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Reproductive Weak Solutions of Magneto-Micropolar Fluid Equations in Exterior Domains

M. DURÁN
Departamento de Ingeniería Matemática
Universidad de Chile
Casilla 170/3, Santiago 3, Chile
mduran@dim.uchile.cl

J. FERREIRA
Departamento de Matemática
UEM/DMA, Av. Colombo 5790
Zona 7, 87020-900, Maringá-PR, Brazil
jorgefe@gauss.dma.uem.br

M. A. ROJAS-MEDAR
Departamento de Matemática Aplicada
UNICAMP-IMECC, C.P. 6065
13081-970, Campinas, SP, Brazil
marko@ime.unicamp.br

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Abstract—We establish the existence of a reproductive weak solution, a so-called periodic weak solution, for the equations of motion of magneto-micropolar fluids in exterior domains in $\mathbb{R}^3$.

Keywords—Navier-Stokes equations, Magneto-micropolar fluid, Reproductive solution, Exterior domain.

1. INTRODUCTION

In several industrial applications, we deal with complicated physical phenomena which are at the root of some difficult, and sometimes strange, effects. Such is the case of the lubrication processes (e.g., see [1–3]), fluid flows in porous media (e.g., see [4]), or MHD generators with neutral fluid seedings in the form of rigid microinclusions (e.g., see [5,6]). To model them, we consider an incompressible, electrically conducting and micropolar fluid, a so-called magneto-micropolar fluid. Many works have been written in the last decades on this subject. Among others we mention Ahmadi and Shahinpoor [7], Eringen [8,9], Bessonov [10], Huilgol [11], and Straughan [12].

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The study of the dynamics of the magneto-micropolar fluid model considered on an exterior domain plays an important and useful role. We often find physical structures in which a bounded body, or obstacle, produces perturbations in the surrounding medium and the spatial volume of the external environment, namely the exterior domain, is extensively much larger than the obstacle. From the modelling point of view, the obstacle may be regarded as a compact domain located in all of $\mathbb{R}^3$. Let K denote this compact subset, and let Ω denote its complement in $\mathbb{R}^3$, that is, $\Omega = K^c$.

It is known that certain dynamical systems may not have periodic solutions because there exist many orbits, or branches of bifurcations, that can be randomly reached by the solution (e.g., see [13]). However, several of these systems are still of the *reproductive type*, in the sense that there exist at least two different times where the solution takes the same value.

We are interested in the study of the existence of reproductive weak solutions for the equations that describe the motion of a viscous incompressible magneto-micropolar fluid in the exterior domain Ω and in the time interval $[0,\infty)$. Such a mathematical model reads: find the three-dimensional fields $(\mathbf{u},\mathbf{w},\mathbf{h}) : \Omega \times (0,\infty) \to \mathbb{R}^9$ and the scalar functions $(p,q) : \Omega \times (0,\infty) \to \mathbb{R}^2$ which satisfy the system of equations

$$\begin{gathered}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - (\mu+\chi)\Delta\mathbf{u} + \nabla\left(p + \frac{1}{2}\mathbf{h}\cdot\mathbf{h}\right) = \chi \operatorname{rot}\mathbf{w} + r\mathbf{h}\cdot\nabla\mathbf{h} + \mathbf{f}, \\
j\,\frac{\partial \mathbf{w}}{\partial t} + j\mathbf{u}\cdot\nabla\mathbf{w} - \gamma\Delta\mathbf{w} + 2\chi\mathbf{w} - (\alpha+\beta)\nabla \operatorname{div}\mathbf{w} = \chi\operatorname{rot}\mathbf{u} + \mathbf{g}, \\
\frac{\partial \mathbf{h}}{\partial t} - \nu\Delta\mathbf{h} + \mathbf{u}\cdot\nabla\mathbf{h} - \mathbf{h}\cdot\nabla\mathbf{u} + \nabla q = \mathbf{0}, \\
\operatorname{div}\mathbf{u} = 0, \qquad \operatorname{div}\mathbf{h} = 0.
\end{gathered} \tag{1.1}$$

Here $\mathbf{u}(x,t), \mathbf{w}(x,t), \mathbf{h}(x,t) \in \mathbb{R}^3$ denote, respectively, the velocity, the microrotational velocity, and the magnetic field of the fluid at point $x \in \Omega$ and time $t \in (0,\infty)$, and $p(x,t), q(x,t) \in \mathbb{R}$ denote the hydrostatic and magnetic pressures at the same place. The values $\mu,\chi,r,\alpha,\beta,\gamma,j$ and ν are constants associated with properties of the material. For physical reasons, we suppose that these constants satisfy $\min\{\mu,\chi,r,j,\gamma,\nu,(\alpha+\beta),\gamma\} > 0$. The vector-valued functions $\mathbf{f}(x,t)$, $\mathbf{g}(x,t) \in \mathbb{R}^3$ are given external fields.

We assume that the following boundary and initial conditions hold:

$$\mathbf{u}(x,t) = \mathbf{w}(x,t) = \mathbf{h}(x,t) = \mathbf{0}, \qquad \text{a.e. in } \partial\Omega, \quad \forall\, t \in (0,\infty), \tag{1.2}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0(x),\ \mathbf{w}(x,0) = \mathbf{w}_0(x),\ \mathbf{h}(x,0) = \mathbf{h}_0(x), \qquad \text{a.e. in } \Omega. \tag{1.3}$$

To complete the system of equations, we prescribe the behaviour of the solutions at infinity. More precisely, we consider the classical homogeneous decay

$$\lim_{|x|\to\infty} \mathbf{u}(x,t) = \lim_{|x|\to\infty} \mathbf{w}(x,t) = \lim_{|x|\to\infty} \mathbf{h}(x,t) = 0, \qquad \forall\, t > 0. \tag{1.4}$$

It is important to remark that to treat (1.4) in our mathematical modelling, no weighted spaces are required. From a physical viewpoint, we note that the classical boundary value condition for the magnetic field, which reads

$$\mathbf{h}\cdot\mathbf{n} = 0 \quad \text{and} \quad \operatorname{rot}\mathbf{h} \wedge \mathbf{n} = \mathbf{0}, \qquad \text{on } \partial\Omega,$$

has been replaced by a homogeneous Dirichlet condition. Then we are considering a nonperfect conductor body K. The unknown function $q(x,t)$, the corresponding magnetic pressure, is concerned with the motion of heavy ions (e.g., see [14]),

$$\nabla q = \frac{-1}{\sigma}\operatorname{rot}\mathbf{j}_0,$$

where $\mathbf{j}_0$ is the density of electric current and $\sigma > 0$ is the constant electric conductivity.

Let $(\mathbf{u}, \mathbf{w}, \mathbf{h})$ be a weak solution of (1.1)–(1.4) (the exact definition will be given later on). Given $T > 0$, if there exists $(\mathbf{u}_0, \mathbf{w}_0, \mathbf{h}_0)$ such that

$$\mathbf{u}(x,T) = \mathbf{u}_0(x), \quad \mathbf{w}(x,T) = \mathbf{w}_0(x), \quad \mathbf{h}(x,T) = \mathbf{h}_0(x), \qquad \text{a.e. in } \Omega, \tag{1.5}$$

then, we call $(\mathbf{u}, \mathbf{w}, \mathbf{h})$ a *reproductive weak solution* of (1.1)–(1.4) at time T. We say that system (1.1)–(1.4) has the *reproductive property* if it is reproductive at every $T > 0$ (see Kaniel and Shinbrot [15] or Takeshita [16] for the case of the Navier-Stokes equations). We note that the above property is a generalization of the notion of periodicity. In this paper, our goal is to prove that system (1.1)–(1.4) is reproductive.

Equation $(1.1)_i$ has the familiar form of the Navier-Stokes equations but it is coupled with equation $(1.1)_{ii}$ and $(1.1)_{iii}$. Equation $(1.1)_{ii}$ describes the motion inside the macrovolumes as they undergo microrotational effects, which are represented by the microrotational velocity vector $\mathbf{w}$. For fluids with no microstructure, this velocity vanishes and we deal with a magnetohydrodynamics system. For Newtonian fluids, where $\chi = 0$, equation $(1.1)_i$ decouples from equation $(1.1)_{ii}$. Equation $(1.1)_{iii}$, which is the equation for $\mathbf{h}$, is the Maxwell system in which the electrical field is determined in *a posteriori* way. It is also important to note that if $\mathbf{h} = \mathbf{0}$, we consider the well-known asymmetric fluid model.

It is now appropriate to cite some earlier works on the initial boundary-value problem (1.1)–(1.3) on a bounded domain, which are related to ours, and also to locate our contribution therein. When the magnetic field is absent ($\mathbf{h} \equiv \mathbf{0}$), the reduced problem was studied by Lukaszewicz [17], Galdi and Rionero [18], Padula and Russo [19], and Conca *et al.* [20]. Lukaszewicz [17] established the global existence of weak solutions for (1.1)–(1.3) under certain assumptions by using linearization and an almost fixed-point theorem. In the same case, by using the same technique, Lukaszewicz [17] also proved the local and global existence, as well as the uniqueness of strong solutions. Again, when $\mathbf{h} \equiv \mathbf{0}$, Galdi and Rionero [18] established results similar to the ones of Lukaszewicz [17]. Finally, we can mention Padula and Russo [19], who studied the uniqueness of the solution in the unbounded domains case.

The full system (1.1)–(1.3) was studied by Galdi and Rionero [18], and they stated, without rigorous proof, results of existence and uniqueness of strong solutions. Rojas-Medar [21], also studied the system (1.1)–(1.3) and established existence and uniqueness of strong solutions by using the spectral Galerkin method, reaching the same level of knowledge as in the case of the classic Navier-Stokes equations. Ahmadi and Shahinpoor [7] studied the stability of solutions of the mentioned system. Boldrini and Rojas-Medar [22] proved existence of weak solutions as well as reproductive weak solutions for system (1.1)–(1.3) in a bounded domain.

In Section 2, we establish the basic mathematical framework to be used and rewrite (1.1)–(1.4) in a more suitable weak form. Also, we state Theorem 2.3, which is our main result. In Section 3, we use the "extending domain method" (see [23,24]) to approximate problem (1.1)–(1.4). Finally, Section 4 is devoted to provide the proof of Theorem 2.3.

2. FUNCTION SPACES AND PRELIMINAIRES

Throughout, the functions are either $\mathbb{R}$ or $\mathbb{R}^3$-valued and we will distinguish between these two situations in our notations. More precisely, the vector-valued functions in $\mathbb{R}^3$ are denoted by $\{\mathbf{u}, \mathbf{w}, \mathbf{h}\}$, while the scalar ones are simply written $\{p, q, u_i, v_j\}$.

We now give the precise definition of the exterior domain Ω where our boundary-valued problem, i.e., problem (1.1)–(1.4), has been formulated: let K be a nonvoid compact subset of $\mathbb{R}^3$ whose boundary ∂K is of class C^2. The exterior domain Ω that we consider is $\Omega = K^c$ and $\partial \Omega = \partial K$.

As we said before, to carry out the mathematical analysis we use the extending domain method. It was introduced by Ladyzhenkaya [23] to study the Navier-Stokes equations in unbounded domains. Before, this method was applied to study a large class of problem of exterior domain, as for example [25]. The main idea of this procedure is as follows: the exterior domain Ω may be approximated by bounded domains $\Omega_k = B_k \cap \Omega$, for every $k \geq 1$, with B_k the ball of radius k centered at the origin. In each interior domain Ω_k, we prove the existence of a reproductive weak solution. For this, we will apply the Galerkin method together with Leray-Schauder's fixed-point theorem (as in [26]). Next, using the estimates given in [23], together with a diagonal argument and Rellich's compactness theorem, we obtain the desirable reproductive weak solution to the original problem (1.1)–(1.4).

In the sequel, we specify several vector-valued function spaces, which are used in what follows. The domains Ω or Ω_k are denoted in a generic way by D. Also, X denotes any function space. Moreover, we consider $\hat{D} = D \times [0, T]$, where T is a strictly positive real constant. Then, we have

$$\begin{aligned}
W^{r,p}(D) &= \{\mathbf{u}; D^\alpha \mathbf{u} \in L^p(D),\ |\alpha| \leq r\},\\
W_0^{r,p}(D) &= \text{Closure of } C_0^\infty(D) \text{ in } W^{r,p}(D),\\
W_0(D) &= \text{Closure of } C_0^\infty(D) \text{ in norm } \|\nabla\phi\|,\\
C_{0,\sigma}^\infty(D) &= \{\varphi \in C_0^\infty(D);\ \operatorname{div}\varphi = 0 \text{ in } D\},\\
J(D) &= \text{Closure of } C_{0,\sigma}^\infty(D) \text{ in norm } \|\nabla\phi\|,\\
H(D) &= \text{Closure of } C_{0,\sigma}^\infty(D) \text{ in norm } \|\phi\|,\\
W_\pi\left(\hat{D}\right) &= \left\{\psi \in C_0^\infty\left(\hat{D}\right);\ \psi(x,T) = \psi(x,0) \text{ in } D\right\},\\
W_{\sigma,\pi}\left(\hat{D}\right) &= \left\{\psi \in C_{0,\sigma}^\infty\left(\hat{D}\right);\ \psi(x,T) = \psi(x,0) \text{ in } D\right\},\\
L_\pi^p(0,T;X) &= \{\mathbf{u} \in L^p(0,T;X);\ \mathbf{u}(x,T) = \mathbf{u}(x,0) \text{ in } D\}.
\end{aligned}$$

We denote by $\|\cdot\|_{L^p(D)}$ and $\|\cdot\|_{W^{r,p}(D)}$ the standard norms of the vector-valued Sobolev spaces $L^p(D)$ and $W^{r,p}(D)$, with $r \geq 0$, $1 \leq p \leq \infty$ (e.g., see [26]). As usual, $W^{r,2}(D) \equiv H^r(D)$, $r > 0$, and otherwise, $W^{0,2}(D) \equiv L^2(D)$. In this last case, the norm and the inner product are denoted solely by $\|\cdot\|$ and $(\cdot,\cdot)$, that is, without subscripts. As was proved by Heywood [25], when D is bounded or an exterior domain, we note that $J(D)$ is equivalent to the space

$$J_0(D) = \{\varphi \in W_0(D);\ \operatorname{div}\varphi = 0, \text{ a.e. in } D\}.$$

Also, it is clear that in the bounded case we have

$$J(\Omega_k) = \left\{\varphi \in H_0^1(\Omega_k);\ \operatorname{div}\varphi = 0, \text{ a.e. in } \Omega_k\right\}.$$

The following inequalities are used henceforth, their proofs can be found in [23].

LEMMA 2.1. *Let $D \subseteq \mathbb{R}^3$ be bounded or unbounded. Then we get the following.*

(a) *For $\mathbf{u} \in W_0(D)$ (or in $J(D)$), we have*

$$\|\mathbf{u}\|_{L^6(D)} \leq C_L \|\nabla \mathbf{u}\|,$$

where $C_L \leq (48)^{1/6}$.

(b) *(Hölder's Inequality.) If each integral makes sense, we have*

$$|((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w})| \leq 3^{1/p+1/r} \|\mathbf{u}\|_{L^p(D)} \|\nabla \mathbf{v}\|_{L^q(D)} \|\mathbf{w}\|_{L^r(D)},$$

where $p, q, r > 0$ and $1/p + 1/q + 1/r = 1$. ∎

LEMMA 2.2. *Suppose that D is a bounded domain in $\mathbb{R}^n$ and its boundary ∂D is of class C^2. Let us take an orthonormal basis $\{\omega^j\}_{j=1}^\infty$ of $L^2(D)$. Then, for any $\varepsilon > 0$, there exists a number N_ε, depending only on ε, such that*

$$\|\mathbf{u}\|^2 \leq \sum_{j=1}^{N_\varepsilon} \left(\mathbf{u}, \omega^j\right)^2 + \varepsilon \|\mathbf{u}\|_{W^{1,p}}^2, \qquad \forall\, \mathbf{u} \in W_0^{1,p}(D),$$

where $p > 2n/n + 2$, if $n \geq 2$, and otherwise $p \geq 1$, if $n = 1$. ∎

We now state our problem rigorously establishing regularity assumptions on the boundary $\partial\Omega$ and on the external forces.

($\mathbf{S}_1$) Let O_0 be a neighbourhood of the origin. Let $O_0 \subsetneq \operatorname{int} K$ and $K \subsetneq B_R$, $R > 0$;
($\mathbf{S}_2$) $\partial\Omega = \partial K \in C^2$;
($\mathbf{S}_3$) $\mathbf{f} \in L^2(0,T;J(\Omega)^*)$, $\mathbf{g} \in L^2(0,T;W_0(\Omega)^*)$, where $J(\Omega)^*$ (respectively, $W_0(\Omega)^*$) is the topological dual of $J(\Omega)$ (respectively, $W_0(\Omega)$).

We denote the classical bilinear and trilinear forms by

$$a(\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^{3} \int_D \frac{\partial v_j}{\partial x_i}\frac{\partial w_j}{\partial x_i}\,dx, \qquad b(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^{3} \int_D u_j \frac{\partial v_i}{\partial x_j}\,w_i\,dx,$$

which are defined for all vector-valued functions $\mathbf{u},\mathbf{v},\mathbf{w}$, for which the integrals make sense.

Now, we can define precisely the notion of a reproductive weak solution for the whole system (1.1)–(1.4).

DEFINITION. *Let $T > 0$. We say that the triple of functions $(\mathbf{u},\mathbf{w},\mathbf{h})$, defined on $\Omega \times (0,T)$, is a reproductive weak solution of (1.1)–(1.4) at time T if and only if there exist $\mathbf{u}_0, \mathbf{h}_0 \in H(\Omega)$ and $\mathbf{w}_0 \in L^2(\Omega)$ such that*

(i) $\mathbf{u}(x,0) = \mathbf{u}_0(x)$, $\mathbf{w}(x,0) = \mathbf{w}_0(x)$, $\mathbf{h}(x,0) = \mathbf{h}_0(x)$, *a.e. in* Ω,
(ii) $\mathbf{u},\mathbf{h} \in L^2(0,T;J(\Omega)) \cap L^\infty_\pi(0,T;H(\Omega))$,
(iii) $\mathbf{w} \in L^2(0,T;W_0(\Omega)) \cap L^\infty_\pi(0,T;L^2(\Omega))$,
(iv) $\mathbf{u},\mathbf{w}$, *and* $\mathbf{h}$ *satisfy the variational equations*

$$\begin{aligned}\int_0^T (\mathbf{u},\varphi_t) + (\mu+\chi)\,a(\mathbf{u},\varphi) + b(\mathbf{u},\varphi,\mathbf{u}) - r\,b(\mathbf{h},\varphi,\mathbf{h}) \\ -(\mathbf{f},\varphi) - \chi\,(\mathbf{w},\operatorname{rot}\varphi)]\,dt = 0,\end{aligned} \tag{2.10}$$

$$\begin{aligned}\int_0^T [j\,(\mathbf{w},\phi_t) + \gamma\,a(\mathbf{w},\phi) + (\alpha+\beta)\,(\operatorname{div}\mathbf{w},\operatorname{div}\phi) + 2\,\chi\,(\mathbf{w},\phi) \\ +j\,b(\mathbf{u},\phi,\mathbf{w}) - (\mathbf{g},\phi) - \chi\,(\mathbf{u},\operatorname{rot}\phi)]\,dt = 0,\end{aligned} \tag{2.11}$$

$$\int_0^T [(\mathbf{h},\psi_t) + \nu\,a(\mathbf{h},\psi) + b(\mathbf{u},\psi,\mathbf{h}) - b(\mathbf{h},\psi,\mathbf{u})]\,dt = 0, \tag{2.12}$$

for all $\varphi,\psi \in W_{\sigma,\pi}(\hat\Omega)$ *and* $\phi \in W_\pi(\hat\Omega)$. ∎

REMARK. It is important to note.

(i) As $\mathbf{u}(\cdot,t), \mathbf{h}(\cdot,t) \in J(\Omega)$ and $\mathbf{w}(\cdot,t) \in W_0(\Omega)$, a.e. in $(0,T)$, we have

$$\mathbf{u}|_{\partial\Omega} = \mathbf{h}|_{\partial\Omega} = \mathbf{w}|_{\partial\Omega} = \mathbf{0}, \qquad \text{a.e. in } (0,T).$$

(ii) By Part (a) of Lemma 2.1

$$\lim_{|x|\to\infty} \mathbf{u}(x,t) = \lim_{|x|\to\infty} \mathbf{w}(x,t) = \lim_{|x|\to\infty} \mathbf{h}(x,t) = \mathbf{0}, \qquad \text{a.e. in } (0,T).$$

(iii) We also see that the pressures are recovered by a standard application of De Rham's theorem. ∎

Finally, we state our main result, which is proved in what follows.

THEOREM 2.3. EXISTENCE OF SOLUTION. *Under Hypotheses* ($\mathbf{S}_1$)–($\mathbf{S}_3$) *problem (1.1)–(1.4) admits at least one reproductive weak solution.* ∎

3. THE INTERIOR PROBLEM

Since (1.1)–(1.4) has been formulated in a variational form, it is natural to use the "extending domain method" (see [23,24]). It consists in building a sequence of functions that converges, in a suitable topological sense, toward the solution of the original exterior problem (1.1)–(1.4). More precisely, we consider the following family of differential problems $\{(P_k)\}_{k\in\mathbb{N}}$, defined on bounded domains $\Omega_k = B_k \cap \Omega$: Find the three-dimensional fields $(\mathbf{v},\mathbf{z},\mathbf{b}) : \Omega_k \times (0,T) \to \mathbb{R}^9$ and the scalar functions $(p,q) : \Omega_k \times (0,T) \to \mathbb{R}^2$ which satisfy the system of equations

$$\begin{gathered}
\mathbf{v}_t - (\mu+\chi)\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\left(p + \frac{r}{2}\mathbf{b}\cdot\mathbf{b}\right) = \chi\,\mathrm{rot}\,\mathbf{z} + r(\mathbf{b}\cdot\nabla)\mathbf{b} + \mathbf{f},\\
\mathbf{z}_t - \gamma\Delta\mathbf{z} - (\alpha+\beta)\nabla\,\mathrm{div}\,\mathbf{z} + j(\mathbf{v}\cdot\nabla)\mathbf{z} + 2\chi\mathbf{z} = \chi\,\mathrm{rot}\,\mathbf{v} + \mathbf{g},\\
\mathbf{b}_t - \nu\Delta\mathbf{b} + (\mathbf{v}\cdot\nabla)\mathbf{b} - (\mathbf{b}\cdot\nabla)\mathbf{v} + \nabla q = \mathbf{0},\\
\mathrm{div}\,\mathbf{v} = 0, \qquad \mathrm{div}\,\mathbf{b} = 0,\\
\mathbf{v} = \mathbf{0}, \quad \mathbf{z} = \mathbf{0}, \quad \mathbf{b} = \mathbf{0}, \qquad \text{on } \partial\Omega_k,\\
\mathbf{v}(\cdot,T) = \mathbf{v}(\cdot,0), \qquad \mathbf{z}(\cdot,T) = \mathbf{z}(\cdot,0), \qquad \mathbf{b}(\cdot,T) = \mathbf{b}(\cdot,0),
\end{gathered} \tag{P$_k$}$$

where $\mathbf{v}_t$ denotes the time derivative and $\partial\Omega_k = \partial\Omega \cup \partial B_k$. It is straightforward to see that the sense of reproductive weak solution for (P_k), also called approximated reproductive solution, is completely similar to the one for (1.1)–(1.4). In fact, we have the following.

DEFINITION. *We say that a triplet of functions* $(\mathbf{v},\mathbf{z},\mathbf{b})$ *defined on* $\Omega_k \times (0,T)$ *is a reproductive weak solution of* (P_k) *at time* T *if only if there exist* $\mathbf{v}_0, \mathbf{b}_0 \in H(\Omega_k)$ *and* $\mathbf{z}_0 \in L^2(\Omega_k)$ *such that*

(i) $\mathbf{v}(x,0) = \mathbf{v}_0(x)$, $\mathbf{z}(x,0) = \mathbf{z}_0(x)$, $\mathbf{b}(x,0) = \mathbf{b}_0(x)$, a.e. *in* Ω_k,
(ii) $\mathbf{v},\mathbf{b} \in L^2(0,T;J(\Omega_k)) \cap L^\infty_\pi(0,T;H(\Omega_k))$,
(iii) $\mathbf{z} \in L^2(0,T;H_0^1(\Omega_k)) \cap L^\infty_\pi(0,T;L^2(\Omega_k))$,
(iv) $\mathbf{v}$, $\mathbf{z}$, *and* $\mathbf{b}$ *satisfy the variational equations*

$$\int_0^T \{(\mathbf{v},\varphi_t) + (\mu+\chi)(\nabla\mathbf{v},\nabla\varphi) + b(\mathbf{v},\varphi,\mathbf{v}) - rb(\mathbf{b},\varphi,\mathbf{b}) - \chi(\mathbf{z},rot\,\varphi) - (\mathbf{f},\varphi)\}\,dt = 0,$$

$$\begin{aligned}\int_0^T \{(\mathbf{z},\phi_t) + \gamma(\nabla\mathbf{z},\nabla\phi) + (\alpha+\beta)(\mathrm{div}\,\mathbf{z},\mathrm{div}\,\phi) + jb(\mathbf{v},\phi,\mathbf{z}) + 2\chi(\mathbf{z},\phi)&\\
- \chi(\mathbf{v},rot\,\phi) - (\mathbf{g},\phi)\}\,dt &= 0,\end{aligned}$$

$$\int_0^T \{(\mathbf{b},\psi_t) + \nu(\nabla\mathbf{b},\nabla\psi) + b(\mathbf{v},\psi,\mathbf{b}) - b(\mathbf{b},\psi,\mathbf{v})\}\,dt = 0,$$

for all $\varphi,\psi \in W_{\sigma,\pi}(\hat\Omega_k)$ *and* $\phi \in W_\pi(\hat\Omega_k)$. ∎

PROPOSITION 3.1. EXISTENCE OF APPROXIMATED REPRODUCTIVE SOLUTION. *Under Hypotheses* (S_1)–(S_3) *problem* (P_k) *admits at least one reproductive weak solution, denoted by* $(\mathbf{v}^k,\mathbf{z}^k,\mathbf{b}^k)$. ∎

The last part of this Section 3 is devoted to establish the proof of Proposition 3.1. We do that using the Galerkin method together with Leray-Schauder's Homotopy Theorem as in [24,26]. We begin by proving a useful *a priori* estimate and by establishing a finite-dimensional iterative scheme to approximate the functions $(\mathbf{v}^k,\mathbf{z}^k,\mathbf{b}^k)$. More precisely, we look for a sequence $(\mathbf{v}^k_m,\mathbf{z}^k_m,\mathbf{b}^k_m)$ defined on a finite $3m$-dimensional space such that for $m \longrightarrow +\infty$ we have

$$\mathbf{v}^k_m \longrightarrow \mathbf{v}^k \quad \text{strong in } L^2(0,T;H(\Omega_k)) \text{ and weak} - * \text{ in } L^\infty_\pi(0,T;H(\Omega_k)), \tag{3.1}$$

$$\mathbf{z}^k_m \longrightarrow \mathbf{z}^k \quad \text{strong in } L^2\left(0,T;L^2(\Omega_k)\right) \text{ and weak} - * \text{in } L^\infty_\pi\left(0,T;L^2(\Omega_k)\right), \tag{3.2}$$

$$\mathbf{b}^k_m \longrightarrow \mathbf{b}^k \quad \text{strong in } L^2(0,T;H(\Omega_k)) \text{ and weak} - * \text{ in } L^\infty_\pi(0,T;H(\Omega_k)). \tag{3.3}$$

As we shall see, the mth-approximating reproductive sequence $\{(\mathbf{v}^k_m,\mathbf{z}^k_m,\mathbf{b}^k_m)\}_{m\geq 1}$ is nothing but the reproductive weak solution of (P_k) restricted on finite-dimensional functional spaces. This fact depends in a crucial way on the following *a priori* estimate.

LEMMA 3.2. *Let $(\mathbf{v}^k, \mathbf{z}^k, \mathbf{b}^k)$ be a weak solution of (P_k). Then it satisfies the estimate*

$$\begin{aligned}\frac{d}{dt}\left(\left\|\mathbf{v}^k\right\|^2 + j\left\|\mathbf{z}^k\right\|^2 + r\left\|\mathbf{b}^k\right\|^2\right) + \mu\, a\left(\mathbf{v}^k, \mathbf{v}^k\right) + \gamma\, a\left(\mathbf{z}^k, \mathbf{z}^k\right) \\ +2r\nu\, a\left(\mathbf{b}^k, \mathbf{b}^k\right) \leq \frac{1}{\mu}\|\mathbf{f}\|^2_{J(\Omega)^*} + \frac{1}{\gamma}\|\mathbf{g}\|^2_{W_0(\Omega)^*}.\end{aligned} \tag{3.4}$$

PROOF. Multiplying $(\mathrm{P}_k)_i$, $(\mathrm{P}_k)_{ii}$, and $(\mathrm{P}_k)_{iii}$ by $\mathbf{v}^k, \mathbf{z}^k$, and $r\mathbf{b}^k$, respectively, and integrating by parts on Ω_k, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|\mathbf{v}^k\right\|^2 + (\mu+\chi)\, a\left(\mathbf{v}^k, \mathbf{v}^k\right) - rb\left(\mathbf{b}^k, \mathbf{b}^k, \mathbf{v}^k\right) = \chi\left(\operatorname{rot}\mathbf{z}^k, \mathbf{v}^k\right) + \left(\mathbf{f}, \mathbf{v}^k\right),$$

$$\frac{1}{2}\frac{d}{dt} j\left\|\mathbf{z}^k\right\|^2 + \gamma\, a\left(\mathbf{z}^k, \mathbf{z}^k\right) + (\alpha+\beta)\left\|\operatorname{div}\mathbf{z}^k\right\|^2 + 2\chi\left\|\mathbf{z}^k\right\|^2 = \chi\left(\operatorname{rot}\mathbf{v}^k, \mathbf{z}^k\right) + \left(\mathbf{g}, \mathbf{z}^k\right),$$

$$\frac{1}{2}\frac{d}{dt} r\left\|\mathbf{b}^k\right\|^2 + r\nu\, a\left(\mathbf{b}^k, \mathbf{b}^k\right) = rb\left(\mathbf{b}^k, \mathbf{v}^k, \mathbf{b}^k\right).$$

Adding these equalities and observing that $b(\mathbf{b}^k, \mathbf{b}^k, \mathbf{v}^k) + b(\mathbf{b}^k, \mathbf{v}^k, \mathbf{b}^k) = 0$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{v}^k\|^2 + j\|\mathbf{z}^k\|^2 + r\|\mathbf{b}^k\|^2\right) + (\mu+\chi)\, a\left(\mathbf{v}^k, \mathbf{v}^k\right) + \gamma\, a\left(\mathbf{z}^k, \mathbf{z}^k\right) + r\nu\, a\left(\mathbf{b}^k, \mathbf{b}^k\right) \\ +(\alpha+\beta)\left\|\operatorname{div}\mathbf{z}^k\right\|^2 + 2\chi\left\|\mathbf{z}^k\right\|^2 = 2\chi\left(\mathbf{z}^k, \operatorname{rot}\mathbf{v}^k\right) + \left(\mathbf{f}, \mathbf{v}^k\right) + \left(\mathbf{g}, \mathbf{z}^k\right).\end{aligned} \tag{3.5}$$

Next, we estimate the right-hand side of (3.5). Using $\|\operatorname{rot}\mathbf{v}\| = \|\nabla\mathbf{v}\|$ and Young's inequality we can deduce

$$2\chi\left(\mathbf{z}^k, \operatorname{rot}\mathbf{v}^k\right) \leq 2\chi\left\|\mathbf{z}^k\right\|\left\|\operatorname{rot}\mathbf{v}^k\right\| \leq 2\chi\left\|\mathbf{z}^k\right\|\left\|\nabla\mathbf{v}^k\right\| \leq \chi\left\|\mathbf{z}^k\right\|^2 + \chi a\left(\mathbf{v}^k, \mathbf{v}^k\right),$$

$$\left(\mathbf{f}, \mathbf{v}^k\right) \leq \|\mathbf{f}\|_{J(\Omega_k)^*}\left\|\nabla\mathbf{v}^k\right\| \leq \frac{1}{2\mu}\|\mathbf{f}\|^2_{J(\Omega)^*} + \frac{\mu}{2}\, a\left(\mathbf{v}^k, \mathbf{v}^k\right),$$

$$\left(\mathbf{g}, \mathbf{z}^k\right) \leq \|\mathbf{g}\|_{H^{-1}(\Omega_k)}\left\|\nabla\mathbf{z}^k\right\| \leq \frac{1}{2\gamma}\|\mathbf{g}\|^2_{W_0(\Omega)^*} + \frac{\gamma}{2}\, a\left(\mathbf{z}^k, \mathbf{z}^k\right).$$

Substituting the above estimates into (3.5) and neglecting some positive terms, we get the desired estimate (3.4), which conclude the proof. ∎

REMARK. The right-hand side of (3.4) does not depend on integer k.

Let $k \in \mathbb{N}$ be fixed and let $\{\varphi^i\}_{i=1}^\infty \subset C^\infty_{0,\sigma}(\Omega_k)$ and $\{\phi^i\}_{i=1}^\infty \subset C^\infty_0(\Omega_k)$ be orthonormal bases in $L^2(\Omega_k)$ and total bases in $J(\Omega_k)$ and $H^1_0(\Omega_k)$, respectively. As m^{th}-approximated reproductive solution of equation (P_k) we choose

$$\mathbf{v}^k_m(x,t) = \sum_{i=1}^m c_{mi}(t)\varphi^i(x), \tag{3.6}$$

$$\mathbf{z}^k_m(x,t) = \sum_{i=1}^m d_{mi}(t)\phi^i(x), \tag{3.7}$$

$$\mathbf{b}^k_m(x,t) = \sum_{i=1}^m e_{mi}(t)\varphi^i(x), \tag{3.8}$$

satisfying $\forall\, i = 1, \ldots, m$, $\forall\, t \in (0,T)$ the system of equations

$$\begin{aligned}\left(\mathbf{v}^k_{m,t}, \varphi^i\right) + (\mu+\chi)\, a\left(\mathbf{v}^k_m, \varphi^i\right) + b\left(\mathbf{v}^k_m, \mathbf{v}^k_m, \varphi^i\right) \\ - r\, b\left(\mathbf{b}^k_m, \mathbf{b}^k_m, \varphi^i\right) = \chi\left(\operatorname{rot}\mathbf{z}^k_m, \varphi^i\right) + \left(\mathbf{f}, \varphi^i\right),\end{aligned} \tag{3.9}$$

$$\begin{aligned}j\left(\mathbf{z}^k_{m,t}, \phi^i\right) + \gamma\, a\left(\mathbf{z}^k_m, \phi^i\right) + (\alpha+\beta)\left(\operatorname{div}\mathbf{z}^k_m, \operatorname{div}\phi^i\right) + jb\left(\mathbf{v}^k_m, \mathbf{z}^k_m, \phi^i\right) \\ + 2\chi\left(\mathbf{v}^k_m, \phi^i\right) = \chi\left(\operatorname{rot}\mathbf{v}^k_m, \phi^i\right) + \left(\mathbf{g}, \phi^i\right),\end{aligned} \tag{3.10}$$

$$\left(\mathbf{b}^k_{m,t}, \varphi^i\right) + \nu\, a\left(\mathbf{b}^k_m, \varphi^i\right) + b\left(\mathbf{v}^k_m, \mathbf{b}^k_m, \varphi^i\right) - b\left(\mathbf{b}^k_m, \mathbf{v}^k_m, \varphi^i\right) = 0. \tag{3.11}$$

Note that if $(\mathbf{v}^k_m, \mathbf{z}^k_m, \mathbf{b}^k_m)$ satisfy (3.9)–(3.11) then these functions verify the inequality (3.4). In fact, it is straightforward to prove the following result.

COROLLARY 3.3. *Let $(\mathbf{v}_m^k, \mathbf{z}_m^k, \mathbf{b}_m^k)$ be a solution of (3.9)–(3.11). Then it satisfies the estimate*

$$\begin{aligned} &\frac{d}{dt}\left(\left\|\mathbf{v}_m^k\right\|^2 + j\left\|\mathbf{z}_m^k\right\|^2 + r\left\|\mathbf{b}_m^k\right\|^2\right) + \mu\, a\left(\mathbf{v}_m^k, \mathbf{v}_m^k\right) + \gamma\, a\left(\mathbf{z}_m^k, \mathbf{z}_m^k\right) \\ &\qquad + 2r\nu\, a\left(\mathbf{b}_m^k, \mathbf{b}_m^k\right) \le \frac{1}{\mu}\|\mathbf{f}\|_{J(\Omega)^*}^2 + \frac{1}{\gamma}\|\mathbf{g}\|_{W_0(\Omega)^*}^2. \end{aligned} \tag{3.12} \blacksquare$$

To simplify notation, we keep some abbreviations

$$\theta_m^k = \left(\mathbf{v}_m^k, \mathbf{z}_m^k, \mathbf{b}_m^k\right), \tag{3.13}$$

$$\left\|\theta_m^k(t)\right\|^2 = \left\|\mathbf{v}_m^k(t)\right\|^2 + j\left\|\mathbf{z}_m^k(t)\right\|^2 + r\left\|\mathbf{b}_m^k(t)\right\|^2. \tag{3.14}$$

$$\text{Let } d_k \text{ denote the diameter of the domain } \Omega_k. \tag{3.15}$$

$$\text{Let } \lambda_k \text{ denote the positive real constant } \frac{2}{d_k^2}\min\{\mu, \gamma, 2r\nu\}. \tag{3.16}$$

Then, we have the following estimate.

LEMMA 3.4. *Let $(\mathbf{v}_m^k, \mathbf{z}_m^k, \mathbf{b}_m^k)$ be a solution of (3.9)–(3.11). Then it satisfies the estimate*

$$e^{\lambda_k T}\left\|\theta_m^k(T)\right\|^2 \le \left\|\theta_m^k(0)\right\|^2 + \int_0^T e^{\lambda_k t}\left(\|\mathbf{f}(t)\|_{J(\Omega)^*}^2 + \|\mathbf{g}(t)\|_{W_0(\Omega)^*}^2\right) dt. \tag{3.17}$$

PROOF. By using Poincaré's inequality, (3.15) and (3.16) in (3.12), we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\left\|\mathbf{v}_m^k\right\|^2 + j\left\|\mathbf{z}_m^k\right\|^2 + r\left\|\mathbf{b}_m^k\right\|^2\right) + \lambda_k\left(\left\|\mathbf{v}_m^k\right\|^2 + \left\|\mathbf{z}_m^k\right\|^2 + \left\|\mathbf{b}_m^k\right\|^2\right) \\ &\qquad \le \frac{1}{\mu}\|\mathbf{f}\|_{J(\Omega)^*}^2 + \frac{1}{\gamma}\|\mathbf{g}\|_{W_0(\Omega_k)^*}^2, \end{aligned} \tag{3.18}$$

or equivalently, by adopting (3.13) and (3.14),

$$\frac{d}{dt}\left(e^{\lambda_k t}\left\|\theta_m^k(T)\right\|^2\right) \le \frac{e^{\lambda_k t}}{\mu}\|\mathbf{f}\|_{J(\Omega)^*}^2 + \frac{e^{\lambda_k t}}{\gamma}\|\mathbf{g}\|_{W_0(\Omega)^*}^2.$$

Integrating from 0 to T, we obtain

$$e^{\lambda_k T}\left\|\theta_m^k(T)\right\|^2 \le \left\|\theta_m^k(0)\right\|^2 + \int_0^T e^{\lambda_k t}\left(\|\mathbf{f}(t)\|_{J(\Omega)^*}^2 + \|\mathbf{g}(t)\|_{W_0(\Omega)^*}^2\right) dt,$$

which is (3.17). ∎

In the sequel, we show that $(\mathbf{v}_m^k, \mathbf{z}_m^k, \mathbf{b}_m^k)$ is a fixed point of the operator Φ^m defined below. Let L^m be the mapping defined by

$$L^m : [0, T] \to \mathbb{R}^{3m}, \tag{3.19}$$

$$t \to L^m(t) = \mathbf{y}(t), \tag{3.20}$$

where $\mathbf{y}(t)$ is computed by

$$\mathbf{y}(t) = \left(c_{m1}(t), \dots, c_{mm}(t), \sqrt{j}\, d_{m1}(t), \dots, \sqrt{j}\, d_{mm}(t), \sqrt{r}\, e_{m1}(t), \dots, \sqrt{r}\, e_{mm}(t)\right),$$

and where the time dependent functions $\{(c_{mi}(t), d_{mi}(t), e_{mi}(t))\}_{i=1}^m$ are the coefficients of the expansion of $(\mathbf{v}_m^k, \mathbf{z}_m^k, \mathbf{b}_m^k)$, as done in (3.6)–(3.8).

Since we have chosen the bases $\{\varphi^i(x)\}_{i=1}^{\infty}$ and $\{\phi^i(x)\}_{i=1}^{\infty}$ orthonormal in $L^2(\Omega_k)$, we have

$$\|\mathbf{y}(t)\|_{R^{3m}} = \left\|\theta_m^k(t)\right\|, \qquad \forall t \in [0,T]. \tag{3.21}$$

Next, we define the operator Φ^m as follows:

$$\Phi^m : \mathbb{R}^{3m} \to \mathbb{R}^{3m}, \tag{3.22}$$

$$\mathbf{x} \to \Phi^m(\mathbf{x}) = \mathbf{y}(T), \tag{3.23}$$

where $\mathbf{x} = (x_i)_{i=1}^{3m}$ and $\mathbf{y}(T) = L^m(T)$ is the vector of coefficients at time T of the solution of (3.9)–(3.11) with initial condition given by

$$\mathbf{v}_0^k(x) = \sum_{i=1}^{m} x_i\, \varphi^i(x), \tag{3.24}$$

$$\mathbf{z}_0^k(x) = \sum_{i=1}^{m} x_{m+i}\, \phi^i(x), \tag{3.25}$$

$$\mathbf{b}_0^k(x) = \sum_{i=1}^{m} x_{2m+i}\, \varphi^i(x). \tag{3.26}$$

It is not difficult to see that Φ^m is continuous, and we claim that Φ^m has at least one fixed point. This will be a consequence of Leray-Schauder's homotopy theorem. To prove this, it is enough to show that for any $\lambda \in [0,1]$, a solution of the equation

$$\lambda\Phi^m(\mathbf{x}(\lambda)) = \mathbf{x}(\lambda), \tag{3.27}$$

has a bound independent of λ. Since $\mathbf{x}(0) = 0$, we restrict the proof to $\lambda \in (0,1]$. In such case (3.27) may be rewritten

$$\Phi^m(\mathbf{x}(\lambda)) = \frac{1}{\lambda}\mathbf{x}(\lambda). \tag{3.28}$$

By the definition of Φ^m and equality (3.21), we deduce from (3.17)

$$e^{\lambda_k T}\left\|\frac{1}{\lambda}\mathbf{x}(\lambda)\right\|_{\mathbb{R}^{3m}}^2 \le \|\mathbf{x}(\lambda)\|_{\mathbb{R}^{3m}}^2 + \int_0^T e^{\lambda_k t}\left(\|\mathbf{f}(t)\|_{J(\Omega)^*}^2 + \|\mathbf{g}(t)\|_{W_0(\Omega)^*}^2\right) dt.$$

Since we impose the condition $\theta_m^k(0) = \theta_m^k(T)$, we obtain

$$\|\mathbf{x}(\lambda)\|_{\mathbb{R}^{3m}}^2 \le \frac{1}{e^{\lambda_k T} - 1}\int_0^T e^{\lambda_k t}\left(\|\mathbf{f}(t)\|_{J(\Omega)^*}^2 + \|\mathbf{g}(t)\|_{W_0(\Omega)^*}^2\right) dt \equiv M, \tag{3.29}$$

$\forall\, \lambda \in (0,1]$. Obviously, this upper bound does not depend on $\lambda \in [0,1]$ and so we have established that the operator Φ^m has at least one fixed point, denoted by $\mathbf{x}(1)$. Also, we remark that $\mathbf{x}(1)$ must satisfy (3.29). In other words, we have proved the following existence result.

Lemma 3.5. Existence of the m^{th}-Approximating Reproductive Solution. *Under Hypotheses* (S_1)–(S_3) *problem (3.9)–(3.11) admits at least one reproductive weak solution,* $(\mathbf{v}_m^k, \mathbf{z}_m^k, \mathbf{b}_m^k)$, $\forall\, m \ge 1$. *Furthermore, this sequence of functions satisfy*

(i) $(\mathbf{v}_m^k, \mathbf{z}_m^k, \mathbf{b}_m^k)$ *is uniformly bounded in* $L^2(0,T;J(\Omega_k)) \times L^2(0,T;H_0^1(\Omega_k)) \times L^2(0,T;J(\Omega_k))$,

(ii) $(\mathbf{v}_m^k, \mathbf{z}_m^k, \mathbf{b}_m^k)$ *is uniformly bounded in* $L_\pi^\infty(0,T;H(\Omega_k) \times L_\pi^\infty(0,T;L^2(\Omega_k)) \times L_\pi^\infty(0,T;H(\Omega_k))$.

PROOF. First, we note that (3.29) gives

$$\left\|\mathbf{v}_m^k(0)\right\| + j\left\|\mathbf{z}_m^k(0)\right\|^2 + \left\|\mathbf{b}_m^k(0)\right\|^2 \leq M.$$

So, by integrating (3.12) from 0 to $t \leq T$ and considering the above estimation, we can deduce for all $m \geq 1$ the following uniform boundness:

$$\operatorname*{sup\,ess}_{t\in[0,T]} \left\|\mathbf{v}_m^k(t)\right\|^2 + j\left\|\mathbf{z}_m^k(t)\right\|^2 + r\left\|\mathbf{b}_m^k(t)\right\|^2 \leq M(\mathbf{f},\mathbf{g}) + M, \tag{3.30}$$

$$C_0 \int_0^T \left(a\left(\mathbf{v}_m^k, \mathbf{v}_m^k\right) + a\left(\mathbf{z}_m^k, \mathbf{z}_m^k\right) + a\left(\mathbf{b}_m^k, \mathbf{b}_m^k\right)\right) ds \leq M(\mathbf{f},\mathbf{g}) + M, \tag{3.31}$$

where $C_0 = \min\{\mu, \gamma, 2r\nu\}$ and

$$M(\mathbf{f},\mathbf{g}) \equiv \int_0^T \left(\frac{1}{\mu}\|\mathbf{f}(t)\|^2_{J(\Omega)^*} + \frac{1}{\gamma}\|\mathbf{g}(t)\|^2_{W_0(\Omega)^*}\right) dt.$$

which are none other than (i) and (ii). ∎

From this lemma, we can directly deduce the following convergence property of the sequence $\{(\mathbf{v}_m^k, \mathbf{z}_m^k, \mathbf{b}_m^k)\}_{m\geq 1}$.

COROLLARY 3.6. CONVERGENCE OF m^{th}-APPROXIMATING REPRODUCTIVE SEQUENCE. *There exist* $\mathbf{v}^k, \mathbf{b}^k \in L^2(0,T;J(\Omega_k)) \cap L^\infty_\pi(0,T;H(\Omega_k))$ *and* $\mathbf{z}^k \in L^2(0,T;H_0^1(\Omega_k)) \cap L^\infty_\pi(0,T;L^2(\Omega_k))$ *such that*

$$\begin{aligned}
\mathbf{v}_m^k &\longrightarrow \mathbf{v}^k \textit{ strong in } L^2(0,T;H(\Omega_k)) \textit{ and weak} - *\textit{in } L^\infty_\pi(0,T;H(\Omega_k)),\\
\mathbf{z}_m^k &\longrightarrow \mathbf{z}^k \textit{ strong in } L^2\left(0,T;L^2(\Omega_k)\right) \textit{ and weak} - * \textit{ in } L^\infty_\pi\left(0,T;L^2(\Omega_k)\right),\\
\mathbf{b}_m^k &\longrightarrow \mathbf{b}^k \textit{ strong in } L^2(0,T;H(\Omega_k)) \textit{ and weak} - * \textit{ in } L^\infty_\pi(0,T;H(\Omega_k)).
\end{aligned}$$

PROOF. Since $J(\Omega_k)$ (respectively, $H_0^1(\Omega_k)$) is compactly embedded in $H(\Omega_k)$ (respectively, $L^2(\Omega_k)$), we may choose a subsequence, which we again denote by $(\mathbf{v}_m^k, \mathbf{z}_m^k, \mathbf{b}_m^k)$, such that

$$\left.\begin{aligned}\mathbf{v}_m^k &\to \mathbf{v}^k\\ \mathbf{b}_m^k &\to \mathbf{b}^k\end{aligned}\right\} \text{ weak in } L^2(0,T;J(\Omega_k)) \text{ and weak} - * \text{ in } L^\infty_\pi(0,T;H(\Omega_k)),$$

$$\mathbf{z}_m^k \to \mathbf{z}^k \text{ weak in } L^2\left(0,T;H_0^1(\Omega_k)\right) \text{ and weak} - * \text{ in } L^\infty_\pi\left(0,T;L^2(\Omega_k)\right).$$

Furthermore, combining Lemma 2.2 (with $n = 3, p = 2$) and (3.31) we see that

$$\left.\begin{aligned}\mathbf{v}_m^k &\to \mathbf{v}^k\\ \mathbf{b}_m^k &\to \mathbf{b}^k\end{aligned}\right\} \text{ strong in } L^2(0,T;H(\Omega_k)),$$

$$\mathbf{z}_m^k \to \mathbf{z}^k \text{ strong in } L^2\left(0,T;L^2(\Omega_k)\right).$$

∎

Finally, we have the following.

PROOF OF PROPOSITION 3.1. Taking the limit as $m \to \infty$ in (3.9)–(3.11) we can easily prove that the approximated reproductive solution $(\mathbf{v}^k, \mathbf{z}^k, \mathbf{b}^k)$ is a reproductive weak solution of problem (P_k). ∎

4. PROOF OF THEOREM 2.3

Let $(\mathbf{v}^k, \mathbf{z}^k, \mathbf{b}^k)$ be a weak solution of (P_k) obtained in Proposition 3.1. Define the extended functions

$$\mathbf{u}^k(x,t) = \begin{cases} \mathbf{v}^k(x,t), & \text{if } x \in \Omega_k,\\ 0, & \text{if } x \in \Omega \setminus \Omega_k,\end{cases} \tag{4.1}$$

$$\mathbf{w}^k(x,t) = \begin{cases} \mathbf{z}^k(x,t), & \text{if } x \in \Omega_k,\\ 0, & \text{if } x \in \Omega \setminus \Omega_k,\end{cases} \tag{4.2}$$

$$\mathbf{h}^k(x,t) = \begin{cases} \mathbf{b}^k(x,t), & \text{if } x \in \Omega_k,\\ 0, & \text{if } x \in \Omega \setminus \Omega_k.\end{cases} \tag{4.3}$$

Then we have the following result.

LEMMA 4.1. *Let $\mathbf{u}^k, \mathbf{w}^k, \mathbf{h}^k$ be defined as above. These functions satisfy*

(i) $\mathbf{u}^k, \mathbf{h}^k \in L^2(0,T;J(\Omega)) \cap L^\infty_\pi(0,T;H(\Omega))$;

(ii) $\mathbf{w}^k \in L^2(0,T;W_0(\Omega)) \cap L^\infty_\pi(0,T;L^2(\Omega))$.

Furthermore, we have

$$\int_0^T \left\|\nabla \mathbf{u}^k\right\|^2 \le \ell_1, \qquad \int_0^T \left\|\nabla \mathbf{w}^k\right\|^2 \le \ell_2, \qquad \int_0^T \left\|\nabla \mathbf{h}^k\right\|^2 \le \ell_3, \tag{4.4}$$

$$\operatorname*{sup\,ess}_{t\in[0,T]} \left\|\mathbf{u}^k\right\| \le \ell_1, \qquad \operatorname*{sup\,ess}_{t\in[0,T]} \left\|\mathbf{w}^k\right\| \le \ell_2, \qquad \operatorname*{sup\,ess}_{t\in[0,T]} \left\|\mathbf{h}^k\right\| \le \ell_3, \tag{4.5}$$

where ℓ_1, ℓ_2, ℓ_3 are strictly positive real constants independ of $k \in \mathbb{N}$.

PROOF. By integrating (3.12) in $[0,T]$, we obtain

$$C_0 \int_0^T \left(\left\|\nabla \mathbf{v}_m^k(t)\right\|^2 + \left\|\nabla \mathbf{z}_m^k(t)\right\|^2 + \left\|\nabla \mathbf{b}_m^k(t)\right\|^2\right) dt \le M(\mathbf{f},\mathbf{g}), \tag{4.6}$$

since $\mathbf{v}_m^k, \mathbf{z}_m^k, \mathbf{b}_m^k$ are reproductive with period T. Consequently, if m goes to $+\infty$ in (4.6), by the lower semicontinuity of the norm with respect to weak convergence, we obtain

$$C_0 \int_0^T \left(\left\|\nabla \mathbf{v}^k(t)\right\|^2 + \left\|\nabla \mathbf{z}^k(t)\right\|^2 + \left\|\nabla \mathbf{b}^k(t)\right\|^2\right) dt \le M(\mathbf{f},\mathbf{g}). \tag{4.7}$$

Analogously, from (3.30) we deduce the uniform estimate

$$\operatorname*{sup\,ess}_{t\in[0,T]} \left\|\mathbf{v}^k(t)\right\|^2 + j\left\|\mathbf{z}^k(t)\right\|^2 + r\left\|\mathbf{b}^k(t)\right\|^2 \le M(\mathbf{f},\mathbf{g}) + M, \tag{4.8}$$

which implies (i) and (ii) and conclude the proof. ∎

Next, we pass to the limits as k goes to infinity. By the uniform estimates (4.4) and (4.5), we have a subsequence $\{(\mathbf{u}^k, \mathbf{w}^k, \mathbf{h}^k)\}$, denoted without any subscript, as well as functions $\mathbf{u}, \mathbf{h} \in L^2(0,T;J(\Omega)) \cap L^\infty_\pi(0,T;H(\Omega))$ and $\mathbf{w} \in L^2(0,T;W_0(\Omega)) \cap L^\infty_\pi(0,T;L^2(\Omega))$ such that

$$\left.\begin{array}{l}\mathbf{u}^k \to \mathbf{u}\\ \mathbf{h}^k \to \mathbf{h}\end{array}\right\} \text{weak in } L^2(0,T;J(\Omega)) \text{ and weak} - * \text{ in } L^\infty_\pi(0,T;H(\Omega)),$$
$$\mathbf{w}^k \to \mathbf{w} \text{ weak in } L^2(0,T;W_0(\Omega)) \text{ and weak} - * \text{ in } L^\infty_\pi\left(0,T;L^2(\Omega)\right).$$

To conclude the proof of Theorem 2.3, it is enough to show that there exists a subsequence $\{\mathbf{u}^{k'}, \mathbf{w}^{k'}, \mathbf{h}^{k'}\}$ such that

$$\begin{aligned}&\left.\begin{array}{l}\mathbf{u}^{k'} \to \mathbf{u}\\ \mathbf{h}^{k'} \to \mathbf{h}\end{array}\right\} \text{strong in } L^2\left(0,T;L^2_{\text{loc}}(\Omega)\right),\\ &\mathbf{w}^{k'} \to \mathbf{w} \text{ strong in } L^2\left(0,T;L^2_{\text{loc}}(\Omega)\right).\end{aligned} \tag{4.9}$$

In fact, once these strong convergences and limits are established, we can easily prove that $(\mathbf{u}, \mathbf{w}, \mathbf{h})$ is the desired reproductive weak solution for (1.1)–(1.4). Indeed, let (φ, ξ, ψ) be any arbitrary test function, Θ a bounded subset of Ω and $k_0 \in \mathbb{N}$ such that: supp φ, supp ξ, and supp ψ are included in $\Theta \subseteq \Omega_{k_0} \subseteq \Omega_k$, $\forall k \ge k_0$. Then, by Lemmas 2.1 and 4.1, we have

$$\begin{aligned}\int_0^T \left((\mathbf{u}^k \cdot \nabla\varphi, \mathbf{u}^k) - (\mathbf{u}\cdot\nabla\varphi, \mathbf{u})\right) dt &\le \int_0^T \left(\left\|\mathbf{u}^k - \mathbf{u}\right\|^2_{L^2(\Theta)} \left\|\mathbf{u}^k\right\|_{L^6(\Omega)} \|\nabla\varphi\|_{L^3(\Theta)}\right.\\ &\quad \left. + \left\|\mathbf{u}^k - \mathbf{u}\right\|^2_{L^2(\Theta)} \|\mathbf{u}\|_{L^6(\Omega)} \|\nabla\varphi\|_{L^3(\Theta)}\right) dt\\ &\le C_L \left(\int_0^T \left\|\mathbf{u}^k - \mathbf{u}\right\|^2_{L^2(\Theta)} dt\right)^{1/2} \left(\int_0^T \left\|\nabla\mathbf{u}^k\right\|^2 dt\right)^{1/2} \operatorname*{sup\,ess}_{t\in[0,T]} \|\nabla\varphi\|_{L^3(\Theta)}\\ &\quad + C_L \left(\int_0^T \left\|\mathbf{u}^k - \mathbf{u}\right\|^2_{L^2(\Theta)} dt\right)^{1/2} \left(\int_0^T \|\nabla\mathbf{u}\|^2 dt\right)^{1/2} \operatorname*{sup\,ess}_{t\in[0,T]} \|\nabla\varphi\|_{L^3(\Theta)}.\end{aligned}$$

Combining this inequality with (4.9), we deduce

$$\int_0^T \left(\left(\mathbf{u}^{k'} \cdot \nabla\varphi, \mathbf{u}^{k'}\right) - (\mathbf{u} \cdot \nabla\varphi, \mathbf{u})\right) dt \longrightarrow 0, \qquad \text{as } k' \to \infty.$$

All the other convergences are proved similarly, and we have established that $(\mathbf{u}, \mathbf{w}, \mathbf{h})$ is one reproductive weak solution of our problem (1.1)–(1.4). ∎

We claim that the convergence properties stated in (4.9) are true.

LEMMA 4.2. *There exists a subsequence* $\{\mathbf{u}^{k'}, \mathbf{w}^{k'}, \mathbf{b}^{k'}\}$ *such that*

$$\left.\begin{array}{l}\mathbf{u}^{k'} \to \mathbf{u} \\ \mathbf{h}^{k'} \to \mathbf{h}\end{array}\right\} \text{strong in } L^2\left(0,T; L^2_{\mathrm{loc}}(\Omega)\right),$$

$$\mathbf{w}^{k'} \to \mathbf{w} \text{ strong in } L^2\left(0,T; L^2_{\mathrm{loc}}(\Omega)\right).$$

PROOF. We restrict our proof to the subsequence $\{\mathbf{u}^{k'}\}_{k'=1}^{\infty}$. The other subsequences, $\{\mathbf{w}^{k'}\}_{k'=1}^{\infty}$ and $\{\mathbf{h}^{k'}\}_{k'=1}^{\infty}$, are treated similarly.

We put $K_j = \bar{\Omega}_j$, then $\{K_j\}_{j=1}^{\infty}$ is an increasing sequence of compact sets, that is: $K_1 \subseteq K_2 \subseteq \cdots \to \Omega$ as $j \to \infty$. For each K_j, we choose $\alpha_j(x) \in C_0^{\infty}(\Omega)$ with the property $0 \le \alpha_j \le 1$, $\alpha_j|_{K_j} \equiv 1$ and supp $\alpha_j \subset \Omega_{j+1}$. It is important to note that $K_j \subset$ supp α_j. Here, and in the remaining, $\|\cdot\|_{\Omega_j} \equiv \|\cdot\|_{L^2(\Omega_j)}$ and d_j denotes the diameter of Ω_j. Then we can construct the desired $\{\mathbf{u}^{k'}\}$ as follows. First, we construct a sequence $\{\alpha_j(x)\mathbf{u}^k(x)\}_{k=1}^{\infty}$, uniformly bounded in $L^2(0,T; H_0^1(\Omega_2))$. Indeed, since $\mathbf{u}^k = \mathbf{0}$ on $\partial\Omega_2$, Poincaré's inequality on Ω_2 yields that $\|\alpha_1\mathbf{u}^k\|_{\Omega_2} \le \|\mathbf{u}^k\|_{\Omega_2} \le d_2^2/2\|\nabla\mathbf{u}^k\|_{\Omega_2}$. Hence, we have

$$\int_0^T \left\|\alpha_1\mathbf{u}^k(t)\right\|_{\Omega_2}^2 dt \le \frac{d_2^2}{2}\int_0^T \left\|\nabla\mathbf{u}^k(t)\right\|^2 dt \le \frac{d_2^2}{2C_0} M(\mathbf{f},\mathbf{g}). \tag{4.10}$$

Moreover,

$$\begin{aligned}\left\|\nabla\left(\alpha_1\mathbf{u}^k\right)\right\|_{\Omega_2} &\le \left\|(\nabla\alpha_1)\cdot\mathbf{u}^k\right\|_{\Omega_2} + \left\|\alpha_1 \nabla\mathbf{u}^k\right\|_{\Omega_2} \\ &\le \left(\frac{d_2}{\sqrt{2}}\|\nabla\alpha_1\|_{L^\infty(\Omega_2)} + \|\alpha_1\|_{L^\infty(\Omega_2)}\right)\left\|\nabla\mathbf{u}^k\right\|_{\Omega_2},\end{aligned}$$

from which, we conclude that

$$-\int_0^T \left\|\nabla\left(\alpha_1\mathbf{u}^k(t)\right)\right\|_{\Omega_2}^2 dt \le \left(\frac{d_2}{\sqrt{2}}\|\nabla\alpha_1\|_{L^\infty(\Omega_2)} + \|\alpha_1\|_{L^\infty(\Omega_2)}\right)^2 \frac{1}{C_0} M(\mathbf{f},\mathbf{g}). \tag{4.11}$$

The estimates (4.10) and (4.11) imply that $\{\alpha_1\mathbf{u}^k\}$ is uniformly bounded in $L^2(0,T; H_0^1(\Omega_2))$. Consequently, there exists a subsequence $\{\alpha_1\mathbf{u}^{1p}\}_{p=1}^{\infty}$ which converges weak in $L^2(0,T; H_0^1(\Omega_2))$ and strong in $L^2(0,T; L^2(\Omega_2))$. Furthermore, as $\alpha_1|_{K_1} \equiv 1$, $\{\mathbf{u}^{1p}\}_{p=1}^{\infty}$ converges strong in $L^2(0,T; L^2(K_1))$. If we repeat the argument, we deal with the sequence $\{\mathbf{u}^{jp}\}_{p=1}^{\infty}$, $j \ge 1$. To end the proof, it is enough to choose the diagonal terms, which we denote by $\{\mathbf{u}^{k'}\}_{k'=1}^{\infty}$, and to remark that the sequence converges for all K_j in $L^2(0,T; L^2(K_j))$.

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