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HYPERGRAPH CYCLE PARTITIONS

TESIS PARA OPTAR AL GRADO DE
DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA

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HYPERGRAPH CYCLE PARTITIONS

The main focus of this thesis is the study of monochromatic cycle partitions in uniform hypergraphs.

The first part deals with Berge-cycles. Extending a result of Rado to hypergraphs, we prove that for all $r, k \in \mathbb{N}$ with $k \geq 2$, the vertices of every $r(k-1)$-edge-coloured countably infinite complete k-uniform hypergraph can be core-partitioned into at most r monochromatic Berge-cycles of different colours. We further describe a construction showing that this result is best possible.

The second part deals with ℓ-cycles. We show that for all $\ell, k, n \in \mathbb{N}$ with $\ell \leq k/2$ the following hypergraph-variant of Lehel's conjecture is true. Every 2-edge-colouring of the k-uniform complete graph on n vertices has at most two disjoint monochromatic ℓ-cycles in different colours that together cover all but a constant number of vertices, where the constant depends on k and ℓ. Furthermore, we can cover all vertices with at most 4 (3 if $\ell \leq k/3$) disjoint monochromatic ℓ-cycles.

The third part deals with tight cycles in 2-edge-colourings of complete 3-uniform hypergraphs. We show that for every $\eta > 0$ there exists an integer n_0 such that every 2-edge-colouring of the 3-uniform complete hypergraph on $n \geq n_0$ vertices contains two disjoint monochromatic tight cycles of distinct colours that together cover all but at most ηn vertices.

Finally, the fourth part deals with tight cycles in a more general setting. We prove that for every $k, r \in \mathbb{N}$, the vertices of every r-edge-coloured complete k-uniform hypergraph can be partitioned into a bounded number (independent of the size of the hypergraph) of monochromatic tight cycles, confirming a conjecture of Gyárfás. We further prove that for every $r, p \in \mathbb{N}$, the vertices of every r-edge-coloured complete graph can be partitioned into a bounded number of p-th powers of cycles, settling a problem of Elekes, D. Soukup, L. Soukup and Szentmiklóssy. In fact we prove a common generalisation of both theorems which further extends these results to all host hypergraphs with bounded independence number.

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HYPERGRAPH CYCLE PARTITIONS

El principal foco de esta tesis es el estudio de particiones monocromáticas por ciclos en hipergrafos uniformes.

La primera parte trata sobre Berge-cycles. Extendiendo un resultado de Rado a hipergrafos, probamos que para todo $r, k \in \mathbb{N}$ con $k \geq 2$, los vértices de todo $r(k-1)$-arista-coloreo del hipergrafo completo k-uniforme de tamaño infinito numerable pueden ser core-particionados en a lo más r Berge-cycles monocromáticos en colores diferentes. También describimos una construcción mostrando que este resultado es ajustado.

La segunda parte trata sobre ℓ-cycles. Mostramos que para todo $\ell, k, n \in \mathbb{N}$ con $\ell \leq k/2$ es cierta la siguiente variante para hipergrafos de la conjetura de Lehel. Todo 2-arista-coloreo del hipergrafo completo k uniforme en n vértices tiene a lo más dos ℓ-cycles monocromáticos disjuntos en colores diferentes, que juntos cubren todos salvo a lo más un número constante de vértices, donde la constante depende de k y ℓ. Más aún, podemos cubrir todos los vértices con a lo más 4 (3 si $\ell \leq k/3$) ℓ-cycles monocromáticos disjuntos.

La tercera parte trata sobre tight cycles en 2-arista-coloreos de hipergrafos completos 3-uniformes. Mostramos que para todo $\eta > 0$ existe un entero n_0 tal que todo 2-arista-coloreo del hipergrafo completo 3-uniforme en $n \geq n_0$ vértices contiene dos tight cycles monocromáticos disjuntos en colores distintos que juntos cubren todos salvo a lo más ηn vértices.

Finalmente, la cuarta parte trata sobre tight cycles en una configuración más general. Probamos que para todo $k, r \in \mathbb{N}$, los vértices de todo r-arista-coloreo de un hipergrafo k-uniforme pueden ser particionados en un número acotado (independiente del tamaño del hipergrafo) de tight cycles monocromáticos, confirmando una conjetura de Gyárfás. También probamos que para todo $r, p \in \mathbb{N}$, los vértices de todo r-arista-coloreo de un grafo completo pueden ser particionados en un número acotado de potencias p-ésimas de ciclos, respondiendo un problema de Elekes, D. Soukup, L. Soukup y Szentmiklóssy. De hecho probamos una generalización común a ambos teoremas, que extiende estos resultados a todos los hipergrafos base con número de independencia acotado.

To my beloved family.

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Contents

Introduction

In the foundational paper "On a problem of formal logic", Frank Ramsey [Ram30] laid the guiding philosophy of what is now known as Ramsey Theory. To better understand the kind of questions that Ramsey Theory aims to answer, we start with the well-known *Theorem on friends and strangers*: in any group of six people we can always find three of them which are either mutual acquaintances or mutual strangers. This apparent coincidence does not hold if we consider only five people, so the appearance of this pattern can be explained as a consequence of considering sufficiently many people. Ramsey's Theorem [Ram30] states that a monochromatic copy of the complete graph K_n on n vertices is guaranteed to appear in any r-edge-colouring of the complete graph K_m, provided that m is sufficiently large in terms of n and r. The least integer m for which Ramsey's Theorem holds is known as the *r-colour Ramsey number* of K_n, denoted by $R_r(K_n)$. In the Theorem on friends and strangers we have two colours (acquaintances and strangers) so the theorem states that $R_2(K_3) = 6$. The focus of classical Ramsey theory is to study Ramsey numbers in more general settings.

Of particular interest for this dissertation is a paper of 1967, where Gerencsér and Gyárfás [GG67] study the Ramsey numbers of paths. They observed, as a side-product, that every 2-edge-colouring of a complete graph (of any size) contains two vertex-disjoint monochromatic paths in different colours covering all the vertices. This statement captures a different pattern, that is, the existence of a partition of the vertices of K_n into few monochromatic subgraphs of a fixed kind, in this case paths, starting the area of monochromatic partitions as a branch of Ramsey Theory. Here we understand few monochromatic paths as a number of paths not depending on n.

Observe that the partition into two monochromatic paths from the previous paragraph provides an upper bound for the Ramsey number of the path P_n on n vertices, namely, it guarantees that $R_2(P_n) \leq 2n$. Even if the best possible upper bound is approximately $3n/2$ (see [GG67]), we can see that monochromatic partitioning can help us to understand classical Ramsey problems. We will see that this relation also appears in the opposite direction, that is, results from classical Ramsey problems can help finding partitions into few monochromatic pieces.

In 1979, Lehel (see [Aye79]) replaced the paths in the observation from [GG67] with cycles, and conjectured that for all $n \in \mathbb{N}$ and every 2-edge-colouring of the complete graph K_n it is possible to partition its vertices into two monochromatic cycles of different colours. Here we consider single vertices and edges as (degenerate) cycles as well. This statement, known as Lehel's conjecture, turned out to be much harder to solve than the path version of Gerencsér and Gyárfás in [GG67]. It was proved to be true for all $n \in \mathbb{N}$ by Bessy and Thomassé [BT10] more than 30 years later, after posititive results for sufficiently large n in [All08, ŁRS98].

A natural extension is to consider $r > 2$ colours and ask if we can still partition the vertices of any r-edge-coloured complete graph K_n with few (independent of n) monochromatic cycles. This question was answered positively by Erdös, Gyárfás and Pyber [EGP91] in 1991, who proved that $25r^2 \log r$ monochromatic cycles suffice to partition the vertex set of every r-edge-coloured complete graph (this number was improved in [GRSS06]). Since then, this problem has been generalised in many directions, gradually giving shape to the area of monochromatic partitions. The cycles have been replaced by paths [Gyá89, Pok14], trees [HK96, FFGT12], k-regular graphs [SSS13], cycle powers [Sár17] and bounded-degree graphs [GS16]. The host graph K_n has also been replaced, allowing sparser graphs, such as complete balanced bipartite graphs [Hax97, Pok17] and multipartite graphs [SS15, LSS17], graphs with large minimum degree [BBG+14, DN17, Let15] and Ore-type conditions [BS16], bounded independence number [BBG+14, Sár11] or few missing edges [GJS97]. We can also allow for the host graph to be random (Erdős-Rényi model) and ask for the analogous versions of the questions already mentioned (see [BD17, KMN+17, KMS18, LL18]). Also, infinite versions have been considered by replacing K_n with the complete graph with vertex set $\mathbb{N}$ [Rad78] or with countably infinite balanced bipartite graphs [Sou15]. The colouring itself has been relaxed, by allowing r-local-colourings [CS16, LS17]. Monochromatic partitions in hypergraphs have also been studied [GS13, Sár14, GS14, ESSS17, FFGT12]. In particular, this dissertation deals with monochromatic cycle partitions in hypergraphs.

It is worth to mention that there are analog results of many of the previous to monochromatic coverings instead of partitions. A great selection of these problems and an overview of the state of the art can be found in the surveys of Fujita, Liu and Magnant [FLM15], and Gyárfás [Gyá16].

0.1 Graph cycles

Lehel's conjecture (now a theorem) states that, for all $n \in \mathbb{N}$, every 2-edge-colouring of K_n admits a partition of the vertex set of K_n into two monochromatic cycles of different colours. This is best possible in the following way. For all sufficiently large n, there exist 2-edge-colourings of K_n with no monochromatic Hamiltonian cycle[1]. Moreover, there is no constant $m \in \mathbb{N}$ such that every 2-edge-colouring of a complete graph yields a monochromatic cycle covering all but m vertices, as the following construction shows.

For $n \in \mathbb{N}$, we consider K_{3n} and a bipartition of its vertices into sets A_1, A_2 with $|A_1| = n$ and $|A_2| = 2n$. We colour every edge intersecting A_1 with colour blue and every other edge with colour red. On the one hand, every blue cycle contains at most $2|A_1|$ vertices. On the other hand, the largest red cycle has at most $|A_2|$ vertices. Therefore, the number of vertices of the largest monochromatic cycle C is $2n$, and C does not cover n vertices.

As already mentioned above, Erdős, Gyárfás and Pyber studied the monochromatic cycle partitioning problem for an arbitrary number of colours, proving the following result.

Theorem 0.1.1 ([EGP91])**.** *For every $r \in \mathbb{N}$ the following holds. Every r-edge-colouring of a complete graph admits a partition of its vertices into at most $25r^2 \log r$ monochromatic cycles.*

The currently best known upper bound for cycle partitions, due to Gyárfás, Ruszinkó, Sárközy and Szemerédi in [GRSS06], is the following.

Theorem 0.1.2 ([GRSS06])**.** *For every $r \in \mathbb{N}$ the following holds. Every r-edge-colouring of a complete graph admits a monochromatic cycle partition of size at most $100r \log r$.*

Regarding lower bounds for the size of monochromatic cycle partitions, Erdős, Gyárfás and Pyber proposed the following conjecture, that, even if is now disproved (see below), constitutes the main reference for most of the subsequent development of monochromatic cycle partitions.

[1]A cycle C in a host graph G is said to be Hamiltonian if the vertex set of C is equal to the vertex set of G.

Conjecture 0.1.3 ([EGP91]). *For every $r, n \in \mathbb{N}$ and every r-edge-colouring of K_n the following holds. The vertex set of K_n can be partitioned into at most r monochromatic cycles.*

Conjecture 0.1.3 would have been best possible in the same way as Lehel's conjecture is best possible: we can extend the construction for two colours to show that, for all $c \in \mathbb{N}$ and sufficiently large $n \in N$, there exist r-edge-colourings of K_n such that every family of $r-1$ vertex-disjoint monochromatic cycles leaves at least c uncovered vertices, as proved in [EGP91] with very similar arguments to the two colour case.

There are several results towards a positive answer to Conjecture 0.1.3. For instance, a result preceding Conjecture 0.1.3 for the countably infinite complete graph $K_{\mathbb{N}}$, due to Rado [Rad78], implies that every r-edge-colouring of $K_{\mathbb{N}}$ admits a partition into r monochromatic cycles in distinct colours. Here we define infinite cycles as two-way infinite paths. As mentioned above, the case $r = 2$ was answered positively in [BT10]. In [GRSS11] the case $r = 3$ was answered asymptotically, that is, for all $\eta > 0$ there exists $n_0 = n_0(\eta) \in \mathbb{N}$ such that every 3-edge-colouring of K_n, with $n \geq n_0$, contains three vertex-disjoint monochromatic cycles covering all but at most ηn vertices. Apart from these results, Conjecture 0.1.3 remained open for almost 25 years until it was disproved for all $r \geq 3$ by Pokrovskiy in [Pok14]. All of Pokrovskiy's counterexamples are r-edge-colourings of a complete graph with the property that there exist r monochromatic vertex-disjoint cycles covering all but one vertex. This led him to propose a slightly relaxed version of Conjecture 0.1.3.

Conjecture 0.1.4 ([Pok14]). *Every r-edge-colouring of a complete graph contains r monochromatic vertex-disjoint cycles covering all but c_r vertices, where c_r is a constant depending only on r.*

Pokrovskiy confirms his conjecture for $r = 3$ and $c_3 = 43000$ in [Pok16].[2] Furthermore, he conjectures that $c_3 = 1$ and that other counterexamples to Conjecture 0.1.3 for $r = 3$ should be very similar to the constructions in [Pok14].

0.2 Hypergraph cycles

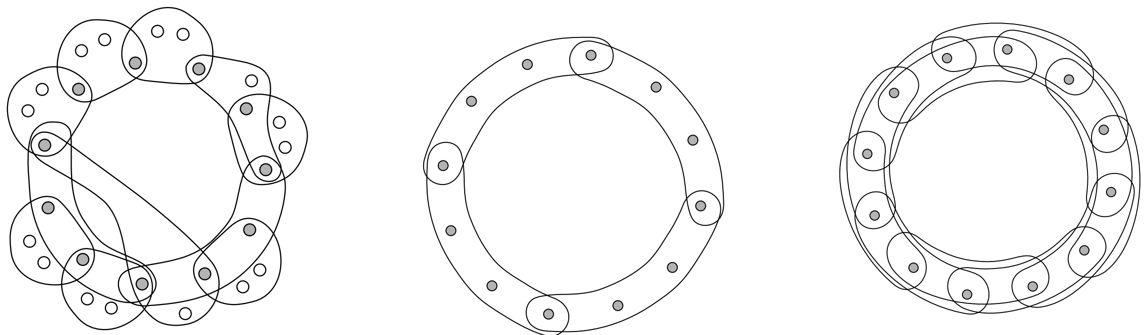
In order to generalise monochromatic partitioning problems to hypergraphs we first need to give a precise definition of the sub-structures that we are looking for. More precisely, to discuss partitions into monochromatic hypergraph cycles requires extending the notion of a graph cycle to the hypergraph setting, and there is no unique way of doing this. Here we only deal with k-uniform host hypergraphs, that is, hypergraphs in which every edge contains exactly k vertices.

The earliest extension of cycles to hypergraphs, due to Berge, is the following. Let $\mathcal{H} = (V, \mathcal{E})$ be a k-uniform hypergraph. A *Berge cycle* in $\mathcal{H}$ is a pair $(X, \mathcal{F})$ where X, called the *core* of the cycle, is a cyclically ordered subset of V and $\mathcal{F} \subseteq \mathcal{E}$ is a subset of edges with $|\mathcal{F}| = |X|$, such that every pair of consecutive vertices is contained in exactly one edge of $\mathcal{F}$. Dorbec, Gravier and Sárközy [DGS08] extended this notion of hypergraph cycles to what we call *t-tight Berge cycles*, by requiring that every set of t consecutive vertices in X, with $2 \leq t \leq k$, is contained in exactly one edge of $\mathcal{F}$, instead of just every consecutive pair as in the Berge cycles. We will refer to 2-tight Berge cycles simply as Berge cycles.

The other classic extension of cycles to hypergraphs consists of the families of *ℓ-cycles*. For a k-uniform hypergraph $\mathcal{H} = (V, E)$ and $1 \leq \ell \leq k-1$, an *ℓ-cycle* is a pair $(X, \mathcal{F})$ where $X \subseteq V$ and $\mathcal{F} \subseteq \mathcal{E}$ are cyclically ordered sets such that every edge of $\mathcal{F}$ contains k consecutive vertices of

[2]An unpublished result of Letzter [Let18] improves c_3 to 60.

X and consecutive edges of $\mathcal{F}$ intersect in exactly ℓ vertices, as shown in Figures 1b and 1c. We also consider sets of exactly $k - \ell$ vertices, and two edges intersecting in 2ℓ vertices (if $2\ell \leq k$), as degenerate ℓ-cycles.[3] We will refer to k-uniform $(k-1)$-cycles as *tight cycles*, and to 1-cycles as *loose cycles*.



(a) A 4-uniform 2-tight Berge cycle. Grey vertices correspond to the core of the cycle.

(b) A 4-uniform 1-cycle or 4-uniform loose cycle.

(c) A 3-uniform 2-cycle, 3-uniform 3-tight Berge cycle or 3-uniform tight cycle.

Figure 1: Different notions of hypergraph cycles.

Observe that all of these notions of hypergraph cycles coincide, as expected, with the definition of graph cycles if we restrict to the case $k = 2$. It is worth mentioning that the family of k-tight Berge cycles and the family of $(k-1)$-cycles are the same family of hypergraphs, that is, tight cycles (see Figure 1c).

0.2.1 Partitions into t-tight Berge cycles

In contrast to a graph cycle $C_G = (V, E)$, where the cyclic ordering of V is an equivalent description of C_G, a t-tight Berge cycle $\mathcal{C}_{\mathcal{H}} = (X, \mathcal{F})$ is not uniquely determined by the ordering of X. The vertices of $\mathcal{C}_{\mathcal{H}}$, that is, the union of its edges, are not necessarily members of X, so we need $\mathcal{F}$ to fully characterise $\mathcal{C}_{\mathcal{H}}$. We will say that a t-tight Berge cycle $\mathcal{C} = (X, \mathcal{F})$ in a host hypergraph $\mathcal{H}$ is Hamiltonian if $X = V(\mathcal{H})$. Accordingly, we say that a family $\mathcal{C}_1 = (X_1, \mathcal{F}_1), \ldots, \mathcal{C}_m = (X_m, \mathcal{F}_m)$ of t-tight Berge cycles *partitions* the vertex set of a host hypergraph $\mathcal{H}$ if the sets X_i, $\mathrm{i} \in [m]$, are pairwise disjoint and $\bigcup_{\mathrm{i} \in [m]} X_\mathrm{i} = V(\mathcal{H})$.

As mentioned in Section 0.1, there are 2-edge-colourings of complete graphs with no monochromatic Hamiltonian cycle. The simplest example is to take any complete graph with at least three vertices, choose a vertex v, colour every edge containing v with blue and every other edge with red. However, if we consider a 3-uniform complete graph hypergraph $\mathcal{K}_n^{(3)}$ on n vertices, colour blue every edge containing a fixed vertex and colour red all the other edges, then the resulting edge-colouring will always contain a blue Hamiltonian (2-tight) Berge cycle. In fact, as Gyárfás, Lehel, Sárközy and Schelp proved in [GLSS08], any procedure to obtain a 2-edge-colouring of the complete 3-uniform hypergraph $\mathcal{K}_n^{(3)}$ on n vertices yields the same result.

Theorem 0.2.1 ([GLSS08]). *Every 2-edge-colouring of $\mathcal{K}_n^{(3)}$, with $n \geq 5$, admits a monochromatic Hamiltonian Berge cycle.*

[3]We can also define degenerate edges as single edges and their subsets, as in the graph case. However, our arguments in Chapter 2 only consider sets of size $k - \ell$ as degenerate cycles.

So now the problem of monochromatic partitions into Berge cycles reveals an additional feature which is not present in the graph case. It is possible for an r-edge-colouring of $\mathcal{K}_n^{(k)}$ to contain a Hamiltonian Berge cycle even if $r > 1$. Omidi [Omi14] (after partial results in [GSS10a, GSS10c, MO17]) proved the following sharp result, which was stated as a conjecture in [GLSS08], and extends Theorem 0.2.1 to every uniformity.

Theorem 0.2.2 ([Omi14]). *For every integer $k \geq 2$ there exists $n_k \in \mathbb{N}$ such that the following holds. Every $(k-1)$-edge-colouring of $\mathcal{K}_n^{(k)}$, with $n \geq n_k$, contains a monochromatic Hamiltonian Berge cycle.*

A generalisation of Theorem 0.2.2 to t-tight Berge cycles was conjectured by Dorbec, Gravier and Sárközy.

Conjecture 0.2.3 ([DGS08]). *For any fixed $2 \leq t \leq k$ there exists $n_{k,t} \in \mathbb{N}$ such that the following holds. Every $(k-t+1)$-edge-colouring of $\mathcal{K}_n^{(k)}$, with $n \geq n_{k,t}$, contains a monochromatic Hamiltonian t-tight Berge cycle.*

Conjecture 0.2.3 was confirmed for $k = 4$ and $t = 3$ in [GSS10b]. If true, Conjecture 0.2.3 is best possible, as shown in [DGS08]. Observe that Conjecture 0.2.3 addresses only Hamiltonian Berge cycles.

In Chapter 1 we study monochromatic partitions into t-tight Berge cycles by considering the countably infinite k-uniform complete hypergraph $\mathcal{K}_{\mathbb{N}}^{(k)}$ as the host combinatorial object. Here we consider a two-way infinite t-tight Berge path[4] as the infinite analogue of t-tight Berge cycles. Our main result of Chapter 1, in joint work with Jan Corsten and Nóra Frankl, is the following generalisation of Theorem 0.2.2.

Theorem 0.2.4. *For every $r \geq 1$ and $k \geq 2$ the following holds. Every $r(k-1)$-edge-colouring of $\mathcal{K}_{\mathbb{N}}^{(k)}$ admits a partition into at most r monochromatic Berge cycles.*

In addition, we extend Conjecture 0.2.3 to an arbitrary number of colours in the following way.

Conjecture 0.2.5. *For any fixed $r > 1$ and $2 \leq t \leq k$ there exists $n_{r,k,t}, c_{r,k,t} \in \mathbb{N}$ such that the following holds. Every $r(k-t+1)$-edge-colouring of $\mathcal{K}_n^{(k)}$, with $n \geq n_{r,k,t}$, contains r core-disjoint monochromatic t-tight Berge cycles covering all but at most $c_{r,k,t}$ vertices.*

We also prove that Conjecture 0.2.5 is best possible, that is, for every positive integers $c, r > 1$ and $2 \leq t \leq k$ there exist infinitely many r-edge-colourings of complete k-uniform hypergraphs such that the following holds. Every family of $r-1$ core-disjoint monochromatic t-tight Berge cycles leaves at least c vertices uncovered by the cores of the family (see Theorem 1.2.1 in Chapter 1).

0.2.2 Partitions into ℓ-cycles

We say that a family of $C_1, \ldots, C_m$ of ℓ-cycles partitions a host hypergraph $\mathcal{H}$ if the ℓ-cycles are vertex-disjoint and the union of the vertices of $C_1, \ldots, C_m$ is the vertex set of $\mathcal{H}$. Observe that, unlike t-tight Berge cycles, it is much more natural to define an ℓ-cycle partition. However, if

[4]A t-tight Berge path is defined as its cycle counterpart but replacing the cyclic ordering of its core with a linear ordering and removing the edge containing the first and the last vertices.

we compare different notions of hypergraph cycle partitioning in terms of the relation between the number of necessary cycles and the number of colours, we strongly believe that k-uniform ℓ-cycles are harder than t-tight Berge cycles for all $\ell \geq 1$ and all $2 \leq t < k$. In fact, we will see in Chapter 2 that for all $r, \ell \geq 1$ and $k \geq 2$, k-uniform ℓ-cycle partitions of r-coloured complete k-uniform hypergraphs may require at least r pieces.

Up to now, most of the work on ℓ-cycle partitions has focused on loose cycles (1-cycles). Such questions were first studied by Gyárfás and Sárközy in [GS13], where they prove the following result.

Theorem 0.2.6 ([GS13]). *For every $k, r \in \mathbb{N}$ there is some $c = c(k, r)$ such that the vertices of every r-edge-coloured complete k-uniform hypergraph can be partitioned into at most c monochromatic loose cycles.*

Later, Sárközy showed in [Sár14] that the constant $c(k, r)$ of Theorem 0.2.6 can be chosen to be $50rk \log(rk)$. An interesting generalisation of Theorem 0.2.6, due to Gyárfás and Sárközy in [GS14], considers edge-colourings of k-uniform hypergraphs with bounded independence number instead of complete k-uniform hypergraphs[5].

Theorem 0.2.7 ([GS14]). *For every $k, r, \alpha \in \mathbb{N}$ there is some $c = c(k, r, \alpha)$ such that the vertices of every r-edge-coloured k-uniform hypergraph with independence number α can be partitioned into at most c monochromatic loose cycles.*

In [Gyá16], Gyárfás conjectured that a result similar to Theorem 0.2.6 holds for tight cycles.

Conjecture 0.2.8 ([Gyá16]). *For every $k, r \in \mathbb{N}$ there is $c = c(k, r)$ such that the vertices of every r-edge-coloured complete k-uniform hypergraph can be partitioned into at most c monochromatic tight cycles.*

If true, Conjecture 0.2.8 is best possible, as showed by Gyárfás in [Gyá16]. A recent result in [ESSS17] for countably infinite uniform complete hypergraphs (due to Elekes, D. Soukup, L. Soukup and Szentmiklóssy) can be seen as an infinite analogue of Conjecture 0.2.8.

Theorem 0.2.9 ([ESSS17]). *Every r-edge-colouring of the countably infinite complete k-uniform hypergraph $\mathcal{K}_{\mathbb{N}}^{(k)}$ admits a partition into at most r monochromatic tight cycles, where two-way infinite tight paths count as tight cycles as well.*

In Chapters 2 and 3 we study monochromatic ℓ-cycle partitions in 2-edge-colourings of uniform hypergraphs. Chapter 2 is joint work with Maya Stein and focuses on k-uniform ℓ-cycles with $\ell \leq k/2$. Our main result is the following.

Theorem 0.2.10. *For every $k \geq 2$ and $\ell \leq k/2$ the following holds. Every 2-edge-colouring of $\mathcal{K}_{\mathbb{N}}^{(k)}$ contains two vertex-disjoint ℓ-cycles in different colours covering all but at most $5(k - \ell) - 1$ vertices.*

We also improve the number of leftover vertices in the case $\ell \leq k/3$. In addition, we show that Theorem 0.2.10 is best possible in the following sense: for every positive integers $c, r > 1$ and $\ell \leq k/2$ there are r-edge-colourings of k-uniform complete hypergraphs such that the largest monochromatic ℓ-cycle leaves at least c uncovered vertices. Observe that $(k - \ell)$ divides the number of vertices of every (non-degenerate) k-uniform ℓ-cycle. Therefore, $k - \ell - 1$ is a lower

[5] Theorem 0.2.7 also extends a previous result for graphs with bounded independence number in [Sár11]. We provide details of the graph version of Theorem 0.2.7 in Chapter 4.

bound for the constant c in Theorem 0.2.10. We conjecture (see Conjecture 2.1.2 in Chapter 2) that, provided that $(k-\ell)$ divides n, any 2-edge-colouring of $\mathcal{K}_n^{(k)}$ admits a monochromatic ℓ-cycle partition of size two, with cycles of different colours.

Chapter 3 deals with tight cycles in 2-edge-colourings of 3-uniform complete graphs. In joint work with Hiệp Hàn and Maya Stein, we prove the following asymptotic result on monochromatic tight cycle partitions.

Theorem 0.2.11. *For every $\eta > 0$ there exists $n_0 \in \mathbb{N}$ such that the following holds. Every 2-edge-colouring of $\mathcal{K}_n^{(3)}$, with $n \geq n_0$, admits two vertex-disjoint monochromatic tight cycles covering all but at most ηn vertices.*

For all sufficiently large n, there exist 2-edge-colourings of $\mathcal{K}_n^{(3)}$ with the following property: the largest monochromatic tight cycle leaves roughly $4n/3$ vertices uncovered (see [HŁP$^+$09]). Therefore, we cannot replace the two monochromatic tight cycles in Theorem 0.2.11 with just one, so in this sense Theorem 0.2.11 is best possible.

Finally, in Chapter 4 we study monochromatic tight cycle in a more general setting, by considering r-edge-colourings of k-uniform hypergraphs with bounded independence number[6] as host hypergraph instead of k-uniform complete hypergraphs. In joint work with Jan Corsten, Nóra Frankl, Jozef Skokan and Alexey Pokrovskiy, we prove the following result.

Theorem 0.2.12. *For every $k, r, \alpha \in \mathbb{N}$, the vertices of every r-edge-coloured hypergraph with independence number α can be partitioned into a constant (depending only on k, r and α) number of monochromatic tight cycles.*

Theorem 0.2.12 generalises Theorem 0.2.7 and confirms Conjecture 0.2.8 by taking $\alpha = 1$. It also answers a question of Grinshpun and Sárközy in [GS16], regarding monochromatic cycle power[7] partitions in 2-edge-colourings of complete graphs (see Chapter 4).

[6]The independence number of a hypergraph $\mathcal{H}$ is the size of largest set of vertices inducing an empty hypergraph.

[7]The p-th power of a cycle is obtained by adding all the edges joining vertices at distance at most p. A suitable generalisation of tight cycle powers is provided in Chapter 4.

Chapter 1

Partitioning countably infinite complete hypergraphs into few Berge cycles

Abstract

We prove that for all $r, k \in \mathbb{N}$ with $k \geq 2$, the vertices of every $r(k-1)$-edge-coloured countably infinite complete k-uniform hypergraph can be core-partitioned into at most r monochromatic Berge cycles of different colours. We further describe a construction showing that this result is best possible. This is joint work with Jan Corsten and Nóra Frankl.

1.1 Introduction

In 1978 Rado [Rad78] studies monochromatic partitions in r-edge-colourings of the countably infinite complete graph $K_{\mathbb{N}}$ instead of finite complete graphs. By considering a two-way infinite path as an analogue of the infinite cycle, he proves that every r-edge-colouring of $K_{\mathbb{N}}$ admits a partition into at most r monochromatic finite or infinite cycles.[1]

The main focus of this chapter is an extension of Rado's theorem to Berge cycles in countably infinite hypergraphs. Recall that Theorem 0.2.2 states that, for sufficiently large n, every $(k-1)$-edge-colouring of $\mathcal{K}_n^{(k)}$ contains a monochromatic Hamiltonian Berge cycle. This result is also sharp, as shown in [GLSS08] and in Theorem 1.2.2 below.

A generalisation of Theorem 0.2.2 to t-tight Berge cycles was proposed as a conjecture (see Conjecture 0.2.3 in the Introduction) by Dorbec, Gravier and Sárközy in [DGS08]. We now restate Conjecture 0.2.3 to its original form.

Conjecture 1.1.1 ([DGS08])**.** *For any fixed $2 \leq c, t \leq r$ satisfying $c + t \leq k + 1$ and sufficiently large n the following holds. Every c-edge-colouring of $\mathcal{K}_n^{(k)}$ admits a monochromatic Hamiltonian t-tight Berge cycle.*

We know that Conjecture 1.1.1 is true for $t = 2$ (see [Omi14]), and also for $k = 5, t = 3$ and $c = 2$ (see [DGS08]). The following weaker result from [DGS08] replaces the sum $c + t$ with the product ct.

[1] In fact, Rado's theorem answers the question for paths instead of cycles, where one-way infinite paths are considered as the infinite analogue of the finite path. However, the cycle version of Rado's Theorem can be proved by considering a slight modification of Rado's original proof.

Theorem 1.1.2 ([DGS08]). *For any fixed $2 \leq c, t \leq r$ satisfying $ct + 1 \leq k$ and $n \geq 2(t+1)kc^2$ the following holds. Every c-edge-colouring of $\mathcal{K}_n^{(k)}$ contains a monochromatic Hamiltonian t-tight Berge cycle.*

For more colours or infinite hypergraphs not much is known in this direction. A recent result from Elekes, D. Soukup, L. Soukup and Szentmiklóssy in [ESSS17] extends Rado's Theorem to the hypergraph setting (see Theorem 0.2.9), by considering edge-colourings of the countably infinite k-uniform hypergraph $\mathcal{K}_{\mathbb{N}}^{(k)}$ and studying monochromatic tight cycle partitions. It is worth to mention that every tight cycle is also a t-tight Berge cycle (for every $2 \leq t \leq k$), so Theorem 0.2.9 provides an upper bound for the t-tight Berge cycle partition number of $\mathcal{K}_{\mathbb{N}}^{(k)}$. Our main result shows that monochromatic Berge cycle partitions require, in general, fewer parts than their tight cycle counterparts.

Theorem 1.1.3. *For all $r, k \in \mathbb{N}$ with $k \geq 2$ and every $r(k-1)$-edge-colouring of $\mathcal{K}_{\mathbb{N}}^{(k)}$ the following holds. The vertices of $\mathcal{K}_{\mathbb{N}}^{(k)}$ can be core-partitioned into at most r monochromatic Berge cycles of different colours.*

We will prove this theorem in Section 1.3. We will rely on the existence of non-trivial ultrafilters and therefore on the axiom of choice. In Section 1.2 we describe a construction for t-tight Berge-cycles showing that Theorem 1.1.3 is best possible. A slight modification of this construction also shows a lower bound for the finite case. We therefore believe that Theorem 1.1.3 should hold in a similar form in the finite case as well.

Conjecture 1.1.4. *For all $r, k, t \in \mathbb{N}$ with $k \geq t \geq 2$, there is some $c = c(k, r, t) \in \mathbb{N}$ such that the following is true for all $n \in \mathbb{N}$. In every $r(k-t+1)$-edge-colouring of $\mathcal{K}_n^{(k)}$, there are at most r monochromatic t-tight Berge cycles, whose cores are disjoint and cover all but c vertices.*

1.2 The constructions

We will prove that Theorem 1.1.3 is best possible, that is, $r-1$ monochromatic Berge cycles do not suffice to partition all the vertices of certain $r(k-1)$-edge-colourings of $\mathcal{K}_{\mathbb{N}}^{(k)}$. This is done by considering the case $t = 2$ of the following result on t-tight Berge cycles, generalising a previous construction for the case $r = 1$ in [DGS08].

Theorem 1.2.1. *For all $r, k, t \in \mathbb{N}$ with $k \geq t \geq 2$, there is an edge-colouring of $\mathcal{K}_{\mathbb{N}}^{(k)}$ with $q = r(k-t+1)+1$ colours in which the vertices cannot be covered by the cores of r monochromatic t-tight Berge cycles.*

Proof. We denote the lexicographical ordering on $\binom{[q]}{r}$ by $\prec$. Partition $\mathbb{N}$ into sets $\{B_I : I \in \binom{[q]}{r}\}$ so that $|B_I| > r \cdot \sum_{J \prec I} |B_J|$ for every $I \in \binom{[q]}{r}$. Note that all B_I's but $B_{q-r+1,\dots,q}$ will be finite.

For $x \in \mathbb{N}$, let $I(x)$ be the r-subset of $[q]$ for which $x \in B_{I(x)}$. We define a q-edge-colouring φ of $\mathcal{K}_{\mathbb{N}}^{(k)}$ as follows. For every $\mathrm{e} \in E(\mathcal{K}_{\mathbb{N}}^{(k)})$ we consider an order $x_{\mathrm{e}}^1, \dots, x_{\mathrm{e}}^k$ of e satisfying $I(x_{\mathrm{e}}^{\mathrm{i}}) \preceq I(x_{\mathrm{e}}^j)$ for all $1 \leq \mathrm{i} < j \leq k$, and define $\varphi(\mathrm{e})$ as an arbitrary member of $[q] \setminus \bigcup_{\mathrm{i} \leq k-t+1} I(x_{\mathrm{e}}^{\mathrm{i}})$.

Assume for contradiction that there are monochromatic t-tight Berge cycles $C_1, \dots, C_r$ with cores $X_1, \dots, X_r$ so that $\bigcup_{\mathrm{i}} X_{\mathrm{i}} = \mathbb{N}$ and let $I \subset [q]$ be a set of size r which contains all colours used by these t-tight Berge cycles.

First observe that $\left|e \cap \bigcup_{J<I} B_J\right| \geq k - t + 1$ for every edge e with $e \cap B_I \neq \emptyset$ and $\varphi(e) \in I$. Therefore, if $e \in E(C_i)$ for some $i \in [r]$ then every t-subset of e containing an element of B_I also contains at least one vertex in $\bigcup_{J<I} B_J$. We conclude that

$$|X_i \cap B_I| \leq \sum_{J<I} |B_I| < |B_I|/r$$

for every $i \in [r]$ and hence $|B_I| = |B_I \cap (\bigcup_i X_i)| < |B_I|$, a contradiction. □

A simple modification of the argument yields the following result which shows that Conjecture 1.1.4 is best possible if it is true.

Theorem 1.2.2. *For all $c, r, k \in \mathbb{N}$ with $k \geq 2$, there is some $n_0 = n_0(c, r, k)$ such that the following is true for every natural number $n \geq n_0$. There is an edge-colouring of $\mathcal{K}_{\mathbb{N}}^{(k)}$ in which the cores of any r monochromatic Berge cycles can cover at most $n - c$ vertices.*

Proof. If for the B_I's instead of $|B_I| > r \cdot \sum_{J<_l I} |B_J|$ we require $|B_I| > c + r \cdot \sum_{J<_l I} |B_J|$, we obtain the desired construction. □

1.3 The upper bound

Our proof is based on the simple proof of Rado's theorem given by Elekes, D. Soukup, L. Soukup and Szentmiklóssy in [ESSS17].

An *ultrafilter* on a set X is a set-system $\mathcal{U} \subseteq 2^X$ satisfying the following properties:

(i) $\emptyset \notin \mathcal{U}$ and $X \in \mathcal{U}$,

(ii) $A \in \mathcal{U}$ and $A \subseteq B \subseteq X \implies B \in \mathcal{U}$,

(iii) $A, B \in \mathcal{U} \implies A \cap B \in \mathcal{U}$,

(iv) $A \subseteq B$ for some $B \in \mathcal{U} \implies A \in \mathcal{U}$ or $B \setminus A \in \mathcal{U}$.

An ultrafilter $\mathcal{U}$ on X is called trivial if there is some $x \in X$ such that $\mathcal{U} = \{A \subseteq X : x \in A\}$ and non-trivial otherwise. A standard application of Zorn's Lemma shows that there exist non-trivial ultrafilters whenever X is infinite. Note that we are assuming the axiom of choice here.

Proof of Theorem 1.1.3. Let $q = r(k-1)$ and let φ be the given q-edge-colouring of $\mathcal{K}_{\mathbb{N}}^{(k)}$. Let $\mathcal{U}$ be a non-trivial ultrafilter on $\mathbb{N}$ and note that $\mathcal{U}$ contains all co-finite sets. We define an edge-multicolouring $\varphi_2 : E(\mathcal{K}_{\mathbb{N}}^{(2)}) \to 2^{[q]}$ by

$$\varphi_2(uv) = \{\varphi(e) : e \in E(\mathcal{K}_{\mathbb{N}}^{(k)}) \text{ and } u, v \in e\}.$$

For a vertex $v \in \mathbb{N}$ and for a colour $c \in [q]$, let $N_2^c(v) := \{u \in \mathbb{N} : c \in \varphi_2(uv)\}$. Now define a vertex-colouring $\chi : \mathbb{N} \to 2^[q]$ by

$$\chi(v) = \{c \in [q] : N_2^c(v) \in \mathcal{U}\}.$$

Partition $[q]$ into sets $A_1, \ldots, A_{k-1}$ of size r. We claim that there is some $i_0 \in [k-1]$ such that $\chi(v) \cap A_{i_0} \neq \emptyset$ for every $v \in \mathbb{N}$. Assuming the contrary, there are vertices $v_1, \ldots, v_{k-1} \in \mathbb{N}$ for which $\chi(v_{\mathrm{i}}) \cap A_{\mathrm{i}} = \emptyset$ for every $\mathrm{i} \in [k-1]$. Let N_{i} be the set of vertices $u \in \mathbb{N} \setminus \{v_1, \ldots, v_{k-1}\}$ for which $\varphi(v_1, \ldots, v_{k-1}, u) \in A_{\mathrm{i}}$. By assumption we have $N_{\mathrm{i}} \notin \mathcal{U}$ for every $\mathrm{i} \in [k-1]$ and consequently $\bigcup_{\mathrm{i}\in[k-1]} N_{\mathrm{i}} \notin \mathcal{U}$. This is a contradiction since $\bigcup_{\mathrm{i}\in[k-1]} N_{\mathrm{i}} = \mathbb{N} \setminus \{v_1, \ldots, v_{k-1}\}$ is co-finite.

We may assume that $A_{\mathrm{i}_0} = [r]$ and delete all other colours. Partition $\mathbb{N}$ into sets $B_1, \ldots, B_r$ such that $\mathrm{i} \in \chi(v)$ for every $v \in B_{\mathrm{i}}$. If B_{i} is finite for some $\mathrm{i} \in [r]$, we write $B_{\mathrm{i}} = \{v_1^{\mathrm{i}}, \ldots, v_{k_{\mathrm{i}}}^{\mathrm{i}}\}$. On the other hand, for every $\mathrm{i} \in [r]$ such that B_{i} is infinite, we write $B_{\mathrm{i}} = \{v_n^{\mathrm{i}} : n \in \mathbb{N}\} \cup \{w_n^{\mathrm{i}} : n \in \mathbb{N}\}$, where $v_1^{\mathrm{i}} = w_1^{\mathrm{i}}$ and $\{v_n^{\mathrm{i}} : n \in \mathbb{N}\} \cap \{w_n^{\mathrm{i}} : n \in \mathbb{N}\} = \{v_1^{\mathrm{i}}\}$. For $\mathrm{i} \in [r]$ and $u, v \in \mathbb{N}$, let $\mathcal{P}^{\mathrm{i}}(u, v)$ be the set of Berge paths in colour i of length[2] at most three with endpoints u, v.

Our purpose is to build hypergraphs inductively in such a way that at every step of the process we have that these hypergraphs are either Berge paths or Berge cycles. In order to find the right edges (and vertices) to add at each of the steps, we will use the following claim.

Claim 1.3.1. *If $u, v \in B_{\mathrm{i}}$ then $\mathcal{P}^{\mathrm{i}}(u, v)$ is infinite.*

Indeed, since $N_2^{\mathrm{i}}(u) \cap N_2^{\mathrm{i}}(v) \in \mathcal{U}$ and for every $w \in N_2^{\mathrm{i}}(u) \cap N_2^{\mathrm{i}}(v)$ either $\{u, v, w\} \subseteq \mathrm{e}$ for some edge $\mathrm{e} \in E(\mathcal{K}_{\mathbb{N}}^{(k)})$ with $\varphi(\mathrm{e}) = \mathrm{i}$, or there exist $\mathrm{e}_1, \mathrm{e}_2 \in E(\mathcal{K}_{\mathbb{N}}^{(k)})$ such that $u, w \in \mathrm{e}_1$, $v, w \in \mathrm{e}_2$ and $\varphi(\mathrm{e}_1) = \varphi(\mathrm{e}_2) = \mathrm{i}$. Therefore Claim 1.3.1 holds.

We start the inductive process as follows. For every $\mathrm{i} \in [r]$, let P_0^{i} be the Berge path $(\{v_1^{\mathrm{i}}\}, \emptyset)$. If $|B_{\mathrm{i}}| \leq 1$ then we set $C_{\mathrm{i}} = (B_{\mathrm{i}}, \emptyset)$ as the monochromatic Berge cycle in colour i, so we assume that $|B_{\mathrm{i}}| \geq 2$ for all $\mathrm{i} \in [r]$. In what follows we consider as induction hypothesis that P_{j-1}^{i} is a Berge path for every $j \in \mathbb{N}$ and $\mathrm{i} \in [r]$.

At each step $j \in \mathbb{N}$ we will choose $\mathrm{i} \in [r]$ such that C_{i} is not already defined and P_{j-1}^{i} has minimum length.

If all the vertices of B_{i} are in the core of the Berge path P_{j-1}^{i} then we take the endpoints u, v of P_{j-1}^{i} and choose (by Claim 1.3.1) $Q \in \mathcal{P}^{\mathrm{i}}(u, v)$ such that the core of Q has empty intersection with the core of $P_{j-1}^{\mathrm{i}'}$ for all $\mathrm{i}' \neq \mathrm{i}$, and intersects the core of P_{j-1}^{i} exactly in $\{u, v\}$. Now we can define $C_{\mathrm{i}} = P_{j-1}^{\mathrm{i}} \cup Q$ as the desired Berge cycle in colour i.

If the core of P_{j-1}^{i} does not cover the vertices of B_{i} then we choose an endpoint $v^{\mathrm{i}} \in B_{\mathrm{i}}$ of P_{j-1}^{i} and the smallest positive integer z for which $v_z^{\mathrm{i}} \in B_{\mathrm{i}}$ (or $w_z^{\mathrm{i}} \in B_{\mathrm{i}}$, if B_{i} is infinite) is not in the core P_{j-1}^{i}. Then, by Claim 1.3.1 we can choose a Berge path $Q \in \mathcal{P}^{\mathrm{i}}(v^{\mathrm{i}}, v_z^{\mathrm{i}})$ such that the core of Q has empty intersection with the core of $P_{j-1}^{\mathrm{i}'}$ for all $\mathrm{i}' \neq \mathrm{i}$, and intersects the core of P_{j-1}^{i} only in u. This is possible since the Berge paths $P_{j-1}^{\mathrm{i}'}$ are finite for all $\mathrm{i}' \in [r]$. Now we set $P_j^{\mathrm{i}} = P_{j-1} \cup Q$, $P_j^{\mathrm{i}'} = P_{j-1}^{\mathrm{i}'}$ for all $\mathrm{i}' \neq \mathrm{i}$ and proceed with step $j + 1$.

Observe that if B_{i} is finite then the monochromatic Berge cycle in colour i will be defined at some step $j \in \mathbb{N}$. On the other hand, for all the infinite sets B_{i} we can define C_{i} as the two-way infinite Berge path $\bigcup_{j\in\mathbb{N}} P_j^{\mathrm{i}}$. This gives us the desired Berge cycle core-partition of $\mathcal{K}_{\mathbb{N}}^{(k)}$. □

[2]The length of a Berge path is the size of its core.

Chapter 2

Partitioning 2-edge-coloured complete k-uniform hypergraphs into monochromatic ℓ-cycles

Abstract

We show that for all $\ell, k, n \in \mathbb{N}$ with $\ell \leq k/2$ and $(k-\ell)$ dividing n the following hypergraph-variant of Lehel's conjecture is true. Every 2-edge-colouring of the k-uniform complete hypergraph on n vertices has at most two vertex-disjoint monochromatic ℓ-cycles in different colours covering all but at most $4(k-\ell)$ vertices. If $\ell \leq k/3$, then at most two vertex-disjoint ℓ-cycles cover all but at most $2(k-\ell)$ vertices. Furthermore, we can cover all vertices with at most 4 (3 if $\ell \leq k/3$) vertex-disjoint monochromatic ℓ-cycles. This is joint work with Maya Stein.

2.1 Introduction

This chapter focuses on monochromatic ℓ-cycle partitions in 2-edge-colourings of complete uniform hypergraphs. Recall that ℓ-cycles are k-uniform hypergraphs with at least three edges, a cyclic ordering of their vertices and a cyclic ordering of their edges such that every edge contains k consecutive vertices, consecutive edges intersect in exactly ℓ vertices, and non-consecutive edges have empty intersection. We also consider two edges intersecting in 2ℓ vertices and vertex sets of size $k-\ell$ as degenerate ℓ-cycles of *length* two and one, respectively. In general, the *length* of an ℓ-cycle is the size of its edge set.

It follows from work of Gyárfás and Sárközy [GS14] that the number of monochromatic loose cycles needed to partition any 2-edge-coloured $\mathcal{K}_n^{(k)}$ is bounded by a function in k. [1] The same authors conjectured [Gyá16, GS14] that any 2-edge-coloured $\mathcal{K}_n^{(k)}$ has two disjoint monochromatic loose paths (a loose path is obtained from a loose cycle by deleting one edge), together covering all but at most $k-2$ vertices, and show this is best possible. This conjecture has recently been confirmed by Lu, Wang and Zhang [LWZ17].

Here we show that for arbitrary $n, k \in \mathbb{N}$, and $\ell \leq k/2$, all but a constant number of vertices of every 2-edge-colouring of $\mathcal{K}_n^{(k)}$ can be covered by two disjoint monochromatic ℓ-cycles.

[1] The actual result in [GS14] is on cycle partitioning in r-coloured complete hypergraphs (for arbitrary $r \geq 2$).

Theorem 2.1.1. *Let $\ell, k, n \in \mathbb{N}$ such that $0 < \ell \leq k/2$ and $k - \ell$ divides n. Let any 2-edge-colouring of $\mathcal{K}_n^{(k)}$ be given.*

(a) There are two vertex-disjoint monochromatic ℓ-cycles in different colours together covering all but at most $4(k-\ell)$ vertices.

(b) If $\ell \leq k/3$, the two ℓ-cycles cover all but at most $2(k-\ell)$ vertices.

Our proof does not use Bessy and Thomassé's theorem, nor does it rely on hypergraph regularity.

We include in our results the condition that $k-\ell$ divides the order of the involved hypergraphs. However, it is clear that by dropping this condition in Theorem 2.1.1 we can partition all but at most $5(k-\ell)-1$ with two monochromatic ℓ-cycles in different colours. We suspect that a partition of all vertices into two cycles should always be possible. (It is not difficult to construct colourings which require at least two disjoint ℓ-cycles for covering all the vertices, so this would be best possible.)

Conjecture 2.1.2. *If $\ell, k, n \in \mathbb{N}$ with $n \equiv 0 \pmod{k-\ell}$, then every 2-edge-colouring of $\mathcal{K}_n^{(k)}$ contains two vertex-disjoint monochromatic ℓ-cycles in different colours covering all vertices.*

An easy argument shows that for $\ell = k/2$ the conjecture is true. In order to see this, take any partition $\mathcal{P}$ of the vertices of $\mathcal{K}_n^{(k)}$ into sets $S_i, i \in [2n/k]$, of size $k/2$. Consider an auxiliary 2-edge-colouring of the complete graph on $\mathcal{P}$, giving $\{S_i, S_j\}$ the colour of $S_i \cup S_j$ in $\mathcal{K}_n^{(k)}$. Bessy and Thomassé's theorem [BT10] yields two graph cycles, which correspond to two disjoint monochromatic ℓ-cycles in different colours in $\mathcal{K}_n^{(k)}$.

Also, we can obtain the following corollary from Theorem 2.1.1.

Corollary 2.1.3. *Let $\ell, k, n \in \mathbb{N}$ such that $0 < \ell \leq k/2$ and $k - \ell$ divides n. Then for any 2-edge-colouring of $\mathcal{K}_n^{(k)}$, one can cover all the vertices of $\mathcal{K}_n^{(k)}$ with four vertex-disjoint monochromatic ℓ-cycles, and if $\ell \leq k/3$, it can be done with three cycles instead of four.*

This follows directly from our main theorem together with the observation that the Ramsey number[2] of the k-uniform ℓ-cycle of length two is $2(k-\ell)$. This can be seen by observing that any 2-edge-colouring of $\mathcal{K} := \mathcal{K}_{2(k-\ell)}^{(k)}$ naturally defines a 2-edge-colouring of $\mathcal{K}^* := \mathcal{K}_{2(k-\ell)}^{(k-2\ell)}$ by giving any edge e^* in $\mathcal{K}^*$ the colour of $V(\mathcal{K}) \setminus e^*$ in $\mathcal{K}$. Then a monochromatic matching of size two in $\mathcal{K}^*$ corresponds to a monochromatic ℓ-cycle of length two in $\mathcal{K}$. Now, results of Alon, Frankl and Lovász [AFL86] imply that the Ramsey number of a 2-edge matching of uniformity r is at most $2r+1$, which, since $2(k-2\ell)+1 < 2(k-\ell)$, is enough for our purposes.

2.2 Partition into a path and a cycle

We will identify a hypergraph $\mathcal{H}$ with its edge set, so when we write $e \in \mathcal{H}$ it refers to the edge e of $\mathcal{H}$. Let us go through some necessary notation.

For an ℓ-path or ℓ-cycle $\mathcal{X}$, we order the k vertices of each edge in such a way that the last ℓ vertices of an edge e_i are the first ℓ vertices of the edge e_{i+1}. For an edge $e = \{v_1, \ldots, v_k\}$, we

[2] The two colour Ramsey number of a k-uniform hypergraph $\mathcal{H}$ is the least integer $R(\mathcal{H})$ for which every blue-red colouring of $\mathcal{K}_{R(\mathcal{H})}^{(k)}$ contains a monochromatic copy of $\mathcal{H}$.

write $V_I(\mathrm{e})$ to denote the vertex set $\{v_\mathrm{i} \in \mathrm{e} : \mathrm{i} \in I\}$, let e^- denote the vertex set $V_{[\ell]}(\mathrm{e})$, let e^+ denote the vertex set $V_{[k]\setminus[k-\ell]}(\mathrm{e})$, and use $\mathring{\mathrm{e}}$ for the set $\mathrm{e} \setminus (\mathrm{e}^- \cup \mathrm{e}^+)$.

An ℓ-path $\mathcal{P}$ of length m is a *blue-red ℓ-path* if there is $m_0 \in [m]$, called *turning point*, such that the ℓ-paths $\{\mathrm{e_i} : \mathrm{i} \in [m_0]\}$ and $\{\mathrm{e_i} : \mathrm{i} \in [m] \setminus [m_0]\}$ are monochromatic and have different colours.

One of the first results in the field of monochromatic partitions was Gerenscér and Gyárfás' observation [GG67] that every two-edge-coloured complete graph has a spanning blue-red path. We extend this observation to ℓ-paths in hypergraphs in the following lemma.

Lemma 2.2.1. *Let $\ell, k, n \in \mathbb{N}$ such that $0 < \ell \le k/2$ and $k-\ell$ divides n. Then every 2-edge-colouring of $\mathcal{K}_n^{(k)}$ contains a blue-red ℓ-path $\mathcal{P}$ with $|V(\mathcal{P})| = n - k + 2\ell$.*

Proof. Take a longest blue-red ℓ-path $\mathcal{P}$ in $\mathcal{K}_n^{(k)}$, with edges $\mathrm{e_i}$ for $\mathrm{i} \in [m]$ and turning point m_0. Assume that all $\mathrm{e_i}$ with $\mathrm{i} \in [m_0]$ are blue and all later edges on $\mathcal{P}$ are red.

If the set Z of all vertices not covered by $\mathcal{P}$ has size $k - 2\ell$, we are done. So assume otherwise; then Z contains at least $2k - 3\ell$ elements. Fix three disjoint sets $Z_0, Z_1, Z_2 \subset Z$ with $|Z_0| = \ell$ and $|Z_1| = |Z_2| = k - 2\ell$. Since $\mathcal{P}$ is maximal, we know that $\mathrm{e}_R = \mathrm{e}_1^- \cup Z_1 \cup Z_0$ is red, $\mathrm{e}_B = \mathrm{e}_m^+ \cup Z_1 \cup Z_0$ is blue, and $m_0 \neq m$.

By colour symmetry, we can assume the edge $\mathrm{e} = \mathrm{e}_{m_0}^+ \cup Z_2 \cup Z_0$ is red. Then $(\mathcal{P} \setminus \{\mathrm{e}_{m_0}\}) \cup \{\mathrm{e}, \mathrm{e}_R\}$ is a blue-red ℓ-path longer than $\mathcal{P}$, which contradicts the maximality of $\mathcal{P}$. □

Observe that the blue-red ℓ-path $\mathcal{P}$ given by Lemma 2.2.1 is as large as possible, since $|V(\mathcal{P})| \equiv \ell \pmod{k - \ell}$ and $k - \ell$ divides n.

Now we can show that there are a monochromatic ℓ-path and a monochromatic ℓ-cycle that together cover almost all the vertices.

Lemma 2.2.2. *Let $\ell, k, n \in \mathbb{N}$ such that $0 < \ell \le k/2$ and $n = n_0(k-\ell)$ for some integer $n_0 \ge 3$. Then every 2-edge-colouring of $\mathcal{K}_n^{(k)}$ contains an ℓ-cycle C and an ℓ-path $\mathcal{P}$ with the following properties:*

1. *C and $\mathcal{P}$ are vertex-disjoint;*
2. *C and $\mathcal{P}$ are each monochromatic but use distinct colours;*
3. *C has at least two edges;*
4. *if $\mathcal{P} \neq \emptyset$, then $|V(C)| + |V(\mathcal{P})| \in \{n - k + 2\ell, n - 2k + 3\ell\}$; and*
5. *if $\mathcal{P} = \emptyset$, then $|V(C)| = n - k + \ell$.*

Proof. By Lemma 2.2.1, there is a blue-red ℓ-path with edges $\mathrm{e_i}$ for $\mathrm{i} \in [m]$ and turning point m_0 that covers all but a set Z of $k - 2\ell$ vertices of $\mathcal{K}_n^{(k)}$. Among such ℓ-paths, choose $\mathcal{P}_{\max}$ such that $\max\{m_0, m - m_0\}$ is as large as possible (i.e. $\mathcal{P}_{\max}$ maximises the length of a monochromatic sub-ℓ-path). By symmetry, we can assume that $\max\{m_0, m - m_0\} = m_0$, that $\mathcal{P}_B = \{\mathrm{e_i} : \mathrm{i} \in [m_0]\}$ is blue and that $\mathcal{P}_R = \{\mathrm{e_i} : \mathrm{i} \in [m] \setminus [m_0]\}$ is red. Since $n \ge 3(k - \ell)$, we know that $m \ge 2$. If $m_0 < m$ and the edge

$$\mathrm{e} := \mathrm{e}_{m_0}^+ \cup Z \cup \mathrm{e}_m^+$$

is blue, then $\left(\mathcal{P}_{\max} \setminus \mathrm{e}_{m_0+1}\right) \cup \{\mathrm{e}\}$ is a blue-red ℓ-path contradicting the choice of $\mathcal{P}_{\max}$. If $m_0 < m$ and the edge e is red, then the ℓ-cycle $C_R = \mathcal{P}_R \cup \mathrm{e}$ together with the ℓ-path $\mathcal{P}_B \setminus \{\mathrm{e}_{m_0}\}$ are as desired.

So we can assume that $m_0 = m$, that is, $\mathcal{P}_{\max}$ is all blue. If also one of the two edges $e_1^+ \cup \mathring{e}_1 \cup e_m^+$, $e_1^+ \cup Z \cup e_m^+$ is blue, we can close $\mathcal{P}_{\max}$, forming an ℓ-cycle that covers all but $e_1^- \cup Z$, or all but $e_1^- \cup \mathring{e}_1$, respectively, which is as desired. So we can suppose both edges $e_1^+ \cup \mathring{e}_1 \cup e_m^+$, $e_1^+ \cup Z \cup e_m^+$ are red. They form an ℓ-cycle with two edges, which together with $\mathcal{P}_{\max} \setminus \{e_1, e_2, e_m\}$ covers all but $e_1^- \cup \mathring{e}_2 \cup \mathring{e}_m$ (note that possibly $m = 2$, in which case e_2 coincides with e_m). So, we found an ℓ-cycle and an ℓ-path which are as required (in particular, either they cover $n - 2k + 3\ell$ vertices, or the ℓ-path is empty and the ℓ-cycle covers $n - k + \ell$ vertices). □

2.3 Proof of Theorem 2.1.1 (a)

This section is devoted to the proof of Theorem 2.1.1 (a).

Consider a monochromatic ℓ-cycle C_B with at least two edges and a disjoint monochromatic ℓ-path $\mathcal{P}_R$ as given by Lemma 2.2.2. Note that if $\mathcal{P}_R$ has at most two edges, we are done, so assume otherwise. By deleting at most two edges from $\mathcal{P}_R$, if necessary, we can assume that

$$|V(C_B \cup \mathcal{P}_R)| = n - 3k + 4\ell.$$

Among all such choices for C_B and $\mathcal{P}_R$ (including those where $\mathcal{P}_R$ is empty), assume we chose C_B and $\mathcal{P}_R$ such that

$$C_B \text{ has as many edges as possible.} \tag{2.3.1}$$

By symmetry, we may assume that C_B is blue and $\mathcal{P}_R$ is red. Say C_B has edges e_i, $i \in [m_c]$ (and thus length $m_c \geq 2$), while $\mathcal{P}_R$ has edges f_j, $j \in [m_p]$ (and thus length $m_p \geq 0$).

Assuming that Theorem 2.1.1 (a) does not hold, we will reach a contradiction by analysing the connections from the first/last edge of $\mathcal{P}_R$ to C_B. If these cannot be used to close up $\mathcal{P}_R$ to an ℓ-cycle, we find a red ℓ-cycle on the same vertices as C_B. In a last step, we will use this new red ℓ-cycle together with $\mathcal{P}_R$ to form one large red ℓ-cycle.

We start by making a couple of easy observations. First of all, note that

$$m_p \geq 2, \tag{2.3.2}$$

as otherwise C_B covers all but at most $4k - 4\ell$ vertices.

Let Z_1, Z_2, Z_3 be mutually disjoint subsets of vertices not covered by $C_B \cup \mathcal{P}_R$ such that $|Z_1| = |Z_2| = |Z_3| = k - 2\ell$. Consider the edges

$$w^t := f_1^- \cup Z_t \cup f_{m_p}^+,$$

for $t = 1, 2, 3$. If any of the edges w^t is red, then C_B together with $\mathcal{P}_R \cup \{w^t\}$ are ℓ-cycles as in the theorem, covering all but $2k - 2\ell$ vertices. So,

$$w^t := f_1^- \cup Z_t \cup f_{m_p}^+ \text{ is blue, for } t = 1, 2, 3. \tag{2.3.3}$$

Consider the edges

$$v_i^t := f_{m_p}^+ \cup Z_t \cup e_i^-$$

for $i \in [m_c]$ and $t, t' \in \{1, 2, 3\}$. If for some triple i, t, t' with $t \neq t'$, both edges v_i^t, $v_{i+1}^{t'}$ are blue, then the ℓ-cycle $\{v_i^t, v_{i+1}^{t'}\} \cup (C_B \setminus \{e_i\})$ together with the ℓ-path $\mathcal{P}_R \setminus \{f_{m_p}\}$ contradicts (2.3.1). So, for each $i \in [m_c]$, and each pair and $t, t' \in \{1, 2, 3\}$ with $t \neq t'$,

$$\text{one of the edges } v_i^t, v_{i+1}^{t'} \text{ is red.} \tag{2.3.4}$$

Similary, for each $\mathrm{i} \in [m_c]$, and each pair and $t, t' \in \{1, 2, 3\}$ with $t \neq t'$, setting

$$u_{\mathrm{i}}^t := f_1^- \cup Z_t \cup \mathrm{e}_{\mathrm{i}}^-,$$

we observe that

$$\text{one of the edges } u_{\mathrm{i}}^t, u_{\mathrm{i}+1}^{t'} \text{ is red.} \tag{2.3.5}$$

We now establish that our ℓ-cycle is a bit longer than our ℓ-path.

Claim 2.3.1. *It holds that $m_c \geq m_p + 2$.*

Proof. Suppose to the contrary that $m_c < m_p + 2$. By (2.3.5), we can assume the edge u_1^1 is red. If the edge v_1^2 is red, too, then $\mathcal{P}_R \cup \{u_1^1, v_1^2\}$ and $C_B \setminus \{\mathrm{e}_1, \mathrm{e}_{m_c}\}$ contradict the choice of C_B and $\mathcal{P}_R$ fulfilling (2.3.1). So the edge v_1^2 is blue, and thus by (2.3.4), the edge v_2^3 is red.

Now, if the edge u_2^1 is red, then $\mathcal{P}_R \cup \{v_2^3, u_2^1\}$ is a red ℓ-cycle of length greater than m_c, which together with the path $C_B \setminus \{\mathrm{e}_1, \mathrm{e}_2\}$ contradicts (2.3.1). Therefore, u_2^1 is blue. But now, since by (2.3.3), the edge w^3 is blue, we found a blue ℓ-cycle, namely $\{v_1^2, w^3, u_2^1\} \cup (C_B \setminus \{\mathrm{e}_1\})$, which together with the red ℓ-path $\mathcal{P}_R \setminus \{f_1, f_{m_p}\}$ contradicts (2.3.1). □

Note that Claim 2.3.1 together with (2.3.2) implies that

$$m_c \geq 4. \tag{2.3.6}$$

Let us now consider the edges

$$g_{\mathrm{i}} := (\mathrm{e}_{\mathrm{i}} \setminus \mathrm{e}_{\mathrm{i}}^+) \cup \mathrm{e}_{\mathrm{i}+2}^+ \quad \text{and} \quad h_{\mathrm{i}} := (\mathrm{e}_{\mathrm{i}} \setminus \mathrm{e}_{\mathrm{i}}^+) \cup \mathrm{e}_{\mathrm{i}+3}^+,$$

for $\mathrm{i} \in [m_c]$ (considering all indices modulo m_c). The advantage of these edges is that on the one hand, each of these edges, if blue, provides a shortcut on C_B (and the vertices left out of C_B can be used for closing up $\mathcal{P}_R$). On the other hand, if all these edges are red, then they form new red ℓ-cycles on the vertex set of C_B.

Let us first show why any of the edges g_{i}, h_{i} would be useful in blue.

Claim 2.3.2. *The edges g_{i} are red for all $\mathrm{i} \in [m_c]$, and if $m_c > 4$, then the edges h_{i} are red for all $\mathrm{i} \in [m_c]$.*

Proof. Suppose that one of these edges g_{i} or h_{i} is blue (the latter only in the case that $m_c > 4$). Then there is a blue cycle C_B' obtained from C_B by replacing the edges $\mathrm{e}_{\mathrm{i}}, \mathrm{e}_{\mathrm{i}+1}, \mathrm{e}_{\mathrm{i}+2}$ with the edge g_{i}, or by replacing the edges $\mathrm{e}_1, \mathrm{e}_{\mathrm{i}+1}, \mathrm{e}_{\mathrm{i}+2}, \mathrm{e}_{\mathrm{i}+3}$ with the edge h_{i}.

Consider the edges $u_{\mathrm{i}+1}^1$ and $v_{\mathrm{i}+1}^2$. If both of these edges are red then the theorem holds, since C_B' together with $\mathcal{P}_R \cup \{u_{\mathrm{i}+1}^1, v_{\mathrm{i}+1}^2\}$ either covers all but $3k - 3\ell$ vertices (if g_{i} is blue); or cover all but $4k - 4\ell$ vertices (if h_{i} is blue). So by symmetry, we can assume that $u_{\mathrm{i}+1}^1$ is blue. Similarly, if the edges $u_{\mathrm{i}+2}^2$ and $v_{\mathrm{i}+2}^3$ are both red then the theorem holds, so at least one of them is blue.

Since $u_{\mathrm{i}+1}^1$ is blue, (2.3.4) implies that $u_{\mathrm{i}+2}^2$ is red, and thus $v_{\mathrm{i}+2}^3$ is blue. Recall that by (2.3.3), the edge w^2 is blue, too, and so, the ℓ-cycle $\{u_{\mathrm{i}+1}^1, w^2, v_{\mathrm{i}+2}^3\} \cup (C_B \setminus \{\mathrm{e}_{\mathrm{i}+1}\})$ together with the ℓ-path $\mathcal{P}_R \setminus \{f_1, f_{m_p}\}$ contradicts (2.3.1). □

Finally, consider the edge sets

$$R_j := \{g_{\mathrm{i}} : \mathrm{i} \equiv j \pmod 3\},$$

for $j = 0, 1, 2$. Notice that R_0, R_1, R_2 are three ℓ-cycles of length $\frac{m_c}{3}$ if $m_c \equiv 0 \pmod 3$, and together form one ℓ-cycle otherwise.

The remainder of the proof is split into several cases, depending on the value of m_c, and which of the edges u_i^t, v_i^t are red. Note that by (2.3.5) (and after possibly renaming the edges on C_B, or the sets Z_t), we may assume that u_1^1 is red. Moreover, by (2.3.4), at least one of the edges v_4^2, v_5^3 is red (if $m_c = 4$, we take indices modulo 4, meaning that one of v_4^2, v_1^3 is red).

In all the cases considered below we use that $m_c \geq 4$ by (2.3.6).

Case 1. $m_c \not\equiv 0 \pmod 3$ and v_4^2 is red.

In this case, consider the red ℓ-cycle formed by $R_0 \cup R_1 \cup R_2$. We can substitute the edge g_1 from this ℓ-cycle with the path $\{u_1^1\} \cup \mathcal{P}_R \cup \{v_4^2\}$ to obtain one red ℓ-cycle which covers all but $2(k - \ell)$ vertices.

Case 2. $m_c \equiv 0 \pmod 3$ and v_4^2 is red.

Consider the auxiliary red ℓ-cycle formed by

$$\{h_4\} \cup (R_2 \setminus \{g_2, g_5\}) \cup \{h_2\} \cup (R_0 \setminus \{g_3\}) \\ \cup \{h_3\} \cup (R_1 \setminus \{g_4\}).$$

Similar as in the previous case, we can substitute the edge g_1 from the auxiliary ℓ-cycle with the path $\{u_1^1\} \cup \mathcal{P}_R \cup \{v_4^2\}$ to obtain one red ℓ-cycle which covers all but $3(k - \ell)$ vertices. (Note that this works fine even if $m_c = 6$.)

Case 3. $m_c \not\equiv 1 \pmod 3$ and v_5^3 is red.

Consider the auxiliary red ℓ-cycle formed by

$$\{h_1\} \cup (R_2 \setminus \{g_2\}) \cup \{h_2\} \cup (R_0 \setminus \{g_3\}) \\ \cup \{h_3\} \cup (R_1 \setminus \{g_1, g_4\})$$

and substitute its edge h_1 with $\{u_1^1\} \cup \mathcal{P}_R \cup \{v_5^3\}$ to obtain one red ℓ-cycle which covers all but $3(k - \ell)$ vertices. (Note that this works even if $m_c = 5$.)

Case 4. $m_c \equiv 1 \pmod 3$, $m_c \neq 4$, and v_5^3 is red.

If m_c is odd, then we can use the cycle spanned by all edges h_i except h_1, and the path $\mathcal{P}_R$, together with edges u_1^1 and v_5^3. This ℓ-cycle covers all but $2(k - \ell)$ vertices. Otherwise, since $m_c > 4$ and $m_c \equiv 1 \pmod 3$, we know that $m_c \geq 10$. So we can use a similar approach as above, using five edges h_i instead of two. More precisely, the red ℓ-path formed by

$$\{h_5\} \cup (R_0 \setminus \{g_3, g_6\}) \cup \{h_2, h_6\} \cup (R_1 \setminus \{g_1, g_4, g_7\}) \\ \cup \{h_3, h_7\} \cup (R_2 \setminus \{g_2, g_5, g_8\})$$

and $\{u_1^1\} \cup \mathcal{P}_R \cup \{v_5^3\}$ covers all but $4(k - \ell)$ vertices.

Case 5. $m_c = 4$, and v_1^3 is red while v_4^2 is blue.

Then by (2.3.4), the edge v_3^1 is red, and we can close $\mathcal{P}_R$ using the edges v_1^3, v_3^1, g_1, g_4. We covered all but $3(k - \ell)$ vertices.

This finishes the proof of the theorem.

2.4 Proof of Theorem 2.1.1 (b)

In this section we give a proof of Theorem 2.1.1 (b), the first part of which follows very much the lines of the proof of Theorem 2.1.1 (a), while the last part is a bit different. In order to avoid repetition, we only sketch the first part, but give all details for the last part. We remark that much of the work can be avoided if we are only interested in two ℓ-cycles covering all but $3(k-\ell)$ vertices instead of the output of Theorem 2.1.1 (b).

For the first part of the proof, the main difference is that now, we use Lemma 2.2.2 to find a monochromatic ℓ-cycle C_B of length $m_c \geq 2$ with edges $\mathrm{e_i}$, $\mathrm{i} \in [m_c]$, and a disjoint monochromatic ℓ-path $\mathcal{P}_R$ of length $m_p \geq 0$ with edges f_j, $j \in [m_p]$ such that

$$|V(C_B \cup \mathcal{P}_R)| = n - 2k + 3\ell, \tag{2.4.1}$$

choosing C_B maximal under the these conditions. That is, now we leave only $2(k-\ell)+\ell$ vertices uncovered in the beginning. Instead of defining Z_1, Z_2, Z_3, we only define Z_1, Z_2 as two disjoint sets of vertices not covered by $C_B \cup \mathcal{P}_R$ with $|Z_1| = |Z_2| = k - 2\ell$. The idea is that now, consecutive edges on C_B only intersect in at most $k/3$ vertices, which means that the interior of such an edge can be used in the same way as one of the sets Z_t. With this we can overcome the difficulty due to having only two sets Z_t to operate with.

Again we easily show that $m_p \geq 2$ (using (2.4.1)), and that edges w^t defined for $t = 1, 2$ and $\mathrm{i} \in [m_c]$, have the same properties as in the proof of Theorem 2.1.1 (a). Now we define v_i^t and u_i^t as in that proof for $t = 1, 2$, and set

$$v_\mathrm{i}^3 := f_{m_p}^+ \cup (\mathrm{e}^\mathrm{i} \setminus \mathrm{e}_\mathrm{i}^+) \text{ and } u_\mathrm{i}^3 := f_1^- \cup (\mathrm{e}^\mathrm{i} \setminus \mathrm{e}_\mathrm{i}^+).$$

It is easy to see that for each $\mathrm{i} \in [m_c]$, and each pair and $t, t' \in \{1, 2, 3\}$ with $t \neq t'$ and $t' \neq 3$, at least one of the edges v_i^t, $v_{\mathrm{i}+1}^{t'}$ is red and at least one of the edges u_i^t, $u_{\mathrm{i}+1}^{t'}$ is red.

For showing that $m_c \geq m_p + 2$ (and thus $m_c \geq 4$), observe that in the proof of Claim 2.3.1 in the proof of Theorem 2.1.1 (a), there is only one time where we need that all three sets Z_t are present, and that is at the very end, when we form the blue ℓ-cycle $\{v_1^2, w^3, u_2^1\} \cup (C_B \setminus \{\mathrm{e}_1\})$. Instead, we can use the ℓ-cycle $\{v_1^3, w^2, u_2^1\} \cup (C_B \setminus \{\mathrm{e}_1\})$.

For the rest of the proof one might define the edges g_i, h_i as in the proof of Theorem 2.1.1 (a), show they are red, and then go through Cases 1-5.[3] However, for establishing that edges h_i are red, we would have to content ourselves with the outcome of two ℓ-cycles covering all but $3(k-\ell)$ vertices. We can do a slightly better than that by arguing as follows.

Consider the edges

$$a := f_1^- \cup \mathring{\mathrm{e}}_3 \cup V_{[2\ell]\setminus[\ell]}(\mathrm{e}_2)$$

and

$$a' := V_{[2\ell]\setminus[\ell]}(f_1) \cup \mathring{\mathrm{e}}_4 \cup V_{[k-\ell]\setminus[k-2\ell]}(\mathrm{e}_5)$$

(note that these edges are symmetric with respect to $\mathrm{e}_3 \cap \mathrm{e}_4$, as Figure 2.1 shows).

[3] To see this goes through we remark that first, near the end of the proof of Claim 2.3.2 of that theorem we used to occupy the set Z_3, by employing the edge $v_{\mathrm{i}+2}^3$. With the new definition of this edge, this works here too. Second, when going through Cases 1-5, we cannot use the edges u_1^1, v_4^2, v_5^3 as before. This problem is easily overcome by first finding out which of v_4^2, v_5^1 is red. Say this is v_5^1 (otherwise rename all edges). Now, if the edge u_2^2 is red, then we are in the same situation as in Case 1, 2 or 5 of the proof of Theorem 2.1.1 (a) (with indices augmented by one). Otherwise, the edge u_2^2 is blue, and thus the edge u_1^3 is red, as otherwise we could augment C_B using these two edges, and destroying one edge of $\mathcal{P}_R$. Now we are in a situation that is very similar to Cases 3 and 4 of Theorem 2.1.1 (a). As in these cases, neither of the edges g_1, h_1 was used, we have no problem finding our red ℓ-cycle using u_1^3 instead of u_1^1.

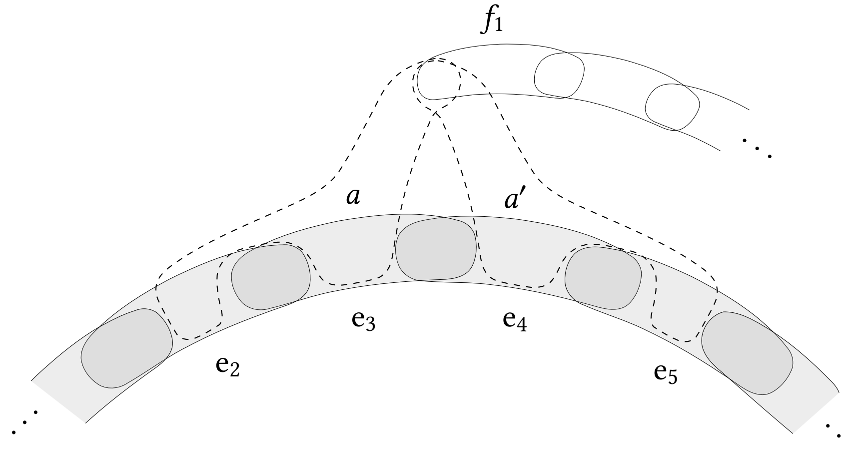


Figure 2.1: Solid gray and solid white edges are blue and red edges, respectively.

If both a and a' are blue then we can replace the edges $e_3, e_4 \in C_B$ with the edges a, a', w^1 and the blue[4] edge $V_{[2\ell]\setminus[\ell](f_1)} \cup Z_2 \cup f^+_{m_p}$ to obtain a blue cycle which together with the red path $\mathcal{P}_R \setminus \{f_1, f_m\}$ contradicts the maximality of C_B. Therefore, we can assume that one of these edges is red, without loss of generality say

$$a \text{ is red.} \tag{2.4.2}$$

Next, consider the edges

$$q_i := V_{[2\ell]\setminus[\ell]}(e_i) \cup \mathring{e}_{i+1} \cup e^+_{i+3},$$

for $i \in [m_c]$. It is easy to see that the edges q_i form an ℓ-cycle, which we will call C_R.

Claim 2.4.1. *We may assume that q_i is red, for all $i \in [m_c]$.*

Proof. Suppose one of these edges, say q_1, is blue. Obtain C'_B from C_B by replacing the edges e_2, e_3, e_4 with the edge q_1.

First assume u^3_3 is blue. Then u^1_4 is red, by our analogue of (2.3.5). Also, v^2_4 is red, as otherwise we can replace e_3 with the edges u^3_3, w^1 and v^2_4, and thus contradicting the maximality of C_B. But now, $\mathcal{P}_R \cup \{u^1_4, v^2_4\}$ is a red ℓ-cycle, which, together with the blue ℓ-cycle C'_B is as desired for the theorem.

So from now on, assume that u^3_3 is red. Then, the edge v^1_3 is blue or we found ℓ-cycles $\mathcal{P}_R \cup \{u^3_3, v^1_3\}$ and C'_B which are as desired for the theorem. Now consider a set $Z'_2 \subseteq Z_2$ of size $k - 3\ell$. By the maximality of C_B and taking into account that v^1_3 is blue, we see that the edge

$$b := f^+_{m_p} \cup Z'_2 \cup V_{[2\ell]\setminus[\ell]}(e_3) \cup e^+_3$$

has to be red (see Figure 2.2).

But then the ℓ-cycles $\mathcal{P}_R \cup \{u^3_3, b\}$ and C'_B give the desired output of the theorem. □

We are now ready to prove Theorem 2.1.1 (b). For this, first assume that v^1_4 is red. Then, by (2.4.2) and by Claim 2.4.1, we know that $(C_R \setminus \{q_1, q_2\}) \cup \mathcal{P}_R \cup \{a, v^1_4\}$ is a red ℓ-cycle, as desired for the theorem.

[4]This edge is blue for the same reason for which w^2 is blue.

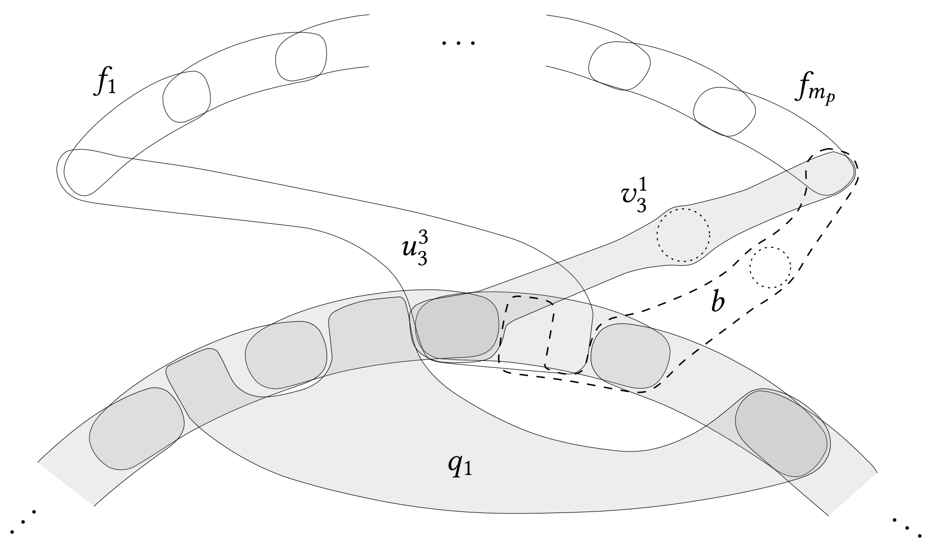


Figure 2.2: Diagram of edges q_1, u_3^3, v_3^1 and b. The dotted circles inside v_3^1 and b are the sets Z_1 and Z_2', respectively.

From now on assume v_4^1 is blue. Then $c := f_{m_p}^+ \cup \mathring{\mathrm{e}}_4 \cup V_{[2\ell]\setminus[\ell]}(\mathrm{e}_5)$ is red, as otherwise the cycle obtained by replacing $\mathrm{e}_4 \in \mathcal{C}_B$ with the edges v_4^1, c yields a contradiction to the maximality of $\mathcal{C}_B$. So, by (2.4.2), and since we chose c so that it meets q_5 in exactly ℓ vertices,

$$\mathcal{C}_R' := \big(\mathcal{C}_R \setminus \{q_2, q_3, q_4\}\big) \cup \mathcal{P}_R \cup \{a, c\}$$

is a red ℓ-cycle covering all vertices, except the $3(k-\ell)$ vertices lying in

$$Z_1 \cup Z_2 \cup W \cup (\mathring{\mathrm{e}}_5 \setminus c) \cup \mathrm{e}_5^+ \cup \mathrm{e}_6^+ \cup \mathrm{e}_7^+,$$

where W is a set of ℓ vertices outside $\mathcal{C}_B \cup \mathcal{P}_R$ disjoint from $Z_1 \cup Z_2$.

Consider $\mathrm{d} := V_{[k-\ell]\setminus[k-2\ell]}(f_{m_p}) \cup Z_2 \cup \mathrm{e}_7^+$. Observe that either the ℓ-cycle

$$\big(\mathcal{C}_R \setminus \{q_2, q_3\}\big) \cup \mathcal{P}_R \cup \{a, \mathrm{d}\}$$

is red, and then it covers all but $2(k-\ell)$ vertices, as desired for the theorem, or the edge d is blue, which we will assume from now on. Then by the maximality of $\mathcal{C}_B$, the edge $\mathrm{d}' := V_{[k-\ell]\setminus[k-2\ell]}(f_{m_p}) \cup Z_1 \cup \mathrm{e}_6^+$ is red. Now consider the edge

$$\mathrm{e} := \mathrm{e}_6^+ \cup Z_2 \cup \mathrm{e}_7^+.$$

If e is blue, consider the blue path e and the red cycle $\mathcal{C}_R'$ to obtain a contradiction to the maximality of $\mathcal{C}_B$ in the choice of $\mathcal{C}_B$ and $\mathcal{P}_R$.

So e is red. Then the ℓ-cycle

$$\big(\mathcal{C}_R \setminus \{q_2, q_3\}\big) \cup \mathcal{P}_R \cup \{a, \mathrm{d}', \mathrm{e}\}$$

is red and covers all but $k-\ell$ vertices, as desired for the theorem.

This concludes the proof of Theorem 2.1.1 (b).

Chapter 3

Almost partitioning 2-edge-coloured complete 3-uniform hypergraphs into two monochromatic tight-cycles

Abstract

We show that for every $\eta > 0$ there exists an integer n_0 such that every 2-edge-colouring of the 3-uniform complete hypergraph on $n \geq n_0$ vertices contains two disjoint monochromatic tight cycles of distinct colours that together cover all but at most ηn vertices. This is joint work with Hiệp Hàn and Maya Stein.

3.1 Introduction

Tight cycles are the most restrictive notion of hypergraph cycle and, up to now, the only known result in tight cycle partitioning is Theorem 0.2.9, where the host hypergraphs are countably infinite uniform complete hypergraphs. In this chapter we study 2-edge-colourings of 3-uniform complete hypergraphs. Inspired by Lehel's conjecture, we show that two disjoint monochromatic tight cycles suffice to cover almost all the vertices of $\mathcal{K}_n^{(3)}$. We will see in Chapter 4 that there exists an absolute constant c such that $2 + c$ monochromatic tight cycles suffice to partition the vertices of every 2-edge-colouring of $\mathcal{K}_n^{(3)}$ (see Theorem 4.1.3 in Chapter 4).

Our main result in this chapter is an approximate result on monochromatic tight cycle partitioning for 2-edge-colourings of complete k-uniform hypergraphs.

Theorem 3.1.1. *For every $\eta > 0$ there exists n_0 such that if $n \geq n_0$ then every two-coloring of the edges of the complete 3-uniform hypergraph $\mathcal{K}_n^{(3)}$ admits two vertex-disjoint monochromatic tight cycles, of distinct colours, which cover all but at most ηn vertices.*
Moreover, we can choose the parity of the length of each of the cycles.

We might be interested in choosing the parity of the cycles for the following reason. If ℓ is even, then any 3-uniform tight cycle on ℓ edges contains a loose cycle. Hence, we can deduce that an analogue of Theorem 3.1.1 holds for loose cycles.

Corollary 3.1.2. *For every $\eta > 0$ there exists n_0 such that if $n \geq n_0$ then every two-coloring of the edges of the complete 3-uniform hypergraph $\mathcal{K}_n^{(3)}$ admits two vertex-disjoint monochromatic loose cycles, of distinct colours, which cover all but at most ηn vertices.*

As shown in Chapter 2 (see Theorem 2.1.1) in a more general way, the error term ηn in Theorem 3.1.1 can be improved, so every 2-edge-colouring of the edges of a 3-uniform complete hypergraph admits two disjoint monochromatic tight cycles which cover all but at most a constant number c of vertices (for some c independent of n).

The proof of Theorem 3.1.1 is inspired by the work of Haxell et al. [HŁP$^+$06, HŁP$^+$09] and relies on an application of the hypergraph regularity lemma [FR02]. This reduces the problem at hand to that of finding, in any two-colouring of the edges of an almost complete 3-uniform hypergraph, two disjoint *monochromatic connected matchings* which cover almost all vertices.

Here, as usual, a *matching* $\mathcal{M}$ in hypergraph $\mathcal{H}$ is a set of pairwise disjoint edges and $\mathcal{M} \subset \mathcal{H}$ is called *connected* if between every pair $\mathrm{e}, f \in \mathcal{M}$ there is a *pseudo-path* in $\mathcal{H}$ connecting e and f, that is, there is a sequence $(\mathrm{e}_1, \ldots, \mathrm{e}_p)$ of not necessarily distinct edges of $\mathcal{H}$ such that $\mathrm{e} = \mathrm{e}_1, f = \mathrm{e}_p$ and $|\mathrm{e}_\mathrm{i} \cap \mathrm{e}_{\mathrm{i}+1}| = 2$ for each $\mathrm{i} \in [p-1]$. (Note that these pseudo-paths may use vertices outside $V(\mathcal{M})$.) Now, we call a connected matching $\mathcal{M}$ in a 2-coloured hypergraph a *monochromatic connected matching* if all edges in $\mathcal{M}$ and all edges on the connecting paths have the same colour.

So, our main contribution reduces to the following result, which might be of independent interest.

Theorem 3.1.3. *For every $\gamma > 0$ there is t_0 such that the following holds for every 3-uniform hypergraph $\mathcal{H}$ with $t > t_0$ vertices and $(1-\gamma)\binom{t}{3}$ edges. Any two-colouring of the edges of $\mathcal{H}$ admits two disjoint monochromatic connected matchings covering at least $(1 - 290\gamma^{\frac{1}{6}})t$ vertices of $\mathcal{H}$.*

We prove Theorem 3.1.3 in Section 3.2. In Section 3.3, we introduce the regularity lemma for hypergraphs and state an embedding result from [HŁP$^+$09]. The proof of Theorem 3.1.1 will be given in Section 3.4.

3.2 Monochromatic connected matchings

Before giving the proof of Theorem 3.1.3 we introduce some notation and auxiliary results.

Let $\mathcal{H}$ denote a k-uniform hypergraph, that is, a pair $\mathcal{H} = (V, E)$ with finite vertex set $V = V(\mathcal{H})$ and edge set $E = E(\mathcal{H}) \subset \binom{V}{k}$, where $\binom{V}{k}$ denotes the set of all k-element sets of V. Often $\mathcal{H}$ will be identified with its edges, that is, $\mathcal{H} \subset \binom{V}{k}$ and for an edge $\{x_1, \ldots, x_k\} \in \mathcal{H}$ we often omit brackets and write $x_1 \ldots x_k$ only. A k-uniform hypergraph C is called an *ℓ-cycle* if there is a cyclic ordering of the vertices of C such that every edge consists of k consecutive vertices, every vertex is contained in an edge and two consecutive edges (where the ordering of the edges is inherited by the ordering of the vertices) intersect in exactly ℓ vertices. For $\ell = 1$ we call the cycle *loose* whereas the cycle is called *tight* if $\ell = k-1$ (and we do not consider other values of ℓ).

A *tight path* is a tight cycle from which one vertex and all incident edges are deleted. The *length* of a path, a pseudo-path or a cycle is the number of edges it contains. As above, two edges in $\mathcal{H}$ are connected if there is a pseudo-path connecting them. Connectedness is an equivalence relation on the edge set of $\mathcal{H}$ and the equivalence classes are called *connected components*.

All hypergraphs $\mathcal{H}$ considered from now on are 3-uniform. We will need the following result concerning the existence of perfect matchings in 3-uniform hypergraphs with high minimum vertex degree.

Theorem 3.2.1 ([HPS09])**.** *For all $\eta > 0$ there is a $n_0 = n_0(\eta)$ such that for all $n > n_0$, $n \in 3\mathbb{Z}$, the following holds. Suppose $\mathcal{H}$ is a 3-uniform hypergraph on n vertices such that every vertex is contained in at least $\left(\frac{5}{9} + \eta\right)\binom{n}{2}$ edges. Then $\mathcal{H}$ contains a perfect matching.*

Denote by $\partial\mathcal{H}$ the *shadow* of $\mathcal{H}$, that is, the set of all pairs xy for which there exists z such that $xyz \in \mathcal{H}$. For a vertex x in a hypergraph $\mathcal{H}$, let $N_{\mathcal{H}}(x) = \{y : xy \in \partial\mathcal{H}\}$. For two vertices x, y, let $N_{\mathcal{H}}(x, y) = \{z : xyz \in \mathcal{H}\}$. Note that if $y \in N_{\mathcal{H}}(x)$ (equivalently, $x \in N_{\mathcal{H}}(y)$) then $N_{\mathcal{H}}(x, y) \neq \emptyset$. We call all such pairs xy of vertices *active*.

Lemma 3.2.2 ([HŁP+06], Lemma 4.1)**.** *Let $\gamma > 0$ and let $\mathcal{H}$ be a 3-uniform hypergraph on $t_{\mathcal{H}}$ vertices and at least $(1-\gamma)\binom{t_{\mathcal{H}}}{3}$ edges. Then $\mathcal{H}$ contains a subhypergraph $\mathcal{K}$ on $t_{\mathcal{K}} \geq (1-10\gamma^{1/6})t_{\mathcal{H}}$ vertices such that every vertex x of $\mathcal{K}$ is in an active pair of $\mathcal{K}$ and for all active pairs xy we have $|N_{\mathcal{K}}(x, y)| \geq (1-10\gamma^{1/6})t_{\mathcal{K}}$.*

We are now ready to prove Theorem 3.1.3.

Proof of Theorem 3.1.3. For given $\gamma > 0$ let $\delta = 10\gamma^{1/6}$ and apply Theorem 3.2.1 with $\eta = 5/36$ to obtain n_0. We choose $t_0 = \max\left\{\frac{2}{\delta}, \frac{n_0}{27\delta}\right\}$.

Suppose we are given a two-coloured 3-uniform hypergraph $\mathcal{H} = \mathcal{H}_{\text{red}} \cup \mathcal{H}_{\text{blue}}$ on $t_{\mathcal{H}} > t_0$ vertices and $(1-\gamma)\binom{t_{\mathcal{H}}}{3}$ edges. Apply Lemma 3.2.2 to $\mathcal{H}$ with parameter γ to obtain $\mathcal{K}$, $t := t_{\mathcal{K}}$ with the properties stated in the lemma. Observe that at most $\delta t_{\mathcal{H}}$ vertices of $\mathcal{H}$ are not vertices of $\mathcal{K}$. We wish to find two monochromatic connected matchings covering all but at most $28\delta t \leq 28\delta t_{\mathcal{H}}$ vertices of $\mathcal{K}$.

Since every vertex is in an active pair in $\mathcal{K}$, we have

$$|N_{\mathcal{K}}(x)| \geq (1-\delta)t \quad \text{for all } x \in V(\mathcal{K}). \tag{3.2.1}$$

Let $\mathcal{K} = \mathcal{K}_{\text{red}} \cup \mathcal{K}_{\text{blue}}$ be the colouring of $\mathcal{K}$ inherited from $\mathcal{H}$. Then a monochromatic component C of $\mathcal{K}$ is a connected component of $\mathcal{K}_{\text{red}}$ or $\mathcal{K}_{\text{blue}}$.

Observation 3.2.3 ([HŁP+09], Observation 8.2)**.** *For every vertex $x \in V(\mathcal{K})$ there exists a monochromatic component C_x such that $|N_{C_x}(x)| \geq (1-\delta)t$.*

For each $x \in V(\mathcal{K})$ choose arbitrarily one component C_x as in Observation 3.2.3. Let $R = \{x \in V(\mathcal{K}) : C_x \text{ is red}\}$ and $B = \{x \in V(\mathcal{K}) : C_x \text{ is blue}\}$, and note that these two sets partition $V(\mathcal{K})$.

Observation 3.2.4 ([HŁP+09], Observation 8.4)**.** *If $|R| \geq 6\delta t$ (or $|B| \geq 6\delta t$, respectively), then there is a red component $\mathcal{R}$ (a blue component $\mathcal{B}$) such that $C_x = \mathcal{R}$ ($C_x = \mathcal{B}$) for all but at most $2\delta t$ vertices $x \in R$ ($x \in B$).*

Set $V_{\text{red}} := \{x \in R : C_x = \mathcal{R}\}$ if $|V_{\text{red}}| \geq 6\delta t$, and set $V_{\text{blue}} := \{x \in B : C_x = \mathcal{B}\}$ if $|B| \geq 6\delta t$. Otherwise, define V_{red} (or V_{blue}, respectively) as the empty set. Our aim is to find two differently coloured disjoint connected matchings in $\mathcal{K}$ that together cover all but $16\delta t \leq 28\delta t - 12\delta t$ vertices of $V_{\text{red}} \cup V_{\text{blue}}$.

We start by choosing a connected matching of maximal size in $\mathcal{R} \cup \mathcal{B}$. This matching decomposes into two disjoint monochromatic connected matchings, $\mathcal{M}_{\text{red}} \subset \mathcal{R}$ and $\mathcal{M}_{\text{blue}} \subset \mathcal{B}$, which together cover as many vertices as possible. Let $V'_{\text{red}} = V_{\text{red}} \setminus V(\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}})$ and $V'_{\text{blue}} = V_{\text{blue}} \setminus V(\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}})$. We may assume that V'_{red} or V'_{blue} has at least $12\delta t$ vertices, as otherwise we are done. By symmetry we may assume that

$$|V'_{\text{red}}| \geq 8\delta t. \tag{3.2.2}$$

Observe that there is no edge xy with $x \in V'_{\text{red}}$ and $y \in V'_{\text{blue}}$ such that $xy \in \partial\mathcal{R} \cap \partial\mathcal{B}$. Indeed, any such edge xy constitutes an active pair (by Lemma 3.2.2) and as $|V'_{\text{red}}| > \delta t + 2$, there must be a vertex $z \in V'_{\text{red}}$ such that xyz is an edge of $\mathcal{K}$. This contradicts the maximality of the matching $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$.

Next, we claim that

$$|V'_{\text{blue}}| \leq 2\delta t. \tag{3.2.3}$$

Assume otherwise. Then, Observation 3.2.3 and the choice of the set V_{red} implies that the number of edges between V'_{red} and V'_{blue} that belong to $\partial\mathcal{R}$ is at least

$$|V'_{\text{red}}| \cdot (|V'_{\text{blue}}| - \delta t) \geq \frac{1}{2}|V'_{\text{red}}| \cdot |V'_{\text{blue}}|.$$

Similarly, there are at least $|V'_{\text{blue}}| \cdot (|V'_{\text{red}}| - \delta t) > \frac{1}{2}|V'_{\text{red}}| \cdot |V'_{\text{blue}}|$ edges between V'_{red} and V'_{blue} that belong to $\partial\mathcal{B}$. As there is no edge xy with $x \in V'_{\text{red}}$ and $y \in V'_{\text{blue}}$ such that $xy \in \partial\mathcal{R} \cap \partial\mathcal{B}$, we have more than $|V'_{\text{red}}| \cdot |V'_{\text{blue}}|$ edges from V'_{red} to V'_{blue}. This yields a contradiction and (3.2.3) follows.

Because of the maximality of $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$, each edge having all its vertices in V'_{red} is blue. Fix one such edge xyz, which exists because of (3.2.2). Obtain V''_{red} from V'_{red} by deleting the at most δt vertices w with $wx \notin \partial\mathcal{R}$. Consider any edge $x'y'z'$ with $x', y', z' \in V''_{\text{red}}$, which also exists because of (3.2.2). As the pairs $xy, xx', x'y'$ are all active, and $|V''_{\text{red}}| > 3\delta t$, there is a vertex $v \in V''_{\text{red}}$ that forms an edge with each of the three pairs, thus giving a pseudo-path in $\mathcal{K}[V''_{\text{red}}]$ from xyz to $x'y'z'$. Denote by $\mathcal{B}''$ the blue component of $\mathcal{K}[V''_{\text{red}}]$ that contains xyz, and let $\mathcal{B}'$ be obtained from $\mathcal{B}''$ by deleting at most 2 vertices and all incident edges, so that $|V[\mathcal{B}']|$ is a multiple of 3. Then, by (3.2.2), we have

$$|V[\mathcal{B}']| \geq |V'_{\text{red}}| - \delta t - 2 \geq 6\delta t. \tag{3.2.4}$$

Let $x \in V[\mathcal{B}']$ be given. At least $|V[\mathcal{B}']| - \delta t$ vertices $y \in V[\mathcal{B}']$ are such that $xy \in \partial\mathcal{R}$, and, for each such y there are at least $|V[\mathcal{B}']| - \delta t$ vertices $z \in V[\mathcal{B}']$ such that $xyz \in \mathcal{B}'$. So, the total number of hyperedges of $\mathcal{B}'$ that contain x is at least

$$\frac{1}{2}(|V[\mathcal{B}']| - \delta t)^2 \;\geq\; \frac{25}{36}\binom{|V[\mathcal{B}']|}{2}.$$

Thus, Theorem 3.2.1 with $\eta = \frac{5}{36}$ yields a perfect matching $\mathcal{M}'_{\text{blue}}$ of $\mathcal{B}'$.

At this point, we have three disjoint monochromatic connected matchings, one in red ($\mathcal{M}_{\text{red}} \subset \mathcal{R}$) and two in blue ($\mathcal{M}_{\text{blue}} \subset \mathcal{B}$ and $\mathcal{M}'_{\text{blue}} \subset \mathcal{B}'$). Together, these matchings cover all but at most $3\delta t + 2$ vertices of $V_{\text{red}} \cup V_{\text{blue}}$ (by (3.2.3) and by (3.2.4)).

Our aim is now to dissolve the blue matching $\mathcal{M}_{\text{blue}}$, and cover its vertices by new red edges, leaving at most $6\delta t$ vertices uncovered. In order to do so, let us first understand where the edges of $\mathcal{M}_{\text{blue}}$ lie.

For convenience, let us call an edge in $\mathcal{K}$ *good* if two different pairs of its vertices $\{a, b\}$ and $\{c, \mathrm{d}\}$ are such that $ab \in \partial\mathcal{R}$ and $c\mathrm{d} \in \partial\mathcal{B}$. Notice that every good red edge is contained in $\mathcal{R}$ and every good blue edge is contained in $\mathcal{B}$.

First, we claim that for every edge $uvw \in \mathcal{M}_{\text{blue}}$,

$$|\{u, v, w\} \cap V_{\text{blue}}| \leq 1. \tag{3.2.5}$$

Indeed, otherwise there is an edge $uvw \in \mathcal{M}_{\text{blue}}$ with $u, v \in V_{\text{blue}}$. By the definition of $\mathcal{B}$, and by (3.2.2), there is an active edge $ua \in \partial\mathcal{B}$ with $a \in V'_{\text{red}}$. As ua is an active pair, as a has very large degree in $\partial\mathcal{R}$, and by (3.2.2), there is an edge $uab \in \mathcal{K}$ with $b \in V'_{\text{red}}$ such that $ab \in \partial\mathcal{R}$. Hence uab is a good edge. Similarly, there is a good edge $vc\mathrm{d}$, with $c, \mathrm{d} \in V'_{\text{red}} \setminus \{a, b\}$. Remove the edge uvw from $\mathcal{M}_{\text{blue}}$ and add edges uab and $vc\mathrm{d}$ to either $\mathcal{M}_{\text{red}}$ or $\mathcal{M}_{\text{blue}}$, according to their colour. The resulting matching covers more vertices than the matching $M_A \cup \mathcal{M}_{\text{blue}}$, a contradiction. This proves (3.2.5).

Next, we claim that there is no edge $uvw \in \mathcal{M}_{\text{blue}}$ with

$$|\{u, v, w\} \cap V_{\text{blue}}| = 1. \tag{3.2.6}$$

Assume otherwise. Then there is an edge $uvw \in \mathcal{M}_{\text{blue}}$ with $u \in V_{\text{blue}}$ and $v, w \in V_{\text{red}}$. As in the proof of (3.2.5), we can cover u with a good edge uab such that $a, b \in V'_{\text{red}}$. Moreover, since vw is an active pair, and v has very large degree in $\partial\mathcal{R}$, there is an edge vwc with $c \in V'_{\text{red}} \setminus \{a, b\}$ and $cv \in \partial R$. Since $vw \in \partial\mathcal{B}$, the edge vwc is good. So we can remove uvw from $\mathcal{M}_{\text{blue}}$ and add edges uab and vwc to $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$, thus covering three additional vertices. This gives the desired contradiction to the choice of $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$, and proves (3.2.6).

Putting (3.2.5) and (3.2.6) together, we know that for every edge $uvw \in \mathcal{M}_{\text{blue}}$ we have $u, v, w \in V_{\text{red}}$. We can assume that $\mathcal{M}_{\text{blue}}$ contains at least two hyperedges, as otherwise we can just forget about $\mathcal{M}_{\text{blue}}$ and we are done. Consider any two edges $u_1v_1w_1, u_2v_2w_2 \in \mathcal{M}_{\text{blue}}$. As before, there are vertices $a, b \in V'_{\text{red}}$ such that edges v_1w_1a, v_2w_2b are good. Now, if there is a red edge u_1u_2c with $c \in V'_{\text{red}}$ and $u_1c \in \partial\mathcal{R}$ then we can remove edges $u_1v_1w_1, u_2v_2w_2$ and add the red edge u_1u_2c to $\mathcal{M}_{\text{red}}$ and edges v_1w_1a, v_2w_2b to $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$, according to their colour, contradicting the choice of $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$. Therefore, for any choice of $u_1v_1w_1, u_2v_2w_2 \in \mathcal{M}_{\text{blue}}$, we have that

$$\text{all edges } u_1u_2c \text{ with } c \in V'_{\text{red}} \text{ and } u_1c \in \partial\mathcal{R} \text{ are blue.} \tag{3.2.7}$$

Moreover, if there is a blue edge u_1u_2x with $x \in \{v_1, w_1, v_2, w_2\}$ then u_1u_2 is an active pair. In that case, we can calculate as before that an edge u_1u_2c with $c \in V'_{\text{red}}$ and $u_1c \in \partial\mathcal{R}$ exists, and by (3.2.7), this edge is blue. The existence of the blue edge u_1u_2x implies that we can link u_1u_2c to $\mathcal{M}_{\text{blue}}$ with a blue tight path. Thus, removing $u_1v_1w_1$ and $u_2v_2w_2$ from $\mathcal{M}_{\text{blue}}$ and adding $v_1w_1a, v_2w_2b, u_1u_2c$ to $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$ (where a, b are as above), we obtain a contradiction to the choice of $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$. So, for any choice of $u_1v_1w_1, u_2v_2w_2 \in \mathcal{M}_{\text{blue}}$, we have that

$$\text{all edges } u_1u_2x \text{ with } x \in \{v_1, w_1, v_2, w_2\} \text{ are red.} \tag{3.2.8}$$

We can now dissolve the edges of $\mathcal{M}_{\text{blue}}$. For this, separate each hyperedge uvw in $\mathcal{M}_{\text{blue}}$ into an edge uv and a single vertex w. Let X be the set of all edges uv, and let Y be the set of all vertices w obtained in this way. Note that every $uv \in X$ is an active pair in $\mathcal{K}$, and therefore forms a hyperedge uvw' with all but at most δt of the vertices $w' \in Y$. Moreover, all but at most δt of these hyperedges uvw' are such that $uw' \in \partial\mathcal{R}$, because of the large degree u has in $\partial\mathcal{R}$. Now we can greedily match all but at most $2\delta t$ edges uv in X with vertices w' in Y such that for every match uvw' we have that uvw' is an edge of $\mathcal{K}$ and $uw \in \partial\mathcal{R}$.

In $\mathcal{K}$, this corresponds to a matching $\mathcal{M}'_{\text{red}}$ covering all but at most $6\delta t$ vertices of $V(\mathcal{M}_{\text{blue}})$. By (3.2.8), all hyperedges of $\mathcal{M}'_{\text{red}}$ are red. Furthermore, since we ensured that every hyperedge in $\mathcal{M}'_{\text{red}}$ contains a pair uw' that forms an edge of $\partial\mathcal{R}$, we know that $\mathcal{M}_{\text{red}}$ and $\mathcal{M}'_{\text{red}}$ belong to the same red component of $\mathcal{K}$. In other words, $\mathcal{M}_{\text{red}} \cup \mathcal{M}'_{\text{red}}$ and $\mathcal{M}'_{\text{blue}}$ are the two monochromatic connected matchings we had to find. □

3.3 Hypergraph regularity

In this section we introduce the regularity lemma for 3-uniform hypergraphs and state an embedding result from [HŁP+09].

Graph regularity. Let G be a graph and let $X, Y \subseteq V(G)$ be disjoint. The *density* of (X, Y) is $\mathrm{d}_G(X, Y) = \frac{\mathrm{e}_G(X,Y)}{|X||Y|}$ where $\mathrm{e}_G(X, Y)$ denotes the number of edges of G between X and Y.

The bipartite graph G on the partition classes X and Y is called $(\mathrm{d}, \varepsilon)$*-regular*, if $|\mathrm{d}_G(X', Y') - \mathrm{d}| < \varepsilon$ holds for all $X' \subseteq X$ and $Y' \subseteq Y$ of size $|X'| > \varepsilon|X|$ and $|Y'| > \varepsilon|Y|$. If $\mathrm{d} = \mathrm{d}_G(X, Y)$ we say that G is ε*-regular*.

Hypergraph regularity. Let $\mathcal{H}$ be a 3-uniform hypergraph. Let $P = P^{12} \cup P^{13} \cup P^{23}$ with $V(P) \subset V(\mathcal{H})$ be a tripartite graph which we also refer to as *triad*. By $\mathcal{T}(P)$ denote the 3-uniform hypergraph on $V(P)$ whose edges are the triangles of P. The *density of $\mathcal{H}$ with respect to P* is

$$\mathrm{d}_{\mathcal{H}}(P) = \frac{|\mathcal{H} \cap \mathcal{T}(P)|}{|\mathcal{T}(P)|}.$$

Similarly, for a tuple $\vec{Q} = (Q_1, \ldots, Q_r)$ of subgraphs of P, we define the *density of $\mathcal{H}$ with respect to $\vec{Q}$* as

$$\mathrm{d}_{\mathcal{H}}(\vec{Q}) = \frac{|\mathcal{H} \cap \bigcup_{\mathrm{i} \in [r]} \mathcal{T}(Q_{\mathrm{i}})|}{|\bigcup_{\mathrm{i} \in [r]} \mathcal{T}(Q_{\mathrm{i}})|}.$$

Let $\alpha, \delta > 0$ and let $r > 0$ be an integer. We say that $\mathcal{H}$ is (α, δ, r)*-regular* with respect to P if, for every r-tuple $\vec{Q} = (Q_1, \ldots, Q_r)$ of subgraphs of P satisfying $|\bigcup_{\mathrm{i} \in [r]} \mathcal{T}(Q_{\mathrm{i}})| > \delta|\mathcal{T}(P)|$, we have $|\mathrm{d}_{\mathcal{H}}(\vec{Q}) - \alpha| < \delta$. If $\alpha = \mathrm{d}_{\mathcal{H}}(P)$ we say that $\mathcal{H}$ is (δ, r)*-regular* with respect to P, and in the same situation, we say P is (δ, r)*-regular* (with respect to $\mathcal{H}$).

If in addition the bipartite graphs P^{12}, P^{13}, P^{23} of an (α, δ, r)-regular $P = P^{12} \cup P^{13} \cup P^{23}$ are $(1/\ell, \varepsilon)$-regular then we say that the pair $(\mathcal{H}, P)$ is an $(\alpha, \delta, \ell, r, \varepsilon)$*-regular complex.*

Finally, a partition of V into $V_0 \cup V_1 \cup \cdots \cup V_t$ is called an *equipartition* if $|V_0| < t$ and $|V_1| = V_2| = \cdots = |V_t|$.

We state the regularity lemma for 3-uniform hypergraphs [FR02] as presented in [RRS06].

Theorem 3.3.1 (Regularity Lemma for 3-uniform Hypergraphs)**.** *For all $\delta, t_0 > 0$, all integer-valued functions $r = r(t, \ell)$, and all decreasing sequences $\varepsilon(\ell) > 0$ there exist constants T_0, L_0 and N_0 such that every 3-uniform hypergraph $\mathcal{H}$ with at least N_0 vertices admits a vertex equipartition*

$$V(\mathcal{H}) = V_0 \cup V_1 \cup \cdots \cup V_t \qquad \textit{with } t_0 \leq t < T_0,$$

and, for each pair i, j, $1 \leq \mathrm{i} < j \leq t$, an edge partition of the complete bipartite graph

$$K(V_{\mathrm{i}}, V_j) = \bigcup_{k \in [\ell]} P_k^{\mathrm{i}j} \qquad \textit{with } 1 \leq \ell < L_0$$

such that

1. *all graphs $P_k^{\mathrm{i}j}$ are $(1/\ell, \varepsilon(\ell))$-regular.*

2. *$\mathcal{H}$ is (δ, r)-regular with respect to all but at most $\delta\ell^3 t^3$ tripartite graphs $P_a^{h\mathrm{i}} \cup P_b^{hj} \cup P_c^{\mathrm{i}j}$.*

Note that the same partitions satisfy the conclusions of Theorem 3.3.1 for the complement of $\mathcal{H}$ as well. Further, as noted in [HŁP⁺09] by choosing a random index $k_{\mathrm{i}j} \in [\ell]$ for each pair (V_{i}, V_j) Markov's inequality yields that with positive probability there are less than $2\delta t^3$ chosen triads which fail to be (δ, r)-regular. Hence one obtains the following.

Observation 3.3.2. *In the partition produced by Theorem 3.3.1 there is a family $\mathcal{P}$ of bipartite graphs $P^{ij} = P^{ij}_{k_{ij}}$ with vertex classes V_i, V_j, where $1 \le i < j \le t$, such that $\mathcal{H}$ is (δ, r)-regular with respect to all but at most $2\delta t^3$ tripartite graphs $P^{hi} \cup P^{hj} \cup P^{ij}$.*

We end this section with a result from [PRRS06] and [HŁP+09] which allows embedding tight paths in regular complexes. In the following, an S-avoiding tight path is one which does not contain any vertex from S. (Note that although Lemma 4.6 from [HŁP+09] is stated slightly differently, its proof actually yields the version below.)

Lemma 3.3.3 ([HŁP+09], Lemma 4.6). *For each $\alpha \in (0,1)$ there exist $\delta_1 > 0$ and sequences $r(\ell)$, $\varepsilon(\ell)$, and $n_1(\ell)$, for $\ell \in \mathbb{N}$, with the following property.*
For each $\ell \in \mathbb{N}$, and each $\delta \le \delta_1$, if $(\mathcal{H}, P)$ is a $(\mathrm{d}_{\mathcal{H}}(P), \delta, \ell, r(\ell), \varepsilon(\ell))$-complex with $\mathrm{d}_{\mathcal{H}}(P) \ge \alpha$ and all of the three vertex classes of P have the same size $n > n_1(\ell)$, then there is a subgraph P_0 on at most $27\sqrt{\delta}n^2/\ell$ edges of P such that, for all ordered pairs of disjoint edges $(\mathrm{e}, f) \in (P \setminus P_0) \times (P \setminus P_0)$ there is $m = m(\mathrm{e}, f) \in [3]$ such that the following holds. For every $S \subseteq V(\mathcal{H}) \setminus (\mathrm{e} \cup f)$ with $|S| < n/(\log n)^2$, and for each ℓ with $3 \le \ell \le (1 - 2\delta^{\frac{1}{4}})n$, there is an S-avoiding tight path from e to f of length $3\ell + m$ in $\mathcal{H}$.

3.4 Proof of Theorem 3.1.1

We follow a procedure suggested by Łuczak in [Łuc99] for graphs and used for tight cycles in 3-uniform hypergraphs in [HŁP+09].

Proof of Theorem 3.1.1. For given $\eta > 0$ we apply Lemma 3.1.3 with $\gamma = (\eta/580)^6$ to obtain t_0. With foresight apply Lemma 3.3.3 with $\alpha = 1/2$ to obtain δ_1, and sequences $r(\ell)$, $\varepsilon(\ell)$, and $n_1(\ell)$. Finally, apply Theorem 3.3.1 with t_0, $r(t, \ell) = r(\ell)$, $\varepsilon(\ell)$, $n_1(\ell)$ and $\delta = \min\{\delta_1/2, \gamma/58, (\eta/16)^4\}$ to obtain constants T_0, L_0 and N_0.

Given a two-colouring $\mathcal{K}_n = \mathcal{H}_{\text{red}} \cup \mathcal{H}_{\text{blue}}$ of the 3-uniform complete hypergraph $\mathcal{K}_n$ on $n > N_0$ vertices. Apply Theorem 3.3.1 with the chosen constants to $\mathcal{H}_{\text{red}}$ to obtain partitions

$$V(\mathcal{K}_n) = V_0 \cup V_1 \cup \cdots \cup V_t \quad \text{and} \quad K(V_i, V_j) = \bigcup_{k \in [\ell]} P^{ij}_k, 1 \le i < j \le t$$

with $t_0 \le t < T_0$, and $\ell < L_0$ which satisfy the properties detailed in Theorem 3.3.1. The partitions satisfy the same properties for $\mathcal{H}_{\text{blue}}$ as it is the complement hypergraph of $\mathcal{H}_{\text{red}}$.

Observation 3.3.2 then yields a family of $(1/\ell, \varepsilon)$-regular bipartite graphs $P^{ij} = P^{ij}_{k_{ij}}$, one for each pair (V_i, V_j), $1 \le i < j \le t$, such that $\mathcal{H}_{\text{red}}$ (and thus also $\mathcal{H}_{\text{blue}}$) is (δ, r)-regular with respect to all but at most $2\delta t^3$ tripartite graphs $P^{ijk} = P^{ij} \cup P^{ik} \cup P^{jk}$. We use this family to construct the reduced hypergraph $\mathcal{R}$ which has the vertex set $[t]$ and the edge set consisting of all triples ijk such that P^{ijk} is (δ, r)-regular. Further, colour the edge ijk red if $\mathrm{d}_{\mathcal{H}_{\text{red}}}(P^{ijk}) \ge 1/2$ and blue otherwise. Then we have a two-colouring of $\mathcal{R} = \mathcal{R}_{\text{red}} \cup \mathcal{R}_{\text{blue}}$, where $\mathcal{R}$ has at least $\binom{t}{3} - 2\delta t^3 > (1-\gamma)\binom{t}{3}$ edges.

Since $t \ge t_0$ Lemma 3.1.3 yields two disjoint monochromatic connected matchings $\mathcal{M}_{\text{red}}$ and $\mathcal{M}_{\text{blue}}$ which cover all but at most $290\gamma^{\frac{1}{6}}t \le \eta t/2$ vertices of $\mathcal{R}$ and in what follows we will turn these connected matchings into disjoint monochromatic tight cycles in $\mathcal{K}_n$.

We start by choosing a red pseudo-path $\mathcal{Q}_{\text{red}} = (\mathrm{e}_1, \ldots, \mathrm{e}_p) \subset \mathcal{R}_{\text{red}}$ which contains the matching $\mathcal{M}_{\text{red}}$. This is possible since $\mathcal{M}_{\text{red}}$ is a connected matching, and so, consecutive matching edges $g_s, g_{s+1} \in \mathcal{M}_{\text{red}}$ are connected by a red pseudo-path of length at most $\binom{t}{3}$. The concatenation of these paths then forms a Q_{red} as desired. In the same manner, choose a blue pseudo-path $Q_{\text{blue}} = (e'_1, \dots, e'_q) \subset \mathcal{R}_{\text{blue}}$ containing the matching $\mathcal{M}_{\text{blue}}$. Note that although $\mathcal{M}_{\text{red}}$ and $\mathcal{M}_{\text{blue}}$ are disjoint, the two paths Q_{red} and Q_{blue} may have vertices in common.

The general idea is to find long tight cycles in different colours is as follows. For each edge $\{i, j, k\} = e_s \in Q_{\text{red}}$ let P^s denote the triad $P^{ij} \cup P^{ik} \cup P^{jk}$ on the partition classes $V_i \cup V_j \cup V_k$ and recall that P^s is (δ, r)-regular (with respect to $\mathcal{H}_{\text{red}}$). Lemma 3.3.3 guarantees that one can find long tight paths in the complex $(\mathcal{H}_{\text{red}}, P^s)$ for each matching edge $e_s \in \mathcal{M}_{\text{red}} \subset Q_{\text{red}}$ which exhaust almost all vertices of $V_i \cup V_j \cup V_k$. Using the connectedness of Q_{red} and Lemma 3.3.3, we then want to connect these long tight paths by short tight paths, hence obtain a tight cycle C_{red} which covers almost all vertices of $\mathcal{K}_n$ spanned by $\mathcal{M}_{\text{red}}$. We wish to do the same with Q_{blue} to obtain a tight cycle C_{blue} which covers almost all vertices of $\mathcal{K}_n$ spanned by $\mathcal{M}_{\text{blue}}$. The two cycles C_{red} and C_{blue} then exhaust most of the vertices of $\mathcal{K}_n$.

To keep the two cycles disjoint, however, the strategy will be slightly less straightforward. First, we will find two disjoint short tight cycles C'_{red} and C'_{blue} in $\mathcal{K}_n$ visiting all triads P^s corresponding to edges $e_s \in \mathcal{M}_{\text{red}}$ and P'^s corresponding to $e'_s \in \mathcal{M}_{\text{blue}}$, respectively. Then, for each edge $e_s \in \mathcal{M}_{\text{red}}$, and each edge $e'_s \in \mathcal{M}_{\text{blue}}$, we will replace parts of the cycles C'_{red}, and of C'_{blue}, i.e., paths corresponding to e_s and e'_s by long tight paths as mentioned above. We now give the details of this idea.

For each $s = 1, \dots, p$, apply Lemma 3.3.3 to the complex $(\mathcal{H}_{\text{red}}, P^s)$ to obtain the subgraph $P_0^s \subset P^s$ of "prohibited" edges and let

$$B_s = (P^s \setminus P_0^s) \cap (P^{s+1} \setminus P_0^{s+1})$$

which is a bipartite graph on the partition classes $V_i \cup V_j$ where $\{i, j\} = e_s \cap e_{s+1}$. We choose mutually distinct edges $f_s, g_s \in B_s$, $s \in [p-1]$ which is possible due to the restriction on $|P_0^s|$ provided n is sufficiently large. Using Lemma 3.3.3 we then find a short tight cycle C'_{red} by concatenating disjoint paths each of length at most 12 between f_1 and f_2, f_2 and f_3 ... between f_{p-1} and g_{p-1} and backwards between g_{p-1} and g_{p-2} ... and finally between g_1 and f_1. Note that the lemma allows the paths to be S-avoiding for any vertex set S of size $|S| < n'/(\log n')^2$ where n' is the size of the partition classes. Therefore, to guarantee the disjointness of the paths, we simply choose S to be the vertices of the paths constructed so far which has size at most $24p$, i.e., independent of $n' > n/2t$. In the same way choose a short tight cycle C'_{blue} disjoint from C'_{red} by including $V(C'_{\text{red}})$ to S in the applications of Lemma 3.3.3.

Let $S' = V(C'_{\text{red}}) \cup V(C'_{\text{blue}})$ which satisfies $|S'| < n'/(\log n')^2$. It is easy to see that for each $e_s \in \mathcal{M}_{\text{red}}$ there are two non-prohibited edges in P^s, connected by a subpath of C'_{red} which is entirely contained in $(\mathcal{H}_{\text{red}}, P^s)$. Hence, by Lemma 3.3.3 we can replace this short path by an S-avoiding path in $(\mathcal{H}_{\text{red}}, P^s)$ which covers all but at most $4\delta^{1/4}n'$ vertices and having any desired parity. Doing this for all $e_s \in \mathcal{M}_{\text{red}}$ and all $e'_s \in \mathcal{M}_{\text{blue}}$ and noting that $n' \le n/t$ we obtain two monochromatic disjoint tight cycles which cover all but at most

$$\begin{aligned}\big(|\mathcal{M}_{\text{red}}| + |\mathcal{M}_{\text{blue}}|\big)4\delta^{1/4}n' + &\big(|V(\mathcal{R})| - |\mathcal{M}_{\text{red}}| + |\mathcal{M}_{\text{blue}}|\big)n' + |V_0| \\ &\le \frac{1}{4}\eta n + \frac{1}{2}\eta n + t \le \eta n\end{aligned}$$

vertices of $\mathcal{K}_n$, and have any desired parity. This finishes the proof of the theorem. □

Chapter 4

Partitioning k-uniform hypergraphs into few monochromatic tight-cycles

Abstract

We prove that for every $k, r \in \mathbb{N}$, the vertices of every r-edge-coloured complete k-uniform hypergraph can be partitioned into a bounded number (independent of the size of the hypergraph) of monochromatic tight cycles, confirming a conjecture of Gyárfás. We further prove that for every $r, p \in \mathbb{N}$, the vertices of every r-edge-coloured complete graph can be partitioned into a bounded number of p-th powers of cycles, settling a problem of Elekes, D. Soukup, L. Soukup and Szentmiklóssy. In fact we prove a common generalisation of both theorems which further extends these results to all host hypergraphs with bounded independence number. This is joint work with Jan Corsten, Nóra Frankl, Alexey Pokrovskiy and Jozef Skokan.

4.1 Introduction

In [EGP91], Erdős, Gyárfás and Pyber proved that the number of monochromatic cycles required to partition the vertices of every r-edge-coloured complete graph on n vertices does not depend on n. Similar problems have been considered for powers of cycles. Given a graph H and a natural number p, the p-th power of H is the graph obtained from H by putting an edge between any two vertices whose distance is at most p. Grinshpun and Sárközy [GS16] proved that the vertices of every 2-edge-coloured complete graph can be partitioned into at most $2^{cp \log p}$ monochromatic p-th powers of cycles, where c is an absolute constant. They conjectured that a much smaller number of pieces should suffice. For more than two colours not much is known. Elekes, D. Soukup, L. Soukup and Szentmiklóssy [ESSS17] proved a similar result using r colours when the host graph is infinite and ask whether it is true for finite graphs.

Problem 4.1.1 ([ESSS17, Problem 6.4][1])**.** *Prove that for every $r, p \in \mathbb{N}$, there is some $c = c(r, p)$ such that the vertices of every r-edge-coloured complete graph can be partitioned into c monochromatic p-th powers of cycles.*

We prove a substantial generalisation of Problem 4.1.1 as a corollary of our main result (see Corollary 4.1.4).

[1] The problem is phrased differently in [ESSS17] but this version is stronger, as Elekes et. al. explain below the problem.

In this chapter we consider similar questions for k-uniform hypergraphs. The k-uniform loose cycle of length m is the k-uniform hypergraph consisting of $m(k-1)$ cyclically ordered vertices and m edges, each consisting of k consecutive vertices, such that consecutive edges intersect in exactly one vertex. The k-uniform tight cycle of length m is the k-uniform hypergraph consisting of m cyclically ordered vertices in which any k consecutive vertices form an edge. Loose and tight paths are defined in a similar way.

Such questions were first studied by Gyárfás and Sárközy [GS13] who showed that for every $k, r \in \mathbb{N}$, there is some $c = c(k, r)$ so that the vertices of every r-edge-coloured complete k-uniform hypergraph can be partitioned into at most c loose cycles. Later, Sárközy [Sár14] showed that $c(k, r)$ can be be chosen to be $50rk \log(rk)$.

In [Gyá16], Gyárfás conjectured that a similar result can be obtained for tight cycles.

Conjecture 4.1.2 ([Gyá16])**.** *For every $k, r \in \mathbb{N}$, there is some $c = c(k, r)$ so that the vertices of every r-edge-coloured complete k-uniform hypergraph can be partitioned into at most c monochromatic tight cycles.*

We shall prove the following generalisation of Conjecture 4.1.2, allowing the host-graph to be any k-uniform hypergraph with bounded independence number. A similar result for graphs was obtained by Sárközy [Sár11], and for loose cycles in hypergraphs by Gyárfás and Sárközy [GS14].

Theorem 4.1.3. *For every $k, r, \alpha \in \mathbb{N}$, there is some $c = c(k, r, \alpha)$ such that the vertices of every r-edge-coloured k-uniform hypergraph G with $\alpha(G) \leq \alpha$ can be partitioned into c monochromatic tight cycles.*

The p-th power of a k-uniform tight cycle of length m is the k-uniform hypergraph consisting of m cyclically ordered vertices, so that any $k + p - 1$ consecutive vertices form a clique. An immediate corollary of Theorem 4.1.3 is the following strengthening.

Corollary 4.1.4. *For every $k, r, p, \alpha \in \mathbb{N}$, there is some $c = c(k, r, p, \alpha)$ such that the vertices of every r-edge-coloured k-uniform hypergraph G with $\alpha(G) \leq \alpha$ can be partitioned into c monochromatic p-th powers of tight cycles.*

Proof. Let $f(k, r, \alpha)$ be the smallest c for which Theorem 4.1.3 is true and let $g(k, r, p, \alpha)$ be the smallest c for which Corollary 4.1.4 is true. We will show that $g(k, r, p, \alpha) \leq f(k, r, \tilde{\alpha})$, where $\tilde{\alpha} = R^{(k)}_{r+1}(k + p - 1, \ldots, k + p - 1, \alpha + 1) - 1$. Suppose now we are given an r-edge-coloured k-uniform hypergraph G with $\alpha(G) \leq \alpha$. Define a $(k+p-1)$-graph H on the same vertex-set whose edges are the monochromatic cliques of size $k + p - 1$ in G. By construction we have $\alpha(H) \leq \tilde{\alpha}$ and thus, by Theorem 4.1.3, there are $f(k, r, \tilde{\alpha})$ monochromatic tight cycles partitioning $V(H)$. To conclude, note that a tight cycle in H naturally corresponds to a p-th power of a tight cycle in G. □

4.2 Notation

In this section we will introduce some basic notation about hypergraphs. Fix a set of vertices V of size n and a natural number $k \geq 2$ for the rest of this section.

Given a k-uniform hypergraph $H^{(k)}$ and a partition $\mathcal{P} = \{V_1, \ldots, V_t\}$ of V we say that $H^{(k)}$ is $\mathcal{P}$-partite if $|\mathrm{e} \cap V_{\mathrm{i}}| \leq 1$ for every $\mathrm{e} \in E(H^{(k)})$ and every $\mathrm{i} \in [t]$. H is s-partite if it is $\mathcal{P}$-partite for some partition $\mathcal{P}$ of V with s parts. We denote by $K^{(k)}(\mathcal{P})$ the complete $\mathcal{P}$-partite k-uniform hypergraph. Furthermore, given some $2 \leq j \leq k-1$ and a j-uniform hypergraph $H^{(j)}$, we define $K^{(k)}(H^{(j)})$ to be the set of all k-cliques in $H^{(j)}$, seen as a k-uniform hypergraph on V.

Given a k-uniform hypergraph $H^{(k)}$ and $\ell \leq k$ distinct vertices $v_1, \ldots, v_\ell \in V(H^{(k)})$, we denote by $\mathrm{Lk}_{H^{(k)}}(v_1, \ldots, v_\ell)$ the $(k-\ell)$-graph on $V(H^{(k)}) \setminus \{v_1, \ldots, v_\ell\}$ with edges $\{\mathrm{e} \in \binom{V(H^{(k)})}{k-\ell} : \mathrm{e} \cup \{v_1, \ldots, v_l\} \in E(H^{(k)})\}$. If, in addition, disjoint sets $V_1, \ldots, V_\ell \subset V(H^{(k)}) \setminus \{v_1, \ldots, v_\ell\}$ are given, we denote by $\mathrm{Lk}_{H^{(k)}}(v_1, \ldots, v_\ell; V_1, \ldots, V_{k-\ell})$ the $(k-\ell)$-partite $(k-\ell)$-graph with parts $V_1, \ldots, V_{k-\ell}$ and edges $\{\mathrm{e} \in K^{(k-\ell)}(V_1, \ldots, V_{k-\ell}) : \mathrm{e} \cup \{v_1, \ldots, v_\ell\} \in E(H^{(k)})\}$. If $H^{(k)}$ is understood, we drop the subscript.

4.3 The proof

The proof idea follows an absorption method introduced in [EGP91]. For complete k-uniform hypergraphs as hosts, the proof can be summarised as follows. First, we find a special monochromatic k-uniform hypergraph H_0 with the following special property. There is some $B \subset V(H_0)$, such that for every $B' \subseteq B$ there is a tight cycle in H_0 with vertices $V(H_0) \setminus B'$. This is explained in section 4.3.2. We then greedily remove vertex-disjoint monochromatic cycles until the set of leftover vertices R is very small in comparison to B.

Finally, we show that the leftover vertices can be absorbed by H_0. More precisely, we show that there are constantly many vertex-disjoint tight cycles with vertices in $R \cup B$ which cover all of R. This is the main difficulty of the paper and will be done in section 4.3.3. Here, we will need basic tools from hypergraph regularity (see section 4.3.1) to build tight cycles in well behaved sub-hypergraphs by concatenation of short tight paths.

In order to prove the main theorem for k-uniform hypergraphs with bounded independence number as hosts, we need to iterate this process a few times. Here the main difficulty is to show that this iteration stops after constantly many steps. This will be done in section 4.3.4.

4.3.1 Finding short paths

The goal of this section is to prove the following lemma, which allows us to find in any dense k-uniform hypergraph G, a dense sub-k-uniform hypergraph $H \subset G$ in which any two non-isolated $(k-1)$-sets are connected by a short tight path of a given prescribed length. For this, we need to use basic tools from hypergraph regularity, but the reader may use Lemma 4.3.1 as a black box if she would like to avoid it.

Before stating the lemma, we need to introduce some notation. Fix some $k \geq 2$ and a partition $\mathcal{P} = \{V_1, \ldots, V_k\}$. We call a tight path in $K^{(k)}(\mathcal{P})$ *positively oriented* if its vertex sequence $(u_1, \ldots, u_m)$ travels through $\mathcal{P}$ in cyclic order, i.e. there is some $j \in [k]$ such that $u_\mathrm{i} \in V_{\mathrm{i}+j}$ for every $\mathrm{i} \in [m]$, where we identify $k+1$ with 1. In this subsection, we will only consider positively oriented tight cycles. In particular, given some $\mathrm{e} \in K^{(k-1)}(\mathcal{P})$, the ordering of e in a tight path starting at e is uniquely determined.

Lemma 4.3.1. *For every* $\mathrm{d} > 0$*, there are constants* $\delta = \delta(\mathrm{d}) > 0$ *and* $\sigma = \sigma(\mathrm{d}) > 0$*, such that the following is true for every partition* $\mathcal{P} = \{V_1, \ldots, V_k\}$ *and every* $\mathcal{P}$*-partite* k*-uniform hypergraph* G *of density at least* d*. There is a* $\mathcal{P}$*-partite sub-*k*-uniform hypergraph* $H \subset G$ *of density at least* δ *such that for every set* $S = S_1 \cup \ldots \cup S_k$ *with* $S_\mathrm{i} \subset V_\mathrm{i}$ *and* $|S_\mathrm{i}| \leq \sigma |V_\mathrm{i}|$ *and any two* $\mathrm{e}, f \in K^{(k-1)}(\mathcal{P})$ *which are disjoint from* S *and have positive co-degree, there is a positively oriented tight path of length*

$\ell \in \{k+2, \ldots, 2k+1\}$ *in* H *which starts at* e, *ends at* f *and avoids* S. [2]

Note that the length of the tight path in Lemma 4.3.1 is uniquely determined by the types of e and f. The *type* of $\mathrm{e} \in K^{(k-1)}(\mathcal{P})$, denoted by $\mathrm{tp}\,(\mathrm{e})$, is the unique index $\mathrm{i} \in [k]$ such that $\mathrm{e} \cap V_{\mathrm{i}} = \emptyset$. Given two $(k-1)$-sets $\mathrm{e}, f \in K^{(k-1)}(\mathcal{P})$, the type of (e, f) is given by $\mathrm{tp}\,(\mathrm{e}, f) := \mathrm{tp}\,(f) - \mathrm{tp}\,(\mathrm{e}) \pmod{k}$. It is easy to see that every tight path in $K^{(k)}(\mathcal{P})$ which starts at e and ends at f has length $\ell k + \mathrm{tp}\,(\mathrm{e}, f)$ for some $\ell \geq 0$. In particular, in lemma 4.3.1, we have $\ell = k + \mathrm{tp}\,(\mathrm{e}, f)$ if $\mathrm{tp}\,(\mathrm{e}, f) \geq 2$ and $\ell = 2k + \mathrm{tp}\,(\mathrm{e}, f)$ otherwise.

4.3.1.1 Hypergraph regularity

We will now introduce the basic concepts of hypergraph regularity in order to state a simple consequence of the strong hypergraph regularity lemma which guarantees a dense regular complex in every large enough k-uniform hypergraph.

For technical reasons, we want to see a 1-uniform hypergraph on some vertex set V as a partition of V in what follows. We call $\mathcal{H}^{(k)} = (H^{(1)}, \ldots, H^{(k)})$ a k-complex if $H^{(j)}$ is a j-uniform hypergraph for every $j \in [k]$ and $H^{(j)}$ underlies $H^{(j+1)}$, i.e. $H^{(j+1)} \subset K^{(k)}(H^{(j)})$ for every $j \in [k-1]$. Note that, in particular, $H^{(j)}$ is $H^{(1)}$-partite for every $j \in [k]$. We call $\mathcal{H}^{(k)}$ s-partite if $H^{(1)}$ consists of s parts.

Now, given some j-uniform hypergraph $H^{(j)}$ and some underlying $(j-1)$-uniform hypergraph $H^{(j-1)}$, we define the *density* of $H^{(j)}$ with respect to $H^{(j-1)}$ by

$$\mathrm{d}\left(H^{(j)}|H^{(j-1)}\right) = \frac{\left|H^{(j)} \cap K^{(j)}(H^{(j-1)})\right|}{\left|K^{(j)}(H^{(j-1)})\right|}.$$

We are now ready to define regularity.

Definition 4.3.2.

- Let $r, j \in \mathbb{N}$ with $j \geq 2$; $\varepsilon, \mathrm{d}_j > 0$, and $H^{(j)}$ be a j-partite j-uniform hypergraph and $H^{(j-1)}$ be an underlying (j-partite) $(j-1)$-uniform hypergraph. We call $H^{(j)}$ $(\varepsilon, \mathrm{d}_j, r)$-regular with respect to $H^{(j-1)}$ if for all $Q_1^{(j-1)}, \ldots, Q_r^{(j-1)} \subset E(H^{(j-1)})$, we have

$$\left|\bigcup_{\mathrm{i} \in [r]} K^{(j)}\left(Q_{\mathrm{i}}^{(j-1)}\right)\right| \geq \varepsilon \left|K^{(j)}\left(H^{(j-1)}\right)\right| \implies \left|\mathrm{d}\left(H^{(j)} \middle| \bigcup_{\mathrm{i} \in [r]} Q_{\mathrm{i}}^{(j-1)}\right) - \mathrm{d}_j\right| \leq \varepsilon.$$

  For short, we say $(\varepsilon, *, r)$-regular for $(\varepsilon, \mathrm{d}\left(H^{(j)}|H^{(j-1)}\right), r)$-regular, and (e, d)-regular for $(\varepsilon, \mathrm{d}, 1)$-regular.

- Let $j, s \in \mathbb{N}$ with $s \geq j \geq 2$, $\varepsilon, \mathrm{d}_j > 0$, and $H^{(j)}$ be an s-partite j-uniform hypergraph and $H^{(j-1)}$ be an underlying (s-partite) $(j-1)$-uniform hypergraph. We call $H^{(j)}$ $(\varepsilon, \mathrm{d}_j)$-regular with respect to $H^{(j-1)}$ if $H^{(j)}[V_1, \ldots, V_j]$ is $(\varepsilon, \mathrm{d})$-regular with respect to $H^{(j-1)}[V_{\mathrm{i}_1}, \ldots, V_{\mathrm{i}_j}]$ for all $1 \leq \mathrm{i}_1 < \ldots < \mathrm{i}_j \leq s$, where $\{V_1, \ldots, V_s\}$ is the vertex partition of $V(H^{(j)})$.

- Let $k, r \in \mathbb{N}$, $\varepsilon, \varepsilon_k, \mathrm{d}_2, \ldots, \mathrm{d}_k > 0$, and $\mathcal{H}^{(k)} = (H_1, \ldots, H_k)$ be a k-partite k-complex. We call $\mathcal{H}^{(k)}$ $(\mathrm{d}_2, \ldots, \mathrm{d}_k, \varepsilon, \varepsilon_k, r)$-regular, if $H^{(j)}$ is $(\varepsilon, \mathrm{d}_j)$-regular with respect to $H^{(j-1)}$ for every $j = 2, \ldots, k-1$ and $H^{(k)}$ is $(\varepsilon_k, \mathrm{d}_k, r)$-regular with respect to $H^{(k-1)}$.

[2] More precisely, $\ell = k + \mathrm{tp}\,(\mathrm{e}, f)$ if $\mathrm{tp}\,(\mathrm{e}, f) \geq 2$ and $\ell = 2k + \mathrm{tp}\,(\mathrm{e}, f)$ otherwise.

The following theorem is a direct consequence of the strong hypergraph regularity as stated in [RS07] (with the exception that we allow for an initial partition of not necessarily equal sizes).

Theorem 4.3.3. *For all integers $k \geq 2$, constants $\varepsilon_k > 0$, and functions $\varepsilon : (0,1) \to (0,1)$ and $r : (0,1) \to \mathbb{N}$, there exists some $\delta = \delta(k, \varepsilon, \varepsilon_k, r) > 0$ such that the following is true. For every partition $\mathcal{P} = \{V_1, \dots, V_k\}$ of some set V and every $\mathcal{P}$-partite k-uniform hypergraph $G^{(k)}$, there are sets $U_\mathrm{i} \subset V_\mathrm{i}$ with $|U_\mathrm{i}| \geq \delta |V_\mathrm{i}|$ for every $\mathrm{i} \in [k]$ and constants $\mathrm{d}_2, \dots, \mathrm{d}_k \geq \delta$ for which there exists some $(\mathrm{d}_2, \dots, \mathrm{d}_k, \varepsilon(\mathrm{d}), \varepsilon_k, r(\mathrm{d}))$-regular k-complex $\mathcal{H}^{(k)}$, so that $H^{(k)} = G^{(k)}[U_1, \dots, U_k]$ and $H^{(1)} = \{U_1, \dots, U_k\}$.*

We will use the following special case of the extension lemma in [CFKO09, Lemma 5] to find short tight paths between almost any two $(k-1)$-sets in a regular complex. Fix a $(\mathrm{d}_2, \dots, \mathrm{d}_k, \varepsilon, \varepsilon_k)$-regular complex $H^{(k)} = (\mathcal{P}, H^{(2)}, \dots, H^{(k)})$, where $\mathcal{P} = \{V_1, \dots, V_k\}$. Let $H_\mathrm{i}^{(k-1)} \subset H^{(k-1)}$ denote the edges of type i and note that the dense counting lemma for complexes [CFKO09, Lemma 6] implies that, for all $\mathrm{i}_0 \in [k]$,

$$\left|H_{\mathrm{i}_0}^{(k-1)}\right| = (1 \pm \varepsilon) \prod_{j=2}^{k-1} \mathrm{d}_j^{\binom{k-1}{j}} \prod_{\mathrm{i} \in [k] \setminus \mathrm{i}_0} |V_\mathrm{i}| \, .$$

Given some $\beta > 0$, we call a pair $(\mathrm{e}, f) \in H_{\mathrm{i}_1}^{(k-1)} \times H_{\mathrm{i}_2}^{(k-1)}$ β-typical for $\mathcal{H}^{(k)}$ if the number of tight paths of length $\ell := k + \mathrm{tp}(\mathrm{i}_1, \mathrm{i}_2)$ in $H^{(k)}$ which start at e and end at f is

$$(1 \pm \beta) \prod_{j=2}^{k} \mathrm{d}_j^{\ell \binom{k}{j} - 2\binom{k-1}{j}} \prod_{\mathrm{i} \in \{\mathrm{i}_1, \dots, \mathrm{i}_2\}} |V_\mathrm{i}| \, ,$$

where $\{\mathrm{i}_1, \dots, \mathrm{i}_2\}$ is understood in cyclic ordering.

Lemma 4.3.4. *Let $k, r, n_0 \in \mathbb{N}$, $\beta, \mathrm{d}_2, \dots, \mathrm{d}_k, \varepsilon, \varepsilon_k > 0$ and suppose that*

$$1/n_0 \ll 1/r, \varepsilon \ll \min\{\varepsilon_k, \mathrm{d}_2, \dots, \mathrm{d}_{k-1}\} \leq \varepsilon_k \ll \beta, \mathrm{d}_k, 1/k.$$

Then the following is true for all positive integers $n \geq n_0$, for all indices $\mathrm{i}_1, \mathrm{i}_2 \in [k]$ and every $(\mathrm{d}_2, \dots, \mathrm{d}_k, \varepsilon, \varepsilon_k, r)$-regular complex $\mathcal{H}^{(k)} = \left(H^{(1)}, \dots, H^{(k)}\right)$ with $|V_\mathrm{i}| \geq n_0$ for all $\mathrm{i} \in [k]$, where $H^{(1)} = \{V_1, \dots, V_k\}$. All but at most $\beta \left|H_{\mathrm{i}_1}^{(k-1)}\right| \left|H_{\mathrm{i}_2}^{(k-1)}\right|$ pairs $(\mathrm{e}, f) \in H_{\mathrm{i}_1}^{(k-1)} \times H_{\mathrm{i}_2}^{(k-1)}$ are β-typical for $\mathcal{H}^{(k)}$.

Combining Theorem 4.3.3 and Lemma 4.3.4 gives Lemma 4.3.1.

Proof-sketch of lemma 4.3.1. Apply Theorem 4.3.3 with suitable constants and delete all $\mathrm{e} \in H^{(k-1)}$ of small co-degree. Let $\mathrm{e} \in H_{\mathrm{i}_1}^{(k-1)}$ and $f \in H_{\mathrm{i}_2}^{(k-1)}$ for some $\mathrm{i}_1, \mathrm{i}_2 \in [k]$ and define

$$X = \left\{g^{(k-1)} \in H_{\mathrm{i}_1+1}^{(k-1)} : \mathrm{e} \cup g^{(k-1)} \in H^{(k)}\right\} \text{ and}$$
$$Y = \left\{g^{(k-1)} \in H_{\mathrm{i}_2-1}^{(k-1)} : f \cup g^{(k-1)} \in H^{(k)}\right\}.$$

Let $\tilde{X} \subset X$ and $\tilde{Y} \subset Y$ be the sets of all those edges in X and Y avoiding S. By Lemma 4.3.4 at least one pair in $\tilde{X} \times \tilde{Y}$ must be typical and by a counting argument not all of the promised paths can touch S. □

4.3.2 Absorption Method

The absorption method was introduced in [EGP91] and has been successfully applied to answer several questions involving partitions of graphs and hypergraphs. A good reference can be found in [Zha16], where Zhao surveys methods that helped to develop Dirac-type problems for hypergraphs. In what follows we present a suitable generalisation for tight cycles of the absorbing structure introduced in [EGP91].

Definition 4.3.5. Let H be an edge-coloured hypergraph and $A, B \subset V(H)$ disjoint subsets. A is called an *absorber* for B if there is a monochromatic tight cycle with vertices $A \cup B'$ for every $B' \subseteq B$.

Lemma 4.3.6. *For every $k, r, \alpha \in \mathbb{N}$, there is some $\beta = \beta(k, r, \alpha) > 0$ such that the following is true for every k-uniform hypergraph H with $\alpha(H) \leq \alpha$. In every r-colouring of $E(H)$ there are disjoint sets $A, B \subset V(H)$ with $|B| \geq \beta|V(H)|$ such that A absorbs B.*

The following hypergraph (see Figure 4.1) will function as our absorber. A very similar hypergraph was used by Gyárfás and Sárközy to absorb loose cycles [GS13, GS14].

Definition 4.3.7. The (k-uniform) crown of order t, $T_t^{(k)}$, is a tight cycle with $n = t(k-1)$ vertices $v_0, \ldots, v_{n-1}$ (the base) and additional vertices $u_0, \ldots, u_{t-1}$ (the rim). Furthermore, for each $\mathrm{i} = 0, \ldots, t-1$, we add the k edges $\{u_\mathrm{i}, v_{(k-1)\mathrm{i}+j}, \ldots, v_{(k-1)\mathrm{i}+j+k-2}\}$, $j = 0, \ldots, k-1$.

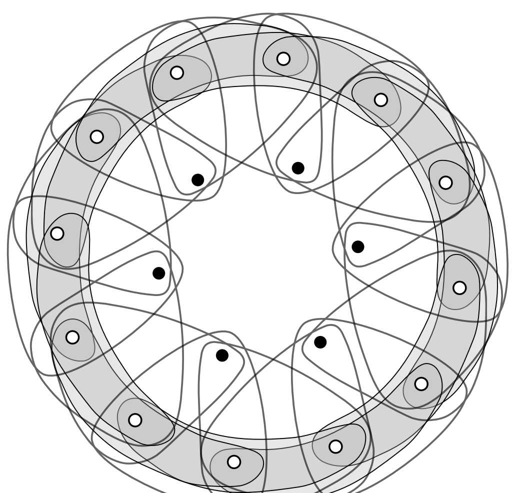


Figure 4.1: The 3-uniform crown of order 6. White vertices are the base of the crown, where the gray edges form the required tight cycle on the base vertices. Black vertices are the rim of $T_6^{(3)}$.

It is easy to see that the base of a crown is an absorber for the rim. To prove Lemma 4.3.6, we therefore only need to show that we can always find monochromatic crowns of linear size. This is a consequence of the following theorem of Conlon, Fox and Sudakov [CFS09].

Theorem 4.3.8. *For every $r, k, \Delta \in \mathbb{N}$, there is some $C = C(r, k, \Delta) > 0$ so that the following is true for all k-graphs $H_1, \ldots, H_r$ with at most n vertices and maximum degree at most Δ, and every $N \geq Cn$. In every edge-colouring of $K_N^{(k)}$ with colours $c_1, \ldots, c_r$, there is some $\mathrm{i} \in [r]$ for which there is a c_i-monochromatic copy of H_i.*

Proof of Lemma 4.3.6. Suppose k, r, α and H are given as in the theorem and that $E(H)$ is coloured with r colours. Let $n = |V(H)|$, $\Delta := \max\left\{2k, \binom{\alpha}{k-1}\right\}$ and $c = 1/C$, where $C = C(r+1, k, \Delta)$ is given by Theorem 4.3.8. Consider now the $(r+1)$-colouring of the edges of $K_n^{(k)}$ in which every edge in $E(H)$ receives the same colour as in H and every other edge receives colour $r+1$. Let $H_{r+1} = K_{\alpha+1}^{(k)}$ and $H_{\mathrm{i}} = T_{cn}^{(k)}$ for all $\mathrm{i} \in [t]$, and note that $\Delta(H_{\mathrm{i}}) \leq \Delta$ for all $\mathrm{i} \in [r+1]$. By choice of Δ, there is no monochromatic H_{r+1} in colour $r+1$ and hence, by choice of c, there is a monochromatic copy of H_{i} for some $\mathrm{i} \in [r]$. □

4.3.3 Absorption Lemma

In this section we prove a suitable absorption lemma for our approach.

Lemma 4.3.9. *For every $\varepsilon > 0$ and $k, r \in \mathbb{N}$, there is some $\gamma = \gamma(k, r, \varepsilon) > 0$ and some $c = c(k, r, \varepsilon)$ such that the following is true. Let H be a k-partite k-uniform hypergraph with parts $B_1, \ldots, B_k$ such that $|B_1| \geq \ldots \geq |B_{k-1}| \geq |B_k|/\gamma$ and $|\operatorname{Lk}(v; B_1, \ldots, B_{k-1})| \geq \varepsilon |B_1| \cdot \cdots \cdot |B_{k-1}|$ for every $v \in B_k$. Then, in every r-colouring of $E(H)$, there are c vertex-disjoint tight cycles covering B_k.*

In the proof, we will need the following simple but slightly technical lemma.

Lemma 4.3.10. *For every $\varepsilon > 0$ there is some $\delta = \delta(\varepsilon) > 0$ and some $C = C(\varepsilon) > 0$ such that the following is true for every $m \in \mathbb{N}$. Let X be set of size m and $\mathcal{F} \subset 2^X$ be a family of subsets such that $|F| \geq \varepsilon m$ for every $F \in \mathcal{F}$. Then there is some $\mathcal{G} \subset \mathcal{F}$ of size $|\mathcal{G}| \leq C$ and a partition $\mathcal{P}$ of $\mathcal{F} \setminus \mathcal{G}$ into sets of size 4 such that $|\bigcap \mathcal{B}| \geq \delta m$ for every $\mathcal{B} \in \mathcal{P}$.*

We will prove Lemma 4.3.10 with $\delta(\varepsilon) = \varepsilon^4/2^6$ and $C(\varepsilon) = 8/\varepsilon^2 + 2/\varepsilon$.

Proof. Define a graph G on $\mathcal{F}$ by $\{F_1, F_2\} \in E(G)$ if and only if $|F_1 \cap F_2| \geq (\varepsilon/2)^2 m$. We claim that $\alpha(G) \leq 2/\varepsilon$. Suppose for contradiction that there is an independent set I of size $2/\varepsilon + 1$. Then we have $|F_0 \setminus \bigcup_{F \in I \setminus \{F_0\}} F| \geq \varepsilon m/2$ for every $F_0 \in I$ and hence $|\bigcup_{F \in I} F| > m$, a contradiction. Hence we find a matching $\mathcal{P}_1$ of all but at most $2/\varepsilon$ vertices. Let $\mathcal{G}_1 = \mathcal{F} \setminus V(\mathcal{P}_1)$ and note that $\mathcal{P}_1$ is a partition of $\mathcal{F} \setminus \mathcal{G}_1$ into sets of size 2. Let $\mathcal{F}_1 = \{F_1 \cap F_2 : \{F_1, F_2\} \in \mathcal{P}_1\}$ and iterate the process once more. □

Proof of Lemma 4.3.9. It suffices to prove the lemma for $r = 1$. Indeed, for each $v \in B_k$, delete all edges containing v which are not in its majority colour and apply the one-colour result (with $\varepsilon' = \varepsilon/r$) for each 'colour class'.

Fix $\varepsilon > 0$, $k \geq 2$ and a k-partite k-uniform hypergraph with parts $B_1, \ldots, B_k$ as in the statement of the lemma. Choose constants $\gamma, \delta_1, \delta_2, \delta_3 > 0$ so that $0 < \gamma \ll \delta_3 \ll \delta_2 \ll \delta_1 \ll \varepsilon$. We begin with a simple but important observation.

Observation 4.3.11. *Let $v_1, \ldots, v_t \in B_k$ be distinct vertices and C be a tight cycle in the hypergraph $K^{(k-1)}(B_1, \ldots, B_{k-1})$ with vertex-sequence $(u_{1,1}, \ldots, u_{1,k-1}, \ldots, u_{t,1}, \ldots, u_{t,k-1})$. Denote by $\mathrm{e}_{s,\mathrm{i}}$ the edge in C starting at $u_{s,\mathrm{i}}$ and suppose that*

(i) $\mathrm{e}_{s,\mathrm{i}} \in \operatorname{Lk}(v_s; B_1, \ldots, B_{k-1})$ *for every $s \in [t]$ and every $\mathrm{i} \in [k-1]$ and*

(ii) $\mathrm{e}_{s,1} \in \operatorname{Lk}(v_{s-1}; B_1, \ldots, B_{k-1})$ *for every $s \in [t]$ (here $v_0 := v_t$).*

Then, $(v_1, u_{1,1}, \ldots, u_{1,k-1}, \ldots, v_t, u_{t,1}, \ldots, u_{t,k-1})$ is the vertex-sequence of a tight cycle in H.

We will proceed in three steps.

Step 1 (Divide into blocks). By Lemma 4.3.10, there is some $C = C(\varepsilon) \in \mathbb{N}$ and a partition $\mathcal{P}$ of all but C $(k-1)$-graphs from $\{\, \mathrm{Lk}(v; B_1, \ldots, B_{k-1}) : v \in B_k\}$ into *blocks* of size 4 with $\mathrm{e}(\mathcal{H}) := |\bigcap_{H \in \mathcal{H}} E(H)| \geq \delta_1 |B_1| \cdots |B_{k-1}|$ for every $\mathcal{H} \in \mathcal{P}$. Remove the C leftover vertices from B_k.

Step 2 (Cover blocks by paths). Think of every block $\mathcal{H}$ now as a graph with edges $E(\mathcal{H}) := \bigcap_{H \in \mathcal{H}} E(H)$. By Lemma 4.3.1, for each $\mathcal{H} \in \mathcal{P}$, there is a subgraph $\mathcal{H}' \subset \mathcal{H}$ such that $\mathrm{e}(\mathcal{H}') \geq \delta_2 |B_1| \cdots |B_{k-1}|$ with the same property as in Lemma 4.3.1. By deleting all the edges $\mathcal{H} \setminus \mathcal{H}'$ we may assume that $\mathcal{H}$ itself has this property. Define an auxiliary graph G with $V(G) = \mathcal{P}$ and $\{\mathcal{H}_1, \mathcal{H}_2\} \in E(G)$ if and only if $\mathrm{e}(\mathcal{H}_1 \cap \mathcal{H}_2) \geq \delta_3 |B_1| \cdots |B_{k-1}|$. Similarly as in the proof of Lemma 4.3.10, we conclude that $\alpha(G) \leq 2/\delta_2$, and hence $V(G)$ can be covered by $2/\delta_1$ vertex-disjoint paths (by Pósa's theorem[3]).

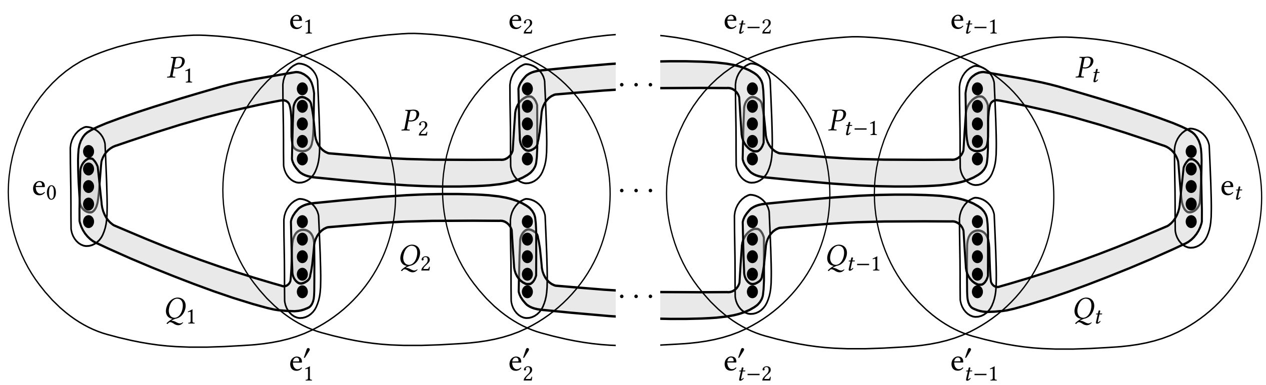


Figure 4.2: Finding cycles in a path of blocks.

Step 3 (Lift to tight cycles). We will lift each of these paths of blocks to a tight cycle in the hypergraph $K^{(k-1)}(B_1, \ldots, B_{k-1})$ of the desired form. Let $\mathbf{P} = (\mathcal{H}_1, \ldots, \mathcal{H}_t)$ be one of the paths. Refer to Figure 4.2 for a helpful picture of the following proof. Choose disjoint edges $\mathrm{e}_0 = \left\{x_1^{(0)}, \ldots, x_{k-1}^{(0)}\right\} \in E(\mathcal{H}_1)$ and $\mathrm{e}_t = \left\{x_1^{(t)}, \ldots, x_{k-1}^{(t)}\right\} \in E(\mathcal{H}_t)$. For each $s \in [t-1]$, further choose two edges $\mathrm{e}_s = \left\{x_1^{(s)}, \ldots, x_{k-1}^{(s)}\right\} \in E(\mathcal{H}_s) \cap E(\mathcal{H}_{s+1})$ and $\mathrm{e}'_s = \left\{y_1^{(s)}, \ldots, y_{k-1}^{(s)}\right\} \in E(\mathcal{H}_s) \cap E(\mathcal{H}_{s+1})$ so that all chosen edges are pairwise disjoint. We identify $x_{\mathrm{i}}^{(0)} = y_{\mathrm{i}}^{(0)}$ and $x_{\mathrm{i}}^{(s)} = y_{\mathrm{i}}^{(s)}$ for every $\mathrm{i} \in [k-1]$, and $\mathrm{e}_0 = \mathrm{e}'_0$ and $\mathrm{e}_t = \mathrm{e}'_t$. Assume without loss of generality, that $x_{\mathrm{i}}^{(s)} \in B_{\mathrm{i}}$ for every $\mathrm{i} \in [k-1]$ and all $s = 0, \ldots, t$.

By construction, there is for every $s \in [t]$ a tight path $P_s \subset \mathcal{H}_s$ of length $2k-3$ which starts at and $(x_2^{(s-1)}, \ldots, x_{k-1}^{(s-1)})$, ends at $(x_1^{(s)}, \ldots, x_{k-2}^{(s)})$ and (internally) avoids all previously used vertices. Similarly, there is for every $s \in [t]$ a tight path $Q_s \subset \mathcal{H}_s$ of length $2k-3$ which starts at and $(y_1^{(s)}, \ldots, y_{k-2}^{(s)})$, ends at $(y_2^{(s-1)}, \ldots, y_{k-1}^{(s-1)})$ and (internally) avoids all previously used vertices. It is now easy to check that the tight cycle in $K^{k-1}(B_1, \ldots, B_{k-1})$ with edge sequence

$$\mathrm{e}'_0 = \mathrm{e}_0, P_1, \mathrm{e}_1, P_2, \mathrm{e}_2, \ldots, P_t, \mathrm{e}_t = \mathrm{e}'_t, Q_t, \ldots, \mathrm{e}_1, Q_1, \mathrm{e}'_0 = \mathrm{e}_0$$

has the desired properties to apply observation 4.3.11.

□

[3] Pósa's theorem actually allows cycles, but for technical reasons we need to work with paths.

4.3.4 Proof of Theorem 4.1.3.

Fix $\alpha, r, n \in \mathbb{N}$ and a k-uniform hypergraph G with $\alpha(G) \le \alpha$. Choose constants $0 < \beta, \gamma, \varepsilon \ll \max\{\alpha, r, k\}^{-1}$ such that $\gamma = \gamma(k, r, \varepsilon)$ works for Lemma 4.3.9 and $\beta = \beta(k, r, \alpha)$ works for Lemma 4.3.6. The proof proceeds in α steps, where the initial step does $k-1$ steps at once.

Step 1, ..., k-1. By Lemma 4.3.6, there is some $B \subset [n]$ of size βn with an absorber $A_{k-1} \subset [n]$. Partition B into $k-1$ sets $B_1^{(k-1)}, \dots, B_{k-1}^{(k-1)}$ of equal sizes. Remove monochromatic tight cycles of maximal lengths from $[n] \setminus (A_{k-1} \cup B)$ until the set R_{k-1} of uncovered vertices in $[n] \setminus (A \cup B)$ satisfies $|R_{k-1}| \le \gamma |B_1^{(k-1)}|$. This is possible, since the Ramsey number of the tight cycle is linear (see section 4.3.2 for more details). Let $R'_{k-1} \subset R_{k-1}$ be the set of vertices v with $|\mathrm{Lk}(v; B_1^{(k-1)}, \dots, B_{k-1}^{(k-1)})| < \varepsilon |B_1^{(k-1)}| \cdots |B_{k-1}^{(k-1)}|$ and let $R''_{k-1} = R_{k-1} \setminus R'_{k-1}$. By Lemma 4.3.9 we can find c_{k-1} vertex-disjoint tight cycles in $B_1^{(k-1)} \cup B_{k-1}^{(k-1)} \cup R''_{k-1}$ covering R''_{k-1}. Remove them and let $B_{\mathrm{i}}^{(k)} \subset B_{\mathrm{i}}^{(k-1)}$, $\mathrm{i} \in [k-1]$, be the set of leftover vertices.

Step j ($j = k, \dots, \alpha$)**.** Suppose we have built, during the previous $j-1$ steps, disjoint sets $B_1^{(j)}, \dots, B_{j-1}^{(j)}, R'_{j-1}$ and absorbers $A_{k-1}, \dots, A_{j-1}$. By lemma 4.3.6 there is some $B_j^{(j)} \subset R'_{j-1}$ of size $\beta |R'_j|$ with an absorber $A_j \subset R'_{j-1}$. Remove monochromatic tight cycles of maximal lengths from $R'_{j-1} \setminus (A_j \cup B_j^{(j)})$ until the set R_j of uncovered vertices in $R'_{j-1} \setminus (A_j \cup B_j^{(j)})$ satisfies $|R_j| \le \gamma |B_j^{(j)}|$. Let $R'_j \subset R_j$ be the set of vertices v with $|\,\mathrm{Lk}(v; B_{t_1}^{(j)}, \dots, B_{t_{k-1}}^{(j)})| < \varepsilon \left|B_{t_1}^{(j)}\right| \cdots \left|B_{t_{k-1}}^{(j)}\right|$ for all $1 \le t_1 < \ldots < t_{k-1} \le j$ and let $R''_j = R_j \setminus R'_j$. By ($\binom{j}{k}$ applications of) Lemma 4.3.9 we can find c_j vertex-disjoint cycles in $B_1^{(j)} \cup \ldots \cup B_j^{(j)} \cup R''_j$ covering R''_j. Remove them and let $B_{\mathrm{i}}^{(j+1)} \subset B_{\mathrm{i}}^{(j)}$, $\mathrm{i} \in [j]$, be the set of leftover vertices.

In the end we are left with disjoint sets $B_1 := B_1^{(\alpha+1)}, \dots, B_\alpha := B_\alpha^{(\alpha+1)}, B_{\alpha+1} := R'_\alpha$ and corresponding absorbers $A_{k-1}, \dots, A_\alpha$ (A_{k-1} absorbs $B_1^{(\alpha+1)}, \dots, B_{k-1}^{(\alpha+1)}$). All other vertices are covered by constantly many cycles.

We will show now that $R'_{\alpha+1} = \emptyset$ finishing the proof. In order to so, we assume the contrary and find an independent set of size $\alpha+1$. Note that $\left|B_j\right| \ge (1-\gamma) \left|B_j^{(\mathrm{i})}\right|$ for every $1 \le j \le \mathrm{i} \le \alpha$ and hence

$$\begin{aligned}
\left|\,\mathrm{Lk}\left(v; B_{\mathrm{i}_1}, \dots, B_{\mathrm{i}_{k-1}}\right)\right| &\le \varepsilon \left|B_{\mathrm{i}_1}^{(\mathrm{i}-1)}\right| \cdots \left|B_{\mathrm{i}_{k-1}}^{(\mathrm{i}-1)}\right| \\
&\le \varepsilon (1-\gamma)^{-(k-1)} \left|B_{\mathrm{i}_1}\right| \cdots \left|B_{\mathrm{i}_{k-1}}\right| \\
&\le 2\varepsilon \left|B_{\mathrm{i}_1}\right| \cdots \left|B_{\mathrm{i}_{k-1}}\right|
\end{aligned}$$

for every $\mathrm{i} \in \{k, \dots, \alpha+1\}$, $1 \le \mathrm{i}_1 < \ldots < \mathrm{i}_{k-1} < \mathrm{i}$ and $v \in B_{\mathrm{i}}$. By the following lemma, there is an independent set of size $\alpha+1$, a contradiction. □

Lemma 4.3.12. *For all $k, r \in \mathbb{N}$ there is some $\varepsilon = \varepsilon(k, r) > 0$ such that the following is true for every k-uniform hypergraph H and all non-empty, disjoint sets $B_1, \dots, B_r \subset V(H)$. If $\left|\,\mathrm{Lk}(v; B_{\mathrm{i}_1}, \dots, B_{\mathrm{i}_{k-1}})\right|$ is at most $\varepsilon \left|B_{\mathrm{i}_1}\right| \cdots \left|B_{\mathrm{i}_{k-1}}\right|$ for all $\mathrm{i} \in \{k, \dots, r\}$, $1 \le \mathrm{i}_1 < \ldots < \mathrm{i}_{k-1} < \mathrm{i}$ and $v \in B_{\mathrm{i}}$, then there is an independent transversal, i.e. an independent set $\{v_1, \dots, v_r\}$ with $v_{\mathrm{i}} \in B_{\mathrm{i}}$ for all $\mathrm{i} \in [r]$.*

We will prove the lemma for $\varepsilon(k, r) = r^{-(k-1)^2}$.

Proof. Let $\delta = r^{-(k-1)}$ and $\varepsilon = \delta^{k-1}$. Choose $v_r \in B_r$ arbitrarily and assume now that $v_r, \ldots, v_{j+1}$ are chosen for some $j \in [r-1]$. Given $s \in \{2, \ldots, k-1\}$ and $\mathbf{i} = (\mathrm{i}_1, \ldots, \mathrm{i}_k)$ with $1 \leq \mathrm{i}_1 < \ldots < \mathrm{i}_{s-1} < \mathrm{i}_s = j < \mathrm{i}_{s+1} < \ldots < \mathrm{i}_k \leq r$, define

$$\overline{B}_j(s, \mathbf{i}) := \left\{u \in B_j : \left| \mathrm{Lk}\left(v_{\mathrm{i}_k}, \ldots, v_{\mathrm{i}_{s+1}}, u; B_{\mathrm{i}_{s-1}}, \ldots, B_{\mathrm{i}_1}\right)\right| \geq \varepsilon/\delta^{k-s} \left|B_{\mathrm{i}_1}\right| \cdots \left|B_{\mathrm{i}_{s-1}}\right|\right\}.$$

Furthermore, given $\mathbf{i} = (\mathrm{i}_1, \ldots, \mathrm{i}_k)$ with $j < \mathrm{i}_2 < \ldots < \mathrm{i}_k \leq r$, define

$$\overline{B}_j(1, \mathbf{i}) := \mathrm{N}\left(v_{\mathrm{i}_k}, \ldots, v_{\mathrm{i}_2}; B_{\mathrm{i}_1}\right),$$

the neighbourhood of $\{v_{\mathrm{i}_2}, \ldots, v_{\mathrm{i}_k}\}$ in B_{i_1}. Note that, by choice of $v_r, \ldots, v_{j+1}$, we have $\left|\overline{B}_j(s, \mathbf{i})\right| < \delta \left|B_j\right|$ for every $s \in \{2, \ldots, k-1\}$ and $\left|\overline{B}_j(1, \mathbf{i})\right| < \varepsilon/\delta^{k-2} \left|B_j\right| = \delta \left|B_j\right|$. Since there are at most $\binom{r-1}{k-1} < 1/\delta$ choices for $(s, \mathbf{i})$, we can choose some $v_j \in B_j \setminus \bigcup_{s,\mathbf{i}} \overline{B}_j(s, \mathbf{i})$. Clearly, at the end of this process, $\{v_1, \ldots, v_r\}$ will be independent. $\square$

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