$$\hat{u}(x) := \inf_{y \in C} \left(\theta(y) + \rho_G(x-y)\right) \tag{1.3}$$

i.e., $\hat{u} = T_{\theta+\psi_C}$, where ψ_C is the indicator function of C (see Section 2 for the definition). A main motivation of the authors for such an investigation is [32, Theorem 1] which states that, under the slope condition $\theta(x) - \theta(y) \leq \rho_G(x-y)$, the function $\hat{u}$ is the (unique) viscosity solution of

$$\rho_{G^o}\left(\nabla u(x)\right) - 1 = 0, \;\; x \in H \setminus C$$

such that $\hat{u}(x) = \theta(x), \;\; x \in C$, where G^o denotes the polar of G (see the next section for the definition); this explains the notation $\hat{u}$ in [32].

In addition to our above interpretation of the function T_f, the set of minimizers $\Pi_f(x)$ of (1.1) will also be interpreted with respect to the optimal time control problem, and it will appear as the minimal time projection set. The study in detail of this mapping $\Pi_f(\cdot)$ will be the subject of the second companion paper [19].

The paper is organized as follows. In Section 2 we discuss the problem (1.2) and we justify that T_f is its infimum value. We also recall various variational concepts, as Fréchet and proximal subdifferentials, as well as some of their main properties needed in the paper.

Section 3 is devoted to explicit estimates and exact expressions, in terms of the data f and G, of the above subdifferentials of the infimum time function T_f when the latter is achieved. Various directional derivatives are also studied. Further properties of the Fréchet subdifferential of T_f constitute the subject of Section 4, mainly when $\Pi_f(x) = \emptyset$. In particular, we provide explicit expressions of the Fréchet subdifferential of T_f in terms of Fréchet normal cone to the sublevel of T_f and in terms of the support function of the set G. Similar results are established in Section 5 with the proximal subdifferential instead of the Fréchet subdifferential. The limiting subdifferential of the infimum time function is studied in the companion paper [18].

2 Preliminaries

Throughout $(X, \|\cdot\|)$ is a *real normed vector space*, X^* is its topological dual, and $\mathbb{B}_X$ is the closed unit ball of X centered at the origin. The dual norm (of $\|\cdot\|$) over X^* will be denoted by $\|\cdot\|_*$. As usual $B(x, \delta)$ is the open ball in X of center $x \in X$ and radius δ and $\mathrm{cl}\, S$ (resp. $\mathrm{int}\, S$, $\mathrm{bd}\, S$) denotes the closure (resp. the interior, the boundary) of a set S in X. It will be convenient, for $\bar{x} \in \mathrm{cl}\, S$, to write $x \xrightarrow{S} \bar{x}$ to mean that $x \to \bar{x}$ with $x \in S$. For a function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ we will denote by $\operatorname*{Argmin}_S \varphi$ or $\operatorname*{Argmin}_{x \in S} \varphi(x)$ the set of minimizers of φ over S; when $S = X$ we will often just write $\operatorname{Argmin} \varphi$.

Given a nonempty closed convex subset K of X, we denote by ρ_K its Minkowski gauge function (Minkowski functional), that is,

$$\rho_K(x) := \inf\{r \geq 0 : \ x \in rK\} \quad \text{for all } x \in X,$$

with the usual convention $\inf \emptyset = +\infty$. It is known and not difficult to see that the function ρ_K is sublinear (that is, positively homogeneous and convex) from X into $\mathbb{R}_+ \cup \{+\infty\}$, where $\mathbb{R}_+$ stands for the set of non-negative real numbers. Further, the equality $\rho_K(0) = 0$ is obvious.

Consider now throughout the paper a *closed bounded convex* subset G of X with $0 \in \mathrm{int}\, G$. For each nonempty closed set C of X and each initial value $x \in X$, consider the optimal control problem

$$\begin{aligned} \text{Minimize} \quad & t \geq 0 \text{ such that } \zeta(t) \in C \text{ and} \\ \text{subject to} \quad & \dot{\zeta}(\tau) \in -G \text{ a.e.} \tau \in [0, t] \text{ and } \zeta(0) = x. \end{aligned}$$

The infimum value function $T_C(\cdot)$, naturally asssociated with this optimal control problem, is then defined by $T_C(x)$ as the infimum of all $t \geq 0$ for which there exists a solution $\zeta(\cdot) = \zeta(\cdot; x)$ of the differential inclusion $\dot{\zeta}(\tau) \in -G$ starting from $x \in X$ at time $\tau = 0$ and such that $\zeta(t; x) \in C$. The value function $T_C(\cdot)$ is generally called the *minimal time function* to reach the target set C. It is easily seen (see the arguments below concerning $T_f(x)$) that the minimal time function above can be rewritten as

$$T_C(x) = \inf_{y \in C} \rho_G(x - y). \tag{2.1}$$

We will put

$$\Pi_C(x) := \operatorname*{Argmin}_C \rho_G(x - \cdot) = \{y \in C : \ \rho_G(x - y) = T_C(x)\} \tag{2.2}$$

and we will say that Π_C is the *minimal time projection set-valued mapping*.

It is worth mentioning some additional properties of the gauge function ρ_G, where G is as above. First, because of the assumption $0 \in \operatorname{int} G$, the function ρ_G is finite on X and Lipschitz continuous on X. Considering the polar set G^o of G defined by

$$G^o := \{x^* \in X^* : \langle x^*, x\rangle \le 1 \;\; \forall x \in G\},$$

it is known (and not difficult to see) that

$$\rho_G(x) = \sigma_{G^o}(x) := \sup_{u^* \in G^o} \langle u^*, x\rangle \quad \text{for all } x \in X. \tag{2.3}$$

Further, using the positive homogeneity of the convex function ρ_G, it is not difficult to see that

$$x^* \in \partial\rho_G(x) = \partial\sigma_{G^o}(x) \Leftrightarrow x^* \in \partial\rho_G(0) \text{ and } \langle x^*, x\rangle = \rho_G(x), \tag{2.4}$$

where $\partial\rho_G(x)$ denotes the usual Fenchel subdifferential of the convex function ρ_G at the point x, that is,

$$\partial\rho_G(x) = \{x^* \in X^* : \langle x^*, x' - x\rangle \le \rho_G(x') - \rho_G(x) \; \forall x' \in X\}.$$

In particular, we then have

$$\partial\rho_G(0) = \partial\sigma_{G^o}(0) = G^o. \tag{2.5}$$

In the case where $\rho_G(x) \ne 0$ the equivalence in (2.4) yields

$$\rho_G(x) \ne 0 \text{ and} x^* \in \partial\rho_G(x) = \partial\sigma_{G^o}(x) \Longrightarrow \langle x^*, \frac{1}{\rho_G(x)} x\rangle = 1 \text{ hence } x^* \in \operatorname{bd} G^o. \tag{2.6}$$

Observe also that for the support function $\sigma_G(\cdot) = \sup_{u \in G}\langle \cdot, u\rangle$ we have with $\rho_G(x) \ne 0$

$$x^* \in G^o \text{ and } \langle x^*, \frac{1}{\rho_G(x)} x\rangle = 1 \Longrightarrow \frac{1}{\rho_G(x)} x \in \partial\sigma_G(x^*). \tag{2.7}$$

We point out that the boundedness of G and the inclusion $0 \in \operatorname{int} G$ ensure the existence of some $\alpha > 0$ and $\beta > 0$ such that $\frac{1}{\beta}\mathbb{B}_X \subset G \subset \frac{1}{\alpha}\mathbb{B}_X$ and hence

$$\alpha\|x\| \le \rho_G(x) \le \beta\|x\| \quad \text{for all } x \in X. \tag{2.8}$$

If G is, in addition, symmetric, then obviously ρ_G is a norm $\|\cdot\|_G$ on X and $T_C(x) = \operatorname{dist}_{\|\cdot\|_G}(x, C)$ is the distance from the point x to the set C associated with the norm $\|\cdot\|_G$; further, in such a case, (2.8) tells us that the norm $\|\cdot\|_G$ is equivalent to the initial norm $\|\cdot\|$. So, if $G = \mathbb{B}_X$ (the closed unit ball with respect to the initial norm $\|\cdot\|$), the function T_C is reduced to the distance function $\operatorname{dist}(\cdot, C)$ with respect to the norm $\|\cdot\|$, and $\Pi_C(x)$ coincides with the projection set $\operatorname{Proj}_C(x)$, that is,

$$\Pi_C(x) = \operatorname{Proj}_C(x) := \{y \in C : \|x - y\| = \operatorname{dist}(x, C)\}.$$

We will also write sometimes d_C in place of $\operatorname{dist}(\cdot, C)$. We observe that in (2.8) we may obviously take

$$\alpha = \left(\sup_{u \in G}\|u\|\right)^{-1} \quad \text{and} \quad \beta = \sup_{u^* \in G^o}\|u^*\|_*.$$

Now let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function which is *proper* in the sense that it is not identically $+\infty$. For any locally (Bochner) integrable mapping

$\dot{\zeta} : [0, +\infty[\to X$ we will denote by $\zeta(\cdot; x)$ its primitive which equals x at 0, that is, $\zeta(t; x) = x + \int_0^t \dot{\zeta}(s)\,ds$. Consider the optimal control problem

$$(\mathcal{P}) \begin{cases} \text{Minimize} \quad t + f(\zeta(t; x)) \quad \text{over } t \geq 0 \\ \text{and over all solutions } \zeta(\cdot; x) \text{ of the differential inclusion} \\ \dot{\zeta}(\tau) \in -G \text{ a.e.} \tau \in [0, t] \text{ with initial condition } \zeta(0) = x, \end{cases}$$

and its infimum value function T_f with $T_f(x)$ defined by the infimum of the latter problem. Observe that for any real $t \geq 0$ and for any solution $\zeta(\cdot)$ of the differential inclusion $\dot{\zeta}(\tau) \in -G$ with initial condition $\zeta(0) = x$, the closedness and convexity assumption of the set $-G$ entails that $\frac{1}{t}\int_0^t \dot{\zeta}(\tau)\,d\tau \in -G$. The equality $\zeta(t; x) = x + t \cdot \frac{1}{t}\int_0^t \dot{\zeta}(\tau)\,d\tau$ assures us that $\zeta(t; x) \in x + t(-G)$. Further, for any $z \in -G$ we see that for the constant mapping $\dot{\zeta}_0(\cdot) = z$ we have $x + tz = \zeta_0(t; x)$. Therefore, starting from $x \in X$ the reachable set $R(t; x)$ at time t of the dynamics involved in $(\mathcal{P})$ is the set $R(t; x) = x + t(-G)$. So, the problem $(\mathcal{P})$ may be reformulated as

$$(\mathcal{P}) \begin{cases} \text{Minimize} \quad t + f(x - tu) \\ \text{over } t \geq 0 \text{ and } u \in G. \end{cases} \tag{2.9}$$

Thus, we can write

$$\begin{aligned} T_f(x) &= \inf_{t \geq 0}[t + \inf_{y \in x + t(-G)} f(y)] \\ &= \inf_{y \in X} \inf_{t \geq 0:\, x - y \in tG} (t + f(y)) \\ &= \inf_{y \in X}[f(y) + \rho_G(x - y)], \end{aligned} \tag{2.10}$$

that is, the function T_f is the infimum convolution (see Moreau [28, 29]) of the functions f and ρ_G. (Remind that the infimum convolution $f \square g$ of f with another function g is defined by $f \square g\,(x) = \inf_{y \in X}[f(y) + g(x - y)]$.) Further, for all $x \in X$ the properness of f ensures that $T_f(x) < +\infty$ and the equality in (2.10) entails $T_f(x) \leq f(x)$.

Denote by ψ_C the indicator function of the closed set C, i.e., $\psi_C(x) = 0$ if $x \in C$ and $\psi_C(x) = +\infty$ otherwise. Obviously, for $f = \psi_C$, the function T_f coincides with T_C as defined above. Throughout, we will say that T_f is the *(generalized) infimum time function* associated with the function f and the dynamics of $(\mathcal{P})$.

Similarly to (2.2) the Argmin of the function $f + \rho_G(x - \cdot)$ or (generalized) minimal time projection in (2.10) will be denoted by $\Pi_f(x)$, that is,

$$\Pi_f(x) := \text{Argmin}\,(f + \rho_G(x - \cdot)) := \{y \in X : f(y) + \rho_G(x - y) = T_f(x)\}. \tag{2.11}$$

We call Π_f the *(generalized) minimal time projection set-valued mapping* associated with the function f and the dynamics of $(\mathcal{P})$.

Throughout the paper, unless otherwise stated $f : X \to \mathbb{R} \cup \{+\infty\}$ is as above an extended real-valued *proper* function and *we will assume that for some real constant* γ

$$f(y) \geq -\rho_G(-y) + \gamma \quad \text{for all } y \in X. \tag{2.12}$$

Then for all $y \in X$ according to the sublinearity of ρ_G

$$f(y) + \rho_G(x - y) \geq \rho_G(x - y) - \rho_G(-y) + \gamma \geq -\rho_G(-x) + \gamma,$$

and this entails that

$$T_f(x) \geq -\rho_G(-x) + \gamma, \text{ hence in particular, } T_f(x) \in \mathbb{R} \text{ for all } x \in X. \tag{2.13}$$

We also observe that

$$\Pi_f(x) \subset \operatorname{dom} f := \{u \in X : f(u) < +\infty\} \quad \text{for all } x \in X.$$

The paper will be focused on several properties of subdifferentials of the infimum time function T_f related to the function f. As pointed out above, for $G = \mathbb{B}_X$, the function T_C related to the set C corresponds to $\operatorname{dist}(\cdot, C)$ and hence in such a case the Fréchet subdifferential of $\operatorname{dist}(\cdot, C)$ at all points of X has been studied in Kruger [21] (see also Borwein and Giles [3] and Bounkhel and Thibault [4]); we refer to [4, 7] for the proximal subdifferential of $\operatorname{dist}(\cdot, C)$ and to [2, 23–25, 34] for its limiting subdifferential. The Fréchet and proximal subdifferentials of the minimal time function T_C in (2.1) have been considered later in finite dimensional space X in [35] and then in the Hilbert setting in [9, 10]. Several results of [9, 10, 35] have been extended in [16]. The limiting subdifferential of T_C is treated in [26]. Colombo, Goncharov and Mordukhovich [8] provided some further strong results concerning subdifferentials of T_C.

Remind that for $x \in \operatorname{dom} f$ and $\varepsilon \geq 0$ the *Fréchet ε-subdifferential* $\partial^{F,\varepsilon} f(x)$ of f at x is the set of $x^* \in X^*$ such that for any $\eta > 0$ there exists a neighborhood U of x such that for all $x' \in U$

$$\langle x^*, x' - x\rangle \leq f(x') - f(x) + (\varepsilon + \eta)\|x' - x\|.$$

When $\varepsilon = 0$ we will write, as usual, $\partial^F f(x)$ instead of $\partial^{F,\varepsilon} f(x)$. Sometimes, a more accurate term is needed in place of $\varepsilon\|x' - x\|$, and this leads to the proximal subdifferential. A continuous linear functional $x^* \in X^*$ is a *proximal subgradient* of f at x provided there exists some constant $r \geq 0$ and some neighborhood U of x such that

$$\langle x^*, x' - x\rangle \leq f(x') - f(x) + r\|x' - x\|^2 \quad \text{for all } x' \in U.$$

The set of all proximal sudgradients of f at x is the *proximal subdifferential* $\partial^P f(x)$ of f at x.

Besides the Fréchet and proximal subdifferentials we will use the (lower) *Dini directional derivative* $d^- f(x; v)$ of f at x in the direction v defined by

$$d^- f(x; v) := \liminf_{w \to v; t \downarrow 0} t^{-1}[f(x + tw) - f(x)].$$

Through the Dini directional derivative one defines the *Dini subdifferential* of f at x by

$$\partial^- f(x) := \{x^* \in X^* : \langle x^*, v\rangle \leq d^- f(x; v) \ \forall v \in X\}.$$

When $x \notin \operatorname{dom} f$ we adopt the convention that all the above subdifferentials at x are empty. When f is convex, then all the above subdifferentials coincide with the Fenchel subdifferential $\partial f(x)$ of f in convex analysis; if, in addition, f is lower semicontinuous (lsc), see, e.g., [28], then

$$x^* \in \partial f(x) \Longleftrightarrow x \in \partial f^*(x^*), \tag{2.14}$$

where $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ denotes the *Legendre-Fenchel conjugate* of f, that is,

$$f^*(x^*) = \sup_{y \in X}[\langle x^*, y\rangle - f(y)].$$

For the closed set C, its *ε-Fréchet* (for $\varepsilon \geq 0$) *normal set* and its *proximal normal cone* at x are defined through its indicator function ψ_C by

$$N^{F,\varepsilon}(C; x) = \partial^{F,\varepsilon} \psi_C(x) \quad \text{and} \quad N^P(C; x) = \partial^P \psi_C(x).$$

The first equality can be translated, for $x \in C$, by writing that $x^* \in N^{F,\varepsilon}(C;x)$ if and only if for each $\eta > 0$ there exists some neighborhood U of x such that

$$\langle x^*, x' - x\rangle \le (\varepsilon + \eta)\|x' - x\| \quad \text{for all } x' \in C \cap U.$$

Analogously, $x^* \in N^P(C;x)$ means that there exist $r > 0$ and a neighborhood U of x such that

$$\langle x^*, x' - x\rangle \le r\|x' - x\|^2 \quad \text{for all } x' \in C \cap U.$$

3 Properties when the Infimum Time Function is Achieved

As we said above we are interested in several subdifferential properties of the infimum time function associated with the function f. We start with the following theorem concerning its Fréchet subdifferential and Fréchet ε-subdifferential when the infimum at the considered point is achieved. We also state the similar results for the proximal and Dini subdifferentials, since the arguments are quite similar.

Theorem 3.1 *Assume that $\bar{y} \in \Pi_f(\bar{x})$. The following hold.*

(a) *For any $\varepsilon \ge 0$*

$$\partial^{F,\varepsilon} T_f(\bar{x}) \subset \partial^{F,\varepsilon} f(\bar{y}) \cap \partial^{F,\varepsilon} \rho_G(\bar{x} - \bar{y}) = \partial^{F,\varepsilon} f(\bar{y}) \cap (\partial\rho_G(\bar{x} - \bar{y}) + \varepsilon\mathbb{B}_{X^*})$$

and

$$\partial^P T_f(\bar{x}) \subset \partial^P f(\bar{y}) \cap \partial\rho_G(\bar{x} - \bar{y}).$$

(b) *For all $v \in X$*

$$d^- T_f(\bar{x}; v) \le \big(d^- f(\bar{y}; \cdot)\Box d^- \rho_G(\bar{x} - \bar{y}; \cdot)\big)(v) \le \big(d^- f(\bar{y}; \cdot)\Box \rho_G\big)(v);$$

further

$$\partial^- T_f(\bar{x}) \subset \partial^- f(\bar{y}) \cap \partial\rho_G(\bar{x} - \bar{y}).$$

(c) *If, in addition, f is convex, then T_f is also convex and the two latter inclusions are equalities, that is,*

$$\partial T_f(\bar{x}) = \partial f(\bar{y}) \cap \partial\rho_G(\bar{x} - \bar{y}).$$

Proof (a) Consider $\varepsilon \ge 0$ and $x^* \in \partial^{F,\varepsilon} T_f(\bar{x})$, and fix any $\eta > 0$. There exists some neighborhood U of zero such that for all $u \in U$

$$\langle x^*, u\rangle \le T_f(\bar{x}+u) - T_f(\bar{x}) + (\varepsilon+\eta)\|u\| = T_f(\bar{x}+u) - f(\bar{y}) - \rho_G(\bar{x}-\bar{y}) + (\varepsilon+\eta)\|u\|$$

and hence on one hand for $y = \bar{y}$ in the definition of $T_f(\bar{x} + u)$ in (2.10)

$$\langle x^*, u\rangle \le f(\bar{y}) + \rho_G(\bar{x} + u - \bar{y}) - f(\bar{y}) - \rho_G(\bar{x} - \bar{y}) + (\varepsilon + \eta)\|u\|,$$

that is,

$$\langle x^*, u\rangle \le \rho_G(\bar{x} - \bar{y} + u) - \rho_G(\bar{x} - \bar{y}) + (\varepsilon + \eta)\|u\|;$$

and on the other hand,

$$\langle x^*, u\rangle \le f(\bar{y} + u) + \rho_G(\bar{x} - \bar{y}) - f(\bar{y}) - \rho_G(\bar{x} - \bar{y}) + (\varepsilon + \eta)\|u\|,$$

that is,

$$\langle x^*, u\rangle \le f(\bar{y}+u) - f(\bar{y}) + (\varepsilon+\eta)\|u\|.$$

This being true for all $\eta > 0$ we obtain $x^* \in \partial^{F,\varepsilon} f(\bar{y}) \cap \partial^{F,\varepsilon}\rho_G(\bar{x}-\bar{y})$. Moreover, the function ρ_G being convex and continuous, it is not difficult to see that

$$\partial^{F,\varepsilon}\rho_G(\bar{x}-\bar{y}) = \partial\rho_G(\bar{x}-\bar{y}) + \varepsilon\mathbb{B}_{X^*},$$

and hence the Fréchet part of assertion (a) of the theorem is established.

The same arguments also hold for the proximal subdifferential.

(b) Fix $v \in X$ and consider any $t > 0$ and any $w, h \in X$. Write

$$\begin{aligned} & t^{-1}[T_f(\bar{x}+tv+th) - T_f(\bar{x})] \\ \le\ & t^{-1}[f(\bar{y}+tw+th) + \rho_G(\bar{x}-\bar{y}+t(v-w)) - f(\bar{y}) - \rho_G(\bar{x}-\bar{y})] \\ \le\ & t^{-1}[f(\bar{y}+tw+th) - f(\bar{y})] + t^{-1}[\rho_G(\bar{x}-\bar{y}+t(v-w)) - \rho_G(\bar{x}-\bar{y})]. \end{aligned}$$

Taking the lower limit as $t \downarrow 0$ and $h \to 0$ in X gives

$$d^-T_f(\bar{x}; v) \le d^-f(\bar{y}; w) + d^-\rho_G(\bar{x}-\bar{y}; v-w),$$

since

$$d^-\rho_G(\bar{x}-\bar{y}; v-w) = \lim_{t\downarrow 0} t^{-1}[\rho_G(\bar{x}-\bar{y}+t(v-w)) - \rho_G(\bar{x}-\bar{y})],$$

according to the convexity of the continuous function ρ_G. The latter inequality being true for all $w \in X$ it follows that

$$d^-T_f(\bar{x}; v) \le \inf_{w\in X}[d^-f(\bar{y}; w) + d^-\rho_G(\bar{x}-\bar{y}; v-w)] = \left(d^-f(\bar{y};\cdot)\Box d^-\rho_G(\bar{x}-\bar{y};\cdot)\right)(v). \tag{3.1}$$

Further, we have that $d^-\rho_G(\bar{x}-\bar{y}; v-w) \le \rho_G(v-w)$ because of the sublinearity of ρ_G, and subsequently

$$\left(d^-f(\bar{y};\cdot)\Box d^-\rho_G(\bar{x}-\bar{y};\cdot)\right)(v) \le \left(d^-f(\bar{y};\cdot)\Box\rho_G(\cdot)\right)(v).$$

Fix now $x^* \in \partial^- T_f(\bar{x})$. We take $w = v$ and $w = 0$ respectively in the inequality of (3.1) to deduce that

$$d^-T_f(\bar{x}; v) \le d^-f(\bar{y}; v) \quad \text{and} \quad d^-T_f(\bar{x}; v) \le d^-\rho_G(\bar{x}-\bar{y}; v),$$

which ensures that $x^* \in \partial^- f(\bar{y})$ and $x^* \in \partial\rho_G(\bar{x}-\bar{y})$. This yields the inclusion of (b).

(c) Under the convexity of f, the function T_f is known to be convex as the infimum convolution of two convex functions (see [28, 29]), and the opposite inclusion of the one of (b) follows from a quite standard argument. It is enough for $x^* \in \partial f(\bar{y}) \cap \partial\rho_G(\bar{x}-\bar{y})$ to write

$$\begin{aligned} \langle x^*, x-\bar{x}\rangle &= \langle x^*, y-\bar{y}\rangle + \langle x^*, x-y-(\bar{x}-\bar{y})\rangle \\ &\le f(y) - f(\bar{y}) + \rho_G(x-y) - \rho_G(\bar{x}-\bar{y}) \\ &= f(y) + \rho_G(x-y) - T_f(\bar{x}). \end{aligned}$$

Taking the infimum over $y \in X$ we arrive at

$$\langle x^*, x-\bar{x}\rangle \le T_f(x) - T_f(\bar{x}) \quad \text{for all} x \in X,$$

which means that $x^* \in \partial T_f(\bar{x})$. This gives the desired opposite inclusion and completes the proof. □

The arguments for the inclusion in (a) follow exactly those of Correa, Jofre and Thibault in [11, Lemma 3.6] where this inclusion was first observed with $\varepsilon = 0$ for any function g in place of ρ_G. Clearly, the arguments with any $\varepsilon \geq 0$ are still valid for any function g, but the statement with ρ_G allows us to keep the presentation of the paper in a unified way. The assertion (a) was also established by Nam and Cuong [31, Proposition 4.4] with $\varepsilon \geq 0$ and with a general function φ in place of ρ_G. Results similar to Theorem 3.1 have been established for the Fréchet and proximal subdifferentials of T_C by Colombo and Wolenski [10, Theorem 3.3] in Hilbert space and by He and Ng [16] for the Dini directional derivative of T_C in Banach space. For the proximal subdifferential inclusion in the assertion (a), we also refer to (88) in [32] with the particular function $\hat{u}$ given by (1.3) in the introduction.

As a first corollary we have an extension of [10, Theorem 4.2] and [8, Proposition 5.9] to the case of T_f in place of T_C.

Corollary 3.1 *Assume in addition to the hypotheses of Theorem 3.1 that the function f is lsc and convex. Then for any $\bar{x} \in X$*

$$\Pi_f(\bar{x}) \subset \partial f^*(x^*) \quad \textit{for all } x^* \in \partial T_f(\bar{x}) \neq \emptyset.$$

Proof We may suppose that $\Pi_f(\bar{x}) \neq \emptyset$. By Proposition 3.1 below, the function T_f is Lipschitz continuous. Since it is further convex under our assumptions, we have $\partial T_f(\bar{x}) \neq \emptyset$. Fix $x^* \in \partial T_f(\bar{x})$ and take any $\bar{y} \in \Pi_f(\bar{x})$. Theorem 3.1 then gives $x^* \in \partial f(\bar{y})$ and by (2.14) this is equivalent to $\bar{y} \in \partial f^*(x^*)$. Consequently, $\Pi_f(\bar{x}) \subset \partial f^*(x^*)$. □

The following proposition shows in particular the Lipschitz continuity of T_f used in the corollary above. The arguments are classical in the sense that they follow the well-known ones with a norm in place of ρ_G.

Proposition 3.1 *The infimum time function T_f verifies for all $x, x' \in X$ the inequality*

$$T_f(x') - T_f(x) \leq \rho_G(x' - x)$$

and hence

$$|T_f(x) - T_f(x')| \leq \max\{\rho_G(x' - x), \rho_G(x - x')\} \leq \beta \|x' - x\|.$$

Proof For any $y \in X$ we have according to the sublinearity of ρ_G

$$f(y) + \rho_G(x' - y) \leq f(y) + \rho_G(x - y) + \rho_G(x' - x)$$

and hence taking the infimum over $y \in X$ we get the first inequality

$$T_f(x') \leq T_f(x) + \rho_G(x' - x)$$

of the proposition. The second part of the statement of the proposition follows directly from the latter inequality and (2.8). □

Due to the Lipschitz continuity of T_f the Dini directional derivative of T_f may be expressed in simpler form than in the original definition:

$$d^- T_f(x; v) := \liminf_{t \downarrow 0} t^{-1}[T_f(x + tv) - T_f(x)] \quad \text{for all } x, v \in X. \tag{3.2}$$

The second corollary below of Theorem 3.1 is related to the case when $T_f(\bar{x}) = f(\bar{x})$.

Corollary 3.2 *Assume $T_f(\bar{x}) = f(\bar{x})$. Then*

$$\partial^{F,\varepsilon} T_f(\bar{x}) \subset \partial^{F,\varepsilon} f(\bar{x}) \cap (G^o + \varepsilon \mathbb{B}_{X^*}) \quad \text{for } \varepsilon \geq 0,$$

and

$$\partial^F T_f(\bar{x}) \subset \partial^F f(\bar{x}) \cap G^o = \partial^F f(\bar{x}) \cap \{x^* \in X^* : \sigma_G(x^*) \leq 1\}.$$

Also

$$\partial^P T_f(\bar{x}) \subset \partial^P f(\bar{x}) \cap G^o = \partial^P f(\bar{x}) \cap \{x^* \in X^* : \sigma_G(x^*) \leq 1\}.$$

If, in addition, f is convex, then the latter inclusions for $\partial^F T_f(\bar{x})$ and $\partial^P T_f(\bar{x})$ are equalities.

Proof The equality assumption $T_f(\bar{x}) = f(\bar{x})$ obviously assures us that $\bar{x} \in \Pi_f(\bar{x})$. Observing that

$$\partial \rho_G(0) = \partial \sigma_{G^o}(0) = G^o,$$

In view of (2.5) Theorem 3.1 implies for $\varepsilon \geq 0$ that

$$\partial^{F,\varepsilon} T_f(\bar{x}) \subset \partial^{F,\varepsilon} f(\bar{x}) \cap (\partial \rho_G(0) + \varepsilon \mathbb{B}_{X^*}) = \partial^{F,\varepsilon} f(\bar{x}) \cap \left(G^o + \varepsilon \mathbb{B}_{X^*}\right).$$

Further, obviously $G^o = \{x^* \in X^* : \sigma_G(x^*) \leq 1\}$. So the inclusions for $\partial^{F,\varepsilon} T_f(\bar{x})$ and $\partial^F T_f(\bar{x})$ are established.

The inclusion concerning $\partial^P T_f(\bar{x})$ holds in a similar way. Finally, the case where f is convex is a consequence of assertion (c) of Theorem 3.1. □

Strengthening the condition $T_f(\bar{x}) = f(\bar{x})$ into $f(\bar{x}) = \inf_X (f + \alpha' \|\bar{x} - \cdot\|)$, the inequalities in (b) of Theorem 3.1 with $\bar{y} = \bar{x}$ become equalities provided that X is finite dimensional and $0 \leq \alpha' < \alpha$, where α is given by (2.8).

Corollary 3.3 *Assume that f is lsc and that $f(\bar{x}) = \inf_X (f + \alpha' \|\bar{x} - \cdot\|)$ with some nonnegative $\alpha' < \alpha$, where α is as in (2.8), and assume that X is finite dimensional. Then for all $v \in X$*

$$d^- T_f(\bar{x}; v) = \min_{w \in X} [d^- f(\bar{x}; w) + \rho_G(v - w)] = \left(d^- f(\bar{x}; \cdot) \square \rho_G\right)(v).$$

Proof The assumption $f(\bar{x}) = \inf_X (f + \alpha' \|\bar{x} - \cdot\|)$ with $\alpha' < \alpha$ and (2.8) assures us that $T_f(\bar{x}) = f(\bar{x})$ and $\bar{x} \in \Pi_f(\bar{x})$. Therefore, assertion (b) of Theorem 3.1 says that for any fixed $v \in X$

$$d^- T_f(\bar{x}; v) \leq \inf_{w \in X} [d^- f(\bar{x}; w) + \rho_G(v - w)]. \tag{3.3}$$

Now in view of (2.8) we see for any $x \in X$ that

$$\begin{aligned} &\{y \in X : \ f(y) + \rho_G(x - y) \leq T_f(x) + 1\} \\ &\subset \{y \in X : \ f(\bar{x}) - \alpha \|x - \bar{x}\| + (\alpha - \alpha')\|\bar{x} - y\| \leq T_f(x) + 1\}. \end{aligned}$$

So, according to the lsc property of f and to the finite dimensional assumption of X the set $\{y \in X : f(y) + \rho_G(x - y) \leq T_f(x) + 1\}$ is compact, and hence the infimum in the definition of $T_f(x)$ is achieved. By (3.2) there exists some sequence $t_k \downarrow 0$ such that

$$d^- T_f(\bar{x}; v) = \lim_k t_k^{-1}[T_f(\bar{x} + t_k v) - T_f(\bar{x})].$$

For each k choose $y_k \in X$ such that $T_f(\bar{x} + t_k v) = f(y_k) + \rho_G(\bar{x} + t_k v - y_k)$. Define $w_k := t_k^{-1}(y_k - \bar{x})$ and write

$$\begin{aligned} t_k^{-1}[T_f(\bar{x} + t_k v) - T_f(\bar{x})] &= t_k^{-1}[f(y_k) + \rho_G(\bar{x} + t_k v - y_k) - f(\bar{x})] \\ &= t_k^{-1}[f(\bar{x} + t_k w_k) - f(\bar{x})] + \rho_G(v - w_k) \\ &\geq -\alpha' \|w_k\| + \alpha \|v - w_k\| \\ &\geq (\alpha - \alpha')\|w_k\| - \alpha\|v\|. \end{aligned} \tag{3.4}$$

This entails by Proposition 3.1 that $(\alpha - \alpha')\|w_k\| \leq (\alpha + \beta)\|v\|$ and hence some subsequence of $(w_k)_k$ converges to some $\bar{w} \in X$. Then we have by (3.4)

$$d^- T_f(\bar{x}; v) \geq d^- f(\bar{x}; \bar{w}) + \rho_G(v - \bar{w}).$$

This combined with (3.3) finishes the proof. □

It is well-known that a vector $v \in X$ belongs to the Bouligand tangent cone $T(C; x)$ of the set C at $x \in C$ when there exists a sequence $v_k \to v$ and a sequence $t_k \downarrow 0$ such that $x + t_k v_k \in C$ for all integers k. The indicator function of the Bouligand tangent cone of C is equal to the (lower) Dini directional derivative of the indicator function of C, that is,

$$\psi_{T(C;\bar{x})}(\cdot) = d^- \psi_C(\bar{x}; \cdot)$$

So, taking $f = \psi_C$ we deduce directly from (b) in Theorem 3.1 and from Corollary 3.3 the following assertions. They have been established in [5] for d_C and then in [16] for T_C.

Corollary 3.4 *Let C be closed subset of X with $\bar{x} \in C$. Then for all $v \in X$*

$$d^- T_C(\bar{x}; v) \leq \inf_{w \in T(C;\bar{x})} \rho_G(v - w).$$

If in addition X is finite dimensional, then the inequality is an equality and the infimum is achieved.

We still suppose that $T_f(\bar{x}) = f(\bar{x})$ and we proceed to estimate the Clarke subdifferential of T_f and the associated directional derivative. For a lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ and $x \in \operatorname{dom} f$ the *Rockafellar directional derivative* (see [33]) is defined by

$$d^{\uparrow} f(x; v) := \inf_{\varepsilon > 0} \limsup_{x' \to_f x; t \downarrow 0} \inf_{\|e\| < \varepsilon} t^{-}[f(x' + tv + te) - f(x')],$$

where $x' \to_f x$ means $x' \to x$ and $f(x') \to f(x)$. The *Clarke subdifferential* of f at x is then defined by

$$\partial^{\uparrow} f(x) := \{x^* \in X^* : \langle x^*, v \rangle \leq d^{\uparrow} f(x; v) \; \forall v \in X\}.$$

If f is Lipschitz continuous near x, then (see [6])

$$d^{\uparrow} f(x; v) = f^o(x; v) := \limsup_{x' \to x; t \downarrow 0} t^{-1}[f(x' + tv) - f(x')].$$

When f is not Lipschitz continuous near x, according to [33] we set

$$f^o(x; v) = \limsup_{\substack{x' \to_f x;\, t \downarrow 0 \\ v' \to v}} t^{-1}[f(x' + tv') - f(x')].$$

For the closed set C and $x \in C$ the more convenient way to define the *Clarke tangent cone* $T^{\uparrow}(C; x)$ corresponds to saying that $v \in T^{\uparrow}(C; x)$ provided for any sequences $x_k \xrightarrow{C} x$ and $t_k \downarrow 0$ there exists some sequence $v_k \to v$ such that $x_k + t_k v_k \in C$ for all k. Similarly to the Bouligand tangent cone, the indicator function of the Clarke tangent cone of C verifies

$$\psi_{T^{\uparrow}(C;x)}(\cdot) = d^{\uparrow}\psi_C(x; \cdot). \tag{3.5}$$

Proposition 3.2 *Assume that f is lsc at $\bar{x}$ and that $f(\bar{x}) = \inf_X (f + \alpha' \|\bar{x} - \cdot\|)$ with some nonnegative $\alpha' < \alpha$, where α is as in (2.8). Then for all $v \in X$*

$$d^{\uparrow} T_f(\bar{x}; v) \le \inf_{w \in X} [d^{\uparrow} f(\bar{x}; w) + \rho_G(v - w)] = \left(d^{\uparrow} f(\bar{x}; \cdot) \Box \rho_G\right)(v).$$

Proof Notice that the assumption $f(\bar{x}) = \inf_X (f + \alpha' \|\bar{x} - \cdot\|)$ with $\alpha' < \alpha$ ensures that $f(\bar{x}) = T_f(\bar{x})$. Fix $\varepsilon > 0$ and choose two sequences $x_k \to \bar{x}$ and $t_k \downarrow 0$ such that

$$\limsup_{x \to \bar{x}; t \downarrow 0} \inf_{\|e\| < \varepsilon} t^{-1}[T_f(x + tv + te) - T_f(x)] = \lim_k \inf_{\|e\| < \varepsilon} t_k^{-1}[T_f(x_k + t_k v + t_k e) - T_f(x_k)].$$

For each k choose some $y_k \in X$ such that $f(y_k) + \rho_G(x_k - y_k) < T_f(x_k) + t_k^2$. Since $f(\bar{x}) \le f(y_k) + \alpha' \|\bar{x} - y_k\|$ and $\rho_G(x_k - y_k) \ge \alpha \|x_k - y_k\|$, we have

$$f(\bar{x}) + (\alpha - \alpha')\|\bar{x} - y_k\| - \alpha \|\bar{x} - x_k\| < T_f(x_k) + t_k^2 \to T_f(\bar{x}) = f(\bar{x}),$$

and hence $y_k \to \bar{x}$. Thus, the inequalities

$$T_f(y_k) \le f(y_k) < T_f(x_k) + t_k^2 - \rho_G(x_k - y_k)$$

yields $y_k \to_f \bar{x}$. Fix any $w \in X$ and write for each $e \in B(0, \varepsilon)$

$$\begin{aligned} &t_k^{-1}[T_f(x_k + t_k v + t_k e) - T_f(x_k)] \\ &\le t_k + t_k^{-1}[T_f(x_k + t_k v + t_k e) - f(y_k) - \rho_G(x_k - y_k)] \\ &\le t_k + t_k^{-1}[f(y_k + t_k w + t_k e) - f(y_k) + \rho_G(x_k - y_k + t_k v - t_k w) - \rho_G(x_k - y_k)], \end{aligned}$$

which yields

$$t_k^{-1}[T_f(x_k + t_k v + t_k e) - T_f(x_k)] \le t_k + t_k^{-1}[f(y_k + t_k w + t_k e) - f(y_k)] + \rho_G(v - w).$$

We then obtain

$$\begin{aligned} &\inf_{\|e\| < \varepsilon} t_k^{-1}[T_f(x_k + t_k v + t_k e) - T_f(x_k)] \\ &\le t_k + \inf_{\|e\| < \varepsilon} t_k^{-1}[f(y_k + t_k w + t_k e) - f(y_k)] + \rho_G(v - w), \end{aligned}$$

and subsequently

$$\begin{aligned} &\limsup_{x \to \bar{x}; t \downarrow 0} \inf_{\|e\| < \varepsilon} t^{-1}[T_f(x + tv + te) - T_f(x)] \\ &\le \limsup_{y \to_f \bar{x}; t \downarrow 0} \inf_{\|e\| < \varepsilon} t^{-1}[f(y + tw + te) - f(y)] + \rho_G(v - w). \end{aligned}$$

Taking the infimum over $\varepsilon > 0$ finally gives

$$d^{\uparrow} T_f(\bar{x}; v) \le \inf_{w \in X} [d^{\uparrow} f(\bar{x}; w) + \rho_G(v - w)]$$

and completes the proof. □

Taking $f = \psi_C$ in the statement of the proposition and using the equality (3.5) between $d^{\uparrow}\psi_C(\bar{x}; \cdot)$ and $\psi_{T^{\uparrow}(C;\bar{x})}(\cdot)$ we directly obtain:

Corollary 3.5 *Let C be closed subset of X with $\bar{x} \in C$. Then*

$$d^{\uparrow} T_C(\bar{x}; v) \le \inf_{w \in T^{\uparrow}(C;\bar{x})} \rho_G(v - w) \ \ \forall v \in X.$$

The inclusion (or description) of the Fréchet ε-subdifferential of T_f when $T_f(\bar{x}) \ne f(\bar{x})$ is different from the case when $T_f(\bar{x}) = f(\bar{x})$ (see Corollary 3.2) in the statement and the arguments as well. We state it only for $\varepsilon = 0$ letting to the reader the case where $\varepsilon > 0$.

Corollary 3.6 *Assume that $T_f(\bar{x}) \ne f(\bar{x})$ and $\Pi_f(\bar{x}) \ne \emptyset$. Then for every $\bar{y} \in \Pi_f(\bar{x})$ one has*

$$\begin{aligned}\partial^F T_f(\bar{x}) &\subset \partial^F f(\bar{y}) \cap N\left(G; \tfrac{\bar{x}-\bar{y}}{\rho_G(\bar{x}-\bar{y})}\right) \cap \{x^* \in X^* : \langle x^*, \bar{x} - \bar{y}\rangle = \rho_G(\bar{x} - \bar{y})\}\\ &= \partial^F f(\bar{y}) \cap N\left(G; \tfrac{\bar{x}-\bar{y}}{\rho_G(\bar{x}-\bar{y})}\right) \cap \operatorname{bd} G^o\end{aligned}$$

and

$$\begin{aligned}\partial^P T_f(\bar{x}) &\subset \partial^P f(\bar{y}) \cap N\left(G; \tfrac{\bar{x}-\bar{y}}{\rho_G(\bar{x}-\bar{y})}\right) \cap \{x^* \in X^* : \langle x^*, \bar{x} - \bar{y}\rangle = \rho_G(\bar{x} - \bar{y})\}\\ &= \partial^P f(\bar{y}) \cap N\left(G; \tfrac{\bar{x}-\bar{y}}{\rho_G(\bar{x}-\bar{y})}\right) \cap \operatorname{bd} G^o\end{aligned}$$

Further, the inclusions are equalities whenever f is convex.

Proof Since $T_f(\bar{x}) \ne f(\bar{x})$, we have $\bar{x} \notin \Pi_f(\bar{x})$ and hence $\bar{x} \ne \bar{y}$. The corollary then follows from Theorem 3.1 and Lemma 3.1 below. □

The next lemma corresponds to [10, Corollary 2.3] which is obtained as a consequence of [10, Proposition 2.2], where the (Legendre-Fenchel) conjugate is used as well as the duality relationship between a convex function and its conjugate. For the convenience of the reader we provide here a direct proof.

Lemma 3.1 *For any $u \ne 0$ one has*

$$\begin{aligned}\partial\rho_G(u) &= N\left(G; \frac{u}{\rho_G(u)}\right) \cap \{u^* \in X^* : \langle u^*, u\rangle = \rho_G(u)\}\\ &= N\left(G; \frac{u}{\rho_G(u)}\right) \cap \operatorname{bd} G^o.\end{aligned}$$

Proof Fix any $u^* \in \partial\rho_G(u)$. Then $\langle u^*, u\rangle = \rho_G(u)$ and $u^* \in \partial\rho_G(0)$ because of (2.4). Further for any $x \in G$ we have $\rho_G(x) \le 1$, and hence $\langle u^*, x\rangle \le \rho_G(x) \le 1$, the first inequality being due to the inclusion $u^* \in \partial\rho_G(0)$. Consequently, $\langle u^*, x - \frac{u}{\rho_G(u)}\rangle \le 0$ for all $x \in G$, and this says that $u^* \in N(G; \frac{u}{\rho_G(u)})$.

Take now any $u^* \in N(G; \frac{u}{\rho_G(u)})$ satisfying $\langle u^*, u\rangle = \rho_G(u)$ and fix any nonzero $x \in X$. Since $\frac{x}{\rho_G(x)} \in G$, we have $\langle u^*, \frac{x}{\rho_G(x)} - \frac{u}{\rho_G(u)}\rangle \le 0$, which ensures

$$\langle u^*, \frac{x}{\rho_G(x)}\rangle \le \langle u^*, \frac{u}{\rho_G(u)}\rangle = 1.$$

Then, for any $x \in X$ we have $\langle u^*, x\rangle \le \rho_G(x)$ and hence

$$\langle u^*, x - u\rangle \le \rho_G(x) - \rho_G(u).$$

So, $u^* \in \partial\rho_G(u)$ and the equality between the first two members of the statement is established.

The equality between the second and third members follows from the fact that the inclusion $u^* \in N(G; \frac{u}{\rho_G(u)})$ is equivalent to the equality $\langle u^*, \frac{u}{\rho_G(u)}\rangle = \sigma_G(u^*)$. □

The next theorem continues with estimations of subdifferentials of T_f at $\bar{x}$ when $\Pi_f(\bar{x})$ is nonempty.

Theorem 3.2 *Assume that $\bar{x} \in X$, $\bar{y} \in \Pi_f(\bar{x})$ and there exist some neighborhood U of $\bar{x}$ and a constant $L > 0$ such that for each $x \in U$ there is a sequence $(y_k)_k$ satisfying $f(y_k) + \rho_G(x - y_k) \to T_f(x)$ and $\limsup_k \|y_k - \bar{y}\| \le L\|x - \bar{x}\|$. Then*

$$\partial^{F,\varepsilon} T_f(\bar{x}) \subset \partial^{F,\varepsilon} f(\bar{y}) \cap \partial^{F,\varepsilon}\rho_G(\bar{x} - \bar{y}) \subset \partial^{F,(2L+1)\varepsilon} T_f(\bar{x}) \qquad \forall\varepsilon \ge 0,$$

and in particular

$$\partial^F T_f(\bar{x}) = \partial^F f(\bar{y}) \cap \partial\rho_G(\bar{x} - \bar{y}).$$

Also, for the proximal subdifferential one has

$$\partial^P T_f(\bar{x}) = \partial^P f(\bar{y}) \cap \partial\rho_G(\bar{x} - \bar{y}).$$

Proof Fix any $\varepsilon \ge 0$. The inclusion $\partial^{F,\varepsilon} T_f(\bar{x}) \subset \partial^{F,\varepsilon} f(\bar{y}) \cap \partial^{F,\varepsilon}\rho_G(\bar{x} - \bar{y})$ is due to Theorem 3.1(a). Now we prove that $\partial^{F,\varepsilon} f(\bar{y}) \cap \partial^{F,\varepsilon}\rho_G(\bar{x} - \bar{y}) \subset \partial^{F,(2L+1)\varepsilon} T_f(\bar{x})$. Let $x^* \in \partial^{F,\varepsilon} f(\bar{y}) \cap \partial^{F,\varepsilon}\rho_G(\bar{x} - \bar{y})$. Fix any $\eta > 0$. There exists $\delta_0 > 0$ such that for all $u, v \in B(0, \delta_0)$ we have

$$\begin{aligned}\langle x^*, u\rangle &\le f(\bar{y} + u) - f(\bar{y}) + (\varepsilon + \eta)\|u\|,\\ \langle x^*, v\rangle &\le \rho_G(\bar{x} - \bar{y} + v) - \rho_G(\bar{x} - \bar{y}) + (\varepsilon + \eta)\|v\|.\end{aligned} \tag{3.6}$$

Choose a positive $\delta < \frac{\delta_0}{1+L}$ such that $B(\bar{x}, \delta) \subset U$. Fix $x \in B(\bar{x}, \delta)$ and consider the sequence $(y_k)_k$ given by the assumption of the theorem satisfying $f(y_k) + \rho_G(x - y_k) \to T_f(x)$ and $\limsup_k \|y_k - \bar{y}\| \le L\|x - \bar{x}\|$. Put $u_k = y_k - \bar{y}$ and $v_k = x - \bar{x} - u_k$. We then have $\limsup_k \|u_k\| \le L\|x - \bar{x}\| < \delta_0$ and $\limsup_k \|v_k\| \le (L + 1)\|x - \bar{x}\| < \delta_0$. Consequently, $u_k, v_k \in B(0, \delta_0)$ for sufficiently large k. Combining the inequalities (3.6) with $u = u_k$ and $v = v_k$, we get for sufficiently large k

$$\langle x^*, x - \bar{x}\rangle \le f(y_k) + \rho_G(x - y_k) - f(\bar{y}) - \rho_G(\bar{x} - \bar{y}) + (\varepsilon + \eta)(\|u_k\| + \|v_k\|).$$

Passing to the limit yields

$$\langle x^*, x - \bar{x}\rangle \le T_f(x) - T_f(\bar{x}) + (\varepsilon + \eta)(2L + 1)\|x - \bar{x}\|.$$

Hence $x^* \in \partial^{F,(2L+1)\varepsilon} T_f(\bar{x})$ and the Fréchet part of the theorem is established. The same arguments also hold for the proximal subdifferential. □

A particular case with $T_f(\bar{x}) = f(\bar{x})$ (but where is involved a general function φ with $\varphi(0) = 0$ in place of ρ_G) was previously considered by Nam in [30, Theorem 2.3]. Again with $T_f(\bar{x}) = f(\bar{x})$, another similar result previously appeared in [36, Theorems 3.1 and 4.1].

In view of the next theorem we need the following lemma.

Lemma 3.2 *Assume that* $\bar{x} \in \operatorname{dom} f$ *and there exists some nonnegative* $\alpha' < \alpha$ *such that* $f(\bar{x}) = \inf_X \big(f + \alpha'\|\bar{x} - \cdot\|\big)$. *(Remind that* α *is given by* (2.8)*). Then the assumptions of Theorem 3.2 hold true with* $\bar{y} = \bar{x}$, $U = X$ *and* $L = \frac{\alpha+\beta}{\alpha-\alpha'}$.

Proof Since for any $y \in X$

$$f(\bar{x}) \le f(y) + \alpha'\|\bar{x} - y\| \le f(y) + \alpha\|\bar{x} - y\| \le f(y) + \rho_G(\bar{x} - y),$$

it follows by (2.10), (2.11) that $T_f(\bar{x}) = f(\bar{x})$ and $\bar{x} \in \Pi_f(\bar{x})$. Fix any $x \in X$ and any sequence $(y_k)_k$ such that $f(y_k) + \rho_G(x - y_k) \to T_f(x)$. Observing that

$$T_f(x) \le f(\bar{x}) + \rho_G(x - \bar{x}) \le f(\bar{x}) + \beta\|x - \bar{x}\|,$$

$$f(y_k) \ge f(\bar{x}) - \alpha'\|y_k - \bar{x}\|,$$

$$\rho_G(x - y_k) \ge \alpha\|y_k - x\| \ge \alpha(\|y_k - \bar{x}\| - \|x - \bar{x}\|),$$

we obtain

$$(\alpha - \alpha') \limsup_k \|y_k - \bar{x}\| \le (\alpha + \beta)\|x - \bar{x}\|.$$

Dividing by $(\alpha - \alpha')$ we complete the proof. □

Combining Theorem 3.2 with Lemma 3.2 and bearing in mind (2.5) yield the following result.

Theorem 3.3 *Assume that* $\bar{x} \in \operatorname{dom} f$ *and there exists some nonnegative* $\alpha' < \alpha$ *such that* $f(\bar{x}) = \inf_X \big(f + \alpha'\|\bar{x} - \cdot\|\big)$. *(Remind that* α *is given by* (2.8)*). Then for any* $\varepsilon \ge 0$ *one has*

$$\partial^{F,\varepsilon} T_f(\bar{x}) \subset \partial^{F,\varepsilon} f(\bar{x}) \cap \big(G^o + \varepsilon\mathbb{B}_{X^*}\big) \subset \partial^{F,\theta(\varepsilon)} T_f(\bar{x})$$

for $\theta(\varepsilon) = \varepsilon\big(1 + 2(\alpha - \alpha')^{-1}(\alpha + \beta)\big)$. *In particular, for the Fréchet subdifferential the equalities*

$$\partial^F T_f(\bar{x}) = \partial^F f(\bar{x}) \cap G^o = \partial^F f(\bar{x}) \cap \{x^* \in X^* : \sigma_G(x^*) \le 1\}$$

hold true. Similarly, for the proximal subdifferential one has

$$\partial^P T_f(\bar{x}) = \partial^P f(\bar{x}) \cap G^o = \partial^P f(\bar{x}) \cap \{x^* \in X^* : \sigma_G(x^*) \le 1\}.$$

The assumption $f(\bar{x}) = \inf_X \big(f + \alpha'\|\bar{x} - \cdot\|\big)$ with $0 \le \alpha' < \alpha$ is essential in Theorem 3.3. Indeed, consider in the space $X = \mathbb{R}$ the set G and the function f defined as:

$$G = [-1, 1], \qquad f(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| \ge 1. \end{cases} \tag{3.7}$$

Clearly $\rho_G(x) = |x|$ and it is easily checked that

$$T_f(x) = \begin{cases} 1 - |x|, & |x| < 1, \\ 0, & |x| \ge 1. \end{cases}$$

So, for the point $\bar{x} = 0$ we have $T_f(\bar{x}) = f(\bar{x})$, but

$$\partial^F T_f(\bar{x}) = \partial^P T_f(\bar{x}) = \emptyset \neq \partial^F f(\bar{x}) \cap G^o = \partial^P f(\bar{x}) \cap G^o = \{0\}.$$

Notice that the greatest α fulfilling (2.8) is $\alpha = 1$, so $0 \leq \alpha' < 1$; then one can see that there is no such α' satisfying the condition $f(\bar{x}) = \inf_X \big(f + \alpha'|\bar{x} - \cdot|\big)$.

In the above mentioned papers [30, 31] results similar to that of Theorem 3.3 are obtained for a more general function $\varphi(\cdot)$ in place of $\rho_G(\cdot)$. However Theorem 3.3 and more generally Theorem 3.2 don't follow from the results of [30, 31]. Indeed, in [30, 31] it is assumed that the function f satisfies a center-Lipschitz condition (or is calm) at $\bar{x}$ on dom f with some constant l, that is

$$|f(\bar{x}) - f(x)| \leq l\|\bar{x} - x\| \quad \forall x \in \operatorname{dom} f.$$

This assumption is more restrictive than the assumption of Theorem 3.3 requiring $f(\bar{x}) = \inf_X \big(f + \alpha'\|\bar{x} - \cdot\|\big)$ with $\alpha' = l$, which can be rewritten as

$$f(\bar{x}) - f(x) \leq l\|\bar{x} - x\| \quad \forall x \in \operatorname{dom} f.$$

See also Theorem 4 and Proposition 1 in [32] for other results.

Taking for f the indicator function of the closed set C and $\bar{x} \in C$, the assumptions of Theorem 3.3 obviously hold with $\alpha' = 0$, and hence we obtain:

Corollary 3.7 *Let C be a closed set of X and $\bar{x} \in C$. Then for any $\varepsilon \geq 0$ one has*

$$\partial^{F,\varepsilon} T_C(\bar{x}) \subset N^{F,\varepsilon}(C; \bar{x}) \cap (G^o + \varepsilon \mathbb{B}_{X^*}) \subset \partial^{F,\theta(\varepsilon)} T_C(\bar{x})$$

for $\theta(\varepsilon) := \varepsilon\left(3 + 2\alpha^{-1}\beta\right)$, and hence

$$\partial^F T_C(\bar{x}) = N^F(C; \bar{x}) \cap G^o = N^F(C; \bar{x}) \cap \{x^* \in X^* : \sigma_G(x^*) \leq 1\}.$$

Also

$$\partial^P T_C(\bar{x}) = N^P(C; \bar{x}) \cap G^o = N^P(C; \bar{x}) \cap \{x^* \in X^* : \sigma_G(x^*) \leq 1\}.$$

For the particular case where $G = \mathbb{B}_X$ one has

$$\partial^F d_C(\bar{x}) = N^F(C; \bar{x}) \cap \mathbb{B}_{X^*}, \tag{3.8}$$

$$\partial^P d_C(\bar{x}) = N^P(C; \bar{x}) \cap \mathbb{B}_{X^*}. \tag{3.9}$$

The equality (3.8) has been first observed and established by Ioffe [17] and Kruger [21] and it has been recently extended to the form above by He and Ng [16, Theorem 4.1] for $\partial^F T_C(\bar{x})$. The more general inclusions of the corollary are similar to those of Mordukhovich and Nam in [26, Theorem 3.4]. The equality (3.9) has been established by Bounkhel and Thibault [4, Theorem 4.1]. The case of T_C has been obtained by He and Ng [16, Theorem 5.1].

In the case of a closed set C, it is well-known that, for any point $y \in C$ realizing the distance $d_C(x)$ from x to C, one has $d_C(x_t) = (1-t)d_C(x)$ where $x_t := (1-t)x + ty$ and $t \in [0, 1]$. This equality has been extended to the function T_C in [26, Lemma 3.1]. The next proposition establishes a similar result for T_f.

Proposition 3.3 *Assume that $\bar{y} \in \Pi_f(\bar{x})$. Then for any $t \in [0, 1]$ and $x_t := (1-t)\bar{x} + t\bar{y}$ one has*

$$T_f(x_t) = T_f(\bar{x}) - t\rho_G(\bar{x} - \bar{y}) = tf(\bar{y}) + (1-t)T_f(\bar{x}).$$

Proof By definition of T_f we may write

$$\begin{aligned} T_f\left((1-t)\bar{x}+t\bar{y}\right) &\le f(\bar{y})+\rho_G\left((1-t)\bar{x}+t\bar{y}-\bar{y}\right) \\ &= f(\bar{y})+(1-t)\rho_G(\bar{x}-\bar{y}) \\ &= T_f(\bar{x})-t\rho_G(\bar{x}-\bar{y}). \end{aligned}$$

On the other hand, for any $y \in X$ we also have

$$\begin{aligned} f(y)+\rho_G(x_t-y) &= f(y)+\rho_G\left((\bar{x}-y)-t(\bar{x}-\bar{y})\right) \\ &\ge f(y)+\rho_G(\bar{x}-y)-t\rho_G(\bar{x}-\bar{y}) \\ &\ge T_f(\bar{x})-t\rho_G(\bar{x}-\bar{y}), \end{aligned}$$

and hence taking the infimum over $y \in X$ gives

$$T_f(x_t) \ge T_f(\bar{x})-t\rho_G(\bar{x}-\bar{y}).$$

Therefore we have the desired equality between the first two members of the statement. The remaining equality follows directly from the latter one. □

4 Further Properties of the Fréchet Subdifferential of the Infimum Time Function

This section provides some further properties enjoyed by elements in the Fréchet ε-subdifferential of the infimum time function. At the opposite of Theorem 3.1 the next first theorem does not require the nonemptiness of the set $\Pi_f(\bar{x})$.

Theorem 4.1 *Let $\varepsilon \ge 0$ and let $\alpha, \beta > 0$ be as in (2.8). The following hold.*
(a) If $x^* \in \partial^{F,\varepsilon} T_f(\bar{x})$, then one has $x^* \in \partial\rho_G(0)+\varepsilon\mathbb{B}_{X^*} = G^o+\varepsilon\mathbb{B}_{X^*}$, and hence $\sigma_G(x^*) \le 1+\alpha^{-1}\varepsilon$.

(b) Assume instead of (2.12) that f is bounded from below. Let $\bar{x} \notin \operatorname{cl}(\operatorname{dom} f)$ and let $x^* \in \partial^{F,\varepsilon} T_f(\bar{x})$. Then $\sigma_G(x^*) \ge 1-\alpha^{-1}\varepsilon$.
Further, if $\varepsilon = 0$ (i.e., $x^* \in \partial^F T_f(\bar{x})$), then $\sigma_G(x^*) = 1$, and subsequently $x^* \in \operatorname{bd} G^o$.

Proof (a) Fix $\eta > 0$ and take a neighborhood U of zero such that for all $u \in U$

$$\langle x^*, u\rangle \le T_f(\bar{x}+u)-T_f(\bar{x})+(\varepsilon+\eta)\|u\|,$$

and hence according to the first inequality of Proposition 3.1

$$\langle x^*, u\rangle \le \rho_G(u)+(\varepsilon+\eta)\|u\| = \rho_G(u)-\rho_G(0)+(\varepsilon+\eta)\|u\|.$$

The function $\rho_G(\cdot)$ being convex and continuous, this entails

$$x^* \in \partial(\rho_G+(\varepsilon+\eta)\|\cdot\|)(0) = \partial\rho_G(0)+\varepsilon\mathbb{B}_{X^*}+\eta\mathbb{B}_{X^*}.$$

Because of the $w(X^*, X)$-closedness of $\partial\rho_G(0)+\varepsilon\mathbb{B}_{X^*}$ we obtain

$$x^* \in \partial\rho_G(0)+\varepsilon\mathbb{B}_{X^*} = G^o+\varepsilon\mathbb{B}_{X^*}.$$

Further, this inclusion combined with $G \subset \frac{1}{\alpha}\mathbb{B}_X$ (see (2.8)) implies the desired inequality $\sigma_G(x^*) \le 1+\alpha^{-1}\varepsilon$.

(b) Assume that $\bar{x} \notin \mathrm{cl}\,(\mathrm{dom}\, f)$ and that f is bounded from below. Denote by μ the real number $\mu := -\inf_X f$ and observe that for all $y \in \mathrm{dom}\, f$

$$f(y) + \rho_G(\bar{x} - y) + \mu \geq \rho_G(\bar{x} - y) \geq \alpha\|\bar{x} - y\| \geq \alpha\, \mathrm{dist}(\bar{x}, \mathrm{dom}\, f),$$

the second inequality being due to (2.8). This gives

$$T_f(\bar{x}) + \mu \geq \alpha\, \mathrm{dist}(\bar{x}, \mathrm{dom}\, f) > 0,$$

since $\bar{x} \notin \mathrm{cl}\,(\mathrm{dom}\, f)$. Therefore, fixing any positive $\delta < 1$ such that

$$\sqrt{\delta}\alpha(1-\delta)\left(T_f(\bar{x}) + \mu\right)^{-1} < 1 \tag{4.1}$$

we can choose some $z \in X$ (depending on δ) such that

$$(1-\delta)^{-1}\left(T_f(\bar{x}) + \mu\right) > f(z) + \mu + \rho_G(\bar{x} - z). \tag{4.2}$$

This inequality entails in particular, on one hand that $f(z) < +\infty$, and hence, since $f(\bar{x}) = +\infty$ by assumption, we have $z \neq \bar{x}$, which assures us that $\rho_G(\bar{x} - z) > 0$ since $\rho_G(x) \geq \alpha\|x\|$ for all $x \in X$. On the other hand, (4.2) combined with the definition of μ yields

$$(1-\delta)^{-1}\left(T_f(\bar{x}) + \mu\right) > \rho_G(\bar{x} - z),$$

which ensures according to (2.8)

$$\alpha\|\bar{x} - z\| \leq (1-\delta)^{-1}\left(T_f(\bar{x}) + \mu\right). \tag{4.3}$$

Observe also that the finiteness of $f(z)$ entails that $z \in \mathrm{dom}\, f$, and hence using (2.8) again we have

$$\alpha^{-1}\rho_G(\bar{x} - z) \geq \|\bar{x} - z\| \geq \mathrm{dist}(\bar{x}, \mathrm{dom}\, f) =: \theta > 0. \tag{4.4}$$

Take now $\eta > 0$ and choose a positive real number $r < 1$ (independent of δ) such that for all $u \in r\mathbb{B}_X$

$$\langle x^*, u\rangle \leq T_f(\bar{x} + u) - T_f(\bar{x}) + (\varepsilon + \eta)\|u\|,$$

and hence by (4.2)

$$\langle x^*, u\rangle \leq T_f(\bar{x} + u) + \mu - (1-\delta)[f(z) + \mu + \rho_G(\bar{x} - z)] + (\varepsilon + \eta)\|u\|. \tag{4.5}$$

Put $s_\delta := r\alpha(1-\delta)\left(T_f(\bar{x}) + \mu\right)^{-1}$ and $t_\delta := s_\delta\sqrt{\delta}$, and notice that $t_\delta < 1$ according to (4.1). Put also $u_\delta = t_\delta(z - \bar{x})$ and observe that $u_\delta \in r\mathbb{B}_X$ because (4.3) ensures $s_\delta\|\bar{x} - z\| \leq r$. Writing

$$\begin{aligned} T_f(\bar{x} + u_\delta) &\leq f(z) + \rho_G(\bar{x} + u_\delta - z) \\ &= f(z) + \rho_G\left((1 - t_\delta)(\bar{x} - z)\right) = f(z) + (1 - t_\delta)\rho_G(\bar{x} - z), \end{aligned}$$

we obtain from (4.5) that

$$t_\delta\langle x^*, z - \bar{x}\rangle \leq (1 - t_\delta)\rho_G(\bar{x} - z) - (1-\delta)\rho_G(\bar{x} - z) + \delta f(z) + \delta\mu + (\varepsilon + \eta)t_\delta\|\bar{x} - z\|$$

and using (4.2) and (2.8) we see that

$$t_\delta\langle x^*, z - \bar{x}\rangle \leq -t_\delta\rho_G(\bar{x} - z) + \frac{\delta}{1-\delta}\left(T_f(\bar{x}) + \mu\right) + \alpha^{-1}t_\delta(\varepsilon + \eta)\rho_G(\bar{x} - z).$$

This yields by (4.4)

$$t_\delta\langle x^*, z - \bar{x}\rangle \leq -t_\delta\rho_G(\bar{x} - z) + \frac{\alpha^{-1}\delta}{\theta(1-\delta)}\left(T_f(\bar{x}) + \mu\right)\rho_G(\bar{x} - z) + \alpha^{-1}t_\delta(\varepsilon + \eta)\rho_G(\bar{x} - z).$$

Since $\rho_G(\bar{x} - z) > 0$ it follows that

$$\langle x^*, \frac{\bar{x} - z}{\rho_G(\bar{x} - z)} \rangle \geq 1 - \alpha^{-1}(\varepsilon + \eta) - \frac{\alpha^{-1}\delta}{t_\delta} \cdot \frac{1}{\theta(1-\delta)} \big(T_f(\bar{x}) + \mu\big).$$

Consequently

$$\sigma_G(x^*) \geq 1 - \alpha^{-1}(\varepsilon + \eta) - \frac{\alpha^{-1}\delta}{t_\delta} \cdot \frac{1}{\theta(1-\delta)} \big(T_f(\bar{x}) + \mu\big).$$

Since

$$\delta/t_\delta = \sqrt{\delta}/s_\delta = \sqrt{\delta} r^{-1}\alpha^{-1}(1-\delta)^{-1} \big(T_f(\bar{x}) + \mu\big) \to 0$$

as $\delta \downarrow 0$, we deduce that $\sigma_G(x^*) \geq 1 - \alpha^{-1}(\varepsilon + \eta)$. Making $\eta \downarrow 0$ assures us that

$$\sigma_G(x^*) \geq 1 - \alpha^{-1}\varepsilon,$$

that is, the inequality of assertion (b) is established.

Finally, the second part of assertion (b) corresponding to the case $\varepsilon = 0$ is a direct consequence of assertion (a) and of the last inequality above. □

From the above theorem we deduce directly the properties below concerning the case of the infimum time function associated with a closed set C. It suffices to take for f the indicator function of C.

Corollary 4.1 *Let C be a nonempty closed subset of X and $\varepsilon \geq 0$.*

(a) *If $x^* \in \partial^{F,\varepsilon} T_C(\bar{x})$, then*

$$x^* \in G^o + \varepsilon \mathbb{B}_{X^*} \quad \text{and} \quad \sigma_G(x^*) \leq 1 + \alpha^{-1}\varepsilon;$$

if in addition $\bar{x} \notin C$, then $\sigma_G(x^*) \geq 1 - \alpha^{-1}\varepsilon$.

(b) *If $x^* \in \partial^F T_C(\bar{x})$ and $\bar{x} \notin C$, then $x^* \in \operatorname{bd} G^o$.*

(c) *In the particular case where $G = \mathbb{B}_X$, for $x^* \in \partial^{F,\varepsilon} T_C(\bar{x})$ one has $\|x^*\|_* \leq 1 + \varepsilon$, and if in addition $\bar{x} \notin C$ then $1 - \varepsilon \leq \|x^*\|_* \leq 1 + \varepsilon$.*

Assertion (c) is due to Kruger (see [21]) and assertion (b) to Mordukhovich and Nam (see [26, Proposition 3.2]).

Now observe that for $\bar{y} \in \operatorname{Argmin}(f + \rho_G(\bar{x} - \cdot)) =: \Pi_f(\bar{x})$ we have for any $x^* \in \partial^F T_f(\bar{x})$ by Theorem 3.1

$$x^* \in \partial \rho_G(\bar{x} - \bar{y}) = \partial \sigma_{G^o}(\bar{x} - \bar{y}),$$

and hence $x^* \in G^o$. If we suppose that $\bar{x} \neq \bar{y}$, then by (2.7)

$$\frac{\bar{x} - \bar{y}}{\rho_G(\bar{x} - \bar{y})} \in \partial \sigma_G(x^*), \text{ i.e.,} \bar{y} \in \bar{x} - \rho_G(\bar{x} - \bar{y}) \partial \sigma_G(x^*),$$

and the latter inclusion still holds for $\bar{y} = \bar{x}$.

Since $T_f(\bar{x}) = f(\bar{y}) + \rho_G(\bar{x} - \bar{y})$ we obtain

$$\bar{y} \in \bar{x} - \big(T_f(\bar{x}) - f(\bar{y})\big) \partial \sigma_G(x^*). \tag{4.6}$$

We can then state the following result.

Theorem 4.2 *Assume that $x^* \in \partial^F T_f(\bar{x})$. One has*

$$\bar{y} \in \Pi_f(\bar{x}) \Longrightarrow \bar{y} \in \bar{x} - \big(T_f(\bar{x}) - f(\bar{y})\big) \partial \sigma_G(x^*).$$

If, in addition, f is bounded from below and $\bar{x} \notin \mathrm{cl}\,(\mathrm{dom}\, f)$, then one has

$$\Pi_f(\bar{x}) = \{\bar{y} \in X : T_f(\bar{x}) \geq f(\bar{y}) \text{ and } \bar{y} \in \bar{x} - \big(T_f(\bar{x}) - f(\bar{y})\big)\,\partial\sigma_G(x^*)\}.$$

Proof Fix any $x^* \in \partial^F T_f(\bar{x})$. The implication $\Longrightarrow$ has been established in (4.6). Suppose now $\bar{x} \notin \mathrm{cl}\,(\mathrm{dom}\, f)$. Take any $\bar{y} \in X$ satisfying the inequality $T_f(\bar{x}) \geq f(\bar{y})$ and the inclusion

$$\bar{y} \in \bar{x} - \big(T_f(\bar{x}) - f(\bar{y})\big)\,\partial\sigma_G(x^*).$$

Choose $\zeta \in \partial\sigma_G(x^*)$ such that

$$\bar{y} = \bar{x} - \big(T_f(\bar{x}) - f(\bar{y})\big)\,\zeta, \text{ i.e., } \big(T_f(\bar{x}) - f(\bar{y})\big)\,\zeta = \bar{x} - \bar{y}. \tag{4.7}$$

Note that the inclusion $\zeta \in \partial\sigma_G(x^*)$ means that $\zeta \in G$ and $\langle x^*, \zeta\rangle = \sigma_G(x^*)$. On the other hand, the inclusion $x^* \in \partial^F T_f(\bar{x})$ entails by (b) in Theorem 4.1 that $x^* \in G^o$ and $\sigma_G(x^*) = 1$. Therefore, $x^* \in G^o$ and $\langle x^*, \zeta\rangle = 1$, and hence

$$\rho_G(\zeta) = \sigma_{G^o}(\zeta) = 1.$$

Combining that equality with (4.7) gives

$$\rho_G(\bar{x} - \bar{y}) = \rho_G((T_f(\bar{x}) - f(\bar{y}))\zeta) = (T_f(\bar{x}) - f(\bar{y}))\rho_G(\zeta) = T_f(\bar{x}) - f(\bar{y}),$$

the second equality being due to the assumption $T_f(\bar{x}) - f(\bar{y}) \geq 0$. So, $T_f(\bar{x}) = f(\bar{y}) + \rho_G(\bar{x} - \bar{y})$, which means $\bar{y} \in \Pi_f(\bar{x})$. The proof is then complete. □

The next corollary has been previously established by Colombo, Goncharov and Mordukhovich in [8, Corollary 3.2].

Corollary 4.2 *Let C be a closed nonempty subset of X and $x^* \in \partial^F T_C(\bar{x})$ with $\bar{x} \notin C$. Then*

$$\Pi_C(\bar{x}) = \big(\bar{x} - T_C(\bar{x})\partial\sigma_G(x^*)\big) \cap C.$$

Proof Taking $f := \psi_C$ we see that the inequality $T_f(\bar{x}) \geq f(\bar{y})$ means that $\psi_C(\bar{y})$ is finite, that is, $\bar{y} \in C$. The corollary then follows directly from Theorem 4.2. □

We proceed now to the study of the Fréchet subdifferential $\partial^F T_f(\bar{x})$ in the case where $\bar{x} \notin \mathrm{dom}\, f$. Remind that for a function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ and $r \in \mathbb{R}$ one denotes by $\{\varphi(\cdot) \leq r\}$ the r-sublevel set of φ, that is,

$$\{\varphi(\cdot) \leq r\} := \{x \in X : \varphi(x) \leq r\}.$$

The proof of the following lemma is quite similar to that of Lemma 3.1 in [4].

Lemma 4.1 *Assume that $\bar{x} \notin \mathrm{dom}\, f$ and $\mathrm{dom}\, f \subset \{T_f(\cdot) \leq T_f(\bar{x})\} =: S$. Then for any $x \notin S$ one has*

$$T_f(x) = T_S(x) + T_f(\bar{x}),$$

reminding that $T_S(x) = \inf_{y \in S} \rho_G(x - y)$.

Proof Fix $x \notin S$. Let us first prove the inequality $\leq$. Let $y \in S$, i.e., $T_f(y) \leq T_f(\bar{x})$. Consider any $\varepsilon > 0$ and choose $y_\varepsilon \in \mathrm{dom}\, f$ such that

$$f(y_\varepsilon) + \rho_G(y - y_\varepsilon) < T_f(y) + \varepsilon \leq T_f(\bar{x}) + \varepsilon.$$

Write by the sublinearity of ρ_G and by the second inequality above

$$\begin{aligned}\rho_G(x-y) &\geq \rho_G(x-y_\varepsilon)-\rho_G(y-y_\varepsilon)\\ &\geq \rho_G(x-y_\varepsilon)+f(y_\varepsilon)-T_f(\bar{x})-\varepsilon,\end{aligned}$$

and hence by definition of T_f

$$\rho_G(x-y)\geq T_f(x)-T_f(\bar{x})-\varepsilon.$$

Taking the infimum over $y\in S$ yields $T_S(x)\geq T_f(x)-T_f(\bar{x})-\varepsilon$, which ensures the desired inequality

$$T_f(x)\leq T_S(x)+T_f(\bar{x}).$$

To establish the converse inequality, fix any $y\in S$. For the continuous function h on $\mathbb{R}$ given by $h(s)=T_f(sx+(1-s)y)$ we have

$$h(0)=T_f(y)\leq T_f(\bar{x})\ \text{and} h(1)=T_f(x)>T_f(\bar{x}).$$

Consequently, there exists some $s_0\in[0,1[$ such that for $z:=s_0x+(1-s_0)y$ we have $T_f(z)=T_f(\bar{x})$. Since ρ_G is positively homogeneous we have

$$\rho_G(x-y)=\rho_G(x-z)+\rho_G(z-y).$$

We may then write

$$\begin{aligned}f(y)+\rho_G(x-y) &= \rho_G(x-z)+\rho_G(z-y)+f(y)\\ &\geq \rho_G(x-z)+T_f(z)=\rho_G(x-z)+T_f(\bar{x}).\end{aligned}$$

The equality $T_f(z)=T_f(\bar{x})$ tells us also that $z\in S$, and hence $\rho_G(x-z)\geq T_S(x)$ according to the definition of T_S. Therefore

$$f(y)+\rho_G(x-y)\geq T_S(x)+T_f(\bar{x}),$$

which yields

$$\inf_{y\in S}[f(y)+\rho_G(x-y)]\geq T_S(x)+T_f(\bar{x}). \tag{4.8}$$

By the assumption $\operatorname{dom} f\subset S$, the left member of (4.8) is equal to

$$\inf_{y\in X}[f(y)+\rho_G(x-y)]=T_f(x),$$

that is, (4.8) corresponds to the desired reverse inequality. The proof of the lemma is then finished. □

When $T_f(\bar{x})\neq f(\bar{x})$, but the infimum time function T_f is achieved at some $\bar{y}$ (hence necessarily different from $\bar{x}$), a first inclusion for $\partial^F T_f(\bar{x})$ (even description if f is in addition convex) has been established in Corollary 3.6 in terms of $\partial f(\bar{y})$ and $\{x^*\in X^*:\sigma_G(x^*)=1\}$. Here, we show that the main ideas in Kruger [21, Proposition 2.16] (see also Bounkhel and Thibault [4, Theorem 3.6]) are also efficient for a full description of the Fréchet subdifferential of T_f at any point $\bar{x}$ outside of $\operatorname{cl}(\operatorname{dom} f)$ without requiring that $\Pi_f(\bar{x})$ be nonempty. The description is given in terms of the Fréchet normal cone to the sublevel set $\{T_f(\cdot)\leq T_f(\bar{x})\}$ and of the same set $\{x^*\in X^*:\sigma_G(x^*)=1\}$. We keep $\alpha,\beta>0$ as given by (2.8).

Theorem 4.3 *Assume that* $\bar{x}\notin\operatorname{cl}(\operatorname{dom} f)$ *and* $\operatorname{dom} f\subset\{T_f(\cdot)\leq T_f(\bar{x})\}$. *Assume also that f is bounded from below. Then for any* $\varepsilon\geq 0$

$$\begin{aligned}\partial^{F,\varepsilon}T_f(\bar{x})\subset N^{F,\varepsilon}\left(\{T_f(\cdot)\leq T_f(\bar{x})\};\bar{x}\right)\cap\left\{x^*\in G^o+\varepsilon\mathbb{B}_{X^*}:\sigma_G(x^*)\geq 1-\alpha^{-1}\varepsilon\right\}\\ \subset\partial^{F,\theta(\varepsilon)}T_f(\bar{x}),\end{aligned}$$

where $\theta(\varepsilon) = \varepsilon\left(\alpha^{-1}\beta + (1+\alpha^{-1}\beta)(3+2\alpha^{-1}\beta)\right)$. *In particular, one has*

$$\partial^F T_f(\bar{x}) = N^F(\{T_f(\cdot) \le T_f(\bar{x})\}; \bar{x}) \cap \{x^* \in X^* : \sigma_G(x^*) = 1\}.$$

Proof Consider $\varepsilon \ge 0$. Fix any $x^* \in \partial^{F,\varepsilon} T_f(\bar{x})$. Theorem 4.1 tells us that $x^* \in G^o + \varepsilon\mathbb{B}_{X^*}$ and $\sigma_G(x^*) \ge 1 - \alpha^{-1}\varepsilon$. On the other hand, for any fixed $\varepsilon' > \varepsilon$ there exists some neighborhood U of $\bar{x}$ in X such that for all $x \in U$

$$\langle x^*, x - \bar{x}\rangle \le T_f(x) - T_f(\bar{x}) + \varepsilon'\|x - \bar{x}\|,$$

and this obviously yields for all $x \in U \cap \{T_f(\cdot) \le T_f(\bar{x})\}$ that

$$\langle x^*, x - \bar{x}\rangle \le \varepsilon'\|x - \bar{x}\|.$$

We then deduce that $x^* \in N^{F,\varepsilon}(\{T_f(\cdot) \le T_f(\bar{x})\}; \bar{x})$. Combining this with what precedes we obtain the first inclusion of the theorem.

Let us prove the second inclusion. Fix any $x^* \in N^{F,\varepsilon}(\{T_f(\cdot) \le T_f(\bar{x})\}; \bar{x})$ with $x^* \in G^o + \varepsilon\mathbb{B}_{X^*}$ and $\sigma_G(x^*) \ge 1 - \alpha^{-1}\varepsilon$. Put

$$\theta'(\varepsilon) = \varepsilon(3 + 2\alpha^{-1}\beta) \text{ and} \theta(\varepsilon) = \alpha^{-1}\beta\varepsilon + (1+\alpha^{-1}\beta)\theta'(\varepsilon)$$

and put also $S := \{T_f(\cdot) \le T_f(\bar{x})\}$. Observe by Corollary 3.7 that $x^* \in \partial^{F,\theta'(\varepsilon)} T_S(\bar{x})$. Fix any $\theta'' > \theta(\varepsilon)$ and choose some $\varepsilon' > \theta'(\varepsilon)$ such that

$$\alpha^{-1}\beta\varepsilon + \varepsilon'(1 + \alpha^{-1}\beta) < \theta''. \tag{4.9}$$

By the inclusions $x^* \in N^{F,\varepsilon}(S; \bar{x})$ and $x^* \in \partial^{F,\theta'(\varepsilon)} T_S(\bar{x})$ we may choose some $\delta > 0$ such that

$$\langle x^*, x - \bar{x}\rangle \le \varepsilon'\|x - \bar{x}\| \quad \text{for all} x \in B(\bar{x}, \delta) \cap S \tag{4.10}$$

and

$$\langle x^*, x - \bar{x}\rangle \le T_S(x) - T_S(\bar{x}) + \varepsilon'\|x - \bar{x}\| \quad \text{for all} x \in B(\bar{x}, \delta). \tag{4.11}$$

Observing that $T_S(\bar{x}) = 0$ because

$$0 \le T_S(\bar{x}) = \inf_{x \in S} \rho_G(x - \bar{x}) \le \rho_G(\bar{x} - \bar{x}) = 0,$$

we deduce from (4.11) and Lemma 4.1 that for all $x \in B(\bar{x}, \delta) \setminus S$

$$\langle x^*, x - \bar{x}\rangle \le T_f(x) - T_f(\bar{x}) + \varepsilon'\|x - \bar{x}\|. \tag{4.12}$$

Consider now any $\eta > 0$ and by the inequality $\sigma_G(x^*) \ge 1 - \alpha^{-1}\varepsilon$ choose some $u \in G$ such that $\langle x^*, u\rangle > 1 - \alpha^{-1}\varepsilon - \eta$. Consider any $x \in B(\bar{x}, \delta) \cap S$ and put

$$t_x := T_f(\bar{x}) - T_f(x) \ge 0.$$

Therefore, by the inequality $\rho_G(u) \le 1$ (due to the inclusion $u \in G$) and by Proposition 3.1 the non-negativity of t_x yields

$$T_f(x + t_x u) \le T_f(x) + t_x\rho_G(u) \le T_f(x) + t_x = T_f(\bar{x}),$$

which gives $x + t_x u \in S$. On the other hand, (2.8) and Proposition 3.1 again entail

$$t_x = T_f(\bar{x}) - T_f(x) \le \rho_G(\bar{x} - x) \le \beta\|\bar{x} - x\| \tag{4.13}$$

and hence according to (2.8) again and to the inequality $\rho_G(u) \le 1$ we obtain

$$\|x + t_x u - \bar{x}\| \le \|x - \bar{x}\| + \alpha^{-1}t_x \le (1 + \alpha^{-1}\beta)\|\bar{x} - x\|. \tag{4.14}$$

Take a positive δ' such that $\delta'(1 + \alpha^{-1}\beta) < \delta$ and fix any $x \in B(\bar{x}, \delta') \cap S$. Then $x + t_x u \in S \cap B(\bar{x}, \delta)$ and by (4.10) and (4.14) we have

$$\langle x^*, x + t_x u - \bar{x}\rangle \le \varepsilon'\|x + t_x u - \bar{x}\| \le \varepsilon'(1 + \alpha^{-1}\beta)\|x - \bar{x}\|,$$

which gives

$$\langle x^*, x - \bar{x}\rangle \le -t_x\langle x^*, u\rangle + \varepsilon'(1 + \alpha^{-1}\beta)\|x - \bar{x}\|,$$

and hence combining this with the inequality $\langle x^*, u\rangle \ge 1 - \alpha^{-1}\varepsilon - \eta$ we see that

$$\langle x^*, x - \bar{x}\rangle \le -(1 - \eta - \alpha^{-1}\varepsilon)t_x + \varepsilon'(1 + \alpha^{-1}\beta)\|x - \bar{x}\|.$$

This being true for all $\eta > 0$, we derive that

$$\begin{aligned}\langle x^*, x - \bar{x}\rangle \quad &\le -(1 - \alpha^{-1}\varepsilon)t_x + \varepsilon'(1 + \alpha^{-1}\beta)\|x - \bar{x}\| \\ &\le T_f(x) - T_f(\bar{x}) + \big(\alpha^{-1}\beta\varepsilon + \varepsilon'(1 + \alpha^{-1}\beta)\big)\|x - \bar{x}\|,\end{aligned}$$

the latter inequality being due to (4.13). Taking also (4.9) and (4.12) into account, we may write

$$\langle x^*, x - \bar{x}\rangle \le T_f(x) - T_f(\bar{x}) + \theta''\|x - \bar{x}\| \quad \text{for all } x \in B(\bar{x}, \delta').$$

Such a positive number δ' being obtained for any $\theta'' > \theta(\varepsilon)$, we get $x^* \in \partial^{F,\theta(\varepsilon)}T_f(\bar{x})$, establishing the inclusion of the second member of the theorem into the third one.

The equalities for $\varepsilon = 0$ follow from the fact that $x^* \in G^o$ is equivalent to the inequality $\sigma_G(x^*) \le 1$. The proof of the theorem is then complete. □

If $f = \psi_C$ for a closed set C and $\bar{x} \notin C$, then obviously f is bounded from below and the assumptions concerning dom f in Theorem 4.3 are fulfilled. Further, in the case where $G = \mathbb{B}_X$ we can take $\alpha = \beta = 1$ and we have

$$\{T_{\psi_C}(\cdot) \le T_{\psi_C}(\bar{x})\} = \{x \in X : d_C(x) \le d_C(\bar{x})\}.$$

Consequently, we derive directly from the above theorem:

Corollary 4.3 *Let C be a closed subset of X and $\bar{x} \notin C$. Then for any $\varepsilon \ge 0$*

$$\begin{aligned}\partial^{F,\varepsilon}T_C(\bar{x}) \subset N^{F,\varepsilon}(\{T_C(\cdot) \le T_C(\bar{x}\}; \bar{x}) \cap \{x^* \in G^o + \varepsilon\mathbb{B}_{X^*} : \sigma(x^*) \ge 1 - \alpha^{-1}\varepsilon\} \\ \subset \partial^{F,\theta(\varepsilon)}T_C(\bar{x}),\end{aligned}$$

where $\theta(\varepsilon)$ as in Theorem 4.3. In particular

$$\partial^F T_C(\bar{x}) = N^F(\{T_C(\cdot) \le T_C(\bar{x}\}; \bar{x}) \cap \{x^* \in X^* : \sigma_G(x^*) = 1\}.$$

If in addition $G = \mathbb{B}_X$, then for $r := d_C(\bar{x}) > 0$ and $C(r) := \{x \in X : d_C(x) \le r\}$ one has

$$\partial^F d_C(\bar{x}) = N^F(C(r); \bar{x}) \cap \{x^* \in X^* : \|x^*\|_* = 1\}.$$

To the best of our knowledge, the second equality of the previous corollary concerning $\partial^F d_C(\bar{x})$ is due to Kruger (see [21, Proposition 2.16] and [4, theorem 3.6]). Through that equality, criteria for the Fréchet subdifferential regularity of d_C have been provided in [4, Theorem 3.8]. For the equality concerning T_C we refer to [10, Theorem 3.1] when the space X is a Hilbert space and to [16, Theorem 4.2] when X is a Banach space. We also refer to Colombo, Goncharov and Mordukhovich [8, Proposition 6.3] for the use of such an equality for criteria ensuring the lower regularity of the minimal time function T_C. The inclusions with the Fréchet approximate subdifferentials of T_C at $\bar{x}$ have been previously established (even with a more accurate term $\theta(\varepsilon)$) in [26, Theorem 4.2].

5 Further Properties of the Proximal Subdifferential of the Infimum Time Function

Remind that the case when $\bar{x} \in \operatorname{dom} f$ has been studied in Theorem 3.3 and Corollary 3.7. Let us examine the case when $\bar{x} \notin \operatorname{cl}(\operatorname{dom} f)$. For $\bar{x}$ outside the closed set C, the relevance of Kruger's ideas (see [21]) in the proof of the equality

$$\partial^F d_C(\bar{x}) = N^F(C(r); \bar{x}) \cap \{x^* \in X^* : \|x^*\|_* = 1\}$$

has been made very apparent in [4, Theorem 4.3] for obtaining a similar formula for $\partial^P d_C(\bar{x})$. In the next theorem we follow the same path to establish a formula for $\partial^P T_f(\bar{x})$ similar to that of Theorem 4.3 when $\bar{x} \notin \operatorname{cl}(\operatorname{dom} f)$.

Theorem 5.1 *Assume that* $\bar{x} \notin \operatorname{cl}(\operatorname{dom} f)$ *and* $\operatorname{dom} f \subset \{T_f(\cdot) \le T_f(\bar{x})\}$. *Assume also that* f *is bounded from below. Then*

$$\partial^P T_f(\bar{x}) = N^P(\{T_f(\cdot) \le T_f(\bar{x})\}; \bar{x}) \cap \{x^* \in X^* : \sigma_G(x^*) = 1\}.$$

Proof Fix any $x^* \in \partial^P T_f(\bar{x})$. Theorem 4.1 tells us that $\sigma_G(x^*) = 1$. On the other hand, there exist some real $r > 0$ and some neighborhood U of $\bar{x}$ in X such that for all $x \in U$

$$\langle x^*, x - \bar{x}\rangle \le T_f(x) - T_f(\bar{x}) + r\|x - \bar{x}\|^2,$$

and this obviously yields for all $x \in U \cap \{T_f(\cdot) \le T_f(\bar{x})\}$ that

$$\langle x^*, x - \bar{x}\rangle \le r\|x - \bar{x}\|^2.$$

Consequently, $x^* \in N^P(\{T_f(\cdot) \le T_f(\bar{x})\}; \bar{x})$ and the inclusion of the left-hand side of the equality of the theorem into the second one is established.

Let us prove the opposite inclusion. Fix any $x^* \in N^P(\{T_f(\cdot) \le T_f(\bar{x})\}; \bar{x})$ with $\sigma_G(x^*) = 1$. Put $S := \{T_f(\cdot) \le T_f(\bar{x})\}$ and observe by Corollary 3.7 that $x^* \in \partial^P T_S(\bar{x})$. The inclusions $x^* \in N^P(S; \bar{x})$ and $x^* \in \partial^P T_S(\bar{x})$ allow us to choose some $r > 0$ and $\delta > 0$ such that

$$\langle x^*, x - \bar{x}\rangle \le r\|x - \bar{x}\|^2 \quad \text{for all } x \in B(\bar{x}, \delta) \cap S \tag{5.1}$$

and

$$\langle x^*, x - \bar{x}\rangle \le T_S(x) - T_S(\bar{x}) + r\|x - \bar{x}\|^2 \quad \text{for all } x \in B(\bar{x}, \delta). \tag{5.2}$$

Observing that $T_S(\bar{x}) = 0$ we deduce from (5.2) and Lemma 4.1 that for all $x \in B(\bar{x}, \delta) \setminus S$

$$\langle x^*, x - \bar{x}\rangle \le T_f(x) - T_f(\bar{x}) + r\|x - \bar{x}\|^2. \tag{5.3}$$

Take now any $\eta > 0$ and by the equality $\sigma_G(x^*) = 1$ choose some $u \in G$ such that $\langle x^*, u\rangle > 1 - \eta$. Consider any $x \in B(\bar{x}, \delta) \cap S$ and put

$$t_x := T_f(\bar{x}) - T_f(x) \ge 0.$$

Then, since $\rho_G(u) \le 1$, by Proposition 3.1 we have

$$T_f(x + t_x u) \le T_f(x) + t_x \rho_G(u) \le T_f(x) + t_x = T_f(\bar{x}),$$

which gives $x + t_x u \in S$. On the other hand, by (2.8) and Proposition 3.1 again

$$t_x = T_f(\bar{x}) - T_f(x) \le \rho_G(\bar{x} - x) \le \beta\|\bar{x} - x\|$$

and hence according to (2.8) again and to the inequality $\rho_G(u) \le 1$ we obtain

$$\|x + t_x u - \bar{x}\| \le \|x - \bar{x}\| + \alpha^{-1} t_x \le (1 + \alpha^{-1}\beta)\|\bar{x} - x\|. \tag{5.4}$$

Choose $\delta' > 0$ such that $\delta'(1+\alpha^{-1}\beta) < \delta$ and fix any $x \in B(\bar{x}, \delta') \cap S$. Then $x + t_x u \in S \cap B(\bar{x}, \delta)$ and by (5.1) and (5.4) we have

$$\langle x^*, x + t_x u - \bar{x}\rangle \le r\|x + t_x u - \bar{x}\|^2 \le r(1+\alpha^{-1}\beta)^2\|x-\bar{x}\|^2,$$

which gives

$$\langle x^*, x - \bar{x}\rangle \le -t_x\langle x^*, u\rangle + r(1+\alpha^{-1}\beta)^2\|x-\bar{x}\|^2,$$

and hence combining this with the inequality $\langle x^*, u\rangle \ge 1-\eta$ we see that

$$\langle x^*, x - \bar{x}\rangle \le (1-\eta)[T_f(x) - T_f(\bar{x})] + r(1+\alpha^{-1}\beta)^2\|x-\bar{x}\|^2.$$

This being true for all $\eta > 0$, we derive that

$$\langle x^*, x - \bar{x}\rangle \le T_f(x) - T_f(\bar{x}) + r(1+\alpha^{-1}\beta)^2\|x-\bar{x}\|^2.$$

Taking also (5.3) into account, we may write

$$\langle x^*, x - \bar{x}\rangle \le T_f(x) - T_f(\bar{x}) + r(1+\alpha^{-1}\beta)^2\|x-\bar{x}\|^2 \quad \text{for all } x \in B(\bar{x}, \delta').$$

Therefore, $x^* \in \partial^P T_f(\bar{x})$ and the proof of the theorem is complete. □

Corollary 5.1 *Let C be a closed subset of X and $\bar{x} \notin C$. Then*

$$\partial^P T_C(\bar{x}) = N^P(\{T_C(\cdot) \le T_C(\bar{x})\}; \bar{x}) \cap \{x^* \in X^* : \sigma_G(x^*) = 1\}.$$

If in addition $G = \mathbb{B}_X$, then for $r := d_C(\bar{x}) > 0$ and $C(r) := \{x \in X : d_C(x) \le r\}$ one has

$$\partial^P d_C(\bar{x}) = N^P(C(r); \bar{x}) \cap \{x^* \in X^* : \|x^*\|_* = 1\}.$$

Clarke, Stern and Wolenski [7, Theorem 3.4] established the above decription of $\partial^P d_C(\bar{x})$ when C is a closed set of a Hilbert space, with a proof based on the Hilbert structure of the space. This equality for $\partial^P d_C(\bar{x})$ has been extended to arbitrary normed vector spaces by Bounkhel and Thibault [4, Theorem 4.3]. The form above for $\partial^P T_C(\bar{x})$ has been given by Colombo and Wolenski [9, Theorem 3.1] in Hilbert space and by He and Ng in [16, Theorem 4.1] in Banach space.

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