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BASES OPTIMALES EN MATROIDES CON INCERTIDUMBRE Y CÓMO ENCONTRARLAS CON CONSULTAS DE COSTO MÍNIMO

TESIS PARA OPTAR AL GRADO DE MAGÍSTER EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MATEMÁTICAS APLICADAS

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

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RESUMEN DE LA MEMORIA PARA OPTAR
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Estudiamos el problema de bases de peso mínimo en matroides en un contexto donde los pesos en los elementos son inciertos. Inicialmente, para cada elemento e de una matroide $(E, \mathfrak{I})$ se conocerá un conjunto no vacío $A_e \subseteq \mathbb{R}$, llamado área de incertidumbre, que contiene los posibles pesos del elemento e. El algoritmo puede escoger un conjunto de elementos $X \subseteq E$ a consultar, de manera que si un elemento e es consultado se obtiene un peso $w_e \in A_e$ con un costo de consulta $c_e \in \mathbb{R}$ asociado. El objetivo es encontrar un conjunto $X \subseteq E$ que, al consultarlo, permita calcular una base de peso mínimo independiente del valor de las aristas no reveladas. A estos conjuntos se les llamará consultas factible; tenemos particular interés en encontrar una de costo mínimo. Esto es de especial interés en aplicaciones donde obtener datos exactos es díficil o costoso, pero datos vagos son de fácil acceso.

El problema adaptativo bajo análisis competitivo fue estudiado anteriormente. En este trabajo consideramos el caso no adaptativo; es decir, cuando los elementos a consultar se eligen todos al mismo tiempo. Formalizamos el problema, definimos las bases de peso mínimo en el contexto incierto, caracterizamos su existencia y demostramos que son las bases de una matroide. Proveemos una caracterización de las consultas factibles de tamaño mínimo, probamos que los complementos de consultas factibles forman una matroide sencilla y esto nos permite idear un algoritmo que encuentra una consulta factible de costo mínimo con una cantidad polinomial tanto de recursos computacionales como de llamadas al oráculo de independencia de la matroide.

OPTIMAL BASES OF UNCERTAINTY MATROIDS AND HOW TO COMPUTE THEM WITH QUERIES OF MINIMUM COST

We study the minimum weight basis problem on matroids when element weights are uncertain. Initially, for each element e in a matroid $(E, \mathcal{I})$ a non-empty set $A_e \subseteq \mathbb{R}$ is known. This set, called uncertainty area, contains the possible weights of the element. The algorithm can choose a set of elements $X \subseteq E$ to query, such that if an element e is queried a weight $w_e \in A_e$ is obtained at some cost $c_e \in \mathbb{R}$ associated to the querying process. Our objective is to find a set $X \subseteq E$ that, when queried, allows us to compute a minimum weight basis. This sets are called feasible queries; furthermore, we are interested in feasible queries of minimum cost. This is of particular interest in applications where exact data is hard or expensive to obtain, but approximations are available.

The adaptative problem under competitive analysis has been previously studied. In this work we consider the non adaptative setting, that is, each query is made at the same time. We formalize the problem, define minimum weight bases in uncertainty matroids, characterize their existence and prove that they are the basis of some matroid. We provide a characterization of feasible queries of minimum size and prove that complements of feasible queries form an uncomplicated matroid. These structural results allow us to design an algorithm that finds feasible query sets of minimum cost in polynomial time and number of calls to the independence oracle.

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Introduction

Most classical set selection problems follow a common setting: there is a ground set where each element has a precise certain weight and we wish to select a subset of elements that satisfies some desired restrictions and optimizes some objective function. More often than not, certainty and precision are very strict assumptions. Exact data can be expensive to obtain, not immediately available, hard to keep updated or even not realistic (for example, in the prescence of measurement inaccuracies).

Applications where uncertainty is key are not hard to come by. A first example occurs when designing connected telecommunication networks: each connection has an expense often associated by the traffic load which is hard to give explicitly, but one can come up with a reasonable approximation. The objective is not clear, should one come up with a network that fares well against all scenarios, one that most of the time works nicely or maybe we should spend on understanding better how such traffic load works?

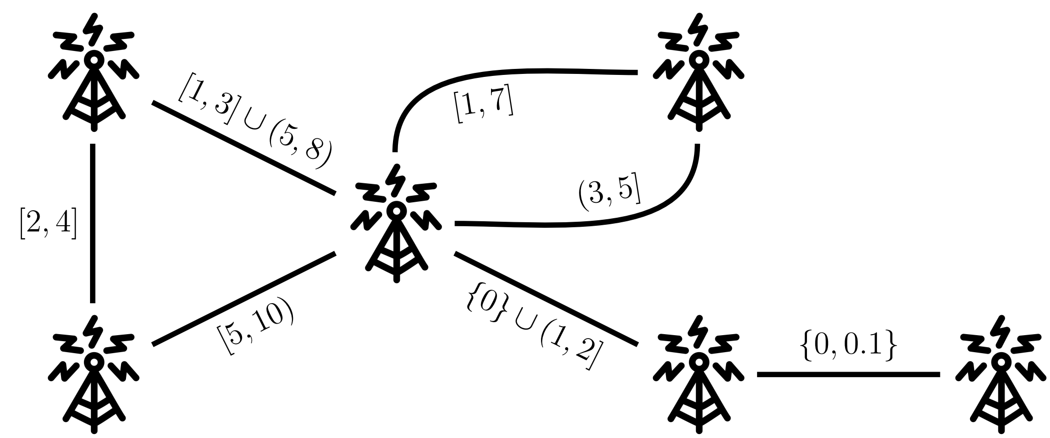


Figure 1: An instance of the connected telecommunication network problem presented above. The traffic load of each connection should lie on the label set.

For a second example consider a set of clients which are interested on specific products, and a salesman that has only a vague idea of how much each client is willing to pay. In this case the objective is clear, the salesman wants to maximize earnings, but it is not clear how he should proceed; make further investigations, come up with prior probabilities on what each client is willing to pay, or maybe pay an expensive consultant to come up with exact data.

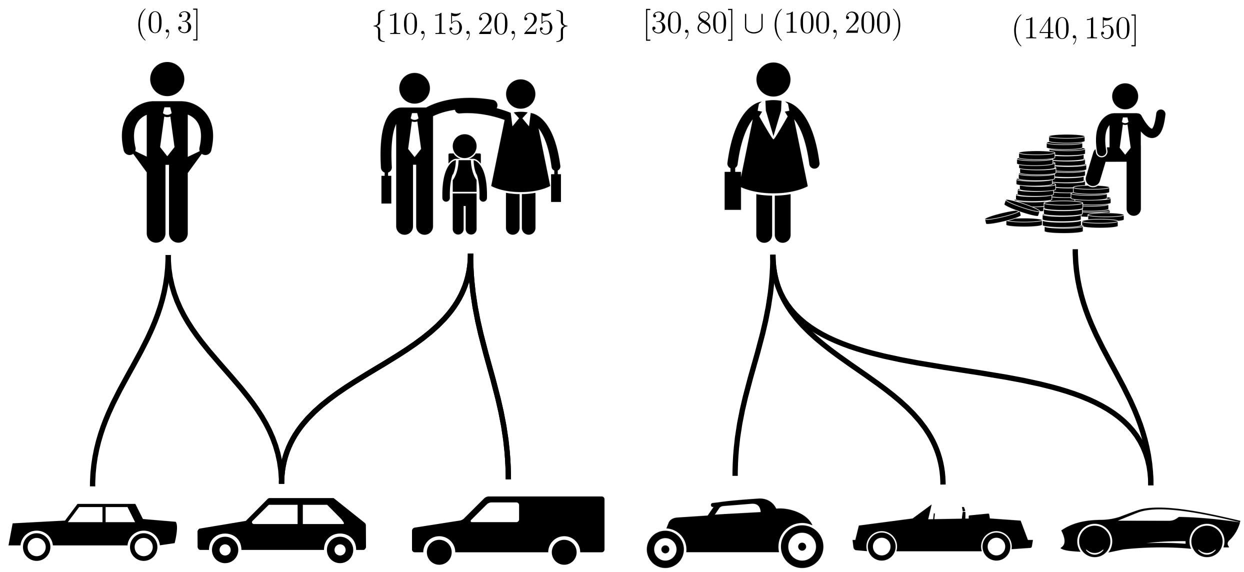


Figure 2: An instance of the salesman problem previously introduced. Each client is presented with a rough budget that the salesman estimates and its product preferences.

In a third application one can imagine a scenario where there is a budget which allows the construction of a fixed number k of hospitals from a preexisting list of potential hospitals. Each potential hospital has an associated benefit which needs exhaustive polling to be made exact but, again, estimates are easier to work out. Ideally we want to select the top k potential hospitals but it is not clear how we can achieve this goal.

These three examples have two things in common: They have some matroidal structure behind them, which is the framework of this work, and they show that it is not clear how to proceed when facing uncertainty.

A first classical approach to combat uncertainty is randomness. Namely, each element gets a probability distribution and we obtain a stochastic programming problem. A first issue with this approach is intractability, when going from deterministic formulations to stochastic ones the difficulty ramps up noticeably, a good example of this are PERT scheduling models [Hag88]. Even though expected weight is a natural objective function, when the process is not heavily repeated, it is not clear that this criterion is at all useful. This issue with expectation made popular multi-staged approaches and the value-at-risk criterion, that is, to ask for solutions such that its weight crosses a certain threshold with high probability.

Results in this paradigm are usually very distribution-specific. For example, if weights are normal and independent any value-at-risk linear program may be reformulated as a non-linear deterministic program. In particular, this happens for the MST problem with normal and independent weights. This result and more insight on stochastic approaches can be found in [KM05].

A second popular approach is robust optimization. In this setting each element can take weights on a closed interval and the objective is to find a solution that fares well against all scenarios. A first measure of faring well is finding a solution that minimizes its maximum weight over all scenarios. Sadly, this approach is not very interesting as the worst scenario is usually achieved when each element is at its upper bound. Instead, the common approach is to pick a solution that minimizes maximal regret, that is, the difference between the weight of the solution and the weight of the best subset that could have been picked had the true weights been known.

Despite the fact that intractability appears frequently when dealing with minmax regret problems, there are interesting results involving matroids in the robust context. In [YKP01] the problem is first introduced in its graphic version characterizing trees that have 0 regret, in [KZ06] a 2-approximation is given for the minmax problem in general matroids and [KZ07] gives an algorithm to compute bases with 0 regret when they exist. An exhaustive review of robust optimization methods can be found in [Kas08].

A third approach that has gained popularity over recent years is the query model. First introduced by Kahan in [Kah91], the query model allows algorithms an extra "query" operation that can be used to obtain exact information at some cost, that is, once an element is queried its true weight is revealed permanently and the objective is to compute an optimal solution.

To assess performance one common choice is to use competitive analysis, that is, to compare the number of queries made by the algorithm against the number of queries made by an adversary who *knows* the true weights beforehand and whose mission is to prove that the solution they provide is correct.

This model allows for extensive variations. We list a few that are interesting and have results related to this work.

- Queries can be adaptive, that is, the algorithm can query elements freely and on the fly; non-adaptive, where the algorithm needs to make all queries at once; or in rounds, where the algorithm can only make queries in predetermined stages.

  An example of adaptive work is [EHK+08] and [EHK16], where the minimum spanning tree and minimum basis problem are respectively solved with unit query costs via optimal competitive analysis algorithms.

- Even tough competitive analysis is a common choice for measuring performance, another option is to compare against the optimum number of queries needed to compute a solution. An example of this is [FMO+07], where they provide a polynomial algorithm for the s-t path problem in the non-adaptive against-optimum setting when paths are given as part of the input.

- One can also allow randomness in the execution of the algorithm. The work done in [MMS17] shows that randomization improves performance for the minimum basis problem with unit query costs.

- One can consider queries that instead of revealing exact data provide a refinement on the current data. Differences of refinement models are explored in [GSS16] for the k-selection and minimum spanning tree problem.

- It is interesting to note that computing the actual weight of a solution is impossible unless every element is queried. An interesting approach is to allow for an additive gap δ, that is, the objective is to output a range $[L, H]$ such that $H - L \leq \delta$ and the actual weight of a solution is comprised within $[L, H]$. Work in this direction is [KT01], where they compute the average number; [FMP+03], where they solve the k-selection problem and; [FMO+07], where they prove that the s-t path problem with additive gap is NP-HARD.

A key tool heavily used in the adaptive/competitive analysis setting are witness sets from which we will draw some inspiration later on. They were first introduced in [BHKR05], and are simply sets which unless queried do not allow any algorithm to make progress. A survey detailing the nuances and recent work in the query model is [EH+15].

This work is set in the query model and our interest is to compute an optimal basis of a matroid while querying a set of minimum cost. We work with non-adaptive queries and compare our solutions against the optimum, as we are interested in minimizing cost.

Our results

We focus in two particular problems:

- The **Uniformly Minimum Basis** problem, where one is concerned in finding a basis that is optimal in all scenarios or deciding that no such basis exists. We call such bases uniformly minimum. This problem was already studied in the case that each area is a closed interval in [KZ07] and an efficient algorithm was given.

  We provide an efficient algorithm for the general case and give some additional structural insight by proving that bases that are optimal in all scenarios form a matroid. We describe the structure of this matroid and characterize the existence of such bases.
- The **Minimum Cost Feasible Query Set** problem. In this problem we are interested in studying feasible query sets, that is, sets of elements such that the revelation of their true weight guarantees the existence of a uniformly minimum basis, no matter the actual true weights. Moreover, we are interested in finding a feasible query set that minimizes cost. To the best of our knowledge, this problem has not been previously studied.

  We provide a characterization of feasible query sets of minimum size, prove that the complement of feasible query sets form an uncomplicated matroid and give an efficient algorithm for solving the Minimum Cost Feasible Query Set problem.

Thesis structure

This thesis is structured into four chapters:

1. The first chapter presents the basics of matroid theory and serves to record-keep some results related to matroids to be used in this work. Readers familiar with matroid theory and its applications in combinatorial optimization may skip this chapter altogether.
2. The second chapter introduces our main objects of interest: uncertainty matroids and uniformly minimum bases. We study generalizations of the traditional blue and red rules for computing minimum spanning trees and provide generalizations for properties already known for weighted matroids. We then prove two interesting results:

(a) The equivalence between the existence of uniformly minimum bases and fully-colored uncertainty matroids, and

(b) A combinatorial description of the matroid of all uniformly minimum bases.

This results end up providing algorithmic solutions to the Uniformly Minimum Basis problem.

3. The third chapter focuses on feasible query sets. We start by approaching problems simpler than the full-fledged Minimum Cost Feasible Query Set problem. We first attack decomposable matroids and show two ways of joining decomposed solutions into more complex ones. We follow by solving the problem in the case uncertainty areas are either all $\{0, 1\}$ or all intervals. Taking inspiration from the previous cases we define an analogue to witness sets for this setting and prove some useful properties. These allow us to provide a characterization of feasible sets of minimum size, give a description of the matroid formed by complements of feasible query sets and conclude an algorithmic solution to the Minimum Cost Feasible Query Set problem.
4. The fourth chapter is about algorithmic implementation. It starts with a brief exposition on the computation model and then discusses the implementation of the specific algorithms deduced on previous chapters.

Chapter 1

Preliminaries

In this first chapter we recall notions from matroid theory and combinatorial optimization needed in this work. The purpose of this chapter is threefold. First, as a way to record-keep known results that will be needed in the following chapters. Second, as a way to fix notation that will be of posterior use. Lastly, as a way to introduce properties that will be later on generalized.

We do not present proofs for well known properties; instead, we point to some general references where proofs and more information is available.

We assume familiarity with the basics of graph theory. Even though we do not work specifically with graphs on the algorithms and propositions presented, most of our examples and motivations arise from graphs. A good reference that covers all the graph theory we use is the second chapter of [KV12].

1.1 Matroids

The theory of matroids was introduced by Whitney in 1935 [Whi35]. In this foundational paper, Whitney conceived matroids as an abstract generalization of matrices while adding some graph theory flavor. This approach is motivated by his earlier work on graph theory and it is palpable as matroid terminology borrows extensively from both linear algebra and graph theory. Since then, matroids have made remarkable appearances in lattice theory, combinatorial theory, geometries and, more relevantly to this work, have played a huge role as a framework for combinatorial optimization problems.

We mostly follow the presentation and notation of matroids used in [Oxl06], but [Wel10] is also a widely used reference. These two books cover a lot of material beyond the scope of this work, so a more succinct and optimization-oriented reference is Chapter 39 of [Sch03].

A first characteristic of matroids is that they can be defined in multiple equivalent ways. We proceed to define matroids in different ways, starting with a linear algebra inspired definition, that is, via independent sets.

Definition 1.1 (Matroids via independent sets) *Let E be a finite set and $\mathfrak{I} \subseteq 2^E$. The tuple $M = (E, \mathfrak{I})$ is a matroid if:*

(I1) $\emptyset \in \mathfrak{I}$,

(I2) If $Y \in \mathfrak{I}$ *and* $X \subseteq Y$, *then* $X \in \mathfrak{I}$,

(I3) If $X, Y \in \mathfrak{I}$ *and* $|Y| > |X|$, *then there is an element* $y \in Y \backslash X$ *with* $X + y \in \mathfrak{I}$.

Conditions (I1) and (I2) make $(E, \mathfrak{I})$ an independence system. Condition (I3) is often called the augmentation axiom. Sometimes we replace (I3) by an equivalent counterpart called the weak augmentation axiom (I3'):

(I3') If $X, Y \in \mathfrak{I}$ are such that $|X \backslash Y| = 1$ and $|Y \backslash X| = 2$, then there is $y \in Y \backslash X$ such that $X + y \in \mathfrak{I}$.

The sets in $\mathfrak{I}$ are called the independent sets of the matroid, sets that are not independent will be called dependent, and each $e \in E$ is referred as an element of the matroid.

We start with a natural class of matroids inherited from linear algebra, namely, the columns of a matrix with independence inherited from the subjacent vector space. This class also give matroids its matrix related name.

Definition 1.2 (Vector Matroids over a Field) *Let $\mathbb{F}$ be a field and $A \in \mathbb{F}^{m \times n}$. Consider $E = [n]$. Then, the set:*

$$\mathfrak{I} = \{I \subseteq E : \{A_{\bullet,i}\}_{i \in I} \text{ is linearly independent on the vector space } \mathbb{F}^m \},$$

is such that $(E, \mathfrak{I})$ is a matroid. We denote such matroid by $M_{\mathbb{F}}[A]$. If M is a matroid such that $M = M_{\mathbb{F}}[A]$ for some field $\mathbb{F}$ and $A \in \mathbb{F}^{m \times n}$, we say that M is a vector matroid over $\mathbb{F}$.

We follow with a pretty simple class of matroids. These are the ones where we consider independent sets simply as sets of size less than a fixed positive integer.

Definition 1.3 (Uniform Matroid) *If S is a finite set and $k \in \mathbb{N}$, then $U_S^k = (S, \mathfrak{I})$ is a matroid, where:*

$$\mathfrak{I} = \{I \subseteq S : |I| \leq k\}.$$

If a matroid M is such that $M = U_S^k$ for some $k \in \mathbb{N}$ and finite set S we say that M is a uniform matroid.

We now give an example of a less natural class of matroids, but an important one. These are the matroids induced by sets systems.

Definition 1.4 (Transversal Matroid) *Let $\mathcal{S} = \{S_i\}_{i=1}^k$ be a collection of finite sets (also called a set system). We say that $T \subseteq \bigcup_{i=1}^{k} S_i$ is a partial transversal if there exists an injective* *function* $\varphi : T \to [k]$ *such that* $t \in S_{\varphi(t)}$ *for each* $t \in T$. *Consider* $E = \bigcup_{i=1}^{k} S_i$, *then the set:*

$$\mathfrak{I} = \{I \subseteq E : I \text{ is a partial transversal}\}$$

is such that $(E, \mathfrak{I})$ *is a matroid that we denote by* $\mathcal{T}[\mathcal{S}]$. *If a matroid* M *is such that* $M = \mathcal{T}[\mathcal{S}]$ *for some collection of finite sets* $\mathcal{S}$, *we say that* M *is a tranversal matroid.*

Independence in matroids plays a similar role to independence in vector spaces. This motivates the study of maximal independent sets as an analogous to bases in vector spaces. It turns out that these sets can also be used to define matroids.

Definition 1.5 (Matroids via bases) *Let* M *be a matroid. A basis is a maximal independent set, and we denote the set of bases by* $\mathfrak{B}$. *One can prove that* $\mathfrak{B}$ *satisfies the following conditions:*

(B1) $\mathfrak{B} \neq \emptyset$,

(B2) If $T_1, T_2 \in \mathfrak{B}$ *and* $x \in T_1 \backslash T_2$, *then there is an element* $y \in T_2 \backslash T_1$ *such that* $T_1 - x + y \in \mathfrak{B}$.

Condition $(B2)$ *is often called the weak basis exchange axiom or, simply, the basis exchange axiom. Moreover, if* E *is a finite set and* $\mathfrak{B}$ *is a collection of subsets of* E *satisfying* $(B1)$ *and* $(B2)$, *then the set:*

$$\mathfrak{I} = \{I \subseteq E : I \subseteq B, \text{ for some } B \in \mathfrak{B}\},$$

is such that $(E, \mathfrak{I})$ *is a matroid having* $\mathfrak{B}$ *as its set of bases. Condition* $(B2)$ *may be replaced by the equivalent strong basis exchange axiom* $(B2')$*:*

(B2') If $T_1, T_2 \in \mathfrak{B}$ *and* $x \in T_1 \backslash T_2$, *then there is an element* $y \in T_2 \backslash T_1$ *such that both* $T_1 - x + y$ *and* $T_2 + x - y$ *belong to* $\mathfrak{B}$.

Similarly to the bases from linear algebra, all matroid bases have the same size. Furthermore, this property characterizes bases, as shown in the next proposition.

Proposition 1.6 *Let* $M = (E, \mathfrak{I})$ *be a matroid,* $I \in \mathfrak{I}$ *and* $B \in \mathcal{B}$ *such that* $|I| = |B|$. *Then,* I *is a basis.*

Our next definition is motivated by graph theory. We define matroids by circuits which are a generalization of cycles in a graph.

Definition 1.7 (Matroids via Circuits) *Let* M *be a matroid. A circuit is a minimal dependent set, and we denote the set of circuits by* $\mathfrak{C}$. *The circuits of a matroid satisfy the following properties:*

(C1) $\emptyset \notin \mathfrak{C}$,

(C2) If $C_1, C_2 \in \mathfrak{C}$ *and* $C_1 \subseteq C_2$, *then* $C_1 = C_2$,

(C3) If $C_1, C_2 \in C$ *are distinct and* $e \in C_1 \cap C_2$, *then there is a circuit* $C_3 \in \mathfrak{C}$ *such that* $C_3 \subseteq (C_1 \cup C_2) - e$.

Property $(C3)$ *is usually called the circuit elimination axiom. Furthermore, if* E *is a finite set and* $\mathfrak{C}$ *is a collection of subsets of* E *satisfying* $(C1)$, $(C2)$ *and* $(C3)$, *then the set:*

$$\mathfrak{I} = \{I \subseteq E : C \nsubseteq I, \textit{ for every } C \in \mathfrak{C}\}$$

is such that $(E, \mathfrak{I})$ *is a matroid having* $\mathfrak{C}$ *as its set of circuits.*

Noting that the cycles of a graph satisfy $(C1)$, $(C2)$ and $(C3)$ we can naturally define matroids induced by graphs.

Definition 1.8 (Graphic Matroid) *If* $G = (V, E)$ *is a graph, we define the matroid* $M(G)$ *by its circuits:*

$$\mathfrak{C} = \{C \subseteq E : C \textit{ is a cycle.}\}$$

If a matroid M *is such that* $M = M(G)$ *for some graph* G, *we say that* M *is a graphic matroid.*

We now define matroids via two concepts, once again, inspired by linear algebra. They are, the rank function and the span of a set. Here we differ a little bit from the terminology in [Oxl06] as they use the term closure instead of span.

Definition 1.9 (Matroids via Rank) *Let* M *be a matroid. We define the rank of a set* $X \subseteq E$ *to be the size of the maximal independent set contained in* X. *The rank function* $r : 2^E \to \mathbb{N}$ *is the function that maps each set to its rank, that is:*

$$r(X) = \max\{|I| : I \in \mathfrak{I} \textit{ and } I \subseteq X\}$$

One can check that the rank function verifies the following conditions:

(R1) *If* $X \subseteq E$, *then* $0 \leq r(X) \leq |X|$,

(R2) *If* $X \subseteq Y$ *and* $Y \subseteq E$, *then* $r(X) \leq r(Y)$,

(R3) *If* $X, Y \subseteq E$, *then* $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

Condition $(R2)$ *is known as monotonocity and* $(R3)$ *as submodularity. More so if* E *is a finite set and* $r : 2^E \to \mathbb{N}$ *is a function that verifies* $(R1)$, $(R2)$ *and* $(R3)$; *then, the set:*

$$\mathfrak{I} = \{I \subseteq E : r(I) = |I|\}$$

is such that $(E, \mathfrak{I})$ *is a matroid with rank function* r.

Definition 1.10 (Matroids via Span) *Let* M *be a matroid. We define the span of a set* $X \subseteq E$ *as the elements that when added to* X *do not augment its rank. The span operator* $\mathrm{span} : 2^E \to 2^E$ *is the operator that maps each set to its span, that is:*

$$\mathrm{span}(X) = \{e \in E : r(X + e) = r(X)\}$$

The span operator on a matroid has the following properties:

(S1) *If* $X \subseteq E$, *then* $X \subseteq \operatorname{span}(X)$.

(S2) *If* $X \subseteq Y$ *and* $Y \subseteq E$, *then* $\operatorname{span}(X) \subseteq \operatorname{span}(Y)$.

(S3) *If* $X \subseteq E$, *then* $\operatorname{span}(X) = \operatorname{span}(\operatorname{span}(X))$.

(S4) *If* $X \subseteq E$, $x \in E$ *and* $y \in \operatorname{span}(X + x) \backslash \operatorname{span}(X)$, *then* $x \in \operatorname{span}(X + y)$.

Furthermore, if E *is a finite set and* $\operatorname{span} : 2^E \to 2^E$ *is an operator that verifies* $(S1)$, $(S2)$, $(S3)$ *and* $(S4)$, *then the set:*

$$\mathfrak{I} = \{I \subseteq E : x \notin \operatorname{span}(I - x), \text{ for all } x \in I\}$$

is such that $(E, \mathfrak{I})$ *is a matroid with span operator* span.

Observation If M is a matroid we denote by $E(M)$, $\mathfrak{I}(M)$, $\mathfrak{B}(M)$, $\mathfrak{C}(M)$, r_M and span_M its elements, independent sets, bases, circuits, rank function and span operator respectively. When there is no risk of confusion, the allusion to M will be omitted.

We now look at some ways in which a matroid induces smaller matroids on a subset of its elements while preserving some of its original structure. These actions are called minor or fundamental operations. We start with two simple operations: deletion and restriction.

Definition 1.11 (Deletion and Restriction) *Let* $M = (E, \mathfrak{I})$ *be a matroid,* X *be a subset of* E *and* $\mathfrak{I} \backslash X = \{I \subseteq E \backslash X : I \in \mathfrak{I}\}$. *Then, the pair* $(E \backslash X, \mathfrak{I} \backslash X)$ *is a matroid. We call this matroid the deletion of* X *from* M *and denote it by* $M \backslash X$. *The following are known characterizations for the bases, circuits, rank and span of* $M \backslash X$ *in terms of* M*:*

$$\begin{aligned}
\mathfrak{B}(M \backslash X) &= \{T \subseteq E : T = B \backslash X \text{ for } B \in \mathfrak{B}(M) \text{ and maximal with respect to inclusion}\},\\
\mathfrak{C}(M \backslash X) &= \{C \subseteq E \backslash X : C \in \mathfrak{C}(M)\},\\
r_{M \backslash X}(F) &= r_M(F) \qquad \forall F \subseteq E \backslash X,\\
\operatorname{span}_{M \backslash X}(F) &= \operatorname{span}_M(F) \backslash X \qquad \forall F \subseteq E \backslash X.
\end{aligned}$$

We also define restriction, which is simply deletion of the complement. That is, the restriction of M *to* X *is the matroid* $(X, \mathfrak{I} \backslash [E \backslash X])$ *and we denote it by* $M|_X$.

Note that the name minor operation comes from graph theory as deletion is one of the two operations allowed in constructing minors of a graph.

The next fundamental operation is less simple than deletion or restriction, but it also comes as analogous of graph minors operations; that is, contraction of elements.

Definition 1.12 (Contraction) *Let* $M = (E, \mathfrak{I})$ *be a matroid and* X *be a subset of* E. *Choose any* $B \in \mathcal{B}(M|_X)$ *and let* $\mathfrak{I}/X = \{I \subseteq E \backslash X : I \cup B \in \mathfrak{I}\}$. *Then, the pair* $(E \backslash X, \mathfrak{I}/X)$ *is a matroid. We call this matroid the contraction of* X *from* M *and denote it by* M/X. *The following are known characterizations for the bases, circuits, rank and span of* M/X *in terms* *of M and B:*

$$\begin{aligned}
\mathfrak{B}(M/X) &= \{T \subseteq E\backslash X : T \cup B \in \mathcal{B}(M)\},\\
\mathfrak{C}(M/X) &= \{C \subseteq E\backslash X : C = C'\backslash X \textit{ for } C' \in \mathfrak{C}(M) \textit{ and minimal with respect to inclusion}\},\\
r_{M/X}(F) &= r_M(F \cup X) - r_M(X) \qquad \forall F \subseteq E\backslash X,\\
\operatorname{span}_{M/X}(F) &= \operatorname{span}_M(F \cup X)\backslash X \qquad \forall F \subseteq E\backslash X.
\end{aligned}$$

It is also natural to consider multiple, or successive, contractions and/or deletions. It turns out that contractions and deletions are commutative and associative; so, the order in which these operations are applied does not matter and we can "merge" successive contractions or deletions. We make this explicit in the following proposition:

Proposition 1.13 *Let $M = (E, \mathfrak{I})$ be a matroid and $S_1, S_2 \subseteq E$ disjoint subsets. Then:*

1. $(M\backslash S_1)\backslash S_2 = M\backslash(S_1 \cup S_2) = (M\backslash S_2)\backslash S_1$.
2. $(M/S_1)/S_2 = M/(S_1 \cup S_2) = (M/S_2)/S_1$.
3. $(M\backslash S_1)/S_2 = (M/S_2)\backslash S_1$.

Using this last result, we can drop the parenthesis when taking minor operations without introducing ambiguity.

Most of the concepts already discussed come borrowed from the familiar settings of linear algebra or graph theory. Matroid duality comes as less familiar, even though it generalizes duality of planar graphs, but it is of fundamental importance in the development of this work.

Definition 1.14 (Duality) *Let $M = (E, \mathfrak{I})$ be a matroid and $\mathcal{B}^*(M) = \{E\backslash B : B \in \mathcal{B}(M)\}$. The set $\mathcal{B}^*(M)$ is the set of bases of a matroid on E. We call such matroid the dual matroid of M, and denote it by M^*. Note that by this definition it is clear that $(M^*)^* = M$.*

The bases of M^* are called cobases of M and they are denoted by $\mathcal{B}^*$. A similar convention is used for other concepts in matroids. For example: independent sets, circuits, rank and span of M^* are called coindependent sets, cocircuits, corank and cospan of M, and they are denoted by $\mathfrak{I}^*(M)$, $\mathfrak{C}^*(M)$, r^*_M and span^*_M respectively.

We now state some results that characterize dual concepts in terms of the concepts in the original matroid:

1. *$I^* \in \mathfrak{I}^*$ if and only if there is a basis $B \in \mathcal{B}$ such that $B \cap I^* = \emptyset$.*
2. *$B^* \in \mathcal{B}^*$ if and only if $E\backslash B^* \in \mathcal{B}$.*
3. *$C^* \in \mathfrak{C}^*$ if and only if $E\backslash C^*$ is a maximal set such that $r(E\backslash C^*) < r(E)$.*
4. *For all subsets X of E:*

$$r^*(X) = |X| - r(M) + r(E\backslash X).$$

5. *For all subsets X of E:*

$$e \in \operatorname{span}^*(X) \text{ if and only if } e \notin \operatorname{span}[(E-e)\backslash X]$$

Duality also tells us that the minor operations of contraction and deletion are dual, that is, if X is any subset of E, then $(M\backslash X)^* = M^*/X$.

We follow by listing some properties and definitions that relate circuits and cocircuits. We start by showing a useful link between circuits and cocircuits that we will sometimes use.

Proposition 1.15 *Let M be a matroid. If C is a circuit and C^* is a cocircuit of M, then $|C \cap C^*| \neq 1$.*

We now pay attention to a special kind of circuits and cocircuits, that is, the ones with only one element.

Definition 1.16 (Loops and Coloops) *Let $M = (E, \mathfrak{I})$ be a matroid. We say that $e \in E$ is a loop if e is in no basis of M. Dually, we say that e is a coloop if it is in no cobasis of M. This statement can be "dualized" obtaining an alternative definition, namely, $e \in E$ is a loop if e is in every cobasis of M and $e \in E$ is a coloop if it belongs to every basis of M. Another equivalent definition can be given via span, that is, e is a loop if $e \in \operatorname{span} \emptyset$. Dually, e is a coloop if $e \in \operatorname{span}^* \emptyset$.*

We also present circuits and cocircuits that arise when adding a single element to a basis or cobasis respectively.

Definition 1.17 (Fundamental Circuit and Cocircuit) *Let M be a matroid. If B is a basis and $e \notin B$, then $B + e$ has a unique circuit called the fundamental circuit of $B + e$. In dual fashion if B^* is a cobasis and $e \notin B^*$, then $B^* + e$ has a unique cocircuit called the fundamental cocircuit of $B^* + e$.*

Elements on fundamental circuits and cocircuits can be exchanged with elements outside of the respective basis or cobasis while preserving them. This is formalized in the following proposition.

Proposition 1.18 *Let M be a matroid and B a basis of M.*

1. *Let $e \notin B$ and consider C the fundamental circuit of $B + e$. For each $f \in C$ the set $B + e - f$ is a basis of M.*
2. *Let $e \in B$ and consider C^* the fundamental cocircuit of $B^c + e$. For each $f \in C^*$ the set $B - e + f$ is a basis of M.*

We have already shown a way to obtain smaller matroids via minor operations. We now show an operation that allows us to form bigger matroids, the direct sum of matroids.

Definition 1.19 (Direct Sum) *Let $M_1 = (E_1, \mathfrak{I}_1)$ and $M_2 = (E_2, \mathfrak{I}_2)$ be matroids such that E_1 and E_2 are disjoint. Let $E = E_1 \cup E_2$, then the set:*

$$\mathfrak{I} = \{I_1 \cup I_2 : I_1 \in \mathfrak{I}_1,\ I_2 \in \mathfrak{I}_2\}$$

is such that $(E, \mathfrak{I})$ is a matroid. This matroid is denoted by $M_1 \oplus M_2$ and called the direct sum between M_1 and M_2. One can relate the set of bases, circuits, rank, span and dual of $M_1 \oplus M_2$ to the ones of M_1 and M_2 as stated by the following propositions:

$$\begin{aligned}
\mathcal{B}(M_1 \oplus M_2) &= \{B_1 \cup B_2 : B_1 \in \mathcal{B}_1(M), B_2 \in \mathcal{B}_2(M) \\
\mathfrak{C}(M_1 \oplus M_2) &= \mathfrak{C}(M_1) \cup \mathfrak{C}(M_2), \\
r_{M_1 \oplus M_2}(F) &= r_{M_1}(F \cap E_1) + r_{M_2}(F \cap E_2), \qquad \forall F \subseteq E \backslash X, \\
\mathrm{span}_{M_1 \oplus M_2}(F) &= \mathrm{span}_{M_1}(F \cap E_1) \cup \mathrm{span}_{M_2}(F \cap E_2), \qquad \forall F \subseteq E \backslash X, \\
(M_1 \oplus M_2)^* &= M_1^* \oplus M_2^*.
\end{aligned}$$

Observation When writing $M = M_1 \oplus M_2$ it is understood, implicitly, that $E(M_1)$ and $E(M_2)$ are disjoint.

An example of using the direct sum to form a "neat" larger class of matroids happens when considering matroids that are the direct sum of uniform matroids.

Definition 1.20 (Partition Matroid) *We say that M is a partition matroid if there exists finite disjoint sets $S_1, \ldots, S_k$ and natural numbers $n_1, \ldots, n_k$ such that $M = \bigoplus_{i=1}^{k} U_{S_i}^{n_i}$. Note that, in this case,*

$$\mathfrak{I}(M) = \{I \subseteq E : |I \cap S_i| \leq n_i,\ \forall i \in [k]\}.$$

In other words, independent sets simply select at most n_i elements from each S_i.

We have already seen some matroid analogues to graph theoretic ideas, so it makes sense to search for an analogue to connectivity. As every graphic matroid is the graphic matroid of a connected graph, it is hard to give a faithful analogue to 1-connectedness. Matroid connectivity gives an analogous to 2-edge-connectivity for matroids. Even tough 2-edge-connectedness is usually defined via separators, and one could do so for matroids (cf. Chapter 8 of [Oxl06]), we do so by defining a connectivity relation.

Definition 1.21 (Matroid Connectivity) *Let $M = (E, \mathfrak{I})$ be a matroid. We say that $e, f \in E$ are connected if there exists a circuit C such that $e, f \in C$,. We write $e\gamma f$ if e and f are connected or $e = f$. It turns out that γ is an equivalence relation. It is then natural to call the equivalence classes of γ the "connected components of M". If E is a connected component we say that M is connected; otherwise, M is disconnected.*

It turns out that each matroid can be decomposed into its connected components in a unique fashion.

Proposition 1.22 *Every matroid is a direct sum of its connected components. Namely, if M is a matroid, then $M = \bigoplus_{\Gamma \in E/\gamma} M|_\Gamma$. Moreover, this decomposition is unique, up to order of the components.*

The next result due to Krogdahl [Kro77], shows that the previous decomposition can be computed efficiently.

Proposition 1.23 *Let $M = (E, \mathfrak{I})$ be a matroid. We can compute the connected components of M in $O(|E|^2)$ time with $O(|E|^2)$ calls to the independence oracle.*

We end our brief discussion of matroid connectivity with a defining characteristic of connected matroids, namely, every pair of elements is the intersection between a circuit and a cocircuit.

Proposition 1.24 *If M is a connected matroid and x, y are distinct elements; then, there exists a circuit C and a cocircuit C^* such that $C \cap C^* = \{x, y\}$.*

1.2 Optimization problems on matroids

Matroids have played a key role in combinatorial optimization, both as a framework for set selection problems and by its close relationship with greedy algorithms.

We mostly consider weight functions that are linear on the elements, as follows:

Definition 1.25 (Weight of a subset) *If E is a finite set and $f : E \to \mathbb{R}$ is a function, we define the evaluation of any subset X of E by $f(X) = \sum_{e \in X} f(e)$*

A classic set selection problem that can be solved efficiently is the minimum spanning tree problem.

Problem 1 (Minimum spanning tree) *Given a connected graph $G = (V, E)$ and a weight function $w : E \to \mathbb{R}$, the minimum spanning tree problem (or MST for short) corresponds to finding a spanning tree T such that $w(T)$ is minimum.*

The MST problem can be generalized to finding a minimum weight basis on weighted matroids as follows.

Definition 1.26 (Weighted matroid) *A weighted matroid is a pair (M, w) such that M is a matroid and $w : E \to \mathbb{R}$ is a weight function.*

Problem 2 *Given a weighted matroid (M, w), define the weight of any subset X of E by $w(X) = \sum_{e \in X} w_e$. The minimum weight basis problem associated to (M, w) is to find a basis $T \in \mathcal{B}(M)$ such that $w(T)$ is minimum.*

The set of minimum weight bases form a new matroid. More so, this matroid can be characterized in terms of the original matroid and "level sets", as shown in the following two propositions.

Proposition 1.27 *Consider a weighted matroid (M, w). Then:*

$$\mathfrak{B}_w = \{T \in \mathfrak{B}(M) : T \textit{ is a basis of minimum weight}\},$$

is the set of basis of a matroid in E, called M_w.

The following result is slightly less known, but a proof can be found in [AK06].

Proposition 1.28 *Consider a weighted matroid (M, w). Denote the ith minimum weight by w_i, the number of different weights by r, and:*

$$E_i = \{e \in E \backslash E^* : w_e = w_i\}, \qquad \forall i \in \{1, ..., r\},$$
$$F_i = \bigcup_{j=1}^{i} E_i, \qquad \forall i \in \{0, ..., r\}.$$

Then, T is a basis of M_w if and only if:

1. *B is a basis of M.*
2. *$|B \cap E_i| = r(F_i) - r(F_{i-1})$ for each $i \in [r]$.*

Moreover $M_w = \bigoplus_{i=1}^{k} M|_{F_i}/F_{i-1}$

One of the more remarkable properties of matroids is their intimate relation with the greedy algorithm. Generally speaking, a greedy procedure picks the best "present" choice and never tracks back. We now describe the specific Greedy Algorithm for weighted matroids.

Definition 1.29 (Best-In Greedy Algorithm) *The greedy algorithm for the weighted matroid (M, w) proceeds as follows:*

Algorithm 1 `Greedy Algorithm`

Input: $\langle M, w\rangle$ where M is a matroid and $w : E \to \mathbb{R}$ is a weight function.
Output: A minimum weight basis.

$T \leftarrow \emptyset$
Sort E by non-increasing weight; $w(e_1) \leq w(e_2) \leq ... \leq w(e_m)$.
for $i \in [m]$ **do**
  if $T + e_i \in \mathfrak{I}(M)$ **then**
    $T \leftarrow T + e_i$
return T

Moreover, using the equivalence between independence and span, it is easy to see that the greedy algorithm always outputs the following base:

$$T_{\texttt{Greedy}} = \{e_i : e_i \notin \operatorname{span}\{e_j : j < i\}\}$$

It turns out that the greedy algorithm solves the minimum weight basis problem for weighted matroids optimally. Furthermore, if the greedy algorithm in an independence system solves the minimum weight basis problem for every weight function, such independence system must be a matroid. This is the famed definition of matroids via the greedy algorithm.

Proposition 1.30 (Matroids via the Greedy Algorithm) *Let M be a matroid. Then the following property holds:*

(G) For all weight functions $w : E \to \mathbb{R}$, the greedy algorithm produces a solution of the minimum weight basis problem associated to (M, w).

Furthermore, if E is a finite set and $\mathfrak{I}$ is a family of subsets of E such that $(E, \mathfrak{I})$ verifies $(I1)$, $(I2)$ and (G), then $(E, \mathfrak{I})$ is a matroid.

We have seen the fact that the greedy algorithm outputs basis of minimum weight on matroids. We end this chapter by noting that every minimum weight basis arises in this way.

Proposition 1.31 *Let (M, w) be a weighted matroid. If T is a basis of minimum weight, then there is a non-increasing ordering of E by weight such that the greedy algorithm outputs T.*

Chapter 2

Uncertainty matroids and uniformly minimum bases

This chapter studies uncertainty matroids and their uniformly minimum bases. It starts by defining these objects and formalizing the querying process that allows the revelation of true weights. We then study generalizations of the blue and red edges that appear when computing minimum spanning trees and their relation with uniformly minimum bases. These allows us to conclude the main results of this chapter;

A **characterization of existence of uniformly minimum bases**: An uncertainty matroid has a uniformly minimum basis if and only if each element is colored.

A **description of the matroid of uniformly minimum bases:** Uniformly minimum bases are the set of bases of a matroid. More so, a basis of the underlying matroid is a uniformly minimum basis if and only if it contains every non-trivial blue element, avoids every non-trivial red element and is of minimum weight in the weighted matroid obtained by contracting non-trivial blue elements and deleting non-trivial red elements.

These two theorems allows us to deduce efficient algorithms for the uniformly minimum basis problem.

2.1 Basic definitions

We first aim to model the computation of minimum weight bases on matroids with uncertainty on their weights. We do so by defining uncertainty matroids, which generalize weighted matroids, but instead of having a weight function one has a family of nonempty subsets of the real numbers indexed by the elements.

Then we introduce the concept of revelation, that is, the process of selecting some elements to be queried in order to "reveal" their true weights. By doing so, one gets another uncertainty matroid where the queried elements can take only one possible weight, eliminating their uncertainty.

Next we introduce uniformly minimum bases, which have minimum weight in every possible revelation. They are uniform in the sense that they do not depend on any specific revelation.

After giving these definitions, we formalize our two central problems. Suppose that we have to pay an element-dependent cost in order to perform queries. How can one find a minimum cost set of elements such that if we reveal them simultaneously, then, no matter what the true weights of the revealed elements are, the existence of a uniformly minimum basis is guaranteed. The second question is how to actually compute such a uniformly minimum basis in case one exists.

We start by defining uncertainty matroids and formalizing the process of querying and revealing elements.

Definition 2.1 (Uncertainty matroid) *An uncertainty matroid is a pair* $(M, \mathcal{A})$ *where* $M = (E, \mathfrak{I})$ *is a matroid and* $\mathcal{A} = \{A_e\}_{e \in E}$ *is a family of non-empty subsets of* $\mathbb{R}$ *indexed by* E*. Each set* A_e *denotes the uncertainty area of an element* e*. If* $e \in E$ *verifies that* $|A_e| = 1$ *we say that such element is trivial, as its true weight is already determined.*

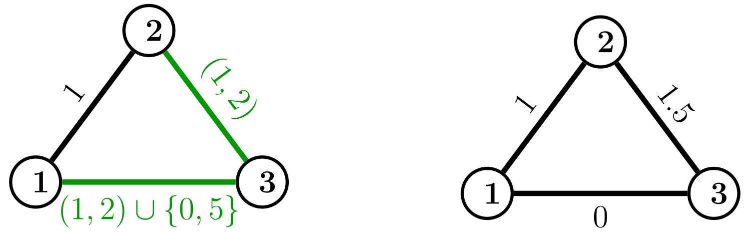
Observation We will assume without loss of generality that each uncertainty area is bounded. This reduction can be made by simply selecting any order preserving bijection between $\mathbb{R}$ and a bounded subset of $\mathbb{R}$ (e.g. $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$).

Definition 2.2 (Revelations and Realizations) *Let* $(M, \mathcal{A})$ *be an uncertainty matroid and* $X \subseteq E$*. A revelation that queries* X *and respects* $\mathcal{A}$ *is a family* $\mathcal{B} = \{B_e\}_{e \in E}$ *that verifies* $B_e = \{w_e\}$ *with* $w_e \in A_e$ *for each* $e \in X$ *and* $B_e = A_e$ *for each* $e \in E \backslash X$*. For* $X \subseteq E$ *we denote the set of revelations that query* X *and respect* $\mathcal{A}$ *by* $\mathcal{R}(X, \mathcal{A})$*. Revelations that query* $X = E$ *and respect* $\mathcal{A}$ *are called realizations of* $\mathcal{A}$*.*

Revelations and uncertainty matroids model the idea of querying the true weight of some elements. Before the revelation it is not certain which weight in the uncertainty area is the "true weight" of the queried element, and after the revelation there is only one possible weight.

It is worth noting that in a realization every element is trivial, so they are not really different to weighted matroids and we usually think of realizations as weight functions on the elements of the matroid.

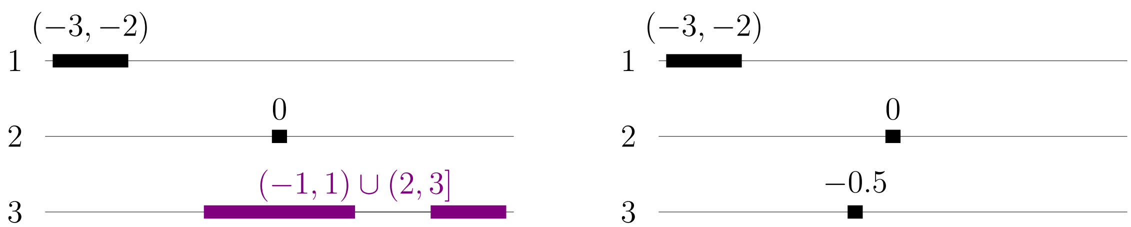
Example If $(M, \mathcal{A})$ is an uncertainty matroid such that $M = M(G)$ for a graph G we will represent $(M, \mathcal{A})$ as the graph G with each edge e labeled by its area of uncertainty A_e. For example if $M = M(K_3)$, $\mathcal{A} = \{A_e\}_{e \in E(K_3)}$, $A_{12} = \{1\}$, $A_{23} = (1, 2)$ and $A_{13} = (1, 2) \cup \{0, 5\}$ then $\mathcal{B} = \{B_e\}_{e \in E(K_3)}$ with $B_{12} = \{1\}$, $B_{23} = \{1.5\}$, $B_{13} = \{0\}$ is a realization of $\mathcal{A}$, as depicted in Fig. 2.1.



(a) $(M, \mathcal{A})$. (b) $(M, \mathcal{B})$

Figure 2.1: Uncertainty matroid $(M, \mathcal{A})$ and one realization. In green we show the set of non-trivial elements.

Example If $(M, \mathcal{A})$ is an uncertainty matroid such that M is uniform we will depict $(M, \mathcal{A})$ as a list of the elements and, next to each element e, a number line representation of its area of uncertainty. For example, if $M = U^2_{[3]}$, $\mathcal{A} = \{A_i\}_{i\in[3]}$, $A_1 = (-3, -2)$, $A_2 = \{0\}$, $A_3 = (-1, 1) \cup (2, 3]$, then $\mathcal{B} = \{B_i\}_{i\in[3]}$ with $B_1 = (-3, -2)$, $B_2 = \{0\}$ and $B_3 = \{-0.5\}$ is a revelation that queries $\{3\}$ and respects $\mathcal{A}$.



(a) $(M, \mathcal{A})$. (b) $(M, \mathcal{B})$.

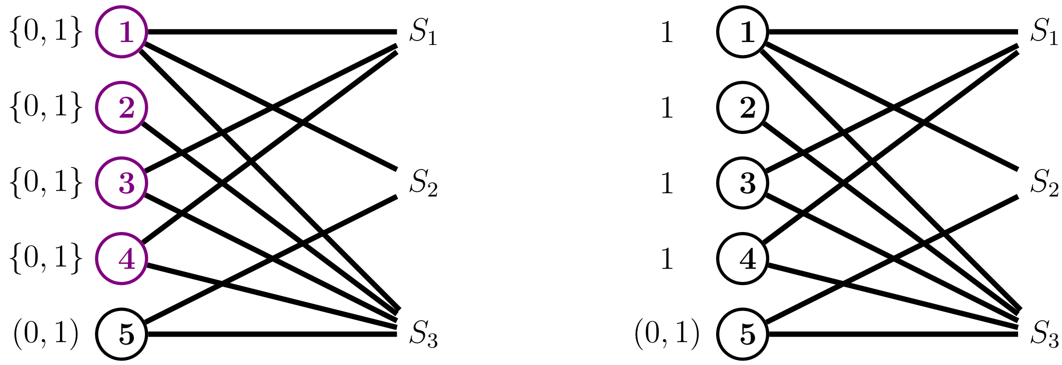
Figure 2.2: Uncertainty matroid $(M, \mathcal{A})$ and one revelation. In violet we show the queried set.

Example If $(M, \mathcal{A})$ is an uncertainty matroid such that M is a transversal matroid we will represent $(M, \mathcal{A})$ as the usual bipartite graph associated with the set sistem while labeling each element by its area of uncertainty. For example, if $S_1 = \{1, 3, 4\}$, $S_2 = \{1, 5\}$, $S_3 = [5]$, $M = M[\{S_i\}_{i\in[3]}]$, $A_i = \{0, 1\}$ for $i \in [4]$, $A_5 = (0, 1)$, then $\mathcal{B} = \{B_i\}_{i\in[3]}$ with $B_i = \{1\}$ for $i \in [4]$ and $B_3 = (0, 1)$ is a revelation that queries $[4]$ and respects $\mathcal{A}$.

The following definition is a generalization of minimum weight bases and it will be central to our study. These are also introduced in the context of robust interval optimization as strong bases (cf. to [KZ07]).

Definition 2.3 (Uniformly minimum bases) *Let $(M, \mathcal{A})$ be an uncertainty matroid. A uniformly minimum basis is a basis of the matroid M that has minimum weight for every realization of $\mathcal{A}$. The uniformly minimum bases of $(M, \mathcal{A})$ are called $(M, \mathcal{A})$-bases or $\mathcal{A}$-bases if M is clear from context. We will often denote uniformly minimum bases by the capital letter T.*

If the uncertainty matroid has only trivial elements, then uniformly minimum bases are exactly minimum weight bases.

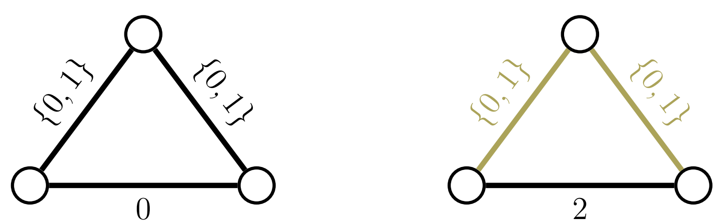


(a) $(M, \mathcal{A})$. (b) $(M, \mathcal{B})$

Figure 2.3: Uncertainty matroid $(M, \mathcal{A})$ and one revelation. In violet we show the queried set.

Even though one can always guarantee the existence of minimum weight bases in weighted matroids, this is not the case in uncertainty matroids. Furthermore, it is not clear when does an uncertainty matroid has a uniformly minimum basis or even whether a specific subset of elements is a uniformly minimum basis.

Example Consider the uncertainty matroid $(M, \mathcal{A}_1)$ and $(M, \mathcal{A}_2)$ shown in Fig. 2.4. There are no $(M, \mathcal{A}_1)$-bases because any such basis T must include exactly one of the edges with area $\{0, 1\}$, and that edge could be revealed as the heaviest in the circuit. This makes it impossible for T to be optimal in this revelation. $(M, \mathcal{A}_2)$ has an $(M, \mathcal{A}_2)$ basis, because in any revelation the edge with label 2 is the heaviest of the circuit, so the only (M, A_2)-basis consists of the other two edges .



(a) $(M, \mathcal{A}_1)$. (b) $(M, \mathcal{A}_2)$.

Figure 2.4: Two uncertainty matroids. In yellow we have marked an $(M_2, \mathcal{A}_2)$-basis.

The following proposition shows that uniformly minimum bases are preserved when revealing elements.

Proposition 2.4 *Let $(M, \mathcal{A})$ be an uncertainty matroid, $X \subseteq E$ and T an $\mathcal{A}$-basis. If $\mathcal{B} \in \mathcal{R}(X, \mathcal{A})$, then T is a $\mathcal{B}$-basis.*

Proof. As T is a minimum basis for every revelation $w \in \mathcal{R}(E, \mathcal{A})$ and $\mathcal{R}(E, \mathcal{B}) \subseteq \mathcal{R}(E, \mathcal{A})$ we have that T is a minimum basis for every revelation $w' \in \mathcal{R}(E, \mathcal{B})$. $\square$

In the case of uncertainty matroids that do not have uniformly minimum bases, we are especially interested in sets $X \subseteq E$ such that every revelation that queries X has a uniformly minimum basis. Note that these sets always exist because, if $X = E$, every element becomes trivial.

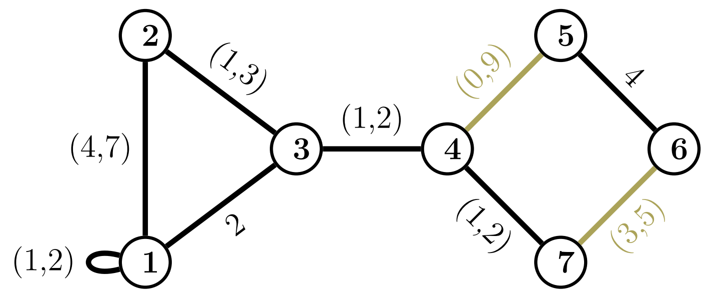
Definition 2.5 (Feasible query sets) *Let $(M, \mathcal{A})$ be an uncertainty matroid. A set $X \subseteq E$ is called a feasible query set, or feasible for short, if for all revelations $\mathcal{B} \in \mathcal{R}(X, \mathcal{A})$ there exists a $\mathcal{B}$-basis.*

Feasible query sets of minimum size (or, if there are costs associated to querying each set, of minimum cost) are sets whose revelation guarantees the existence of uniformly minimum bases (regardless of the true weight of non-revealed elements). They provide a first characterization of existence of uniformly minimum weight bases, namely, that they exists if and only if $\emptyset$ is a feasible query set. We pose two natural questions that are the central focus of our study:

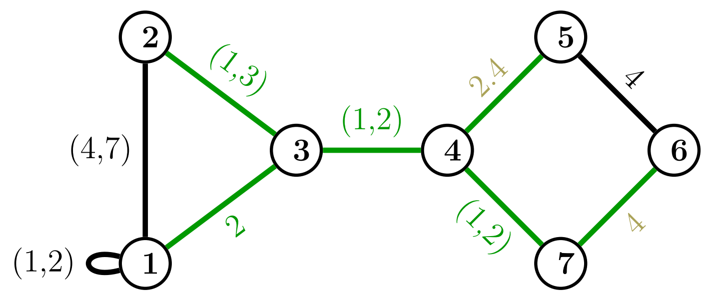
Problem 3 (Uniformly Minimum Basis) *Given an uncertainty matroid $(M, \mathcal{A})$. The uniformly minimum basis problem (or UMB for short) corresponds to finding an $\mathcal{A}$-basis or deciding that no $\mathcal{A}$-basis exists.*

Problem 4 (Minimum Cost Feasible Query Set) *Given an uncertainty matroid $(M, \mathcal{A})$ and a cost function on the elements $c : E \to \mathbb{R}$. The minimum cost feasible query set problem (or MCFQS for short) corresponds to computing a feasible query set X such that $c(X)$ is minimum. Note that the MCFQS with unit costs corresponds to computing a feasible query set X such that $|X|$ is minimum.*

Example



(a) In yellow we show a feasible query set.



(b) A revelation of the feasible query set marked in yellow. Once revealed we can find a uniformly minimum basis, this time marked in green.

Figure 2.5: An uncertainty matroid and a feasible query set.

2.2 Blue elements

In this section we provide a generalization of the so called blue rule for computing minimum spanning trees.

Blue Rule: If e is the lightest edge of a cutset of a graph G (allowing ties), then e belongs to at least one MST of G.

The blue rule has played a fundamental role in the development of MST algorithms. It is the driving force in Prim's, Boruvka's and the Round Robin algorithm, so it is a natural starting point when searching for algorithms that solve the MCFQS and UMB problem. The rule also provides some structural insight as elements that satisfy the blue rule are "safe to contract", in the following sense: if e is a blue rule edge, then T is an MST of G/e if and only if $T + e$ is an MST of G. We give a direct analogous to this structural insight and some useful characterizations of the blue rule for uncertainty matroids.

More discussion of the traditional blue rule can be found in chapter 6 of [Tar83]. The idea of blue elements under uncertainty has already been introduced in the inverval case by [YKP01] and [KZ07] under the name of weak elements.

We start by defining blue elements in uncertainty matroids.

Definition 2.6 (Blue elements) *Let $(M, \mathcal{A})$ be an uncertainty matroid and $e \in E$. We say that e is blue if in each realization, e is in a basis of minimum weight. That is, $\forall w \in \mathcal{R}(E, \mathcal{A})$, $\exists T$ w-basis such that $e \in T$.*

If each element is trivial, then an element is blue if and only if there is a basis of minimum weight that contains it. Moreover, if the matroid is graphic, one can prove that an element is blue if and only if satisfies the traditional blue rule.

Note that the definition of blue elements is dependent on the uncertainty matroid. If there is risk of confusion, we will make this dependency explicit, writing "e is blue in $(M, \mathcal{A})$" instead of "e is blue".

Example Every coloop of an uncertainty matroid is blue, since they are in every basis. In the same manner, no loop is blue, because they are not included in any basis.

The next proposition indicates that blue elements in uncertainty matroids verify the same "safe to contract" property as blue edges in weighted graphs.

Proposition 2.7 *Let $(M, \mathcal{A})$ be an uncertainty matroid and $e \in E$ blue. Then:*

$$T \text{ is an } (M/e, \mathcal{A} - A_e)\text{-basis if and only if } T + e \text{ is an } (M, \mathcal{A})\text{-basis.}$$

Proof. For the direct implication observe that T is a basis of M/e and e is not a loop of M, therefore $T + e$ is a basis of M. Let us suppose that $T + e$ is not an $(M, \mathcal{A})$-basis, then there exists a basis $T' \in \mathfrak{B}(M)$ and a realization $w \in \mathcal{R}(E, \mathcal{A})$ such that $w(T') < w(T+e)$. As e is blue in $(M, \mathcal{A})$ there exists T'' an $(M, w)-$basis such that $e \in T''$, so $w(T'') = w(T')$. Notice that $T'' - e$ is a basis of M/e. Since $w(T'' - e) < w(T)$ this contradicts the fact that T is $(M/e, \mathcal{A} - A_e)$-basis.

For the converse, suppose that T is not an $(M/e, \mathcal{A} - A_e)$-basis. Then we have T' a basis of M/e and a realization $w \in \mathcal{R}(E - e, \mathcal{A} - A_e)$ such that $w(T') < w(T)$. Extending w to $\hat{w} \in \mathcal{R}(E, \mathcal{A})$ by selecting any $\hat{w}_e \in A_e$ we have that $\hat{w}(T' + e) < \hat{w}(T + e)$. Since $T' + e$ is a basis of M we have contradicted that T is an $(M, \mathcal{A})$-basis. □

We have not yet used any property that requires comparing specific areas of the elements. The following definitions and notation will prove useful when dealing with comparisons.

Definition 2.8 *Let $(M, \mathcal{A})$ be an uncertainty matroid. For each $e \in E$ we will denote* $\sup A_e$ *by $U_e^{\mathcal{A}}$ and* $\inf A_e$ *by $L_e^{\mathcal{A}}$. This allows us to define the following sets:*

$$\begin{aligned}
\texttt{low}^{\mathcal{A}}(e) &= \{f \in E - e : U_f^{\mathcal{A}} \le L_e^{\mathcal{A}}\},\\
\texttt{high}^{\mathcal{A}}(e) &= \{f \in E - e : U_e^{\mathcal{A}} \le L_f^{\mathcal{A}}\},\\
\texttt{mid}^{\mathcal{A}}(e) &= \{f \in E - e : A_f \cap (L_e^{\mathcal{A}}, U_e^{\mathcal{A}}) \neq \emptyset\},\\
\texttt{both}^{\mathcal{A}}(e) &= \{f \in E \backslash \texttt{mid}^{\mathcal{A}}(e) - e : A_f \cap (-\infty, L_e^{\mathcal{A}}] \neq \emptyset \ \wedge \ A_f \cap [U_e^{\mathcal{A}}, \infty) \neq \emptyset\},\\
F^{\mathcal{A}}(e) &= \{f \in E - e : L_f^{\mathcal{A}} < U_e^{\mathcal{A}}\},\\
F^{*\mathcal{A}}(e) &= \{f \in E - e : L_e^{\mathcal{A}} < U_f^{\mathcal{A}}\}.
\end{aligned}$$

If the family of areas is clear from the context, we will usually omit superscripts.

Note that in every realization the elements of $\texttt{low}(e)$ are "lighter" than e. Similarly, each element of $\texttt{high}(e)$ is "heavier" than e, no matter the realization. The elements of $\texttt{both}(e)$ and $\texttt{mid}(e)$ have some realizations where e is heavier than them and some other realizations where e is lighter, the difference being that elements of $\texttt{both}(e)$ can never be revealed inside of (L_e, U_e). Additionally $E - e$ is partitioned by these sets when e is non-trivial. We formalize some of these properties in the following proposition:

Proposition 2.9 *Let $(M, \mathcal{A})$ be an uncertainty matroid and $e \in E$. We have the following:*

1. $E - e = \texttt{low}(e) \cup \texttt{mid}(e) \cup \texttt{high}(e) \cup \texttt{both}(e)$.
2. $F(e) = \{f \in E - e : L_f < U_e\} = E \backslash \texttt{high}(e) - e$.
3. $F^*(e) = \{f \in E - e : L_e < U_f\} = E \backslash \texttt{low}(e) - e$.

Moreover, if e is non-trivial we have the following strengthened versions:

4. *$E - e$ is partitioned by $\texttt{low}(e)$, $\texttt{mid}(e)$, $\texttt{high}(e)$ and $\texttt{both}(e)$.*
5. $F(e) = E \backslash \texttt{high}(e) - e = \texttt{low}(e) \cup \texttt{mid}(e) \cup \texttt{both}(e)$.
6. $F^*(e) = E \backslash \texttt{low}(e) - e = \texttt{high}(e) \cup \texttt{mid}(e) \cup \texttt{both}(e)$.

Proof.

1. As every set on the right hand side is included in $E - e$, the inclusion $\texttt{low}\,(e) \cup \texttt{mid}\,(e) \cup \texttt{high}\,(e) \cup \texttt{both}\,(e) \subseteq E - e$ is clear.

   For the other inclusion, suppose that $f \in E - e$ and $f \notin \texttt{low}\,(e) \cup \texttt{high}\,(e) \cup \texttt{mid}\,(e)$. Then, $L_e < U_f$ and, since $A_f \cap (L_e, U_e) = \emptyset$, we get $A_f \cap [U_e, \infty) \neq \emptyset$. Similarly, $L_f < U_e$ and using that $A_f \cap (L_e, U_e) = \emptyset$, we conclude that $A_f \cap (-\infty, U_e] \neq \emptyset$. Consequently, $f \in \texttt{both}\,(e)$, which implies the proposition.
2. $g \notin F(e)$ if and only if $g \in \{f \in E - e : L_f \geq U_e\} = \texttt{high}\,(e)$ which implies the property.
3. $g \notin F^*(e)$ if and only if $g \in \{f \in E - e : L_e \geq U_f\} = \texttt{low}\,(e)$ which implies the property.
4. It is clear that, if $f \in \texttt{low}\,(e)$, then $A_f \cap (L_e, \infty) = \emptyset$, which implies that e is neither in $\texttt{mid}\,(e)$ nor $\texttt{both}\,(e)$. Similarly, if $f \in \texttt{high}\,(e)$, then $A_f \cap (-\infty, U_e)$ which implies that f is not in $\texttt{mid}\,(e)$ nor $\texttt{both}\,(e)$. As $\texttt{mid}\,(e) \cap \texttt{both}\,(e) = \emptyset$, we only need to prove that $\texttt{low}\,(e) \cap \texttt{high}\,(e) = \emptyset$. Suppose that there is some $f \in \texttt{low}\,(e) \cap \texttt{high}\,(e)$. Then, $L_e = U_e = L_f = U_f$ which contradicts the fact that e is non-trivial.
5. $F(e) = (\texttt{low}\,(e) \cup \texttt{mid}\,(e) \cup \texttt{high}\,(e) \cup \texttt{both}\,(e)) \setminus \texttt{high}\,(e) = \texttt{low}\,(e) \cup \texttt{mid}\,(e) \cup \texttt{both}\,(e)$.
6. $F^*(e) = (\texttt{low}\,(e) \cup \texttt{mid}\,(e) \cup \texttt{high}\,(e) \cup \texttt{both}\,(e)) \setminus \texttt{low}\,(e) = \texttt{high}\,(e) \cup \texttt{mid}\,(e) \cup \texttt{both}\,(e)$. □

Example Consider the uncertainty matroid $(U^3_{[6]}, \mathcal{A})$ as depicted in Fig. 2.6. Using the definitions, we get that $\texttt{low}\,(1) = \{4\}$, $\texttt{high}\,(1) = \{6\}$, $\texttt{mid}\,(1) = \{2, 3\}$ and $\texttt{both}\,(e) = \{5\}$.

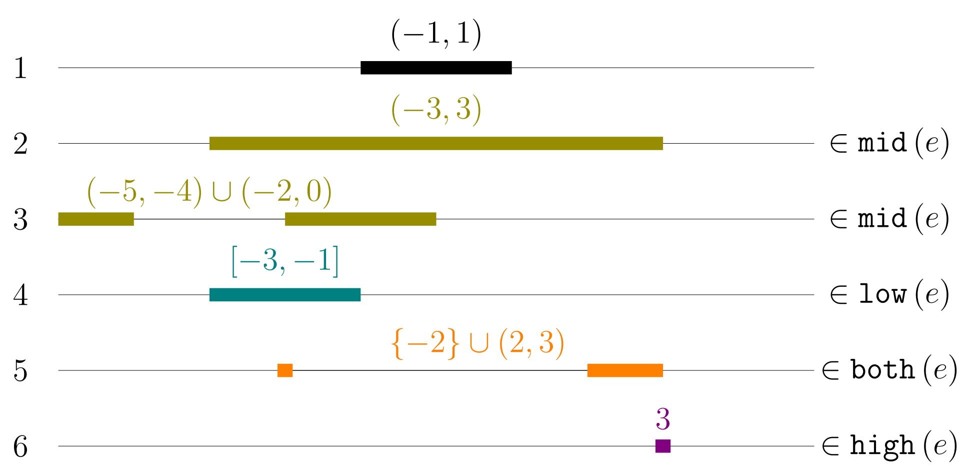


Figure 2.6: Uncertainty matroid $(U^3_{[6]}, \mathcal{A})$. We have marked the set $\texttt{mid}\,(1)$ in olive, $\texttt{low}\,(1)$ in teal, $\texttt{both}\,(1)$ in orange and $\texttt{high}\,(1)$ in violet.

As blue elements are in a minimum weight basis for any realization, this should occur even when considering "hard" realizations that make it difficult for a particular element e to be included in some minimum weight basis. Intuitively, this occurs when e is revealed close to U_e and every other edge $f \neq e$ near L_f. Surprisingly, the converse also holds, providing a characterization of blueness. This property was already noted for intervals in [YKP01] and [KZ07].

Since $F(e)$ are the elements that are lighter than e in this "worst case" realization, we first state this property in terms of the span of $F(e)$.

Proposition 2.10 *Let $(M, \mathcal{A})$ be an uncertainty matroid. Then $e \in E$ is blue if and only if $e \notin \operatorname{span} F(e)$.*

Proof. For the direct implication, let $K = \min_{f \in F(e)} U_e - L_f > 0$. For each $f \in E - e$ we can choose $\varepsilon_f \geq 0$ such that $L_f + \varepsilon_f \in A_f$. Moreover, we can guarantee that $\varepsilon_f < \frac{K}{2}$. In a similar way, we can choose $\varepsilon_e \in [0, K/2)$ such that $U_e - \varepsilon_e \in A_e$. We then consider the following realization $w \in \mathcal{R}(E, \mathcal{A})$:

$$w_f = \begin{cases} U_e - \varepsilon_e & \text{if } f = e, \\ L_f + \varepsilon_f & \text{if } f \neq e. \end{cases}$$

Note that if $f \in F(e)$ then $w_f < w_e$. Indeed:

$$w_f = L_f + \varepsilon_f < L_f + \frac{K}{2} \leq L_f + \frac{U_e - L_f}{2} = U_e - \frac{U_e - L_f}{2} \leq U_e - \frac{K}{2} < U_e - \varepsilon_e = w_e.$$

Suppose now that $e \in \operatorname{span} F(e)$. Then, there exists a circuit $C \subseteq F(e) + e$ such that e is the heaviest element of C. This implies that e is not in any w-basis, which contradicts that e was blue.

For the converse, first note that e is not a loop, as we would have $e \in \operatorname{span} \emptyset \subseteq \operatorname{span} F(e)$. Let us suppose that e is not blue. Then, there exists a realization $w \in \mathcal{R}(E, \mathcal{A})$ such that e is not in any w-basis. Choose T to be any w-basis and C the fundamental circuit of $T + e$. As $e \notin \operatorname{span} F(e)$ we have that $C \subsetneq F(e) + e$. Since e is not a loop we have that $|C| \geq 2$ and that $F(e) \backslash C \neq \emptyset$, select any $f \in C \backslash F(e)$, then:

$$w_f \geq L_f \geq U_e \geq w_e$$

We conclude that $T - f + e$ is a w-basis that contains e. $\square$

We now restate the property in terms of being selected in a minimum weight basis when using the worst case weight function. As one could expect we only have to focus a weight function that fixes e to U_e and every other element to L_f, instead of realizations that are close to L_f and U_e. Note that this is not necessarily a realization as L_f or U_e could not belong to A_f or A_e respectively.

Proposition 2.11 *Let $(M, \mathcal{A})$ be an uncertainty matroid. Then $e \in E$ is blue if and only if e is in a minimum weight basis for the following weight function:*

$$w_f = \begin{cases} U_e & \textit{if } f = e, \\ L_f & \textit{if } f \neq e. \end{cases}$$

Proof. We sort the edges by weight in increasing order, breaking ties in favor of e, i.e. e is considered lighter than any other element of the same weight. Notice that:

$$F(e) = \{f \in E - e : L_f < U_e\} = \{f \in E - e : w_f < w_e\} = \{e_1, \ldots, e_{k-1}\}$$

Since e is blue, we have that $e \notin \operatorname{span}\{e_1, \ldots, e_{k-1}\}$. Then, when executing the greedy algorithm under the aforementioned order, e_k is selected.

For the converse, as e is in a minimum weight base there exists an ordering of E such that $w(e_1) \leq \cdots \leq w(e_m)$ and the greedy algorithm selects $e = e_k$. Using this, we have:

$$\{e_1, \ldots, e_{k-1}\} \supseteq \{f \in E - e : w_f < w_e\} = F(e)$$

Since $e \notin \operatorname{span}\{e_1, \ldots, e_{k-1}\}$, we have that $e \notin \operatorname{span} F(e)$. $\square$

Example Using this last property we can test if a certain element is blue, by simply executing the greedy algorithm on the uncertainty matroid with the "worst case" weight function for the particular element.

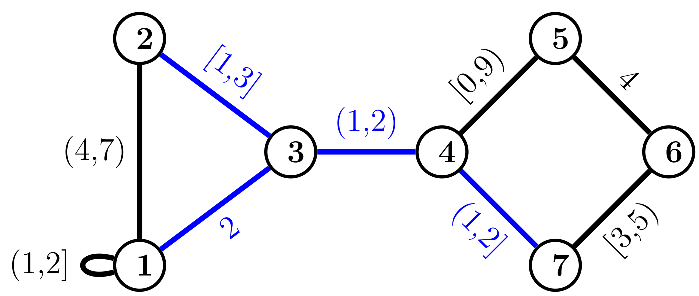


Figure 2.7: One uncertainty matroid and all of its blue elements.

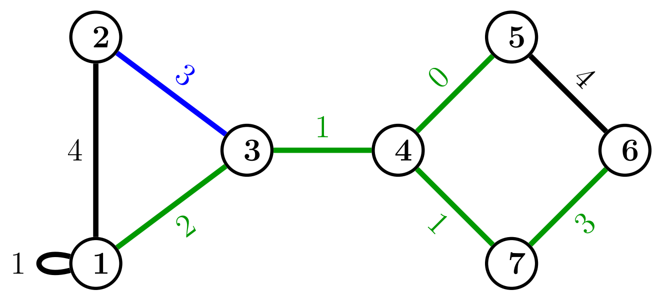


Figure 2.8: A certificate of the blueness of edge $e = 23$. In green we show a set that when adding e forms an MST for its worst case weight function and, by Proposition 2.11, the edge e must be blue.

Blue elements will play the following role in solving the MCFQS problem. Intuitively, they are a first example of elements that do not need to be revealed since one could devise an algorithm that consists on successive "safe contractions" eliminating all blue edges in each iteration. Most importantly, they are key in understanding the UMB problem. In fact, as we will see in the following section, they will provide useful characterizations of the existence of uniformly minimum bases and, ultimately, lead to an algorithm to compute them.

2.3 Red elements

In this section we give a generalization of the red rule for computing minimum spanning trees.

Red Rule: If e is the heaviest edge of a cycle of a graph G (allowing ties), then e avoids at least one MST of G.

Even though the red rule is not as popular in the development of fast MST algorithms, it is still a key component of some of them, the most famous example being Kruskal's algorithm. The rule gives an analogous structural insight but, instead of "safe contraction", it provides "safe deletion" of red elements. That is, if e is a red rule edge, then T is an MST of $G - e$ if and only if T is an MST of G. We give an analogous result to this structural insight and some characterizations for the red rule in uncertainty matroids.

Similarly to the blue rule, an in depth discussion of the classical red rule can be found on [Tar83].

The red rule and the generalization proposed here are dual to the blue rule and its generalization. This will be especially apparent in the proofs of this section, where the arguments and ideas presented will be the same as the ones used in the previous section, but with dual arguments instead.

Definition 2.12 (Red and colored elements) *Let $(M, \mathcal{A})$ be an uncertainty matroid and $e \in E$. We say that e is red if, in each realization, e avoids a basis of minimum weight. That is, $\forall w \in \mathcal{R}(E, \mathcal{A})$, $\exists T$ w-basis such that $e \notin T$. If an element is blue or red we say that it is colored.*

If each element is trivial, an element is red if and only if there is a basis of minimum weight that avoids it. Moreover, if the matroid is graphic, one can prove that an element is red if and only if satisfies the traditional red rule.

Similarly to the definition of blue elements, the definition of red elements is also dependent on the specific uncertainty matroid. If there is risk of confusion, we will make this dependency explicit, writing "e is red in $(M, \mathcal{A})$" instead of "e is red".

We now state and prove the "safe to delete" property of red elements.

Proposition 2.13 *Let $(M, \mathcal{A})$ be an uncertainty matroid and $e \in E$ red. Then:*

$$T \text{ is an } (M - e, \mathcal{A} - A_e)\text{-basis if and only if } T \text{ is an } (M, \mathcal{A})\text{-basis.}$$

Proof. For the direct implication observe that, as e is red, there is a basis $T^- \in \mathfrak{B}(M)$ that avoids e. Furthermore, $T^- \in \mathfrak{B}(M - e)$. Since $T, T^- \in \mathfrak{B}(M - e)$, then $|T^-| = |T|$ and T is also a basis of M. Let us suppose that T is not an $(M, \mathcal{A})$-basis. Then, there exists a basis $T' \in \mathfrak{B}(M)$ and a realization $w \in \mathcal{R}(E, \mathcal{A})$ such that $w(T') < w(T)$. As e is red in $(M, \mathcal{A})$ there exists T'', an $(M, w)-$basis, such that $e \notin T''$. So, $w(T'') = w(T')$. Notice that T'' is a basis of $M - e$. Since $w(T'' - e) < w(T)$ this contradicts the fact that T is $(M - e, \mathcal{A} - A_e)$-basis.

For the converse, it is clear that T is a basis of $M - e$. Suppose that T is not an $(M-e, \mathcal{A}-A_e)$-basis. Then, we have T' a basis of $M-e$ and a realization $w \in \mathcal{R}(E-e, \mathcal{A}-A_e)$ such that $w(T') < w(T)$. Extending w to $\hat{w} \in \mathcal{R}(E, \mathcal{A})$ by selecting any $\hat{w}_e \in A_e$ we have that $\hat{w}(T') < \hat{w}(T)$. Since $|T'| = |T|$ we have that T' is a basis of M which contradicts that T is an $(M, \mathcal{A})$-basis. □

We now provide dual characterizations to the ones presented for the blue elements. We consider a realization that favors e when selecting minimum weight bases, that is, a realization with e close to L_e and every other element $f \neq e$ is near U_e. Analogously to the blue case, we first state this property in terms of the span of $F^*(e) = \{e \in E - e : L_e < U_f\}$, since they are the elements that are heavier than e in this "best case" realization.

Proposition 2.14 *Let $(M, \mathcal{A})$ be an uncertainty matroid. Then, $e \in E$ is red if and only if $e \notin \operatorname{span}^* F^*(e)$.*

Proof. For the direct implication, define $K = \min_{f \in F^*(e)} U_f - L_e > 0$. For each $f \in E - e$ we can choose $\varepsilon_f \in [0, K/2)$ such that $U_f - \varepsilon_f \in A_f$ similarly we can pick $\varepsilon_e \in [0, K/2)$ such that $L_e + \varepsilon_e \in A_e$. We then consider the following realization $w^* \in \mathcal{R}(E, \mathcal{A})$:

$$w^*_f = \begin{cases} L_e + \varepsilon_e & \text{if } f = e, \\ U_f - \varepsilon_f & \text{if } f \neq e. \end{cases}$$

Note that if $f \in F^*(e)$ then $w^*_f > w^*_e$, indeed:

$$w^*_f = U_f - \varepsilon_f > U_f - \frac{K}{2} \geq U_f - \frac{U_f - L_e}{2} = L_e + \frac{U_f - L_e}{2} \geq L_e + \frac{K}{2} > U_e - \varepsilon_e = w^*_e$$

Suppose now that $e \in \operatorname{span}^* F^*(e)$, then there exists a cocircuit $C^* \subseteq F^*(e) + e$ such that e is the lightest element of C^*, this implies that e is in every w-basis, contradicting that e was red.

For the converse first note that e is not a coloop, as we would have $e \in \operatorname{span}^* \emptyset \subseteq \operatorname{span}^* F^*(e)$. Let us suppose that e is not red, then there exists a realization $w \in \mathcal{R}(E, \mathcal{A})$ such that e is in every w-basis. Choose T to be any w-basis and C^* the fundamental circuit of $T^c + e$, as $e \notin \operatorname{span} F^*(e)$ we have that $C^* \subsetneq F^*(e) + e$. Since e is not a coloop we have that $|C^*| \geq 2$ and that $F^*(e) \backslash C^* \neq \emptyset$, select any $f \in C^* \backslash F^*(e)$, then:

$$w_e \geq L_e \geq U_f \geq w_f$$

We conclude that $T + f - e$ is a w-basis that contains e. □

We now prove the equivalence between red elements and elements not selected in a minimum weight basis for their best case weight function.

Proposition 2.15 *Let $(M, \mathcal{A})$ be an uncertainty matroid. Then $e \in E$ is red if and only if e is not in some minimum weight basis for the following weight function:*

$$w^*_f = \begin{cases} L_e & \text{if } f = e, \\ U_f & \text{if } f \neq e. \end{cases}$$

Proof. We sort the edges by weight in ascending fashion, breaking ties against e, i.e. e is considered heavier than any other element of the same weight. Notice that:

$$F^*(e) = \{f \in E - e : L_e < U_f\} = \{f \in E - e : w^*_f < w^*_e\}$$

Since e is red, we have that $e \notin \operatorname{span}^*\{e_{k+1}, \ldots, e_m\}$ or equivalently $e \in \operatorname{span}\{e_1, \ldots, e_{k-1}\}$, then when executing the greedy algorithm under the aforementioned order e_k is not selected.

For the converse, as e avoids a minimum weight base there exists an ordering of E such that $w^*(e_1) \leq \cdots \leq w^*(e_m)$ and the greedy algorithm under this ordering does not select $e = e_k$. Using this, we have $e \in \operatorname{span}\{e_1, \ldots, e_{k-1}\}$ and consequently $e \notin \operatorname{span}^*\{e_{k+1}, \ldots, e_m\}$. Since

$$\{e_{k+1}, \ldots, e_m\} \supseteq \{f \in E - e : w^*_f < w^*_e\} = F^*(e)$$

we have that $e \notin \operatorname{span}^* F^*(e)$. $\square$

Example Similarly to the blue case one can test whether a particular element is red by executing the greedy algorithm on the uncertainty matroid with the "best case" weight function for the particular element.

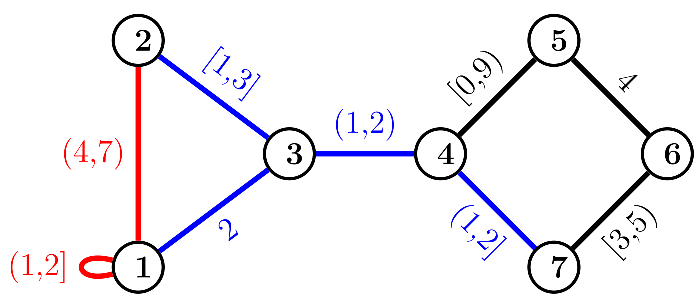


Figure 2.9: One uncertainty matroid and all of its blue and red elements. Note that there are elements that are not colored.

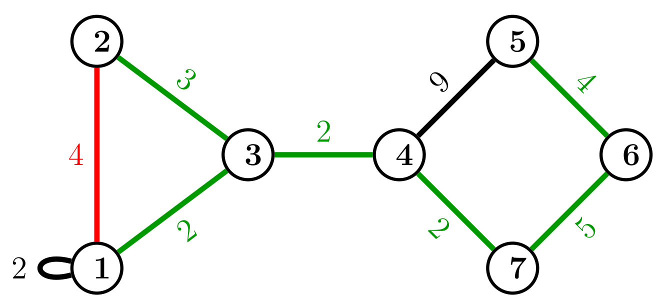


Figure 2.10: A certificate of the redness of edge $e = 12$. In green we show an MST for the best case weight function that avoids e and, by Proposition 2.15, edge e must be red.

Colored elements are intimately related to the MCFQS and UMB problems. We have already seen how blue elements intuitively do not need to be revealed. The same argument can be made for red elements, where one could imagine an algorithm that works by successive "safe deletion" arriving at an uncertainty matroid with no red elements. The relation between colored elements and uniformly minimum will be throughly explored in the following sections.

2.4 Properties of uniformly minimum bases

In this section we explore some properties of uniformly minimum bases. First, we analyze the relationship between colored edges and uniformly minimum bases. We prove that every uniformly minimum basis contains every non-trivial blue element and avoids every non-trivial red element. Moreover, when uniformly minimum bases do exist they provide an additional characterization of blue and red elements. We end the section with two propositions on the structure of uniformly minimum bases, namely, that they can only differ on trivial elements and that they are well-behaved with respect to the direct sum.

Most of the proofs in this section work via an exchange argument similar to the one first presented in proposition 2.10. The key idea of this argument is that if an element is not colored we can select another element in a circuit or a cocircuit and perform some exchange. If we ask for extra properties of the elements, like non-triviality, we can arrive at stronger conclusions.

Consider the following "strict" version of the graph blue rule.

Blue Rule (Strict version): If e is the lightest edge of a cutset in a graph G (disallowing ties), then e is in *every* MST of G.

This provides a strengthened version of the blue rule. Similarly, one can consider a strict version of the red rule.

Red Rule (Strict version): If e is the heaviest edge of a circuit in a graph G (disallowing ties), then e is not in *any* MST of G.

We generalize this results, but replace the notion of strictness in the colored elements by non-triviality. Intuitively, non-trivial edges are "strict" since they cannot always participate in ties, as one can consider realizations where they have different weight (effectively breaking the tie).

Proposition 2.16 *Let $(M, \mathcal{A})$ be an uncertainty matroid and $e \in E$ a blue and non-trivial element. If T is an $\mathcal{A}$-basis, then $e \in T$.*

Proof. Let T be an $\mathcal{A}$-basis such that $e \notin T$. Consider C the fundamental circuit of $T + e$, as e is blue, we have that e is not a loop and $C \not\subseteq F(e) + e$ (since $e \notin \operatorname{span} F(e)$). Considering that e is non-trivial there exists a realization $w \in \mathcal{R}(E, \mathcal{A})$ such that $U_e > w_e$. Then, by selecting $f \in C \backslash F(e)$, we have:

$$w_f \geq L_f \geq U_e > w_e,$$

This implies that $w(T - f + e) < w(T)$, which contradicts the fact that T is an $\mathcal{A}$-basis. $\square$

Proposition 2.17 *Let $(M, \mathcal{A})$ be an uncertainty matroid, and $e \in E$ a red and non-trivial element. If T is an $\mathcal{A}$-basis, then $e \notin T$.*

The proof for the red elements works in the same way but using fundamental cocircuits instead of fundamental circuits.

Proof. Let T be an $\mathcal{A}$-basis such that $e \in T$. Consider C^* the fundamental cocircuit of $T^c + e$, as e is red, we have that e is not a coloop and $C^* \not\subseteq F^*(e) + e$ (since $e \notin \operatorname{span} F^*(e)$). Considering that e is non-trivial there exists a realization $w \in \mathcal{R}(E, \mathcal{A})$ such that $w_e > L_e$, then selecting $f \in C^* \backslash F^*(e)$ we have:

$$w_e \leq L_e \leq U_f < w_f,$$

This implies that $w(T^c + e - f) > w(T^c)$ and $w(T - e + f) < w(T)$, which contradicts the fact that T is an $\mathcal{A}$-basis. $\square$

From these propositions we find that all uniformly minimum bases share all non-trivial blue elements and avoid all non-trivial red elements. It is now natural to ask the relation between general colored elements and uniformly minimum bases (independently of non-triviality). In the graphic case, blue and red edges provide the existence of an MST that respectively contains and avoid such edges. Furthermore, this property characterizes blue and red edges in graphs. This same characterization can be obtained for colored elements in uncertainty matroids as shown by the next results.

Proposition 2.18 *Let $(M, \mathcal{A})$ be an uncertainty matroid, such that an $\mathcal{A}$-basis exists. Then $e \in E$ is blue if and only if there exists an $\mathcal{A}$-basis such that $e \in T$.*

Proof. Suppose that T is an $\mathcal{A}$-basis such that $e \in T$. Since each $\mathcal{A}$-basis is a w-basis for every realization $w \in \mathcal{R}(E, \mathcal{A})$, we have that e is blue.

We now prove the converse. Note that if e is nontrivial the result follows from Proposition 2.16. We then assume that e is trivial. Let T be an $\mathcal{A}$-basis, if $e \in T$ we have nothing to prove, so assume that $e \notin T$. Again, consider C the fundamental circuit of $T + e$. As before, e is blue, we have that e is not a loop and there exists some $f \in C \backslash F(e)$. If f is non-trivial we would have $w \in R(E, \mathcal{A})$ such that $w_f > L_f$, and, therefore:

$$w_f > L_f \geq U_e = w_e.$$

So $w(T + e - f) < w(T)$, arriving at a contradiction. Moreover, if $L_f > U_e$, then for any $w \in R(E, \mathcal{A})$ we would have:

$$w_f = L_f > U_e = w_e,$$

which leads to the same contradiction. The only remaining case is that e and f are trivial elements of equal weight, so $w(T) = w(T + e - f)$ for each $w \in \mathcal{R}(E, \mathcal{A})$, implying that $T + e - f$ is also an $\mathcal{A}$-basis. $\square$

Proposition 2.19 *Let $(M, \mathcal{A})$ be an uncertainty matroid such that an $\mathcal{A}$-basis exists. Then $e \in E$ is red if and only if it exists an $\mathcal{A}$-basis such that $e \notin T$.*

Again, the red proof follows the same ideas as the blue proof, but working with fundamental cocircuits instead of fundamental circuits.

Proof. Suppose that T is an $\mathcal{A}$-basis such that $e \notin T$. Since each $\mathcal{A}$-basis is a w-basis for every realization $w \in \mathcal{R}(E, \mathcal{A})$, we have that e is red.

For the converse, note that if e is nontrivial the result follows from Proposition 2.17. We then assume that e is trivial. Let T be an $\mathcal{A}$-basis, if $e \notin T$ we have nothing to prove, so assume that $e \in T$. Consider C^* the fundamental cocircuit of $T^c + e$. As e is red, we have that e is not a coloop and there exists some $f \in C^* \backslash F(e)$. If f is non-trivial we would have $w \in R(E, \mathcal{A})$ such that $w_f < U_f$, and:

$$w_f < U_f \leq L_e = w_e.$$

So $w(T^c + e - f) > w(T^c)$, arriving at a contradiction, since $w(T - e + f) < w(T)$. Moreover, if $L_e > U_f$, for any $w \in R(E, \mathcal{A})$ we would have:

$$w_f \leq U_f < L_e = w_e,$$

which leads to the same contradiction. The only remaining case is that e and f are trivial edges of equal weight, so $w(T) = w(T - e + f)$ for each $w \in \mathcal{R}(E, \mathcal{A})$, implying that $T + e - f$ is also an $\mathcal{A}$-basis. $\square$

With further examination of the proofs of Propositions 2.18 and 2.19, we find that if one tries to swap colored elements while keeping uniformly minimum bases, the elements involved must be trivial. This is not only the case when trying to swap colored elements. Furthermore, any two uniformly minimum bases can differ only on trivial elements, as shown in the next proposition.

Proposition 2.20 *Let $(M, \mathcal{A})$ be an uncertainty matroid. If T_1 and T_2 are two $\mathcal{A}$-bases, then every element of $T_1 \Delta T_2$ is trivial.*

Proof. Without loss of generality, we can assume that there is $e \in T_1 \backslash T_2$ non-trivial. Strong basis exchange allows us to pick $f \in T_2 \backslash T_1$, such that, $T_1 - e + f$ and $T_2 + e - f$ are basis of M. As e is non-trivial, there exists a realization $w \in \mathcal{R}(E, \mathcal{A})$ such that $w_e > w_f$ or $w_e < w_f$. If $w_e > w_f$, then $w(T_1 - e + f) < w(T_1)$, which contradicts the fact that T_1 is an $\mathcal{A}$-basis. On the other hand, if $w_f > w_e$, then $w(T_2 + e - f) < w(T_2)$, this is also a contradiction, since T_2 is an $\mathcal{A}$-basis. $\square$

We end this section with an expected result, namely, that uniformly minimum bases behave similarly to bases when considering matroids that can be expressed as the direct sum of k matroids. Recall that, for matroid bases, we have that T is a basis of the direct sum if and only if the restriction of T on each summand is a basis. A similar proposition holds for uncertainty matroids and uniformly minimum bases, but we need to work out a way to distribute the family of areas between the matroids in the direct sum. We do so by defining restrictions of families of areas.

Definition 2.21 (Restriction) *Let E be a finite set, $\mathcal{A} = \{A_e\}_{e \in E}$ a family of sets indexed by E and $F \subseteq E$. We define $\mathcal{A}$ restricted to F by:*

$$\mathcal{A}|_F = \{A_e \in \mathcal{A} : e \in F\}.$$

We can now state the result.

Proposition 2.22 *Let $(M, \mathcal{A})$ be an uncertainty matroid such that $M = \bigoplus_{i=1}^{k} M_i$ and $E_i = E(M_i)$ for each $i \in [k]$. Then, T is an $(M, \mathcal{A})$-basis if and only if $T \cap E_i$ is an $(M_i, \mathcal{A}|_{E_i})$-basis for every $i \in [k]$.*

Proof. Suppose that $T \cap E_i$ is not an $(M, \mathcal{A})$-basis for some $i \in [k]$. Then, there exists T' basis of M_i and a realization $w \in \mathcal{R}(E_i, \mathcal{A}|_{E_i})$ such that $w(T') < w(T \cap E_i)$. We can extend such w to $\hat{w} \in \mathcal{R}(E, \mathcal{A})$ by selecting $\hat{w}_e \in A_e$ arbitrarily, then:

$$\hat{w}(T' \cup (T \backslash E_i)) = \hat{w}(T') + \hat{w}(T \backslash E_i) < \hat{w}(T \cap E_i) + \hat{w}(T \backslash E_i) = \hat{w}(T),$$

as $T' \cup (T \backslash E_i)$ is a basis of M, we have contradicted the fact that T is an $(M, \mathcal{A})$-basis.

For the converse, suppose that T is not an $(M, \mathcal{A})$-basis, then there exists T' basis of M and a realization $w \in \mathcal{R}(E, \mathcal{A})$ such that $w(T') < w(T)$. As every $T \cap E_i$ is an $(M, w|_{E_i})$-basis, we have that:

$$w(T) = \sum_{i=1}^{k} w(T \cap E_i) \leq \sum_{i=1}^{k} w(T' \cap E_i) = w(T'),$$

which contradicts the previous inequality. □

This section was a first effort on addressing the UMB problem. Specifically, the part that involves finding uniformly minimum bases. We know now that we do not need to pay attention to red non-trivial elements and can include from the start every blue non-trivial element in our uniformly minimum basis. We also concluded that we can consider only connected components of the matroid as uniformly minimum basis behave well with direct sums. The question remains on what to do with non-colored non-trivial elements and trivial elements. These questions will be the focus of following sections.

2.5 Colorings and existence of uniformly minimum bases

The main objective of this section is to provide a solution for the decision part of the UMB problem. Specifically, we characterize uncertainty matroids that have uniformly minimum bases as the ones that are fully colored.

An outline of the proof is as follows. We first prove that if uniformly minimum bases exist there can not be non-colored elements (using the circuit and cocircuit of which a supposed non-colored element would participate). For the converse, we first prove the interesting fact that contracting blue non-trivial elements and deleting red non-trivial elements preserves colorings. Then, we contract and delete every non-trivial edge, arriving at an uncertainty matroid with trivial areas, where the result is easier to prove.

A small complication encountered in some of the proofs presented occurs when trying to pick proper realizations with weights close to L_e or U_e. This is not really a problem, since colors only depend on infima $\{L_e\}_{e\in E}$ and suprema $\{U_e\}_{e\in E}$ as made explicit by Propositions 2.11 and 2.15. We start by showing that colors are the same when replacing A_e by $[L_e, U_e]$ and vice-versa. This simplifies some proofs in this section.

Definition 2.23 *Let $(M, \mathcal{A})$ be an uncertainty matroid. We define the closure of $\mathcal{A}$ as:*

$$\operatorname{cl} \mathcal{A} = \{[L_e, U_e] : e \in E\}$$

Proposition 2.24 *Let $(M, \mathcal{A})$ be an uncertainty matroid. If $\mathcal{B} = \{B_e\}_{e\in E}$ is a family of areas such that $L_e^{\mathcal{A}} = L_e^{\mathcal{B}}$ and $U_e^{\mathcal{A}} = U_e^{\mathcal{B}}$ for each $e \in E$, then $(M, \mathcal{A})$ and $(M, \mathcal{B})$ have the same colors. In particular $(M, \mathcal{A})$ and $(M, \operatorname{cl} \mathcal{A})$ have the same colors.*

Proof. Note that $F^{\mathcal{A}}(e) = F^{\mathcal{B}}(e)$ and $F^{*\mathcal{A}}(e) = F^{*\mathcal{B}}(e)$. By propositions 2.10 and 2.14 we have that $(M, \mathcal{A})$ and $(M, \mathcal{B})$ have the same colors. $\square$

We now prove one direction of the characterization.

Proposition 2.25 *Let $(M, \mathcal{A})$ be an uncertainty matroid such that an $\mathcal{A}$-basis exists. Then, every element of E is colored.*

Proof. Suppose that there exists $e \in E$ non-colored and choose any $\mathcal{A}$-basis T. Since e is not red nor blue, by Propositions 2.10 and 2.11 we have that $e \in \operatorname{span} F(e) \cap \operatorname{span}^* F^*(e)$. As in the proof of proposition 2.10 and 2.14, we can choose realizations $w, w^* \in \mathcal{R}(E, \mathcal{A})$ such that for every $f \in F(e)$ we have that $w_f < w_e$ and $w^*_e < w^*_g$ for every $g \in F^*(e)$. We conclude that e is not in any w-basis and it is in every w^*-basis. This is a contradiction since T is both a w-basis and a w^*-basis. $\square$

The next propositions shows that contraction of non-trivial blue elements preserves colors. Moreover red elements are always preserved when contracting blue elements, even when the elements being contracted are trivial.

Proposition 2.26 *Let $(M, \mathcal{A})$ be an uncertainty matroid and $e, f \in E$ blue elements in $(M, \mathcal{A})$ such that e is non-trivial. We have that f is blue in $(M/e, \mathcal{A} - A_e)$.*

PROOF. Consider the weight function $w : E \to \mathbb{R}$ given by:

$$w_x = \begin{cases} U_f & \text{if } f = x, \\ L_x & \text{if } f \neq x. \end{cases}$$

Since f is blue in $(M, \operatorname{cl} \mathcal{A})$, there exists some (M, w)-basis T such that $f \in T$. We start by proving that $e \in T$, suppose not, then we can consider C the fundamental circuit of $T + e$. As e is blue, we have that $e \notin \operatorname{span} F(e)$ and e is not a loop. Hence, we can select $g \in C \backslash F(e)$ such that $g \neq e$. Since e is non-trivial:

$$w_g \geq L_g \geq U_e > L_e = w_e.$$

It follows that $w(T - g + e) < w(T)$, which contradicts the fact that T is a w-basis.

As $e, f \in T$ we have that $T - e$ is a $(M/e, w|_{E-e})$-basis such that $f \in T - e$, using Proposition 2.11 it follows that f is blue in $(M/e, \mathcal{A} - A_e)$. □

Proposition 2.27 *Let $(M, \mathcal{A})$ be an uncertainty matroid and $e, f \in E$ be blue and red elements in $(M, \mathcal{A})$, respectively. Then f is red in $(M/e, \mathcal{A} - A_e)$.*

PROOF. Consider the realization $w \in \mathcal{R}(E, \operatorname{cl} \mathcal{A})$ given by:

$$w_x = \begin{cases} U_f & \text{if } f = x, \\ L_x & \text{if } f \neq x. \end{cases}$$

As f is red in $(M, \operatorname{cl} \mathcal{A})$, there exists some (M, w)-basis T such that $f \notin T$. If $e \in T$ then $T - e$ is an $(M/e, w|_{E-e})$-basis and by Proposition 2.15 we have that f is red in $(M, \mathcal{A})$, we may assume that $e \notin T$. Since e is blue in $(M, \operatorname{cl} \mathcal{A})$, there exists some (M, w)-basis T' such that $e \in T$. As (M, w)-basis are basis of a matroid (because they are basis of minimum weight with respect to w) we can find $e' \in T \backslash T'$ such that $T - e' + e$ is an (M, w)-basis, from here we can proceed as before but working with $T - e' + e$ instead of T. □

Dually, deletion of non-trivial red elements also preserves colors. Similarly, blue elements are preserved even when deleting trivial red elements. Both proofs follow exactly the same ideas as the blue case, but using dual concepts.

Proposition 2.28 *Let $(M, \mathcal{A})$ be an uncertainty matroid and $e, f \in E$ red elements in $(M, \mathcal{A})$ such that e is non-trivial. We have that f is red in $(M - e, \mathcal{A} - A_e)$.*

PROOF. Consider the weight function $w^* : E \to \mathbb{R}$ given by:

$$w^*_x = \begin{cases} L_f & \text{if } f = x, \\ U_x & \text{if } f \neq x. \end{cases}$$

Since f is red in $(M, \operatorname{cl} \mathcal{A})$, there exists some (M, w^*)-basis T such that $f \notin T$. We first prove that $e \notin T$, suppose not, we can then take C^* the fundamental cocircuit of $T^c + e$. As e is red, we have that $e \notin \operatorname{span}^* F^*(e)$ and e is not a coloop. Then, we can select $g \in C^* \nsubseteq F^*(e)$ such that $g \neq e$. Since e is non-trivial:

$$w_e^* = U_e > L_e \geq U_g \geq w_g^*,$$

it follows that $w^*(T^c - g + e) > w^*(T^c)$ and $w^*(T - e + g) < w^*(T)$, which contradicts the fact that T is a (M, w^*)-basis.

As $e, f \notin T$ we have that T is a basis of minimum weight in $M - e$ such that $f \notin T - e$, using proposition 2.15 it follows that f is red in $(M - e, \mathcal{A} - A_e)$. $\square$

Proposition 2.29 *Let $(M, \mathcal{A})$ be an uncertainty matroid and $e, f \in E$ be red and blue elements in $(M, \mathcal{A})$, respectively. Then, f is blue in $(M - e, \mathcal{A} - A_e)$.*

Proof. Consider the realization $w^* \in \mathcal{R}(E, \operatorname{cl} \mathcal{A})$ given by:

$$w_x^* = \begin{cases} L_f & \text{if } f = x, \\ U_x & \text{if } f \neq x. \end{cases}$$

As f is blue in $(M, \operatorname{cl} \mathcal{A})$, there exists some (M, w^*)-basis T such that $f \in T$. If $e \notin T$ then T is an $(M - e, w|_{E-e})$-basis and by Proposition 2.11 we have that f is blue in $(M, \mathcal{A})$, we may assume that $e \in T$. Since e is red in $(M, \operatorname{cl} \mathcal{A})$, there exists some (M, w)-basis T' such that $e \notin T$. We can find $e' \in T' \backslash T$ such that $T - e + e'$ is an (M, w)-basis, from here we can proceed as before but working with $T - e + e'$ instead of T. $\square$

We now prove the main result of this section.

Theorem 2.30 *Let $(M, \mathcal{A})$ be an uncertainty matroid. There exists an $\mathcal{A}$-basis if and only if every element is colored.*

Proof. We only need to prove the converse and we proceed by induction on the number of non-trivial edges k. If $k = 0$ an $\mathcal{A}$-basis is simply a basis of minimum weight, which clearly exists. Suppose that $k > 0$, and let $e \in E$ be any non-trivial element. If e is blue, we have that every element of $E - e$ is colored in $(M/e, \mathcal{A} - A_e)$, since colors were preserved. By inductive hypothesis, we have an $(M/e, \mathcal{A} - A_e)$-basis T and by proposition 2.7 we have that $T + e$ is an $(M, \mathcal{A})$-basis. On the other hand, if e is red, every element of $E - e$ is colored in $(M - e, \mathcal{A} - A_e)$, by inductive hypothesis, we have an $(M - e, \mathcal{A} - A_e)$-basis T and using proposition 2.13 we conclude that T is an $(M, \mathcal{A})$-basis. $\square$

It is interesting to note that the previous proof only uses the fact that non-trivial elements are colored. We can then deduce the following strengthened version of theorem 2.30

Theorem 2.31 *Let $(M, \mathcal{A})$ be an uncertainty matroid. There exists an $\mathcal{A}$-basis if and only if every non-trivial element is colored.*

We end this section with a small remark regarding colorings. It may seem at first that blueness and redness are mutually exclusive, this is not true for general elements, as shown in fig. 2.11, but it is true for non-trivial elements. When $\mathcal{A}$-bases do exist this remark is a direct corollary of propositions 2.16 and 2.17, but it also holds in the more general setting.

Proposition 2.32 *Let $(M, \mathcal{A})$ be an uncertainty matroid. If e is red and blue, then e is trivial.*

Proof. Suppose for the sake of contradiction that e is non-trivial and pick $w \in \mathcal{R}(E, \mathcal{A})$ such that $w_e < U_e$. As e is red there exists some w-basis T without e. Let C be the fundamental circuit of $T + e$, since e is blue, there exists some $f \neq e$ such that $f \in C \backslash F(e)$. Then:

$$w_f \geq L_f \geq U_e > w_e,$$

therefore $w(T - f + e) < w(T)$ which contradicts the fact that T was a w-basis. $\square$

Example Consider an uncertainty matroid $(M, \mathcal{A})$ as depicted in fig. 2.11. Since any set of three elements is an $\mathcal{A}$-basis, every element is colored red and blue.

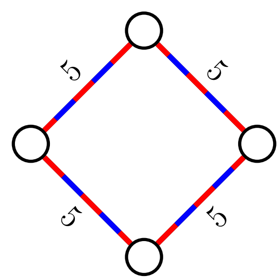


Figure 2.11: All of the elements of $(M, \mathcal{A})$ are red and blue.

This section partially solved the UMB problem by giving a characterization of when uniformly minimum bases exist. Furthermore this characterization is easily testable as one only needs to check the color of each element, this can be accomplished by executing the greedy algorithm and appropriate weight functions.

2.6 A solution to the UMB problem

We end this chapter by solving the UMB problem fully via an important result on the structure of uniformly minimum bases, namely, that they are the bases of some matroid. This result is a generalization of the fact that optimal bases of weighted matroids are the bases of a matroid and it will provide an algorithm for computing uniformly minimum bases when they exist.

Proposition 2.33 *Let $(M, \mathcal{A})$ be an uncertainty matroid such that an $\mathcal{A}$-basis exists. The following set:*

$$\mathfrak{B} = \{T \subseteq E : T \textit{ is an } \mathcal{A}\textit{-basis}\}$$

is the set of bases of a matroid with elements in E, we will denote such matroid by $\texttt{Mat}(M, \mathcal{A})$.

Proof. We only need to check the exchange axiom of bases as $\mathcal{B}$ is non-empty by hypothesis. Let T_1, T_2 two $\mathcal{A}$-bases and $e \in T_1 \backslash T_2$, as both are bases of M, by strong exchange we have $f \in T_2 \backslash T_1$ such that $T_1 - e + f$ and $T_2 + e - f$ are bases of M. By proposition 2.20 we have that e and f are trivial, moreover for any realization $w \in \mathcal{R}(E, \mathcal{A})$ we must have $w_e = w_f$, since if $w_e > w_f$ we have $w(T_1 + f - e) < w(T_1)$ contradicting the fact that T_1 is a w-basis, similarly if $w_e < w_f$ we have that $w(T_2 + e - f) < w(T_2)$ which is also a contradiction. Therefore, for every realization $w \in \mathcal{R}(E, \mathcal{A})$ we get $w(T_1) = w(T_1 - e + f)$, concluding that $T_1 - e + f$ is also an $\mathcal{A}$-basis. □

Note that $\texttt{Mat}(M, \mathcal{A})$ is finer than M in the sense that independence in $\texttt{Mat}(M, \mathcal{A})$ implies independence in M, this observation allows us to conclude that the set of non-trivial blue elements must be independent in M, since non-trivial blue elements are included in every uniformly minimum basis.

As every uniformly minimum basis is a basis of minimum weight for any realization, we have that every uniformly minimum basis is fully colored blue. At first sight it may seem that any blue basis, that is, a basis where every element is blue, is an uniformly minimum basis. This converse does not hold, even when asking for blue bases that contain every non-trivial blue element, as shown in Fig. 2.12.

In the end of this section we aim to give a useful characterization of bases and independence of the uniformly minimum bases matroid. Note that if uniformly minimum bases exist one can contract blue non-trivial elements and delete red elements while preserving colors in each contraction or deletion. Since every element is colored one arrives at a matroid that has only trivial elements, this matroid is key in comprehending uniformly minimum bases, as we already know what to do with non-trivial elements.

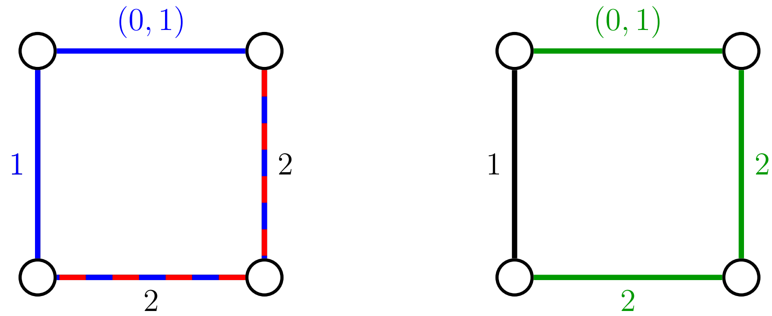
Definition 2.34 (Trivial Matroid) *Let $(M, \mathcal{A})$ be an uncertainty matroid. Consider:*

$$N_B = \{e \in E : e \textit{ is non-trivial and blue}\}$$
$$N_R = \{e \in E : e \textit{ is non-trivial and red}\}$$

If $(M, \mathcal{A})$ has an $\mathcal{A}$-basis, we define $M^t = M/N_B \backslash N_R$ and consider $w : E(M^t) \to \mathbb{R}$ the weight function that maps each trivial element e to $L_e = U_e$. We call the weighted matroid (M^t, w^t) the trivial weighted matroid associated to $(M, \mathcal{A})$.

Example Consider $(M, \mathcal{A})$ an uncertainty matroid as depicted in Fig. 2.12. Note that every element is blue, but even though the non-trivial elements and the elements with area 2 form a blue basis they are not an $\mathcal{A}$-basis, as replacing any element with area 2 by the element with area 1 would improve the weight of any realization.

Note that since colored elements allowed safe deletion and safe contraction minimality of weight is preserved in each contraction or deletion arriving at a minimum weight basis of the trivial weighted matroid. This property added to the already mentioned relations



(a) $(M, \mathcal{A})$.

(b) A blue basis that is not an $\mathcal{A}$-basis.

Figure 2.12: A blue basis that is not an $\mathcal{A}$-basis.

with colored non-trivial elements allows us to prove a useful characterization of uniformly minimum bases.

Theorem 2.35 *Let $(M, \mathcal{A})$ be an uncertainty matroid such that an $\mathcal{A}$-basis exists and (M^t, w^t) its trivial weighted matroid. Then T is an $\mathcal{A}$-basis if and only if:*

1. $N_B \subseteq T$,
2. $N_R \cap T = \emptyset$,
3. $T \backslash N_B$ *is a minimum weight basis of* (M^t, w^t).

Proof. $\mathcal{A}$-bases always contain every blue non-trivial element by Proposition 2.16 and avoid each red non-trivial element by Proposition 2.17. We can delete each element in N_R while preserving colors in each deletion using Propositions 2.28 and 2.29, more so Proposition 2.13 concludes that T is an $(M \backslash N_R, \mathcal{A} \backslash \{A_e\}_{e \in N_R})$-basis. We now contract each element in N_B while preserving colors by Propositions 2.26 and 2.27 repeated aplication of Proposition 2.7 concludes that $T \backslash N_B$ is an (M^t, w^t)-basis.

We show the converse by induction on the number of non-trivial elements k. If $k = 0$, using condition 3 we get that T is a minimum weight basis of (M^t, w^t). Noting that $M^t = M$ and $\mathcal{A} = \{w^t(e)\}_{e \in E}$ we conclude that T is an $(M, \mathcal{A})$-basis.

If $k > 0$ select any non-trivial element $e \in E$. If e is blue, we have that $e \in T$. As colors are preserved when contracting e, it follows by inductive hypothesis that $T - e$ is an $(M/e, \mathcal{A} - A_e)$-basis, and using Proposition 2.7 we conclude that T is an $(M, \mathcal{A})$-basis. If e is red it cannot be blue, as it is non-trivial, then $e \notin T$. Since deletion of e also preserves colors, by inductive hypothesis we have that T is an $(M - e, \mathcal{A} - A_e)$-basis, Proposition 2.13 allows us to conclude that T is an $(M, \mathcal{A})$-basis. $\square$

Note that this theorem in conjunction with the existential characterization of uniformly minimum bases provides an algorithmic solution to the UMB problem. One can first decide if a uniformly minimum basis exist by checking if it is fully colored, if it is, one can contract all blue non-trivial elements, delete every red non-trivial element compute a minimum weight basis of the trivial weighted matroid by using the greedy algorithm and return the output of the greedy algorithm and every blue non-trivial element.

We now use this theorem to provide additional exploration on the structure of the matroid introduced in this section, specifically, we give a characterization of its bases depending on the rank function of the original matroid and express it as the direct sum of simpler matroids.

Proposition 2.36 *Let $(M, \mathcal{A})$ be a matroid with areas such that an $\mathcal{A}$-basis exists and (M^t, w^t) its trivial weighted matroid. Consider the set of trivial weights $\{w^t_e : e \in E(M^t)\}$, and order them increasingly $w_1 < w_2 < ... < w_r$. We define the following sets:*

$$E_i = \{e \in E(M^t) : w^t_e = w_i\}, \quad \forall i \in \{1, ..., r\},$$
$$F_i = \bigcup_{j=1}^{i} E_i, \qquad \forall i \in \{0, ..., r\}.$$

Then T is an $\mathcal{A}$ basis if and only if:

1. *T is a basis of M,*
2. *$N_B \subseteq T$,*
3. *$N_R \cap T = \emptyset$,*
4. *$|T \cap E_i| = r_M(F_i \cup N_B) - r_M(F_{i-1} \cup N_B)$ for each $i \in [r]$.*

Moreover $\mathtt{Mat}(M, \mathcal{A}) = M^t_{w^t} \oplus U^{|N_B|}_{N_B} \oplus U^0_{N_R} = \left[\bigoplus_{i=1}^{r} M|_{F_i}/F_{i-1}\right] \oplus U^{|N_B|}_{N_B} \oplus U^0_{N_R}$.

Proof. First note that as every element is colored the set of trivial elements is $E \backslash N_B \backslash N_R$. By Proposition 2.35 we get condition 2. and 3. and since $T \backslash N_B \in \mathcal{B}(M^t)$ and $T \cap N_R = \emptyset$ we have that T is a basis of M, hence we also get 1. We now check condition 4., since $T \backslash N_B$ is a minimum weight basis of (M^t, w^t), using Proposition 1.28 we have for each $i \in [r]$:

$$\begin{aligned}
|T \cap E_i| &= |(T \backslash N_B) \cap E_i| \\
&= r_{M/N_B \backslash N_R}(F_i) - r_{M/N_B \backslash N_R}(F_{i-1}) \\
&= r_M(F_i \cup N_B) - r_M(N_B) - [r_M(F_{i-1} \cup N_B) - r_M(N_B)] \\
&= r_M(F_i \cup N_B) - r_M(F_{i-1} \cup N_B)
\end{aligned}$$

For the converse we only need to check that $T \backslash N_B$ is a minimum weight basis of (M^t, w^t). Note that:

$$\begin{aligned}
|(T \backslash N_B) \cap E_i| &= |T \cap E_i| \\
&= r_M(F_i \cup N_B) - r_M(F_{i-1} \cup N_B) \\
&= r_M(F_i \cup N_B) - r_M(N_B) - [r_M(F_{i-1} \cup N_B) - r_M(N_B)] \\
&= r_{M/N_B \backslash N_R}(F_i) - r_{M/N_B \backslash N_R}(F_{i-1})
\end{aligned}$$

we conclude by noting that $T \backslash N_B$ is a basis of $M/N_B \backslash N_R$ and using Proposition 1.28.

We now prove the decomposition part of the proposition. By Proposition 1.28 it suffices to prove that $\texttt{Mat}(M, \mathcal{A}) = M^t_{w^t} \oplus M|_{N_B} \oplus U^0_{N_B}$, by Proposition 2.35 we have:

$$\mathcal{B}(\texttt{Mat}(M, \mathcal{A})) = \{B \cup N_B : B \in \mathcal{B}(M^t_{w^t}))\}$$

which implies that $\texttt{Mat}(M, \mathcal{A}) = M^t_{w^t} \oplus U^{|N_B|}_{N_B} \oplus U^0_{N_R}$. □

The solution of the UMB problem marks the end of this chapter. This is not only important as it solves one of the two central problems presented at the beginning, but this chapter provided us with some useful techniques an machinery that will provide a starting point in dealing with the MCFQS problem, in particular blue/red ideas will provide motivation when tackling this new problem.

Chapter 3

Feasible query sets

This chapter focuses on feasible query sets and the MCFQS problem. We start by studying how feasible query sets behave on the parts of decomposable matroids, providing a way to combine solutions on its decomposition. Then, we study two particular cases of the MCFQS problem that will provide inspiration later on:

MCFQS problem with unit costs on interval areas: In this case, the unique feasible query set of minimum size is the set of non-colored elements.

MCFQS problem with unit costs on $\{0,1\}$ areas: In this case, every feasible query set of minimum size consists of all elements but one in each connected component.

We then introduce an analogue to witness sets by studying sets that intersect every feasible query set. We use this to define critical pairs as pairs of elements that intersect every feasible query set but no single element does. This allows us to define a critical relation, and state the two main results of this chapter:

A **characterization of feasible query sets of minimum size:** Each feasible query set of minimum size consists of all elements but one in each equivalence class of the critical relation.

A **description of the cofeasible query set matroid:** Complements of feasible query sets are the independent sets of an uncomplicated partition matroid that depends only on equivalence classes of the critical relation.

3.1 Feasible query sets in decomposable matroids

We start the search of a solution for the MCFQS problem by showing two ways of combining feasible query sets of simpler matroids with areas into feasible query sets of a more complex uncertainty matroid.

The first method applies when considering an uncertainty matroid that can be expressed as the direct sum of simpler matroids. In this case, it suffices to simply find a feasible query set in each simple matroid and join them together afterwards.

The second property explored, applies when the elements of the uncertainty matroid being studied can be partitioned into two sets such that the elements of one set are heavier than the ones in other set for every realization. In such case, we can find a feasible query set for the “light” set, contract them, and then find a feasible query set for the “heavy” set.

Both of these properties rely heavily on analogous facts for uniformly minimum bases. These are; Proposition 2.22; and a generalization of the property that to find a minimum weight basis, one can simply choose a threshold, select an optimal basis of edges below the threshold, contract them, and then select an optimal basis of the edges above the threshold in the contracted matroid.

We first state and provide proof of the property relating directed sums.

Proposition 3.1 *Let $(M, \mathcal{A})$ be an uncertainty matroid such that $M = \bigoplus_{i=1}^{k} M_i$ and $E_i = E(M_i)$. Then X is a feasible query set for $(M, \mathcal{A})$ if and only if $X \cap E_i$ is a feasible query set for $(M_i, \mathcal{A}|_{E_i})$ for each $i \in [k]$.*

Proof. For the forward implication, let $\mathcal{B}_i \in \mathcal{R}(X \cap E_i, \mathcal{A}|_{E_i})$ be a revelation of $X \cap E_i$ and for each $e \in X \backslash E_i$ pick any $w_e \in A_e$, then we define $\mathcal{B} = \mathcal{B}_i \cup \{w_e\}_{e \in X \backslash E_i}$. Since $\mathcal{B}$ is a revelation of X and X is feasible, there exists some $(M, \mathcal{B})$-basis T, by Proposition 2.22 we have that T_i is an $(M, \mathcal{B}|_{E_i})$-basis. Finally, since $\mathcal{B}|_{E_i} = \mathcal{B}_i$, we have that $X \cap E_i$ is feasible for $(M_i, \mathcal{A}|_{E_i})$.

We now prove the converse. Consider $\mathcal{B} \in \mathcal{R}(X, \mathcal{A})$ and note that $\mathcal{B}|_{E_i} \in \mathcal{R}(X \cap E_i, \mathcal{A}|_{E_i})$. As $X \cap E_i$ is feasible in $(M_i, \mathcal{A}|_{E_i})$ for each $i \in [k]$, it follows that for each $i \in [k]$ there exists T_i an $(M_i, \mathcal{B}|_{E_i})$-basis. Note that $T = \bigcup_{i=1}^{k} T_i$ verifies that $T \cap E_i$ is an $(M_i, \mathcal{B}|_{E_i})$-basis for each $i \in [k]$, and by Proposition 2.22 we conclude that T is an $(M, \mathcal{B})$-basis and that X is feasible in $(M, \mathcal{A})$. $\square$

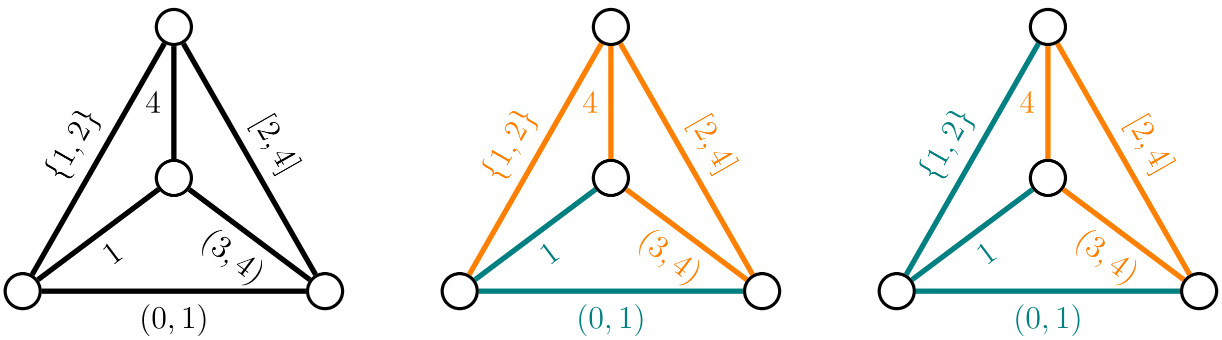
Note that since every matroid can be written as the direct sum of connected matroids, the previous property shows that connectivity isn't such an strong assumption to make.

In the rest of this section we will consider matroids with areas $(M, \mathcal{A})$ that are partitioned into “light” and “heavy” elements.

Definition 3.2 (Light and heavy matroid) *Let $(M, \mathcal{A})$ be an uncertainty matroid. We say that L, H is a light/heavy partition if $\max_{e \in L} U_e \leq \alpha \leq \min_{e \in H} L_e$ for some $\alpha \in \mathbb{R}$, $\{e \in E : L_e = U_e = \alpha\} = \emptyset$ and $L \cup H = E$. Starting from this, we define the light and heavy matroid respectively as $M_L = M|_L$ and $M_H = M/L$.*

Example Consider an uncertainty matroid as in Fig. 3.1 and the two candidate light/heavy partitions provided. Let L be the set of teal elements in (b) and H be the set of orange elements in (b). Even though these sets verify $\max_{e\in L} U_e \leq \min_{e\in H} L_e$, they are not a light/heavy partition, as there is a trivial edge with area 1 that touches the only possible threshold $\alpha = 1$. This is a problem since it is not clear if this edge is light or heavy.

On the other hand consider L as the set of teal elements in (c) and H as set of the orange elements in (c), again we have that $\max_{e\in L} U_e \leq \min_{e\in H} L_e$, but, even though there are two sets touching the only possible threshold of $\alpha = 2$, namely the sets $\{1, 2\}$ and $[2, 4]$, there is no problem as the other there is no doubt that $\{1, 2\}$ must be light and $[2, 4]$ must be heavy.



(a) $(M, \mathcal{A})$.

(b) The sets in teal/orange are not a light/heavy partition.

(c) The sets in teal/orange are a light/heavy partition.

Figure 3.1: An uncertainty matroid and two candidate light/heavy partitions.

We aim to prove that it suffices to find feasible query sets for the light and heavy matroids with areas. We start by proving that colors are preserved. That is, if an element is heavy, then it is colored the same in the heavy matroid with areas and in the original uncertainty matroid; this same color-preserving property occurs for light elements. The proofs depend mainly on Propositions 2.10, 2.14 and some span or cospan calculations.

Proposition 3.3 *Let $(M, \mathcal{A})$ be an uncertainty matroid and L, R a light/heavy partition. Then:*

1. *$e \in L$ is blue for $(M_L, \mathcal{A}|_L)$ if and only if it is blue for $(M, \mathcal{A})$.*
2. *$e \in H$ is blue for $(M_H, \mathcal{A}|_H)$ if and only if it is blue for $(M, \mathcal{A})$.*
3. *$e \in L$ is red for $(M_L, \mathcal{A}|_L)$ if and only if it is red for $(M, \mathcal{A})$.*
4. *$e \in H$ is red for $(M_H, \mathcal{A}|_H)$ if and only if it is red for $(M, \mathcal{A})$.*

Proof. 1. Note that:

$$\operatorname{span}_{M_L} F^{\mathcal{A}|_L}(e) = \operatorname{span}_{M\setminus H} F^{\mathcal{A}|_L}(e) = \operatorname{span}_M F^{\mathcal{A}|_L}(e)\setminus H = \operatorname{span}_M F^{\mathcal{A}}(e)\setminus H,$$

for the last equality note that if $e \in L$, then $F^{\mathcal{A}|_L}(e) = F^{\mathcal{A}}(e)$. As $e \notin H$ we have that $e \notin \operatorname{span}_{M_L} F^{\mathcal{A}|_L}(e)$ if and only if $\operatorname{span}_M F^{\mathcal{A}}(e)$. Using Proposition 2.10, we conclude that e is blue for $(M_L, \mathcal{A}|_L)$ if and only if it is blue for $(M, \mathcal{A})$.

2. As for every $e \in H$ we have $F^{\mathcal{A}}(e) = F^{\mathcal{A}|_H}(e) \cup L$, then:

$$\operatorname{span}_{M_H} F^{\mathcal{A}|_H}(e) = \operatorname{span}_{M/L} F^{\mathcal{A}|_H}(e) = \operatorname{span}_M [F^{\mathcal{A}|_H}(e) \cup L] \backslash L = \operatorname{span}_M F^{\mathcal{A}}(e) \backslash L.$$

   Since $e \notin L$, we have that $e \notin \operatorname{span}_{M_H} F^{\mathcal{A}|_H}(e)$ if and only if $e \notin \operatorname{span}_M F^{\mathcal{A}}(e)$. By Proposition 2.10, we conclude that e is blue for $(M_H, \mathcal{A}|_H)$ if and only if it is blue for $(M, \mathcal{A})$.

3. Since $F^{*\mathcal{A}|_L}(e) \cup H = F^{*\mathcal{A}}(e)$ for any $e \in L$, we have:

$$\operatorname{span}^*_{M_L} F^{*\mathcal{A}|_L}(e) = \operatorname{span}_{(M \backslash H)^*} F^{*\mathcal{A}|_L}(e) = \operatorname{span}_{M^*/H} F^{*\mathcal{A}|_L}(e)$$

$$= \operatorname{span}_{M^*} [F^{*\mathcal{A}|_L}(e) \cup H] \backslash H = \operatorname{span}^*_M F^{*\mathcal{A}|_L}(e) \backslash H.$$

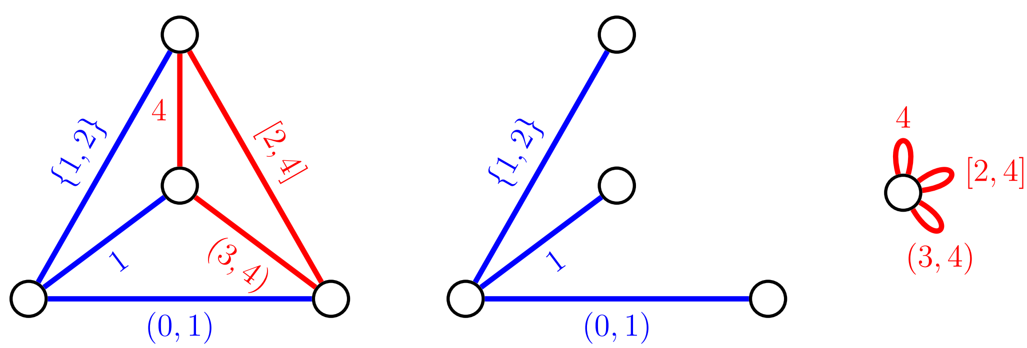
   Seeing that $e \notin H$, we conclude that $e \notin \operatorname{span}^*_{M_L} F^{*\mathcal{A}|_L}(e)$ if and only if $e \notin \operatorname{span} *_M F^{*\mathcal{A}|_L}(e)$. Proposition 2.14 implies that e is red for $(M_L, \mathcal{A}|_L)$ if and only if it is red for $(M, \mathcal{A})$.

4. Note that for any $e \in H$ we have $F^{*\mathcal{A}}(e) = F^{*\mathcal{A}|_H}(e)$, then:

$$\operatorname{span}^*_{M_H} F^{*\mathcal{A}|_H}(e) = \operatorname{span}_{(M/L)^*} F^{*\mathcal{A}|_H}(e) = \operatorname{span}_{M^* \backslash L} F^{*\mathcal{A}|_H}(e) = \operatorname{span}_{M^*} F^{*\mathcal{A}}(e) \backslash L.$$

   As $e \in L$, $e \notin \operatorname{span}^*_{M_H} F^{*\mathcal{A}|_H}(e)$ if and only if $\operatorname{span}_{M^*} F^{*\mathcal{A}}(e)$. It follows by Proposition 2.14 that e is red for $(M_H, \mathcal{A}|_H)$ if and only if it is red for $(M, \mathcal{A})$. □

Example Consider the uncertainty matroid previously discussed in Fig. 3.1. We exemplify the coloring preserving property on its light and heavy matroid, as depicted in Fig. 3.2.



(a) The original matroid with areas, edges are colored accordingly.

(b) The light matroid, edges are colored accordingly.

(c) The heavy matroid, edges have been colored accordingly.

Figure 3.2: Three uncertainty matroids, colored accordingly.

We now state and prove the threshold property mentioned at the beginning of this section.

Proposition 3.4 *Let $(M, \mathcal{A})$ be an uncertainty matroid and L, H a light/heavy partition. Then T_L is an $(M_L, \mathcal{A}|_L)$-basis and T_H is an $(M_H, \mathcal{A}|_H)$-basis if and only if $T_L \cup T_H$ is an $(M, \mathcal{A})$-basis.*

Proof. Since colors are preserved by Proposition 3.3, using Theorem 2.30, the existence of $\mathcal{A}$-bases in $(M, \mathcal{A})$ implies that uniformly minimum bases exists in $(M_L, \mathcal{A}|_L)$ and $(M_H, \mathcal{A}|_H)$, and vice-versa. As all the uniformly minimum bases involved exist, we only need to prove that $\mathtt{Mat}(M, \mathcal{A}) = \mathtt{Mat}(M_L, \mathcal{A}|_L) \oplus \mathtt{Mat}(M_H, \mathcal{A}|_H)$.

Consider r, $\{E_i\}_{i=1}^r$, $\{F_i\}_{i=0}^r$ as in Proposition 2.36 and $l = \max\{i \in [r] : E_i \subseteq L\}$. Using Proposition 2.36 on $\mathtt{Mat}(M_L, \mathcal{A}|_L)$ we have:

$$\begin{aligned}
\mathtt{Mat}(M_L, \mathcal{A}|_L) &= \left[\bigoplus_{i=1}^{l} M_L|_{F_i}/F_{i-1}\right] \oplus U_{N_B\cap L}^{|N_B\cap L|} \oplus U_{N_R\cap L}^{0} \\
&= \left[\bigoplus_{i=1}^{l} M|_{F_i}/F_{i-1}\right] \oplus U_{N_B\cap L}^{|N_B\cap L|} \oplus U_{N_R\cap L}^{0}
\end{aligned}$$

Using the same proposition on $\mathtt{Mat}(M_H, \mathcal{A}_H)$ we get:

$$\begin{aligned}
\mathtt{Mat}(M_H, \mathcal{A}|_H) &= \left[\bigoplus_{i=l+1}^{r} M_H|_{(F_i\setminus L)}/(F_{i-1}\setminus L)\right] \oplus U_{N_B\cap H}^{|N_B\cap H|} \oplus U_{N_R\cap H}^{0} \\
&= \left[\bigoplus_{i=l+1}^{r} M/L|_{(F_i\setminus L)}/(F_{i-1}\setminus L)\right] \oplus U_{N_B\cap H}^{|N_B\cap H|} \oplus U_{N_R\cap H}^{0} \\
&= \left[\bigoplus_{i=l+1}^{r} M|_{F_i}/F_{i-1}\right] \oplus U_{N_B\cap H}^{|N_B\cap H|} \oplus U_{N_R\cap H}^{0}
\end{aligned}$$

Finally applying Proposition 2.36 on $\mathtt{Mat}(M, \mathcal{A})$, we can conclude after the following computation:

$$\begin{aligned}
\mathtt{Mat}(M, \mathcal{A}) &= \left[\bigoplus_{i=1}^{r} M|_{F_i}/F_{i-1}\right] \oplus U_{N_B}^{|N_B|} \oplus U_{N_R}^{0} \\
&= \left[\bigoplus_{i=1}^{r} M|_{F_i}/F_{i-1}\right] \oplus U_{N_B\cap L}^{|N_B\cap L|} \oplus U_{N_B\cap H}^{|N_B\cap H|} \oplus U_{N_R\cap L}^{0} \oplus U_{N_R\cap H}^{0} \\
&= \left[\bigoplus_{i=1}^{l} M|_{F_i}/F_{i-1}\right] \oplus U_{N_B\cap L}^{|N_B\cap L|} \oplus U_{N_R\cap L}^{0} \oplus \left[\bigoplus_{i=l+1}^{r} M|_{F_i}/F_{i-1}\right] \oplus U_{N_B\cap H}^{|N_B\cap H|} \oplus U_{N_R\cap H}^{0} \\
&= \mathtt{Mat}(M_L, \mathcal{A}|_L) \oplus \mathtt{Mat}(M_H, \mathcal{A}|_H)
\end{aligned}$$

□

The threshold property for uniformly minimum bases will allow us to deduce an analogous version for feasible query sets. This proof will work similarly as when Proposition 2.22 allowed us to deduce Proposition 3.1, that is, we fix revelations of the feasible query sets and start working with the respective uniformly minimum bases that arise.

Proposition 3.5 *Let $(M, \mathcal{A})$ be an uncertainty matroid and L, R a light/heavy partition. Then X_L is feasible in $(M_L, \mathcal{A}|_L)$ and X_H is feasible in $(M_H, \mathcal{A}|_H)$ if and only if $X = X_L \cup X_H$ is feasible in $(M, \mathcal{A})$.*

Proof. Let $\mathcal{B} \in \mathcal{R}(X, \mathcal{A})$ be a revelation of X, then $\mathcal{B}|_L \in \mathcal{R}(X_L, \mathcal{A}|_L)$ and $\mathcal{B}|_H \in \mathcal{R}(X, \mathcal{A}|_H)$. Since X_L and X_H are feasible we have an $(M_L, \mathcal{B}|_L)$-basis T_L and an $(M_H, \mathcal{B}|_H)$-basis T_H. By Proposition 3.4, we have that $T_L \cup T_H$ is an $(M, \mathcal{B})$-basis, implying that X is feasible in $(M, \mathcal{A})$.

For the converse, let $\mathcal{B}_L \in \mathcal{R}(X_L, \mathcal{A}|_L)$,$\mathcal{B}_H \in \mathcal{R}(X_H, \mathcal{A}|_H)$ and define $\mathcal{B} = \mathcal{B}_L \cup \mathcal{B}_H$. Note that $\mathcal{B} \in \mathcal{R}(X, \mathcal{A})$ and, since X is feasible in $(M, \mathcal{A})$, there exists T an $(M, \mathcal{B})$-basis. By Proposition 3.4 we have that $T \cap L$ is an $(M_L, \mathcal{B}_L)$-basis and $T \cap H$ is an $(M_H, \mathcal{B}_H)$-basis, implying that they are both feasible. $\square$

This section provided us with two techniques to build larger feasible query sets from smaller ones. More so, since the techniques are characterizations, one can use them to build minimum size feasible query sets from those of simpler uncertainty matroid. We have also found the expected remarks that matroid connectivity is a "free" assumption, and that areas of uncertainty that cannot be separated by some threshold are troublesome.

3.2 Two illustrative cases: Interval and $\{0, 1\}$ areas

In this section we study two particular type of areas that provide additional insight in how we can arrive to a solution of the general MCFQS problem.

First, we restrict our attention to uncertainty matroids that have interval areas, that is, every area of uncertainty is an interval in $\mathbb{R}$. We prove that, in this case, there is a unique feasible query set of minimum size; namely, the set of non-colored non-trivial elements.

Our second study case will be uncertainty matroids where each area is exactly $\{0, 1\}$. We restrict ourselves to connected matroids and use Proposition 3.1 to conclude the general case. For connected matroids with $\{0, 1\}$ areas, we prove that every minimum size feasible query set is exactly every element of the matroid except for one.

We start this section by studying uncertainty matroids with interval areas. We first prove that non-colored non-trivial elements must be revealed, that is, they are in every feasible query set. The main idea involved in this proof is that non-colored elements belong in a light circuit and in a heavy cocircuit as they are not blue nor red respectively and since they are non-trivial we can select the weight of the element in two "extreme" manners, causing it to be the heaviest of a circuit in some realizations and the lightest of a cocircuit in others.

Proposition 3.6 *Let $(M, \mathcal{A})$ be an uncertainty matroid with interval areas and e a non-colored non-trivial element. Then $e \in X$ for every $X \subseteq E$ feasible query set.*

Proof. First note that in the interval case $\texttt{both}(e) = \emptyset$, since for every $f \neq e$ such that $A_f \cap (-\infty, L_e] \neq \emptyset$ and $A_f \cap [U_e, \infty) \neq \emptyset$ we must have $A_f \cap (L_e, U_e) \neq \emptyset$ as A_f is an inverval. Suppose that there is some feasible query set X such that $e \notin X$, then $E - e$ is a feasible query set.

As e is non-colored, by Propositions 2.10 and 2.14 we have that $e \in \operatorname{span} F(e) \cap \operatorname{span}^* F^*(e)$, in particular, we have a circuit $C \subseteq F(e) + e$ and a cocircuit $C^* \subseteq F^*(e) + e$ such that $e \in C \cap C^*$. Choose $w \in \mathcal{R}(E - e, \mathcal{A})$ so that $w_f \in A_f$ for each $f \in \mathtt{low}^{\mathcal{A}}(e) \cup \mathtt{both}^{\mathcal{A}}(e)$ and for every $f \in \mathtt{mid}^{\mathcal{A}}(e)$ we choose $w_f \in (L_e, U_e)$, we also pick $\varepsilon = \frac{1}{2} \min_{f \in \mathtt{mid}^{\mathcal{A}}(e)} \{U_f - w_f, w_f - L_f\} > 0$. Consider the following revelation $\mathcal{B} \in \mathcal{R}(E - e, \mathcal{A})$:

$$B_f = \begin{cases} w_f & \text{if } f \neq e, \\ A_e & \text{if } f = e. \end{cases}$$

As $E - e$ is a feasible query set, we have that there is a $\mathcal{B}$-basis T. If $e \in T$ we consider the realization $\hat{w} \in \mathcal{R}(E, \mathcal{B})$ by fixing $\hat{w}_e = U_e - \varepsilon$. Since $F^{\hat{w}}(e) = \mathtt{low}^{\mathcal{A}}(e) \cup \mathtt{mid}^{\mathcal{A}}(e)$ we have that e is the heaviest element in C arriving at a contradiction. On the other hand if $e \notin T$ the realization $\tilde{w} \in \mathcal{R}(E, \mathcal{A})$ that arises by fixing $\tilde{w}_e = L_e + \varepsilon$ has e as the cheapest element in C^*. Since $F^{*\tilde{w}}(e) = \mathtt{high}^{\mathcal{A}}(e) \cup \mathtt{mid}^{\mathcal{A}}(e)$, this is also a contradiction. □

We now prove that it is always sufficient to reveal non-colored non-trivial elements. If we query all non-colored non-trivial elements they become trivial, and, by Theorem 2.31, we only need to worry about the remaining (previously colored) elements. Therefore, it suffices to show that these elements preserve their colors.

Proposition 3.7 *Let $(M, \mathcal{A})$ be an uncertainty matroid, $X \subseteq E$, $e \notin X$ and $\mathcal{B} \in \mathcal{R}(X, \mathcal{A})$. If e is blue (resp. red) in $(M, \mathcal{A})$, then it is blue (resp. red) in $(M, \mathcal{B})$.*

Proof. Since $\mathcal{B} \in \mathcal{R}(X, \mathcal{A})$, we get that $U_e^{\mathcal{A}} = U_e^{\mathcal{B}}$, $L_f^{\mathcal{A}} \leq L_f^{\mathcal{B}}$ and $U_f^{\mathcal{A}} \geq U_f^{\mathcal{B}}$ for every $f \in E - e$. Therefore:

$$F^{\mathcal{B}}(e) = \{f \in E - e : L_f^{\mathcal{B}} < U_e^{\mathcal{B}}\} \subseteq \{f \in E - e : L_f^{\mathcal{A}} < U_e^{\mathcal{A}}\} = F^{\mathcal{A}}(e)$$
$$F^{*\mathcal{B}}(e) = \{f \in E - e : L_e^{\mathcal{B}} < U_f^{\mathcal{B}}\} \subseteq \{f \in E - e : L_e^{\check{\mathcal{B}}} < U_f^{\check{\mathcal{B}}}\} = F^{*\check{\mathcal{B}}}(e)$$

If e is blue in $(M, \mathcal{A})$, then, $e \notin \operatorname{span} F^{\mathcal{A}}(e)$. Therefore, $e \notin \operatorname{span} F^{\mathcal{B}}(e)$ and e is blue in $(M, \mathcal{B})$. Otherwise, e is red in $(M, \mathcal{A})$, then, $e \notin \operatorname{span}^* F^{*\mathcal{A}}(e)$. Therefore, $e \notin \operatorname{span}^* F^{*\mathcal{B}}(e)$ and e is red in $(M, \mathcal{B})$. □

We can now conclude that the set of non-trivial non-colored elements is the only minimum sized feasible query set.

Theorem 3.8 *Let $(M, \mathcal{A})$ be an uncertainty matroid with interval areas. Then the only minimum size feasible query set is:*

$$X = \{e \in E : e \textit{ is non-colored and non-trivial}\}.$$

Proof. Let F be any minimum size feasible query set, then, by Proposition 3.6 $X \subseteq F$.

We now prove that X is a feasible query set. Let $\mathcal{B} \in \mathcal{R}(X, \mathcal{A})$, by proposition 3.7 every non-trivial element of $(M, \mathcal{B})$ is colored. By Theorem 2.31, we conclude that X is a feasible query set and $|F| \leq |X|$.

We can now conclude that $F = X$ for every minimum size feasible query set F. $\square$

It is interesting to note that Theorem 3.8 provides an algorithm for finding feasible query sets of minimum size. We can test the color of each element via the greedy algorithm, by using the worst and best case weight function. Then, the minimum size feasible query set is the set of non-trivial non-colored elements. Furthermore, this can be done efficiently in terms of computational resources and calls to the independence oracle.

Example Consider an uncertainty matroid as shown in Fig. 3.3. After doing the previous procedure we arrive at a feasible query set of minimum size.

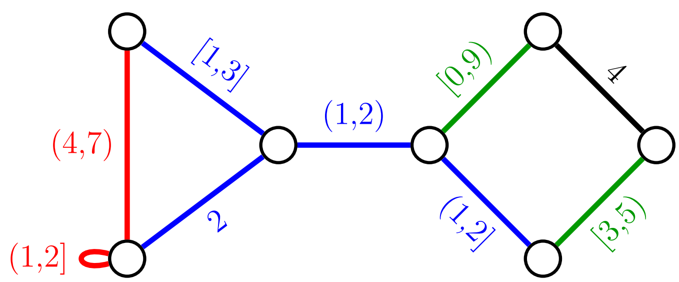


Figure 3.3: An uncertainty matroid. Red and blue elements are colored accordingly. In green we show a feasible query set of minimum size.

It is also interesting to see that this provides a solution to the MCFQS problem with interval areas. One must query every non-colored non-trivial element independently of their cost, and only query colored elements or trivial elements if they are of negative cost. Again, this can be solved algorithmically.

Note that some parts of this proof work in a more general setting. For example, if $\texttt{both}\,(e) = \emptyset$ for each element e, the proof of Proposition 3.6 works in the same way. Sadly, the full theorem isn't true for general uncertainty matroids, as they may have multiple feasible query sets of minimum size or non-colored elements that don't need to be revealed. We show this phenomena in the following example.

Example Consider an uncertainty matroid $(M, \mathcal{A})$ as shown in Fig. 3.4. Note that every element is non-colored, but if we reveal any two elements, we can sort the elements by weight as follows: first the elements revealed to be 0, then the non-revealed element, and finally elements revealed to be 1. We can use the greedy algorithm under the aforementioned order, obtaining a uniformly optimum basis, this implies that any two elements form a feasible query set.

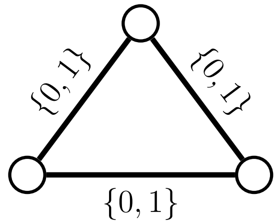


Figure 3.4: Every element is non-colored and non-trivial, but every pair of elements is a feasible query set of minimum size.

We now restrict our attention to connected matroids in which every area of uncertainty is $\{0,1\}$. As the previous example is a connected matroid where each area is $\{0,1\}$ the analysis that worked for interval areas won't work. Despite that, the key idea of selecting extreme weights of non-colored elements to make them the heaviest of a circuit and the lightest of a cocircuit will still work, albeit it will need some modifications.

One of the problems encountered in working with $\{0,1\}$ areas is that when trying to select extreme weights there is no space to do it. Specifically, as one needs the elements to be the "strict" heaviest and the "strict" lighter, ties between elements become troublesome. Note that this problem doesn't exist in uncertainty matroids with interval areas as one can always pick weights such that ties are broken.

Even so, we will proceed in similar fashion by making use of an important fact of connected matroids, that is, in a connected matroid every pair of elements is exactly the intersection of a circuit and a cocircuit. As in the previous case with non-colored elements, we will make use of such circuit and cocircuit to conclude that something must be revealed, in this case, one element of each pair.

Proposition 3.9 *Let $(M, \mathcal{A})$ be an uncertainty matroid and $x, y \in E$. If M is connected and every area is $\{0,1\}$, then $\{x,y\} \cap X \neq \emptyset$, for every feasible query set $X \subseteq E$.*

Proof. Suppose there is some feasible query set X such that $X \cap \{x,y\} = \emptyset$, then it must be that $E \backslash \{x,y\}$ is feasible. As the matroid is connected, by Proposition 1.24 we have C circuit and C^* cocircuit such that $C \cap C^* = \{x,y\}$. Define a revelation $\mathcal{B} \in \mathcal{R}(E\backslash\{x,y\}, \mathcal{A})$ as follows:

$$B_f = \begin{cases} 0 & \text{if } f \in C\backslash\{x,y\}, \\ 1 & \text{if } f \in E\backslash C, \\ \{0{,}1\} & \text{if } f \in \{x,y\}, \end{cases}$$

since $E\backslash\{x,y\}$ is feasible, we have some $\mathcal{B}$-basis T.

If $x \in T$, consider the realization $\hat{w} \in \mathcal{R}(E, \mathcal{B})$ that arises by fixing $\hat{w}_x = 1$ and $\hat{w}_y = 0$. Note that x is the heaviest element in C arriving at a contradiction, since T is a w-basis. It must then be that $x \notin T$, but now we can consider the realization $\check{w} \in \mathcal{R}(E, \mathcal{B})$ that emerges by selecting $\check{w}_x = 0$ and $\check{w}_y = 1$. Note that x is the lightest element in C^* arriving once again at a contradiction, as T is a w-basis. $\square$

The previous proposition indicates that every feasible query set touches every pair of elements, so leaving any two elements unrevealed would not be enough. We now prove that one doesn't need to reveal every element, arriving at a feasible query set of minimum size.

Proposition 3.10 *Let $(M, \mathcal{A})$ be an uncertainty matroid such that M is connected and every area is $\{0, 1\}$. Then $E - e$ is a feasible query set of minimum size for any $e \in E$.*

Proof. Let $\mathcal{B} \in \mathcal{R}(E-e, \mathcal{A})$ and define $E_0 = \{f \in E : B_f = \{0\}\}$, $E_1 = \{f \in E : B_f = \{1\}\}$. As we need to show that some $\mathcal{B}$-basis exists, by Theorem 2.30, it suffices to prove that every element is colored. Note that any $f \in E_0$ has $F(f) = \emptyset$, then $e \notin \operatorname{span} F(f)$, since e is not a loop because of connectivity. Similarly, any $f \in E_1$ has $F^*(f) = \emptyset$, then $e \notin \operatorname{span} F(f)$, as e is not a loop because of connectivity. Using Propositions 2.10 and 2.14 we conclude that every element in E_0 is blue and evety element in E_1 is red. Suppose that e wasn't colored, we have $e \in \operatorname{span} F(e) \cap \operatorname{span}^* F^*(e) = \operatorname{span} E_0 \cap \operatorname{span}^* E_1$, since $F(e) = E_0$ and $F^*(e) = E_1$. Then, there is a circuit $C \subseteq E_0 + e$ and a cocircuit $C^* \subseteq E_1 + e$ such that $e \in C \cap C^*$, but since $E_0 \cap E_1 = \emptyset$, we must have $C \cap C^* = \{e\}$ which is impossible by Proposition 1.15.

We now prove that $E - e$ is of minimum size. Suppose that there is a feasible query set F such that $|F| < |E - e|$, then there is a pair $x, y \in E \backslash F$ which contradicts Proposition 3.9. □

Once again, this provides an algorithm to select a feasible query set of minimum size. Moreover, we can extend this idea to not necessarily connected matroids by simply selecting all but one element in each connected component, as the following proposition indicates:

Theorem 3.11 *Let $(M, \mathcal{A})$ be an uncertainty matroid such that every area is $\{0, 1\}$. Consider $\{K_i\}_{i=1}^{l}$ the connected components of M and choose $e_i \in K_i$ for each $i \in [l]$, then $X = E \backslash \{e_i\}_{i=1}^{l}$ is a feasible query set of minimum size.*

Proof. By Proposition 3.10 we get that $K_i - e_i$ is a feasible query set of $(M|_{K_i}, \mathcal{A}|_{K_i})$ and using Proposition 3.1 we conclude that $E \backslash \{e_i\}_{i=1}^{l}$ is a feasible query set of $(M, \mathcal{A})$.

Suppose that there is some feasible query set F such that $|F| < |X|$, therefore $|F \cap K_i| < |K_i| - 1$ for some $i \in [l]$. As $F \cap K_i$ is a feasible query set of $(M|_{K_i}, \mathcal{A}|_{K_i})$ by Proposition 3.1, we have contradicted that $K_i - e_i$ is a feasible query set of minimum size. □

As one could expect, this result also holds for uncertainty matroids that instead of having only $\{0, 1\}$ areas have only $\{L, U\}$ areas with $L, U \in \mathbb{R}$. One can also expand the same idea of selecting extreme weights in a circuit and a cocircuit to uncertainty matroids that have a gap between their minimum and suprema. That is, uncertainty matroids where there exists some $\alpha \in \mathbb{R}$ such that $\max_{e \in E} L_e < \alpha < \min_{e \in E} U_e$.

This last result combined with repeated uses of Proposition 3.5 allows to find feasible query sets in uncertainty matroids with areas are of the form $\{n, n+1\}$ for multiple values of $n \in \mathbb{Z}$. One can proceed in the following manner, consider just the areas $\{n, n+1\}$ for the smallest n and apply Theorem 3.11 finding a minimum size feasible query set, then contract this areas and repeat the procedure for the next smallest n. Once more, this idea provides an algorithm that allows the computation of feasible query sets of minimum size in this class of uncertainty matroids.

Example Consider an uncertainty matroid as in Fig. 3.5 (a). In (b) we show L_1, H_1 a light/heavy partition of M and obtain a minimum size feasible query set for M_{L_1} using Theorem 3.11 shown in (c). We continue in (d) by showing L_2, H_2 a light/heavy partition of M_{H_1}. In (e) and (f) we show a minimum size feasible query set for M_{L_2} and M_{H_2} respectively. Repeated application of Proposition 3.5 allows us to conclude that all the elements selected are a minimum size feasible query set for the original uncertainty matroid.

Note that these algorithmic remarks are just examples of larger classes of uncertainty matroids where one can find feasible query sets of minimum size by just using the fact that interval and $\{0, 1\}$ areas can be solved with polynomial number of calls to the independence oracle and the methods presented in Section 3.1. We don't go in full extent of the even larger classes of uncertainty matroid that could be solved via these techniques.

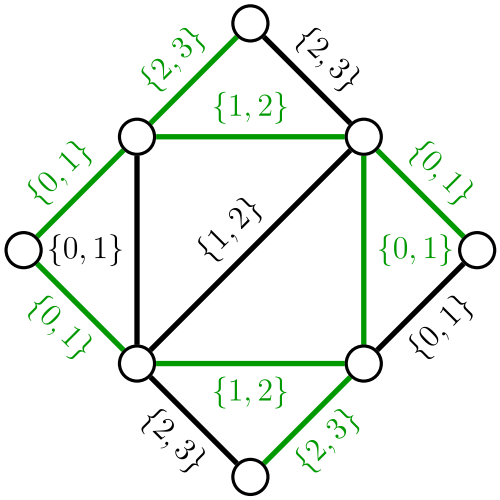
3.3 An analogue to witness sets

A key idea that appears in the previous section is to find small sets that intersect feasible query sets. In the case of interval areas any non-colored non-trivial element intersected every feasible query set, and in $\{0, 1\}$ areas each pair in the same connected component intersected every feasible query set.

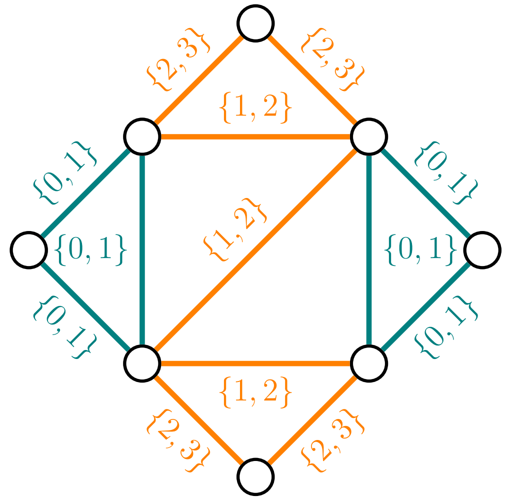
These sets work similarly to witness sets in the adaptative setting. Namely, they are a great tool to prove that proposed solutions are "small" in each context. In the adaptative context, witness sets allow for a good competitive analysis algorithm, that is, the witness algorithm; while in the MCFQS context, small sets that intersect every feasible query set provide good lower bounds on the size of the sets we have to query.

In this section we prove two results that will play a huge role in the search of a solution for the MCFQS problem:

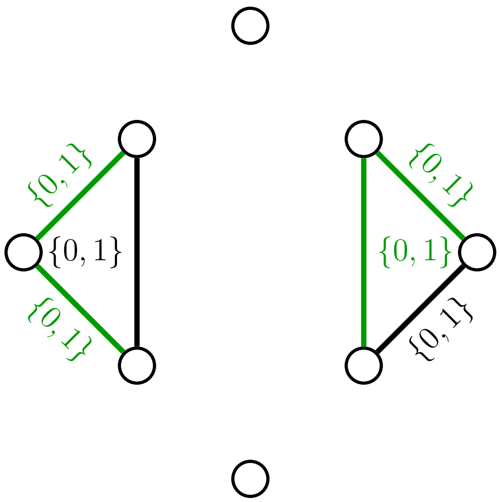
1. We provide a two-way characterization of sets that intersect every feasible query sets. First, as the sets one can select extreme weights of some element to make it the heaviest of a circuit and the lightest of a cocircuit, exploiting to full extent the idea that allowed us to solve uncertainty matroids with interval and $\{0, 1\}$ areas. Second, as the sets that have a special circuit in the contraction of light elements and deletion of heavy elements, this will provide us with an easy to test characterization.
2. We also prove that sets which intersect every feasible query set can always be selected to have at most two elements. This property is extremely good as it will provide us with good lower bounds for proving minimality. Also, we have already dealt with the problems that arise with elements and pairs that intersect every feasible query set in uncertainty matroids with interval and $\{0, 1\}$ areas respectively.



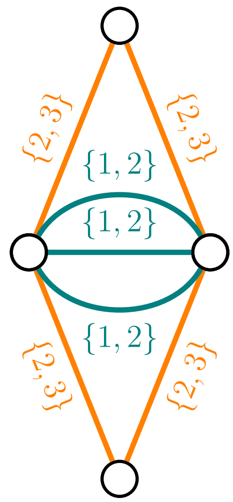
(a) $(M, \mathcal{A})$ and a minimum sized feasible query set.



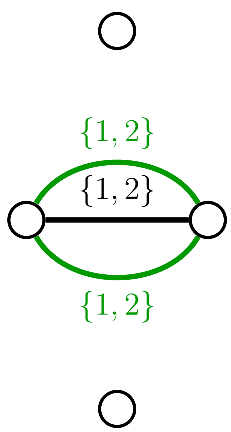
(b) L_1 in teal and H_1 in orange.



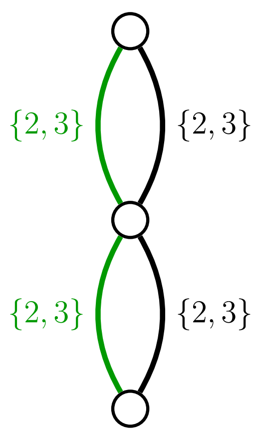
(c) A feasible query set for M_{L_1}.



(d) L_2 in teal and H_2 in orange.



(e) A feasible query set for M_{L_2}.



(f) A feasible query set for M_{H_2}.

Figure 3.5: An example of an uncertainty matroid where the MCFQS problem with unit costs can be solved by repeated application of Proposition 3.5 and Theorem 3.11.

We now state the two-way characterization of sets that intersect every feasible query set:

Proposition 3.12 *Let $(M, \mathcal{A})$ be an uncertainty matroid, $F \subseteq E$ and for each $e \in E$ denote $M/\mathtt{low}(e)\backslash\mathtt{high}(e)$ by M'_e. The following statements are equivalent:*

1. *$F \cap X \neq \emptyset$ for every feasible query set X.*
2. *There exists a non-trivial element $e \in F$ and $Y \subseteq \mathtt{both}(e)\backslash F$ such that:*

   $$e \in \text{span}[\mathtt{low}(e) \cup \mathtt{mid}(e) \cup Y \cup (F \cap \mathtt{both}(e))] \cap \text{span}^*[\mathtt{high}(e) \cup \mathtt{mid}(e) \cup \overline{Y} \cup (F \cap \mathtt{both}(e))],$$

   where $\overline{Y} = \mathtt{both}(e)\backslash F \backslash Y$.
3. *There exists a non-trivial element $e \in F$ and $C \in \mathfrak{C}(M'_e)$ such that $e \in C$ and $C \cap [\mathtt{mid}(e) \cup (F \cap \mathtt{both}(e))] \neq \emptyset$.*

Before proving this proposition we give some intuition on why these statements are equivalent.

Note that the second statement is simply the selecting extreme weights argument made in the previous section. That is, given a "good" revelation of the elements in Y, the span will provide a light circuit and span* will provide a heavy cocircuit. The matroid M'_e hasn't appeared before, but note that if `low` and `high` are non-empty we can reason in the same spirit as Proposition 3.5, meaning that we can contract `low`, delete `high` and keep the "essence" of the problem. The third statement just says that e is still "competing" to be the heaviest of some circuit in M'_e with some other elements, so, we should need additional elements to be revealed as which one is going to be colored red is still uncertain.

Proof. We first prove that 1 and 2 are equivalent. Note that since $E\backslash F$ isn't feasible there is a revelation $\mathcal{B} \in \mathcal{R}(E\backslash F, \mathcal{A})$ such that there is no $(M, \mathcal{B})$-basis and by Theorem 2.31 we must have an element e that is non-colored and non-trivial in $(M, \mathcal{B})$. Note that $e \in F$ as every element in $E\backslash F$ is trivial in $(M, \mathcal{B})$.

Consider the revelation $\check{\mathcal{B}} \in \mathcal{R}(\mathtt{both}(e)\backslash F, \mathcal{A})$ such that $\check{B}_f = \begin{cases} \check{B}_f & \text{if } f \in \mathtt{both}(e)\backslash F, \\ A_f & \text{if } f \notin \mathtt{both}(e)\backslash F. \end{cases}$ Since $\mathcal{B} \in \mathcal{R}(E\backslash F, \check{B})$ and e is non-colored in $(M, \mathcal{B})$, by Proposition 3.7, we get that e is also non-colored in $(M, \check{\mathcal{B}})$.

Select Y as the elements revealed to be lighter than or equal to L_e, that is, $Y = \{f \in \mathtt{both}(e)\backslash F : \check{B}_f \leq L_e\}$. Note that:

$$\mathtt{low}^{\check{\mathcal{B}}}(e) = \mathtt{low}^{\mathcal{A}}(e) \cup Y \qquad \mathtt{mid}^{\check{\mathcal{B}}}(e) = \mathtt{mid}^{\mathcal{A}}(e)$$
$$\mathtt{high}^{\check{\mathcal{B}}}(e) = \mathtt{high}^{\mathcal{A}}(e) \cup \overline{Y} \qquad \mathtt{both}^{\check{\mathcal{B}}}(e) = F \cap \mathtt{both}^{\mathcal{A}}(e)$$

Since e is not blue in $(M, \check{B})$:

$$e \in \text{span}[\mathtt{low}^{\check{\mathcal{B}}}(e) \cup \mathtt{mid}^{\check{\mathcal{B}}}(e) \cup \mathtt{both}^{\check{\mathcal{B}}}(e)] = \text{span}[\mathtt{low}^{\mathcal{A}}(e) \cup \mathtt{mid}^{\mathcal{A}}(e) \cup Y \cup (F \cap \mathtt{both}^{\mathcal{A}}(e))],$$

and as e is not red in $(M, \check{B})$:

$e \in \operatorname{span}^*[\texttt{high}^{\check{B}}(e) \cup \texttt{mid}^{\check{B}}(e) \cup \texttt{both}^{\check{B}}(e)] = \operatorname{span}^*[\texttt{high}^{\mathcal{A}}(e) \cup \texttt{mid}^{\mathcal{A}}(e) \cup \overline{Y} \cup (F \cap \texttt{both}^{\mathcal{A}}(e))]$.

For the converse consider two revelations $w^+, w^- \in \mathcal{R}(E, \mathcal{A})$ as follows:

$$w_f^+ = \begin{cases} w_f^+ \in (L_e, U_e) \cap A_f & \text{if } f \in \texttt{mid}\,(e), \\ w_f^+ \in (-\infty, L_e] \cap A_f & \text{if } f \in Y \cup [F \cap \texttt{both}\,(e)], \\ w_f^+ \in [U_e, \infty) \cap A_f^+ & \text{if } f \in \overline{Y}, \\ w_e^+ \in A_e \cap (L_e, U_e] & \text{if } f = e. \end{cases}$$

$$w_f^- = \begin{cases} w_f^- \in [U_e, \infty) \cap A_f^+ & \text{if } f \in F \cap \texttt{both}\,(e)], \\ w_e^- \in A_e \cap [L_e, U_e) & \text{if } f = e, \\ w_f^+ & \text{otherwise.} \end{cases}$$

Note that w^+ and w^- only differ on $F \cap \texttt{both}\,(e)$ and e. As $e \in \operatorname{span}[\texttt{low}\,(e) \cup \texttt{mid}\,(e) \cup Y \cup (F \cap \texttt{both}\,(e))]$ it is the heaviest element in a circuit in (M, w^+), therefore it is in no (M, w^+)-basis. Similarly, since $e \in \operatorname{span}^*[\texttt{high}\,(e) \cup \texttt{mid}\,(e) \cup \overline{Y} \cup (F \cap \texttt{both}\,(e))]$ it is the lightest element in a cocircuit in (M, w^-), hence it is in every (M, w^-)-basis. To conclude suppose that there is a feasible query set X such that $F \cap X = \emptyset$, we then pick the following revelation $\mathcal{B} \in \mathcal{R}(X, \mathcal{A})$:

$$B_f = \begin{cases} A_f & \text{if } f \notin X, \\ w_f^+ & \text{if } f \in X. \end{cases}$$

Select any $\mathcal{B}$-basis T, as $w^+, w^- \in \mathcal{R}(E, \mathcal{B})$ we have that T is a w^+-basis and a w^--basis, this implies that $e \in T$ and $e \notin T$ which is a contradiction.

We now prove that 2 and 3 are equivalent. Recall that $e \in \operatorname{span}^* Q$ if and only if $e \notin \operatorname{span}(E \backslash Q - e)$ for any set $Q \subseteq E$, using this we get that 2 is equivalent to $e \in \operatorname{span}[\texttt{low}\,(e) \cup \texttt{mid}\,(e) \cup Y \cup (F \cap \texttt{both}\,(e))] \backslash \operatorname{span}[\texttt{low}\,(e) \cup Y]$, we prove equivalence with this last statement.

For the direct implication note that $e \in \operatorname{span}[\texttt{low}\,(e) \cup \texttt{mid}\,(e) \cup Y \cup (F \cap \texttt{both}\,(e))] \backslash \texttt{low}\,(e) = \operatorname{span}_{M_e'}[\texttt{mid}\,(e) \cup Y \cup (F \cap \texttt{both}\,(e))]$ and $e \notin \operatorname{span}[\texttt{low}\,(e) \cup Y] \backslash \texttt{low}\,(e) = \operatorname{span}_{M_e'} Y$. Therefore there is a circuit $C \in \mathfrak{C}(M_e')$ such that $C \subseteq \texttt{mid}\,(e) \cup Y \cup F \cap \texttt{both}\,(e) + e$ and $e \in C$. If $C \cap [\texttt{mid}\,(e) \cup (F \cap \texttt{both}\,(e))] = \emptyset$, we get that $C \subseteq Y + e$, implying that $e \in \operatorname{span}_{M_e'} Y$ which is not possible.

We now prove the converse. Choose $Y = C \cap [\texttt{both}\,(e) \backslash F]$ and note that:

$$\begin{aligned} e \in \operatorname{span}_{M_e'}(C - e) &= \operatorname{span}_{M_e'}[(C \cap \texttt{mid}\,(e)) \cup (C \cap (\texttt{both}\,(e) \backslash F)) \cup (C \cap \texttt{both}\,(e) \cap F)] \\ &\subseteq \operatorname{span}_{M_e'}[\texttt{mid}\,(e) \cup Y \cup (F \cap \texttt{both}\,(e))]. \end{aligned}$$

If $e \in \operatorname{span}_{M_e'}(Y)$ we get some circuit $D \in \mathfrak{C}(M_e')$ such that $D \subseteq Y + e$, since $C \cap [\texttt{mid}\,(e) \cup (F \cap \texttt{both}\,(e))] \neq \emptyset$ we get $D \subsetneq C$ which contradicts the minimality of C. Then:

$$\begin{aligned} & e \in \operatorname{span}_{M_e'}[\texttt{mid}\,(e) \cup Y \cup (F \cap \texttt{both}\,(e))] \backslash \operatorname{span}_{M_e'} Y \\ & \quad = (\operatorname{span}[\texttt{low}\,(e) \cup \texttt{mid}\,(e) \cup Y \cup (F \cap \texttt{both}\,(e))] \backslash \texttt{low}\,(e)) \backslash (\operatorname{span}[\texttt{low}\,(e) \cup Y] \backslash \texttt{low}\,(e)) \\ & \quad = (\operatorname{span}[\texttt{low}\,(e) \cup \texttt{mid}\,(e) \cup Y \cup (F \cap \texttt{both}\,(e))] \backslash \operatorname{span}[\texttt{low}\,(e) \cup Y]) \backslash \texttt{low}\,(e) \\ & \quad \subseteq \operatorname{span}[\texttt{low}\,(e) \cup \texttt{mid}\,(e) \cup Y \cup (F \cap \texttt{both}\,(e))] \backslash \operatorname{span}[\texttt{low}\,(e) \cup Y] \end{aligned}$$

□

Example Consider an uncertainty matroid as in Fig. 3.6 (a). Let F be the elements with area of uncertainty $\{1, 3\}$ and pick any $e \in F$. In (b) we show M'_e for any of the elements and note that there exists the circuit required in Proposition 3.12. In (c) we color the elements in teal if they are in `low`, olive if they are in `mid`, orange if they are in `high` and violet if they are in `both` so it is easier to visualize the span and cospan part of Proposition 3.12. We conclude that F intersects with every feasible query sets (so one of them must be queried). Consider then f as the element of area $\{0, 5\}$. Doing the same as before, we conclude that f belongs to every feasible query set, so it must be queried (see Fig. 3.6 (d), (e) and (f)).

In the uncertainty matroids with $\{0, 1\}$ areas we saw an example of pairs of elements which intersected every feasible query set, but no single element needs to be revealed. A natural question is to ask for trios that intersect every feasible query set, but no pair does. More generally, it is reasonable to ask for big sets that intersect every feasible query set and are minimal with respect to this property. The next proposition shows that these sets can not exist as every set that intersects every feasible query set contains a pair that does.

Proposition 3.13 *Let $(M, \mathcal{A})$ be an uncertainty matroid, $F \subseteq E$ such that $|F| \geq 2$ and $F \cap X \neq \emptyset$ for every feasible query set X. Then, there exists distinct $e, f \in F$ such that $\{e, f\} \cap X \neq \emptyset$ for every X feasible query set.*

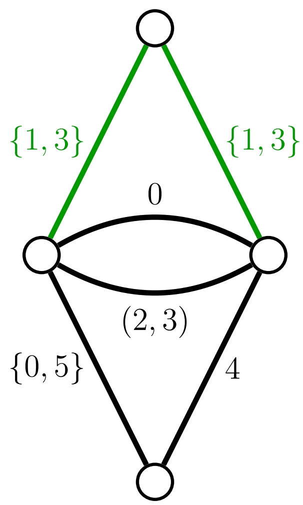
Proof. If $F \cap X \neq \emptyset$ for every feasible query set X by Proposition 3.12 there exists non-trivial $e \in F$, $C \in \mathfrak{C}(M'_e)$ such that $e \in C$ and $C \cap [\texttt{mid}(e) \cup (F \cap \texttt{both}(e))] \neq \emptyset$.

If $C \cap (F \cap \texttt{both}(e)) = \emptyset$, then $C \cap \texttt{mid}(e) \neq \emptyset$ and by Proposition 3.12 we have that $\{e\} \cap X \neq \emptyset$ for every X feasible query set, picking any $f \in F$ such that $f \neq e$ we conclude that $\{e, f\} \cap X \neq \emptyset$ for every feasible query set X. If $C \cap (F \cap \texttt{both}(e)) \neq \emptyset$, select any $g \in C \cap (F \cap \texttt{both}(e))$, once again, Proposition 3.12 let us conclude that $\{e, g\} \cap X \neq \emptyset$ for every X feasible query set. □

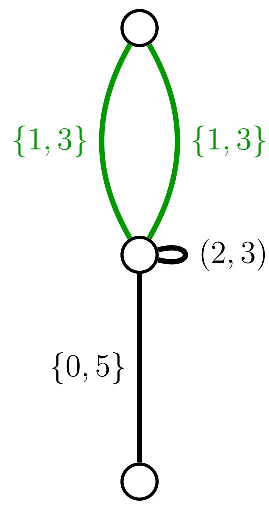
This section gave us some key generalizations of the insight already presented in the illustrative cases. We worked out the selecting extreme weights idea to its full extent by providing the two-way characterization. More so, we found out that sets that intersect each feasible query set cannot be too big. This is similar to what happened with $\{0, 1\}$ areas, therefore, we will try to emulate the same "all but one" approach on the next section.

3.4 Critical pairs and a solution to the MCFQS problem

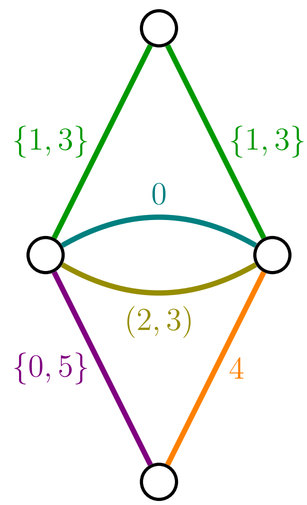
In this section we provide a solution to the MCFQS problem by giving an easy to compute characterization of feasible query sets of minimum size and showing that complements of feasible query sets form a matroid. Note that elements that intersect every feasible set provide a "core set" that is in every feasible query set, so, we do not need to worry about these elements, as they must be revealed. A next natural step is to consider pairs of elements that intersect every feasible query sets, but no element is in every query set. We call such



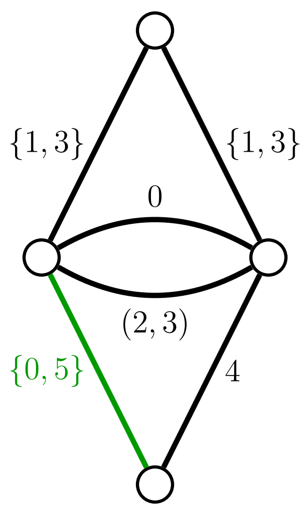
(a) $(M, \mathcal{A})$ and a pair of elements that intersect every feasible query set.



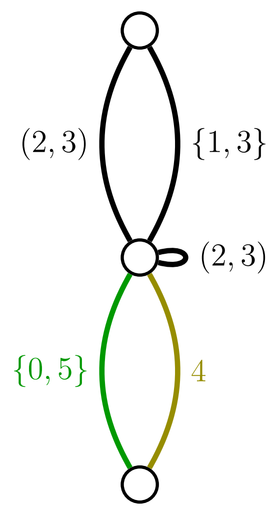
(b) The matroid M'_e. In green there is a circuit that intersects $\texttt{both}\,(e) \cap F$.



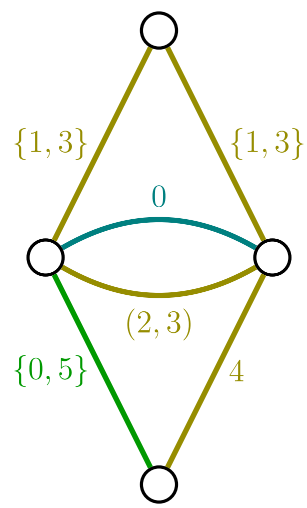
(c) Elements colored according to their belonging in $\texttt{low}\,(e)$, $\texttt{mid}\,(e)$, $\texttt{high}\,(e)$ or $\texttt{both}\,(e)$



(d) $(M, \mathcal{A})$ and an element that belongs to every feasible query set.



(e) The matroid M'_f. In color there is a circuit that intersects $\texttt{mid}\,(f)$.



(f) Elements colored according to their belonging in $\texttt{low}\,(f)$, $\texttt{mid}\,(f)$, $\texttt{high}\,(f)$ or $\texttt{both}\,(f)$.

Figure 3.6: An application of Proposition 3.12.

pairs critical. Note that the previous section guarantees that our analysis stops here, that is, we do not need to worry about trios or bigger sets of elements.

We start by studying critical pairs. We characterize them and prove that they induce an equivalence relation on the complement of the core set previously mentioned. More so, they provide good lower bounds on feasible query sets. Specifically, one must query all elements but one in each equivalence class of the critical relation. We then prove two of the main results of this work:

- All feasible query sets of minimum size arise by revealing all elements but one in each equivalence class, and
- The complements of feasible query sets are the independent sets of a partition matroid.

We end the section and chapter by deducing again the results for illustrative cases as corollaries.

We now define critical pairs, the critical relation and prove some properties about them.

Definition 3.14 (Core set and critical pairs) *Let $(M, \mathcal{A})$ be an uncertainty matroid and:*

$$\begin{aligned} \texttt{core} &= \{e \in E : e \in X \textit{ for every feasible query set } X\} \\ \overline{\texttt{core}} &= E \backslash \texttt{core} \end{aligned}$$

We say that $e, f \in \overline{\texttt{core}}$ are critical if $\{e, f\} \cap X \neq \emptyset$ for every feasible query set X. We write $e \sim f$ if e, f are critical or $e = f$.

Proposition 3.15 *Let $(M, \mathcal{A})$ be an uncertainty matroid. Then:*

1. *Let $e, f \in \overline{\texttt{core}}$, $e \sim f$ if and only if $A_e = A_f = \{L_e, U_e\}$ with $L_e < U_e$ and there is a circuit $C \in \mathfrak{C}(M'_e)$ such that $e, f \in C$.*
2. *$\sim$ is an equivalence relation on $\overline{\texttt{core}}$.*
3. *If X is feasible and $\Gamma \in \overline{\texttt{core}}/\sim$, then $|X \cap \Gamma| \geq |\Gamma| - 1$.*

Proof. 1. Let $e, f \in \overline{\texttt{core}}$ such that $e \sim f$. As $\{e, f\}$ intersects every feasible query set, by Proposition 3.12, we get a circuit $C \in \mathfrak{C}(M'_e)$ such that $e \in C$ and $C \cap [\texttt{mid}\,(e) \cup (\{e, f\} \cap \texttt{both}\,(e))]$. Note that $C \cap \texttt{mid}\,(e) = \emptyset$, otherwise Proposition 3.12 allows us to conclude that $\{e\} \cap X$ for every feasible query set which contradicts the fact that $e \in \overline{\texttt{core}}$. Then $C \cap \{e, f\} \cap \texttt{both}\,(e) \neq \emptyset$, consequently there is a circuit such that $e, f \in C$ and $f \in \texttt{both}\,(e)$. Since $\sim$ is symmetric the same argument over $f \sim e$ allows us to conclude that $e \in \texttt{both}\,(f)$.

   We now prove the converse. As e is a non-trivial element such that there is a circuit $C \in \mathfrak{C}(M'_e)$ and $f \in C \cap \{e, f\} \cap \texttt{both}\,(e)$ by Proposition 3.12 we conclude that $\{e, f\}$ intersects every feasible query set.

2. Since $\sim$ is clearly symmetric and reflexive, we only prove that it is transitive. Let $e, f, g \in \overline{\texttt{core}}$ such that $e \sim f$ and $f \sim g$. By the previous item $A_e = A_f = A_g =$

$\{L_e, U_e\}$, in particular $M'_e = M'_f = M'_g \doteq M'$. The previous item also allows us to conclude that there are two circuits $C^1, C^2 \in \mathfrak{C}(M')$ such that $e, f \in C^1$ and $f, g \in C^2$. Then e, f and g are in the same connected component in M', therefore there is a circuit $C \in \mathfrak{C}(M')$ such that $e, g \in C$ and using the previous item we conclude that $e \sim g$.

3. Suppose that there is some feasible query set X such that $|X \cap \Gamma| < |\Gamma| - 1$, then, there are $e, f \in \Gamma \backslash X$ such that $e \neq f$, since e, f are critical we have that $\{e, f\} \cap X \neq \emptyset$ which is not possible. □

Observation Note that if e is trivial then $e \in \overline{\texttt{core}}$ and its equivalence class is simply $\{e\}$.

We are ready to state and prove the first main result of this section:

Theorem 3.16 *Let $(M, \mathcal{A})$ be an uncertainty matroid and for each $\Gamma \in \overline{\texttt{core}}/\sim$ select any $e_\Gamma \in \Gamma$. Then:*

$$X = \left[\bigcup_{\Gamma \in \overline{\texttt{core}}/\sim} (\Gamma - e_\Gamma) \right] \cup \texttt{core} = E \backslash \{e_\Gamma : \Gamma \in \overline{\texttt{core}}/\sim\},$$

is a feasible query set of minimum size. More so, every feasible query set of minimum size arises this way.

Proof. We first prove that X is a feasible query set. Note that if $E \backslash X = \emptyset$, we get that $X = E$ which is clearly a feasible query set, therefore we can assume that $|E \backslash X| \geq 1$. Suppose that X is not a feasible, then $F \subseteq X$ is not a feasible for any F, hence $E \backslash X = \{e_\Gamma : \Gamma \in \overline{\texttt{core}}/\sim\}$ intersects every feasible query set. We proceed by cases:

Suppose that $E \backslash X = \{e\}$ for some $e \in E$ and consider $\mathcal{B} \in \mathcal{R}(X, \mathcal{A})$ such that there is no $\mathcal{B}$-basis. As e is the only non-trivial element it must be non-colored by Theorem 2.31. Applying Propositions 2.10 and 2.14 we get that $e \in \operatorname{span} F^{\mathcal{B}}(e) \cap \operatorname{span}^* F^{*\mathcal{B}}(e)$. Let $Y = \{f \in \texttt{both}\,(e) : B_f \leq L_e\}$, note that:

$$F^{\mathcal{B}}(e) \subseteq \texttt{low}^{\mathcal{A}}(e) \cup \texttt{mid}^{\mathcal{A}}(e) \cup Y.$$
$$F^{*\mathcal{B}}(e) \subseteq \texttt{high}^{\mathcal{A}}(e) \cup \texttt{mid}^{\mathcal{A}}(e) \cup \overline{Y}.$$

Therefore $e \in \operatorname{span}[\texttt{low}^{\mathcal{A}}(e) \cup \texttt{mid}^{\mathcal{A}}(e) \cup Y] \cap \operatorname{span}^*[\texttt{high}^{\mathcal{A}}(e)] \cup \texttt{mid}^{\mathcal{A}}(e) \cup \overline{Y}]$ and by Proposition 3.12 we get that $e \notin \overline{\texttt{core}}$ which is not possible.

We are only left with the case $|E \backslash X| \geq 2$. By Proposition 3.13 there are distinct elements $e, f \in E \backslash X$ such that $\{e, f\} \cap X \neq \emptyset$ for every feasible query set X, since $e, f \in \overline{\texttt{core}}$ we conclude that $e \sim f$ which is a contradiction.

We now prove minimality of size. By Proposition 3.15.3, for any feasible query set F:

$$|F| = |F \cap \texttt{core}| + \sum_{\Gamma \in \overline{\texttt{core}}/\sim} |F \cap \Gamma| \geq |F \cap \texttt{core}| + |F \cap \overline{\texttt{core}}| - |\overline{\texttt{core}}/\sim| = |F| - |\overline{\texttt{core}}/\sim|,$$

which implies that X is of minimum size. Moreover, every feasible query set F must at least reveal all but one element of each Γ, that is, every set of minimum size must arise this way. $\square$

Note that this theorem provides a solution to the MCFQS problem with unit costs. Since a set is a feasible query set if and only if it contains a feasible query set of minimum size, this theorem also tells us that the structure of feasible query sets is a very specific one. Namely, they select every element except at most one of each equivalence class. This structural idea is shown in the fact that cofeasible query sets form a partition matroid that selects at most one on each equivalence class.

Definition 3.17 (Cofeasible matroid) *Given $(M, \mathcal{A})$ an uncertainty matroid, we define the matroid of cofeasible query sets $\texttt{Cofeas}(M, \mathcal{A}) = (E, \mathcal{F})$. Where the independent sets are given by:*

$$\mathcal{F} = \{I \in 2^E : E \backslash I \text{ is feasible}\}$$

Theorem 3.18 *Let $(M, \mathcal{A})$ be an uncertainty matroid. $\texttt{Cofeas}(M, \mathcal{A})$ is a partition matroid and $\texttt{Cofeas}(M, \mathcal{A}) = \left[\bigoplus_{\Gamma \in \overline{\texttt{core}}/\sim} U^1_\Gamma\right] \oplus U^0_{\texttt{core}}$.*

Proof. Note that $E \backslash I$ is feasible if and only if contains a feasible query set of minimum size. By Theorem 3.16 we get:

$$\begin{aligned}
\mathfrak{I} &= \left\{ I \in 2^E : \begin{array}{ll} |(E\backslash I) \cap \Gamma| \geq |\Gamma| - 1 & , \forall \Gamma \in E/\sim \\ |(E\backslash I) \cap \texttt{core}| = |\texttt{core}| & \end{array} \right\} \\
&= \left\{ I \in 2^E : \begin{array}{ll} |I \cap \Gamma| \leq 1 & , \forall \Gamma \in E/\sim \\ |I \cap \texttt{core}| = 0 & \end{array} \right\},
\end{aligned}$$

that is $\texttt{Cofeas}(M, \mathcal{A})$ is a partition matroid and $\texttt{Cofeas}(M, \mathcal{A}) = \left[\bigoplus_{\Gamma \in \overline{\texttt{core}}/\sim} U^1_\Gamma\right] \oplus U^0_{\texttt{core}}$. $\square$

Since cofeasible query sets form a matroid we have arrived at a solution of the MCFQS problem. We simply execute a variant of the greedy algorithm to find a maximum cost independent set on the cofeasible query sets matroid and, by taking complements, we arrive at a feasible query set of minimum cost.

We now prove some corollaries of Theorem 3.16.

First we show that, if there are not pairs of element with identical areas of uncertainty of size two, then there is a unique minimum size feasible query set.

Corolary 3.19 *Let $(M, \mathcal{A})$ be an uncertainty matroid. If there is not a pair $e, f \in E$ such that $A_e = A_f = \{L_e, U_e\}$ with $L_e < U_e$. Then $\texttt{core}$ is the unique minimum size feasible query set.*

Proof. By Proposition 3.15.1 we get that $e \not\sim f$ for every pair. Then $\Gamma = \{e_\Gamma\}$ for each equivalence class of $\sim$ and by Theorem 3.16 every feasible query set of minimum size is exactly $\texttt{core}$. $\square$

We end the section by retrieving the illustrative cases as corollaries of Theorem 3.16. We start by generalizing the result obtained for matroids with interval areas:

Corolary 3.20 *Let $(M, \mathcal{A})$ be an uncertainty matroid such that $\texttt{both}(e) = \emptyset$ for every $e \in E$. Then, the only minimum size feasible query set is:*

$$X = \texttt{core} = \{e \in E : e \textit{ is non-colored and non-trivial}\}$$

Proof. By Corollary 3.19 we have that the only feasible query set of minimum size is $\texttt{core}$. Using Propositions 3.12, 2.10 and 2.14 we have:

$$\begin{aligned}\texttt{core} &= \{e \in E :\ e \in X \text{ for every feasible query set } X\}\\ &= \{e \in E : e \text{ is non-trivial and } e \in \operatorname{span}[\texttt{low}(e) \cup \texttt{mid}(e)] \cap \operatorname{span}^*[\texttt{high}(e) \cup \texttt{mid}(e)]\}\\ &= \{e \in E : e \text{ is non-trivial and } e \in \operatorname{span} F(e) \cap \operatorname{span}^* F^*(e)\}\\ &= \{e \in E : e \text{ is non-colored and non-trivial}\}\end{aligned}$$ $\square$

Since $\texttt{both}(e) = \emptyset$ for each e if all areas are intervals this is effectively a generalization of the illustrative result for interval areas.

Finally, we retrieve the result for $\{0, 1\}$ areas:

Corolary 3.21 *Let $(M, \mathcal{A})$ be an uncertainty matroid such that every area is $\{0, 1\}$. Consider $\{K_i\}_{i=1}^l$ the connected components of M and choose $e_i \in K_i$ for each $i \in [l]$, then $X = E \backslash \{e_i\}_{i=1}^l$ is a feasible query set of minimum size.*

Proof. Note that for each $e \in E$ we have $\texttt{mid}(e) = \emptyset$, therefore by Proposition 3.12 we have that $E = \overline{\texttt{core}}$. As $M'_e = M$ for each $e \in \overline{\texttt{core}}$ using Proposition 3.15 we have that $e \sim f$ if and only if e and f are connected. We conclude by Theorem 3.16 and noting that connected components are exactly the equivalence classes of $\sim$. $\square$

The solution of the MCFQS problem marks the end of the third chapter. More so, we found out that the MCFQS problem is simply finding a maximum cost independent set on a simple partition matroid. We also constructed a powerful tool, that is, the characterization of minimum sized feasible query sets given by Theorem 3.16 reobtaining previous results as simple corollaries.

Chapter 4

Algorithmic solutions to the UMB and MCFQS problems

In this chapter we discuss our computational model and algorithms for the UMB and MCFQS problems. The main results of this chapter are;

An **algorithm for the general MCFQS problem:** We give an algorithm that uses $O(|E|^3)$ time and calls to the independence oracle.

A **coloring-based algorithm for the general UMB problem:** We give an algorithm that uses $O(|E|^2 \log |E|)$ time and $O(|E|^2)$ calls to the independence oracle.

A **simple way of converting algorithms that solve the UMB problem with closed interval areas to ones that solve the general UMB problem**: This shows that the regret-base algorithm presented in [KZ07] solves the general UMB problem in $O(|E| \log |E|)$ time and $O(|E|)$ calls to the independence oracle.

4.1 Model of computation

We start this chapter with a small remark regarding the model of computation we are going to use. We express algorithms in pseudocode and we are interested in counting elementary steps and calls to the independence oracle.

Some examples of elementary steps are variable assignments, accessing variables, conditionals (if, else if, else), for loops, while loops and simple arithmetic operations (addition, subtraction, multiplication, division and comparison). We call the aggregate of elementary steps time.

We assume that each matroid $(E, \mathfrak{I})$ is given via oracular access, that is, if $I \subseteq E$ we can determine if $I \in \mathfrak{I}$ in $O(1)$ time by asking an oracle. Since the time it takes to actually determine matroid independence can differ wildly from matroid to matroid, so we also keep track of the number of calls to this independence oracle. Sometimes we give results specific to graphic or uniform matroids, in such case we assume direct acceass to the graph or the rank respectively.

We also assume access to some oracles related to the family of areas $\mathcal{A} = \{A_e\}_{e \in E}$, namely:

1. **Infima and suprema oracles:** Given $e \in E$ we can compute L_e and U_e in $O(1)$ time.
2. **Intersection oracle:** Given $e \in E$ and $a, b \in \mathbb{R}$ we can determine if $A_e \cap (a, b) \neq \emptyset$ in $O(1)$ time.

We call such oracles area oracles. Note that these allows us to compute the sets `low`, `mid`, `both`, `high` as follows:

Proposition 4.1 *Let $(M, \mathcal{A})$ be an uncertainty matroid. We can precompute* `low`(e), `mid`(e), `high`(e), `both`(e) *for all $e \in E$ totaling $O(|E|^2)$ time and $O(|E|^2)$ calls to area oracles.*

Proof. It suffices to show that for each pair $e, f \in E$ we can check if f belongs to `low`(e), `mid`(e), `high`(e) or `both`(e) in $O(1)$ calls to area oracles and $O(1)$ time.

1. We can check if $f \in$ `low`(e) by simply asking L_e to the infima oracle, U_f to the supremum oracle and checking if $U_f \leq L_e$.
2. Similarly, to check if $f \in$ `high`(e), we ask U_e and L_f to their respective oracles and check if $U_e \leq L_f$.
3. To check if $f \in$ `mid`(e), we ask L_e, U_e to the infima and suprema oracle and then we ask if $A_f \cap (L_e, U_e) \neq \emptyset$ to the intersection oracle.
4. To verify if $f \in$ `both`(e) we first make sure that $f \notin$ `mid`(e) and then ask the area oracles for L_e, U_e, L_f, U_f and answer affirmatively if $L_f \leq L_e$ and $U_f \geq U_e$.

Each of these procedures can be done in $O(1)$ time with $O(1)$ calls to area oracles. □

We choose to use oracles so that we do not fix any specific representation of families of areas, this is reasonable as we are not really interested in the time cost of the set operations involved. More so, if one picks interval areas represented by their lower and upper bounds, both types of oracles can be replaced by computations that take $O(1)$ time. Similarly, if areas are finite and represented by a list of elements one can precompute all infima and suprema in linear time dispensing of the oracles. Additionally each call to the intersection oracle can be computed in $O(|A_e|)$ time for each $e \in E$ and $a, b \in \mathbb{R}$ as needed, but as we will only use lower and upper bound oracles to precompute `mid`(e) for each e, this will not yield a worse bound than when considering the oracular model.

We end by stating that we will not keep track of calls to area oracles as we will only need them to precompute `low`(e), `mid`(e), `high`(e) and `both`(e) which will always take $O(|E|^2)$ calls to area oracles.

4.2 Algorithmic solutions to the MCFQS problem

In this section we give some algorithmic solutions for the MCFQS problem. In the rest of this chapter we will denote $|E|$ by m and if there is a graphic matroid involved we denote $|V|$ by n.

We start by giving a coloring based algorithm that only works when $\texttt{both}\,(e) = \emptyset$ for each $e \in E$, in particular, it works when each area is an interval. Recall the worst and best case weight functions already mentioned in propositions 2.11 and 2.15:

Definition 4.2 (Worst and best case weight function) *Let $(M, \mathcal{A})$ be an uncertainty matroid and $e \in E$. We define the worst case weight $w : E \to \mathbb{R}$ function for e as:*

$$w_f = \begin{cases} U_e & \textit{if } f = e, \\ L_f & \textit{if } f \neq e. \end{cases}$$

Similarly, we define the best case weight function $w^* : E \to \mathbb{R}$ for e as:

$$w^*_f = \begin{cases} L_e & \textit{if } f = e, \\ U_f & \textit{if } f \neq e. \end{cases}$$

These functions allow us to design procedures that decide blueness and redness efficiently:

Algorithm 2 `blue`

Input: $\langle M, \mathcal{A}, e\rangle$ where $(M, \mathcal{A})$ is an uncertainty matroid and $e \in E(M)$
Output: `TRUE` if e is blue, `FALSE` otherwise.

```
Compute the worst case weight function w for e.
T ← Greedy(M, w)                         ▷ Break ties in favor of e when comparing.
if e ∈ T then
    return TRUE
else if e ∉ T then
    return FALSE
```

Algorithm 3 `red`

Input: $\langle M, \mathcal{A}, e\rangle$ where $(M, \mathcal{A})$ is an uncertainty matroid and $e \in E(M)$
Output: `TRUE` if e is red, `FALSE` otherwise.

```
Compute the best case weight function w* for e.
T ← Greedy(M, w*)                        ▷ Break ties against e when comparing.
if e ∈ T then
    return FALSE
else if e ∉ T then
    return TRUE
```

Observation Breaking ties in favor of e means that e is the first element of weight w_e processed by the greedy algorithm. On the other hand breaking ties against e means that e is the last element of weight w_e^* processed by the greedy algorithm.

Proposition 4.3 *For any uncertainty matroid $(M, \mathcal{A})$ the procedures* `red` *and* `blue` *correctly decide redness and blueness. In doing so they make $O(m)$ calls to the independence oracle and use $O(m \log m)$ time. If the matroid is graphic or uniform these procedures only use $O(m\alpha(m, n))$ or $O(m)$ time respectively.*

Proof. Correctness is given by propositions 2.11 and 2.15. Time complexity and number of calls to the independence oracle are given by the greedy algorithm. If the matroid is graphic, one can replace `Greedy` by any algorithm that computes an MST, in particular, it can be done in $O(m\alpha(m, n))$ time using an algorithm of Chazelle [Cha00]. If the matroid is uniform of rank r any r-selection algorithm will do, in particular, it can be done in $O(m)$ time by using an algorithm of Blum, Floyd, Pratt, Rivest and Tarjan [BFP+73]. □

Recall from corollary 3.20 that when `both` $(e) = \emptyset$ for all $e \in E$, the set of non-colored non-trivial elements is the unique minimum sized feasible query set. Since we are searching for minimum cost feasible query sets we query non-colored non-trivial elements plus elements with negative cost, as shown in the following algorithm:

Algorithm 4 `Coloring based MCFQS algorithm`

Input: $\langle M, \mathcal{A}, c\rangle$ where $(M, \mathcal{A})$ is an uncertainty matroid such that `both` (e) is empty for each $e \in E(M)$ and $c : E \to \mathbb{R}$ is a cost function.
Output: A minimum cost feasible query set.

```
Q ← E;
for e ∈ E(M) do
    if blue(M, A, e) or red(M, A, e) then
        Q ← Q − e;
return Q ∪ {e ∈ E : c_e < 0};
```

Proposition 4.4 *For any uncertainty matroid $(M, \mathcal{A})$ such that* `both` (e) *is empty for each $e \in E(M)$ and any cost function $c : E \to \mathbb{R}$ the coloring based algorithm computes a minimum cost feasible query set. In doing so it makes $O(m^2)$ calls to the independence oracle and uses $O(m^2 \log m)$ time, If the matroid is graphic or uniform we only need $O(m^2\alpha(m, n))$ or $O(m^2)$ time respectively.*

Before proving the proposition, we state a small lemma that will aid us in proving optimality:

Lemma 4.5 *Let $(M, \mathcal{A})$ be an uncertainty matroid and $c : E \to \mathbb{R}$ any cost function on the elements. Then $c($*`core`$\cap \{e \in E : c_e \geq 0\}) \leq c(F \cap \{e \in E : c_e \geq 0\})$ *for any feasible set F.*

Proof. By definition of $\texttt{core}$ we know that $\texttt{core} \subseteq F$, therefore $\texttt{core} \cap \{e \in E : c_e \geq 0\} \subseteq F \cap \{e \in E : c_e \geq 0\}$ and the lemma follows. □

We are now ready to prove proposition 4.4.

Proof. Let Q be as in the final iteration of the algorithm. Note that Q is the set of non-colored non-trivial elements, so it is a feasible query set by corollary 3.20 therefore $Q \cup \{e \in E : c_e < 0\}$ is a feasible query set. We now prove that the output is of minimum cost, note that for any feasible set F we have:

$$\begin{aligned} c(Q \cup \{e \in E : c_e < 0\}) &= c(Q \cap \{e \in E : c_e \geq 0\}) + c(\{e \in E : c_e < 0\}) \\ &= c(\texttt{core} \cap \{e \in E : c_e \geq 0\}) + c(\{e \in E : c_e < 0\}) \\ &\leq c(F \cap \{e \in E : c_e \geq 0\}) + c(e \in E : c_e < 0) \\ &= c(F) + c(\{e \in E : c_e < 0\} \backslash F) \\ &\leq c(F) \end{aligned}$$

where we used lemma 4.5. Since we ask if and element is blue or red m times, we make $O(m^2)$ calls to the independence oracle, using up to $O(m^2 \log m)$ time. If the matroid is graphic or uniform we only need $O(m^2 \alpha(m, n))$ or $O(m^2)$ time respectively. □

Example The coloring based algorithm simply colors each element and outputs elements that are non-colored or have negative as shown in fig 4.2. Note that it is not necessary that each area is an interval, simply that $\texttt{both}\,(e) = \emptyset$ for each $e \in E$.

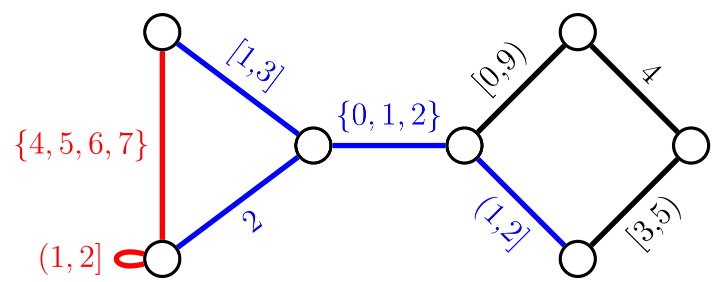


Figure 4.1: An uncertainty matroid $(M, \mathcal{A})$. Red and blue elements are colored accordingly.

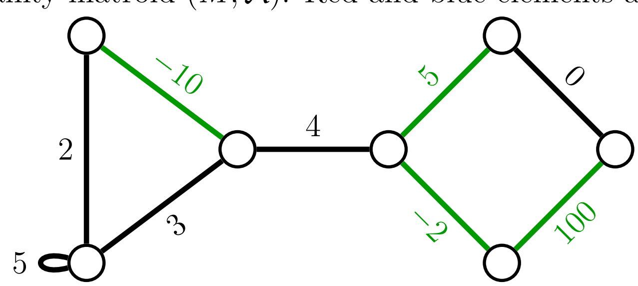


Figure 4.2: The matroid M and the querying costs of each element. In green we show a minimum cost feasible query set.

It is interesting to note that by theorem 3.16 there is a core of elements that must be queried no matter what. It is useful to have a procedure that efficiently computes these elements, as the following algorithm does:

Algorithm 5 `core`

Input: $\langle M, \mathcal{A} \rangle$ where $(M, \mathcal{A})$ is an uncertainty matroid.
Output: The intersection of every feasible query set.

$F \leftarrow \emptyset$;
for $e \in E(M)$ **do**
  Compute $\texttt{low}(e)$, $\texttt{mid}(e)$ and $\texttt{high}(e)$;
  $K \leftarrow \{f \in E(M'_e) : f$ and e are connected in $M'_e\}$;
  if $K \cap \texttt{mid}(e) = \emptyset$ **then**
    $F \leftarrow F + e$;
return F;

Proposition 4.6 *For any uncertainty matroid $(M, \mathcal{A})$ the* `core` *algorithm outputs the set $\bigcap\{F : F$ is a feasible query set$\}$. In doing so it makes $O(m^3)$ calls to the independence oracle and takes $O(m^3)$ time. If the matroid is graphic or uniform we only need $O(m(m+n))$ or $O(m^2)$ time respectively.*

Proof. Using proposition 3.12:

$$\begin{aligned}
&\bigcap\{F : F \text{ is a feasible query set}\},\\
&= \{e \in E : \text{There exists } C \in \mathfrak{C}(M'_e) \text{ such that } C \cap \texttt{mid}(e) \neq \emptyset\},\\
&= \{e \in E : \text{There exists } f \in \texttt{mid}(e) \text{ such that } e \text{ and } f \text{ are connected in } M'_e\},
\end{aligned}$$

hence the `core` procedure outputs the intended set. Using the connected components algorithm for general matroids one can determine K with $O(m^2)$ calls to the independence oracle and $O(m^2)$ time. If the matroid is graphic one can use the biconnected components algorithm to compute K in $O(m+n)$ time. If the matroid is uniform of rank r, then $M'_e = U^{r-|\texttt{low}(e)|}_{E(M'_e)}$. Note that $r - |\texttt{low}(e)| < E(M'_e)$ if and only if $|\texttt{high}(e)| < |E| - r$, so we can determine K in $O(m)$ time. □

More so, there are some cases where revealing these obligatory elements is enough, the coloring-based algorithm is an example of such a case. We extend this idea to matroids with areas that have no critical pairs. Once again, one shall also reveal elements of negative cost, as shown in the next algorithm:

Algorithm 6 `MCFQS algorithm with no critical pairs`

Input: $\langle M, \mathcal{A}, c \rangle$ where $(M, \mathcal{A})$ is an uncertainty matroid such that there are no critical pairs and $c : E \to \mathbb{R}$ is a cost function.
Output: A minimum cost feasible query set.

return $\texttt{core}(M, \mathcal{A}) \cup \{e \in E : c_e < 0\}$;

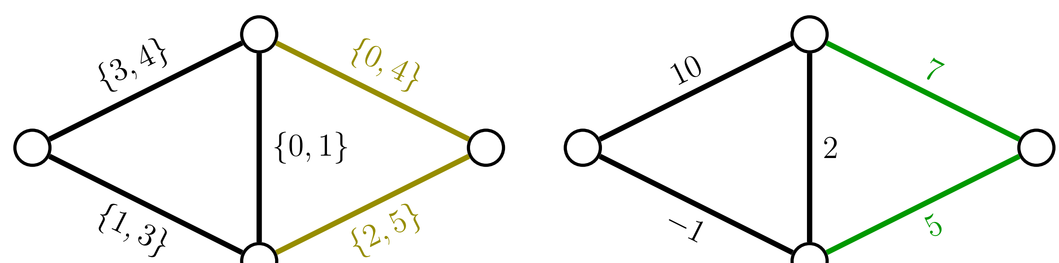
Proposition 4.7 *For any uncertainty matroid $(M, \mathcal{A})$ such that there are no critical pairs and any cost function $c : E \to \mathbb{R}$ the previous algorithm computes a minimum cost feasible query set. In doing so it makes $O(m^3)$ calls to the independence oracle and uses $O(m^3)$ time. If the matroid is graphic or uniform we only need $O(m(m+n))$ or $O(m^2)$ time respectively.*

Proof. Since $\texttt{core}(M, \mathcal{A})$ is a feasible query set, we have that $\texttt{core}(M, \mathcal{A}) \cup \{e \in E : c_e < 0\}$ is also a feasible query set. We use lemma 4.5 to prove that it has minimum cost. Note that for each feasible set F:

$$\begin{aligned} c(\texttt{core}(M, \mathcal{A}) \cup \{e \in E : c_e < 0\}) &= c(\texttt{core}(M, \mathcal{A}) \cap \{e \in E : c_e \geq 0\}) + c(\{e \in E : c_e < 0\}) \\ &\leq c(F \cap \{e \in E : c_e \geq 0\}) + c(\{e \in E : c_e < 0\}) \\ &= c(F) + c(\{e \in E : c_e < 0\} \backslash F) \\ &\leq c(F) \end{aligned}$$

□

Example The algorithm with no critical pairs simply outputs the core set and elements of negative cost. An example is found in fig 4.3.



(a) An uncertainty matroid. In olive the core set is marked.

(b) The previous matroid with the querying costs. In green we show a minimum cost feasible query set.

Figure 4.3: An example when no critical pairs are present.

We finish this section by giving the full-fledged MCFQS algorithm, the only refinement needed with respect to the previous version is that we need to manage critical pairs. Theorem 3.16 tells us to discard an element for each equivalence class of the critical relation, so we discard the one with maximum cost. Note that the previous idea is simply the greedy algorithm on the cofeasible query sets matroid. Once again, we also need to add each element with negative cost, obtaining the following algorithm:

Algorithm 7 `MCFQS algorithm`

Input: $\langle M, \mathcal{A}, c\rangle$ where $(M, \mathcal{A})$ is an uncertainty matroid and $c : E \to \mathbb{R}$ a cost function.
Output: A minimum cost feasible query set.

for $e \in E(M)$ **do**
  Compute $\texttt{low}\,(e)$, $\texttt{mid}\,(e)$ and $\texttt{high}\,(e)$;
$Q \leftarrow \texttt{core}(M, \mathcal{A})$; $E' \leftarrow \{e \in E \backslash Q : |A_e| = 2\}$;
while $E' \neq \emptyset$ **do**
  Select any $e \in E'$;
  $\Gamma \leftarrow \{f \in E(M'_e) : f$ and e are connected in $M'_e\}$;
  Choose any $e_\Gamma \in \operatorname{argmax}\{c_f : f \in K\}$;
  $Q \leftarrow Q \cup (\Gamma - e_\Gamma)$;
  $E' \leftarrow E' \backslash \Gamma$;
return $Q \cup \{e \in E : c_e < 0\}$;

Proposition 4.8 *For any uncertainty matroid $(M, \mathcal{A})$ and any cost function $c : E \to \mathbb{R}$ the MCFQS algorithm computes a minimum cost feasible query set. In doing so it makes $O(m^3)$ calls to the independence oracle and uses $O(m^3)$ time. If the matroid is graphic or uniform we only need $O(m(m+n))$ or $O(m^2)$ time respectively.*

Before proving Proposition 4.8 we note that detecting if $|A_e| = 2$ for $e \in E$, can be done by simply asking the area oracles if $A_e \cap (L_e, U_e) = \emptyset$.

Proof. We first prove correctness. Let Q be as in the final iteration of the `MCFQS` algorithm, we start by proving that $Q \cup \{e \in E : c_e < 0\}$ is a feasible query set. By Proposition 3.15 in each iteration Γ is an equivalence class of $\sim$, more so every equivalence class appears once as every element of E' is eventually deleted. Then $Q = \texttt{core} \cup \bigcup_{\Gamma \in \overline{\texttt{core}}/\sim} (\Gamma - e_\Gamma)$ and by Theorem 3.16 is a feasible query set, consequently $Q \cup \{e \in E : c_e < 0\}$ is also a feasible query set.

Before proving optimality we introduce a small lemma.

Lemma 4.9 *For each $\Gamma \in \overline{\texttt{core}}/\sim$ and F feasible query set, one has $c(Q \cap \Gamma \cap \{e \in E : c_e \geq 0\}) \leq c(F \cap \Gamma \cap \cap \{e \in E : c_e \geq 0\})$.*

We can assume that $\max\limits_{e \in \Gamma} c_e \geq 0$, as in the contrary both sets are empty. We also know that:

$$|F \cap \Gamma \cap \{e \in E : c_e \geq 0\}| \geq |\Gamma \cap \{e \in E : c_e \geq 0\}| - 1,$$

otherwise $|F \cap \Gamma| < |\Gamma| - 1$ contradicting Proposition 3.15. If $F \cap \Gamma \cap \{e \in E : c_e \geq 0\} = \Gamma \cap \{e \in E : c_e \geq 0\}$ it is clear that $c(Q \cap \Gamma \cap \{e \in E : c_e \geq 0\}) \leq c(F \cap \Gamma \cap \{e \in E : c_e \geq 0\})$, therefore we can assume that $F \cap \Gamma \cap \{e \in E : c_e \geq 0\} = \Gamma \cap \{e \in E : c_e \geq 0\} - f$ for some $f \in \Gamma \cap \{e \in E : c_e \geq 0\}$. Let $e_\Gamma = \operatorname{argmax}\{c_f : f \in \Gamma\}$, then if $c(Q \cap \Gamma \cap \{e \in E : c_e \geq 0\}) > c(F \cap \Gamma \cap \cap \{e \in E : c_e \geq 0\})$ we would get $c_f > c_{e_\Gamma}$ which is not possible.

We can now prove that $Q \cup \{e \in E : c_e < 0\}$ has minimum cost by using the previous lemma. Note that for any feasible query set F :

$$
\begin{aligned}
& c(Q \cup \{e \in E : c_e < 0\}) \\
& = c(\{e \in E : c_e \geq 0\} \cap \texttt{core}) + c(Q \cap \overline{\texttt{core}} \cap \{e \in E : c_e \geq 0\}) + c(\{e \in E : c_e < 0\}) \\
& = c(\{e \in E : c_e \geq 0\} \cap \texttt{core}) + \sum_{\Gamma \in \overline{\texttt{core}}/\sim} c(Q \cap \Gamma \cap \{e \in E : c_e \geq 0\}) + c(\{e \in E : c_e < 0\}) \\
& \leq c(\{e \in E : c_e \geq 0\} \cap \texttt{core}) + \sum_{\Gamma \in \overline{\texttt{core}}/\sim} c(F \cap \Gamma \cap \{e \in E : c_e \geq 0\}) + c(\{e \in E : c_e < 0\}) \\
& = c(\{e \in E : c_e \geq 0\} \cap \texttt{core}) + c(F \cap \overline{\texttt{core}} \cap \{e \in E : c_e \geq 0\}) + c(\{e \in E : c_e < 0\}) \\
& = c(F) + c(\{e \in E : c_e < 0\} \backslash F) \\
& \leq c(F)
\end{aligned}
$$

The complexity analysis is similar to the `core` procedure. Note that each element is selected at most once, as they are surely deleted in line 9, so the while loop repeats at most m times. In each while loop we compute a connected component in $O(m^2)$ time and $O(m^2)$ calls to the independence oracle, and take linear time to find the argmax naively. If the matroid is graphic we can improve the search of the connected component of e by using the biconnected components algorithm, this takes $O(m+n)$ time.If the matroid is uniform of rank r, then $M'_e = U^{r-|\texttt{low}(e)|}_{E(M'_e)}$. Note that $r - |\texttt{low}(e)| < E(M'_e)$ if and only if $|\texttt{high}(e)| < |E| - r$, so the matroid is either connected or has no circuits, so the connected components are computed in $O(m)$ time. □

	Coloring based	No critical pairs	General
Uniform matroids	$O(m^2)$ time	$O(m^2)$ time	$O(m^2)$ time
Graphic matroids	$O(m^2\alpha(m,n))$ time	$O(m(m+n))$ time	$O(m(m+n))$ time
Matroids	$O(m^2)$ calls and $O(m^2 \log m)$ time	$O(m^3)$ calls and $O(m^3)$ time	$O(m^3)$ calls and $O(m^3)$ time

Table 4.1: A table summarizing the performance of each discussed implementation.

Example The general algorithm computes each equivalence class of the critical relation and outputs the union of the following sets:

1. The core set,
2. All elements not in the core set but the most costly element of each equivalence class, and
3. The elements of negative cost.

We give an example in Figs. 4.4 and 4.5

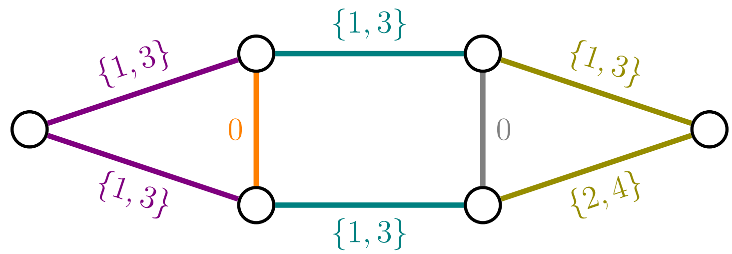


Figure 4.4: An uncertainty matroid $(M, \mathcal{A})$. In olive we have the core set, every other color represents a different equivalence class of the critical relation.

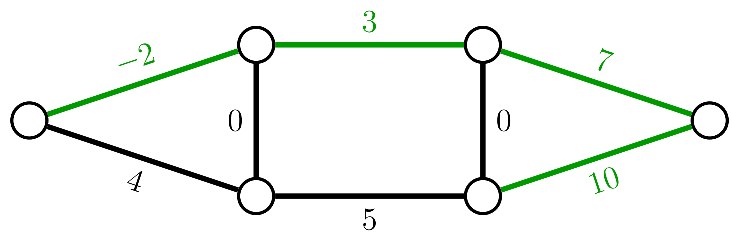


Figure 4.5: The matroid M and the querying costs of each element. In green we show a minimum cost feasible query set.

4.3 Algorithmic solutions to the UMB problem

Once the a feasible query set of minimum cost is found and subsequently the revelation of the queried elements is performed, we are still left with the task of finding a uniformly minimum basis. More generally we give a coloring based algorithm to the UMB problem, that is based on the following key properties:

1. The equivalence between existence of uniformly minimum bases and fully colored uncertainty matroids.

2. The fact that every uniformly minimum basis is a basis that contains all non-trivial blue elements, avoids each red element and is of minimum weight when restricted to the trivial matroid.

The coloring based UMB algorithm will simply check these two properties.

Algorithm 8 `Coloring based UMB algorithm`

Input: $\langle M, \mathcal{A} \rangle$ where $(M, \mathcal{A})$ is an uncertainty matroid.
Output: An uniformly minimum basis if it exists otherwise `FALSE`.

$B \leftarrow \emptyset$; $R \leftarrow \emptyset$;
$N \leftarrow \{e \in E(M) : e \text{ is non-trivial}\}$
for $e \in N$ **do**
    if `blue`$(M, \mathcal{A}, e)$ **then**
        $B \leftarrow B + e$;
    else if `red`$(M, \mathcal{A}, e)$ **then**
        $R \leftarrow R + e$;
    else
        return `FALSE`
Compute the trivial weighted matroid (M^t, w^t);
return $B \cup$ `greedy`(M^t, w^t);

Proposition 4.10 *For any uncertainty matroid $(M, \mathcal{A})$ the UMB algorithm finds a uniformly minimum basis or decides that none exist. In doing so it makes $O(m^2)$ calls to the independence oracle and uses $O(m^2 \log m)$ time. If the matroid is graphic or uniform we only need $O(m^2\alpha(m,n))$ or $O(m^2)$ time respectively.*

Proof. The algorithm simply checks if every element is colored and outputs non-trivial blue elements and a minimum weight basis of the trivial weighted matroid so correctness is given by Theorems 2.30 and 2.35. The for loops calls the independence oracle $O(m^2)$ times and uses $O(m \log m)$ time, the greedy algorithm on the trivial weighted matroid uses $O(m)$ calls and $O(m \log m)$ time totaling the desired complexity. If the matroid is graphic, we can improve the for loop to use $O(m^2\alpha(m,n))$ time and the greedy algorithm can be replaced to use $O(m\alpha(m,n))$. If the matroid is uniform, we can improve the for loop to use $O(m^2)$ time and the greedy algorithm can be replaced to use $O(m)$ time. $\square$

Example Consider a revelation of the minimum cost feasible query set of Fig. 4.4. We have colored accordingly, and in this case the blue elements form a uniformly minimum basis.

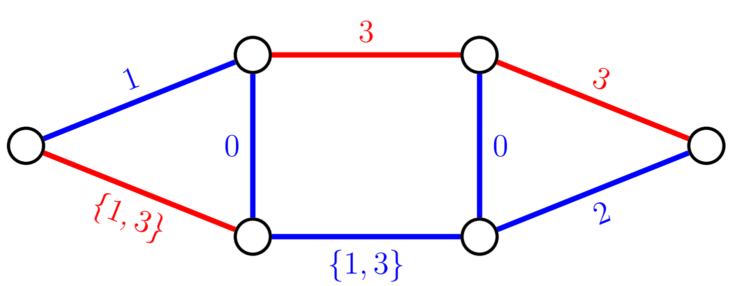


Figure 4.6: A revelation of the minimum cost feasible query set of Fig. 4.4.

It is interesting to note that in [KZ07] a fast algorithm for the UMB problem with closed interval areas is proposed. They show that uniformly minimum bases (or necessarily optimal bases in [KZ07]) are simply bases with 0 regret and propose a regret based algorithm:

Algorithm 9 `Regret based UMB algorithm`

Input: $\langle M, \mathcal{A} \rangle$ where $(M, \mathcal{A})$ is an uncertainty matroid with closed interval areas.
Output: A UOB if it exists, otherwise `FALSE`.

for $e \in E$ **do**
  $w_e \leftarrow \frac{L_e + U_e}{2}$
$T \leftarrow$ `greedy`(M, w);
for $e \in T$ **do**
  $w'_e \leftarrow U_e$
for $e \notin T$ **do**
  $w'_e \leftarrow L_e$
$T' \leftarrow$ `greedy`(M, w');
if $w'(T') < w'(T)$ **then**
  return `FALSE`;
else
  return T;

It turns out that this algorithm also solves the general UMB problem. Furthermore any algorithm that solves the UMB problem for closed intervals works in the general setting with little modification, as shown by the next proposition.

Proposition 4.11 *Let* `ALG` *be any algorithm that solves the UMB problem for closed intervals, then* `ALG'` $\doteq$ `ALG`$(M, \operatorname{cl} \mathcal{A})$ *solves the general UMB problem.*

Proof. First we note that by Proposition 2.24 and Theorem 2.30, the existence of uniformly minimium bases in $(M, \mathcal{A})$ is equivalent to their existence in $(M, \operatorname{cl} \mathcal{A})$. More so, as $(M, \mathcal{A})$ and $(M, \operatorname{cl} A)$ have the same trivial matroids and same colors by Theorem 2.35 we conclude that they have the same uniformly minimum bases. $\square$

This final chapter ends as we provided solutions to the uncertainty problems presented at the beginning by executing the algorithms provided. Given any uncertainty matroid and any element-dependent cost function, we can now compute a set of minimum cost that, when queried, allows us to compute a uniformly minimum basis. Furthermore, we do so efficiently in number of calls to the independence oracle and time.

Conclusion

In the first part of this work we studied uniformly minimum bases and uncertainty on matroids in general terms and later on we focused on solving the MCFQS problem with general areas. We provided efficient algorithmic solutions and found interesting structural results and insights along the way.

We started this work by introducing coloring ideas and provided an interesting equivalence between the existence of uniformly minimum bases and fully colored matroids. On a similar structural note, we proved that uniformly minimum bases are the bases of a matroid and provided some new characterizations in this context.

We provided an analogous of witness sets, namely, sets that intersected every feasible sets and characterized them in terms of the underlying matroid. Key insight on such sets allowed us to detail the structure of feasible query sets of minimum size, which is one of our main results. This result lets us conclude that cofeasible query sets form a partition matroid that only depends on the critical pair relation, which ends up proposing an algorithm to the MCFQS problem.

Even though the MCFQS was solved in full extent, a lot of interesting future work is available. We give three lines of work that were not explored in this thesis.

A first non-explored line of work is to make accurate comparisons between the different solutions provided by each uncertainty approach. How do adaptative competitive analysis relate to non-adaptative against optimum solutions and what happens when we introduce an additive gap are examples of natural questions to ask. An interesting approach in trying to link non-adaptativeness with adaptativeness is to consider algorithms that must query in limited rounds, either by limiting number of queries per round or the rounds themselves.

It is also interesting to consider a variant of the UMB and MCFQS problems that replaces matroids with some other independence system. There is a plethora of well-structured and compelling set systems to use; matchings, matroid intersections and k-systems among others. One could try to generalize the approach presented in this work and see if it provides good approximation algorithms for other independence systems, hoping for analogous results to the ones in combinatorial optimization.

A final non-explored approach is to consider mixed criteria optimization that accounts for both element weight and query costs. The approach utilized considers two "budgets", one for the queries involved and one for the actual weight of the solution, it may be useful to consider only one budget that accounts for both weights and costs. One could imagine scenarios where this provides improvements; for example, if the cost of queries is too expensive in relation to element weight, it may then be useful to remain uncertain at the cost of a worse solution while being free of querying costs. One could also consider a compound approach in which the significance of costs and weights is measured. In this context our work gives a lot of significance to obtaining solutions that are uniformly minimum, while if we allowed non-uniformly solutions savings could be made in querying costs.

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