ROTATIONALLY SYMMETRIC SOLUTIONS TO THE CAHN-HILLIARD EQUATION

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ABSTRACT. This paper is devoted to construction of new solutions to the Cahn-Hilliard equation in $\mathbb{R}^d$. Staring from a Delaunay unduloid D_τ with parameter $\tau \in (0, \tau^*)$ we find for each sufficiently small ε a solution u of this equation which is periodic in the direction of the x_d axis and rotationally symmetric with respect to rotations about this axis. The zero level set of u approaches as $\varepsilon \to 0$ the surface D_τ. We use a refined version of the Lyapunov-Schmidt reduction method which simplifies very technical aspects of previous constructions for similar problems.

1. INTRODUCTION

The Cahn-Hilliard equation

$$
\begin{aligned}
u_t &= -\Delta(\varepsilon^2 \Delta u - F'(u)) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega, \\
\frac{\partial}{\partial \nu}(\varepsilon^2 \Delta u - F'(u)) &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
$$

(1.1)

where F is a *double-well potential*, is a model that describes the process of phase separation of two components of a binary alloy. Here $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a bounded domain represents the place where the isolation of the components takes place, and ν, as usual, denotes the outer normal on $\partial\Omega$. The function u represents the concentration of one of the components and ε is the range of intermolecular forces. The double-well potential $F(u)$ corresponds to the free energy density at low temperatures, and in this paper we will take

$$
F(u) = \frac{1}{4}\left(1 - u^2\right)^2, \quad F'(u) = u^3 - u.
$$

From now on we will denote $F'(u) = -f(u)$.

Equation (1.1) can be derived from the gradient flow of the Helmholtz-free energy functional

$$
E_\varepsilon(u) = \int_\Omega \left(F(u(x)) + \frac{1}{2}\varepsilon^2 |\nabla u(x)|^2 \right) dx
$$

(1.2)

in $H^{-1}(\Omega)$ subject to the average concentration to be constant, i.e.

$$
\frac{1}{|\Omega|}\int_\Omega u \, dx = m,
$$

(1.3)

where $m \in [-1, 1]$ (see [16], [17] for details). Note that constant functions $u \equiv \pm 1$ are minimizers of this functional subject to $m = \pm 1$.

Stationary solutions of (1.1) satisfy the Euler-Lagrange equation (with $f(u) = -F'(u)$)

$$\varepsilon^2 \Delta u + f(u) = \delta_\varepsilon \qquad \text{in } \Omega,$$

(1.4)
$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega,$$

$$\frac{1}{|\Omega|} \int_\Omega u \, dx = m$$

where δ_ε is a Lagrange multiplier.

Using Γ-convergence approach Modica [23] showed that minimizers u_ε of (1.2) subject to constraint (1.3) Γ-converge, as $\varepsilon \to 0$, to the function $1 - 2\chi_{A_0}$, where χ_{A_0} is the characteristic function of an open set $A_0 \subset \Omega$. Moreover $\partial A_0 \cap \Omega$ is locally a surface of constant mean curvature (CMC surface for short). Geometrically the set A_0 minimizes the perimeter functional $\text{Per}_\Omega(A)$ among the sets $A \subset \Omega$ whose volume is fixed. A generalisation of these results was given by Sternberg [26]. Furthermore Hutchinson and Tonegawa [18] studied limits of general critical points (1.2) and showed that their limits are locally minimal or CMC surfaces. On the other hand it is known [20] that if a set $A \subset \Omega$ is an *isolated* mimimizer of the perimeter functional subject to the constant volume constraint then there exists a sequence of minimizers u_ε of (1.2) which Γ converges to A. This result can be used to construct solutions to (1.4) at least in dimension 2 , see [8]. The most complete construction is due to Pacard and Ritoré [24] who proved the following: if M is a compact Riemannian manifold and N is a non degenerate minimal or CMC sub manifold of M which divides M into 2 disjoint components then for all sufficiently small ε there exist critical points of (1.2) whose 0 level set converges to N. The counterpart of this theory for the time dependent problem (1.1) was developed among others by Alikakos, Bates and Chen [1] who proved that as $\varepsilon \to 0$ the time evolution of interfaces is governed by the Helle-Shaw problem-of course CMC surfaces are stationary points of the flow. More detailed description of the Cahn-Hilliard flow and key spectral tools can be found for instance in [3], [5], [4], [2], [7] and the references therein. Additional examples of stationary solutions for the singular perturbation problem in a bounded domain have been constructed in [28], [27], [6].

Scaling variables $x \mapsto x/\varepsilon$ in (1.4) and letting ε tend to 0 leads in a natural way to the following problem:

(1.5)
$$\Delta u + f(u) = \delta, \quad \text{in } \mathbb{R}^d.$$

In dimension $d = 1$ there is an obvious solution of this problem when $\delta = 0$, namely the unique odd and monotonically increasing heteroclinic solution H of the ODE, which satisfies

(1.6)
$$H'' + f(H) = 0, \quad \text{in } \mathbb{R},$$
$$H(\pm\infty) = \pm 1.$$

If $\mathsf{a} \in \mathbb{R}^d$ is a unit vector and $b \in \mathbb{R}$ then the function

$$u(x) = H(\mathsf{a} \cdot x + b),$$

is also a solution of (1.5) with $\delta = 0$. When $\delta \neq 0$ there exist radially symmetric solutions to (1.5), see [25]. In both cases the level sets of the solutions are CMC surfaces, in the former case their mean curvature is 0 and in the latter case it is positive number equal to $\frac{1}{d-1}\frac{1}{R_0}$, where R_0 is the radius of the level set of the solution. The radially symmetric solutions in $\mathbb{R}^{d-1}$ can be lifted trivially to $\mathbb{R}^d$ giving solutions whose nodal sets are cylinders, which again are CMC surfaces.

Dilating of the independent variable by a (large) factor $\varepsilon^{-1} > 0$

$$\mathsf{x} \longmapsto \varepsilon^{-1}\mathsf{x},$$

we obtain an equivalent form of (1.5):

(1.7)
$$\varepsilon \Delta u + \frac{1}{\varepsilon} f(u) = \ell_\varepsilon, \quad \text{in } \mathbb{R}^d.$$

where we have denoted $\frac{\delta}{\varepsilon} = \ell_\varepsilon$. Clearly, if u_ε is a solution of (1.7) then $v(\mathsf{x}) = u_\varepsilon(\varepsilon\mathsf{x})$ is a solution of (1.5). On the other hand, if v is a solution of (1.5) then $u_\varepsilon(\mathsf{x}) = v\left(\frac{\mathsf{x}}{\varepsilon}\right)$ is a solution of (1.7). In particular this means that while phase transition of the solutions of (1.5) are of order 1, for the solutions of (1.7) they are of order ε. Thus the latter are more "concentrated". In the sequel we will focus on solving (1.7). From what we

have said above about the singular perturbation problem it is clear that level sets of these solutions should converge, as ε tends to 0, to CMC surfaces in $\mathbb{R}^3$. In fact we expect (on the basis of formal calculations in section 2.3) that the Lagrange multiplier

$$\ell_\varepsilon = -\frac{1}{2}H_\Sigma \int_{\mathbb{R}} H'(s)^2\, ds + \mathcal{O}(\varepsilon),$$

where Σ is the surface of the phase transition and H_Σ is its mean curvature.

We will now introduce a family of embedded CMC surfaces which are good candidates to be the limits of the nodal surfaces. We recall that Delaunay *unduloids* [14], [15] are a one parameter family D_τ, $\tau \in (0,1)$ of embedded, periodic CMC surfaces of revolution in $\mathbb{R}^3$. When the real parameter τ tends to 1 the surfaces D_τ approach the straight cylinder while when $\tau \to 0$ they become an array of identical spheres arranged along the x_3 axis. It turns out that Delaunay surfaces can be constructed in any dimension $d > 3$ and from now on by D_τ, $\tau \in (0,\tau_*)$ we will denote the family of Delaunay surfaces in $\mathbb{R}^d$. We note that the parameter τ_* satisfies:

$$(1.8) \qquad\qquad\qquad \tau_* = \frac{2}{(d-1)(d-2)^{(d-2)}}.$$

Again, in the limit $\tau \to \tau_*$ the surfaces D_τ approach the straight cylinder, see [19] or Section 2.1 for details.

It is convenient to "normalize" the Delaunay surface and suppose that the mean curvature of D_τ is 1 for all $\tau \in (0,\tau_*)$. We will also denote by N_τ the vector field normal to D_τ. Let us notice that the surface D_τ divides the space into two disjoint components $\Omega_\tau^\pm$, such that $\mathbb{R}^d \setminus D_\tau = \Omega_\tau^+ \cup \Omega_\tau^-$, where N_τ points towards Ω_τ^+. By changing the orientation of D_τ if necessary we can chose N_τ in such a way that Ω_τ^+ contains the x_d axis.

Our result is:

Theorem 1.1. *For all $\tau \in (0,\tau_*)$ there exits $\varepsilon_\tau > 0$ such that for all $\varepsilon \in (0,\varepsilon_\tau)$ the problem*

$$(1.9) \qquad\qquad\qquad \varepsilon \Delta u + \frac{1}{\varepsilon}f(u) = \ell_\varepsilon \quad \text{in } \mathbb{R}^d$$

has a solution $u_{\tau,\varepsilon}$, which is one-periodic in the direction of the x_d-axis and rotationally symmetric with respect to rotations about the same axis. As $\varepsilon \to 0$ we have $\ell_\varepsilon = 1 + \mathcal{O}(\varepsilon)$, and $u_{\tau,\varepsilon}$ satisfies

$$u_{\tau,\varepsilon} \to 1 \ \text{as } \varepsilon \to 0 \ \text{in } \Omega_\tau^+,$$
$$u_{\tau,\varepsilon} \to -1 \ \text{as } \varepsilon \to 0 \ \text{in } \Omega_\tau^-,$$

uniformly over compacts.

Remark 1.1. *In this paper we took $f(u) = u - u^3$, which is the standard nonlinearity for the Cahn-Hilliard equation. Theorem 1.1 holds for more general nonlinearities of bistable, balanced type, namely $f \in C^3$ such that $f(u) = -F'(u)$ where F is a double well, even potential with non degenerate wells at ± 1. Rather straightforward modifications required in the proof of the more general setting are left to the reader.*

We will explain now the implementation of the Lyapunov-Schmidt reduction we used in this paper and discuss the differences between our approach and the older implementations which can be found in [24] and [11], [12]. Let us first recall the standard Lyapunov-Schmidt reduction method in its abstract version (see [9]). Given Banach spaces X, Y and a linear operator $A\colon X \to Y$ and a continuous, nonlinear operator $N\colon X \to Z$, we are to solve the problem:

$$(1.10) \qquad\qquad\qquad Ax - N(x) = 0.$$

Let

$$\mathcal{N}(A) = Y \subset X, \qquad \mathcal{R}(A) = W \subset Z,$$

and let π_Y, π_W be the projections on the corresponding subspaces. There exists a bounded linear operator $K\colon W \to \mathcal{R}(I - \pi_Y)$ (the right inverse of A) such that $AK = I$ on W and $KA = I - \pi_Y$, and moreover the

equation (1.10) is equivalent to the equation

$$x = y + z, \qquad y \in Y, \quad z \in \mathcal{R}(I - \pi_Y)$$

(1.11)
$$z - K\pi_W N(y + z) = 0,$$

$$(I - \pi_W)N(y + z) = 0.$$

In applications the Lyapunov-Schmidt method consists of reducing (1.10) to (1.11), solving the first equation for z with y given (which usually can be done by a fixed point argument) and replacing this solution in the second equation to obtain *the reduced problem*

$$(1.12) \qquad\qquad (I - \pi_W)N(y + z(y)) = 0.$$

In practice several complications may arise and we will illustrate this considering a related to our problem which was treated by Pacard and Ritoré [24], and in many aspects it is similar to problem we consider in this paper. Let M be a compact, closed manifold of dimension n and $N \subset M$ a minimal $n - 1$ dimensional sub manifold which divides M into two disjoint components. Consider the problem

$$(1.13) \qquad\qquad \varepsilon^2 \Delta_M u + u(1 - u^2) = 0, \qquad \text{on } M.$$

We say that N is non degenerate if the Jacobi operator of N

$$J_N = \Delta_N + |A_N|^2 + \mathrm{Ric}_g(\nu_N, \nu_N)$$

has empty kernel (Δ_N is the Laplace-Beltrami operator on N, $|A_N|^2$ is the norm of the second fundamental form, Ric_g is the Ricci tensor on M and ν_N is the normal vector to N). The result proven in [24] is: given a non degenerate, minimal sub manifold N of M for each sufficiently small ε there exists a solution u_ε of (1.13) such that the zero level set of u_ε approaches N as $\varepsilon \to 0$. Moreover, u_ε converges to ± 1 uniformly over compacts of the two disjoint components of $M \setminus N$.

Let us explain now the implementation of the Lyapunov-Schmidt reduction in [24]. It is expected that for $x \in M$ near N we should have $u_\varepsilon(x) = H(\varepsilon^{-1}\mathrm{dist}\,(x, N)) + \varphi$, where $\mathrm{dist}\,(\cdot, N)$ is the signed distance function on M, H is the unique odd, monotonically increasing solution of $-H'' = H(1 - H^2)$ in $\mathbb{R}$ and φ is a small perturbation. The problem to solve for φ amounts to inverting the linearized operator around $H(\varepsilon^{-1}\mathrm{dist}\,(x, N))$ which has form

$$L = \Delta_M + f'(H(\varepsilon^{-1}\mathrm{dist}\,(\cdot, N))).$$

It is known that the norm of L^{-1} is large due to local translational invariance of the problem. Thus we need to perturb N as well. To describe this perturbation we consider a manifold N_h to be a normal graph over N described by a smooth and small function $h \colon N \to \mathbb{R}$. Furthermoe we let $t_h(x) = \mathrm{dist}\,(x, N_h)$ to be the signed distance from N_h. Then we look for a solution of the form

$$u = H\left(\frac{t_h}{\varepsilon}\right) + \varphi.$$

Now both h and φ are unknowns. The problem to solve for φ is

$$L_h \varphi = \mathcal{F}(h, \varphi),$$

where L_h is the linearized operator around $H\left(\frac{t_h}{\varepsilon}\right)$. The Lyapunov-Schmidt reduction strategy amounts to projection of the above equation onto the function $H'\left(\frac{t_h}{\varepsilon}\right)$ and its complement, denote this last projection by π_h. This leads to a problem for φ

$$\pi_h L_h \varphi = \pi_h \mathcal{F}(h, \varphi),$$

which we solve first for a given h, and the problem for h

$$(1.14) \qquad\qquad J_{N_h} h = \mathcal{G}(h),$$

which we solve next (J_{N_h} is the Jacobi operator of N_h). Let us discuss (1.14). We notice that the expression of J_{N_h} in local coordinates will depend in general on h and its derivatives up to order 3, while the Jacobi operator is itself only a second order operator. This loss of regularity was dealt with in [24] using a regularisation procedure. In a series of papers [11], [12], [13] del Pino, Kowalczyk and Wei introduced a slightly different approach to circumvent this problem. It amounts to considering perturbation in the normal direction of the

fixed manifold N so that $u = H\left(\frac{t+h}{\varepsilon}\right) + \ldots$, where now t is the signed distance from N and h is a smooth, unknown function defined on N. Equation (1.14) takes form

$$(1.15) \qquad\qquad J_N h = \mathcal{G}(h),$$

and the problem of the loss of regularity is thus avoided. The problem is now reduced to finding a fixed point of $J_N \circ \mathcal{G}(h)$, using for example Banach fixed point theorem. To do this we need to know that $\mathcal{G}$ is at least Lipschitz in h. In both implementations of the Lyapunov-Schidt reduction described above this is rather complicated technical point since $\mathcal{G}$ depends in a non explicit, non local and non linear way on h. This is mainly due to the fact that the linearized operator L_h still depends on h through the potential $f'(H(\frac{t+h}{\varepsilon}))$.

In this paper we propose still another modification to the method. The idea is simple: instead of working with an approximation of the form $u = H\left(\frac{t+h}{\varepsilon}\right) + \ldots$ with h unknown we will improve the initial approximation to $w(t,y) = H\left(\frac{t}{\varepsilon}\right) + \ldots$, t being the signed distance to N and $y \in N$ in such a way that we do not need to "move" N anymore. In other words h will be determined with some sufficient precision before setting up the Lyapunov-Schmidt reduction, which with this modification will look like the abstract setting described at the beginning. This way we avoid both the loss of regularity and technical difficulties due to complicated character of the nonlinear function $\mathcal{G}(h)$. This is described in detail in section 3.1.

2. Preliminaries

2.1. **The surfaces of Delaunay.** The Delaunay unduloids D_τ, $\tau \in (0, \tau_*)$ are CMC surfaces of revolution in $\mathbb{R}^d$. Thus for instance in $\mathbb{R}^3$ one can parametrize them in the form

$$\mathbf{x}(t, \theta) = (\rho(t)\cos\theta, \rho(t)\sin\theta, t),$$

where $\rho(t)$ solves

$$\rho_{tt} - \frac{1}{\rho}(1 + \rho_t^2) - (1 + \rho_t^2)^{3/2} = 0.$$

However, in this paper we will use mostly isothermal coordinates of $D_\tau \subset \mathbb{R}^d$:

$$(2.1) \qquad\qquad X_\tau(s, \Theta) = \frac{1}{2}(\tau e^{\sigma_\tau(s)}\Theta, \kappa_\tau(s)), \quad (s, \Theta) \in \mathbb{R} \times S^{d-2},$$

where functions $(\sigma_\tau, \kappa_\tau)$ are the unique solutions of the following system of ODEs:

$$(2.2) \qquad \begin{aligned} (\partial_s\sigma_\tau)^2 + \frac{1}{4}\tau^2(e^{\sigma_\tau} + e^{(2-d)\sigma_\tau})^2 &= 0, \quad \partial_s\sigma_\tau(0) = 0, \quad \sigma_\tau(0) < 0, \\ \partial_s\kappa_\tau - \frac{1}{4}\tau^2(e^{\sigma_\tau} + e^{(2-d)\sigma_\tau}) &= 0, \quad \kappa_\tau(0) = 0. \end{aligned}$$

We will now summarize some basic facts about the Delaunay surfaces and their isothermal parametrization. The function σ_τ is periodic, and consequently the surfaces D_τ are one-periodic along the x_d-axis: namely if $2T_\tau$ denotes the minimal period then

$$D_\tau = D_\tau + 2T_\tau \mathbf{e}_d.$$

Clearly we have the relation

$$T_\tau = \frac{1}{4}\kappa_\tau(2s_\tau),$$

where $2s_\tau$ is the minimal period of σ_τ.

The Jacobi operator $\mathcal{J}_\tau$ of D_τ is defined by:

$$(2.3) \qquad\qquad \mathcal{J}_\tau := \Delta_{D_\tau} + |A_\tau|^2,$$

where Δ_{D_τ} is the Laplace-Beltrami operator on D_τ and $|A_\tau|^2$ is the square of the norm of the second fundamental form of D_τ. The Jacobi operator is of fundamental importance in this paper and to understand well its properties we will first consider the special case $d = 3$. In the isothermal coordinates $(s, \theta) \in \mathbb{R} \times S^1$ its expression is given by:

$$(2.4) \qquad\qquad L_\tau = \frac{1}{\tau^2 e^{2\sigma_\tau}}\left\{\partial_s^2 + \partial_\theta^2 + \tau^2\cosh(2\sigma_\tau)\right\}.$$

The Jacobi fields on D_τ, which are elements of the kernel of L_τ are of three types:

(1) *Jacobi fields arising from infinitesimal translations.* For any $\mathbf{e} \in \mathbb{R}^3$, $|\mathbf{e}| = 1$ the constant Killing field associated to translations

$$\mathbf{x} \longmapsto \mathbf{e}$$

induces the following Jacobi fields

$$\Phi_\tau^{T,\mathbf{e}} = \mathbf{e} \cdot N_\tau,$$

where N_τ is the unit normal vector to D_τ. The coordinate vectors $\mathbf{e}_j$, $j = 1, 2, 3$ generate three linearly independent Jacobi fields $\Phi_\tau^{T,\mathbf{e}_j}$ corresponding to translations of D_τ in the directions of the coordinate axis. We note that in the isothermal coordinates

$$\Phi_\tau^{T,\mathbf{e}_3} = \Phi_\tau^{T,\mathbf{e}_3}(s), \quad \Phi_\tau^{T,\mathbf{e}_j} = \Phi_\tau^{T,\mathbf{e}_j}(s,\theta), \quad j = 1, 2.$$

It is important to notice that the Jacobi fields $\Phi_\tau^{T,\mathbf{e}_j}$ are bounded.

(2) *Jacobi fields arising from infinitesimal rotations.* Let $\mathbf{e} \in \mathbb{R}^3$, $|\mathbf{e}| = 1$ be such that $\mathbf{e} \cdot \mathbf{e}_3 = 0$. The Killing vector field corresponding to the rotation about the vector $\mathbf{e}$ is:

$$\mathbf{x} \longmapsto (\mathbf{x} \cdot \mathbf{e})\mathbf{e}_3 - (\mathbf{x} \cdot \mathbf{e}_3)\mathbf{e}.$$

We define the Jacobi field associated to this vector field by:

$$\Phi_\tau^{R,\mathbf{e}} = [(\mathbf{x} \cdot \mathbf{e})\mathbf{e}_3 - (\mathbf{x} \cdot \mathbf{e}_3)\mathbf{e}] \cdot N_\tau.$$

There are clearly two linearly independent Jacobi fields associated to the rotations. They are:

$$\Phi_\tau^{R,\mathbf{e}_1}, \quad \Phi_\tau^{R,\mathbf{e}_2},$$

and they correspond to rotations about the coordinate axis. Note that in isothermal coordinates functions $\Phi_\tau^{R,\mathbf{e}_j}$, $j = 1, 2$ grow linearly as functions of s.

(3) *Jacobi field associated with the variation of the Delaunay parameter.* We define:

$$\Phi_\tau^D = -\partial_\tau X_\tau \cdot N_\tau.$$

This Jacobi field is somewhat harder to write explicitly however it can be shown that the function $\Phi_\tau^D(s)$ is linearly growing.

In summary, the Jacobi operator L_τ has at least 6 explicit Jacobi fields which are either linearly growing or bounded. By a result of Mazzeo and Pacard [22] we know that these are *all* Jacobi fields with at most linear growth. To explain this let us observe that by separation of variables the equation $J_\tau \varphi = 0$ separates into a sequence of problems

$$L_{\tau,j}\varphi = 0, \quad L_{\tau,j} = \frac{1}{\tau^2 e^{2\sigma_\tau}} \left\{ \partial_s^2 + \tau^2 \cosh(2\sigma_\tau) - j^2 \right\}, \quad |j| = 0, 1, \ldots.$$

Then we have:

Proposition 2.1 ([22]). *The homogeneous problem $L_{\tau,j}\varphi = 0$ has the following solutions:*

(1) *one periodic and one linearly growing solution when $j = 0$ or $|j| = 1$;*
(2) *two solutions $\varphi_{\tau,j}^\pm(s)$ which satisfy:*

$$\varphi_{\tau,j}^\pm(s + s_\tau) = e^{\pm\zeta_{\tau,j} s_\tau} \varphi_{\tau,j}^\pm(s),$$

with

$$\gamma_{\tau,j} = \operatorname{Re} \zeta_{\tau,j} > 0,$$

when $|j| > 2$. The numbers $\zeta_{\tau,j}$ are the indicial roots of the operators $L_{\tau,j}$ and correspond to the behavior of the solutions of the homogeneous problem at $\pm\infty$.

This basic facts can be generalized for the Jacobi operator of Delaunay surfaces in $\mathbb{R}^d$, $d > 3$. We will summarize them now and refer the reader to [19] for details. We have the following at most linearly growing Jacobi fields:

(1) The are d bounded, periodic Jacobi fields arising from infinitesimal translations. They will be denoted by $\Phi_\tau^{T,\mathbf{e}_j}$, $j = 1, \ldots, d$.

(2) There are $d - 1$ Jacobi fields arising from infinitesimal rotations in the direction of x_j axis, $j = 1, \ldots, d - 1$. We will denote them by $\Phi_\tau^{R,\mathbf{e}_j}$, $j = 1, \ldots, d - 1$. and they correspond to rotations about the coordinate axis. Note that in isothermal coordinates functions $\Phi_\tau^{R,\mathbf{e}_j}$, grow linearly as functions of s.

(3) There is one Jacobi field associated with the variation of the Delaunay parameter $\Phi_\tau^D = -\partial_\tau X_\tau \cdot N_\tau$, and it is linearly growing.

2.2. Fermi coordinates and shifted Fermi coordinates near a CMC surface.

Let Σ be an embedded CMC surface in $\mathbb{R}^d$ and let H_Σ denote its mean curvature. By N we will denote its unit normal. We will assume that there exists a tubular neighborhood $\mathcal{N}_\delta$ of Σ of width 2δ in which we can introduce local system of coordinates (Fermi coordinates) $(y, z) \in \Sigma \times (-\delta, \delta)$ by setting:

$$\mathbf{x} \longmapsto (y, z), \quad \text{where } \mathbf{x} = y + zN(y).$$

We suppose that this map, which we denote by Y, is a diffeomorphism from $\mathcal{N}_\delta$ to $\Sigma \times (-\delta, \delta)$ whenever δ is taken sufficiently small. In the sequel we will use the inverse of this map

$$Y^{-1} \colon \Sigma \times (-\delta, \delta) \longrightarrow \mathcal{N}_\delta$$
$$(y, z) \longmapsto \mathbf{x}.$$

Given a function $w \colon \mathcal{N}_\delta \to \mathbb{R}^d$ we define its pullback Y^*w to $\Sigma \times (-\delta, \delta)$ by the diffeomorphism Y as:

$$Y^*w(y, z) = w \circ Y^{-1}(y, z).$$

For technical reasons we will chose later the size of the tubular neighbourhood δ depending on ε but for now on we just take δ small.

Next we will define *shifted* Fermi coordinates. To do this we let $h \colon \Sigma \to \mathbb{R}$ be a given smooth function such that the map

$$\mathbf{x} \longmapsto (y, t), \quad \text{where } \mathbf{x} = y + (t + h(y)) N(y),$$

is a diffeomorphism from $\mathcal{N}_\delta$ into $\Sigma \times (-\delta, \delta)$. We will denote this map by Y_h and by Y_h^{-1} we will denote its inverse, finally by Y_h^*w we will denote the pullback of $w \colon \mathcal{N}_\delta \to \mathbb{R}^d$ by Y_h:

$$Y_h^*w(y, t) = w \circ Y_h^{-1}(y, t).$$

It will be convenient to have at hand expressions for the Laplacian in Fermi and shifted Fermi coordinates. To derive them by Σ_z we will denote the surface $\Sigma + zN$ i.e. the original surface Σ shifted in the direction of the normal by z. Locally near Σ we have

$$\Delta = \Delta_{\Sigma_z} + \partial_z^2 - H_{\Sigma_z}\partial_z.$$

We denote $z\mathbb{B}_{\Sigma,z} = \Delta_{\Sigma_z} - \Delta_\Sigma$. The operator $\mathbb{B}_{\Sigma,z}$ is a second order differential operator. To expand the curvature term we use the well known formula:

$$H_{\Sigma_z} = \sum_{j=1}^{d-1} \frac{\mathbf{k}_j}{1 - z\mathbf{k}_j} = H_\Sigma + z|A_\Sigma|^2 + z^2 \sum_{j=1}^{d-1} \mathbf{k}_j^3 + \mathcal{O}(z^3) = H_\Sigma + z|A_\Sigma|^2 + z^2\mathbb{Q}_{\Sigma,z}.$$

where $\mathbf{k}_j$ are the principal curvatures of Σ. In summary we have:

$$\Delta = \Delta_\Sigma + \partial_z^2 - (H_\Sigma + z|A_\Sigma|^2)\partial_z + z\mathbb{B}_{\Sigma,z} + z^2\mathbb{Q}_{\Sigma,z}.$$

The reason we expanded the Laplacian in this way will become clear later on. From this it is easy to obtain a formula for the Laplacian in the shifted Fermi coordinates:

$$(2.1) \qquad \Delta = \Delta_\Sigma + (1 + |\nabla_\Sigma h|^2)\partial_t^2 - (H_\Sigma + \Delta_\Sigma h + (t + h)|A_\Sigma|^2)\partial_t + (t + h)\mathbb{B}_{\Sigma,t+h} + (t + h)^2\mathbb{Q}_{\Sigma,t+h}.$$

Anticipating the content of the next section we introduce the stretched shifted Fermi coordinate

$$\mathtt{t} = \frac{t}{\varepsilon}, \quad \mathtt{y} = y.$$

Formal consideration will show that an approximate solution w_ε of the Cahn-Hilliard can be obtained if we assume that it is a function of the form:

$$Y_h^* w_\varepsilon(y, t) = H\left(\frac{t}{\varepsilon}\right) + o(1),$$

where H is the heteroclinic solution of (1.6)

As before we have a diffeomorphism $Y_{\varepsilon,h}$ and its inverse $Y_{\varepsilon,h}^{-1}\colon \Sigma \times (-\frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon}) \to \mathcal{N}_\delta$, and for any function $w\colon \mathcal{N}_\delta \to \mathbb{R}^k$ we define its pullback by $Y_{\varepsilon,h}$ by:

$$Y_{\varepsilon,h}^* w(\mathsf{y}, \mathsf{t}) = w \circ Y_{\varepsilon,h}^{-1}(\mathsf{y}, \mathsf{t}).$$

Taking onto account formula (2.1) we get

(2.2) $\Delta = \Delta_\Sigma + \varepsilon^{-2}(1 + |\nabla_\Sigma h|^2)\partial_\mathsf{t}^2 - \varepsilon^{-1}(H_\Sigma + \Delta_\Sigma h + (\varepsilon\mathsf{t} + h)|A_\Sigma|^2)\partial_\mathsf{t} + (\varepsilon\mathsf{t} + h)\mathbb{B}_{\Sigma,\varepsilon\mathsf{t}+h} + (\varepsilon\mathsf{t} + h)^2\mathbb{Q}_{\Sigma,\varepsilon\mathsf{t}+h}.$

2.3. **Formal expansion of the solution of the Cahn-Hilliard equation concentrating on Σ.** For the purpose of formal calculations we will assume that the the solution of (1.7) near Σ is a function w, which depends on the stretched and shifted Fermi coordinates (y, t), in the following way

(2.3) $$Y_{\varepsilon,h}^* w(\mathsf{y}, \mathsf{t}) = U(\mathsf{t}) + \varepsilon^2 \psi_0(\mathsf{y}, \mathsf{t}),,$$

for some functions U and ψ_0 which we will determine. Moreover, we will assume that

$$h = \varepsilon^2 h_0,$$

where h_0 is a constant to be chosen.

To determine U and ψ_0 we write the error $N_\varepsilon(w) - \ell_\varepsilon := \varepsilon\Delta w + \frac{1}{\varepsilon}w(1 - w^2) - \ell_\varepsilon$ in the local coordinates

(2.4)
$$\begin{aligned}
N_\varepsilon(w) - \ell_\varepsilon = \varepsilon\Big\{ & \varepsilon^{-2}\partial_\mathsf{t}^2 U - \varepsilon^{-1}H_\Sigma\partial_\mathsf{t} U + \varepsilon^{-2}f(U) - \varepsilon^{-1}\ell_\varepsilon \\
& + \partial_\mathsf{t}^2\psi_0 - \varepsilon H_\Sigma\partial_\mathsf{t}\psi_0 + f'(U)\psi_0 - (\mathsf{t} + \varepsilon h_0)|A_\Sigma|^2\partial_\mathsf{t} U \\
& + \frac{1}{\varepsilon}f(w) - \frac{1}{\varepsilon}f(U) - \varepsilon f'(U)\psi_0 \\
& + \varepsilon^2\Delta_\Sigma\psi_0 - \varepsilon^2(\mathsf{t} + \varepsilon h_0)|A_\Sigma|^2\partial_\mathsf{t}\psi_0) \\
& + \big((\varepsilon\mathsf{t} + h)\mathbb{B}_{\Sigma,\varepsilon\mathsf{t}+h} + (\varepsilon\mathsf{t} + h)^2\mathbb{Q}_{\Sigma,\varepsilon\mathsf{t}+h}\big)(U + \varepsilon^2\psi_0)\Big\}.
\end{aligned}$$

In order to get as small as possible this approximation we have to get rid of the first three terms of the right hand side of the expression above, since they show the lower powers in ε. To write things compactly let:

$$S_0(w) := \partial_\mathsf{t}^2 w - \varepsilon H_\Sigma\partial_\mathsf{t} w + f(w),$$
$$L_0 w := \partial_\mathsf{t}^2 w - \varepsilon H_\Sigma\partial_\mathsf{t} w + f'(U)w.$$

With this notation and the ansatz (2.3) we can write the problem in the form:

$$S_0(U) + \varepsilon^2 L_0\psi_0 + Q_0(U + \varepsilon^2\psi_0) = \varepsilon\ell_\varepsilon.$$

where $Q(U + \varepsilon^2\psi_0)$ represents the rest of the terms in (2.4), and we also have to determine the Lagrange multiplier ℓ_ε.

Thus we take the Lagrange multiplier ℓ_ε to be a number such that the following ODE has a unique, monotonically increasing solution U

(2.5)
$$\begin{aligned}
S_0(U) = U'' - \varepsilon H_\Sigma U' + f(U) &= \varepsilon\ell_\varepsilon, \quad \text{in } \mathbb{R}, \\
f(U(\pm\infty)) &= \varepsilon\ell_\varepsilon.
\end{aligned}$$

This function is easy to find by perturbing the heteroclinic solution H.

Note that since f is odd symmetric we have $U(\pm\infty) = \pm 1 + \sigma_\varepsilon$, where

$$f(\pm 1 + \sigma_\varepsilon) = \varepsilon\ell_\varepsilon.$$

Also, we have

$$\ell_\varepsilon = \ell_0 + \mathcal{O}(\varepsilon), \quad \ell_0 = -\frac{1}{2}H_\Sigma\int_{\mathbb{R}} H'(s)^2\, ds,$$

and
$$U(\mathtt{t}) = H(\mathtt{t}) + \mathcal{O}(\varepsilon).$$

Next, we will determine the $\mathcal{O}(\varepsilon^2)$ correction ψ_0. Ignoring terms of order $\mathcal{O}(\varepsilon^3)$ we get the following equation to solve:

$$(2.6) \qquad \partial_\mathtt{t}^2 \psi_0 - \varepsilon H_\Sigma \partial_\mathtt{t} \psi_0 + (1 - 3U^2)\psi_0 = (\mathtt{t} + \varepsilon h_0)|A_\Sigma|^2 \partial_\mathtt{t} U.$$

It is convenient to consider, more generally, an ODE (with the right hand side possibly depending on the variable $\mathtt{y}$) of the form:

$$(2.7) \qquad \partial_\mathtt{t}^2 \varphi - \varepsilon H_\Sigma \partial_\mathtt{t} \varphi + (1 - 3U^2)\varphi = g(\mathtt{y}, \mathtt{t}).$$

A solution of this problem can be found by the variation of parameters formula. Indeed, the fundamental set of the ODE is spanned by the functions

$$(2.8) \qquad \partial_\mathtt{t} U = \mathcal{O}\big((\cosh \mathtt{t})^{\eta^\pm}\big), \quad W(\mathtt{t}) = \mathcal{O}\big((\cosh \mathtt{t})^{\nu^\pm}\big), \quad \mathtt{t} \to \pm\infty.$$

with

$$(2.9) \qquad \eta^\pm = \frac{1}{2}\Big(\varepsilon H_\Sigma - \sqrt{-4\iota(\pm\infty) + \varepsilon^2 H_\Sigma^2}\Big), \quad \nu^\pm = \frac{1}{2}\Big(\varepsilon H_\Sigma + \sqrt{-4\iota(\pm\infty) + \varepsilon^2 H_\Sigma^2}\Big),$$

and

$$\iota(\infty) = 1 - 3(1 + \sigma_\varepsilon)^2, \quad \iota(-\infty) = 1 - 3(-1 + \sigma_\varepsilon)^2.$$

We can assume that the Wronskian at 0 is 1. If the right hand side of (2.7) satisfies

$$(2.10) \qquad \int_\mathbb{R} g(\mathtt{t}, \mathtt{y}) \partial_\mathtt{t} U(\mathtt{t}) e^{-\varepsilon H_\Sigma \mathtt{t}} \, d\mathtt{t} = 0, \quad \forall \mathtt{y} \in \Sigma$$

we can write $\varphi = \mathcal{G}(g)$ where

$$(2.11) \qquad \mathcal{G}(v)(\mathtt{t}, \mathtt{y}) = -\partial_\mathtt{t} U(\mathtt{t}) \int_0^\mathtt{t} W(s) e^{-\varepsilon H_\Sigma s} v(s, \mathtt{y}) \, ds + W(\mathtt{t}) \int_{-\infty}^\mathtt{t} \partial_s U(s) e^{-\varepsilon H_\Sigma s} v(s, \mathtt{y}) \, ds.$$

Note that the orthogonality condition (2.10) guarantees that the function $\mathcal{G}(g)$ is exponentially decaying whenever g is exponentially decaying (in $\mathtt{t}$). To be more precise let us assume for instance that

$$|g(\mathtt{y}, \mathtt{t})|(\cosh \mathtt{t})^\mu \le C,$$

with $\mu \in (\eta + \varepsilon H_\Sigma, -\eta]$, where $\eta = \max\{\eta^+, \eta^-\} < 0$. Then we have

$$|\varphi(\mathtt{y}, \mathtt{t})|(\cosh \mathtt{t})^\mu \le C,$$

as well.

Using this we can determine the function ψ_0. To do this we need to chose h_0 such that (2.10) is satisfied, in other words:

$$\int_\mathbb{R} (\mathtt{t} + \varepsilon h_0) \partial_\mathtt{t} U(\mathtt{t})^2 e^{-\varepsilon H_\Sigma \mathtt{t}} \, d\mathtt{t} = 0,$$

hence

$$h_0 = \frac{-\int_\mathbb{R} \mathtt{t} \partial_\mathtt{t} U(\mathtt{t})^2 e^{-\varepsilon H_\Sigma \mathtt{t}} \, d\mathtt{t}}{\varepsilon \int_\mathbb{R} \partial_\mathtt{t} U(\mathtt{t})^2 e^{-\varepsilon H_\Sigma \mathtt{t}} \, d\mathtt{t}} = \mathcal{O}(1),$$

With this choice we define:

$$\psi_0(\mathtt{y}, \mathtt{t}) = \mathcal{G}\big((\mathtt{t} + \varepsilon h_0)\partial_\mathtt{t} U\big)|A_\Sigma|^2.$$

Note that we have:

$$(|\mathtt{t}| + \varepsilon|h_0|)|\partial_\mathtt{t} U(\mathtt{t})|(\cosh \mathtt{t})^\mu < C, \quad 0 < \mu < -\eta,$$

and as a consequence

$$|\psi_0(\mathtt{t}, \mathtt{y})|(\cosh \mathtt{t})^\mu < C, \quad 0 < \mu < -\eta.$$

Sometimes it is convenient to derive a more refined estimate taking into account the fact that the leading order term on the right hand side is $\mathtt{t} \partial_\mathtt{t} U(\mathtt{t}) = \mathcal{O}\big(|\mathtt{t}|(\cosh \mathtt{t})^{\eta^\pm}\big)$, $|\mathtt{t}| \to \pm\infty$. Thus, we consider (2.7) assuming that

$$(2.12) \qquad |g(\mathtt{y}, \mathtt{t})|(1 + |\mathtt{t}|)^\beta (\cosh \mathtt{t})^\mu \le C,$$

with $\mu \in (\eta + \varepsilon H_\Sigma, -\eta]$, and $\beta \in \mathbb{R}$. By a simple argument we have as well:

$$(2.13) \qquad |\mathcal{G}(g)(\mathsf{t}, \mathsf{y})|(1 + |\mathsf{t}|)^\beta (\cosh \mathsf{t})^\mu \leq C.$$

3. DELAUNAY SOLUTIONS OF THE CAHN-HILLIARD EQUATION

3.1. **The Lyapunov-Schmidt reduction.** While our formal considerations in the above section were valid for any embedded CMC surface Σ in $\mathbb{R}^d$ in what follows we will focus on a special example when $\Sigma = D_\tau$, i.e. it is a Delaunay unduloid. Since we are interested in functions which are periodic in the direction of the x_d axis with the minimal period equal to that of D_τ we will introduce the manifold $\mathring{D}_\tau$ which is obtained by identifying the set $D_\tau \cap \{x_d = 0\}$ with the set $D_\tau \cap \{x_d = 2T_\tau\}$. The set $\mathring{D}_\tau$ is homeomorphic to the $d - 1$ dimensional torus $\mathbb{T}^{d-1}$.

Fist we note that the approximate solution $w = U + \varepsilon^2 \psi_0$ is so far only defined in $\mathcal{N}_\delta$, which is a tubular neighborhood of $\mathring{D}_\tau$. To extend w to the whole space let us define

$$\mathbb{H}(\mathsf{x}) = \begin{cases} 1 + \sigma_\varepsilon & \text{if } \mathsf{x} \in \mathring{D}_\tau^+, \\ -1 + \sigma_\varepsilon & \text{if } \mathsf{x} \in \mathring{D}_\tau^-, \end{cases}$$

where $\mathring{D}_\tau^+$, $\mathring{D}_\tau^-$ denote, respectively, the interior and the exterior of $\mathring{D}_\tau$.

Let us notice that the function w approaches $\mathbb{H}$ exponentially. Indeed, we have

$$|Y_{\varepsilon,h}^* w(\mathsf{y}, \mathsf{t}) - Y_{\varepsilon,h}^* \mathbb{H}(\mathsf{y}, \mathsf{t})| \leq C_\mu e^{-\mu|\mathsf{t}|}, \quad \mathsf{y} \in \mathring{D}_\tau, \quad \mathsf{t} \in \left(-\frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon}\right),$$

for any $\mu \in (0, -\eta)$. To make the definition of w^* precise we let χ to be a cutoff function such that $\chi(s) = 1$, when $|s| \leq \frac{1}{2}$ and $\chi(s) = 0$, when $|s| \geq 1$. Next, we define a cutoff function χ^* supported in $\mathcal{N}_\delta$ by:

$$Y_{\varepsilon,h}^* \chi^*(\mathsf{t}) = \chi\left(\frac{\varepsilon \mathsf{t}}{\delta}\right).$$

We can define the global approximate solution w^* by

$$(3.1) \qquad w^*(\mathsf{x}) = w(\mathsf{x})\chi^*(\mathsf{x}) + \mathbb{H}(\mathsf{x})\big[1 - \chi^*(\mathsf{x})\big].$$

Now we look for a solution of the equation (1.7) in the form

$$u = w^* + \varphi$$

where φ is a small (in a way to be specified) function. Thus our problem can be stated: find $\varphi \colon \mathbb{R}^{d-1} \times S_{2T_\tau}^1 \to \mathbb{R}$, which is one-periodic with period $2T_\tau$ in the x_d direction, such that

$$(3.2) \qquad N_\varepsilon(w^* + \varphi) = \ell_\varepsilon, \quad \text{in } \mathbb{R}^{d-1} \times S_{2T_\tau}^1,$$

where $S_{2T_\tau}^1$ is a circle of radius $2T_\tau$,

$$N_\varepsilon(u) = \varepsilon \Delta u + \frac{1}{\varepsilon} u(1 - u^2),$$

and ℓ_ε is the Lagrange multiplier defined in (2.5). Let us recall that we want our solution to be rotationally symmetric. That is, if by $\mathcal{R}_\theta$ we denote the rotation of $\mathbb{R}^d$ about the x_d axis by angle θ then we should have:

$$(w^* + \varphi)(\mathsf{x}) = (w^* + \varphi)(\mathcal{R}_\theta \mathsf{x}).$$

Since we already have (by definition)

$$w^*(\mathsf{x}) = w^*(\mathcal{R}_\theta \mathsf{x}),$$

than as a result we will have $\varphi(\mathsf{x}) = \varphi(\mathcal{R}_\theta \mathsf{x})$ as well, as can be seen easily from the proceeding construction.

Since the function φ appearing in (3.2) is expected to be small it is natural to expand the nonlinear operator N_ε and write:

$$L_\varepsilon \varphi = -N_\varepsilon(w^*) - Q_\varepsilon(\varphi) + \ell_\varepsilon$$

where

$$L_\varepsilon \varphi = DN_\varepsilon(w^*)\varphi, \quad Q_\varepsilon(\varphi) = N(w^* + \varphi) - N_\varepsilon(w^*) - L_\varepsilon \varphi.$$

The strategy, based on the Lyapunov-Schmidt reduction is clear. Indeed, we expect that due to the d dimensional bounded (and periodic) kernel of the Jacobi operator $\mathcal{J}_\tau$ which is associated to translations of

D_τ in the directions of the coordinate axis e_j, $j = 1, \ldots, d$, the linear operator L_ε should have d dimensional kernel spanned, roughly speaking, by the functions $Z_{\tau,\varepsilon}^{T,\mathsf{e}_j}$ where

$$(3.3) \qquad Y_{\varepsilon,h}^* Z_{\tau,\varepsilon}^{T,\mathsf{e}_j}(\mathsf{y}, \mathsf{t}) = \mathsf{V}(\mathsf{y}, \mathsf{t}) \Phi_\tau^{T,\mathsf{e}_j}(\mathsf{y}), \quad j = 1, \ldots, d, \quad \mathsf{V}(\mathsf{y}, \mathsf{t}) = \partial_\mathsf{t} w = \partial_\mathsf{t} U(\mathsf{t}) + \varepsilon^2 \partial_\mathsf{t} \psi_0(\mathsf{y}, \mathsf{t}).$$

Notice also that in general any function Z_ε such that

$$Y_{\varepsilon,h}^* Z_\varepsilon(\mathsf{y}, \mathsf{t}) = \Phi(\mathsf{y}) \mathsf{V}(\mathsf{y}, \mathsf{t}),$$

is also "almost" in the kernel of L_ε, in the sense that $Y_{\varepsilon,h}^*(L_\varepsilon Z_\varepsilon) = o(1)$. We introduce a linear subspace of $L^2(\mathring{D}_\tau \times \mathbb{R})$ of functions that are orthogonal to Z_ε by:

$$(3.4) \qquad \mathcal{X} = \left\{ \varphi \in L^2(\mathring{D}_\tau \times \mathbb{R}) \,\middle|\, \int_\mathbb{R} \varphi(\mathsf{y}, \mathsf{t}) \mathsf{V}(\mathsf{y}, \mathsf{t}) \, d\mathsf{t} = 0 \right\}.$$

By Π we denote the orthogonal projection on $\mathcal{X}$. We set $\varphi = \varphi^\| + \varphi^\perp$, where

$$Y_{\varepsilon,h}^* \varphi^\| \in \mathcal{X}, \quad Y_{\varepsilon,h}^* \varphi^\perp = Z\mathsf{V} \in \mathcal{X}^\perp.$$

Finally we split our problem into two equations:

$$(3.5) \qquad \Pi \circ Y_{\varepsilon,h}^* [N_\varepsilon(w^* + \varphi^\| + \varphi^\perp) - \ell_\varepsilon] = 0,$$

$$(3.6) \qquad (\mathrm{Id} - \Pi) \circ Y_{\varepsilon,h}^* [N_\varepsilon(w^* + \varphi^\| + \varphi^\perp) - \ell_\varepsilon] = 0.$$

When solving (3.5) we use the fact that the associated linear operator is coercive on $\mathcal{X}$. To solve (3.6) we will make use of the theory of solvability of the Jacobi operator $\mathcal{J}_{\mathring{D}_\tau}$. An additional, somewhat technical, step which we have omitted in this informal discussion is to "transfer" the original problem from the space of functions defined on $\mathbb{R}^{d-1} \times S_{2T_\tau}^1$ to a space of functions defined on $\mathring{D}_\tau \times \mathbb{R}$. We will explain these details in section 3.3 but first we will introduce and study a linear operator which is essentially the expression of L_ε in the Fermi coordinates of $\mathring{D}_\tau$.

3.2. **Linear theory for a model problem.** In this section we will develop the necessary theory to deal with the operator L_ε. To this end we will consider the operator

$$(3.7) \qquad \mathbb{L}_\varepsilon = \varepsilon \Delta_{\mathring{D}_\tau} - (H_{\mathring{D}_\tau} + \varepsilon(\mathsf{t} + \varepsilon h_0)\chi(\varepsilon \mathsf{t}/\delta)|A_{\mathring{D}_\tau}|^2)\partial_\mathsf{t} + \varepsilon^{-1}\partial_\mathsf{t}^2 + \varepsilon^{-1}f'(w),$$

where $\chi(s)$ is a cutoff function supported in $(-1, 1)$ and equal to 1 in $(-1/2, 1/2)$. Note that this operator is defined for functions $\phi \colon \mathring{D}_\tau \times \mathbb{R} \to \mathbb{R}$ (and not just functions defined on $\mathring{D}_\tau \times (-\frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon})$). It is clear that $Y_{\varepsilon,h}^* L_\varepsilon \approx \mathbb{L}_\varepsilon$. Although the function $w = U + \varepsilon^2 \psi_0$ depends on both variables (y, t) in some sense the operator $\mathbb{L}_\varepsilon$ separates variables. To see this, with $\partial_\mathsf{t} w = \partial_\mathsf{t}(U + \varepsilon^2 \psi_0) = \mathsf{V}$, we consider functions of the form:

$$\varphi(\mathsf{y}, \mathsf{t}) = \mathsf{V}(\mathsf{y}, \mathsf{t}) Z(\mathsf{y}).$$

Observe that, by construction of $w = U + \varepsilon^2 \psi_0$, combining equations (2.5) and (2.6) multiplied by ε^2 we get

$$\varepsilon^{-1}\partial_\mathsf{t}^2 w - (H_{\mathring{D}_\tau} + \varepsilon(\mathsf{t} + \varepsilon h_0)|A_{\mathring{D}_\tau}|^2)\partial_\mathsf{t} w + \varepsilon^{-1}f(w) = \ell_\varepsilon - \varepsilon^3(\mathsf{t} + \varepsilon h_0)|A_{\mathring{D}_\tau}|^2)\partial_\mathsf{t}\psi_0 + \varepsilon^{-1}Q(\varepsilon\psi_0),$$

where $Q(v) = f(U + v) - f(U) - f'(U)v$. Differentiating this equation in t we get for $\mathsf{V} = \partial_\mathsf{t} w$:

$$(3.8) \qquad \begin{aligned} \varepsilon^{-1}\partial_\mathsf{t}^2 \mathsf{V} - (H_{\mathring{D}_\tau} + \varepsilon(\mathsf{t} + \varepsilon h_0)|A_{\mathring{D}_\tau}|^2)\partial_\mathsf{t}\mathsf{V} - \varepsilon|A_{\mathring{D}_\tau}|^2 \mathsf{V} + \varepsilon^{-1}f'(w)\mathsf{V} &= -\varepsilon^3|A_{\mathring{D}_\tau}|^2\partial_\mathsf{t}\psi_0 \\ &\quad - \varepsilon^3(\mathsf{t} + \varepsilon h_0)|A_{\mathring{D}_\tau}|^2\partial_\mathsf{t}^2\psi_0 - \varepsilon^{-1}\partial_\mathsf{t} Q(\varepsilon^2\psi_0). \end{aligned}$$

From this, using the definition of $\mathbb{L}_\varepsilon$ in (3.7) we get:

$$(3.9) \qquad \mathbb{L}_\varepsilon(\varphi) = \mathbb{L}_\varepsilon(\mathsf{V}Z) = \varepsilon \mathsf{V} \mathcal{J}_{\mathring{D}_\tau} Z + B_\varepsilon(Z)$$

where $\mathcal{J}_{\mathring{D}_\tau}$ is the Jacobi operator on $\mathring{D}_\tau$ and

$$\begin{aligned} B_\varepsilon(Z) = &-\varepsilon(\mathsf{t} + \varepsilon h_0)(1 - \chi^*)|A_{\mathring{D}_\tau}|^2 Z \partial_\mathsf{t}\mathsf{V} + 2\varepsilon\nabla_{\mathring{D}_\tau}\mathsf{V} \cdot \nabla_{\mathring{D}_\tau} Z \\ &+ \varepsilon Z \Delta_{\mathring{D}_\tau}\mathsf{V} - \varepsilon^3 \left[(\mathsf{t} + \varepsilon h_0)|A_{\mathring{D}_\tau}|^2\partial_\mathsf{t}^2\psi_0 + |A_{\mathring{D}_\tau}|^2\partial_\mathsf{t}\psi_0 \right] Z \\ &+ \varepsilon^{-1} \left[f'(w)\partial_\mathsf{t} w - f'(U)\partial_\mathsf{t} w - \varepsilon^2 f''(U)\partial_\mathsf{t} U\psi_0 \right] Z. \end{aligned}$$

We note that

(3.10) $B_\varepsilon(Z) = \mathcal{O}(\varepsilon^3)\|Z\|_{\mathcal{C}^1(\mathring{D}_\tau)}.$

Identity (3.8) and its consequence (3.9) is the key calculation which allows to use the usual Lyapunov-Schmidt reduction scheme, as w explained in the introduction. Indeed, if we had taken as the approximate solution only the function U then differentiating the equation (2.5) for $U' = \partial_t U$ we would have gotten

$$\varepsilon^{-1}\partial_t^2 U' - H_{\mathring{D}_\tau}\partial_t U' + \varepsilon^{-1}f'(U)U' = 0.$$

This equation, unlike (3.8), does not carry any information about the geometry of $\mathring{D}_\tau$ besides its mean curvature which is constant. Following the method of [24] or [11], [12], [13] we would have to perturb the surface $\mathring{D}_\tau$ additionally introducing new unknown functions in our problem. With the approach presented here this is no longer necessary and the Lyapunov-Schmidt procedure in this version is simpler. Recalling that the linearization of the mean curvature operator is the Jacobi operator which depends on the second fundamental form, we see that the operator $\mathbb{L}_\varepsilon$ is naturally compatible with the geometric context of our problem. To put it differently: the operator $\mathbb{L}_\varepsilon$ is, up to negligible terms, the correct linearization of the Cahn-Hilliard operator near a solution whose zero level set is the constant curvature surface $\mathring{D}_\tau$.

To develop invertibility theory for $\mathbb{L}_\varepsilon$ we will we employ two basic facts. First, we observe that on the subspace:

$$\mathcal{Y} := \left\{ \varphi(\mathbf{y},\mathbf{t}) = \mathtt{V}(\mathbf{y},\mathbf{t})Z(\mathbf{y}) \mid \int_{\mathring{D}_\tau} Z(\mathbf{y})\Phi_\tau^{T,\mathbf{e}_j}(\mathbf{y})\,d\mathbf{y} = 0, j = 1,\ldots,d \right\},$$

we have

$$\langle \mathbb{L}_\varepsilon\varphi, \varphi \rangle \geq C\varepsilon\|\varphi\|_{L^2(\mathring{D}_\tau \times \mathbb{R})}^2.$$

Second, when we consider $\varphi \in \mathcal{X}$ (space $\mathcal{X}$ is defined in (3.4)) and $g \in L^2(\mathring{D}_\tau \times \mathbb{R})$ such that φ is a bounded solution of the problem

$$\mathtt{L}_\varepsilon\varphi = g, \qquad \mathtt{L}_\varepsilon = \varepsilon\Delta_{\mathring{D}_\tau} + \varepsilon^{-1}\partial_{\mathtt{tt}}^2 + \varepsilon^{-1}f'(H),$$

then we have

(3.11) $\|\varphi\|_{L^2(\mathring{D}_\tau \times \mathbb{R})} \leq C\varepsilon\|g\|_{L^2(\mathring{D}_\tau \times \mathbb{R})}.$

To prove this estimate we use a contradiction argument which relies on the fact that from

$$\int_{\mathbb{R}} |v'|^2 - f'(H)v^2 \geq C\|v\|_{L^2(\mathbb{R})}^2, \quad \text{if } \int_{\mathbb{R}} v(t)H'(t)\,dt = 0,$$

it follows that the bilinear form

$$\mathtt{B}_\varepsilon(\varphi) := \langle \mathtt{L}_\varepsilon\varphi, \varphi \rangle,$$

is coercive on $\mathcal{X}$. In the same way as (3.11) is shown one can prove:

(3.12) $\|\varphi\|_{H^1(\mathring{D}_\tau \times \mathbb{R})} \leq C\|g\|_{L^2(\mathring{D}_\tau \times \mathbb{R})}.$

The loss of the factor of ε in this last estimate is related with the scaling of the problem. We refer the reader to [11] or [13] where results similar to estimates (3.11) and (3.12) were proven.

At the same time we can use (3.12) and the coercivity of the bilinear form $\mathtt{B}_\varepsilon(\varphi)$ to solve the equation

(3.13) $\Pi_{\mathcal{X}}\mathbb{L}_\varepsilon\varphi = g,$

where $\Pi_{\mathcal{X}}$ is the projection on $\mathcal{X}$, in $\mathcal{X}$. To do this we write $\mathbb{L}_\varepsilon = \mathtt{L}_\varepsilon + (\mathbb{L}_\varepsilon - \mathtt{L}_\varepsilon)$ and use a perturbation argument. The solution will still satisfy estimate (3.12). The perturbation argument is as follows: for $g \in \mathcal{X}$ we solve

$$\mathtt{L}_\varepsilon\phi = g + cH',$$

where $c = \frac{\int gH'\,dt}{\int (H')^2\,dt}$. Then we define a map

$$G_{\mathcal{X}}(g) = \phi - V\frac{\int \phi V\,dt}{\int V^2\,dt}.$$

Note that $G_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X}$. Next we check

$$\|\Pi_{\mathcal{X}}\mathbb{L}_\varepsilon G_{\mathcal{X}}(g) - g\|_{L^2(\mathring{D}_\tau \times \mathbb{R})} \leq o(1)\|g\|_{L^2(\mathring{D}_\tau \times \mathbb{R})}.$$

Therefore $\Pi_{\mathcal{X}}\mathbb{L}_\varepsilon G_{\mathcal{X}}$ is invertible as a map from $\mathcal{X}$ to itself and we can define

$$(\Pi_{\mathcal{X}}\mathbb{L}_\varepsilon)^{-1} = G_{\mathcal{X}}(\mathbb{L}_\varepsilon G_{\mathcal{X}})^{-1}.$$

Moreover it is rather straightforward to show that a solution to (3.13) will satisfy estimate (3.12).

We will use these observations to solve the following model equation:

$$\tag{3.14} \mathbb{L}_\varepsilon \varphi = g(\mathtt{y}, \mathtt{t}),$$

where we will assume initially that $g \in L^p(\mathring{D}_\tau \times \mathbb{R})$. We look for a solution in the form $\varphi = \varphi^{\|} + \varphi^{\perp}$, where

$$\varphi^{\|} \in \mathcal{X}, \quad \varphi^{\perp} = Z\mathtt{V} \in \mathcal{X}^{\perp}.$$

We write

$$\mathbb{L}_\varepsilon \varphi = \Pi_{\mathcal{X}}\mathbb{L}_\varepsilon(\varphi^{\|} + \varphi^{\perp}) + \Pi_{\mathcal{X}^{\perp}}\mathbb{L}_\varepsilon(\varphi^{\|} + \varphi^{\perp}),$$

and then we need to solve

$$\Pi_{\mathcal{X}}\mathbb{L}_\varepsilon\varphi^{\|} = \Pi_{\mathcal{X}}g - \Pi_{\mathcal{X}}\mathbb{L}_\varepsilon\varphi^{\perp}$$
$$\Pi_{\mathcal{X}^{\perp}}\mathbb{L}_\varepsilon\varphi^{\perp} = \Pi_{\mathcal{X}^{\perp}}g - \Pi_{\mathcal{X}^{\perp}}\mathbb{L}_\varepsilon\varphi^{\|}.$$

The idea is that terms $\Pi_{\mathcal{X}}\mathbb{L}_\varepsilon\varphi^{\perp}$ and $\Pi_{\mathcal{X}^{\perp}}\mathbb{L}_\varepsilon\varphi^{\|}$ are of smaller order because $\mathbb{L}_\varepsilon V = o(1)$ so that the coupling between the two equations is rather weak. Another important point is that

$$\mathbb{L}_\varepsilon\varphi^{\perp} = \varepsilon V \mathcal{J}_{\mathring{D}_\tau} Z + B_\varepsilon(Z),$$

where $B_\varepsilon(Z)$ is small (see (3.10)). We decompose accordingly $g = g^{\|} + g^{\perp}$, $g^{\perp} = \Xi\mathtt{V}$ and $B_\varepsilon(Z) = B_\varepsilon^{\|}(Z) + B_\varepsilon^{\perp}(Z)$, $B_\varepsilon^{\perp}(Z) = \Upsilon_\varepsilon(Z)\mathtt{V}$ and look for a solution of the system:

$$\Pi_{\mathcal{X}}\mathbb{L}_\varepsilon\varphi^{\|} = g^{\|} - B_\varepsilon^{\|}(Z),$$

$$\tag{3.15} \varepsilon\mathcal{J}_{\mathring{D}_\tau}Z + \Upsilon_\varepsilon(Z) = \Xi - \int_{\mathbb{R}}(\mathbb{L}_\varepsilon\varphi^{\|})\mathtt{V}\,d\mathtt{t} + \sum_{j=1}^{d} c_j \Phi_\tau^{T,\mathbf{e}_j}.$$

Note that in the second equation we have introduced Lagrange multipliers c_j to be determined.

Although the two equations in (3.15) are coupled but this coupling is weak and we can solve the system without any difficulty using invertibility of $\Pi_{\mathcal{X}}$ and $\mathcal{J}_{\mathring{D}_\tau}$, a fixed point argument and estimates (3.10), (3.11) and (3.12). We leave this to the reader.

Given that we can solve (3.15) our purpose is to find suitable estimates for the solution of the problem

$$\mathbb{L}_\varepsilon\varphi = g(\mathtt{y}, \mathtt{t})$$

on $\mathcal{X}$ assuming that

$$g(\mathtt{y}, \mathtt{t}) = \mathcal{O}(e^{-\mu|\mathtt{t}|}), \quad |\mathtt{t}| \to \infty.$$

In particular we would like to know that $\varphi(\mathtt{t}, \mathtt{y}) = \mathcal{O}(e^{-\mu|\mathtt{t}|})$ as well. This is straightforward by comparison principle once we know for example that φ is bounded. Thus the main issue is to obtain L^∞ control for φ. We will go a little further now and show how to control *a priori* certain weighted Hölder norms of $\varphi^{\|}$ and $\varphi^{\perp}$ (see decomposition in (3.15)).

In order to simplify the argument and avoid keeping track of negative powers of ε appearing on the right hand side of various estimates we will rescale the $\mathtt{y}$ variable. Thus we will introduce:

$$\tilde{\mathtt{y}} = \frac{\mathtt{y}}{\varepsilon}, \quad \tilde{\mathtt{t}} = \mathtt{t}.$$

We will denote $\mathring{D}_{\tau,\varepsilon} = \frac{1}{\varepsilon}\mathring{D}_{\tau}$. We consider the manifold $\mathring{D}_{\tau,\varepsilon} \times \mathbb{R}$ equipped with the product metric and the associated Levi-Civita connection. We define weighted Hölder norms on $\mathring{D}_{\tau,\varepsilon} \times \mathbb{R}$:

$$\|u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})} = \sup_{\tilde{\mathsf{t}}\in\mathbb{R}}(\cosh\tilde{\mathsf{t}})^{\mu}\|u\|_{\mathcal{C}^{0,\alpha}(\mathring{D}_{\tau,\varepsilon}\times(\tilde{\mathsf{t}}-1,\tilde{\mathsf{t}}+1))},$$

(3.16) $$\|u\|_{\mathcal{C}^{1,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})} = \|u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})} + \|\nabla_{\mathring{D}_{\tau,\varepsilon}\times\mathbb{R}}u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})},$$

$$\|u\|_{\mathcal{C}^{2,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})} = \|u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})} + \|\nabla_{\mathring{D}_{\tau,\varepsilon}\times\mathbb{R}}u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})} + \|\nabla^2_{\mathring{D}_{\tau,\varepsilon}\times\mathbb{R}}u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})}.$$

Above $\nabla_{\mathring{D}_{\tau,\varepsilon}\times\mathbb{R}}$ and $\nabla^2_{\mathring{D}_{\tau,\varepsilon}\times\mathbb{R}}$ respectively denote the gradient and the Hessian on the manifold $\mathring{D}_{\tau,\varepsilon} \times \mathbb{R}$.

Given functions u and g on : $\mathring{D}_{\tau} \times \mathbb{R}$ and we introduce functions

$$\tilde{u}(\tilde{\mathsf{y}},\tilde{\mathsf{t}}) = u(\varepsilon\tilde{\mathsf{y}},\tilde{\mathsf{t}}), \quad \tilde{w}(\tilde{\mathsf{y}},\tilde{\mathsf{t}}) = w(\varepsilon\tilde{\mathsf{y}},\tilde{\mathsf{t}}), \quad \tilde{g}(\tilde{\mathsf{y}},\tilde{\mathsf{t}}) = \varepsilon g(\varepsilon\tilde{\mathsf{y}},\tilde{\mathsf{t}}).$$

and set

$$\tilde{\mathbb{L}}_{\varepsilon}\tilde{u} = \Delta_{\mathring{D}_{\tau,\varepsilon}}\tilde{u} + \partial^2_{\tilde{\mathsf{t}}}\tilde{u} + f'(\tilde{w})\tilde{u} + \varepsilon\tilde{q}\partial_{\tilde{\mathsf{t}}}\tilde{u},$$

where

$$\tilde{q} := H_{\mathring{D}_{\tau}} + \varepsilon(\tilde{\mathsf{t}} + \varepsilon h_0)\chi(\varepsilon\tilde{\mathsf{t}}/\delta)|A_{\mathring{D}_{\tau}}(\varepsilon\tilde{\mathsf{y}})|^2$$

is a bounded function. The linear problem we consider is:

(3.17) $$\Pi_{\mathcal{X}}\tilde{\mathbb{L}}_{\varepsilon}\tilde{u} = \tilde{g}, \quad \text{in } \mathring{D}_{\tau,\varepsilon} \times \mathbb{R},$$

where now we assume simply that the right hand side satisfies the orthogonality condition (c.f. (2.11)):

$$\int_{\mathbb{R}} \tilde{g}(\tilde{\mathsf{y}},\tilde{\mathsf{t}})\mathsf{V}(\varepsilon\tilde{\mathsf{y}},\tilde{\mathsf{t}})\, d\tilde{\mathsf{t}} = 0.$$

For this problem we can derive *a priori* estimates using the method of [11], [12], [13]. From this, by a perturbation argument, we will be able get corresponding estimates for $\varphi^{\|}$ satisfying the first equation in (3.15). Let us explain briefly the main steps.

Step 1. We consider a problem of the form:

(3.18) $$\Delta_y\phi + \partial^2_t\phi + f'(H(t))\phi = 0, \quad \text{in } \mathbb{R}^{d-1} \times \mathbb{R}.$$

The following result is known ([11], [12]):

Lemma 3.1. *Let ϕ be a bounded solution of (3.18). Then $\phi = cH'(\mathsf{t})$, with some constant c.*

Step 2. Consider now equation (3.17) and assume that $\tilde{g} \in \mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon} \times \mathbb{R})$, with $\mu \in (0, |\eta|)$. The we have:

Lemma 3.2. *There exists a constant $C > 0$ such that for all sufficiently small ε any bounded solution of (3.17) satisfies:*

(3.19) $$\|\tilde{u}\|_{\mathcal{C}^{2,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})} \leq C\|\tilde{g}\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau,\varepsilon}\times\mathbb{R})}.$$

A proof of this lemma, which relies on Lemma 3.1 and a contradiction argument, follows the same lines as the proof of Lemma 5.2 in [13] (see also similar results in [11], [12]).

Now we should go back to the original variables. We define weighted Hölder norms on $\mathring{D}_{\tau} \times \mathbb{R}$ similarly as in (3.16):

$$\|u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau}\times\mathbb{R})} = \sup_{\mathsf{t}\in\mathbb{R}}(\cosh\mathsf{t})^{\mu}\|u\|_{\mathcal{C}^{0,\alpha}(\mathring{D}_{\tau}\times(\mathsf{t}-1,\mathsf{t}+1))},$$

(3.20) $$\|u\|_{\mathcal{C}^{1,\alpha}_{\mu}(\mathring{D}_{\tau}\times\mathbb{R})} = \|u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau}\times\mathbb{R})} + \|\nabla_{\mathring{D}_{\tau}\times\mathbb{R}}u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau}\times\mathbb{R})}$$

$$\|u\|_{\mathcal{C}^{2,\alpha}_{\mu}(\mathring{D}_{\tau}\times\mathbb{R})} = \|u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau}\times\mathbb{R})} + \|\nabla_{\mathring{D}_{\tau}\times\mathbb{R}}u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau}\times\mathbb{R})} + \|\nabla^2_{\mathring{D}_{\tau}\times\mathbb{R}}u\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_{\tau}\times\mathbb{R})}.$$

We note that if for a given function $u\colon \mathring{D}_{\tau} \times \mathbb{R} \to \mathbb{R}$ we set $\tilde{u}(\tilde{\mathsf{y}},\tilde{\mathsf{t}}) = u(\varepsilon\tilde{\mathsf{y}},\tilde{\mathsf{t}})$ then we have

(3.21) $$\|\tilde{u}\|_{\mathcal{C}^{\ell,\alpha}_{\mu}(\mathring{D}_{\tau_{\varepsilon}}\times\mathbb{R})} = \sum_{0\leq k+m\leq\ell} \varepsilon^m\|\partial^k_{\mathsf{t}}D^m_{\mathring{D}_{\tau}}u\|_{\mathcal{C}^0_{\mu}(\mathring{D}_{\tau}\times\mathbb{R})} + \sum_{0\leq k+m\leq\ell} \varepsilon^{m+\alpha}[\partial^k_{\mathsf{t}}D^m_{\mathring{D}_{\tau}}u]_{\alpha,\mu,\mathring{D}_{\tau}\times\mathbb{R}},$$

where $[\cdot]_{\alpha,\mu,\mathring{D}_\tau \times \mathbb{R}}$ is the weighted Hölder seminorm. Consequently by $\mathcal{E}_\mu^{\ell,\alpha}(\mathring{D}_\tau \times \mathbb{R})$ we denote the space of functions on $\mathring{D}_\tau \times \mathbb{R}$ with the norm

$$\|u\|_{\mathcal{E}_\mu^{\ell,\alpha}(\mathring{D}_\tau \times \mathbb{R})} := \sum_{0 \leq k+m \leq \ell} \varepsilon^m \|\partial_{\mathsf{t}}^k D_{\mathring{D}_\tau}^m u\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})}.$$

With this definition we have

$$\|u\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})} = \|u\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})},$$

while with the notation of (3.21) we get

$$C^{-1}\|\tilde{u}\|_{\mathcal{C}_\mu^{\ell,\alpha}(\mathring{D}_{\tau_\varepsilon} \times \mathbb{R})} \leq \|u\|_{\mathcal{E}_\mu^{\ell,\alpha}(\mathring{D}_\tau \times \mathbb{R})} \leq C\varepsilon^{-\alpha}\|\tilde{u}\|_{\mathcal{C}_\mu^{\ell,\alpha}(\mathring{D}_{\tau_\varepsilon} \times \mathbb{R})}.$$

From this and Lemma 3.2 it follows for $\ell = 0, 1, 2$:

$$(3.22) \qquad \|u\|_{\mathcal{E}_\mu^{\ell,\alpha}(\mathring{D}_\tau \times \mathbb{R})} \leq C\varepsilon^{1-\alpha}\|g\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})}.$$

The procedure described above shows that we can control the size of $\mathcal{E}_\mu^{\ell,\alpha}(\mathring{D}_\tau \times \mathbb{R})$ norm of $\varphi^{\|}$ in (3.15) obtaining:

$$(3.23) \qquad \|\varphi^{\|}\|_{\mathcal{E}_\mu^{\ell,\alpha}(\mathring{D}_\tau \times \mathbb{R})} \leq C\varepsilon^{1-\alpha}(\|g^{\|}\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})} + \varepsilon^2\|g^{\perp}\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})}).$$

Finally, we notice that for the second equation in (3.15) using elliptic theory we can get Hölder estimates and since $\varphi^{\perp} = Z\mathsf{V}$ we find:

$$(3.24) \qquad \|\varphi^{\perp}\|_{\mathcal{C}_\mu^{\ell,\alpha}(\mathring{D}_\tau \times \mathbb{R})} \leq C\varepsilon^{-1}\|g^{\perp}\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})}.$$

3.3. The linear problem in the whole space.
Now we will use the theory outlined above to solve the following problem:

$$(3.25) \qquad \varepsilon\Delta\varphi + \frac{1}{\varepsilon}f'(w^*)\varphi = g(\mathbf{x}), \quad \text{in } \mathbb{R}^{d-1} \times S_{2T_\tau},$$

From what we have said above it is in general not possible to find a solution with a reasonably bounded norm unless the right hand side satisfies some extra conditions, or equivalently, we need to introduce Lagrange multipliers that correspond to natural invariances of the problem. Thus, we will solve

$$(3.26) \qquad \varepsilon\Delta\varphi + \frac{1}{\varepsilon}f'(w^*)\varphi = g(\mathbf{x}) + \chi^* \sum_{j=1}^{d} \mathsf{c}_j Z_{\tau,\varepsilon}^{T,\mathbf{e}_j}, \quad \text{in } \mathbb{R}^{d-1} \times S_{2T_\tau}$$

where

$$Y_{\varepsilon,h}^* \chi^*(\mathsf{t}) = \chi(\varepsilon\mathsf{t}/\delta), \quad Y_{\varepsilon,h}^* Z_{\tau,\varepsilon}^{T,\mathbf{e}_j}(\mathbf{y},\mathsf{t}) = \mathsf{V}(\mathbf{y},\mathsf{t})\Phi_\tau^{T,\mathbf{e}_j}(\mathbf{y}), \quad j = 1,\ldots,d$$

The idea is to solve (3.26) by *gluing* a solution defined near $\mathring{D}_\tau$ and another one defined away from $\mathring{D}_\tau$. To describe this construction rigorously we need some preparation. We introduce the function $q(\mathbf{x})$ as follows:

$$q(\mathbf{x}) = \begin{cases} f'(1 + \sigma_\varepsilon), & \text{dist}(\mathbf{x}, D_\tau) > \delta/2, \\ f'(-1 + \sigma_\varepsilon), & \text{dist}(\mathbf{x}, D_\tau) < -\delta/2, \end{cases}$$

and otherwise $q(\mathbf{x})$ is a smooth function such that $\min\{f'(1 + \sigma_\varepsilon), f'(-1 + \sigma_\varepsilon)\} < q(\mathbf{x}) \leq \max\{f'(1 + \sigma_\varepsilon), f'(-1 + \sigma_\varepsilon)\}$. Note that $q(\mathbf{x}) = -2 + \mathcal{O}(\varepsilon)$. Finally, we need another cutoff function $\widetilde{\chi}$ such that $\widetilde{\chi}\chi^* = \chi^*$ (take for instance $Y_{\varepsilon,h}^* \widetilde{\chi}(\mathsf{t}) = \chi(\varepsilon\mathsf{t}/2\delta)$ and chose δ smaller so that the Fermi coordinates are defined in $\mathcal{N}_{2\delta}$). We want to find a solution of (3.26) in the form $\varphi = \chi^*\check{\varphi} \circ Y_{\varepsilon,h} + \psi$, where the function $\check{\varphi}$ solves:

$$(3.27) \qquad \mathbb{L}_\varepsilon\check{\varphi} = \left(\widetilde{\chi}\{g + \chi^* \sum_{j=1}^{d} \mathsf{c}_j Z_{\tau,\varepsilon}^{T,\mathbf{e}_j} + (\mathbb{L}_\varepsilon - L_\varepsilon)\check{\varphi} - [\chi^*, L_\varepsilon]\check{\varphi} + \varepsilon^{-1}[q - f'(w^*)]\psi\}\right), \quad \text{in } \mathring{D}_\tau \times \mathbb{R},$$

and the function ψ solves

$$(3.28) \quad \varepsilon\Delta\psi + \varepsilon^{-1}[(1 - \chi^*)f'(w^*) + \chi^* q]\psi = (1 - \chi^*)\{g + \chi^* \sum_{j=1}^{d} \mathsf{c}_j Z_{\tau,\varepsilon}^{T,\mathbf{e}_j} - [\chi^*, L_\varepsilon]\check{\varphi}\}, \quad \text{in } \mathbb{R}^{d-1} \times S_{2T_\tau}.$$

It is clear that multiplying (3.27) by χ^* and adding the equations (3.27)–(3.28) and using the fact that $\widetilde{\chi}\chi^* = \chi^*$ we get the solution to our problem. In the above and in what follows we abuse slightly notation writing for instance $\check{\varphi}$ as a function defined on $\mathring{D}_\tau \times \mathbb{R}$ and as a function defined on $\mathbb{R}^{d-1} \times S_{2T_\tau}$. It is understood that in the latter case we take $\check{\varphi} \circ Y_{\varepsilon,h}$. To avoid complicated notions we will omit the composition with $Y_{\varepsilon,h}$ or $Y_{\varepsilon,h}^{-1}$ whenever it does not cause confusion. Thus the commutator $[\chi^*, L_\varepsilon]\check{\varphi}$ in $\mathbb{R}^{d-1} \times S_{2T_\tau}$ is

$$[\chi^*, L_\varepsilon]\check{\varphi} = 2\varepsilon\nabla\check{\varphi}\circ Y_{\varepsilon,h}\nabla\chi^* + \varepsilon\check{\varphi}\circ Y_{\varepsilon,h}\Delta\chi^*,$$

while in $\mathring{D}_\tau \times R$ we have to first express L_ε in local coordinate $(\mathbf{y}, \mathbf{t})$ (written as $Y_{\varepsilon,h}^* L_\varepsilon$) and calculate $[\chi^*, Y_{\varepsilon,h}^* L_\varepsilon]\check{\varphi}$.

The function g on the right hand side of this equation satisfies the following general assumptions on its asymptotic behaviour:

$$\|(g\chi^*)^\|\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})} \le C,$$

(3.29)
$$\|(g\chi^*)^\perp\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})} \le C$$

$$\|(1 - \chi^*)g\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})} \le Ce^{-c_0/\varepsilon}.$$

In addition we assume that g is rotationally symmetric about the x_d axis, namely if by $\mathcal{R}_\theta$ we denote the rotation of $\mathbb{R}^d$ about the x_d axis by angle θ then $g(\mathcal{R}_\theta \mathbf{x}) = g(\mathbf{x})$.

In order to solve this coupled system we need to make sure that all terms on the right hand side that involve $\check{\varphi}$ and ψ are small in suitable weighted Hölder and Hölder norms respectively. It is at this point that we need to chose the parameter δ in the definition of the tubular neighbourhood $\mathcal{N}_\delta$ small and dependent on ε. Thus we take $\delta(\varepsilon) = \varepsilon^{2/3}$. This means in particular that for $\mathbf{x} \in \mathcal{N}_\delta$ we have $\varepsilon\mathbf{t}(\mathbf{x}) = \mathcal{O}(\varepsilon^{2/3})$. For reasons that will become clear soon we will also chose the Hölder exponent α in the definition of $\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})$ and $\mathcal{C}^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})$ to be in the interval $(0, \frac{1}{10})$. Finally, the parameter μ will be always taken in the interval $(0, |\eta|)$.

Considering equation (3.28) we have the following:

Lemma 3.3. *If u is a solution of*

$$\varepsilon\Delta u - \frac{1}{\varepsilon}[(1 - \chi^*)f'(w^*) + \chi^* q]u = g, \quad in \ \mathbb{R}^{d-1} \times S_{2T_\tau},$$

then we have an a priori estimate:

(3.30)
$$\|u\|_{\mathcal{C}^{\ell,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})} \le C\varepsilon^{1-\ell-\alpha}\|g\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})}.$$

The proof of this lemma is straightforward and it is omitted, for similar results see for instance [10]. From this we get readily an *a priori* estimate for (3.28):

(3.31) $\|\psi\|_{\mathcal{C}^{\ell,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})} \le C\varepsilon^{1-\ell-\alpha}\|(1-\chi^*)g\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})} + e^{-c\varepsilon^{-1/4}}\left(\sum_{j=1}^{d}|\mathsf{c}_j| + \|\check{\varphi}\|_{\mathcal{C}_\mu^{1,\alpha}(\mathring{D}_\tau \times \mathbb{R})}\right).$

From the theory developed in the previous section we can also obtain an *a priori* estimate for (3.27). If we write

$$\mathfrak{g} = \widetilde{\chi}\{g + \chi^*\sum_{j=1}^{d}\mathsf{c}_j Z_{\tau,\varepsilon}^{T,\mathbf{e}_j} + (\mathbb{L}_\varepsilon - L_\varepsilon)\check{\varphi} - [\chi^*, L_\varepsilon]\check{\varphi} + \varepsilon^{-1}[q - f'(w^*)]\psi\},$$

then we have using (3.23)

(3.32) $\|\check{\varphi}^\|\|_{\mathcal{E}_\mu^{\ell,\alpha}(\mathring{D}_\tau \times \mathbb{R})} \le C\varepsilon^{1-\alpha}(\|\mathfrak{g}^\|\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})} + \varepsilon^2\|\mathfrak{g}^\perp\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})}),$

and using (3.24):

(3.33) $\|\check{\varphi}^\perp\|_{\mathcal{C}_\mu^{\ell,\alpha}(\mathring{D}_\tau \times \mathbb{R})} \le C\varepsilon^{-1}\|\mathfrak{g}^\perp\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})}.$

We note that the weighted norms we use for $\check{\phi}^\|$ and $\check{\phi}^\perp$ are scaled differently with ε. This slight nuisance is a result of our choice of the original scaling of the Cahn-Hilliard equation. We observe as well that with our definitions $\|\cdot\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})} = \|\cdot\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})}$.

We will now estimate $\|\mathfrak{g}^{\|}\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})}$. To do this we observe that, with $\check\phi = \check\phi^{\|} + \check\phi^{\perp}$, we have:

$$\big\|\big(\widetilde\chi(\mathbb{L}_\varepsilon - L_\varepsilon)\check\varphi\big)^{\|}\big\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} \le C\delta(\varepsilon)(\varepsilon^{-1}\|\check\varphi^{\|}\|_{\mathcal{E}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon\|\check\varphi^{\perp}\|_{\mathcal{C}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})})$$

$$\big\|\big(\widetilde\chi[\chi^*, L_\varepsilon]\check\varphi\big)^{\|}\big\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} \le C\delta(\varepsilon)^{-1}(\|\check\varphi^{\|}\|_{\mathcal{E}_\mu^{1,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \mathcal{O}(e^{-c\varepsilon^{-1/3}})\|\check\varphi^{\perp}\|_{\mathcal{C}_\mu^{1,\alpha}(\mathring{D}_\tau\times\mathbb{R})}),$$

$$\big\|\big(\widetilde\chi\varepsilon^{-1}[q - f'(w^*)]\psi\big)^{\|}\big\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} \le C\varepsilon^{-1}\|\psi\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})}.$$

Next we estimate the orthogonal complement of these functions

$$\big\|\big(\widetilde\chi(\mathbb{L}_\varepsilon - L_\varepsilon)\check\varphi\big)^{\perp}\big\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} \le C(\|\check\varphi^{\|}\|_{\mathcal{E}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^2\|\check\varphi^{\perp}\|_{\mathcal{C}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})})$$

$$\big\|\big(\widetilde\chi[\chi^*, L_\varepsilon]\check\varphi\big)^{\perp}\big\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} \le \mathcal{O}(e^{-c\varepsilon^{-1/3}})\big(\|\check\varphi^{\|}\|_{\mathcal{E}_\mu^{1,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \|\check\varphi^{\perp}\|_{\mathcal{C}_\mu^{1,\alpha}(\mathring{D}_\tau\times\mathbb{R})}),$$

$$\big\|\big(\widetilde\chi\varepsilon^{-1}[q - f'(w^*)]\psi\big)^{\perp}\big\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} \le C\varepsilon^{-1}\|\psi\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})}$$

We can estimate the parameters c_j by projection of $\mathfrak{g}$ onto $Z_{\tau,\varepsilon}^{T,\mathbf{e}_j}$. Using the above estimate we get:

$$|\mathsf{c}_j| \le C\big\{\|(\widetilde\chi\, g)^{\perp}\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \big\|\big(\widetilde\chi(\mathbb{L}_\varepsilon - L_\varepsilon)\check\varphi\big)^{\perp}\big\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})}$$

$$+ \big\|\big(\widetilde\chi[\chi^*, L_\varepsilon]\check\varphi\big)^{\perp}\big\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \big\|\big(\widetilde\chi\varepsilon^{-1}[q - f'(w^*)]\psi\big)^{\perp}\big\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})}\big\}$$

$$\le C\big\{\|(\widetilde\chi\, g)^{\perp}\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \|\check\varphi^{\|}\|_{\mathcal{E}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^2\|\check\varphi^{\perp}\|_{\mathcal{C}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})}$$

$$+ O(e^{-c\varepsilon^{-1/3}})\big(\|\check\varphi^{\|}\|_{\mathcal{E}_\mu^{1,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \|\check\varphi^{\perp}\|_{\mathcal{C}_\mu^{1,\alpha}(\mathring{D}_\tau\times\mathbb{R})}) + \varepsilon^{-1}\|\psi\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})}\big\}$$

$$\le C\big\{\|(\widetilde\chi\, g)^{\perp}\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \|\check\varphi^{\|}\|_{\mathcal{E}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^2\|\check\varphi^{\perp}\|_{\mathcal{C}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^{-1}\|\psi\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})}\big\}$$

Now we use estimates (3.32)–(3.33). After rearranging terms suitably and using $\varepsilon^{1-\alpha}\delta^{-1}(\varepsilon) = o(1)$ to absorb $\check\varphi^{\|}$ in the first inequality below we get

$$\|\check\varphi^{\|}\|_{\mathcal{E}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})} \le C\varepsilon^{1-\alpha}\big\{\|(\widetilde\chi\, g)^{\|}\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^2\|(\widetilde\chi\, g)^{\perp}\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})}$$

(3.34)

$$+ \varepsilon^{-1}\|\psi\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})} + \delta(\varepsilon)\|\check\varphi^{\perp}\|_{\mathcal{C}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})}\big\},$$

$$\|\check\varphi^{\perp}\|_{\mathcal{C}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})} \le C\varepsilon^{-1}\big\{\|(\widetilde\chi\, g)^{\perp}\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \|\check\varphi^{\|}\|_{\mathcal{E}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^{-1}\|\psi\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})}\big\}.$$

From (3.31) we get as well for $\ell = 0, 1, 2$:
(3.35)
$$\|\psi\|_{\mathcal{C}^{\ell,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})} \le C\varepsilon^{1-\ell-\alpha}\big\{\|(1 - \chi^*)g\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^3)}$$

$$+ e^{-c\varepsilon^{-1/8}}(\|\widetilde\chi g\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \|\check\varphi^{\|}\|_{\mathcal{E}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \|\check\varphi^{\perp}\|_{\mathcal{C}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \|\psi\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})})\big\}.$$

Using the fact that $\delta(\varepsilon)\varepsilon^{-\alpha} = o(1)$ to absorb term $\delta(\varepsilon)\|\check\varphi^{\perp}\|_{\mathcal{C}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})}$ appearing on the right hand side of the first inequality in (3.34) and combining these estimates we get
(3.36)
$$\|\check\varphi^{\|}\|_{\mathcal{E}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})} \le C\varepsilon^{1-\alpha}\Big\{\|(\widetilde\chi g)^{\|}\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^{-1}\delta(\varepsilon)\|(\widetilde\chi\, g)^{\perp}\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})}$$

$$+ \varepsilon^{-3-\alpha}\delta(\varepsilon)\|(1 - \chi^*)g\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})}\Big\},$$

$$\|\check\varphi^{\perp}\|_{\mathcal{C}_\mu^{2,\alpha}(\mathring{D}_\tau\times\mathbb{R})} \le C\varepsilon^{-1}\Big\{\|(\widetilde\chi g)^{\perp}\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^{1-\alpha}|(\widetilde\chi\, g)^{\|}\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})} + \varepsilon^{-2-\alpha}\|(1 - \chi^*)g\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})}\Big\},$$

and
(3.37)
$$\|\psi\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})} \le C\varepsilon^{-1-\alpha}\|(1 - \chi^*)g\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})} + \mathcal{O}(e^{-c\varepsilon^{-1/8}})\|\widetilde\chi g\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau\times\mathbb{R})}.$$

Using these *a priori* estimates we can solve the system (3.27)–(3.28) by a standard fixed point argument. To do this we replace the functions $\check\varphi^{\|}, \check\varphi^{\perp}, \psi$ on the right hand side of the system by known functions $\check\Phi^{\|}, \check\Phi^{\perp}, \Psi$ which satisfy estimates of the same type as (3.36)–(3.37) but with constants bigger that those appearing in (3.36)–(3.37). Then we have a map

$$(\check\Phi^{\|}, \check\Phi^{\perp}, \Psi) \longmapsto (\check\varphi^{\|}, \check\varphi^{\perp}, \psi),$$

from a certain ball in the space $\mathcal{E}_\mu^{2,\alpha}(\mathring{D}_\tau \times \mathbb{R}) \times \mathcal{C}_\mu^{2,\alpha}(\mathring{D}_\tau \times \mathbb{R}) \times \mathcal{C}^{2,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})$ into itself. This and the Lipschitz character of this map being evident from the way we have derived *a priori* estimates allows for an application of the Banach fixed point theorem. We leave the details to the reader and simply state this as a result for (3.26).

Lemma 3.4. *For each sufficiently small ε there exists a solution of (3.26) in the form $\varphi = \chi^* \check{\varphi} \circ Y_{\varepsilon,h} + \psi$ such that estimates (3.36)–(3.37) hold.*

3.4. **Proof of Theorem 1.1.** Now we can finish solving the nonlinear problem

$$(3.38) \qquad\qquad L_\varepsilon \varphi = \ell_\varepsilon - N_\varepsilon(w^*) - Q_\varepsilon(\varphi).$$

As we saw above we need to modify this equation by introducing Lagrange multipliers. Thus we will consider:

$$(3.39) \qquad\qquad L_\varepsilon \varphi = \ell_\varepsilon - N_\varepsilon(w^*) - Q_\varepsilon(\varphi) + \chi^* \sum_{j=1}^d \mathsf{c}_j Z_{\tau,\varepsilon}^{T,\mathbf{e}_j}.$$

To solve this problem we use a fixed point argument and the linear theory in Lemma 3.4 above. The first task is to calculate the size of the error of the approximation $\ell_\varepsilon - N_\varepsilon(w^*)$. This is straightforward using the definition of w^* and formula (2.2). We recall here that $h = \varepsilon^2 h_0$, where h_0 is a constant and consequently this last formula simplifies significantly. We can write:

$$\ell_\varepsilon - N_\varepsilon(w^*) = \chi^*[\ell_\varepsilon - N_\varepsilon(w)] + [N(w^*) - \chi^* N(w) - (1-\chi^*)N(\mathbb{H})] \equiv A_1 + A_2,$$

since $\ell_\varepsilon = N_\varepsilon(\mathbb{H})$ in $\mathrm{supp}\,(1-\chi^*)$. Using exponential decay of $w - (\pm 1 + \sigma_\varepsilon)$ when $\mathtt{t} \to \pm\infty$ we get easily:

$$(3.40) \qquad\begin{aligned} \|Y_{\varepsilon,h}^* \widetilde{\chi} A_2\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})} &\le C_0 e^{-c_\mu \varepsilon^{-1/3}}, \\ \|(1-\chi^*)A_2\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})} &\le C_0 e^{-\theta \varepsilon^{-1}}. \end{aligned}$$

To estimate A_1 some standard calculations which we will omit are needed (c.f Section 2.3). As a result we get

$$(3.41) \qquad\begin{aligned} \|[Y_{\varepsilon,h}^* \widetilde{\chi} A_1]^{\|}\|_{\mathcal{E}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})} &\le C_0 \varepsilon^3, \\ \|[Y_{\varepsilon,h}^* \widetilde{\chi} A_1]^{\perp}\|_{\mathcal{C}_\mu^{0,\alpha}(\mathring{D}_\tau \times \mathbb{R})} &\le C_0 \varepsilon^3, \\ \|(1-\chi^*)A_1\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})} &\le C_0 e^{-\theta \varepsilon^{-1}}, \end{aligned}$$

where C_0, c_μ and θ are positive constants. Now we use the linear theory developed in the previous section to solve the nonlinear problem (3.39). Thus we write $\varphi = \chi^* \check{\varphi} \circ Y_{\varepsilon,h} + \psi$, and further decompose $\check{\varphi} = \check{\varphi}^{\|} + \check{\varphi}^{\perp}$ where $\check{\varphi}^{\|} \in \mathcal{X} \cap \mathcal{C}_\mu^{2,\alpha}(\mathring{D}_\tau \times \mathbb{R})$, $\varphi^{\perp} \in \mathcal{Y} \cap \mathcal{C}_\mu^{2,\alpha}(\mathring{D}_\tau \times \mathbb{R})$ and $\psi \in \mathcal{C}^{2,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})$. To set up a fixed point scheme we fix functions $\widetilde{\varphi}^{\|}$, $\widetilde{\varphi}^{\perp}$ and $\widetilde{\psi}$ in these sets such that

$$(3.42) \qquad\begin{aligned} \|\widetilde{\varphi}^{\|}\|_{\mathcal{E}_\mu^{2,\alpha}(\mathring{D}_\tau \times \mathbb{R})} &\le K\varepsilon^{4-\alpha}, \\ \|\widetilde{\varphi}^{\perp}\|_{\mathcal{C}_\mu^{2,\alpha}(\mathring{D}_\tau \times \mathbb{R})} &\le K\varepsilon^2, \\ \|\widetilde{\psi}\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}^{d-1} \times S_{2T_\tau})} &\le K e^{-\bar{\theta}\varepsilon^{-1}}, \end{aligned}$$

where K is a large constant to be chosen and $\bar{\theta} \in (\theta/2, \theta)$ is a constant. Let us denote the right hand side of (3.39) by $\mathfrak{g}$. It is evident that under the assumptions (3.42), and with a suitable choice of the constants $\alpha > 0$ and $\mu \in (0, |\eta|)$ we can solve the problem (3.39) for functions $(\check{\varphi}^{\|}, \check{\varphi}^{\perp}, \psi)$ which again satisfy (3.42). Thus we have a non-linear map

$$(\widetilde{\varphi}^{\|}, \widetilde{\varphi}^{\perp}, \widetilde{\psi}) \longmapsto (\check{\varphi}^{\|}, \check{\varphi}^{\perp}, \psi),$$

of this set into itself. To show that this map is a contraction is straightforward, using the quadratic nature of the nonlinear function $Q(\varphi)$. At the end we have a solution of the problem:

$$(3.43) \qquad \varepsilon \Delta u + \frac{1}{\varepsilon} f(u) = \ell_\varepsilon + \sum_{j=1}^d \chi^* \mathsf{c}_j Z_{\tau,\varepsilon}^{T,\mathbf{e}_j}, \quad \text{in } \mathbb{R}^{d-1} \times S_{2T_\tau}$$

where $Z_{\tau,\varepsilon}^{T,\mathbf{e}_j}$ is the (approximate) element of the kernel of the linear operator $\mathbb{L}_\varepsilon$ associated with translation in the direction of the x_j axis, see (3.3). To show that in fact

$$\mathsf{c}_j = 0, \quad j = 1, \ldots, d,$$

we need:

Lemma 3.5 (Balancing formula). *Let $X = \sum a_j \partial_{x_j}$ be the infinitesimal generator of translations or rotations in $\mathbb{R}^d$. For any $\mathcal{C}^2(\mathbb{R}^d)$ function it holds:*

$$(3.44) \qquad \operatorname{div}\left[\left(\frac{\varepsilon}{2}|\nabla u|^2 - \frac{1}{\varepsilon}F(u)\right)X(u) - \varepsilon X(u)\nabla u\right] = -[\varepsilon\Delta u + \frac{1}{\varepsilon}F'(u)]X(u).$$

We will take $X_j = \partial_{x_j}$ for some $1 \leq j \leq d$ and integrate the balancing formula over the cylinder $\mathcal{C}_R = B_R \times S_{2T_\tau}$. Using (3.43) and Green's theorem we get:

$$\underbrace{\int_{\partial\mathcal{C}_R}\left(\frac{\varepsilon}{2}|\nabla u|^2 - \frac{1}{\varepsilon}F(u) + \ell_\varepsilon u\right)n_j\,dS}_{I_R} - \underbrace{\int_{\partial\mathcal{C}_R}\partial_{x_j}u\partial_n u\,dS}_{II_R} = -\underbrace{\int_{\mathcal{C}_R}\left(\sum_{j'=1}^{d}\chi^*\mathsf{c}_{j'}Z_{\tau,\varepsilon}^{T,\mathbf{e}_{j'}}\right)\partial_{x_j}u}_{III_R}\,.$$

The first integral I_R is 0 on the top and the bottom of $\mathcal{C}_R$ and on the other hand, using the asymptotic behavior of the solution we get finally $\lim_{R\to\infty}I_R = 0$. In the second integral the integrals over the top and the bottom of $\mathcal{C}_R$ cancel because u is periodic. Then, from exponential decay of the derivatives of u we get $\lim_{R\to\infty}II_R = 0$. Finally, we note that

$$\partial_{x_j}u \approx Z_{\tau,\varepsilon}^{T,\mathbf{e}_j},$$

hence

$$III_R = \mathsf{c}_j\int_{\mathring{D}_\tau\times\mathbb{R}}|Z_{\tau,\varepsilon}^{T,\mathbf{e}_j}|^2 + o(1)\sum_{j'=1}^{d}\mathsf{c}_{j'},$$

from which we get immediately $\mathsf{c}_j = 0$, $j = 1, \ldots, d$.

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